

TRIANGULATING POLYGONS

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Universitat Politècnica de Catalunya

Based on slides by Vera Sacristán

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An **internal diagonal** is any segment...

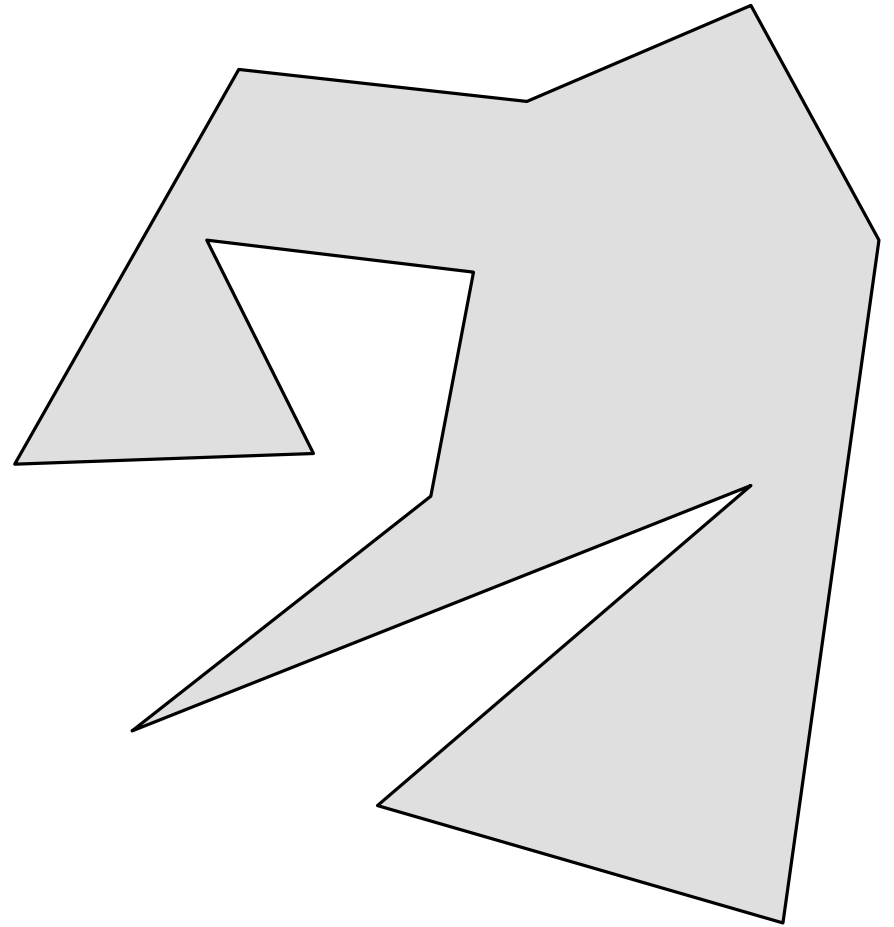
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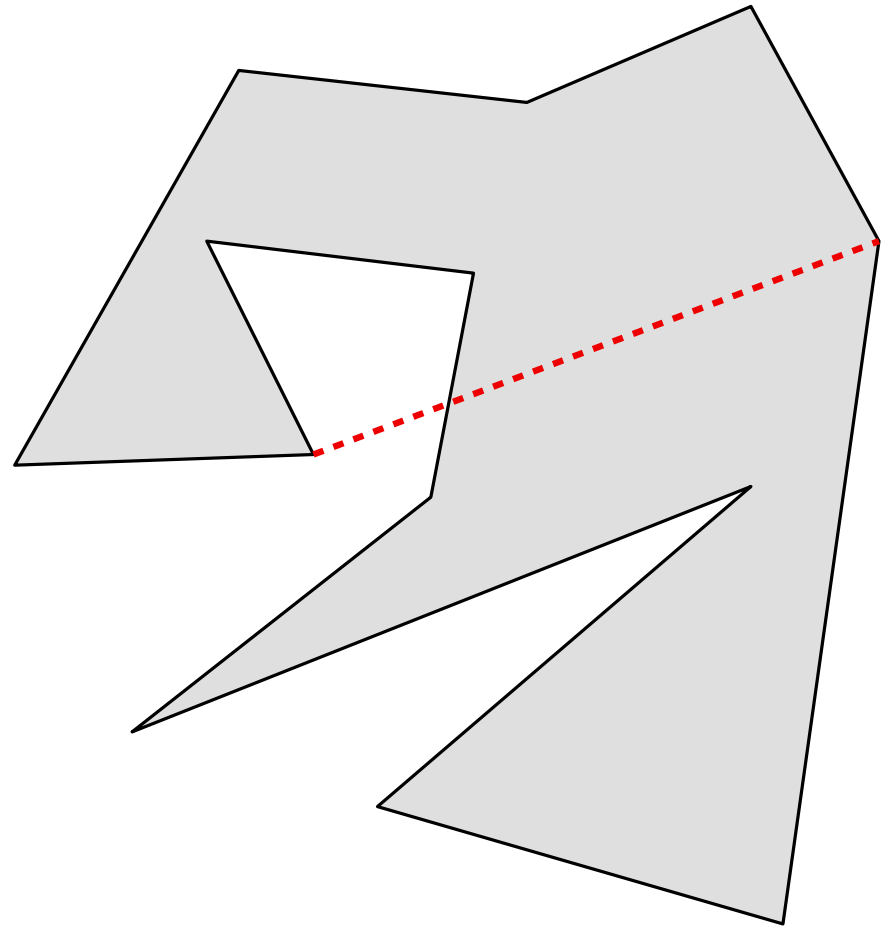


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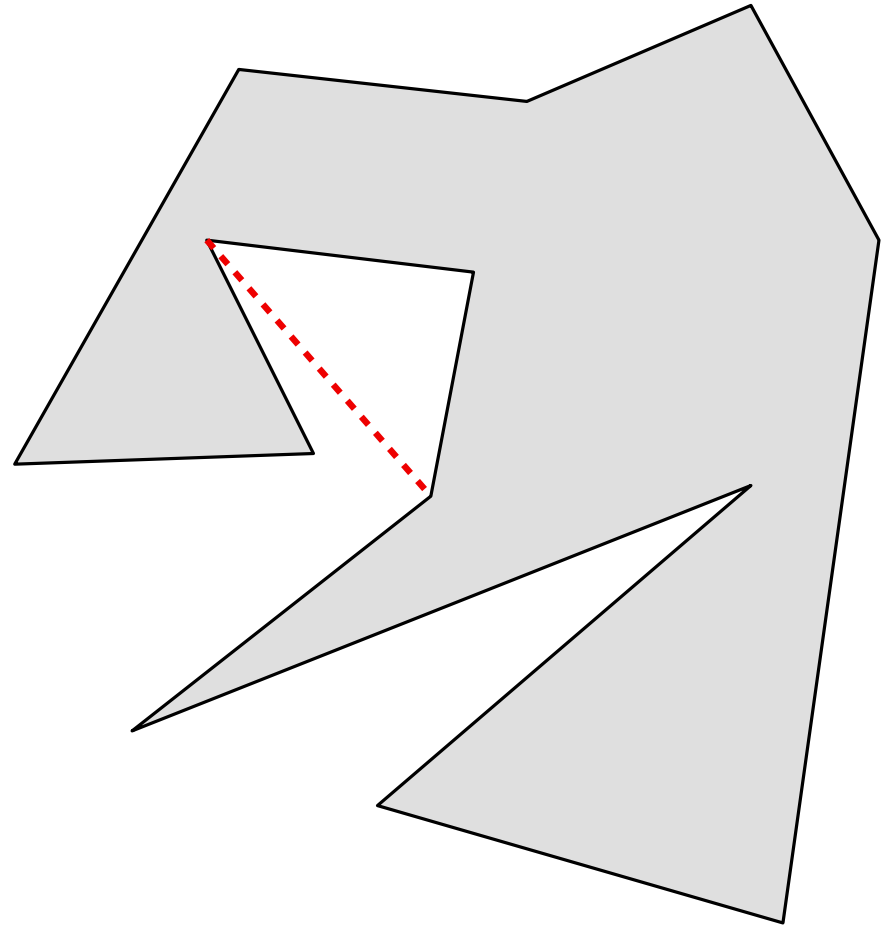


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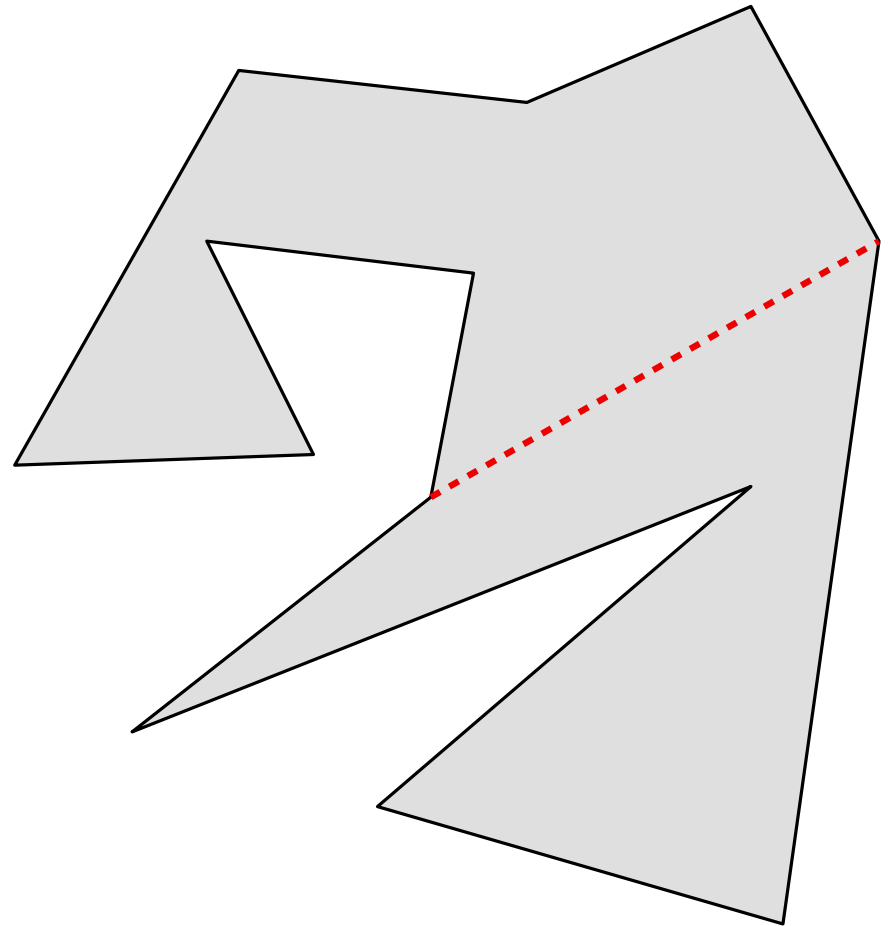


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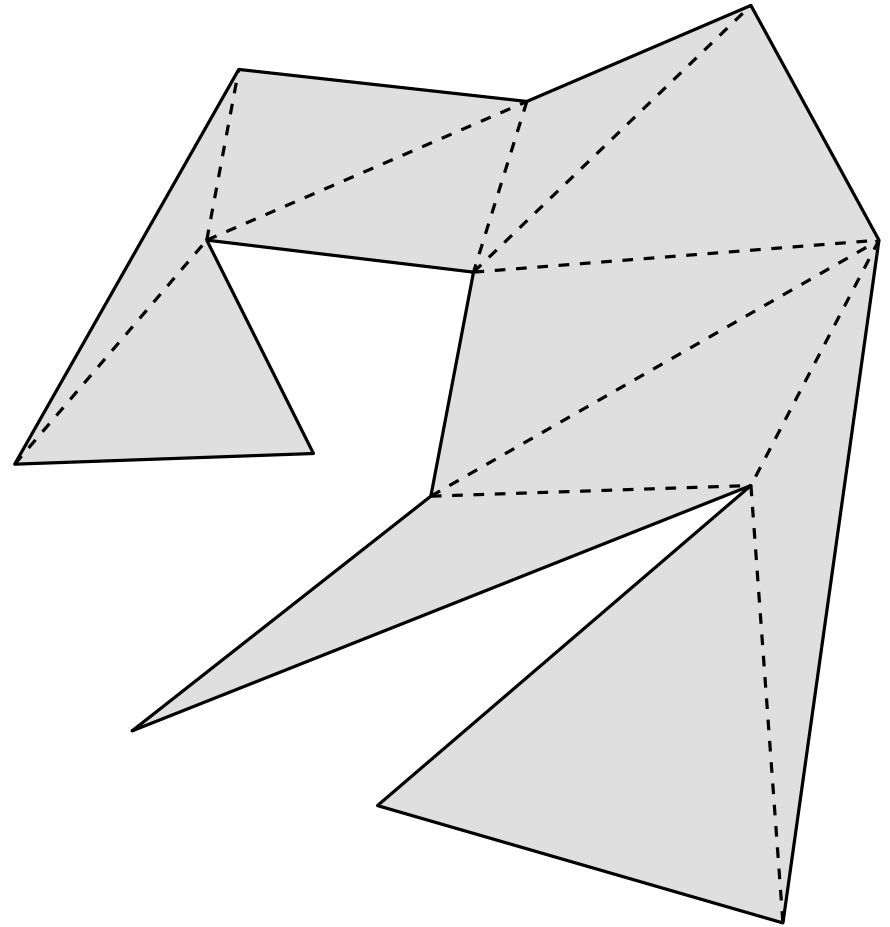


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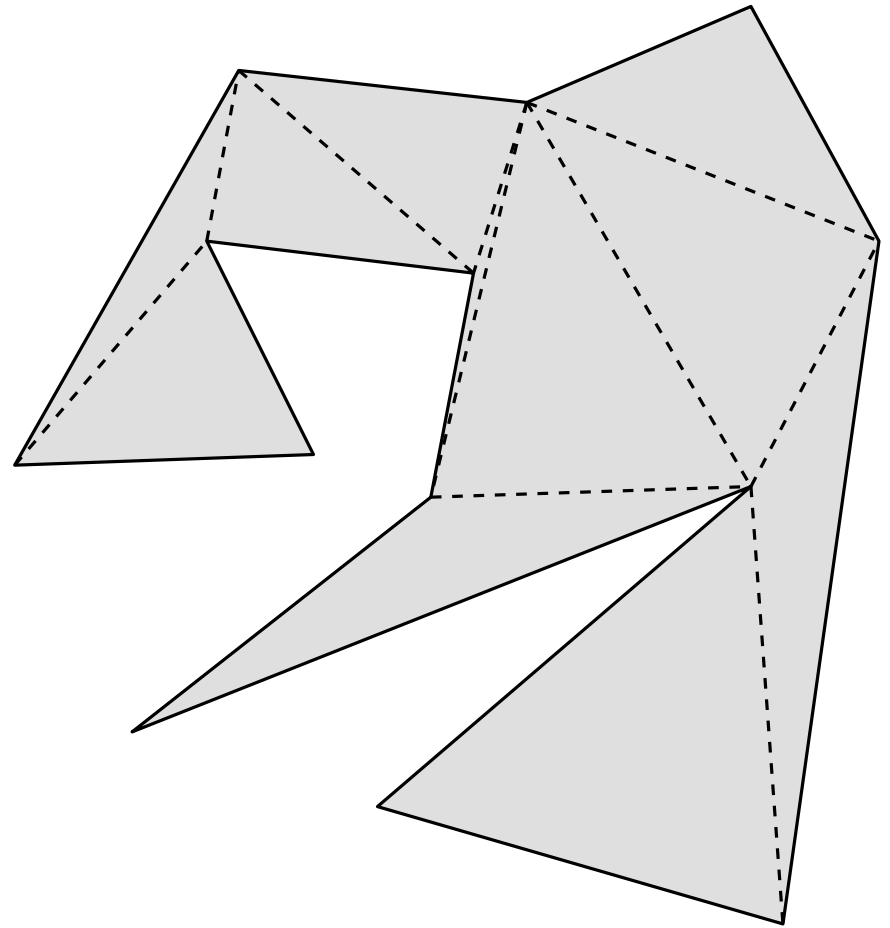


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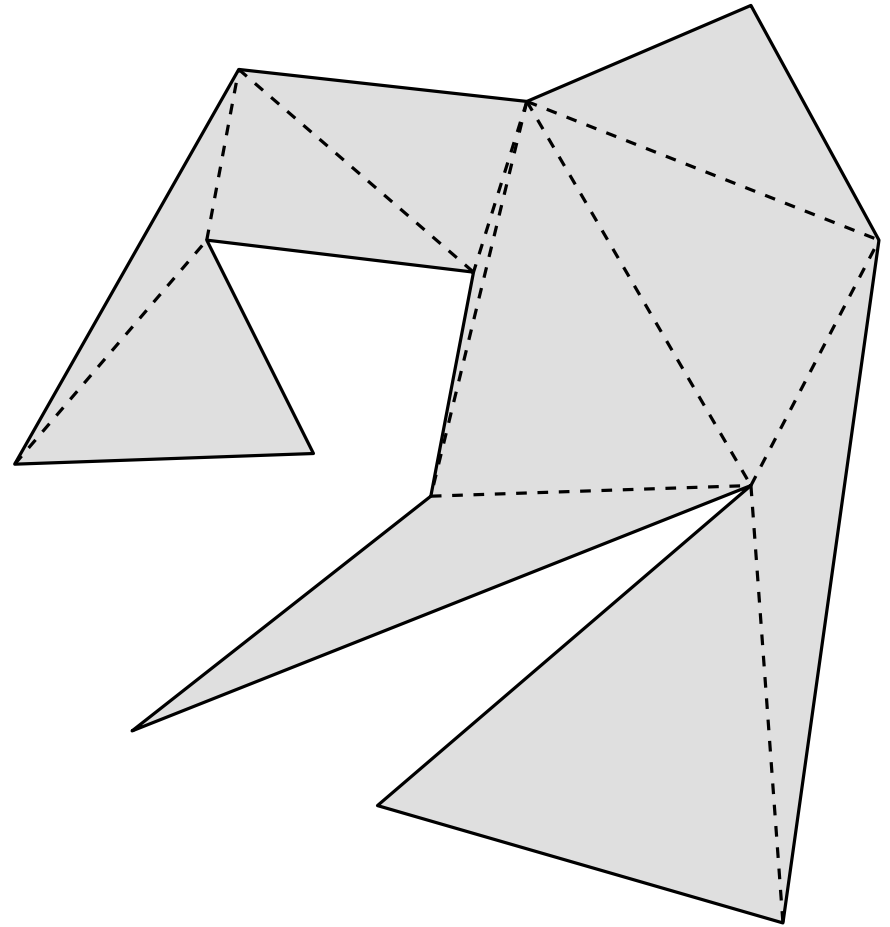
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Application example: area computation



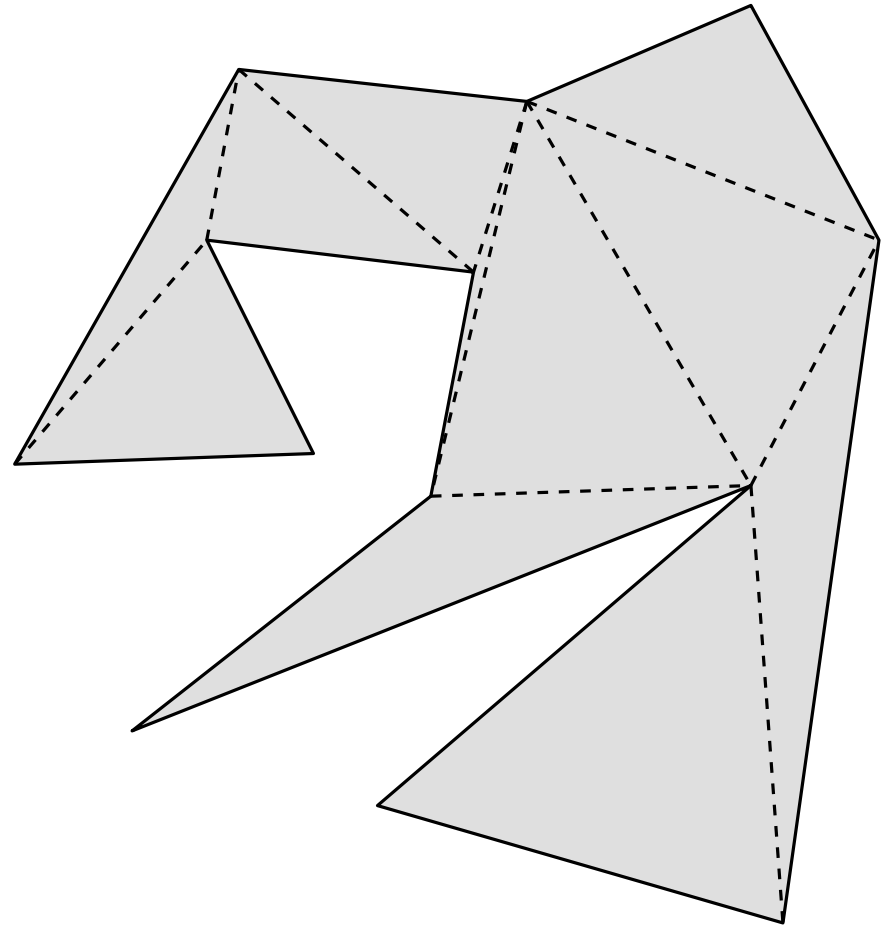
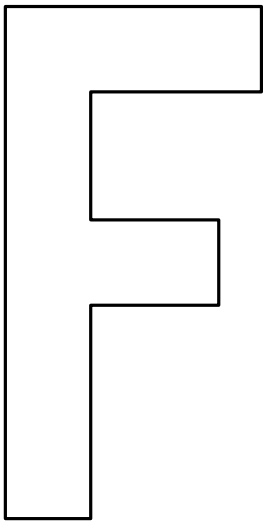
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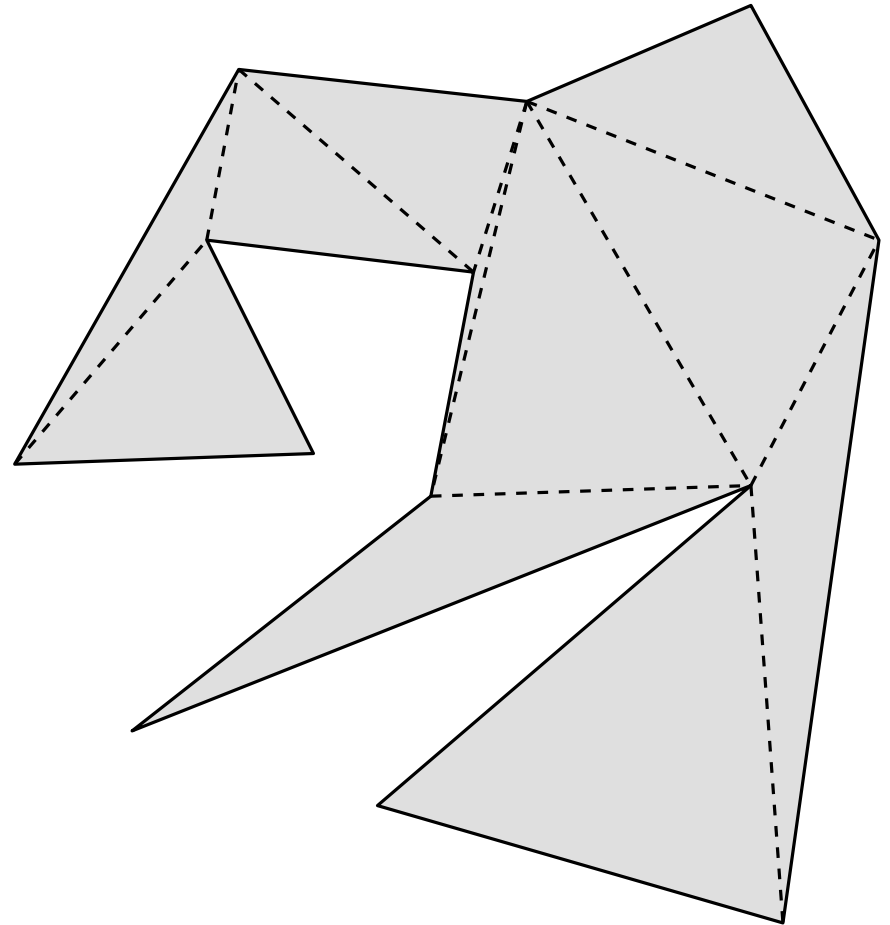
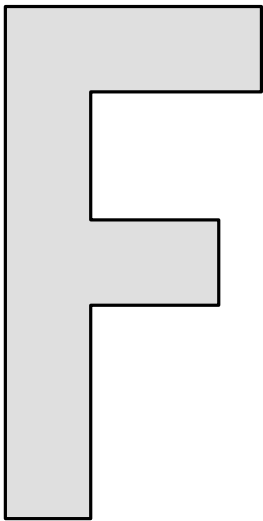
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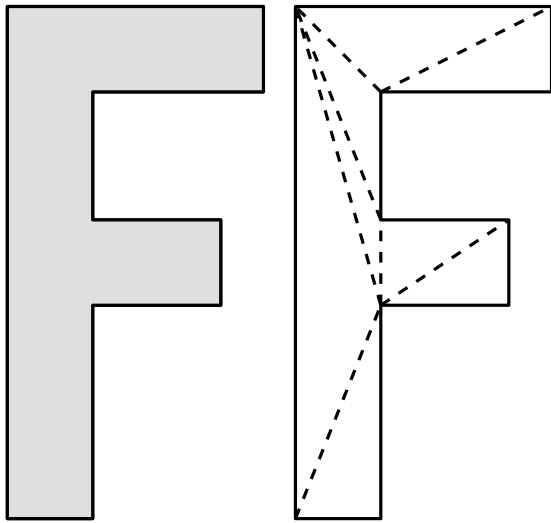
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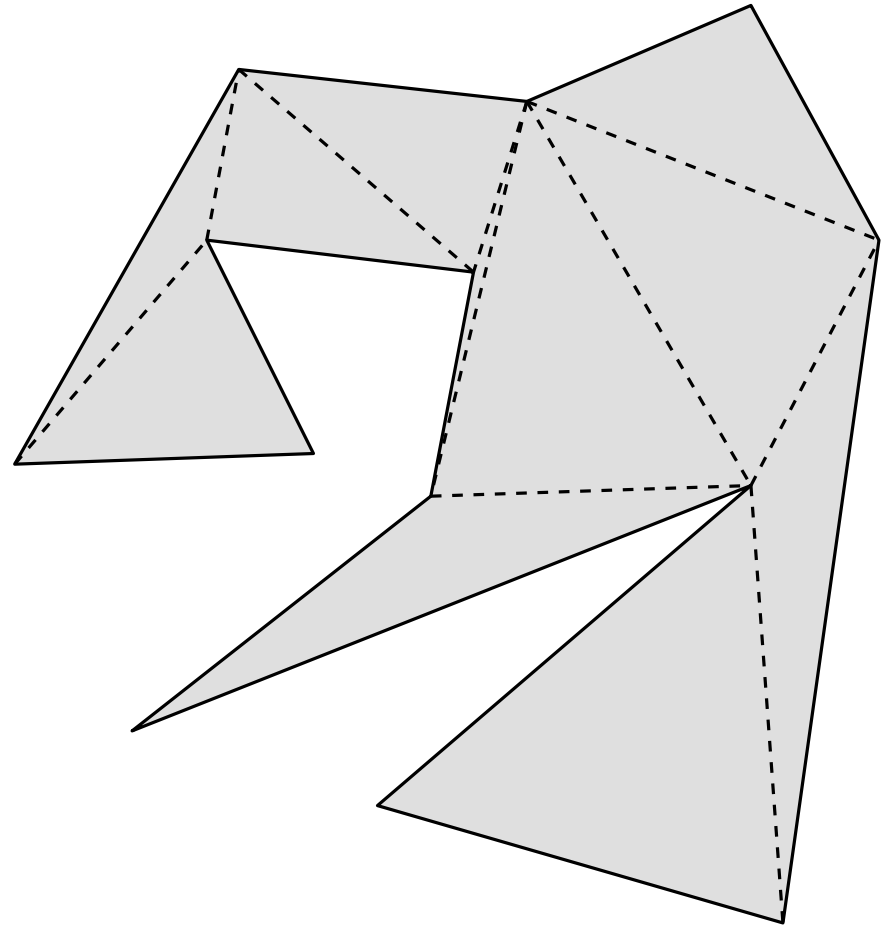
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Trivial if triangulated



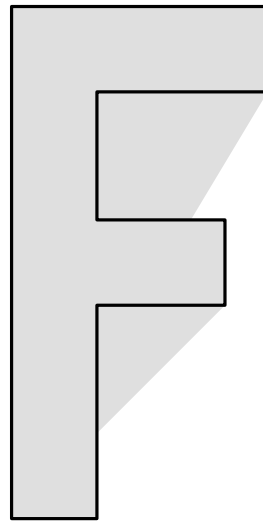
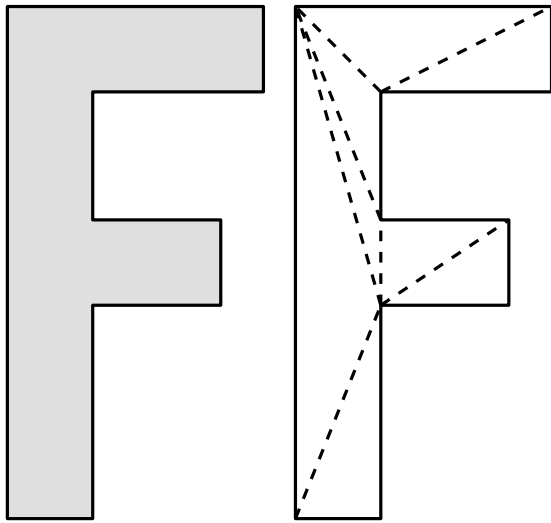
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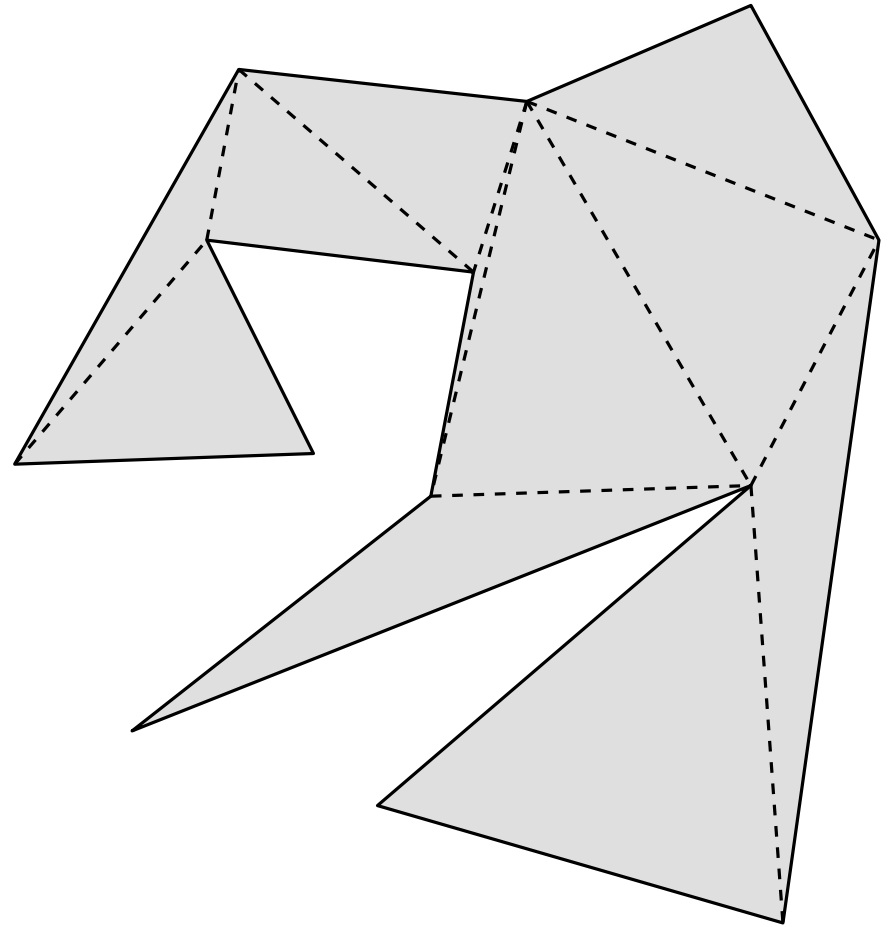
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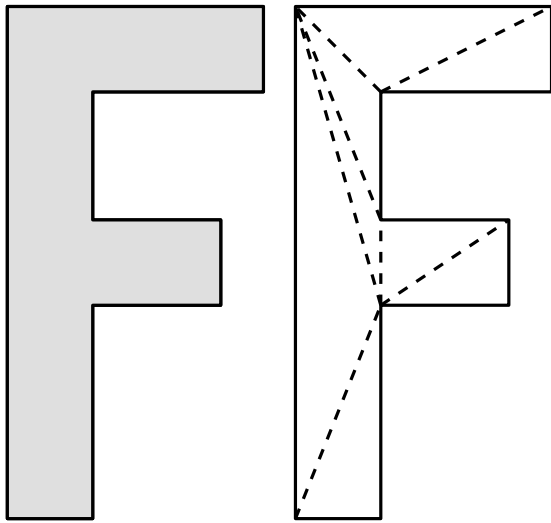
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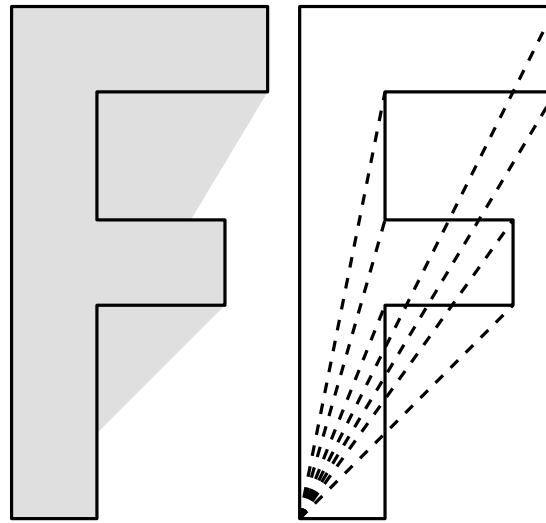
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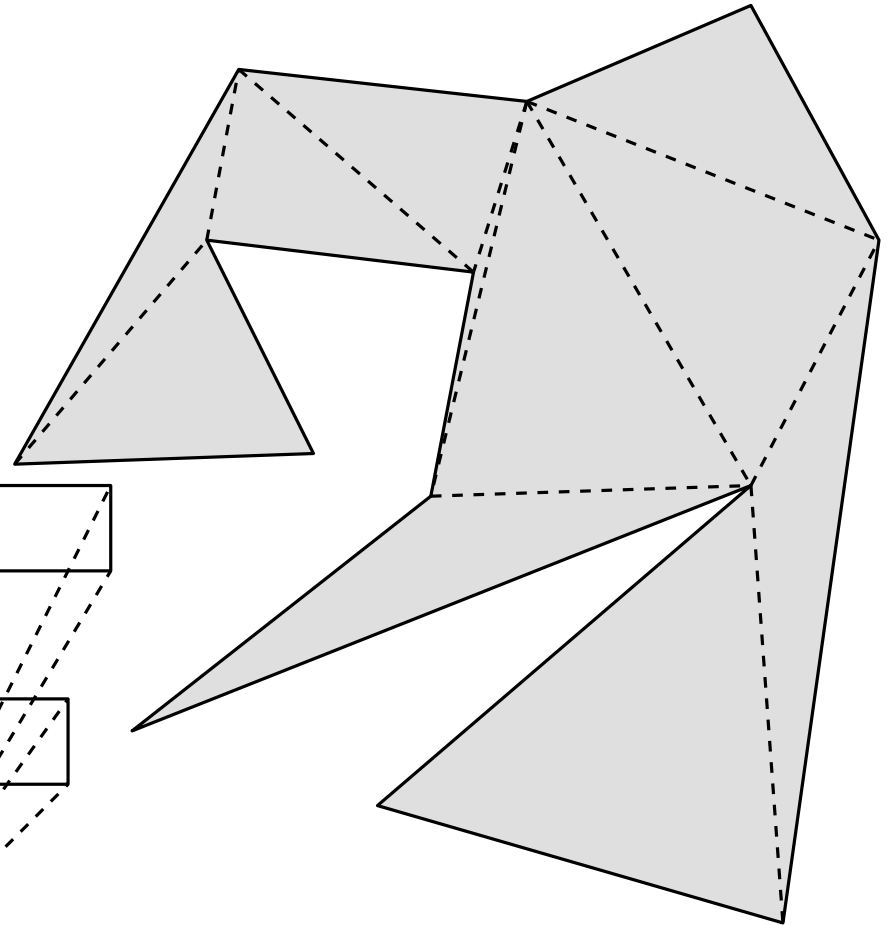
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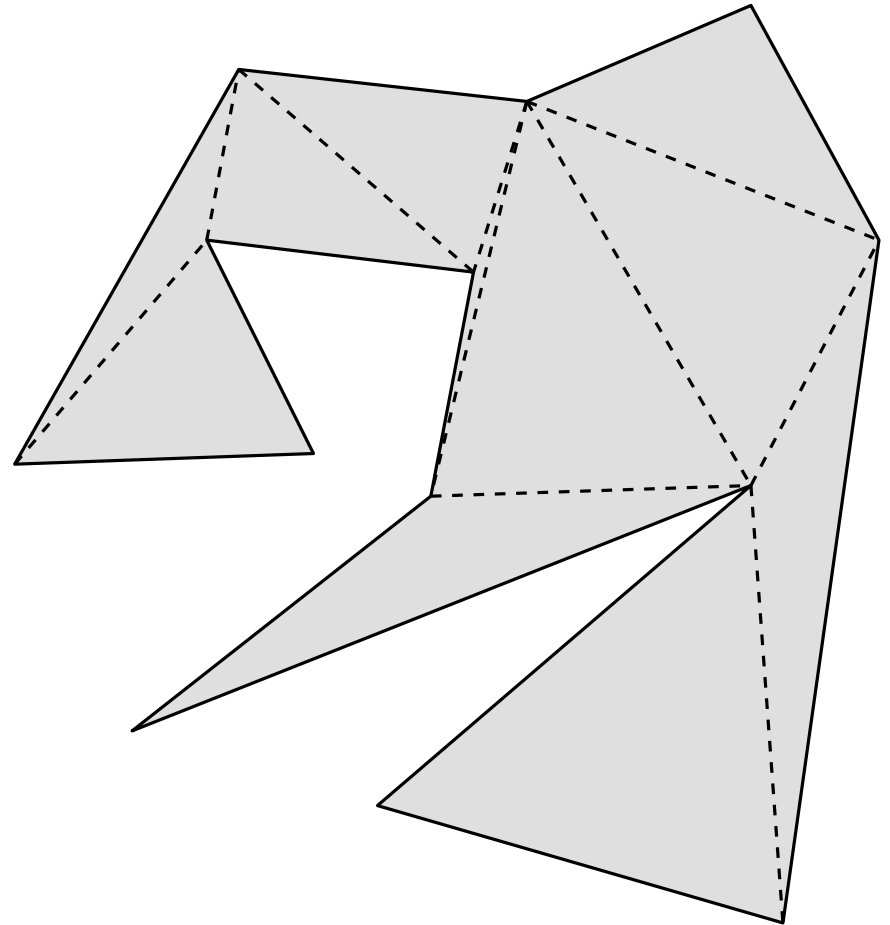
Doesn't work for *any* triangles



TRIANGULATING POLYGONS

Lecture outline

1. Every polygon can be triangulated
2. Properties of polygon triangulations
3. Algorithms
 - (a) Triangulating by inserting diagonals
 - (b) Triangulating by subtracting ears
 - (c) Triangulating convex polygons
 - (d) Triangulating monotone polygons
 - (e) Monotone partitioning



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Every polygon admits a triangulation

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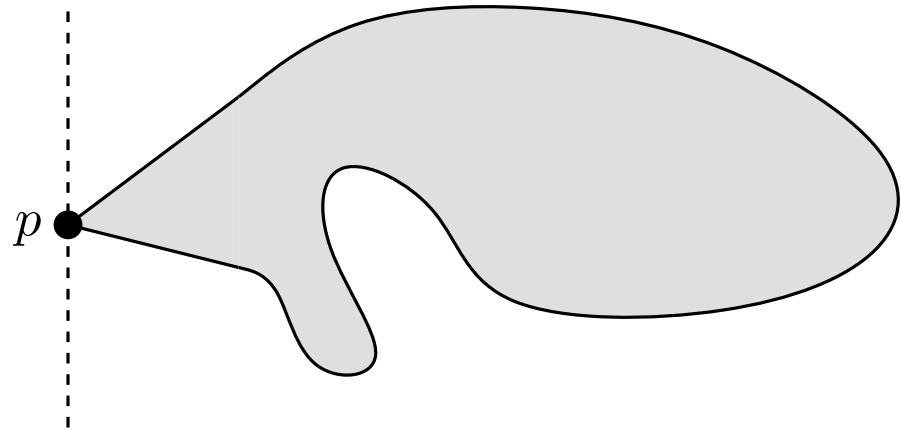
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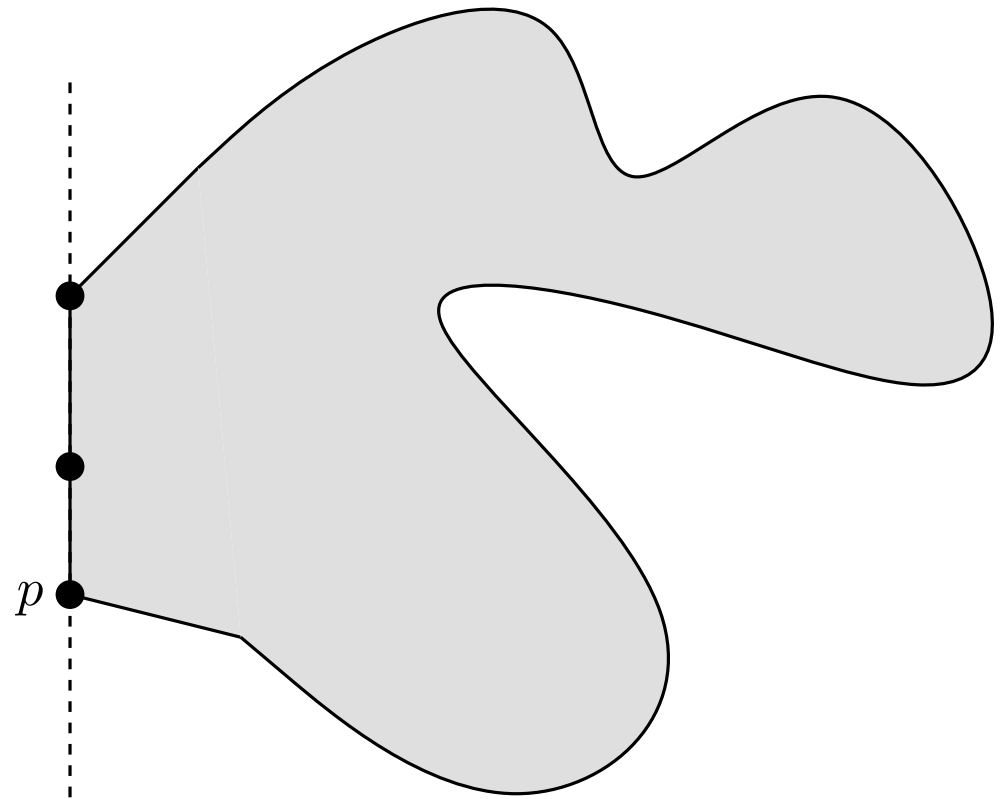


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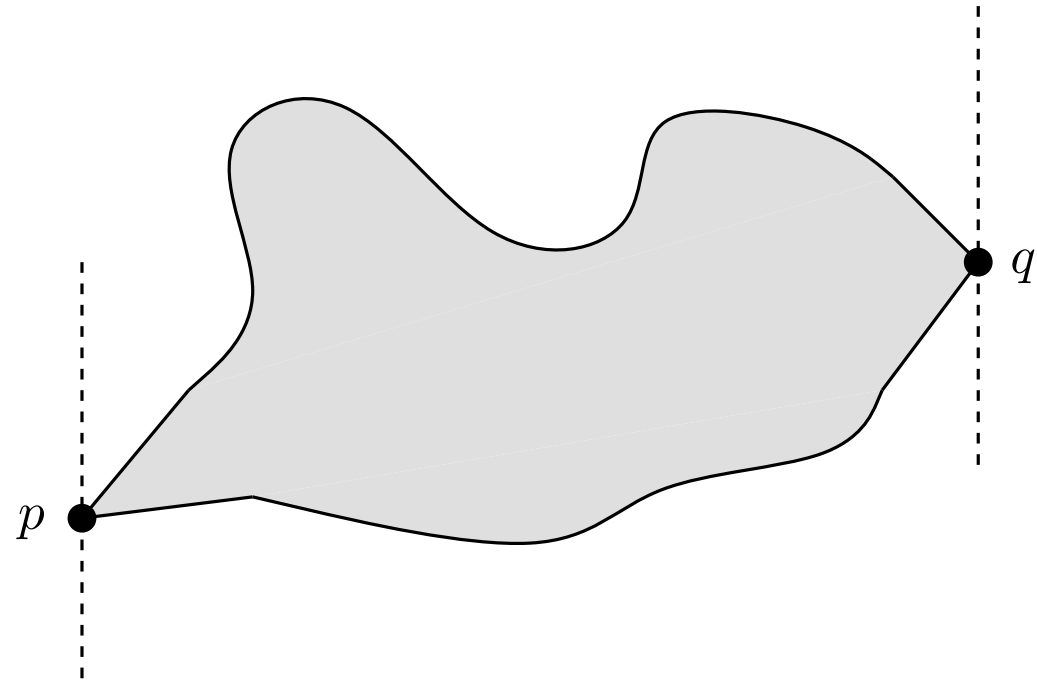
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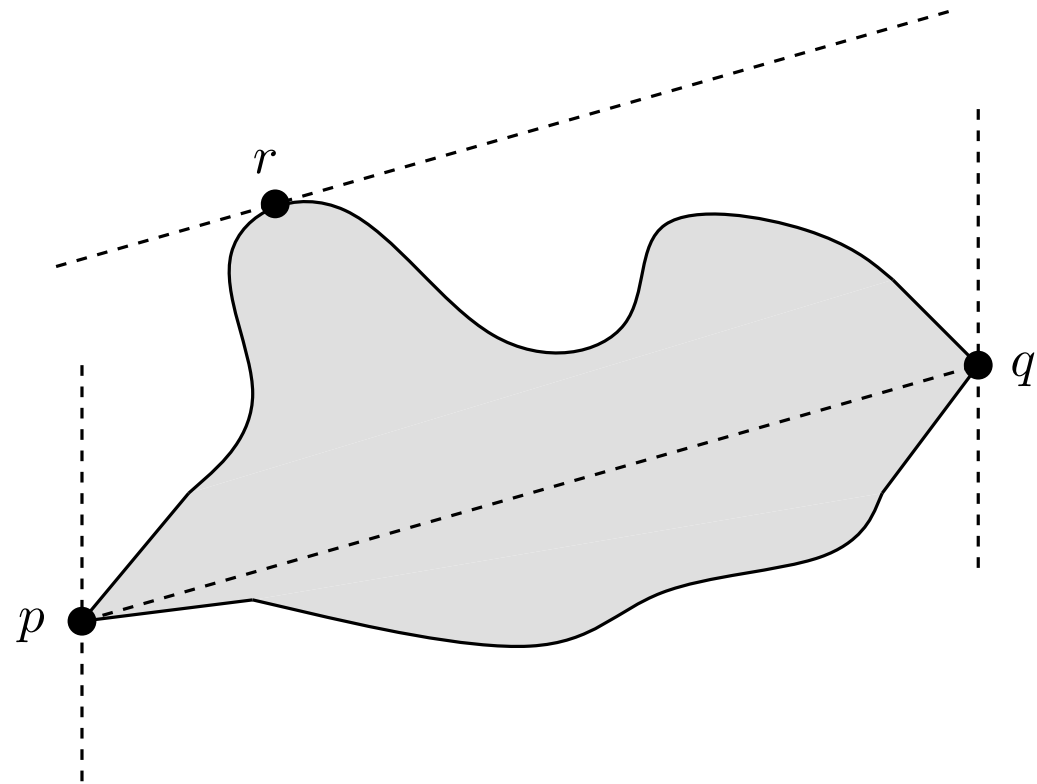
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Finally, there is at least one vertex which is extreme in the direction orthogonal to pq and does not coincide with any of the above. This third vertex r is necessarily convex.



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Lemma 2. Every n -gon with $n \geq 4$ has at least one internal diagonal.

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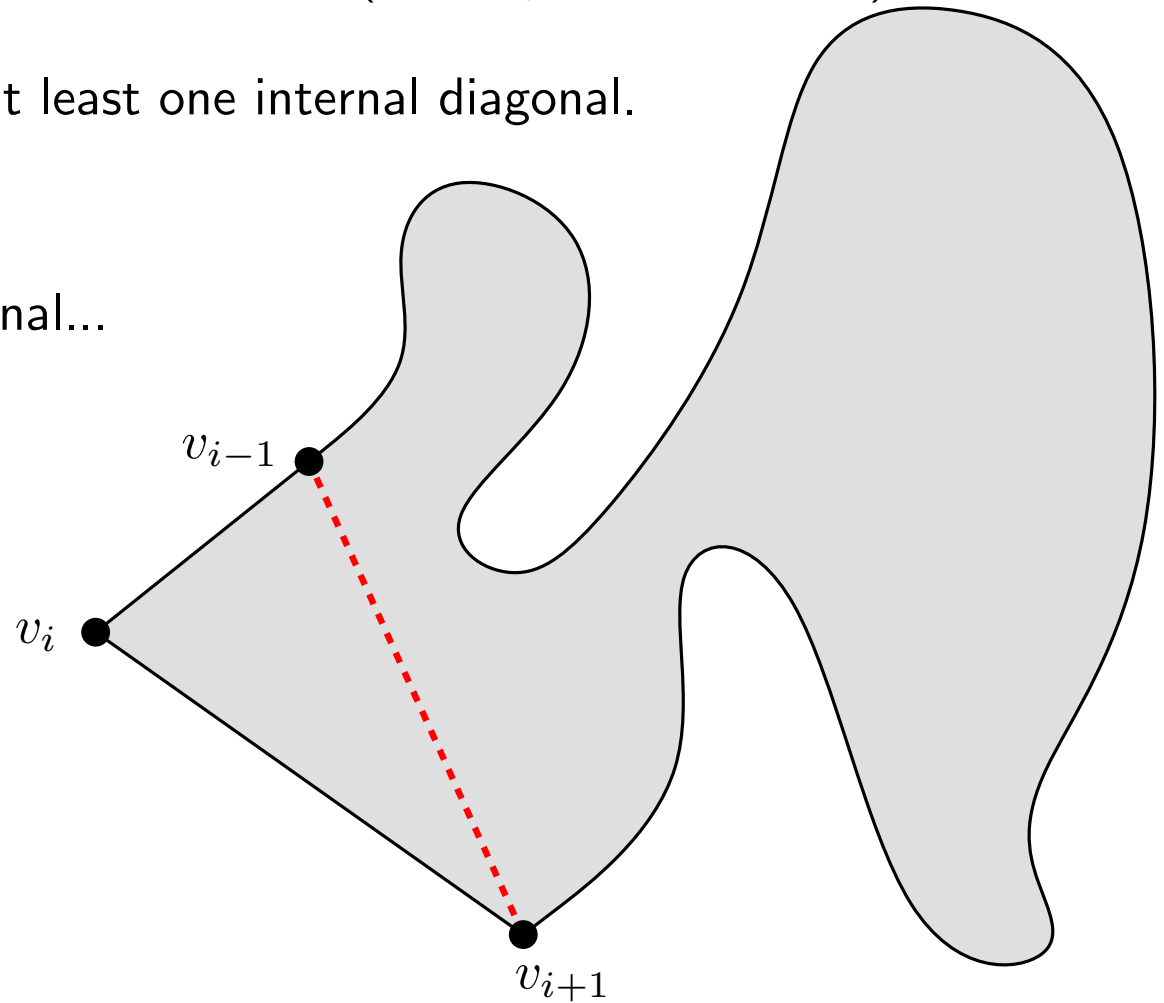
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Then, either $v_{i-1}v_{i+1}$ is an internal diagonal...



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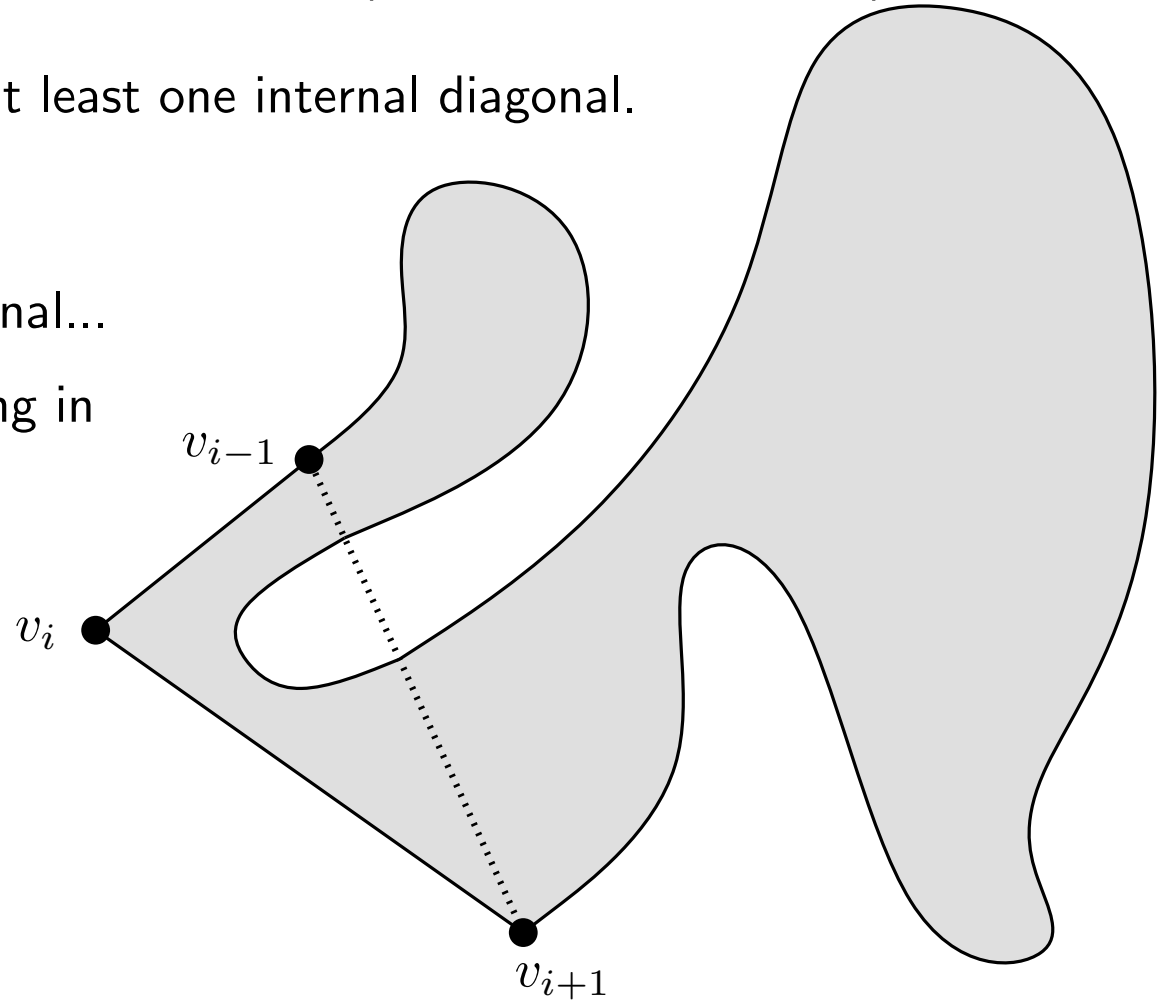
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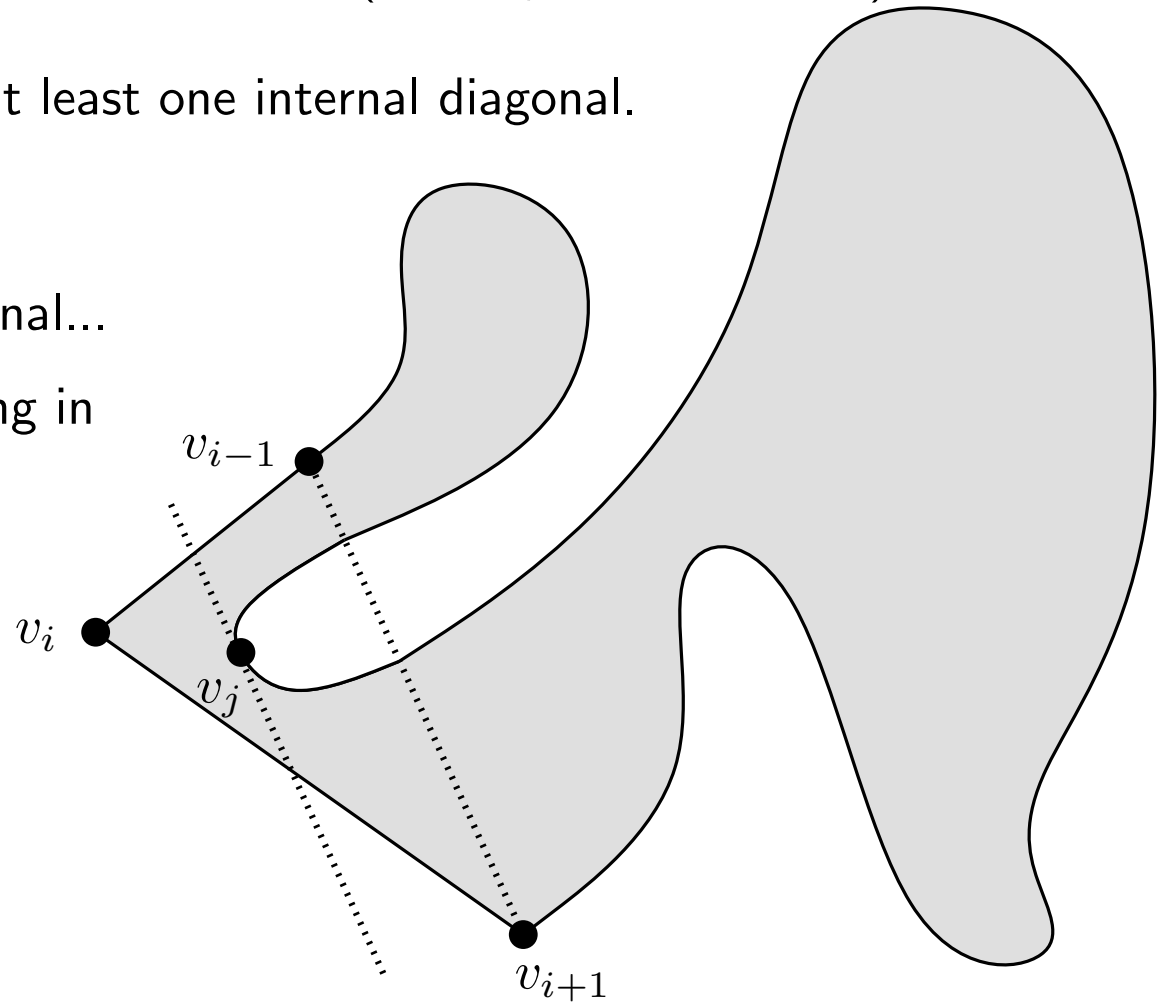
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In this case, among all the vertices lying in the triangle, let v_j be the farthest one from the segment $v_{i-1}v_{i+1}$. Then v_iv_j is an internal diagonal (it cannot be intersected by any edge of the polygon).



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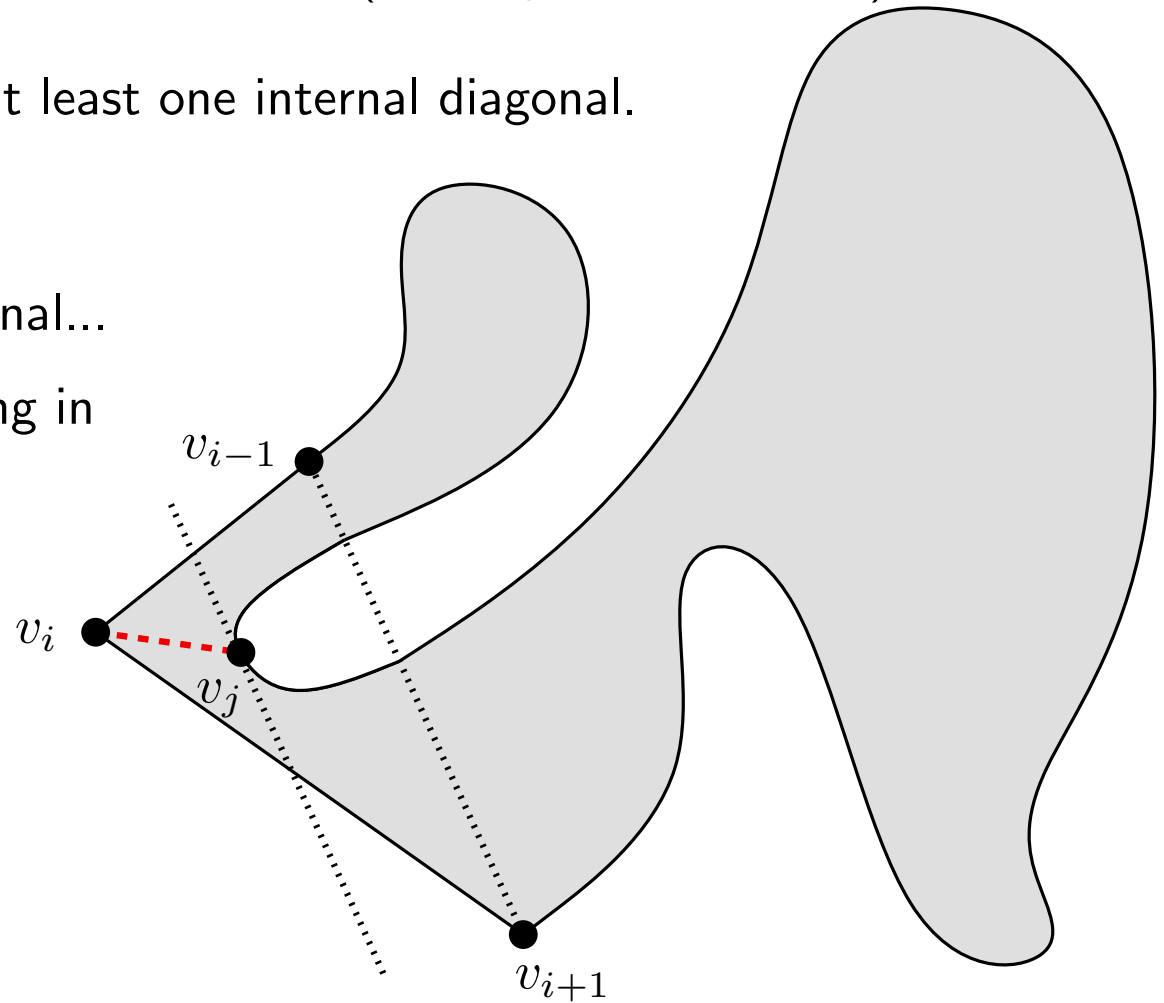
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Corollary. Every polygon can be triangulated. (By induction.)

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Properties of the triangulations of polygons

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Let P be a simple polygon with n vertices (a.k.a. n -gon).

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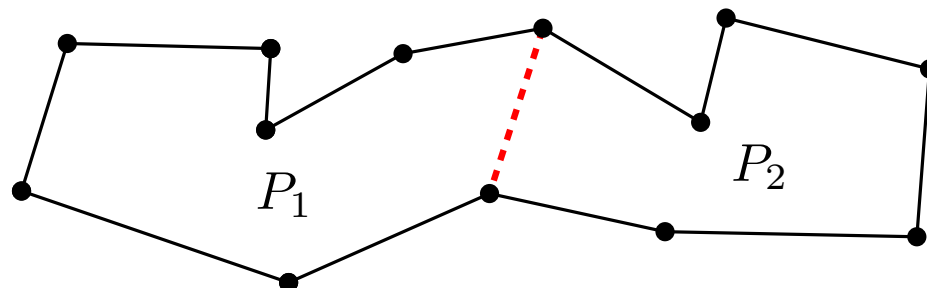
Proof by induction.

Base case: When $n = 3$, the number of diagonals is $d = 0 = n - 3$.

Inductive step: Consider a diagonal of a triangulation T of P , decomposing P into two subpolygons: a $(k + 1)$ -gon P_1 and an $(n - k + 1)$ -gon P_2 . By inductive hypothesis, the number of diagonals of the triangulations induced by T in P_1 and P_2 are:

$$\begin{aligned}d_1 &= k + 1 - 3, \\d_2 &= n - k + 1 - 3,\end{aligned}$$

therefore, $d = d_1 + d_2 + 1 = k + 1 - 3 + n - k + 1 - 3 + 1 = n - 3$.



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Again, the proof is by induction.

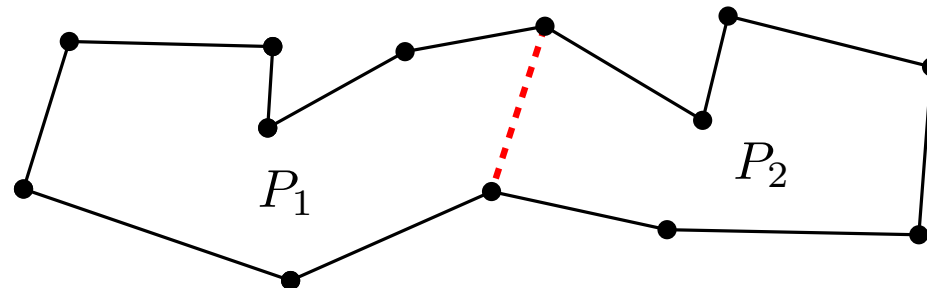
Base case: When $n = 3$, the number of triangles is $t = 1 = n - 2$.

Inductive step: With the same conditions of the previous proof,

$$\begin{aligned}t_1 &= k + 1 - 2, \\t_2 &= n - k + 1 - 2,\end{aligned}$$

hence,

$$t = t_1 + t_2 = k + 1 - 2 + n - k + 1 - 2 = n - 2.$$



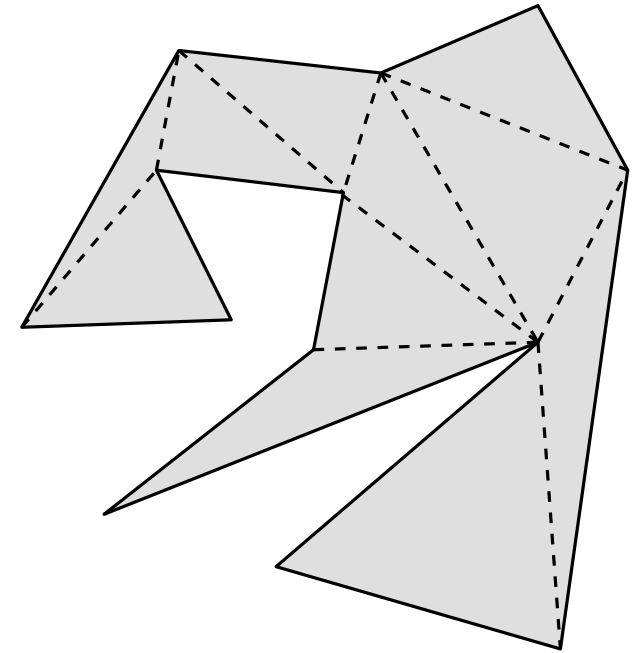
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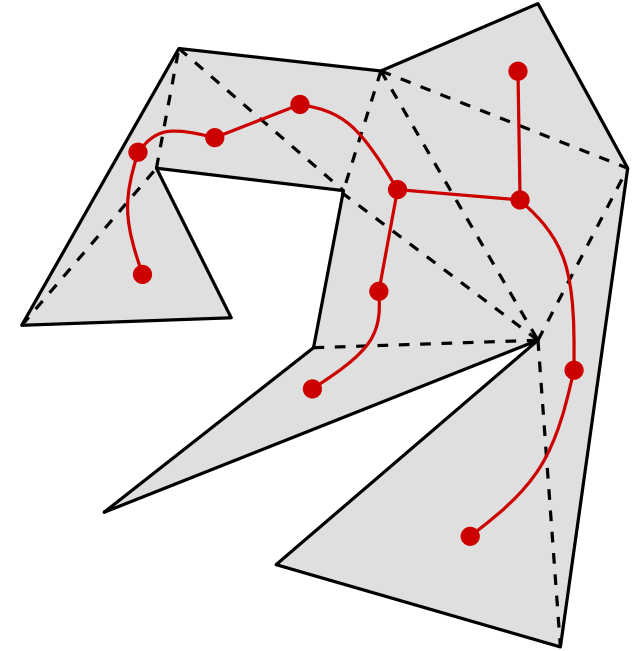
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Given a triangulation of P , its **dual graph** has one vertex for each triangle, and one edge connecting two vertices whenever their corresponding triangles are adjacent.

Property 3. The dual graph of any triangulation of P is a tree.



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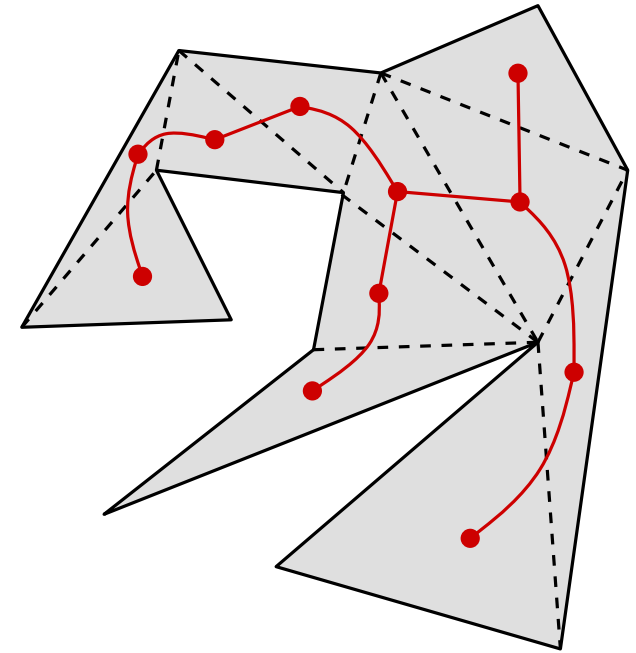
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We want to prove that this graph is a tree, that is, connected and acyclic. The graph is trivially connected.

About the acyclicity: Notice that each edge of the dual graph “separates” the two endpoints of the internal diagonal of P shared by the two adjacent triangles. If the graph had a cycle, it would enclose the endpoint(s) of the diagonals intersected by the cycle and, therefore, it would enclose points belonging to the boundary of the polygon, contradicting the hypothesis that P is simple (and thus, without holes).



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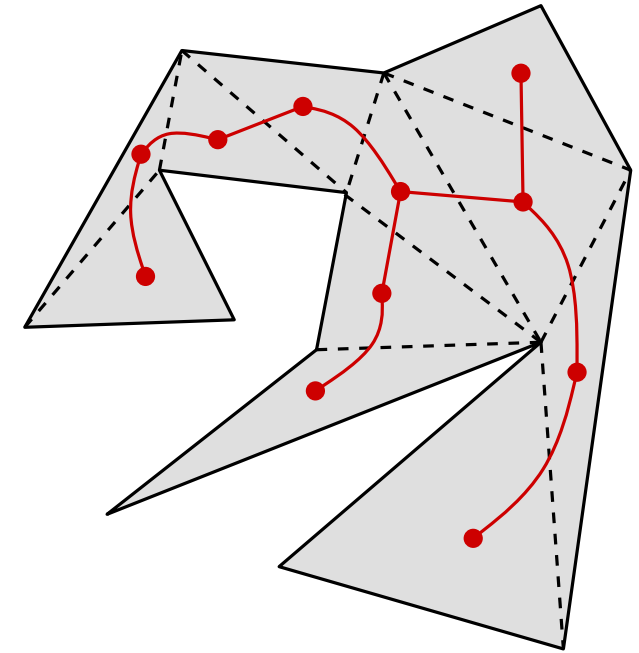
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Corollary. Every n -gon with $n \geq 4$ has at least two non-adjacent **ears**.



Ear: convex vertex such that its two neighboring vertices form an internal diagonal

ALGORITHMS FOR POLYGON TRIANGULATION

TRIANGULATING POLYGONS

Tringulating a polygon by subtracting ears

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Input: $v_1, \dots, v_n$, sorted list of the vertices of a simple polygon P .

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Procedure

1. Sequentially explore the vertices until you find an ear
2. Crop it
3. Recurse on remaining polygon

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Finding an ear: $O(n^2)$.

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Overall running time:

$$T(n) = O(n^2) + O((n-1)^2) + O((n-2)^2) + \dots + O(1) = O(n^3).$$

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Improved procedure

Initialization

1. Detect all convex vertices
2. Detect all ears

Next step

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2. Update the information of the convex vertices
3. Update the information of the ears

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 $O(n^2)$ Only once

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Test. How to decide whether a given segment $v_i v_j$ is an internal diagonal?

TRIANGULATING POLYGONS

Tringulating a polygon by inserting diagonals

Input: $v_1, \dots, v_n$, sorted list of the vertices of a simple polygon P .

Output: List of internal diagonals of P , $v_i v_j$, determining a triangulation of P .

Procedure:

1. Find an internal diagonal
2. Decompose the polygon into two subpolygons
3. Proceed recursively

Test. How to decide whether a given segment $v_i v_j$ is an internal diagonal?

Is it a diagonal?

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Check $v_i v_j$ against all segments $v_k v_{k+1}$ for intersection.

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Is it internal?

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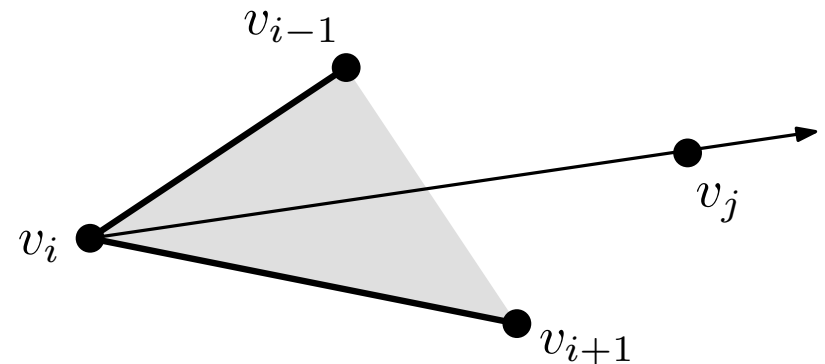
Test. How to decide whether a given segment $v_i v_j$ is an internal diagonal?

Is it a diagonal?

Check $v_i v_j$ against all segments $v_k v_{k+1}$ for intersection.

Is it internal?

Apply orientation test. For instance, if P in counter clockwise order, the oriented line $\overrightarrow{v_i v_j}$ should leave v_{i-1} to its left and v_{i+1} to its right.



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Test. How to decide whether a given segment $v_i v_j$ is an internal diagonal? $O(n)$

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Brute-force solution:

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Brute-force solution:

Apply the test to each candidate segment.

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Brute-force solution: $O(n^3)$

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Testing each candidate takes $O(n)$ time, and there are $\binom{n}{2}$ of them.

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Apply the test to each candidate segment.

Applying previous results:

1. Find a convex vertex, v_i .
2. Detect whether $v_{i-1} v_{i+1}$ is an internal diagonal.
3. If so, report it.

Else, find the farthest v_k from the segment $v_{i-1} v_{i+1}$, lying in the triangle $v_{i-1} v_i v_{i+1}$.

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Search. How to find an internal diagonal? $O(n)$

Partition. How to partition the polygon into two subpolygons?

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From the diagonal found, create the sorted list of the vertices of the two subpolygons.

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Total running time of the algorithm: $O(n^2)$

It finds $n - 3$ diagonals and each one is found in $O(n)$ time.

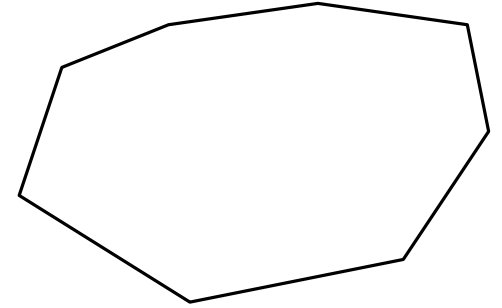
TRIANGULATING POLYGONS

Is it possible to triangulate a polygon more efficiently?

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Triangulating a convex polygon

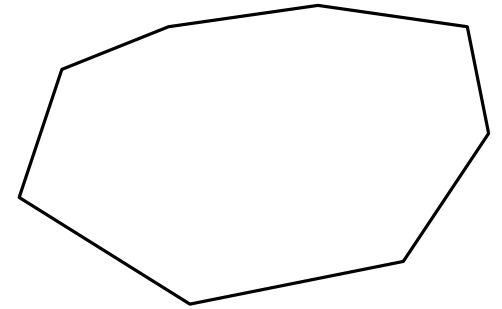


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Trivially done in $O(n)$ time.

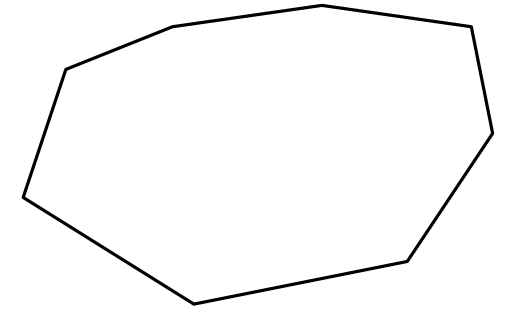
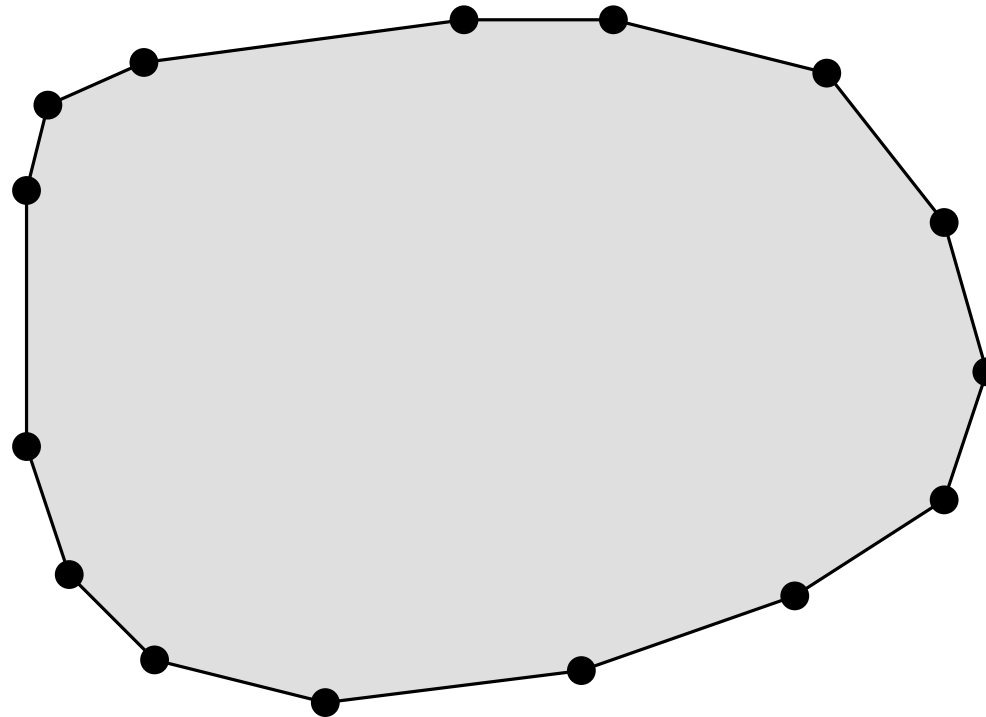


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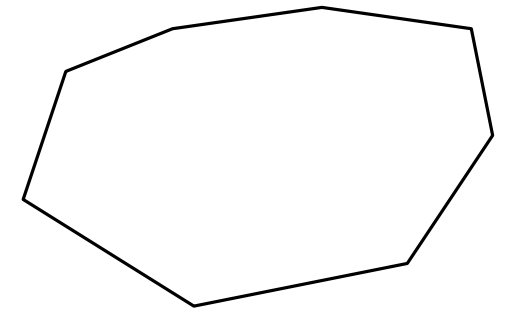
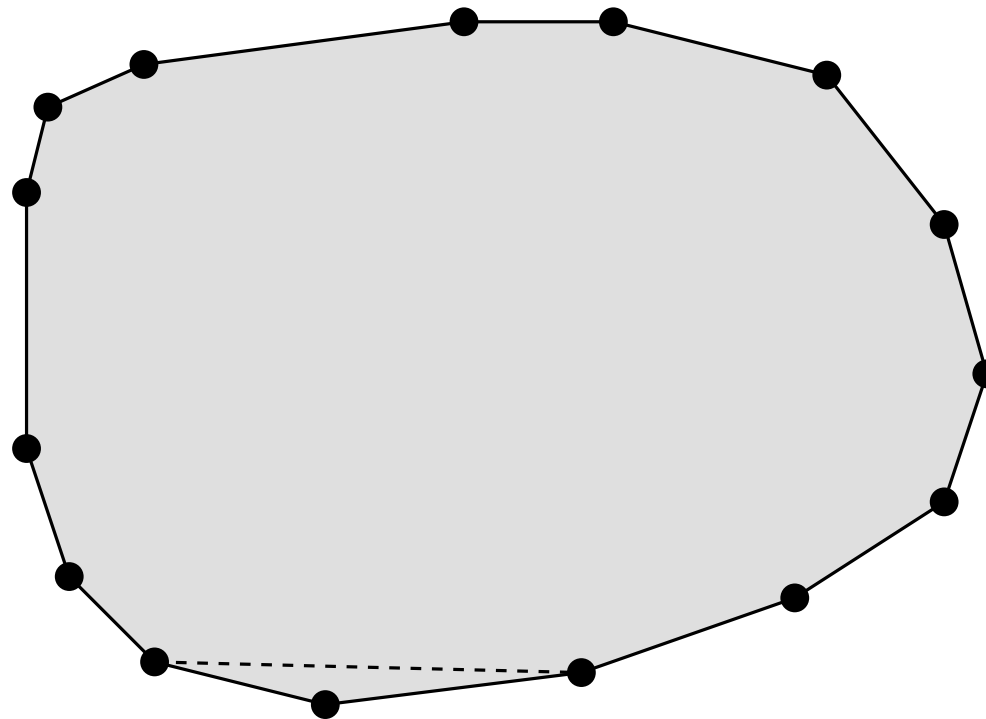


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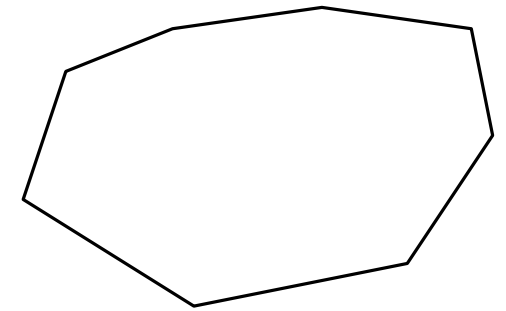
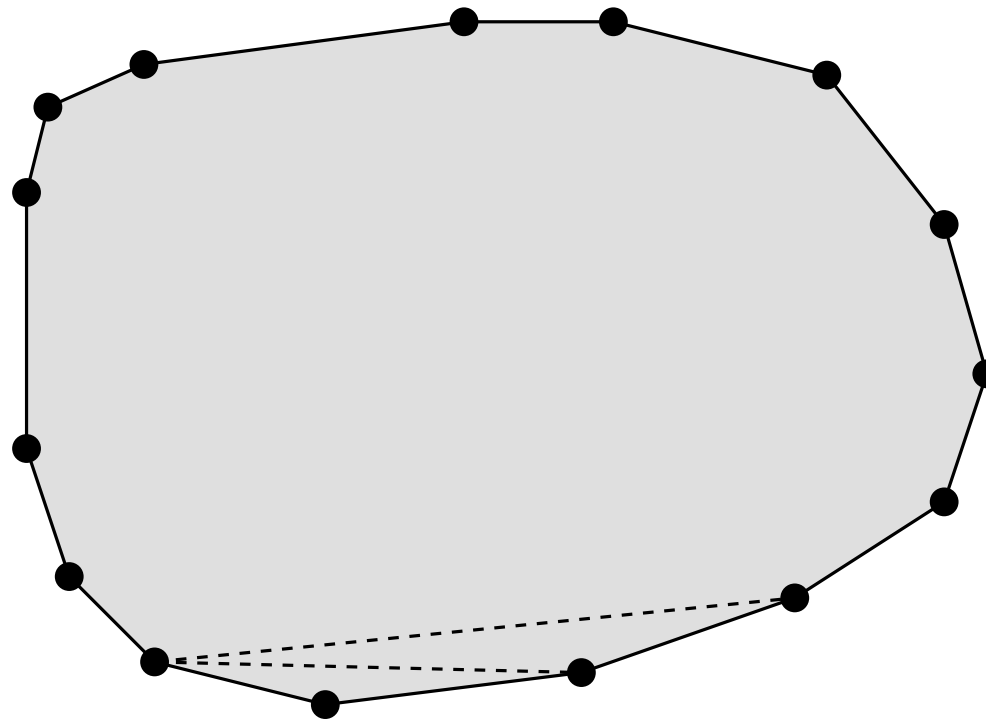


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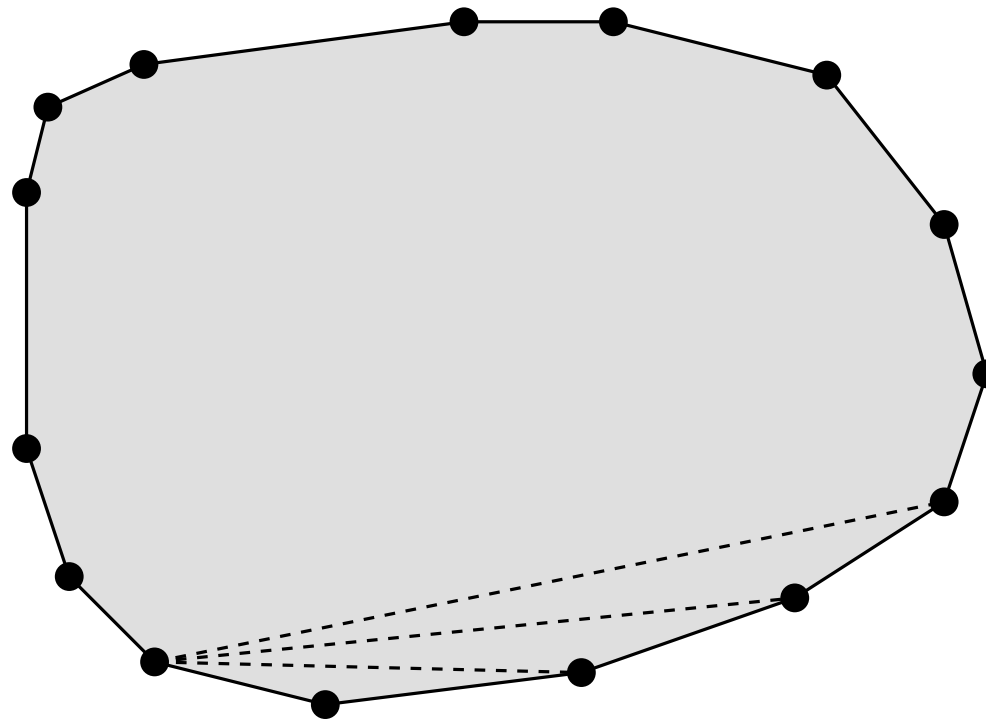


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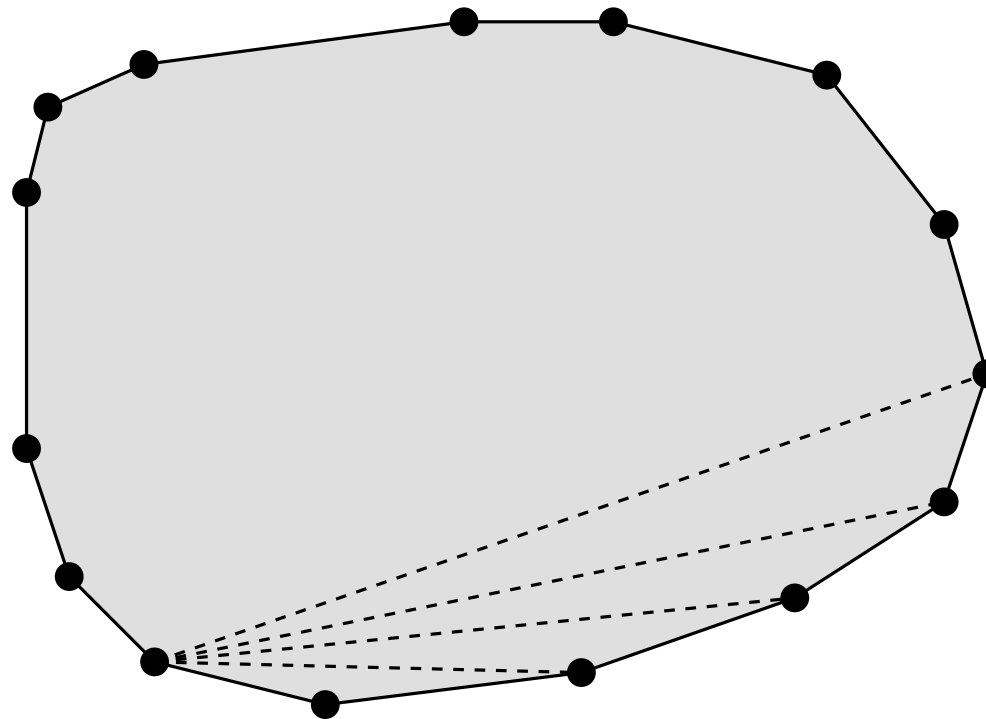


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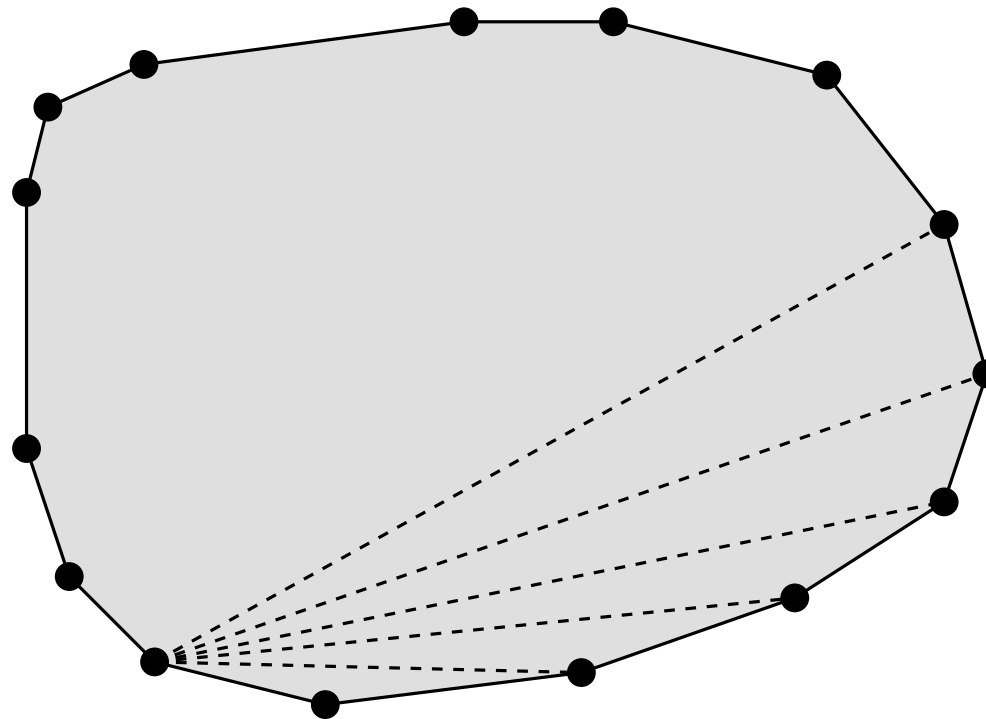


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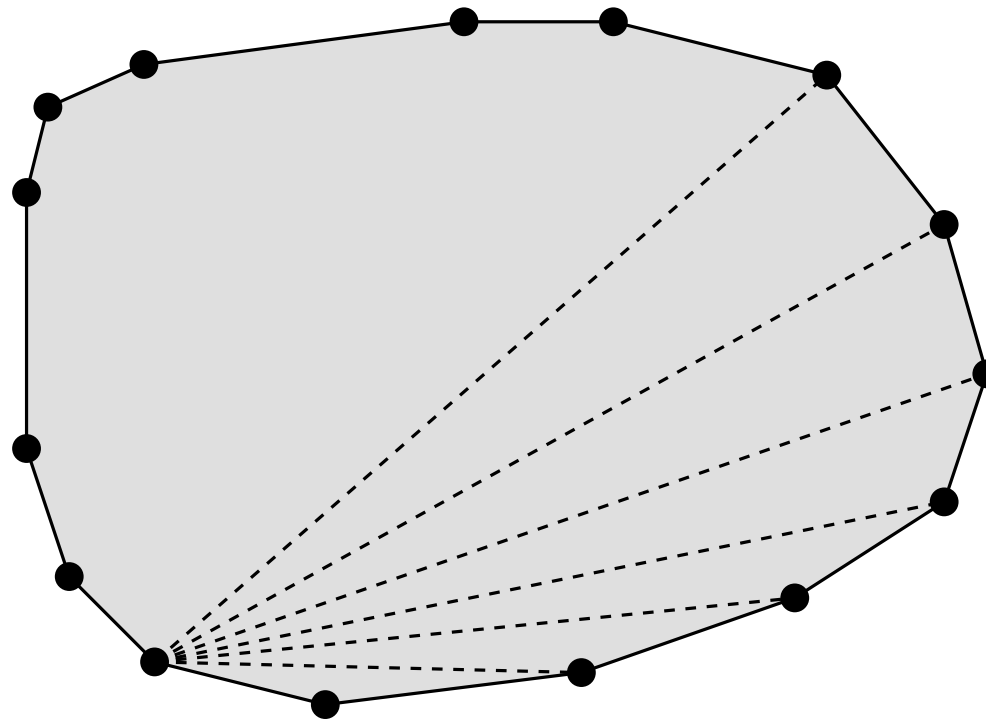


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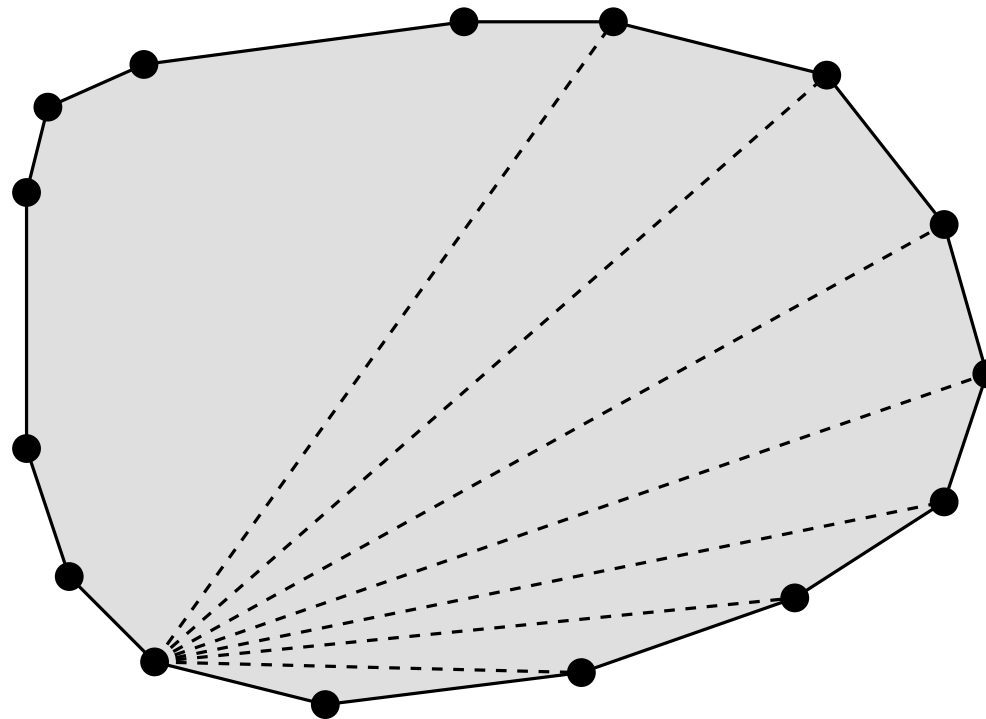


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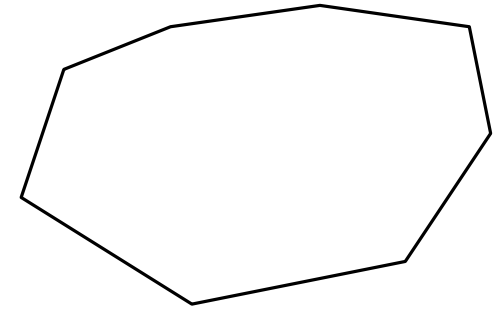
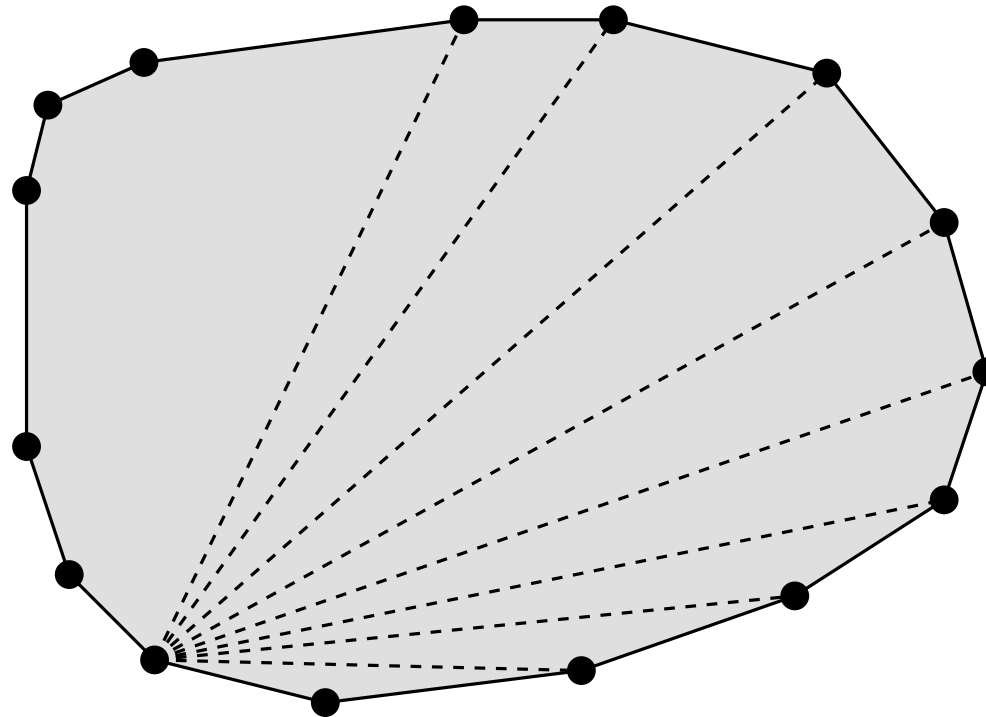


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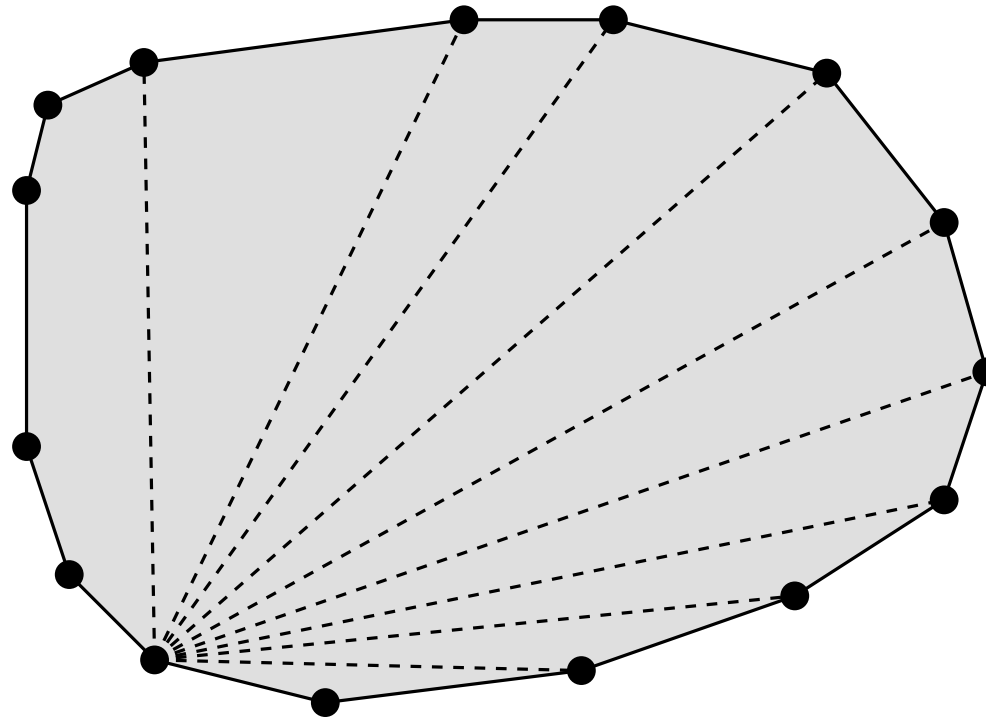


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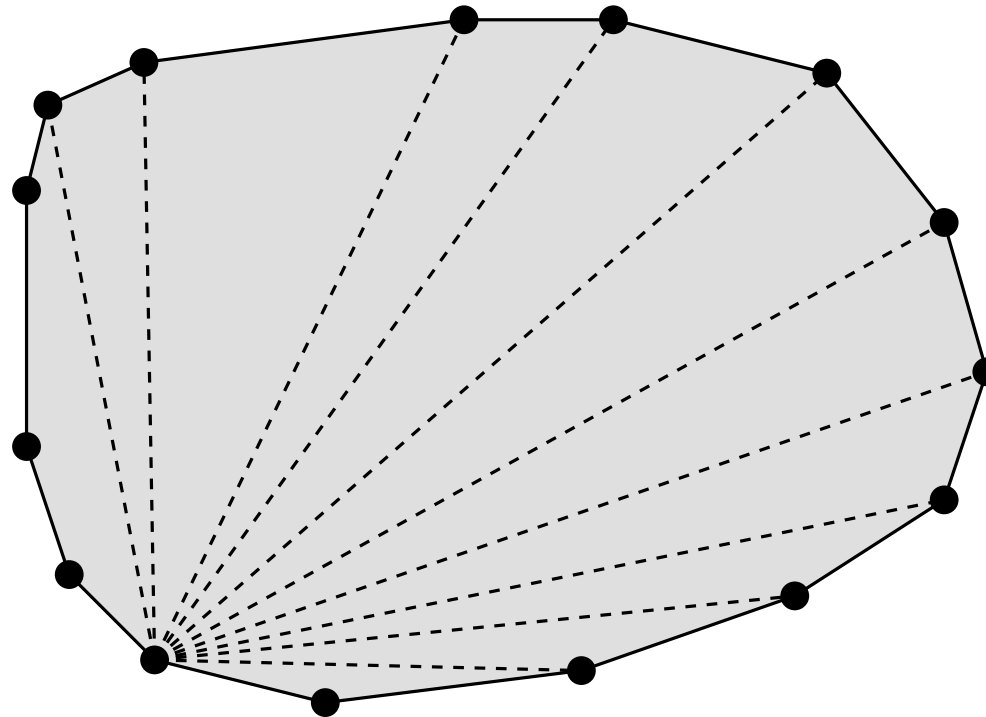


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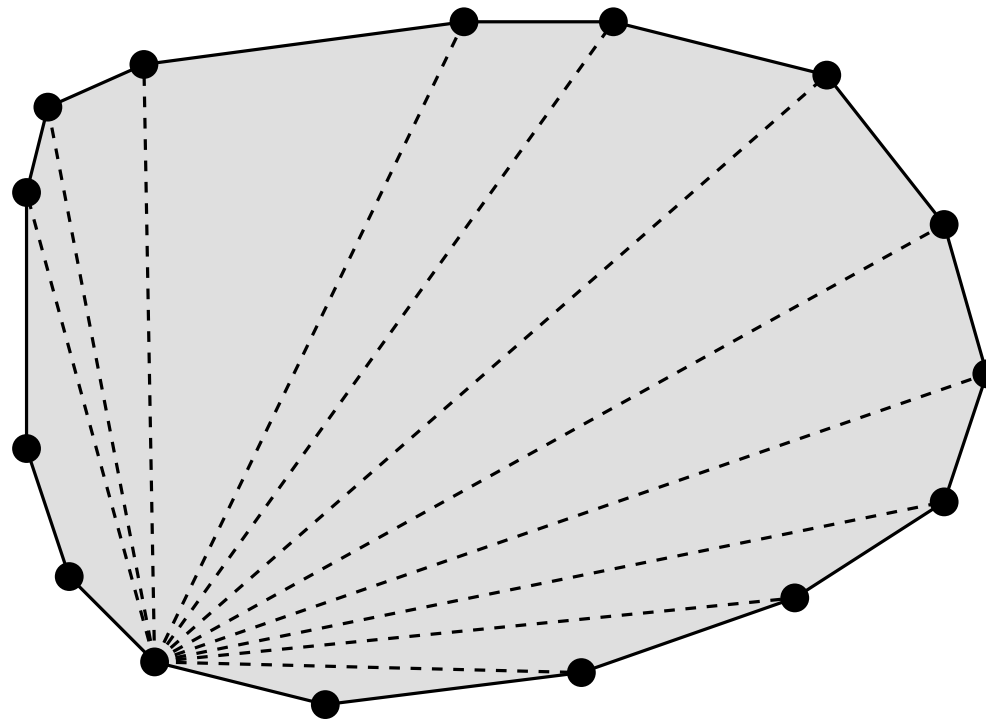


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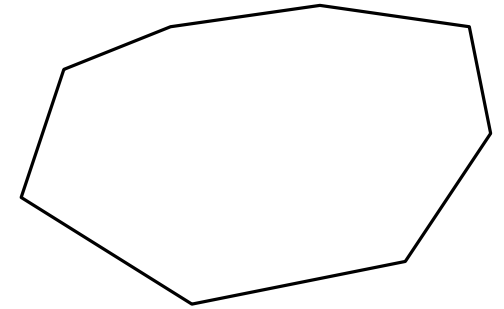
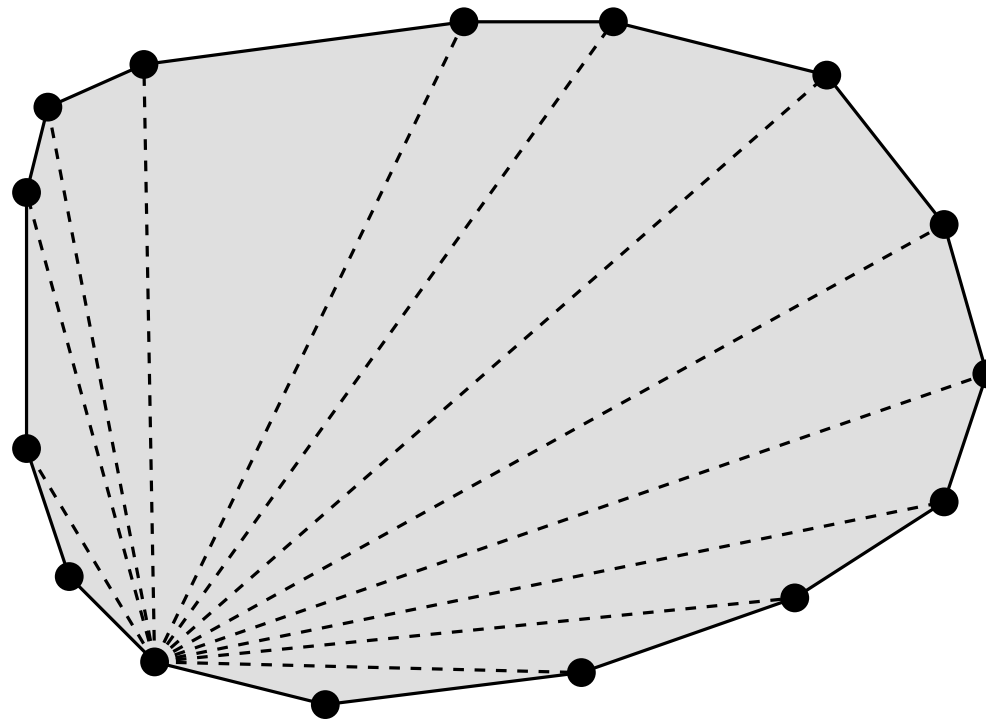


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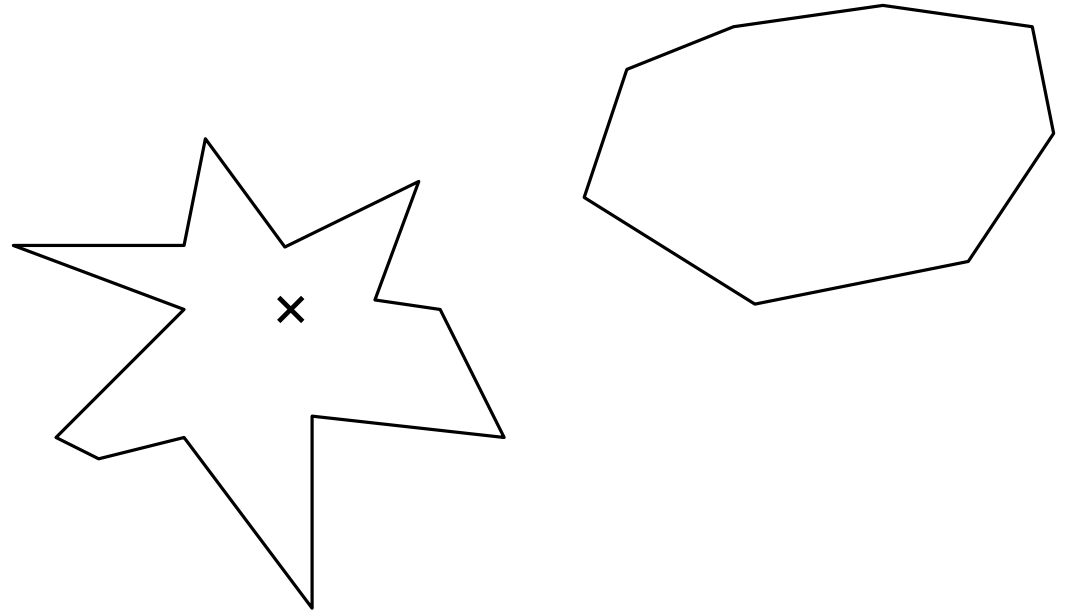
Is it possible to triangulate a polygon more efficiently?

Triangulating a convex polygon

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Triangulating a star-shaped polygon

Can be done in $O(n)$ time. Posed as problem.



TRIANGULATING POLYGONS

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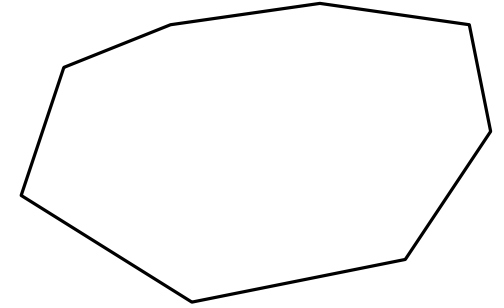
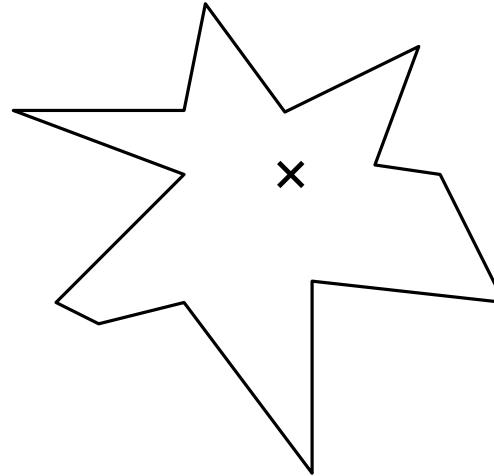
Trivially done in $O(n)$ time.

Triangulating a star-shaped polygon

Can be done in $O(n)$ time. Posed as problem.

Triangulating a monotone polygon

It can also be done in $O(n)$ time. In the following we will see how.



TRIANGULATING POLYGONS

Monotone polygon

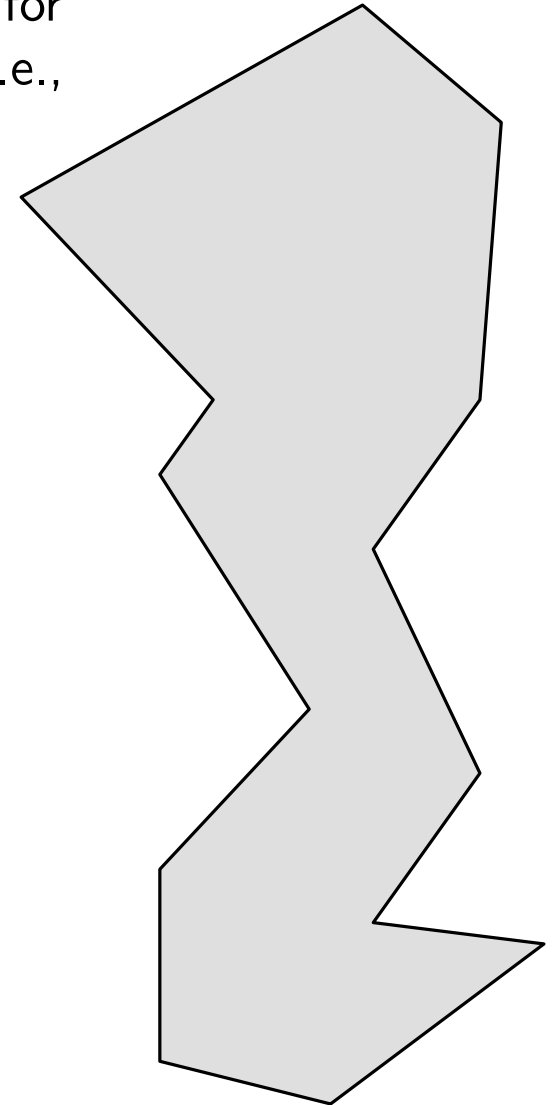
A polygon P is called **monotone** with respect to a direction r if, for every line r' orthogonal to r , the intersection $P \cap r'$ is connected (i.e., it is a segment, a point or the empty set).

TRIANGULATING POLYGONS

Monotone polygon

A polygon P is called **monotone** with respect to a direction r if, for every line r' orthogonal to r , the intersection $P \cap r'$ is connected (i.e., it is a segment, a point or the empty set).

A y -monotone polygon

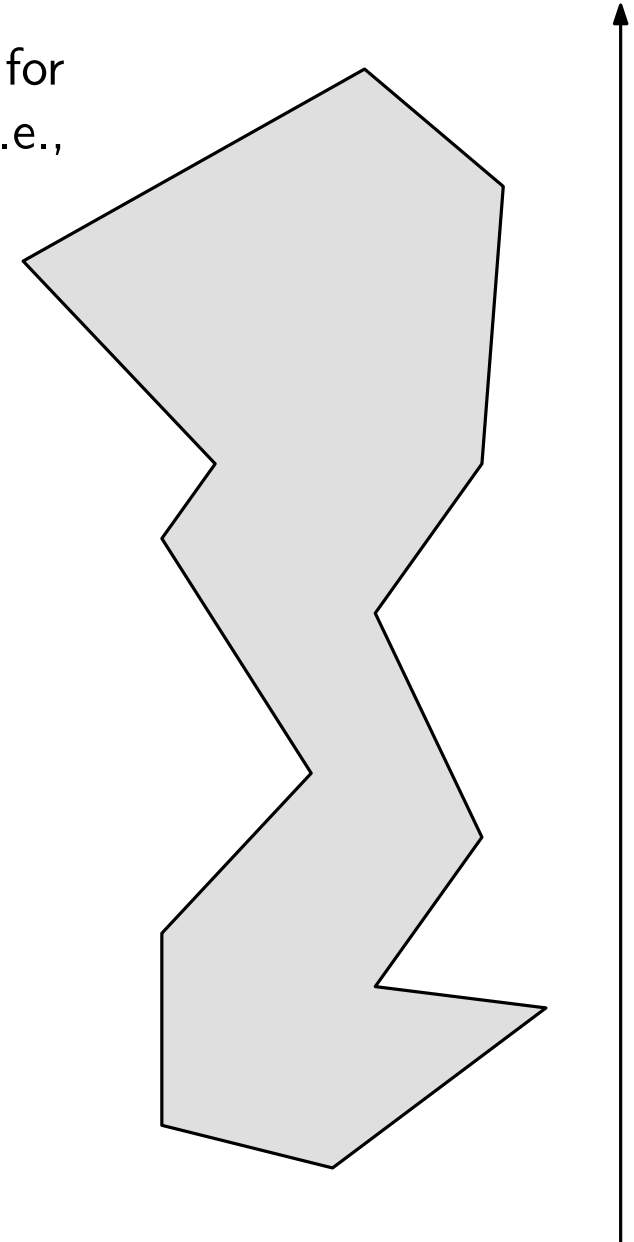


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A y -monotone polygon _{y}

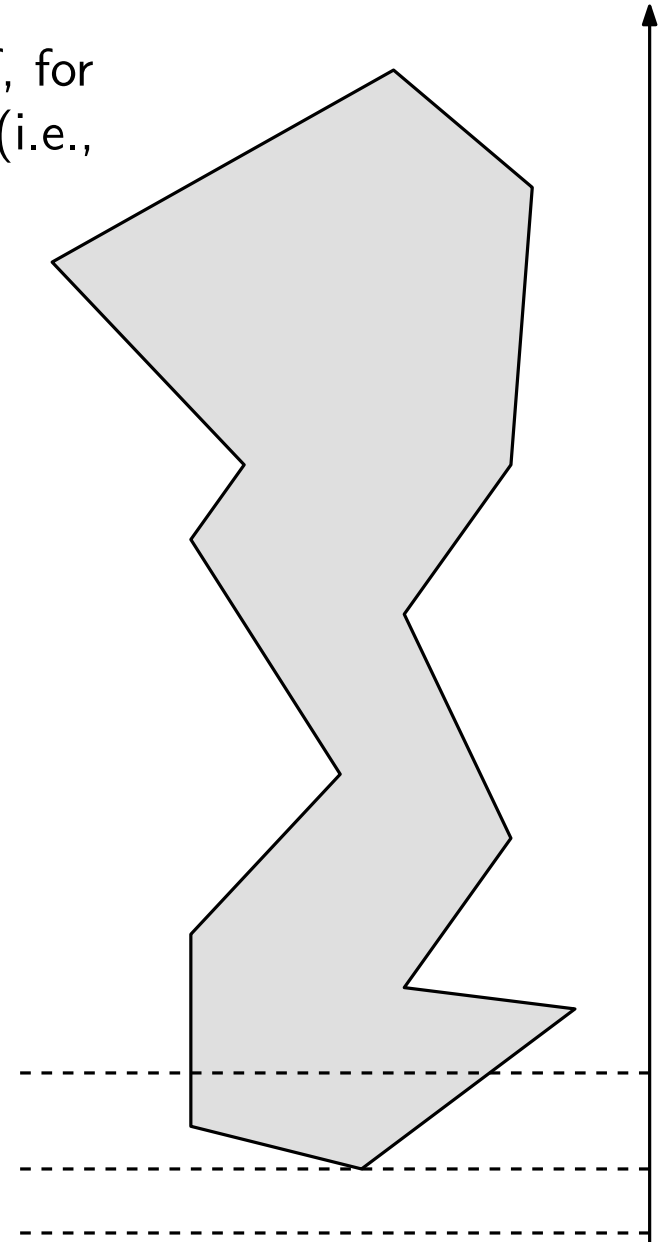


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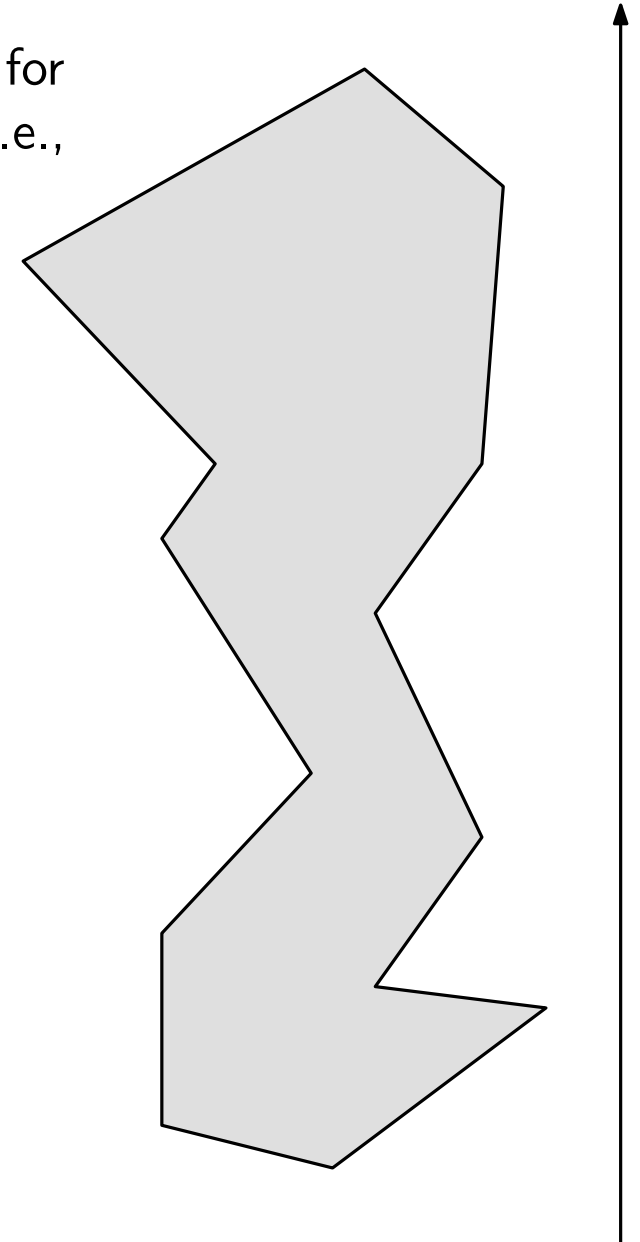
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Local characterization

A **cusp** is a reflex vertex v of the polygon such that its two incident edges both lie to the same side of the horizontal line through v .

A y -monotone polygon _{y}



TRIANGULATING POLYGONS

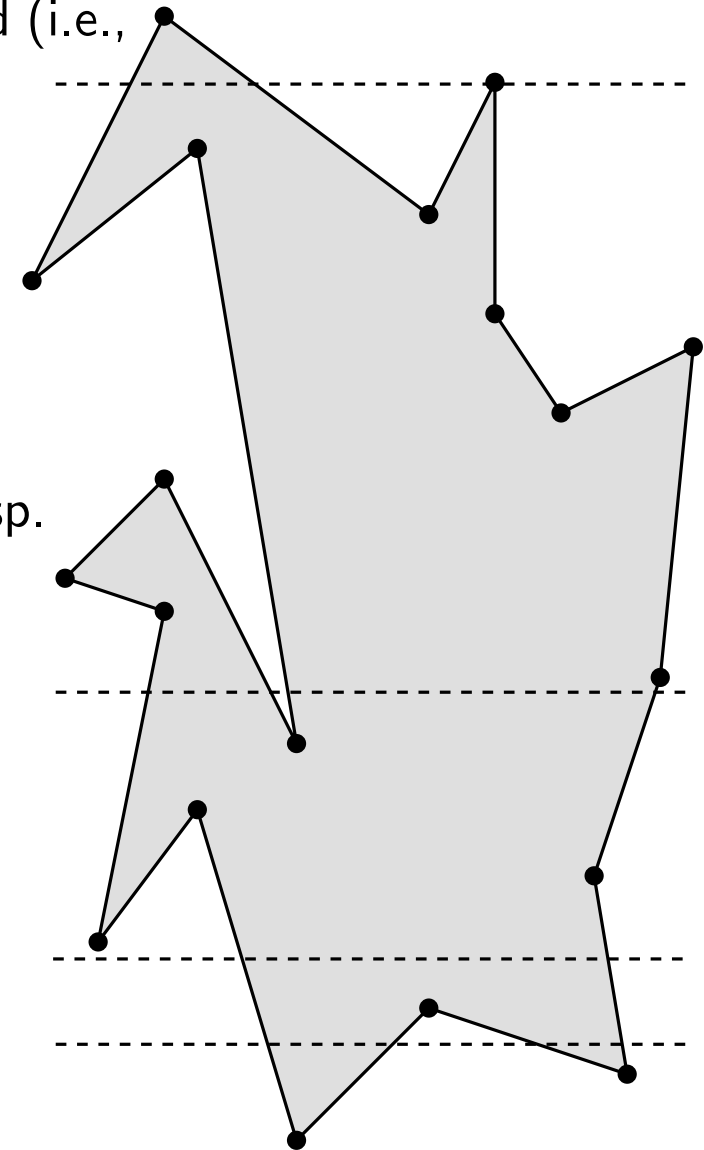
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Local characterization

A **cusp** is a reflex vertex v of the polygon such that its two incident edges both lie to the same side of the horizontal line through v .

A polygon is y -monotone if and only if it does not have any cusp.



TRIANGULATING POLYGONS

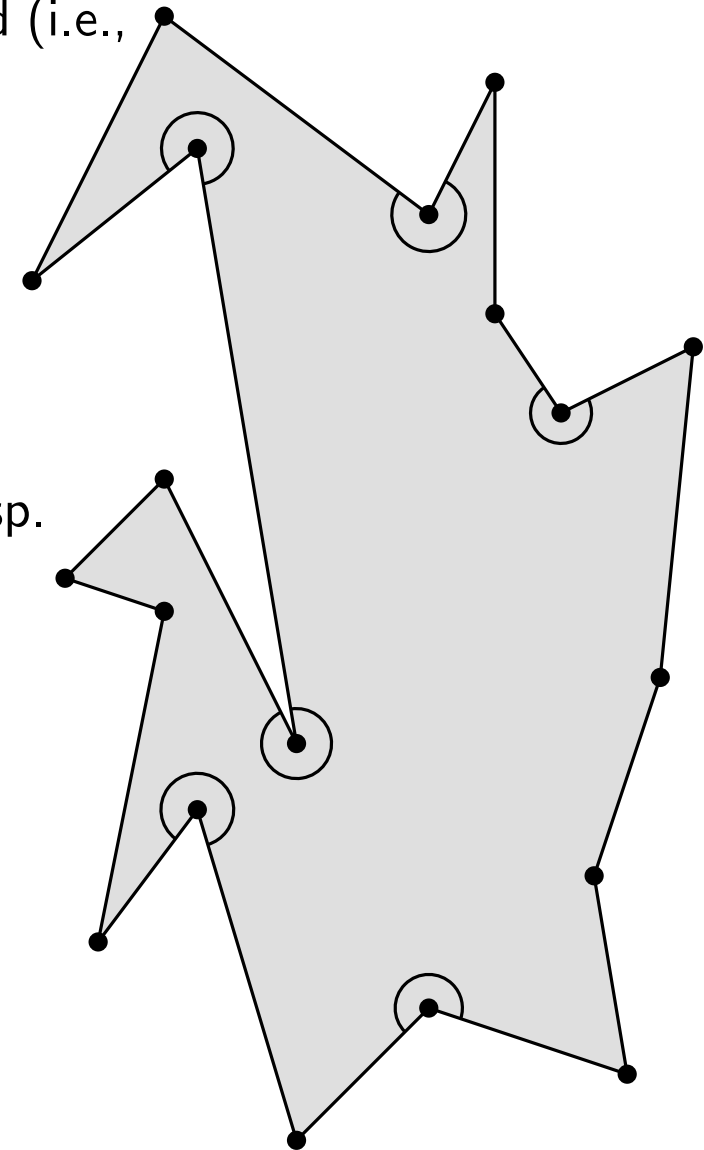
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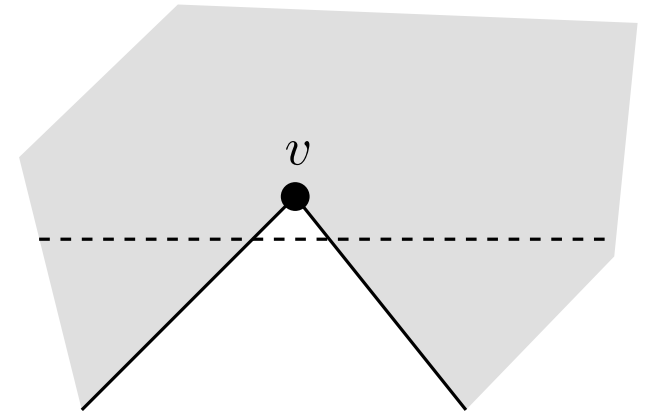
Local characterization

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A polygon is y -monotone if and only if it does not have any cusp.

Proof:

If the polygon has a local maximum cusp v , an infinitesimal downwards translation of the horizontal line through v would intersect the polygon in at least two connected components.



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Local characterization

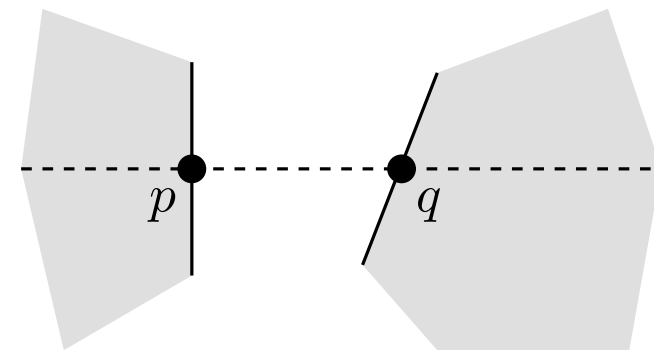
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A polygon is y -monotone if and only if it does not have any cusp.

Proof:

If the polygon has a local maximum cusp v , an infinitesimal downwards translation of the horizontal line through v would intersect the polygon in at least two connected components.

If the polygon is not y -monotone, let r be a horizontal line intersecting the polygon in two or more connected components. Consider two consecutive components, with facing endpoints p and q as in the figure. The polygon boundary needs to connect p and q . No matter whether it goes above or below the horizontal line, it will have a cusp.



TRIANGULATING POLYGONS

Monotone polygon

A polygon P is called **monotone** with respect to a direction r if, for every line r' orthogonal to r , the intersection $P \cap r'$ is connected (i.e., it is a segment, a point or the empty set).

Local characterization

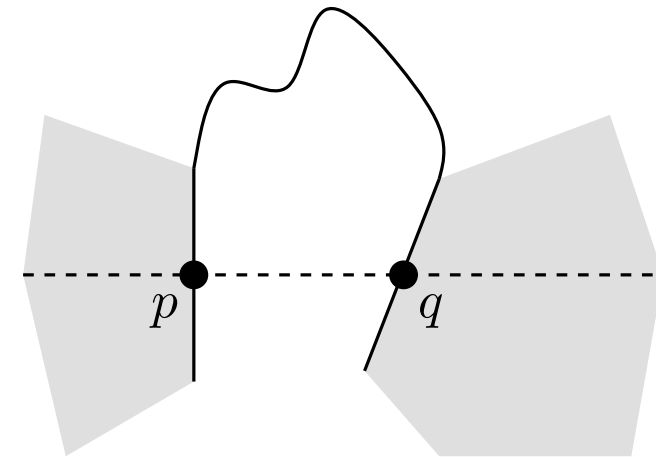
A **cusp** is a reflex vertex v of the polygon such that its two incident edges both lie to the same side of the horizontal line through v .

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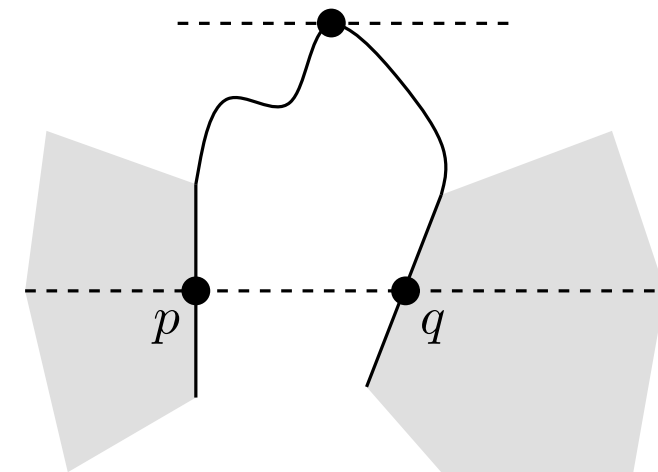
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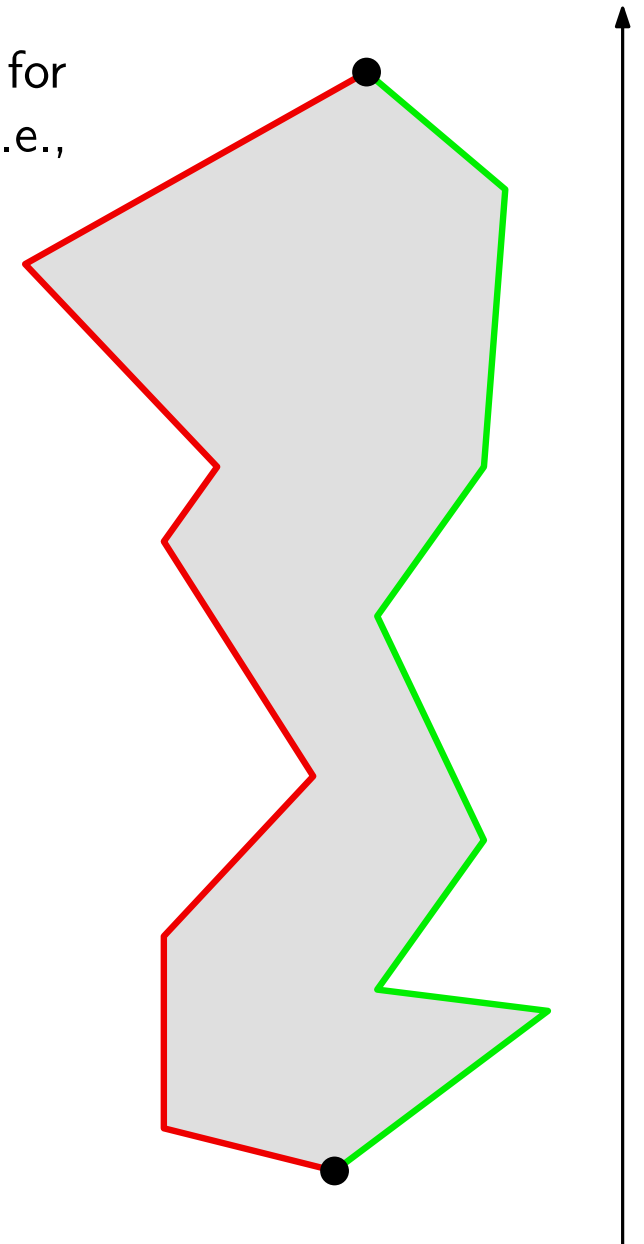
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A polygon is y -monotone if and only if it does not have any cusp.

Corollary

If a polygon is y -monotone, then it can be decomposed into two y -monotone non-intersecting chains sharing their endpoints.

A y -monotone polygon



TRIANGULATING POLYGONS

Triangulating a monotone polygon

TRIANGULATING POLYGONS

Triangulating a monotone polygon

The vertices of the polygon P are processed by decreasing order of their y -coordinate.

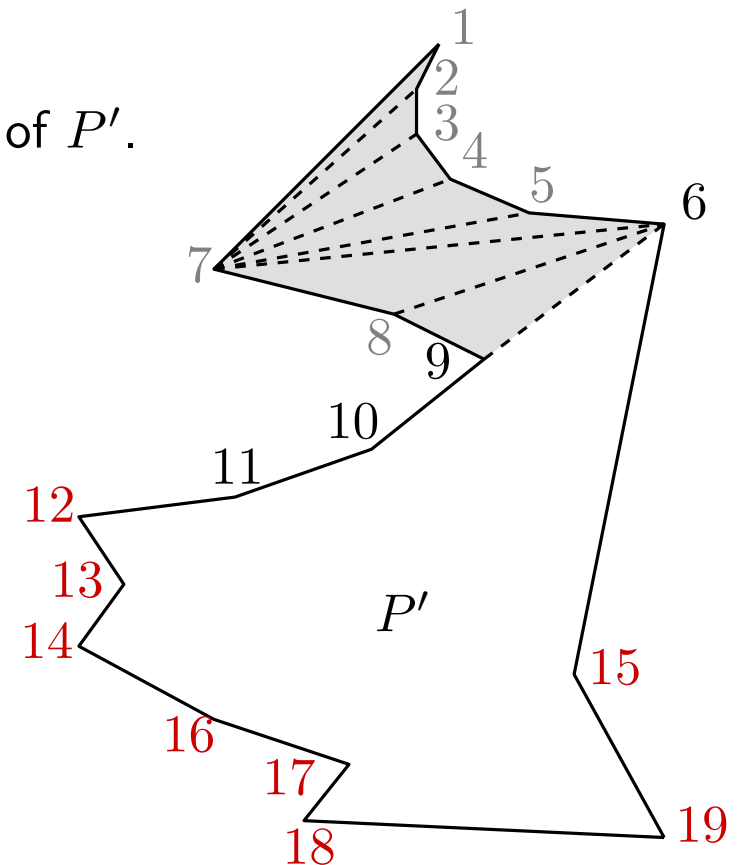
TRIANGULATING POLYGONS

Triangulating a monotone polygon

The vertices of the polygon P are processed by decreasing order of their y -coordinate.

During the process a queue Q is used to store the vertices that have already been visited but are still needed in order to generate the triangulation. Characteristics of Q :

- The topmost (i.e., largest y -coordinate) vertex in Q , is a convex vertex of the subpolygon P' still to be triangulated.
- All the remaining vertices in Q are reflex.
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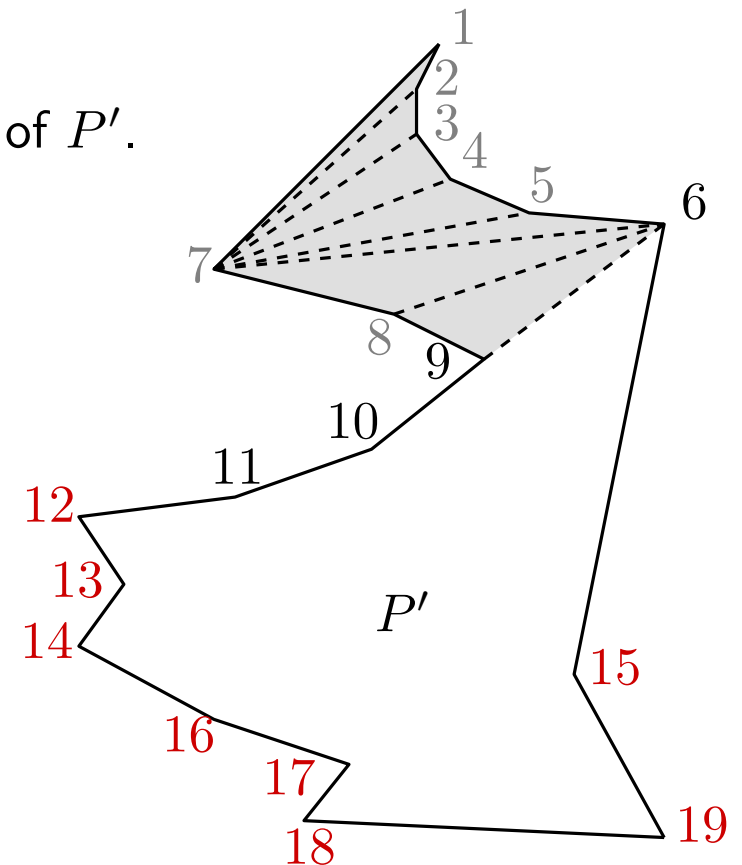
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Processing a vertex v_i :



TRIANGULATING POLYGONS

Triangulating a monotone polygon

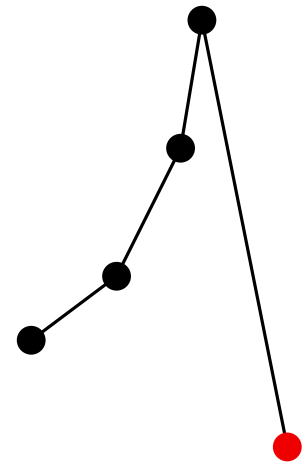
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Processing a vertex v_i :

- If v_i belongs to the opposite chain, report the diagonals connecting v_i to every vertex of Q and delete them all from Q , except the last one. Add v_i to Q .



TRIANGULATING POLYGONS

Triangulating a monotone polygon

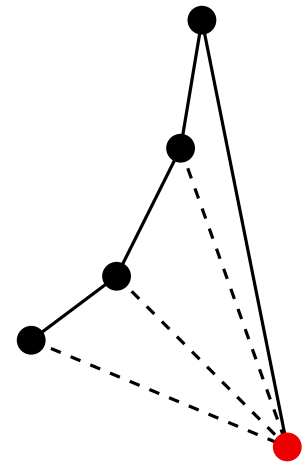
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TRIANGULATING POLYGONS

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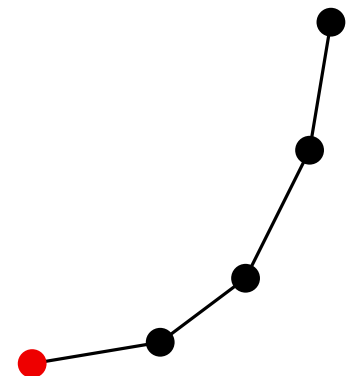
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- If v_i belongs to the opposite chain, report the diagonals connecting v_i to every vertex of Q and delete them all from Q , except the last one. Add v_i to Q .
- If v_i belongs to the same chain and produces a reflex turn, add v_i to Q .



TRIANGULATING POLYGONS

Triangulating a monotone polygon

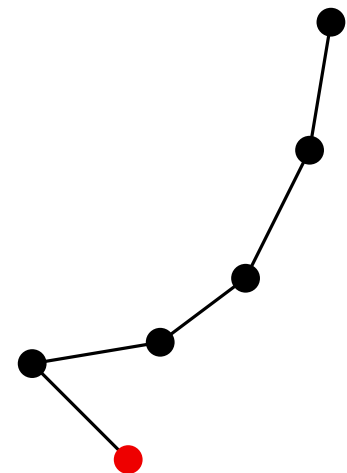
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Processing a vertex v_i :

- **Opposite chain** If v_i belongs to the opposite chain, report the diagonals connecting v_i to every vertex of Q and delete them from Q , except the last one. Add v_i to Q .
- **Add** If v_i belongs to the same chain and produces a reflex turn, add v_i to Q .
- **Ear** If v_i belongs to the same chain and makes a convex turn, report the diagonal connecting v_i to the penultimate element of Q , delete the last element of Q , and process v_i again.



TRIANGULATING POLYGONS

Triangulating a monotone polygon

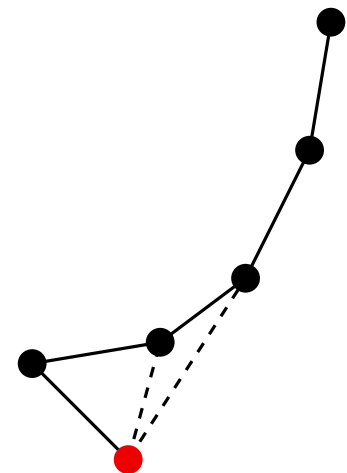
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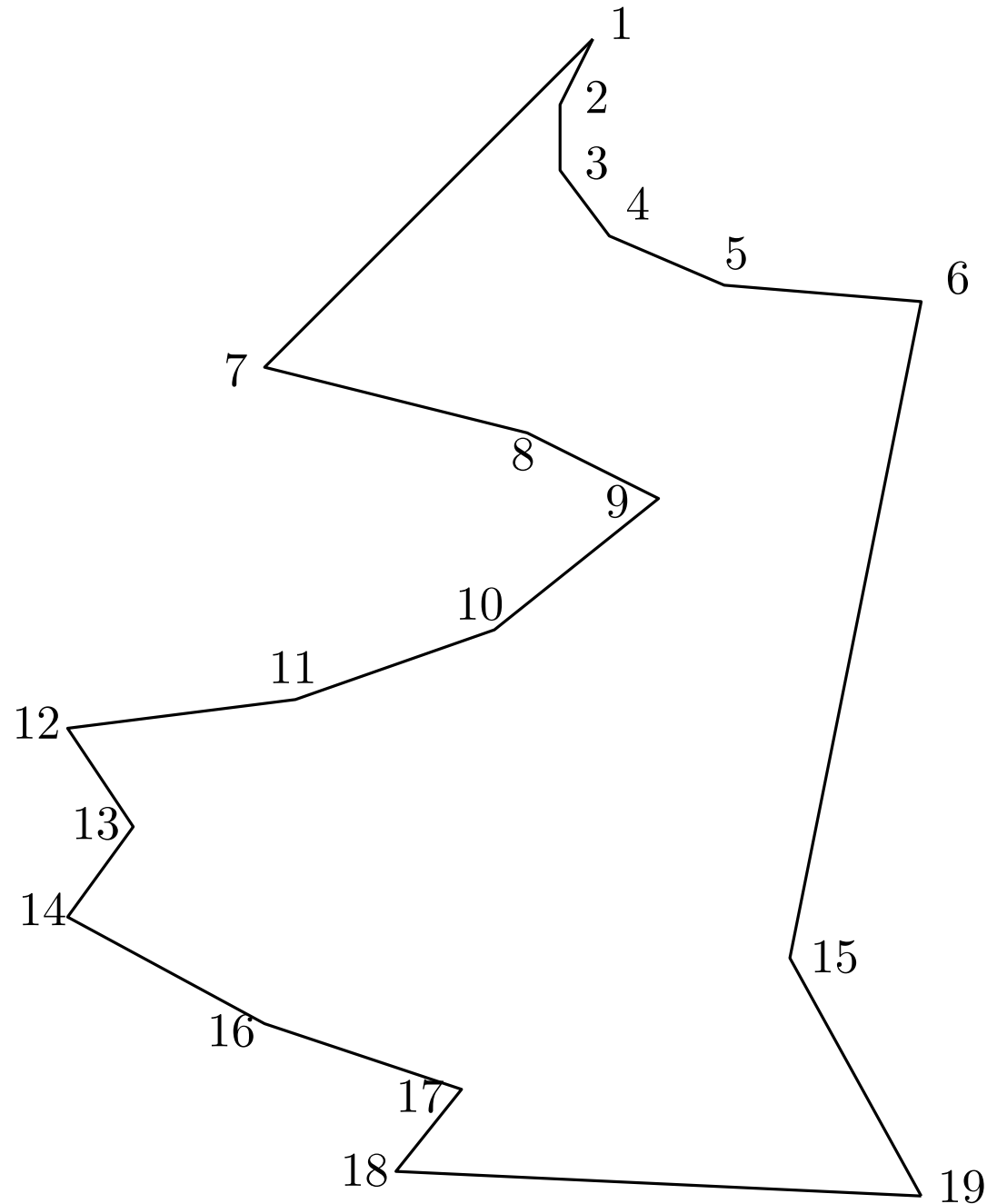
Processing a vertex v_i :

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TRIANGULATING POLYGONS

Triangulating a monotone polygon



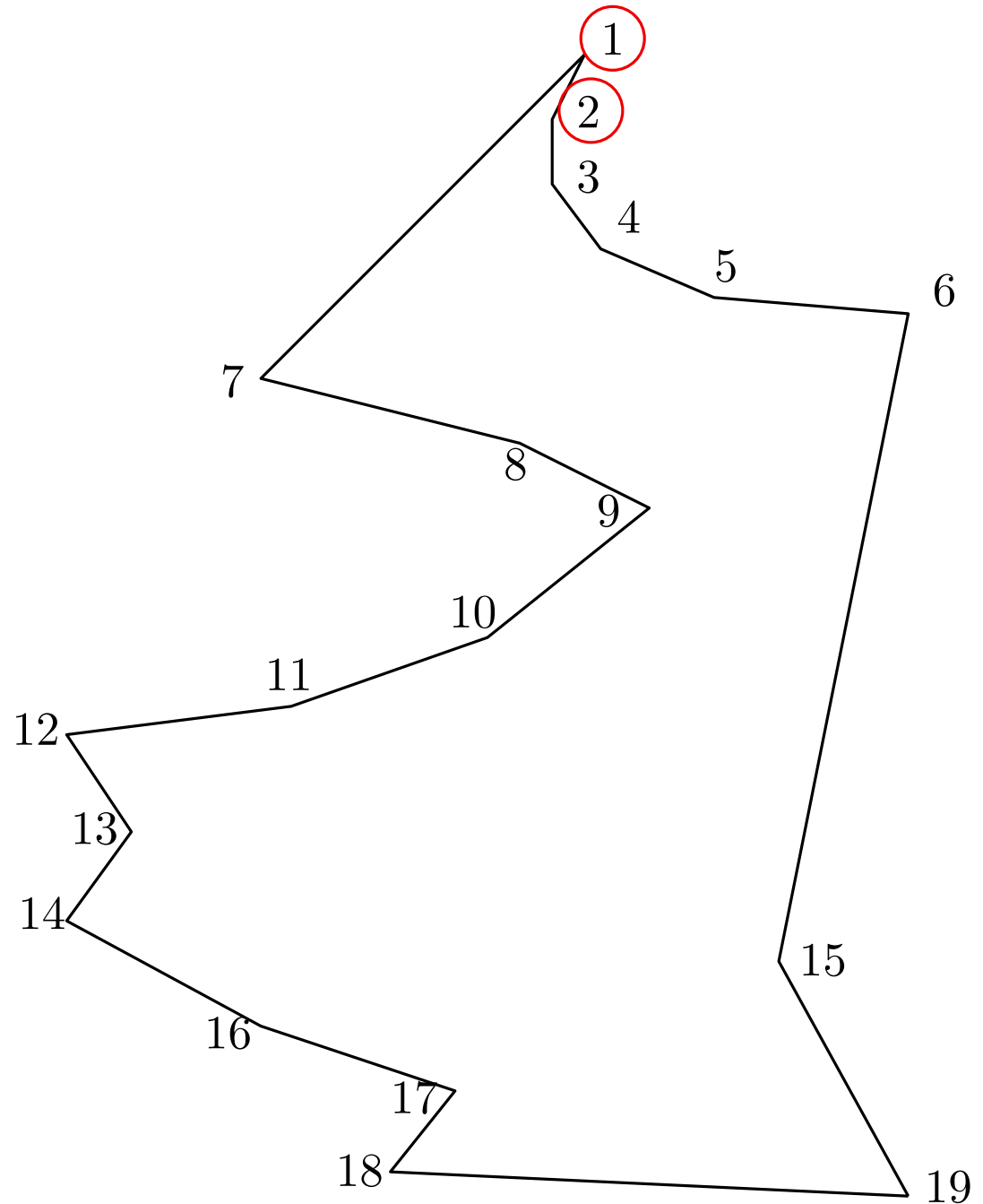
TRIANGULATING POLYGONS

Triangulating a monotone polygon

Start

Queue state

1, 2



TRIANGULATING POLYGONS

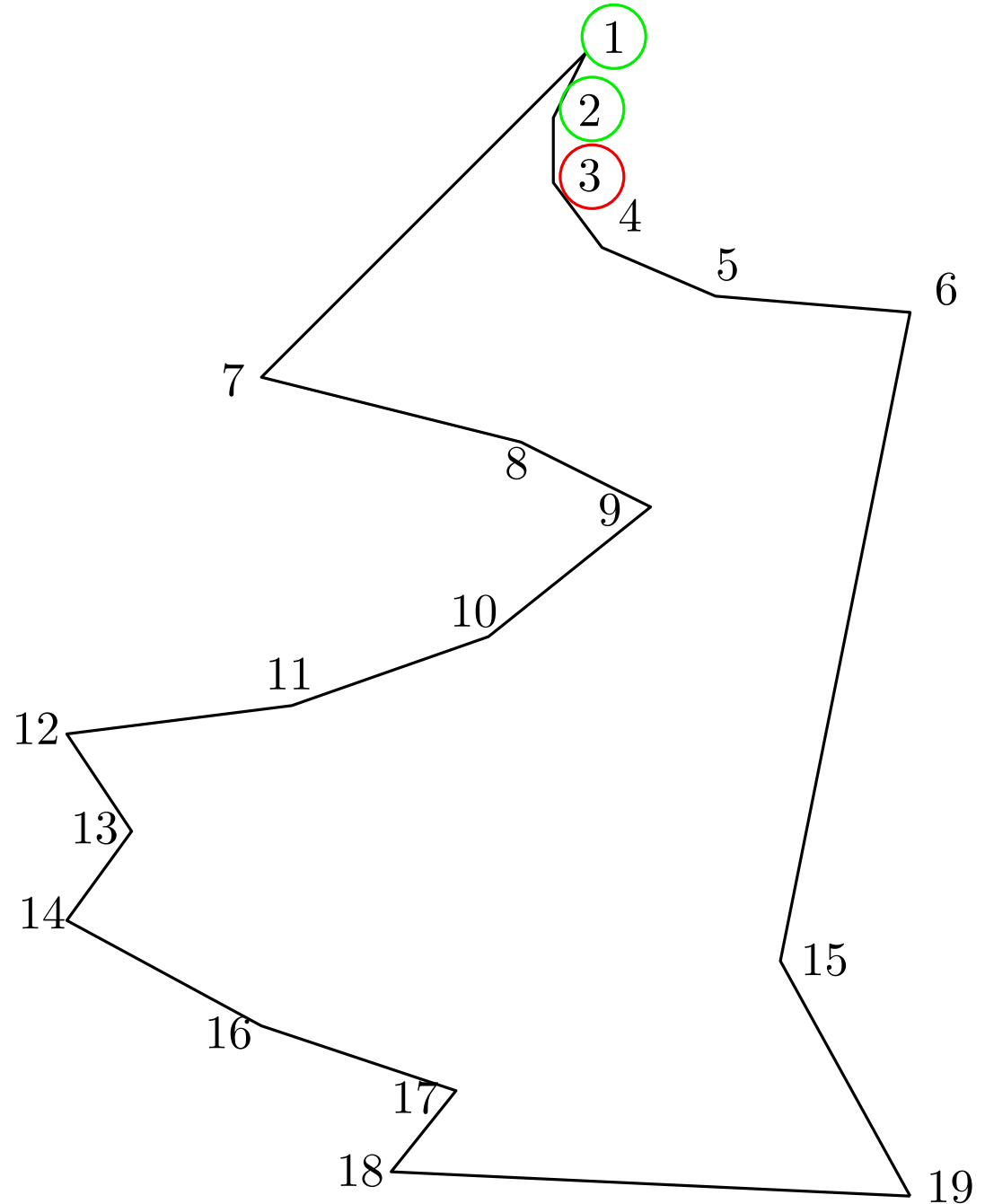
Triangulating a monotone polygon

Current vertex: 3

Add

Queue state:

1, 2, 3



TRIANGULATING POLYGONS

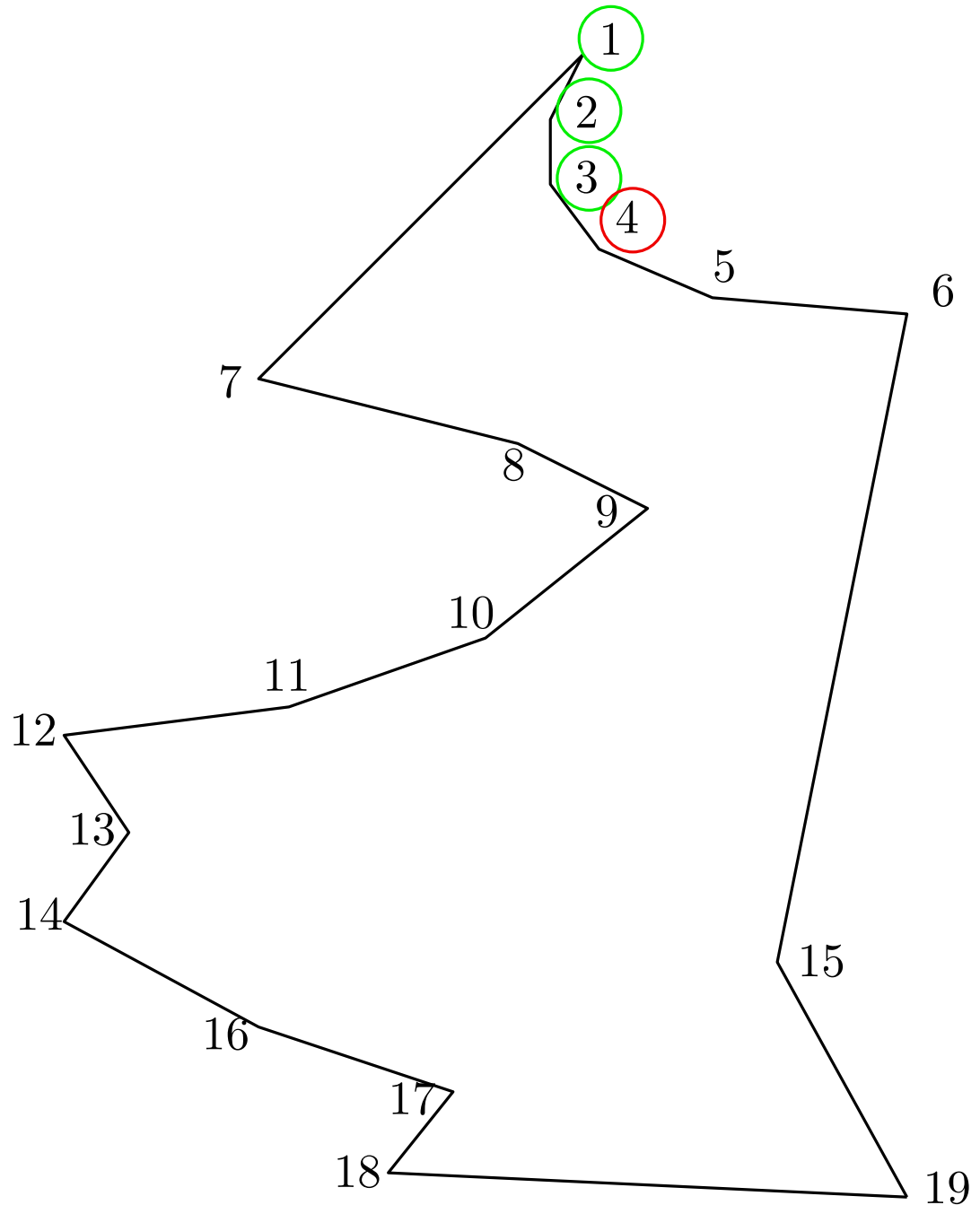
Triangulating a monotone polygon

Current vertex: 4

Add

Queue state:

1, 2, 3, 4



TRIANGULATING POLYGONS

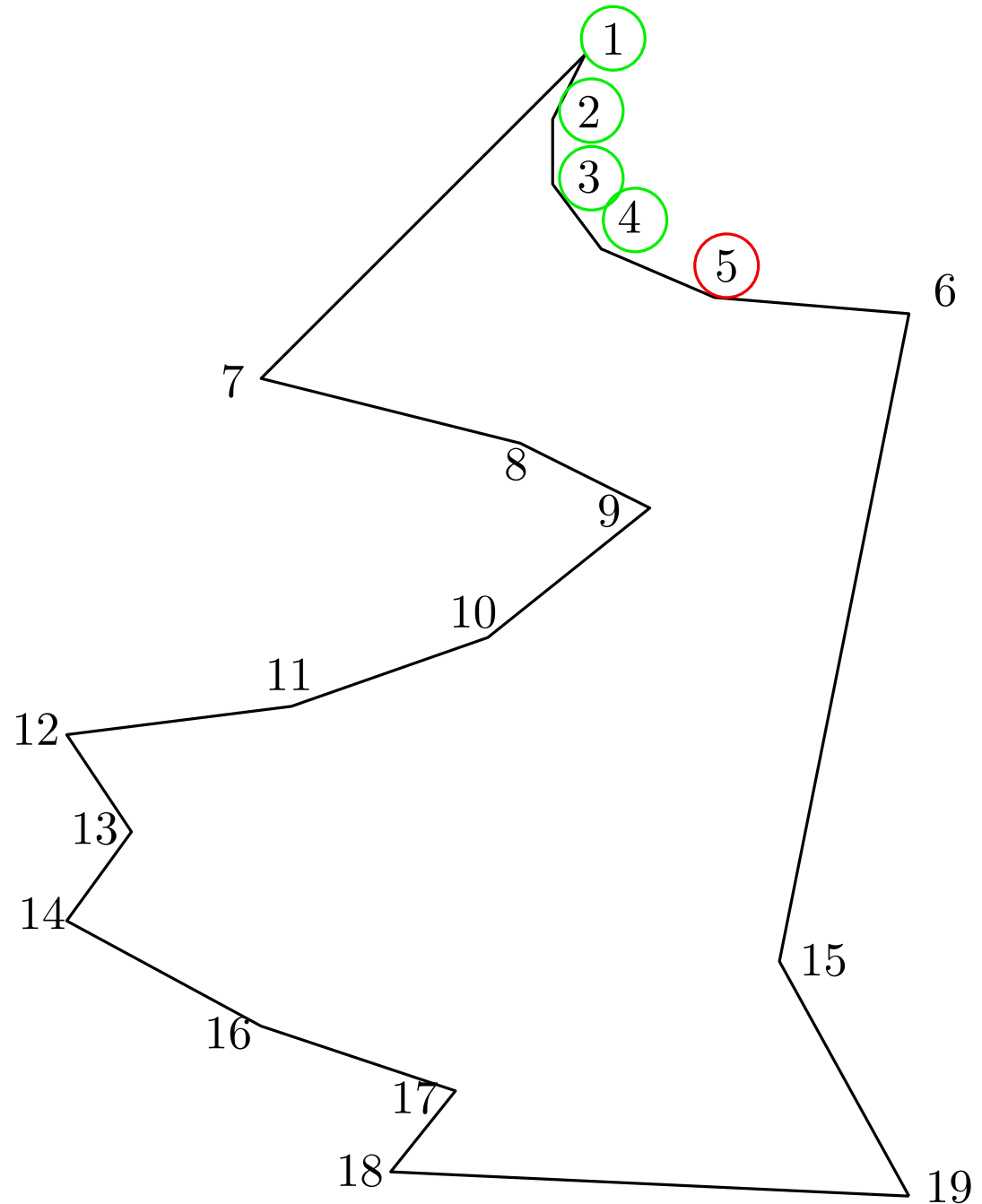
Triangulating a monotone polygon

Current vertex: 5

Add

Queue state:

1, 2, 3, 4, 5



TRIANGULATING POLYGONS

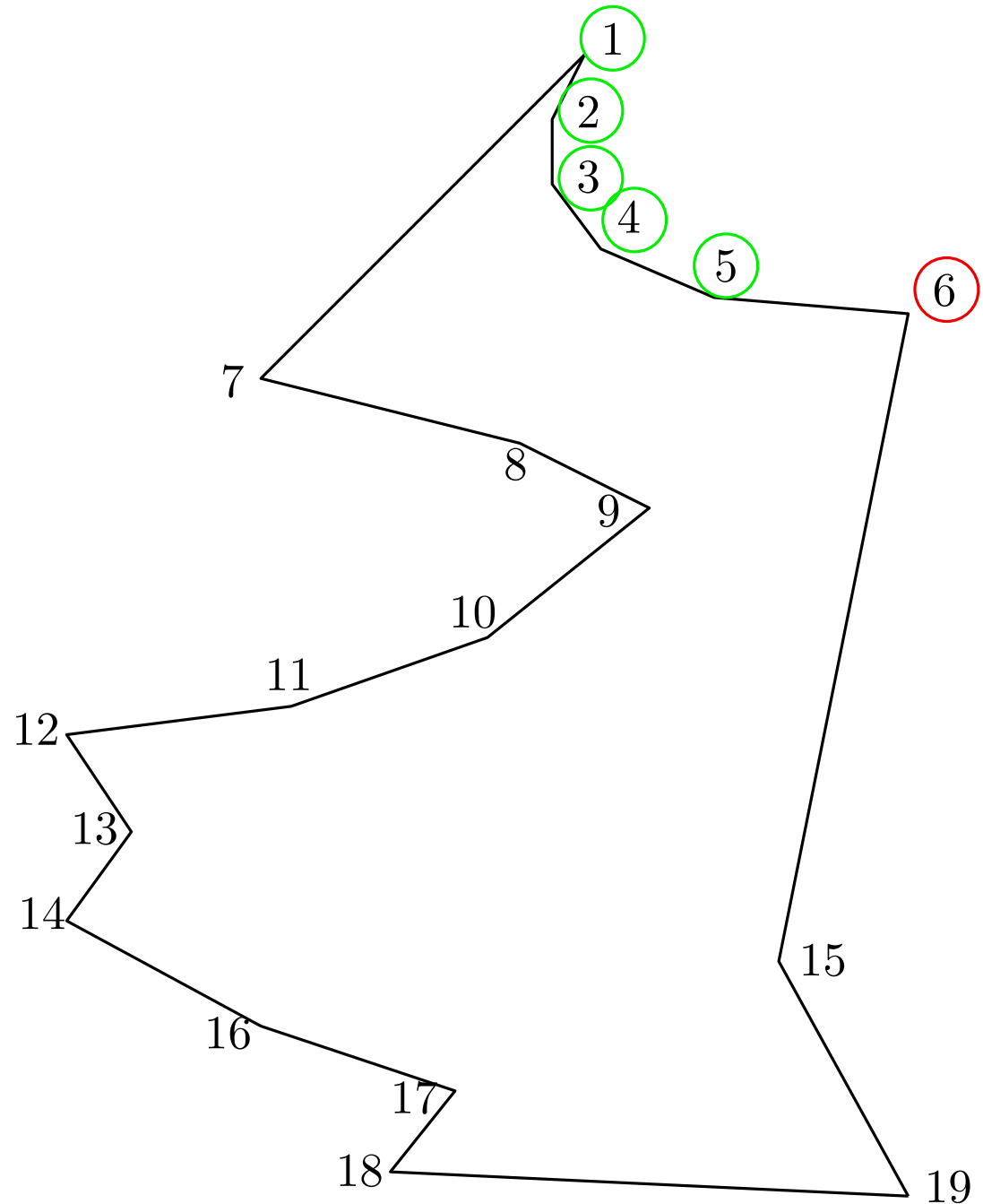
Triangulating a monotone polygon

Current vertex: 6

Add

Queue state:

1, 2, 3, 4, 5, 6



TRIANGULATING POLYGONS

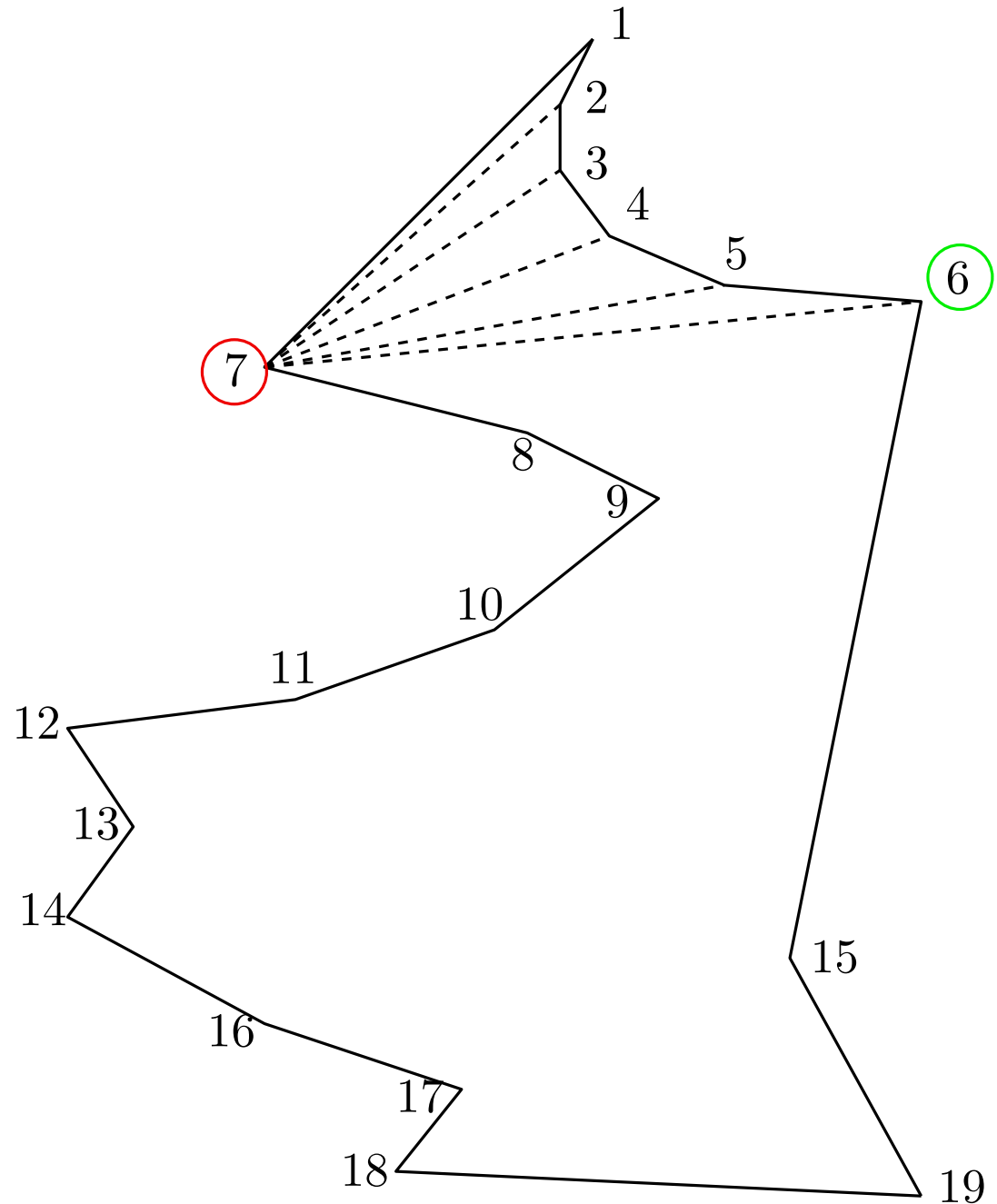
Triangulating a monotone polygon

Current vertex: 7

Opposite chain

Queue state:

6, 7



TRIANGULATING POLYGONS

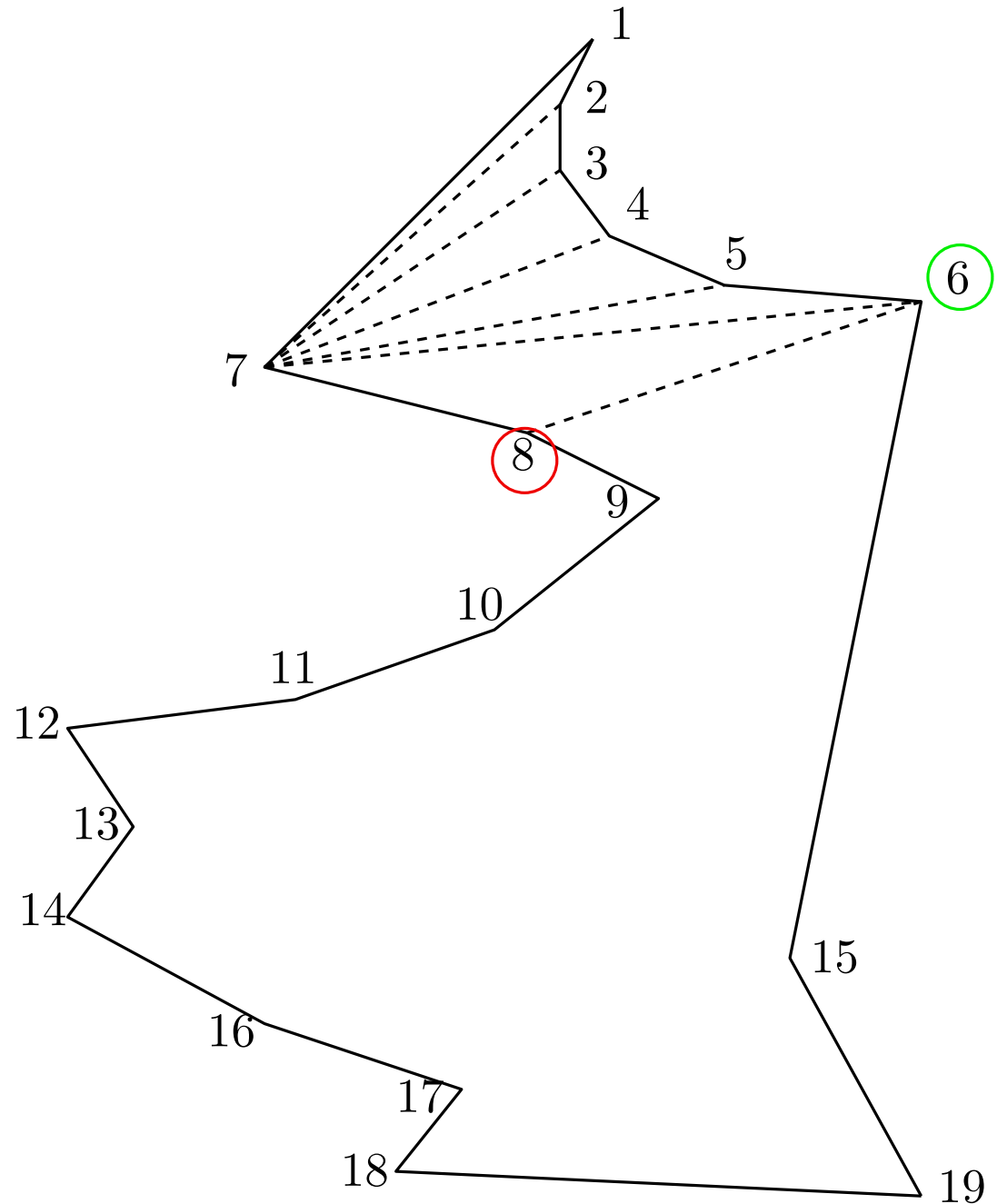
Triangulating a monotone polygon

Current vertex: 8

Ear

Queue state:

6, 8



TRIANGULATING POLYGONS

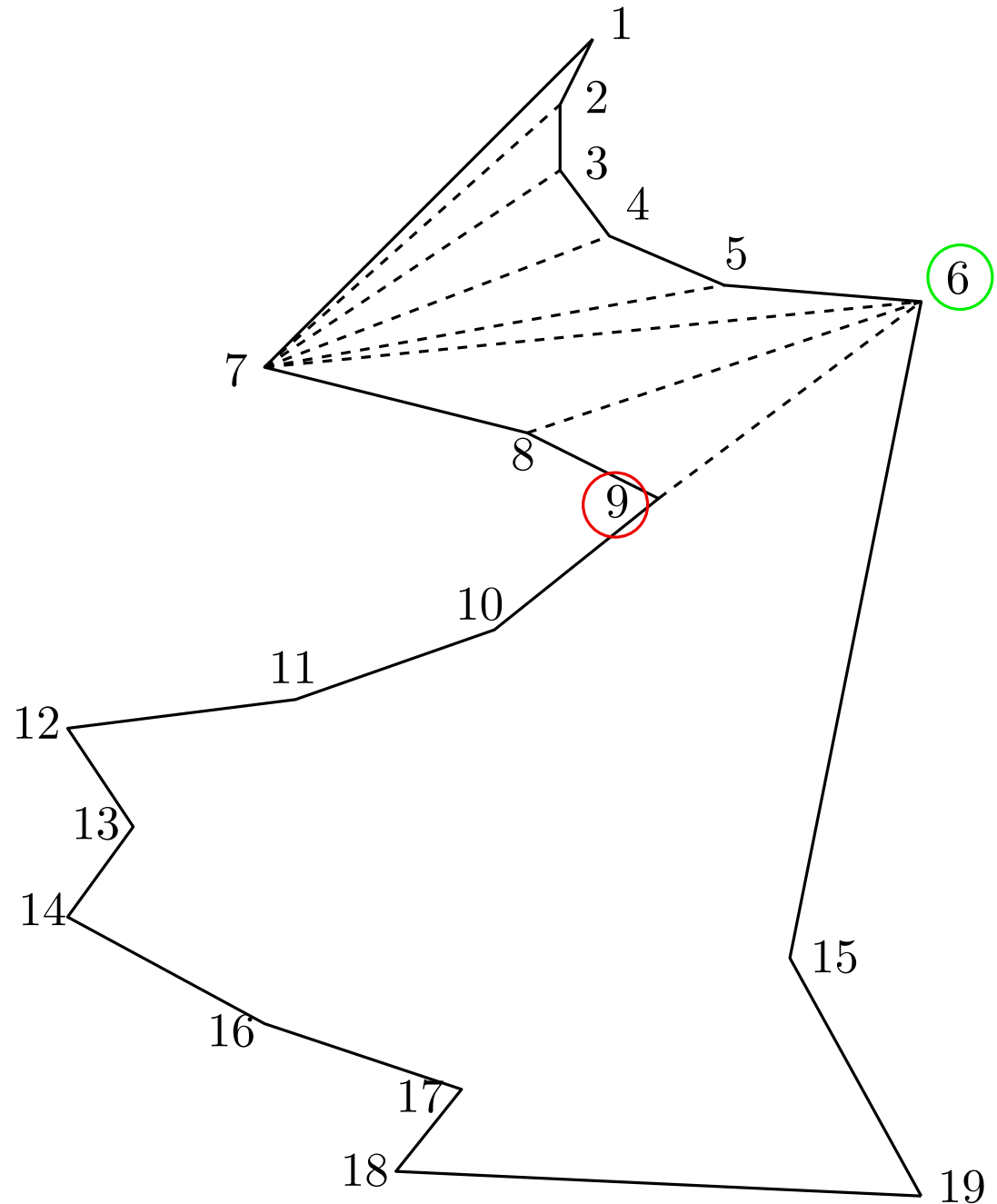
Triangulating a monotone polygon

Current vertex: 9

Ear

Queue state:

6, 9



TRIANGULATING POLYGONS

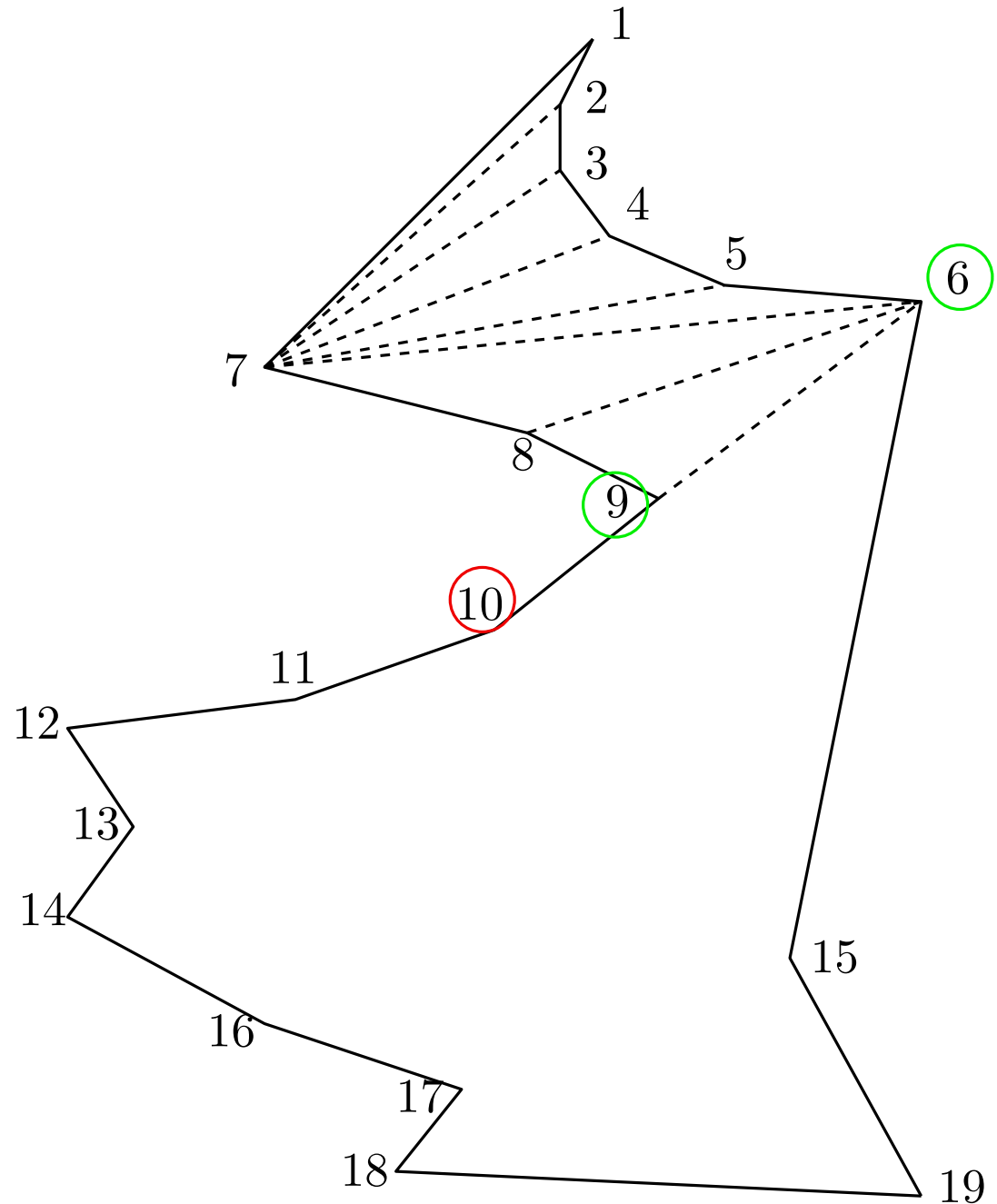
Triangulating a monotone polygon

Current vertex: 10

Add

Queue state:

6, 9, 10



TRIANGULATING POLYGONS

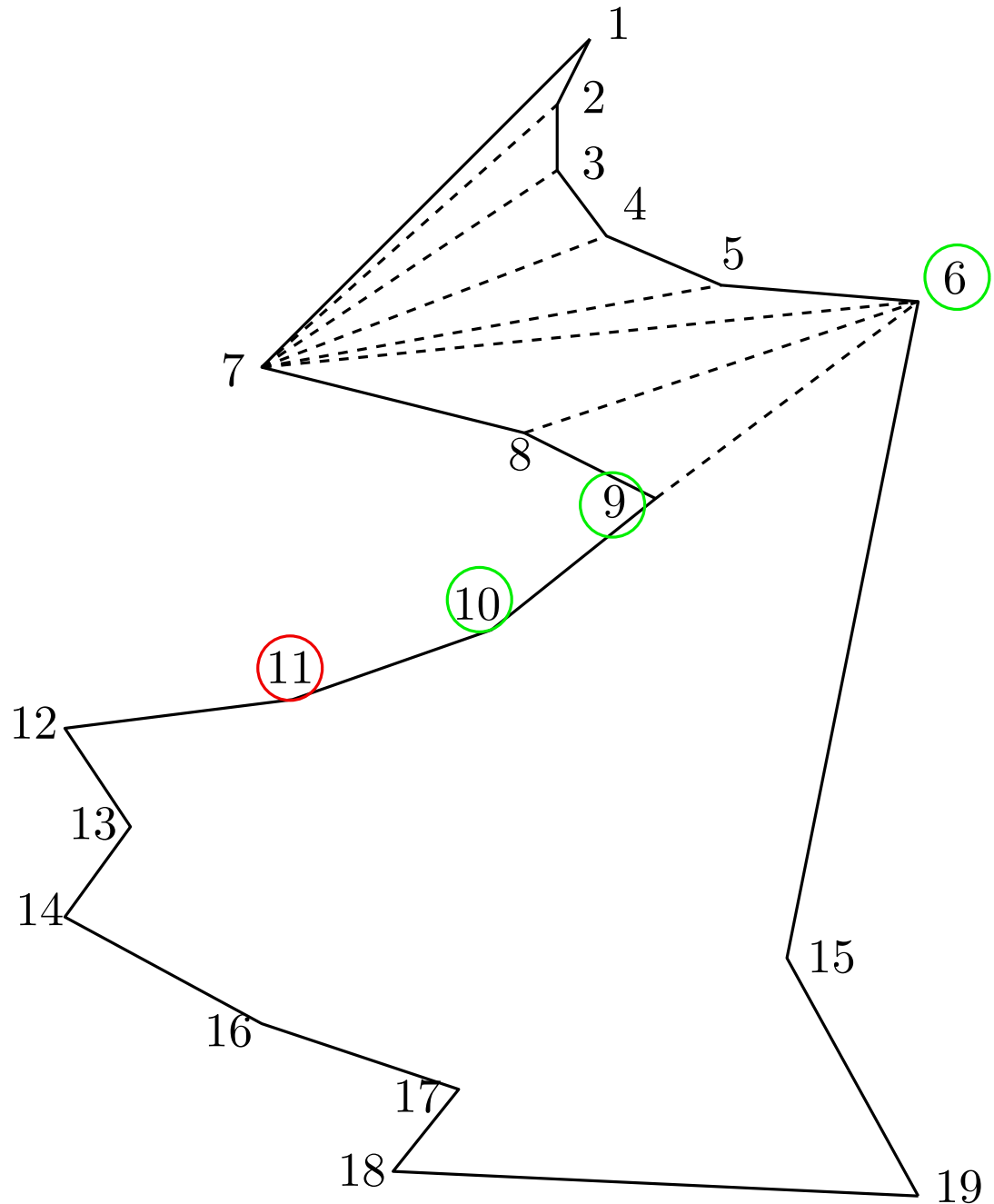
Triangulating a monotone polygon

Current vertex: 11

Add

Queue state:

6, 9, 10, 11



TRIANGULATING POLYGONS

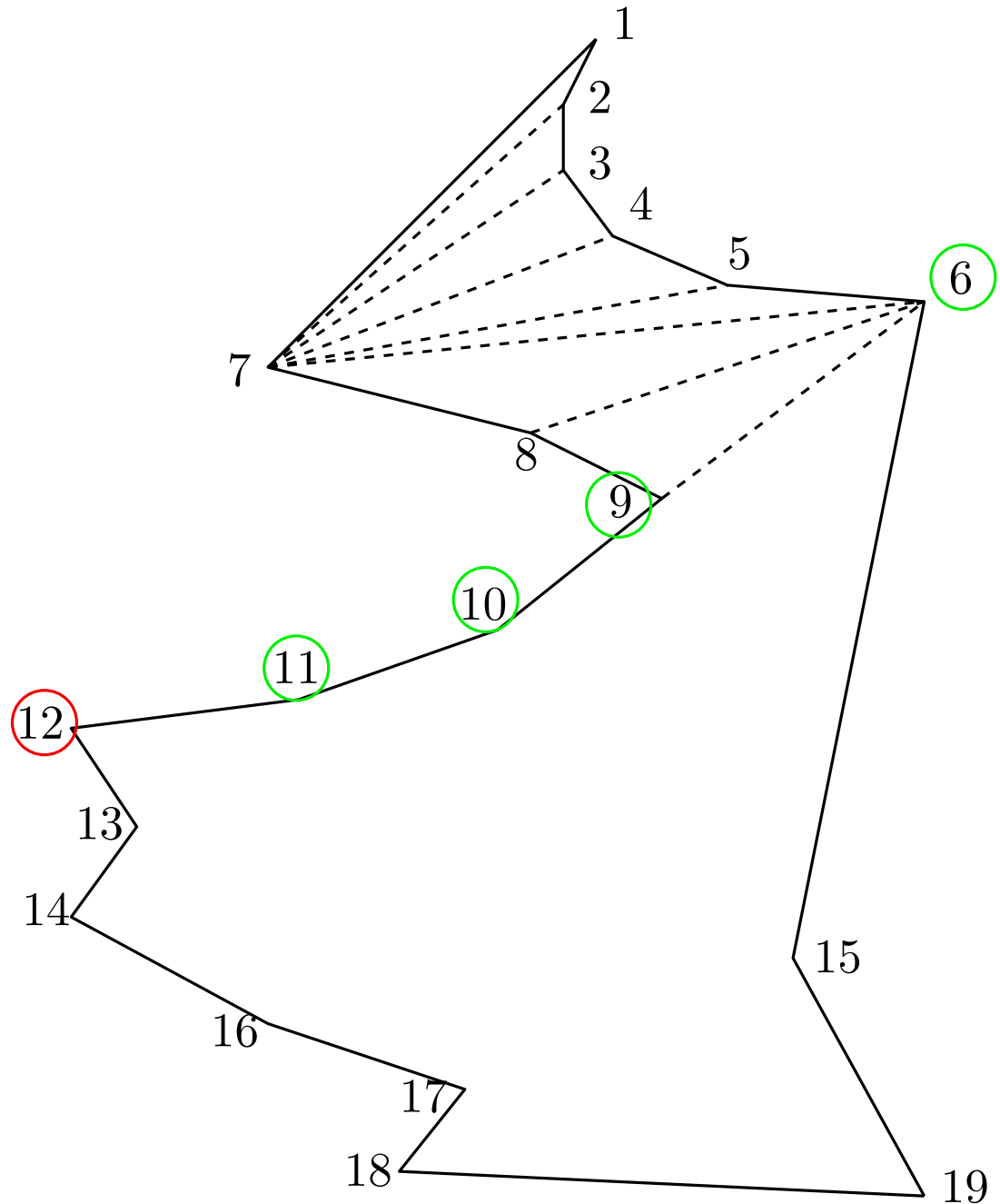
Triangulating a monotone polygon

Current vertex: 12

Add

Queue state:

6, 9, 10, 11, 12



TRIANGULATING POLYGONS

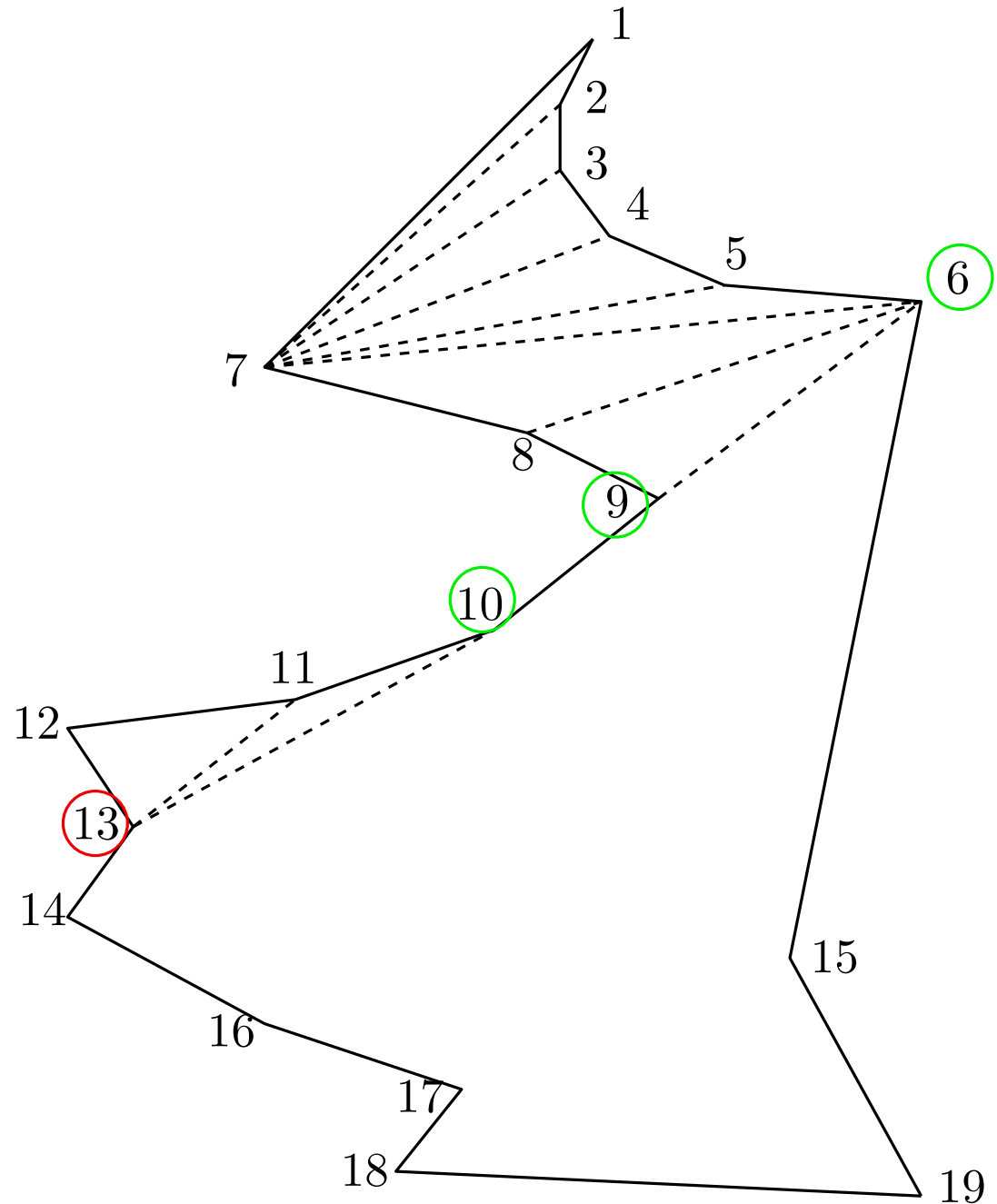
Triangulating a monotone polygon

Current vertex: 13

Ear

Queue state:

6, 9, 10, 13



TRIANGULATING POLYGONS

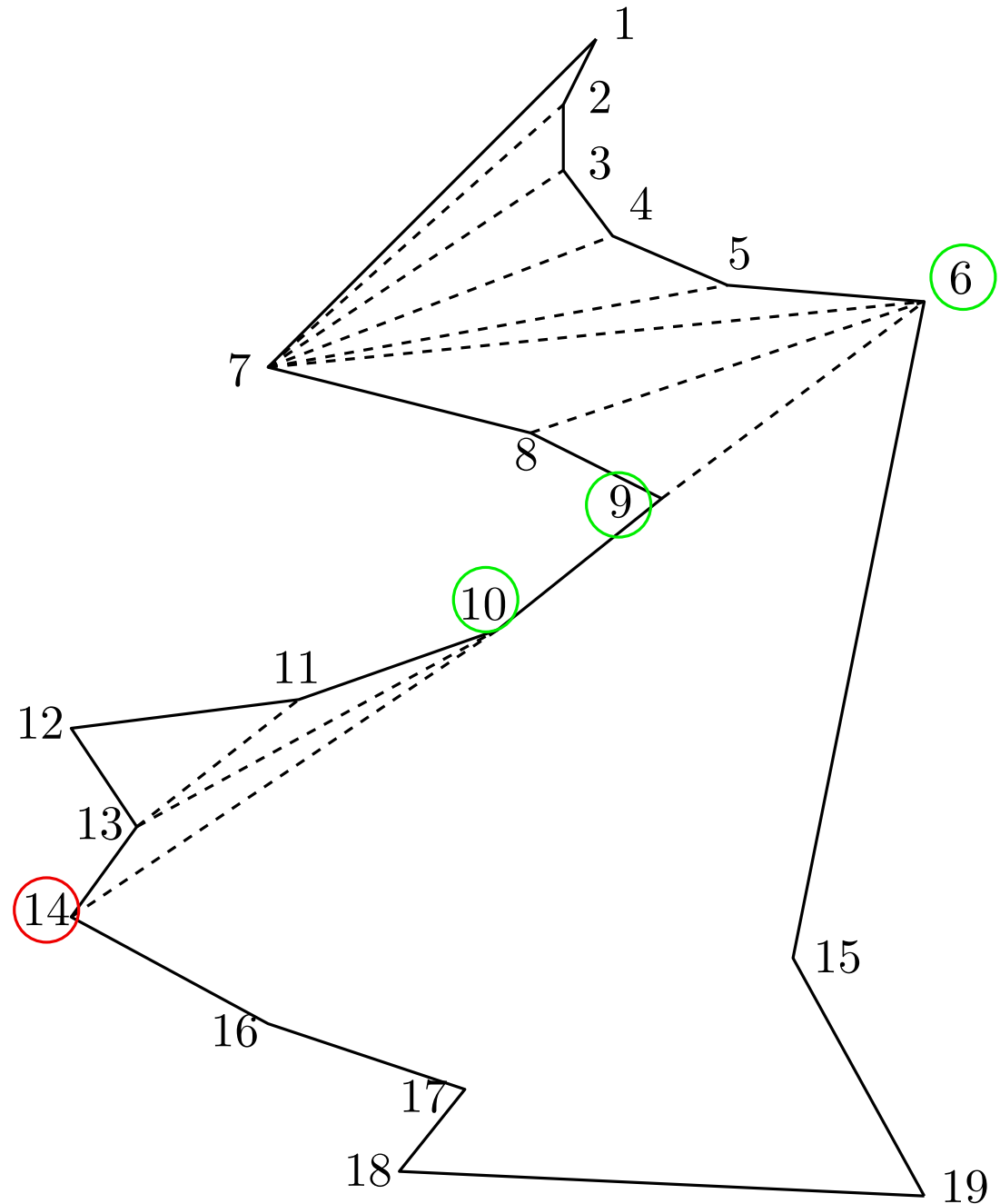
Triangulating a monotone polygon

Current vertex: 14

Ear

Queue state:

6, 9, 10, 14



TRIANGULATING POLYGONS

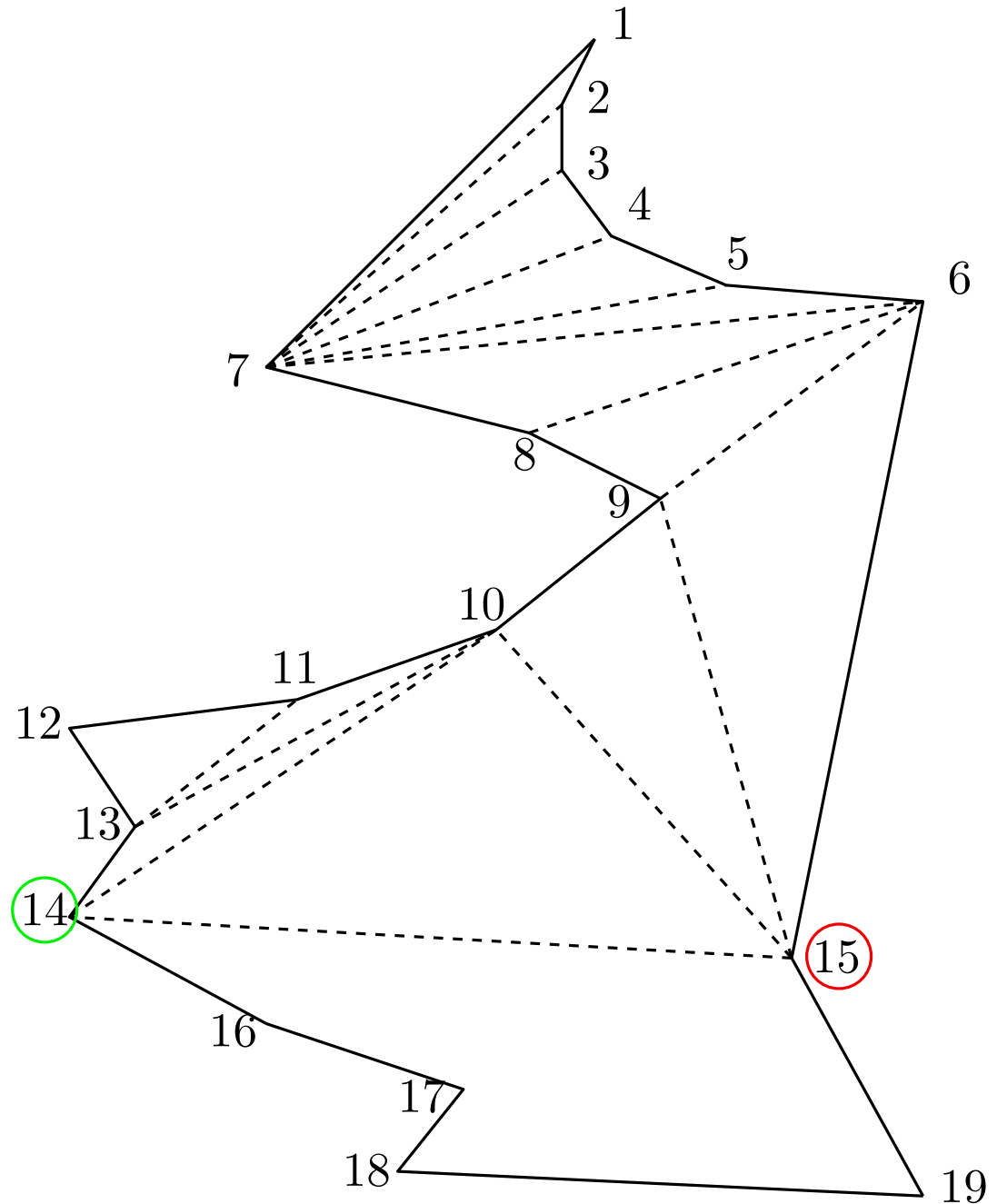
Triangulating a monotone polygon

Current vertex: 15

Opposite chain

Queue state:

14, 15



TRIANGULATING POLYGONS

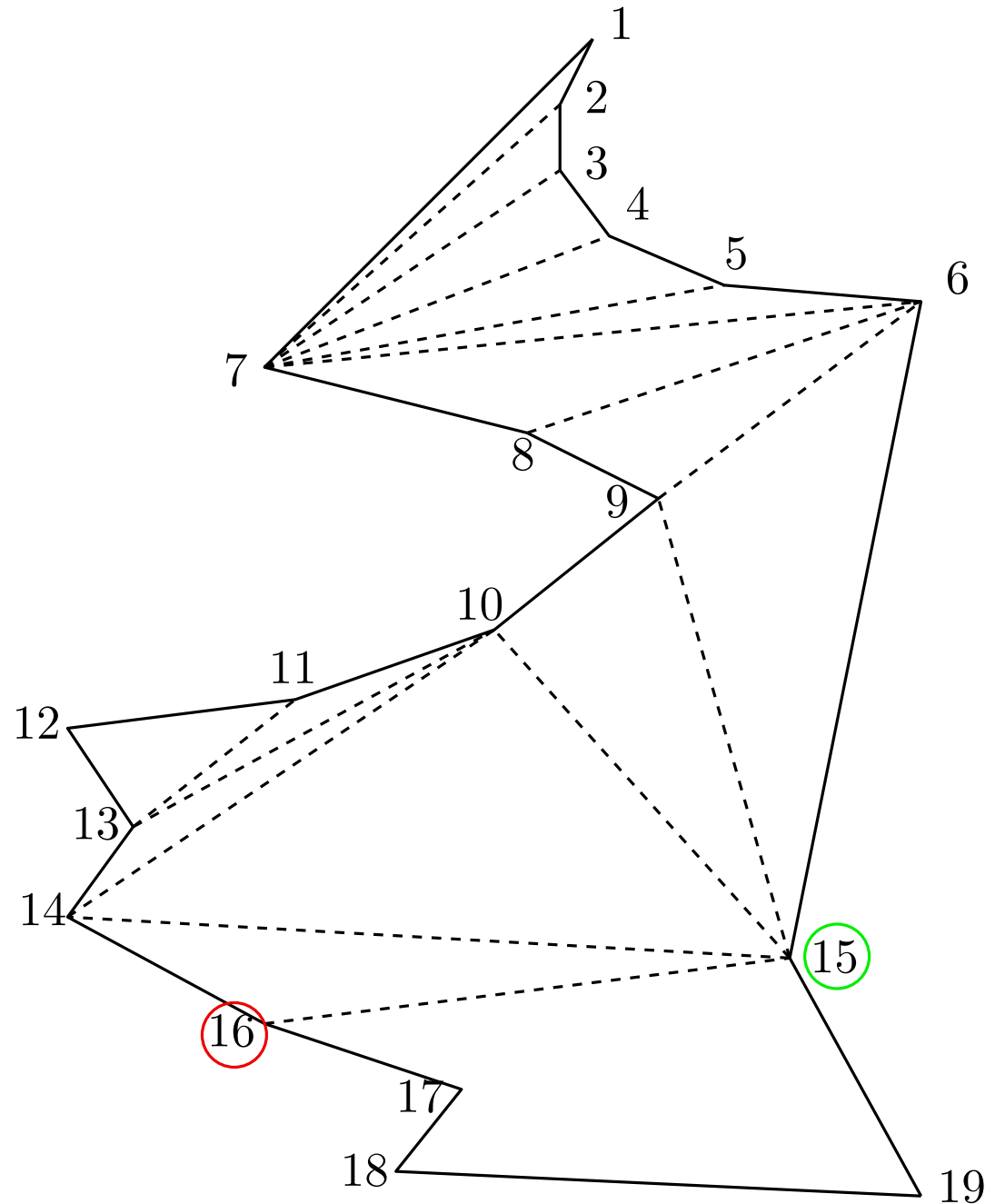
Triangulating a monotone polygon

Current vertex: 16

Opposite chain

Queue state:

15, 16



TRIANGULATING POLYGONS

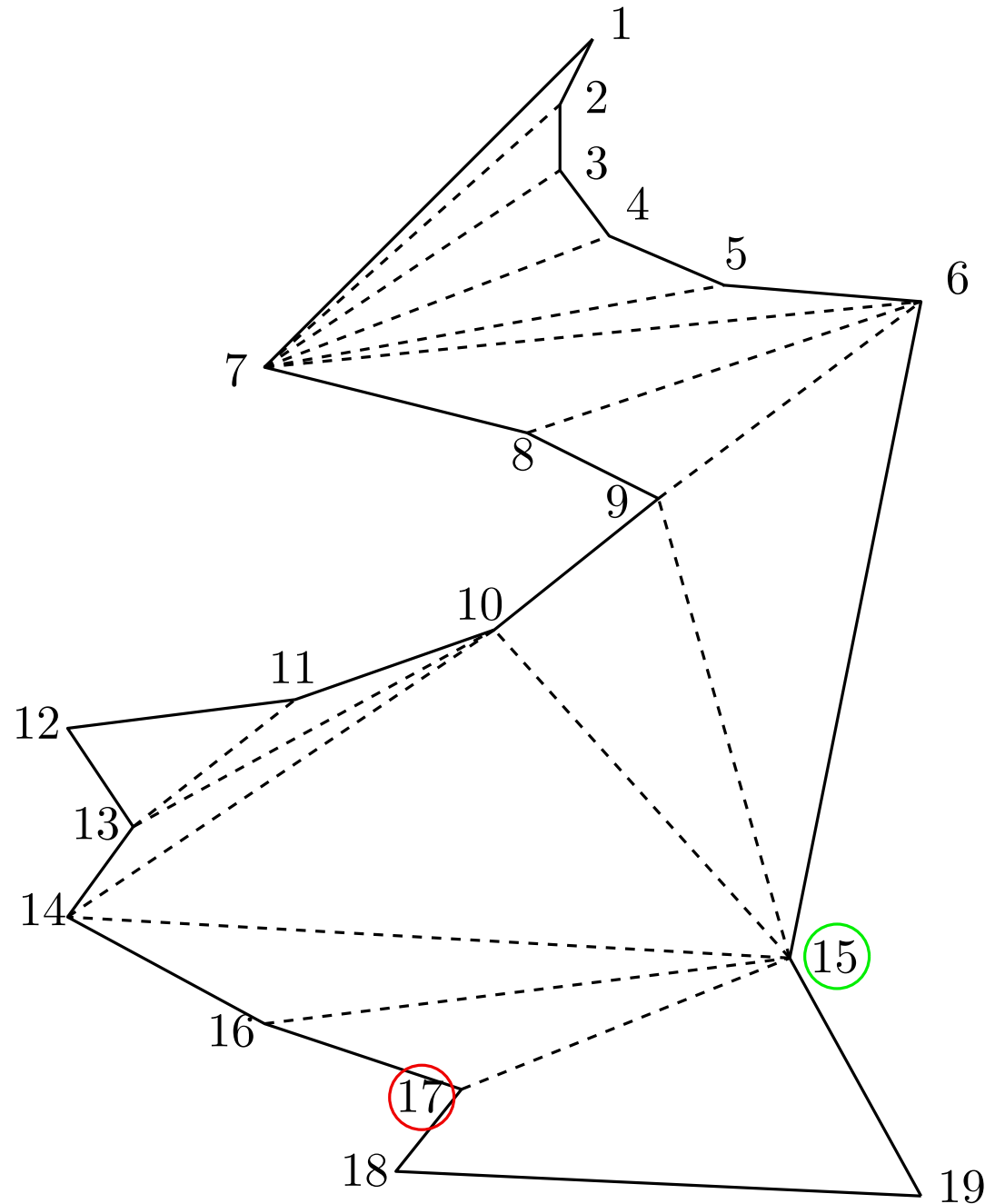
Triangulating a monotone polygon

Current vertex: 17

Ear

Queue state:

15, 17



TRIANGULATING POLYGONS

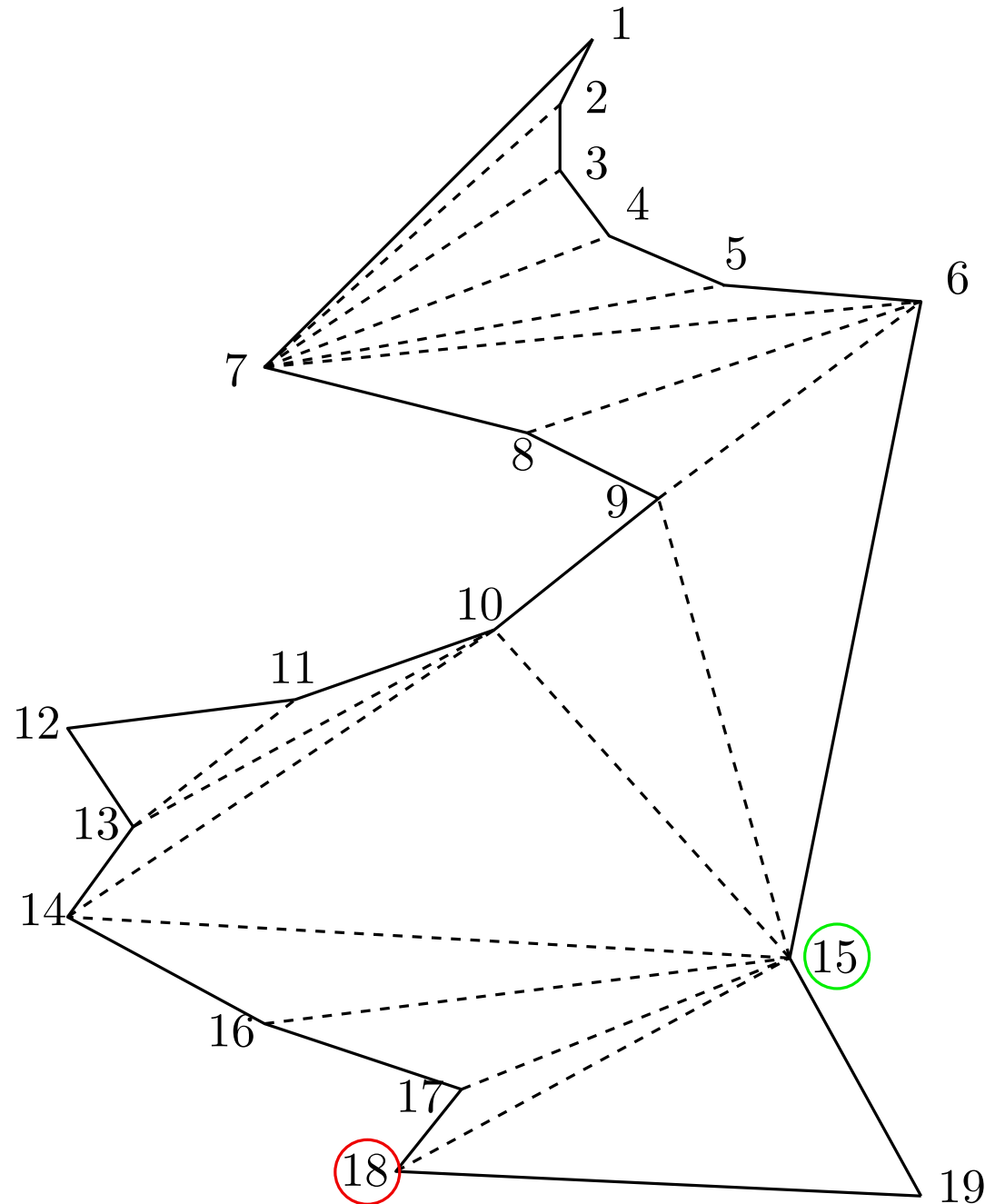
Triangulating a monotone polygon

Current vertex: 18

Ear

Queue state:

15, 18



TRIANGULATING POLYGONS

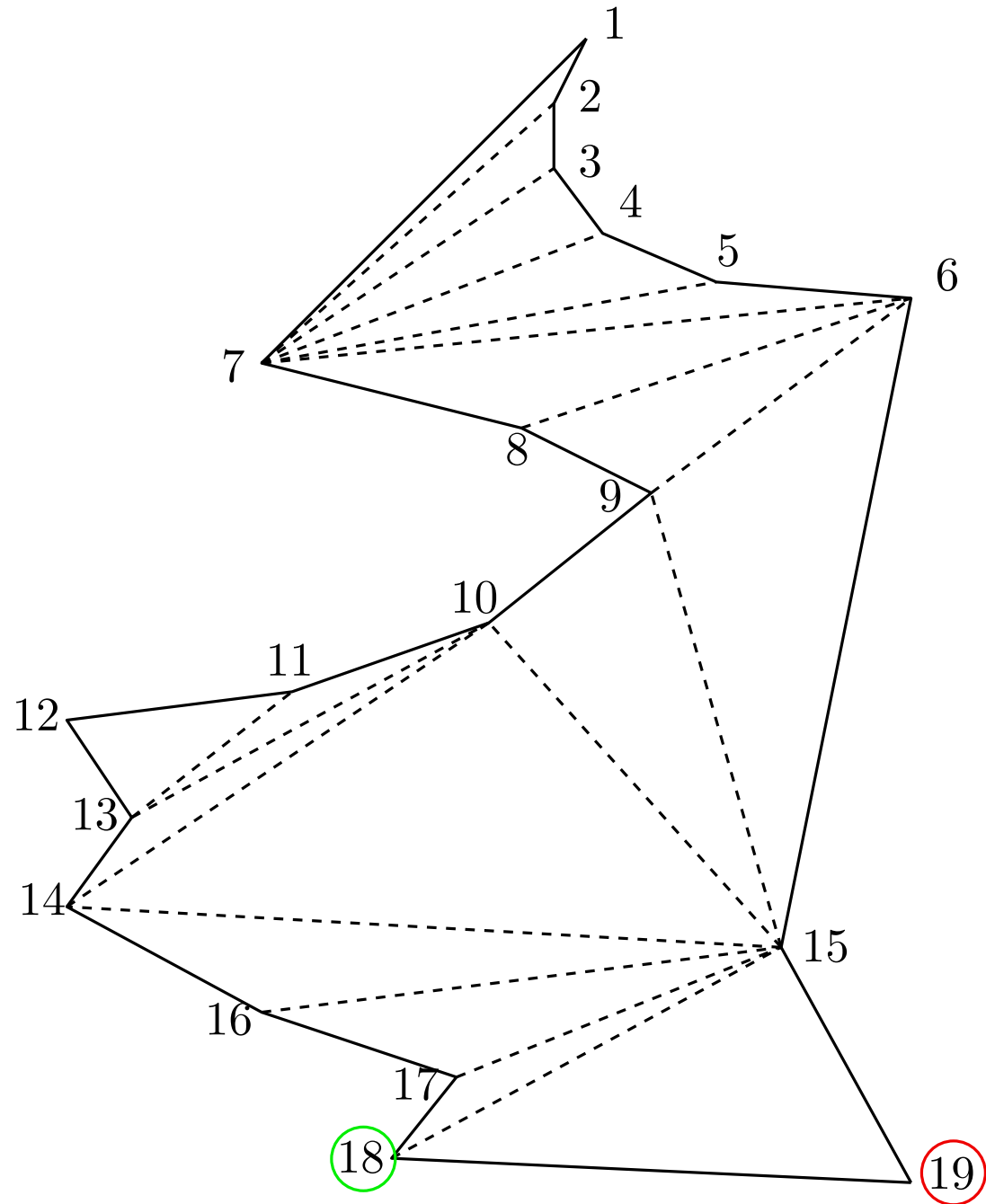
Triangulating a monotone polygon

Current vertex: 19

Opposite chain

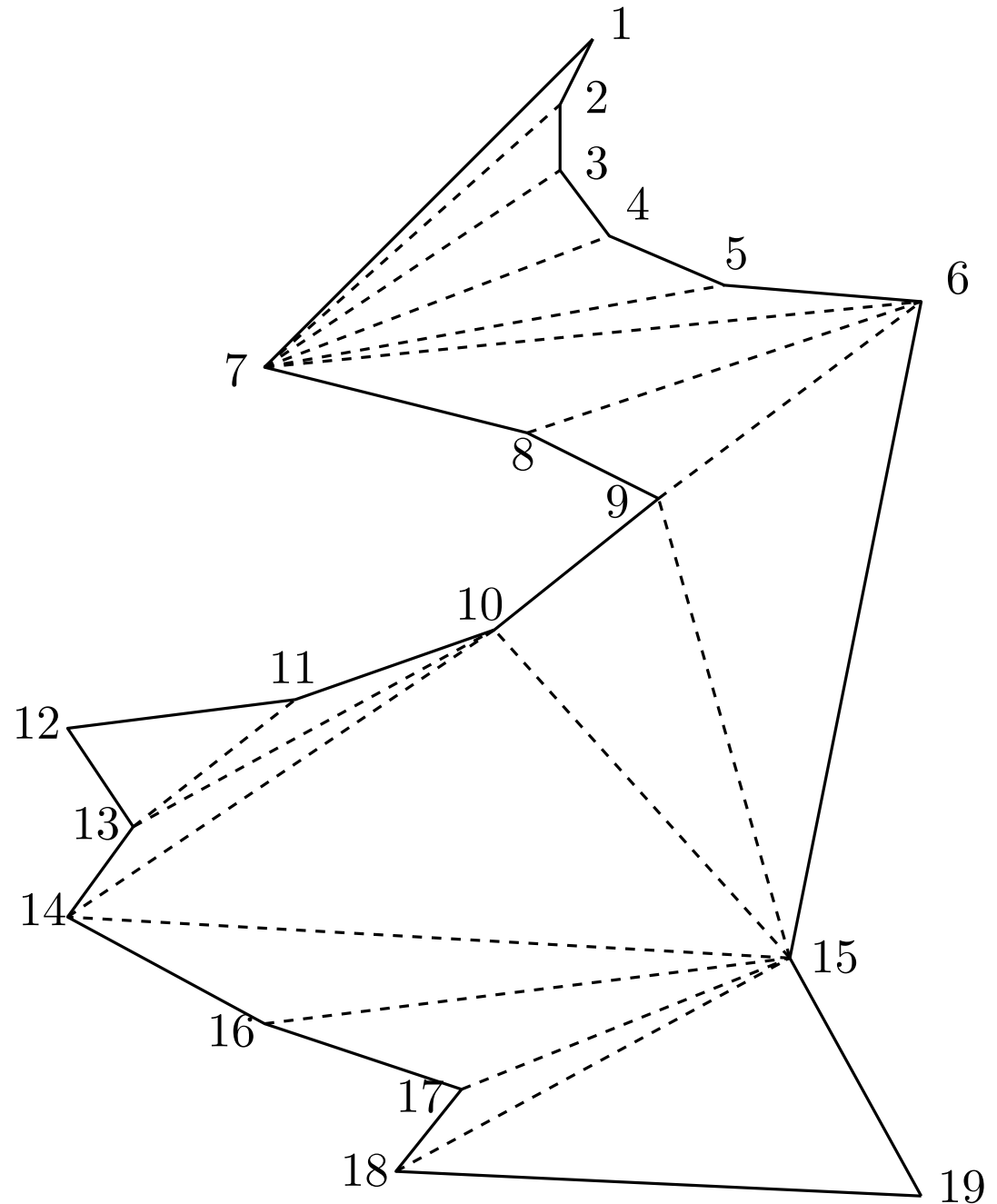
Queue state:

18, 19



TRIANGULATING POLYGONS

Triangulating a monotone polygon



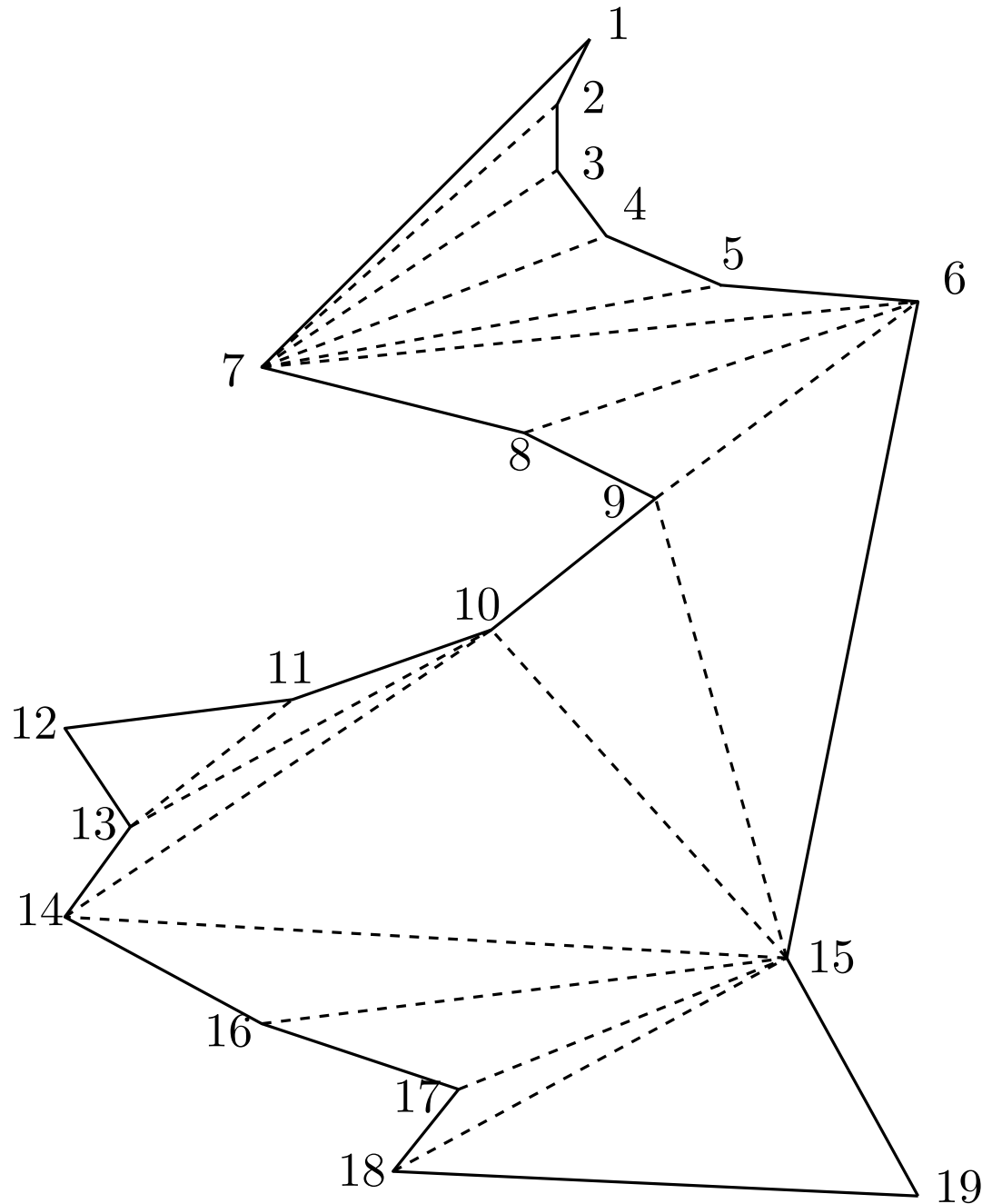
End

TRIANGULATING POLYGONS

Triangulating a monotone polygon

Running time: $O(n)$

Each vertex is removed from the queue Q in $O(1)$ time.



TRIANGULATING POLYGONS

Summarizing

Running time for triangulating a polygon:

- $O(n^2)$ by subtracting ears
- $O(n^2)$ by inserting diagonals

If the polygon is convex:

- $O(n)$ trivially

If the polygon is monotone:

- $O(n)$ scanning the monotone chains in order

TRIANGULATING POLYGONS

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Is it possible to be more efficient more general polygons?

TRIANGULATING POLYGONS

Summarizing

Running time for triangulating a polygon:

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- $O(n^2)$ by inserting diagonals

If the polygon is convex:

- $O(n)$ trivially

If the polygon is monotone:

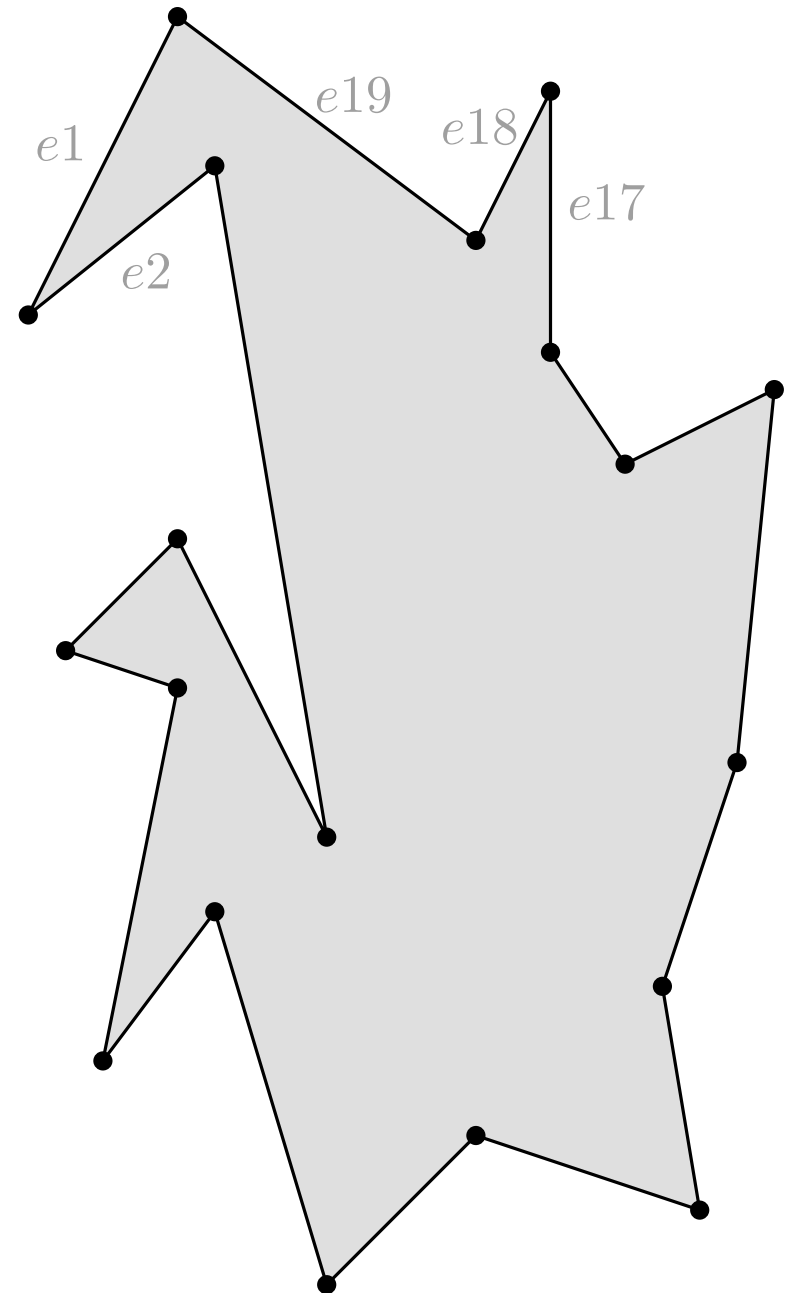
- $O(n)$ scanning the monotone chains in order

Is it possible to be more efficient more general polygons? Yes!

1. Decompose the polygon into monotone subpolygons
2. Triangulate the monotone subpolygons

TRIANGULATING POLYGONS

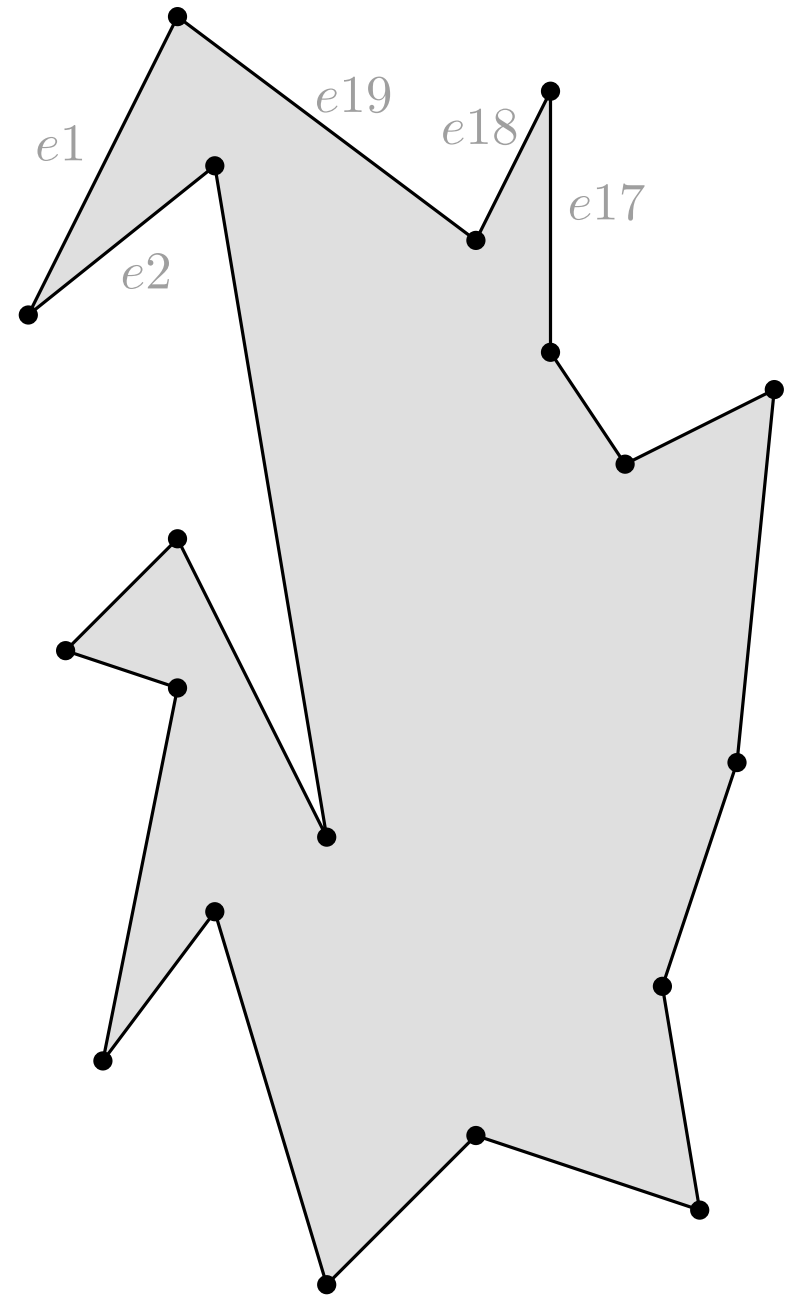
Monotone partition



TRIANGULATING POLYGONS

Monotone partition

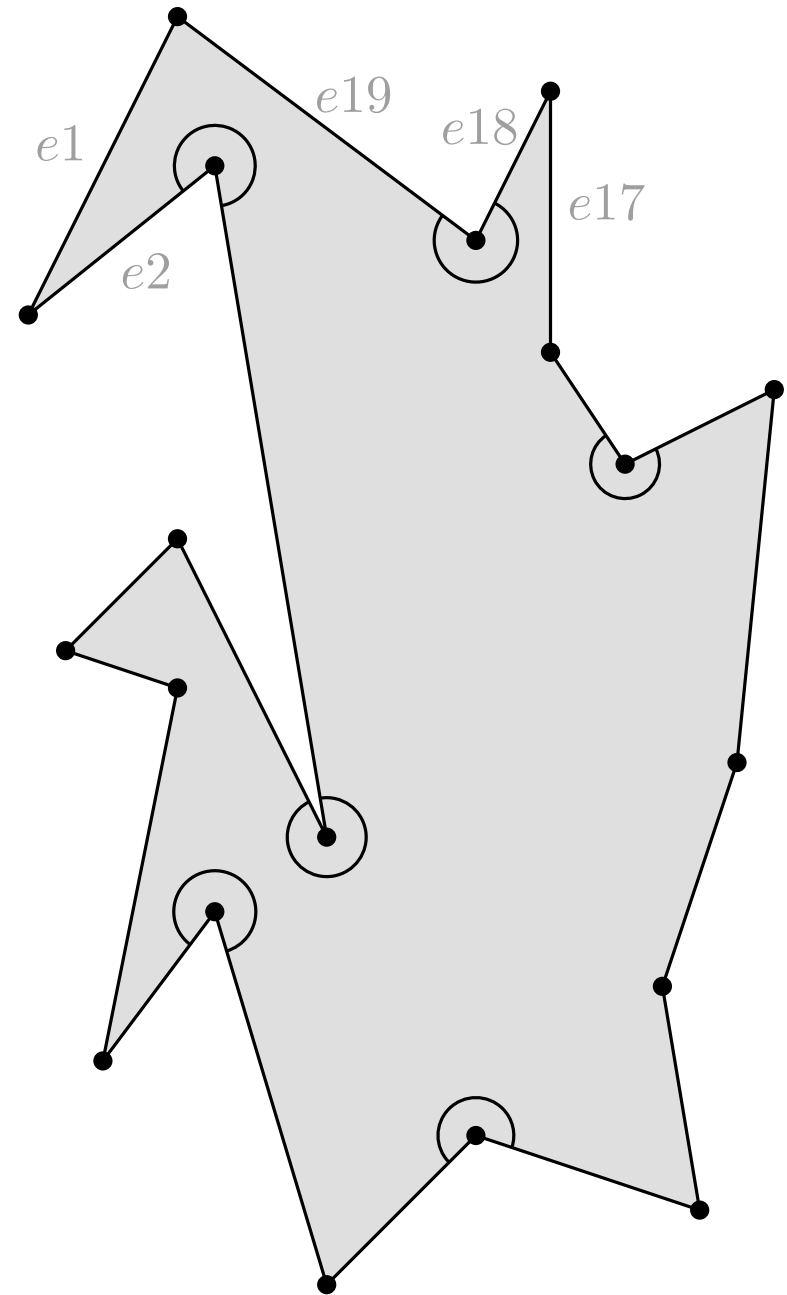
In order to create a monotone partition of a polygon, all cusps need to be “broken” by internal diagonals.



TRIANGULATING POLYGONS

Monotone partition

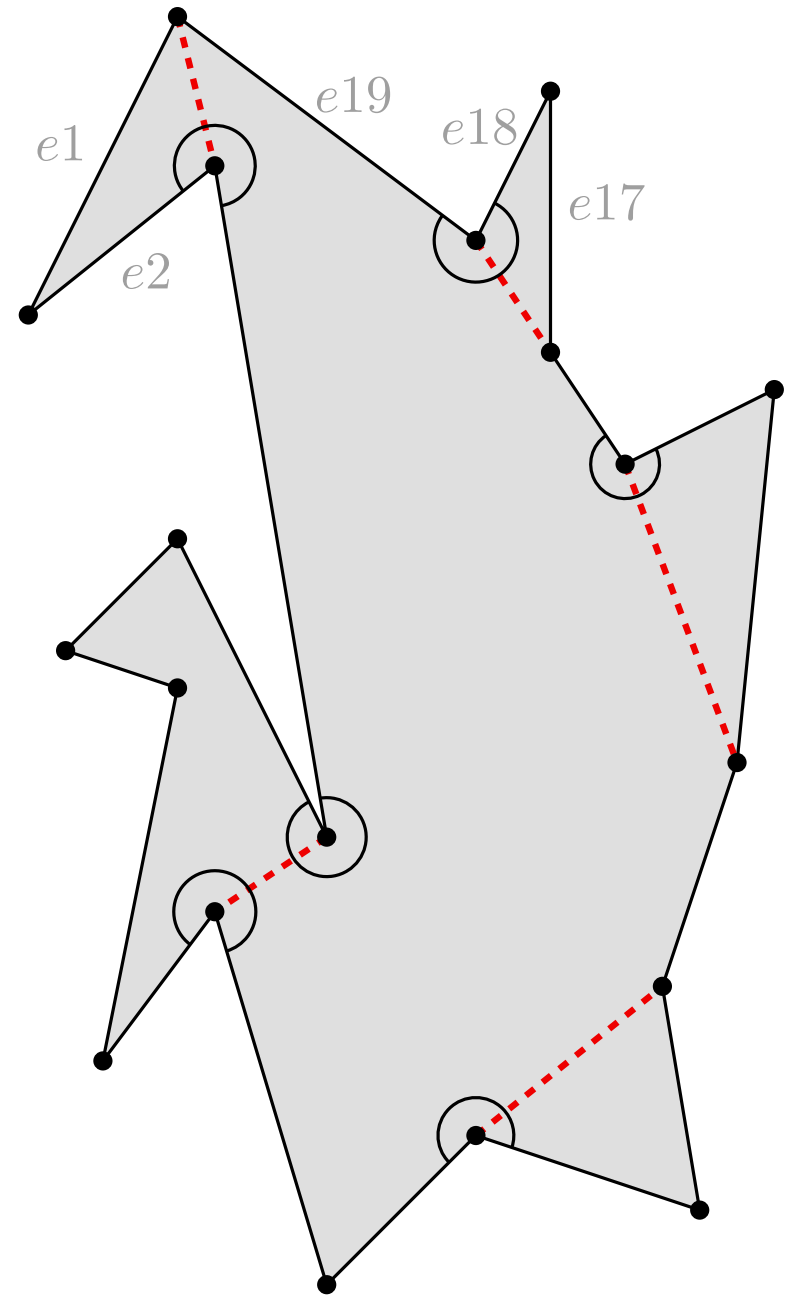
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TRIANGULATING POLYGONS

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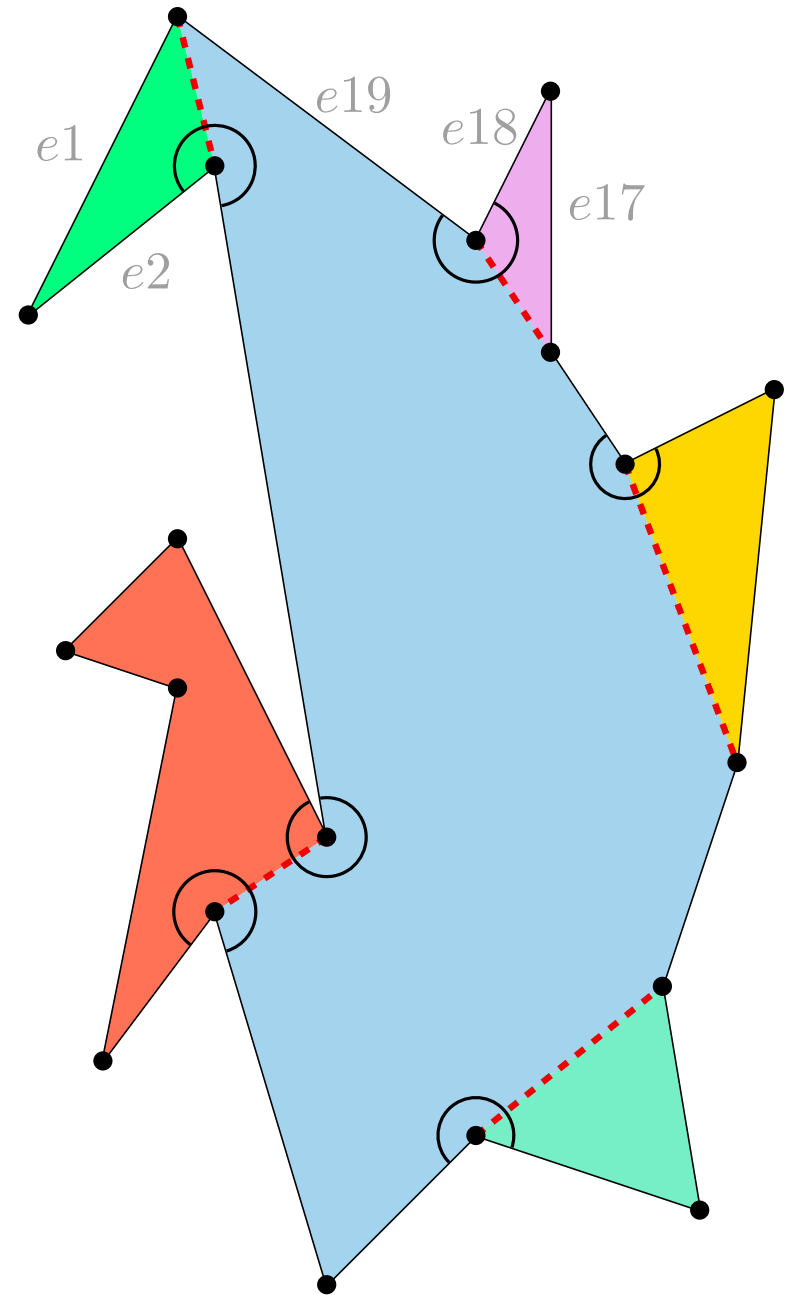
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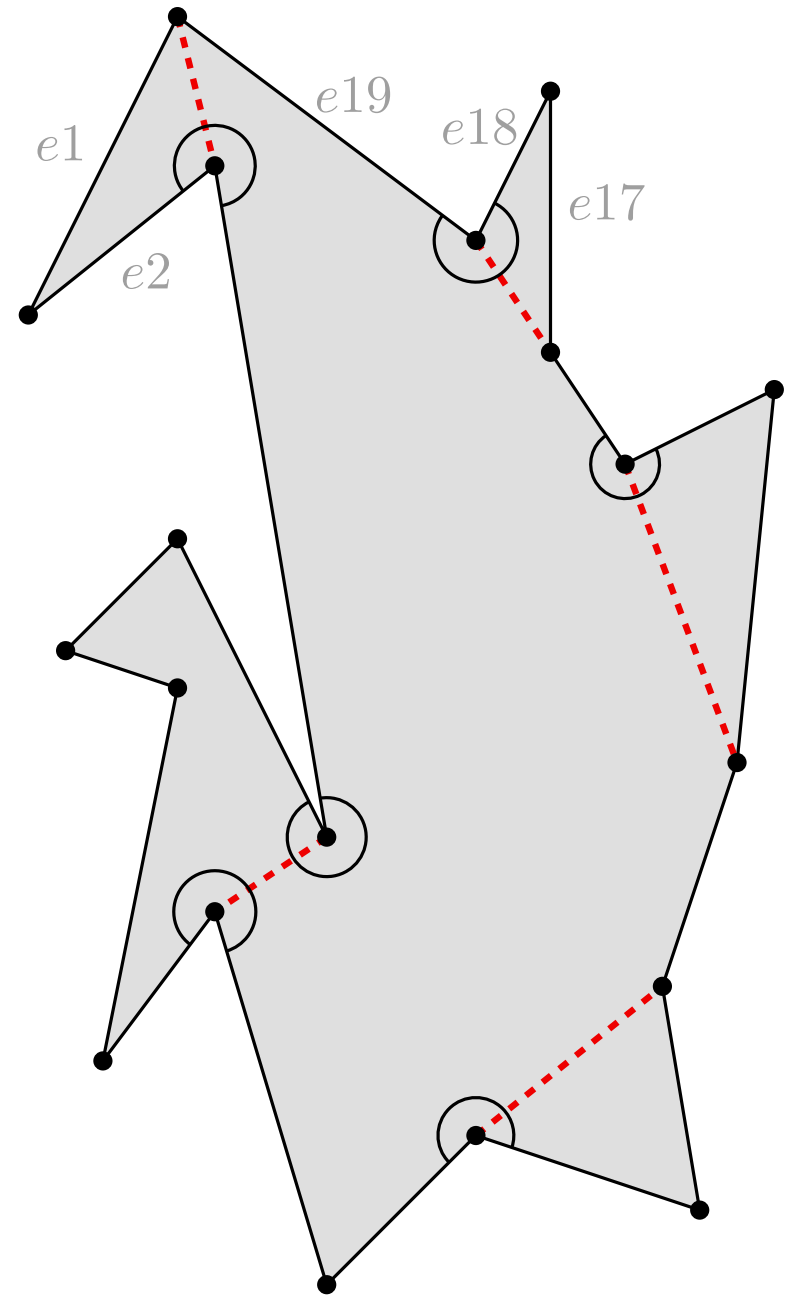


TRIANGULATING POLYGONS

Monotone partition

In order to create a monotone partition of a polygon, all cusps need to be “broken” by internal diagonals.

This can be done starting from a **trapezoidal decomposition** of the polygon.

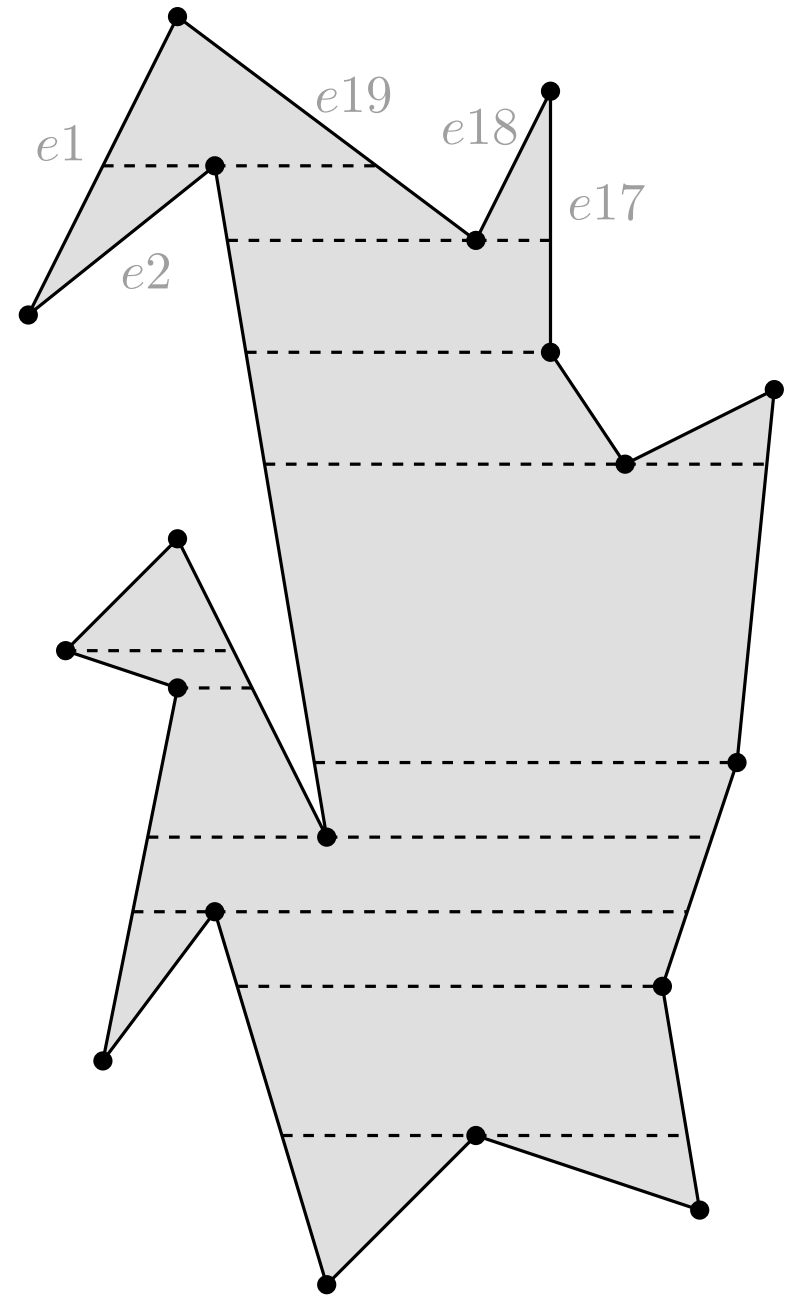


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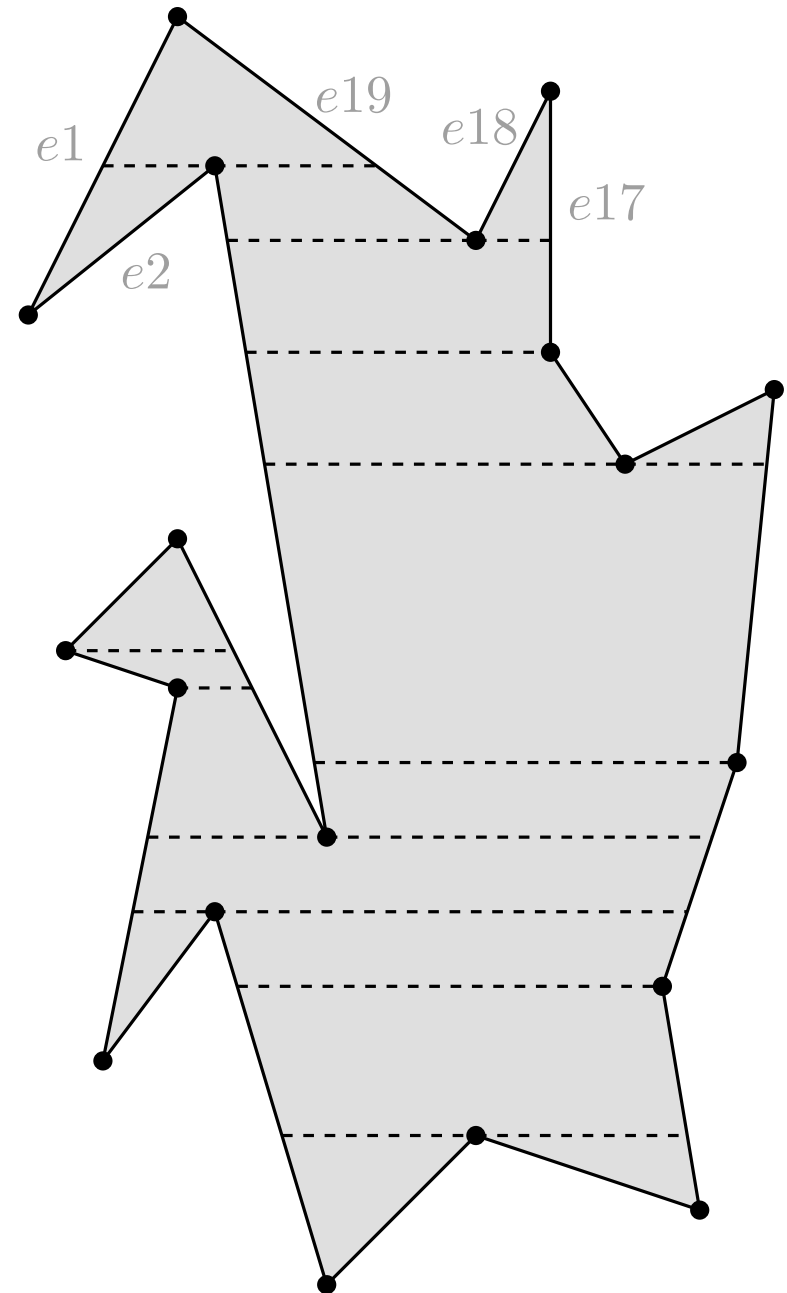
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Connect each cusp with the opposite vertex in its trapezoid (the upper trapezoid if the cusp is a local maximum, or the lower one if it is a local minimum).



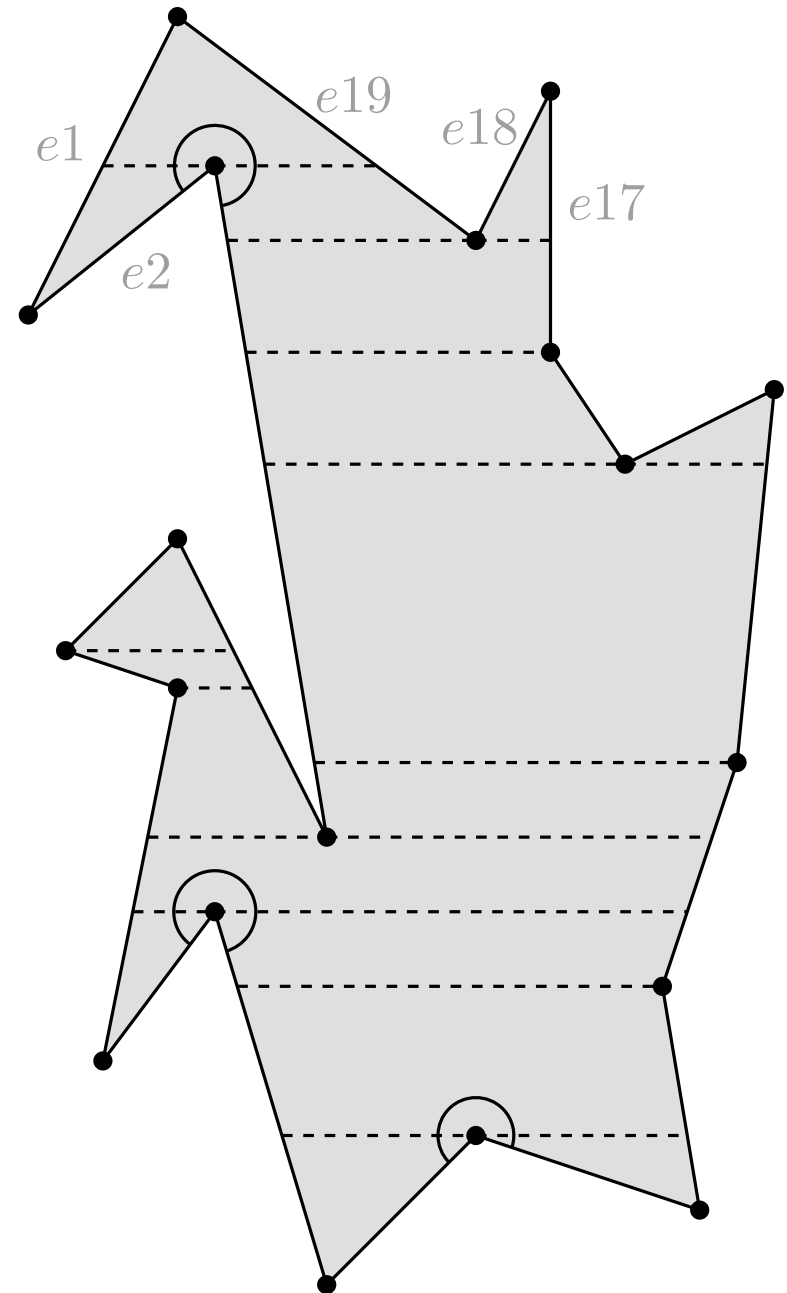
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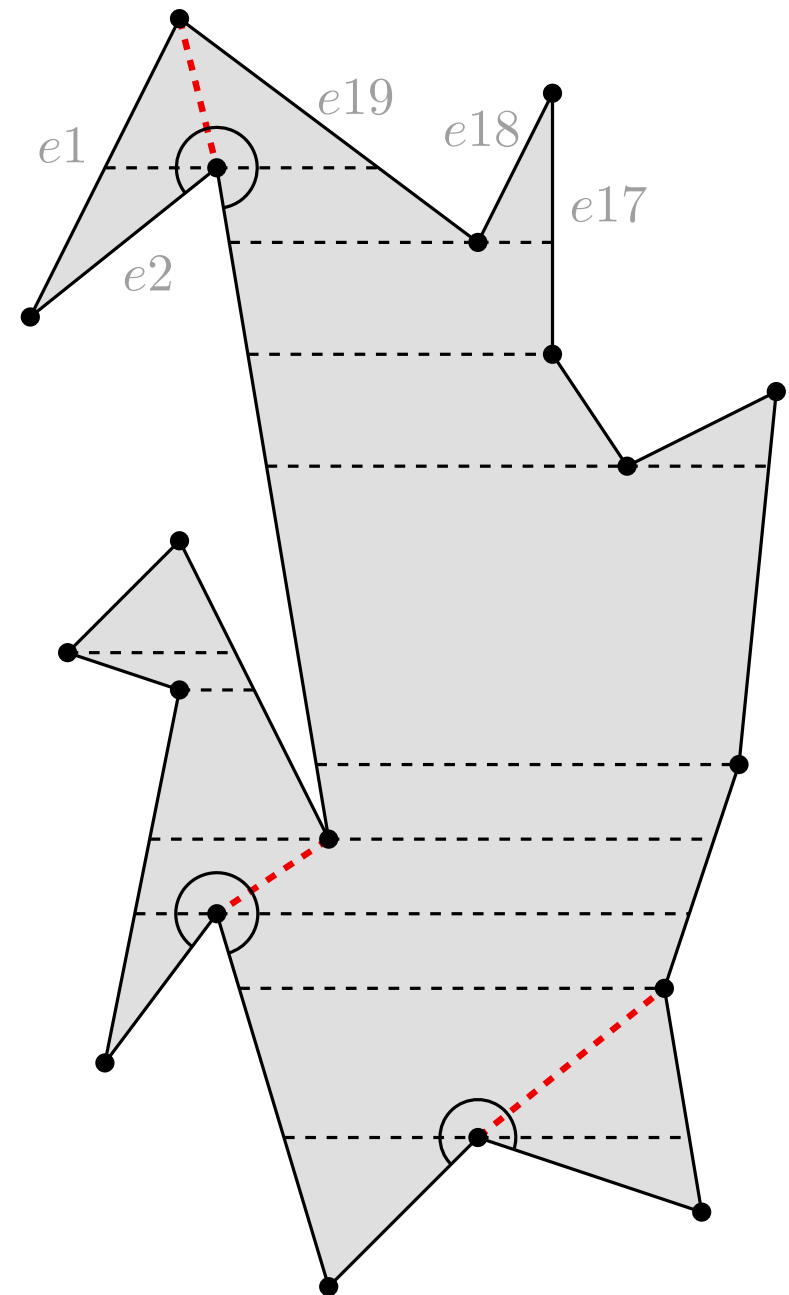
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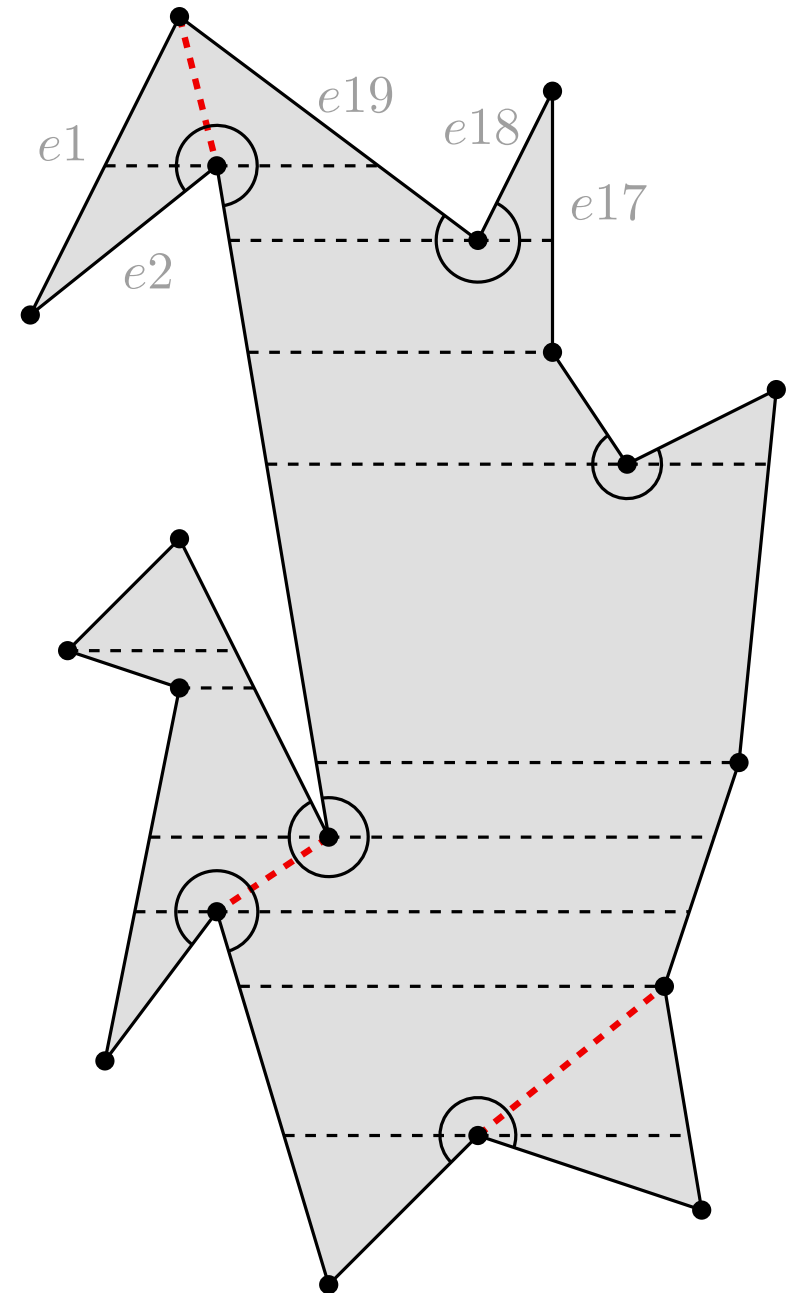
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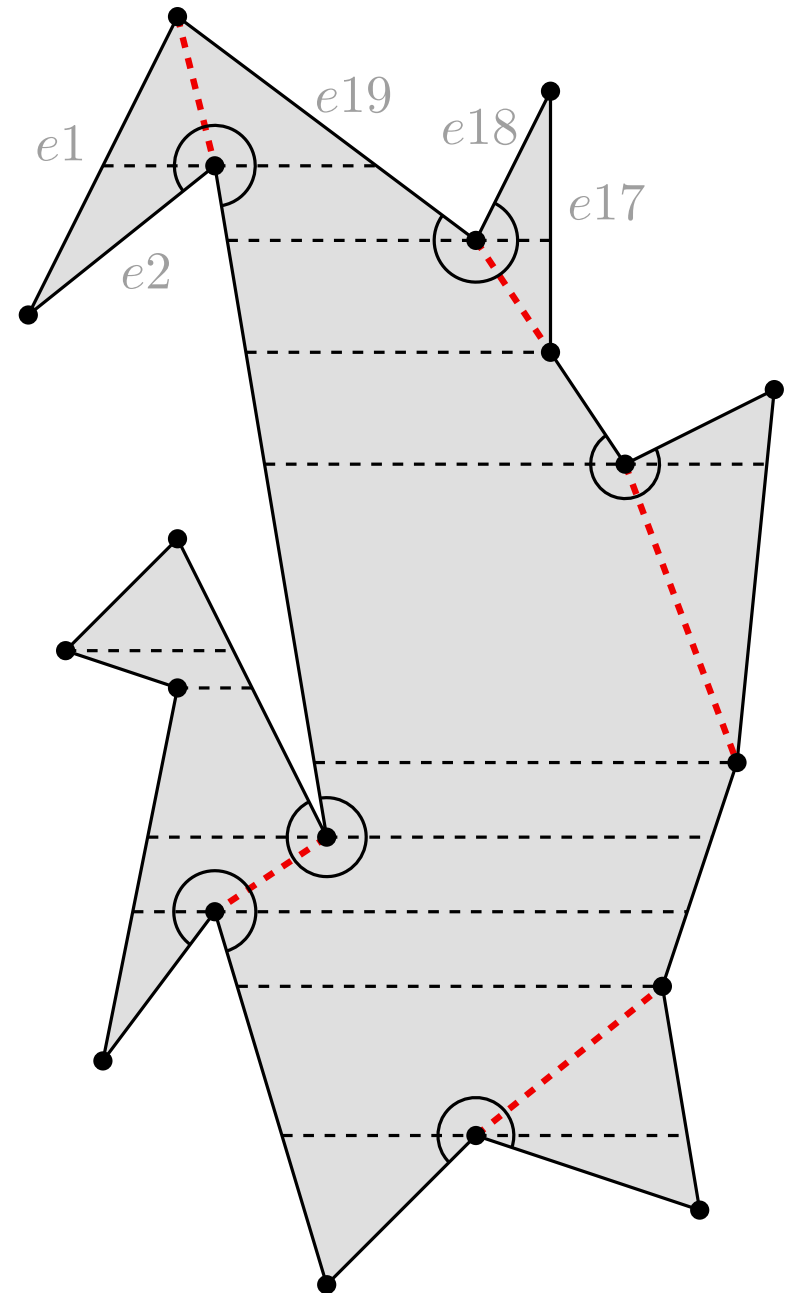
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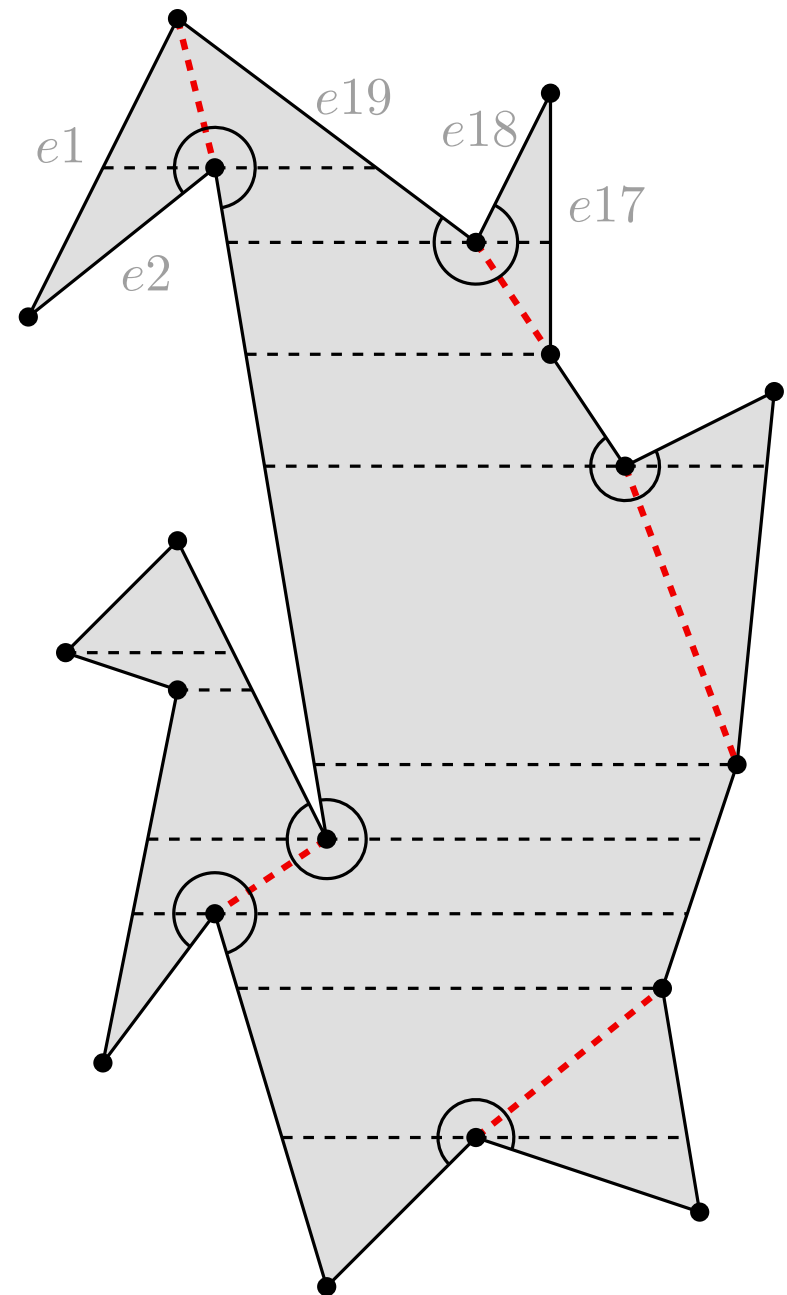
This can be done starting from a **trapezoidal decomposition** of the polygon.

Connect each cusp with the opposite vertex in its trapezoid (the upper trapezoid if the cusp is a local maximum, or the lower one if it is a local minimum).

This gives rise to a correct algorithm:

- The diagonals do not intersect, because they belong to different trapezoids.
- The polygon ends up decomposed into monotone subpolygons.

We can compute this with a **sweep line algorithm**



TRIANGULATING POLYGONS

Monotone partition

Sweep line algorithm

TRIANGULATING POLYGONS

Monotone partition

Sweep line algorithm

A straight line (horizontal, in this case) scans the object (the polygon) and allows detecting and constructing the desired elements (cusps, trapezoids, diagonals), leaving the problem solved behind it. The sweeping process is discretized.

TRIANGULATING POLYGONS

Monotone partition

Sweep line algorithm

A straight line (horizontal, in this case) scans the object (the polygon) and allows detecting and constructing the desired elements (cusps, trapezoids, diagonals), leaving the problem solved behind it. The sweeping process is discretized.

Recall the essential elements of a sweep line algorithm:

- Events queue

Priority queue keeping the information of the events (stop points). In our problem: vertices of the polygon, sorted by y -coordinate (all known in advance).

- Sweep line

Data structure storing the information of the portion of the object intersected by the sweep line. It gets updated at each event. In our problem: edges of the polygon intersected by the sweep line, sorted by x -coordinate, as well as the list of active trapezoids and their upper vertices.

TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

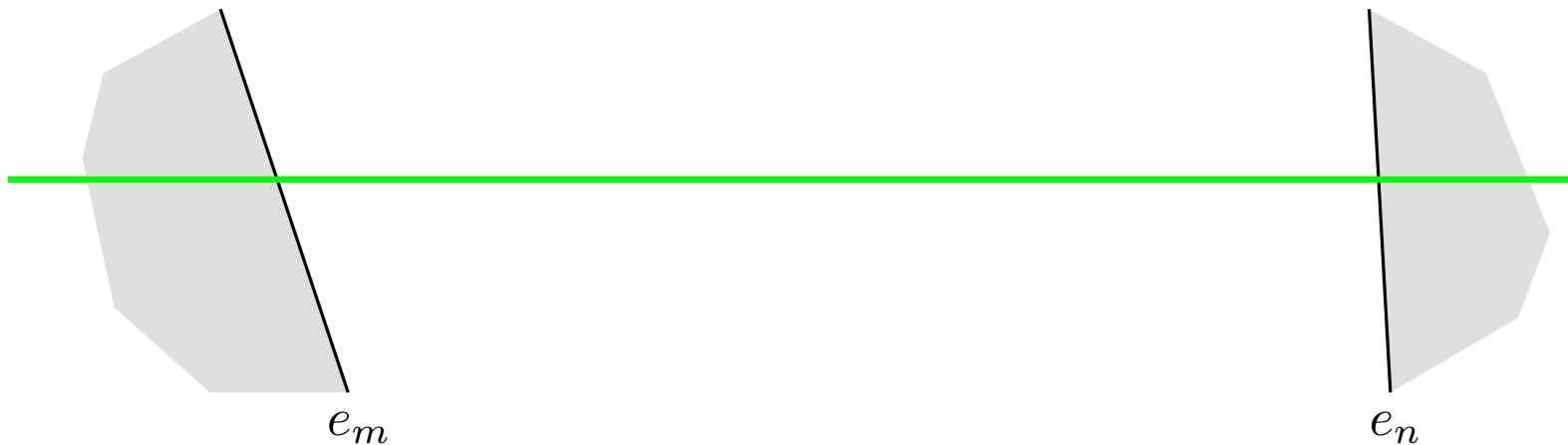
- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

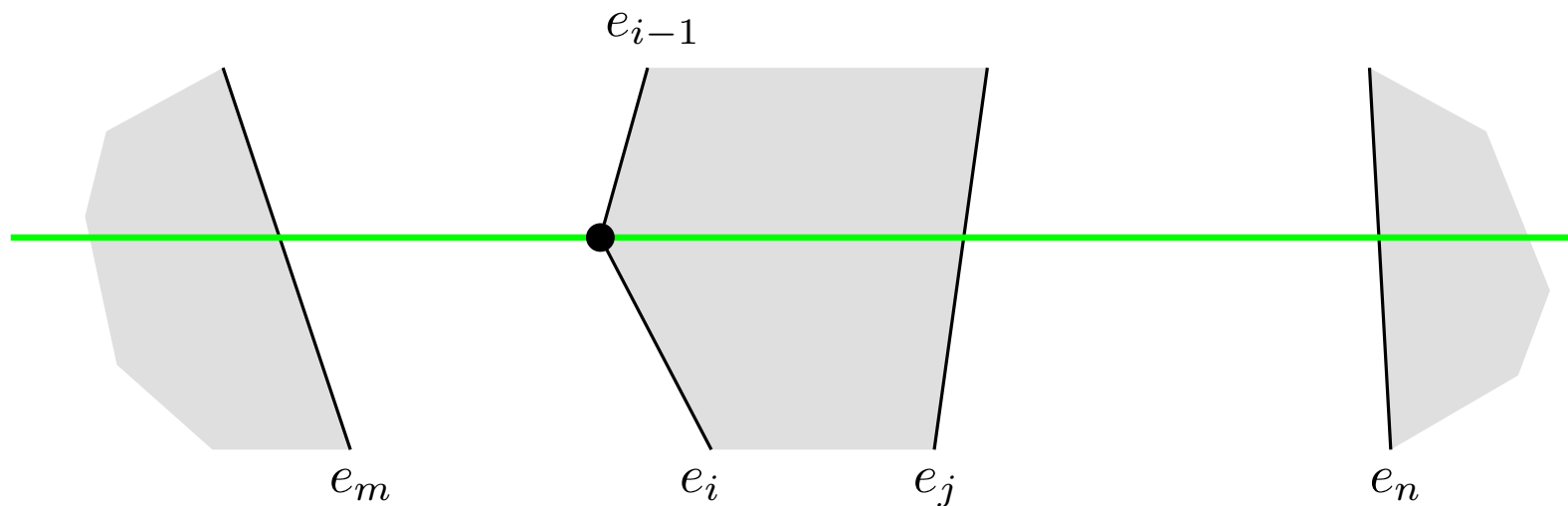


TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

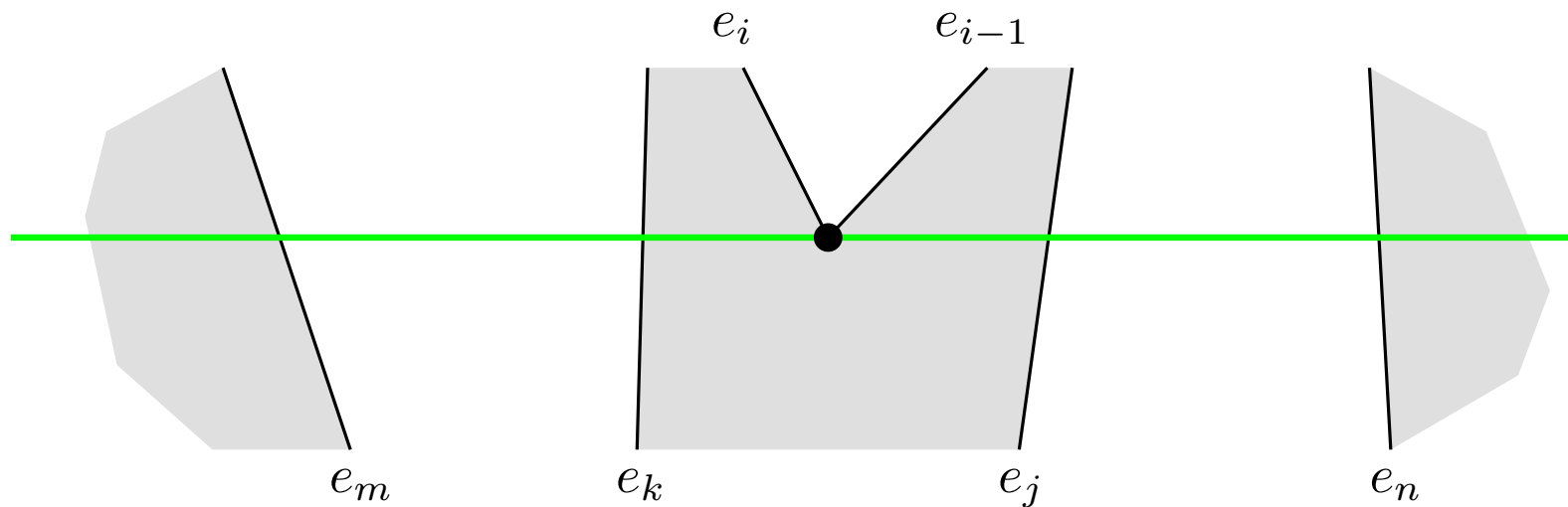


TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

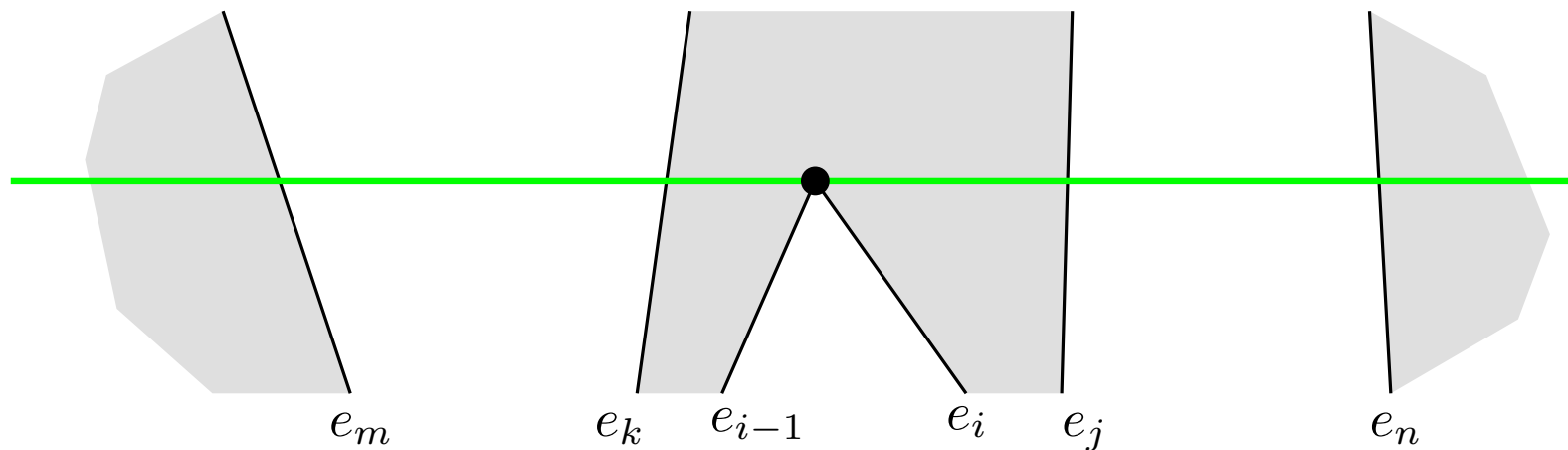


TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

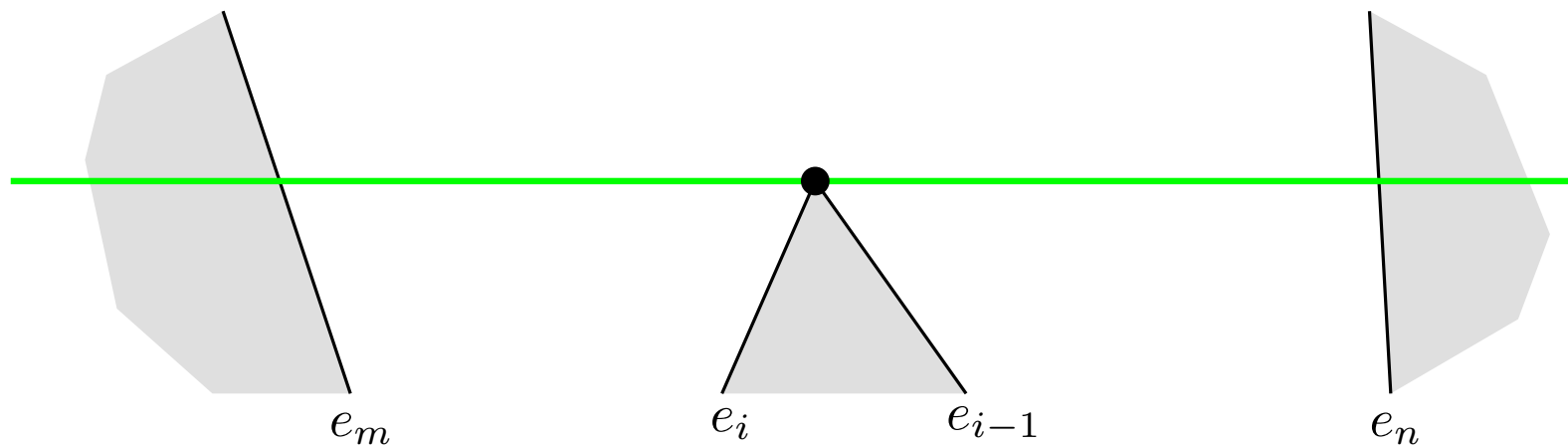


TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

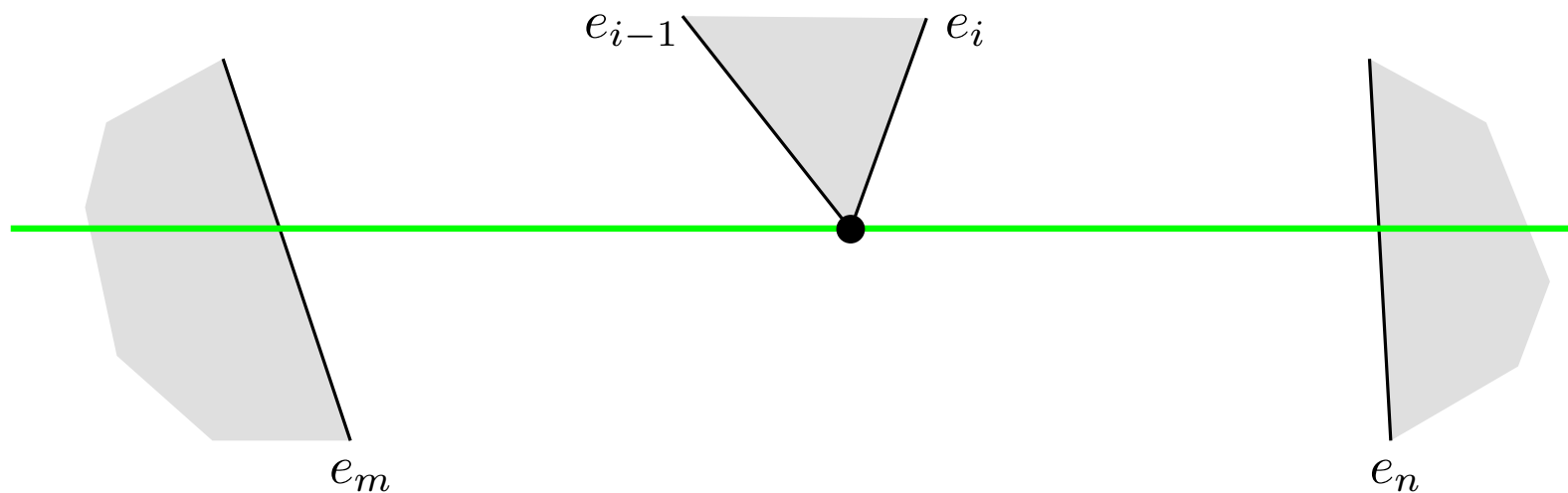


TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
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- Initial vertex: insert
- Final vertex: delete



TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

In addition, eventually update the information of the upper vertex of the starting trapezoid.

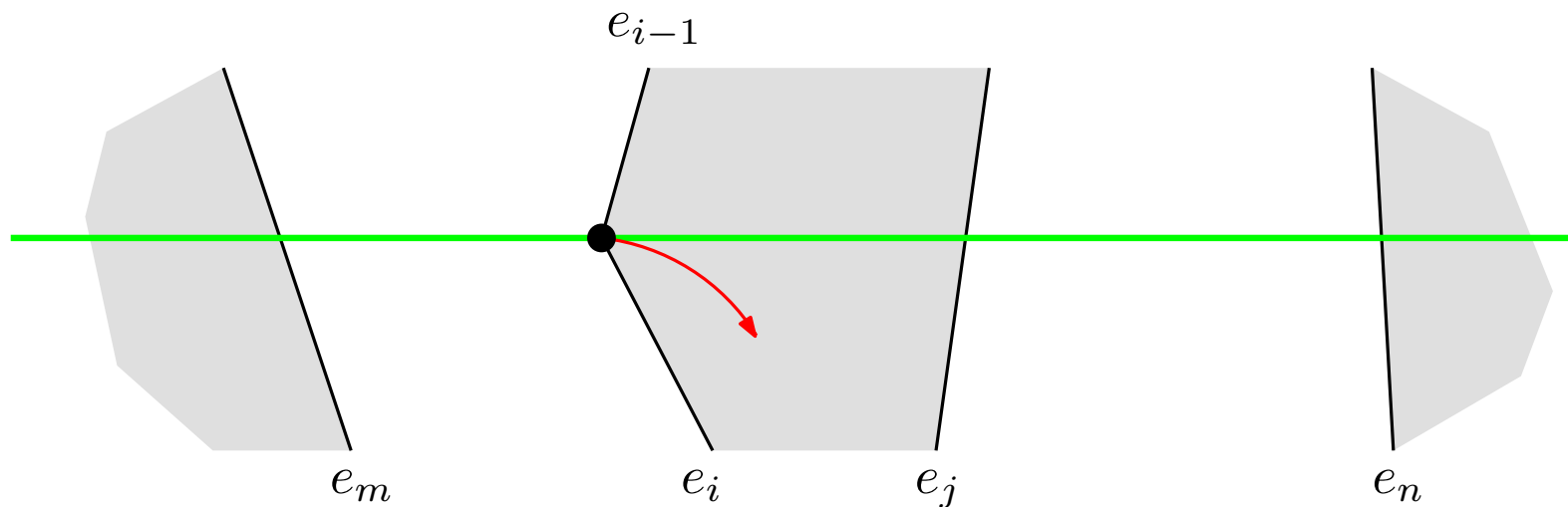
TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

In addition, eventually update the information of the upper vertex of the starting trapezoid.



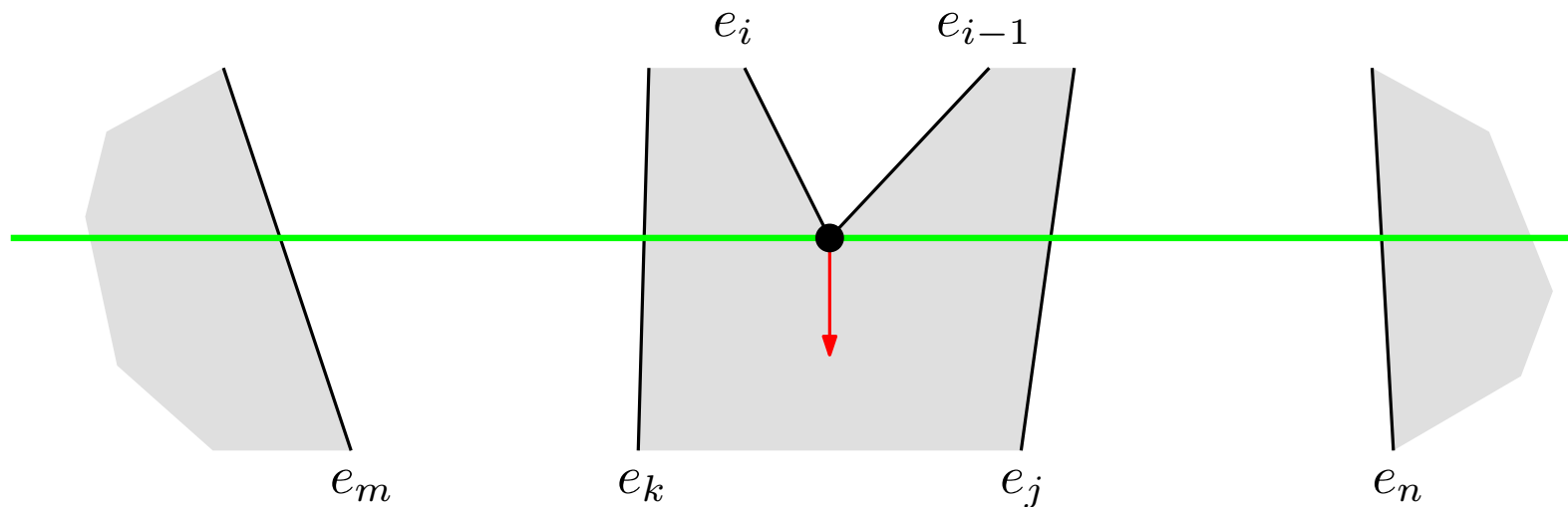
TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

- Passing vertex: replace
- Local minimum cusp: delete
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- Initial vertex: insert
- Final vertex: delete

In addition, eventually update the information of the upper vertex of the starting trapezoid.



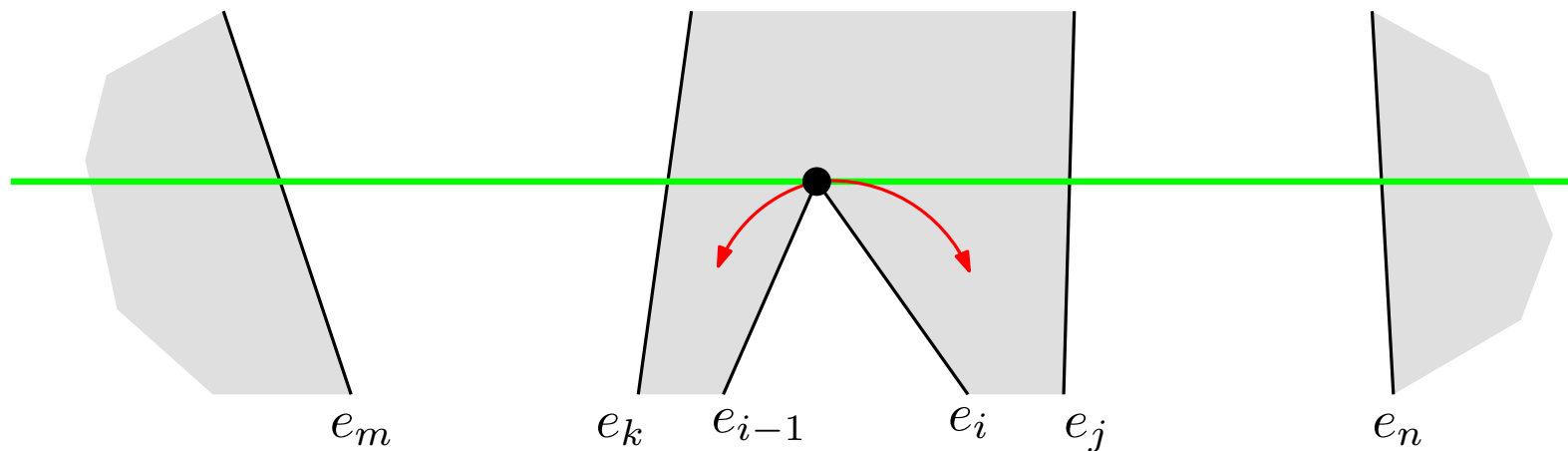
TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

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- Local minimum cusp: delete
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- Final vertex: delete

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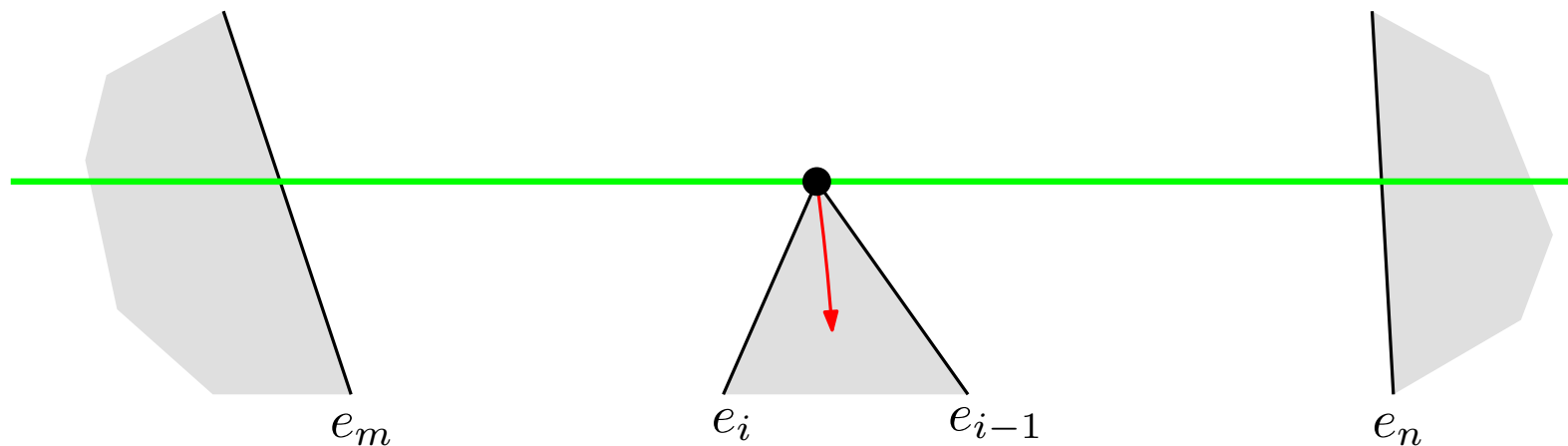
TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

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In addition, eventually update the information of the upper vertex of the starting trapezoid.



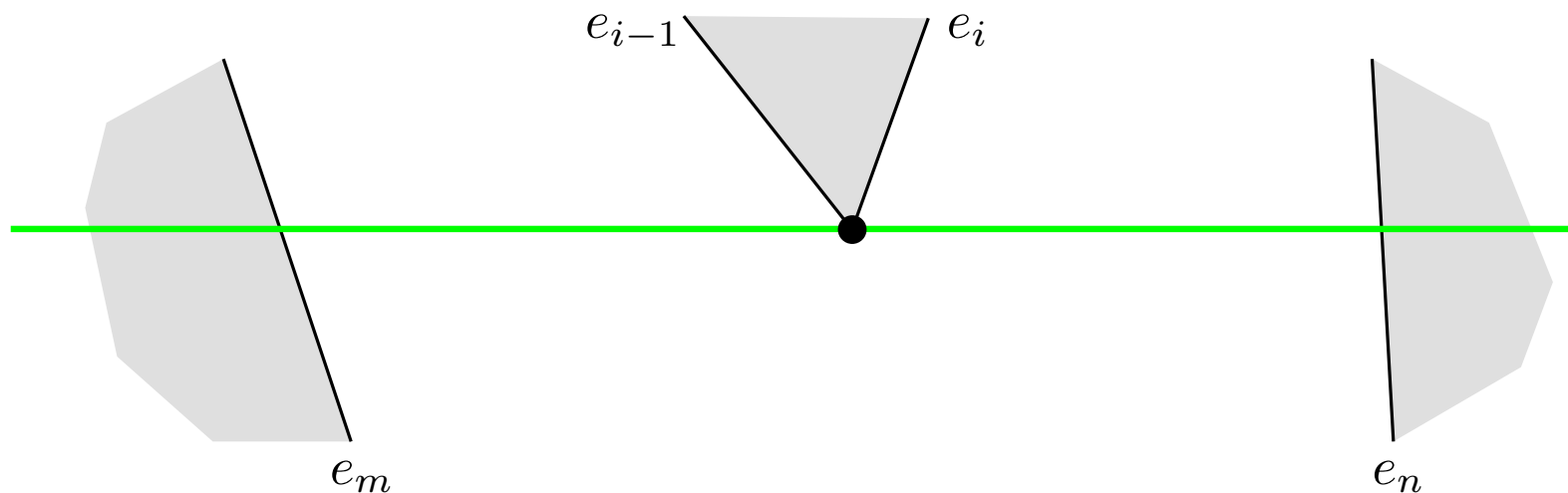
TRIANGULATING POLYGONS

Monotone partition

Updating the sweep line:

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- Local maximum cusp: insert
- Initial vertex: insert
- Final vertex: delete

In addition, eventually update the information of the upper vertex of the starting trapezoid.



TRIANGULATING POLYGONS

Monotone partition

As for diagonals:

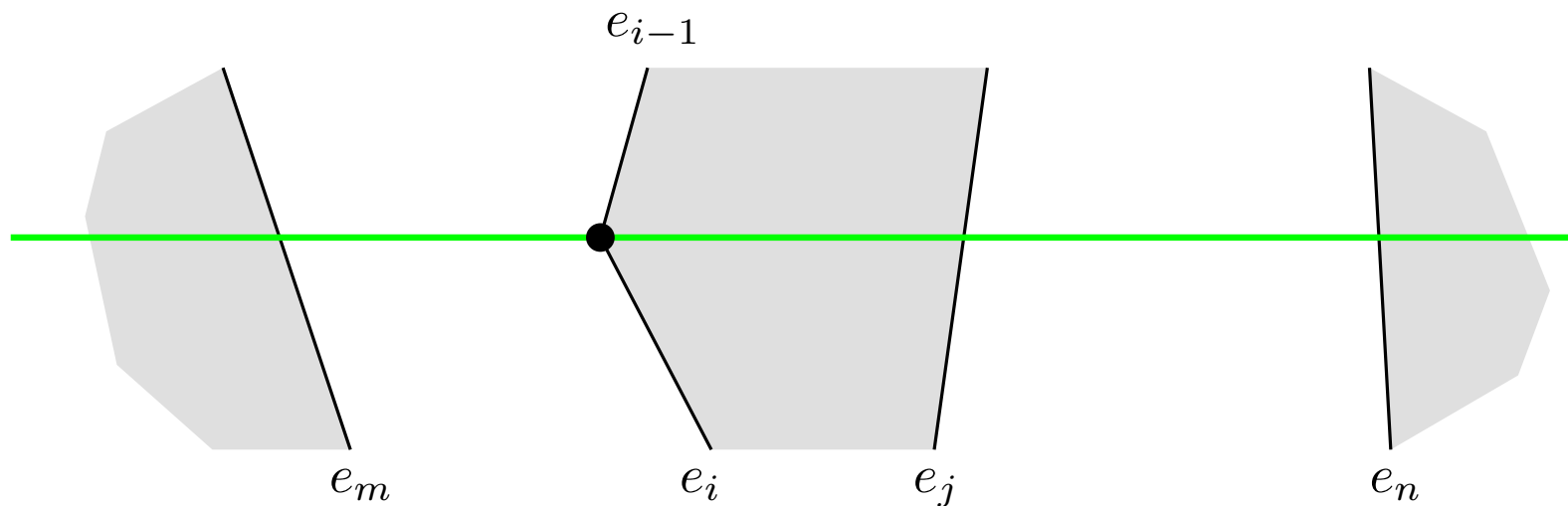
- Passing vertex: nothing
- Local minimum cusp: wait until it can be connected to final vertex of starting trapezoid
- Local maximum cusp: connect it with the initial vertex of the ending trapezoid
- Initial vertex: nothing
- Final vertex: nothing

TRIANGULATING POLYGONS

Monotone partition

As for diagonals:

- Passing vertex: nothing
- Local minimum cusp: wait until it can be connected to final vertex of starting trapezoid
- Local maximum cusp: connect it with the initial vertex of the ending trapezoid
- Initial vertex: nothing
- Final vertex: nothing

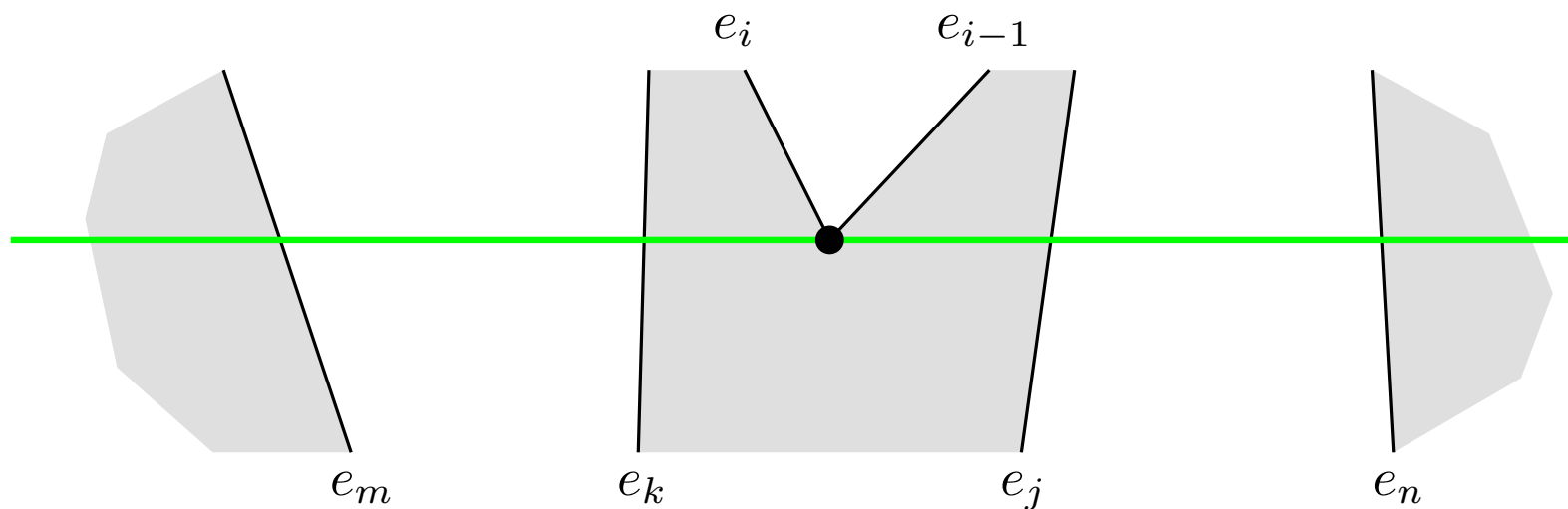


TRIANGULATING POLYGONS

Monotone partition

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- Passing vertex: nothing
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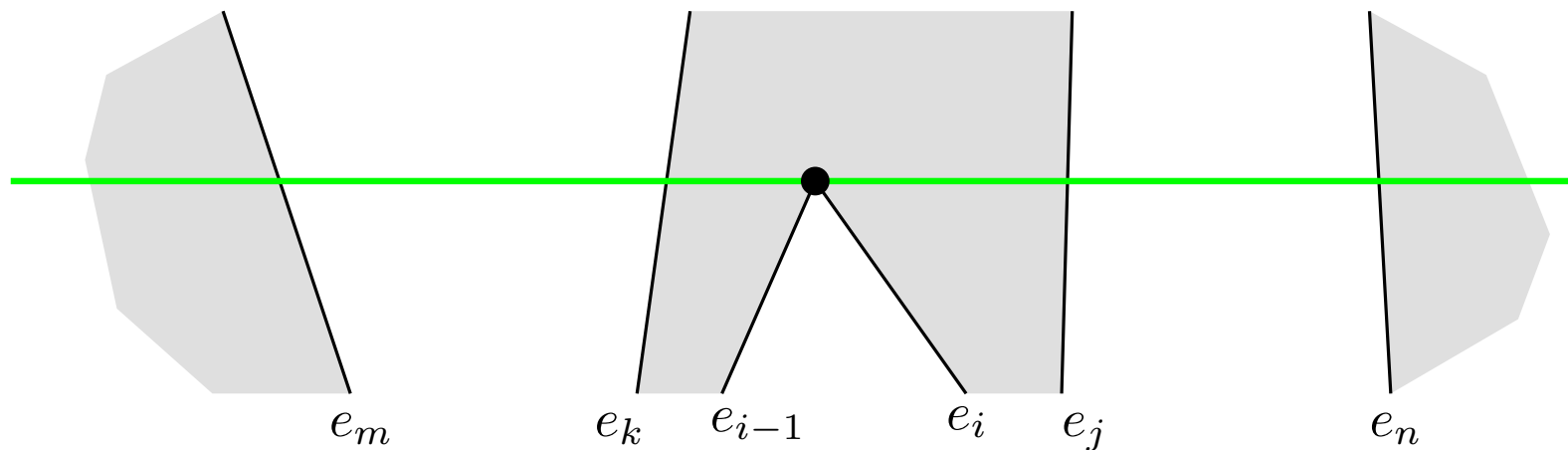


TRIANGULATING POLYGONS

Monotone partition

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- Local minimum cusp: wait until it can be connected to final vertex of starting trapezoid
- Local maximum cusp: connect it with the initial vertex of the ending trapezoid
- Initial vertex: nothing
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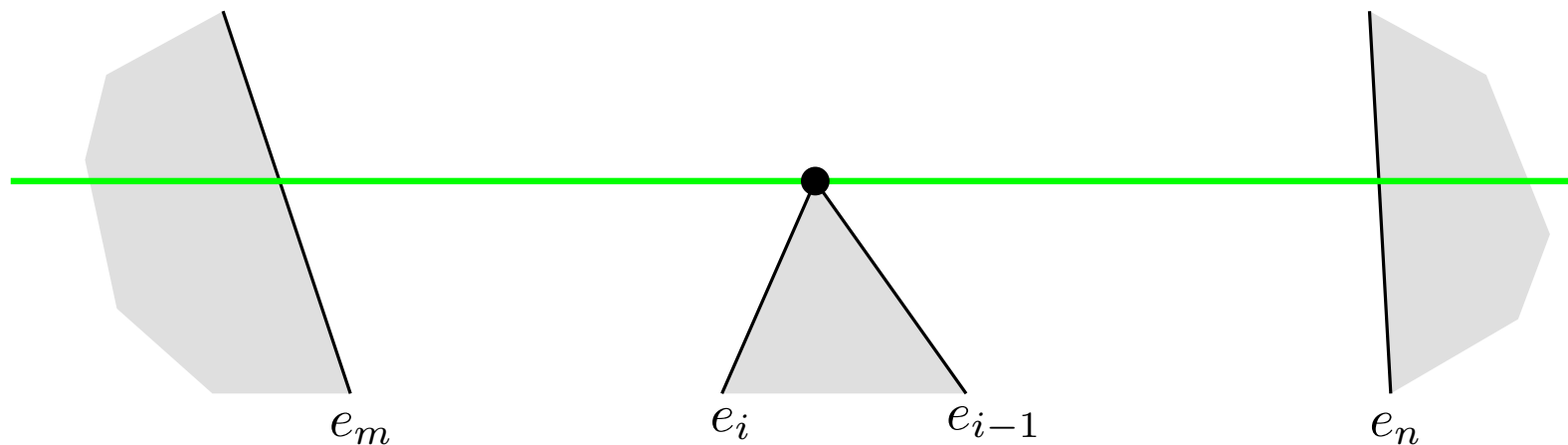


TRIANGULATING POLYGONS

Monotone partition

As for diagonals:

- Passing vertex: nothing
- Local minimum cusp: wait until it can be connected to final vertex of starting trapezoid
- Local maximum cusp: connect it with the initial vertex of the ending trapezoid
- Initial vertex: nothing
- Final vertex: nothing

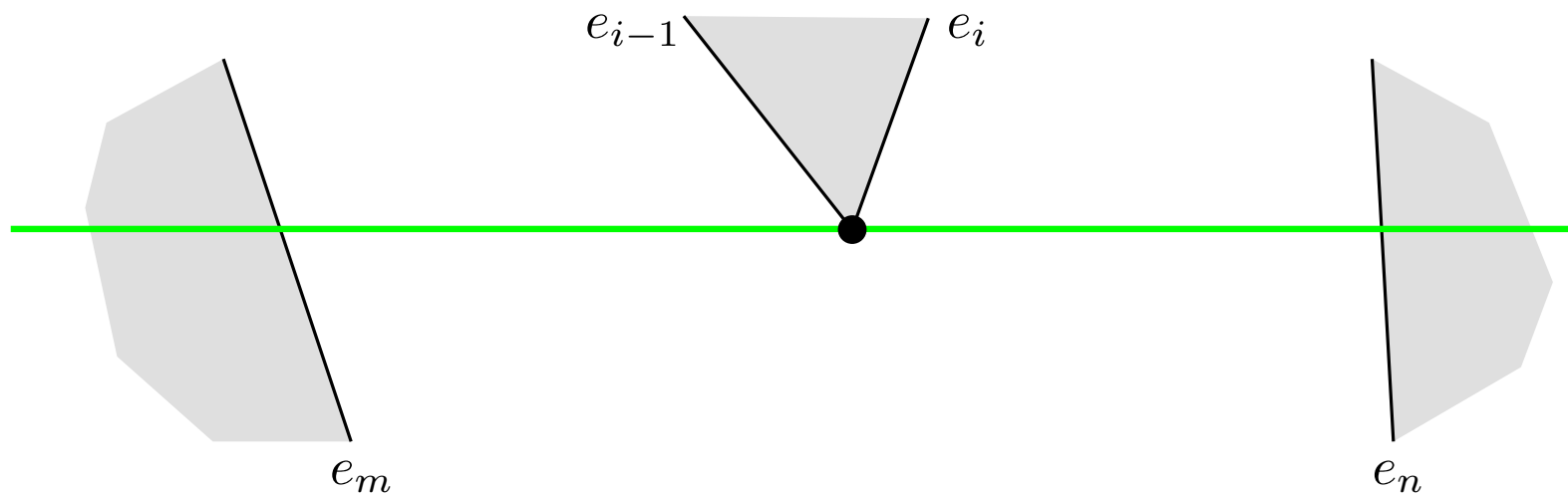


TRIANGULATING POLYGONS

Monotone partition

As for diagonals:

- Passing vertex: nothing
- Local minimum cusp: wait until it can be connected to final vertex of starting trapezoid
- Local maximum cusp: connect it with the initial vertex of the ending trapezoid
- Initial vertex: nothing
- Final vertex: nothing



TRIANGULATING POLYGONS

Monotone partition

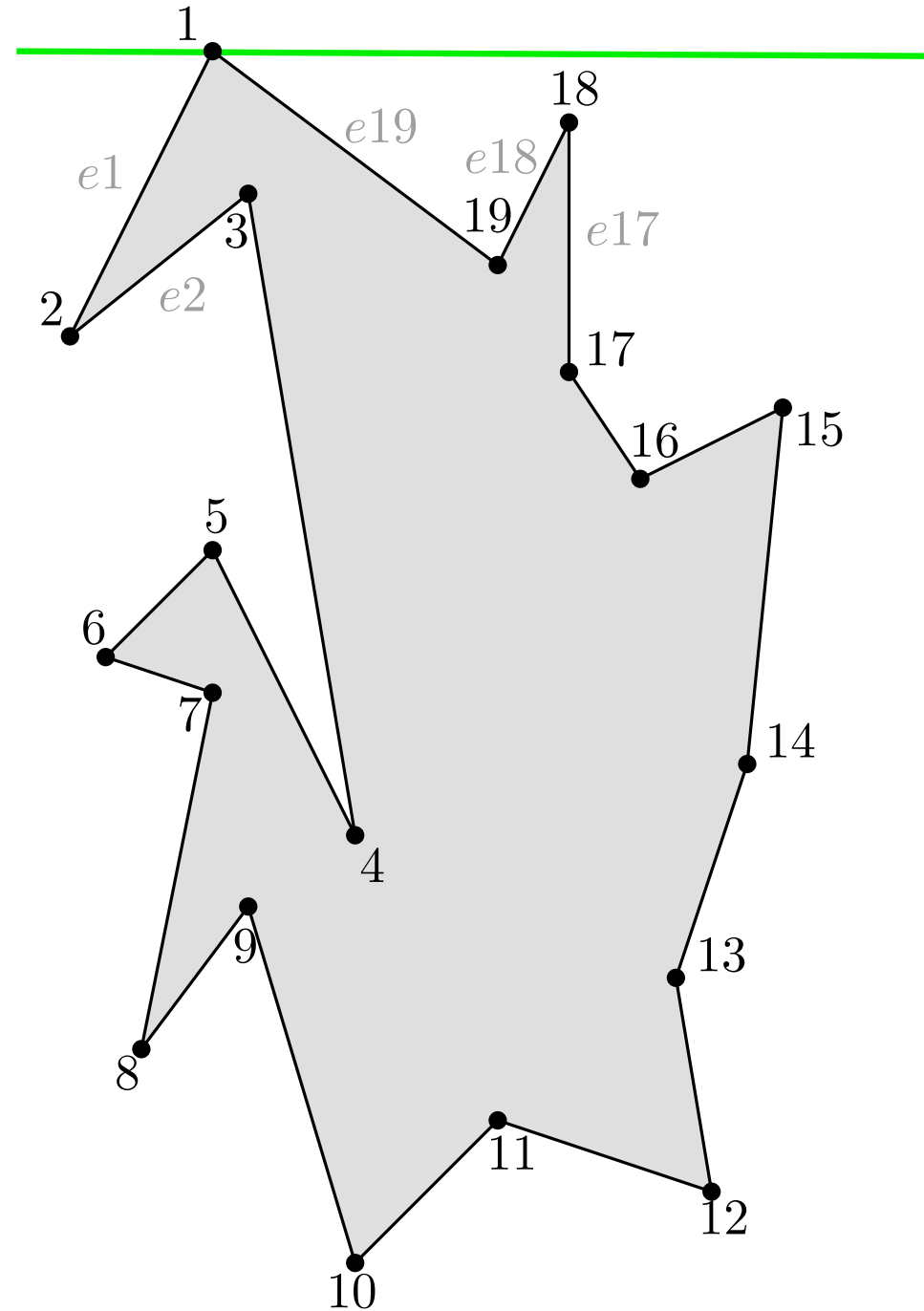
vertex

1

sweep line

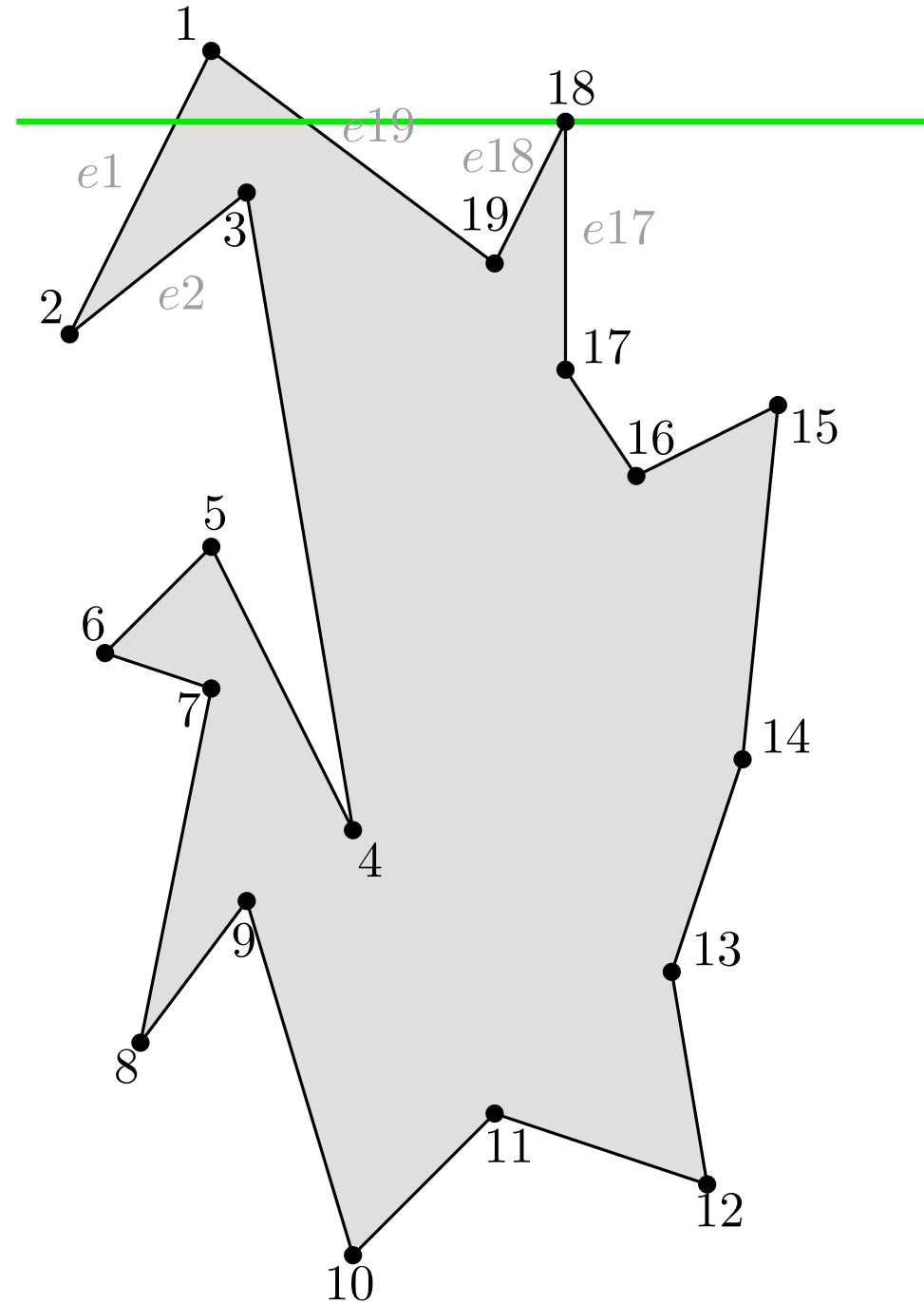
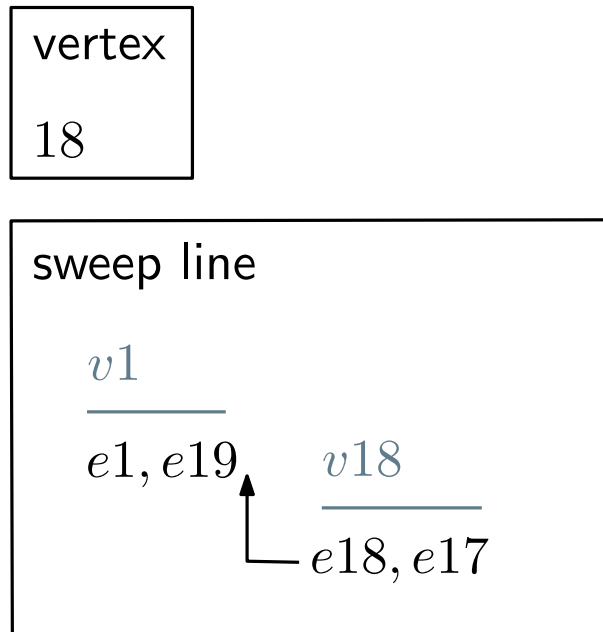
$v1$

$e1, e19$



TRIANGULATING POLYGONS

Monotone partition



TRIANGULATING POLYGONS

Monotone partition

vertex

3

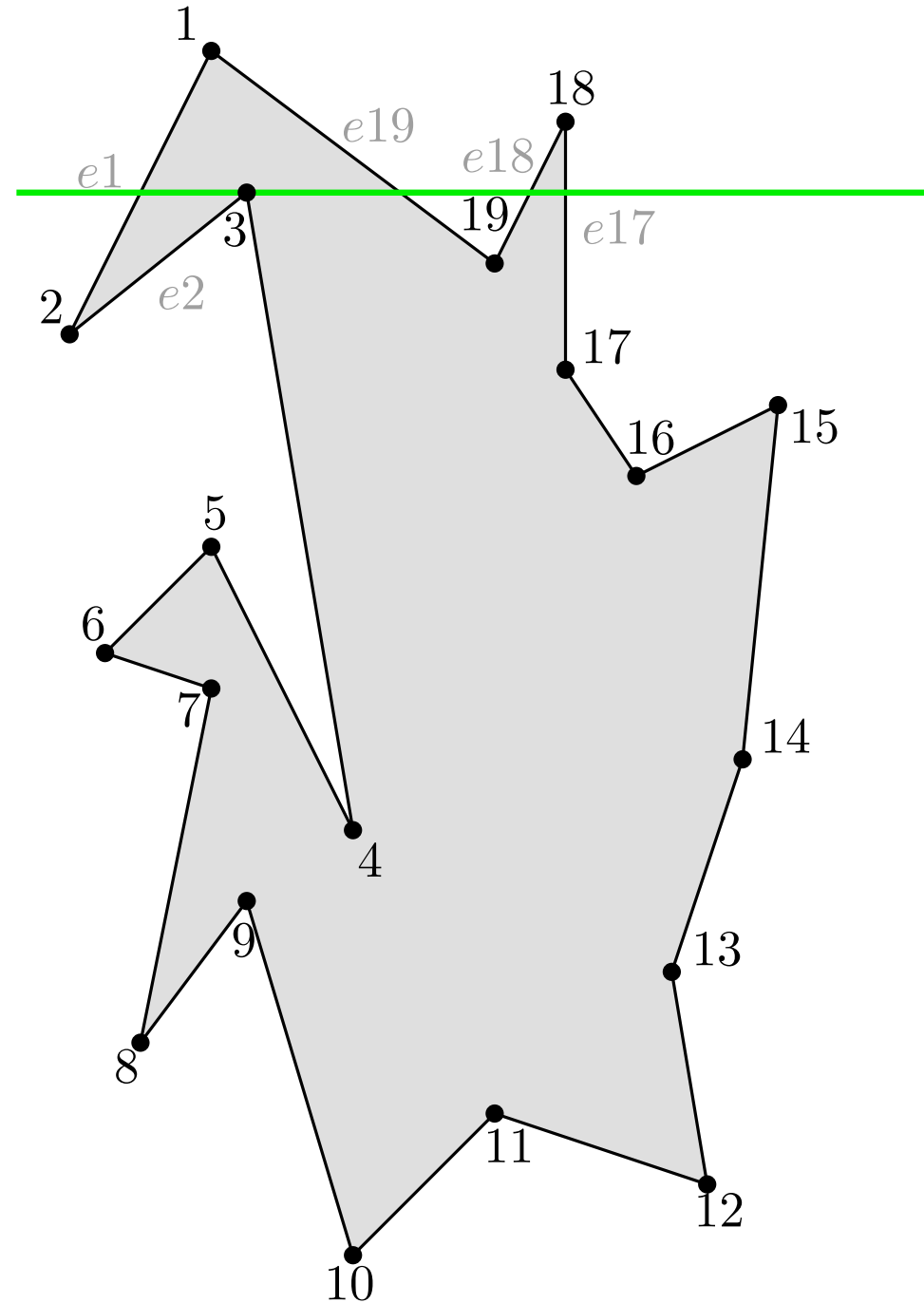
sweep line

~~$v1$~~

$v18$

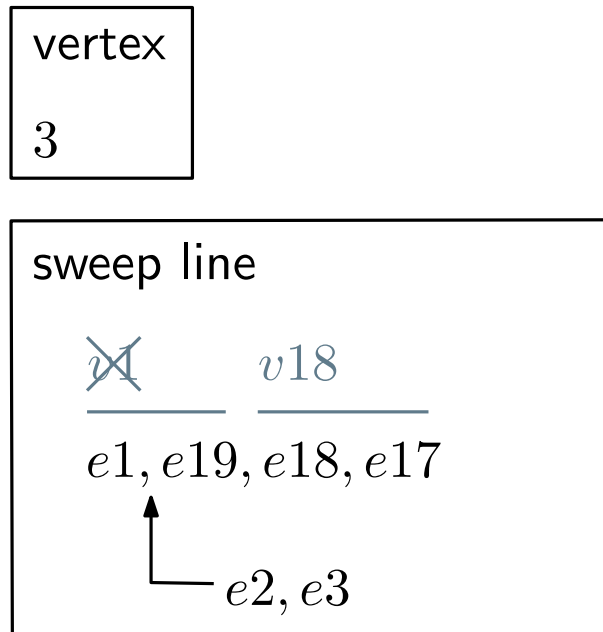
$e1, e19, e18, e17$

$e2, e3$

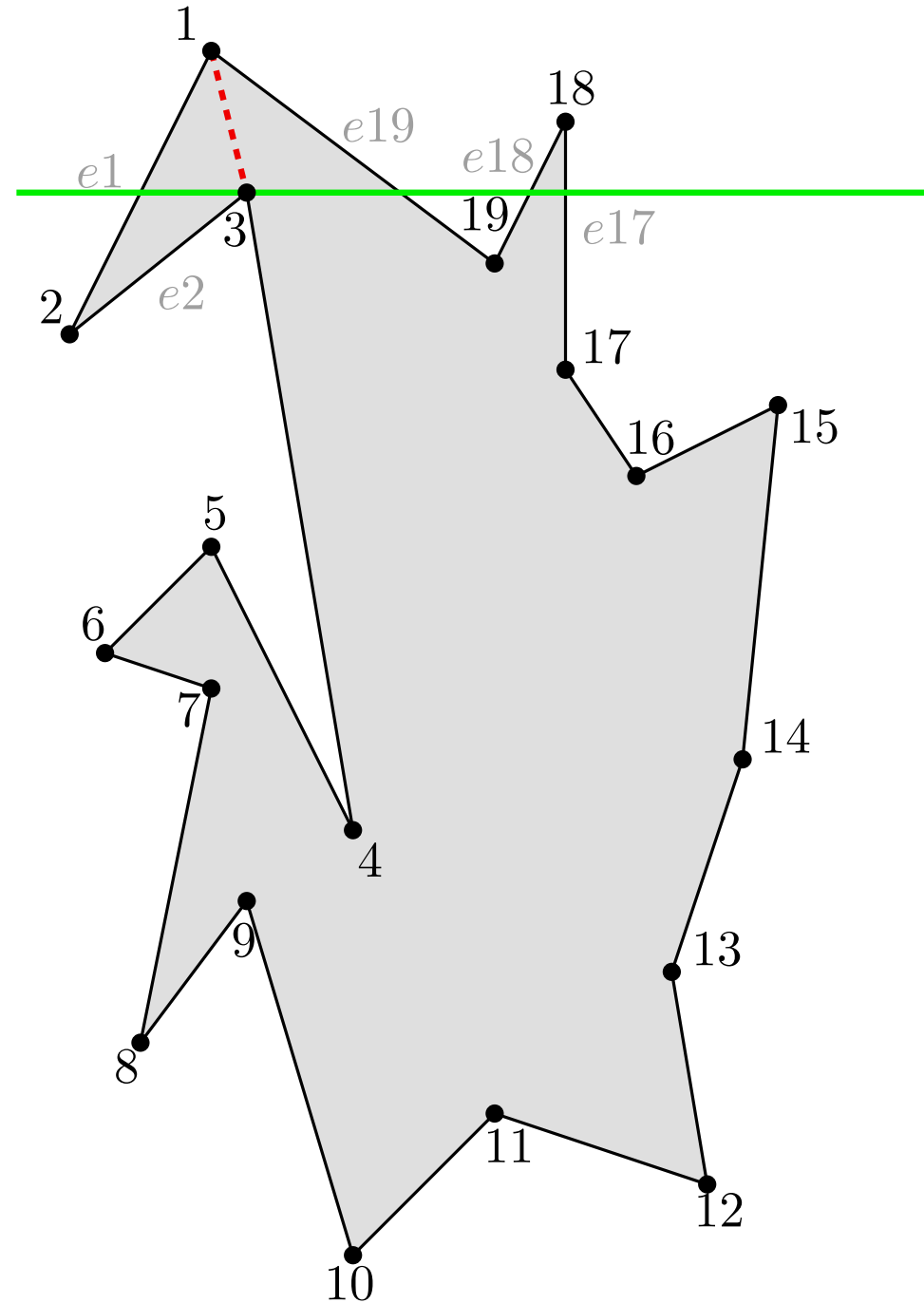


TRIANGULATING POLYGONS

Monotone partition



Diagonal $v1 - v3$



TRIANGULATING POLYGONS

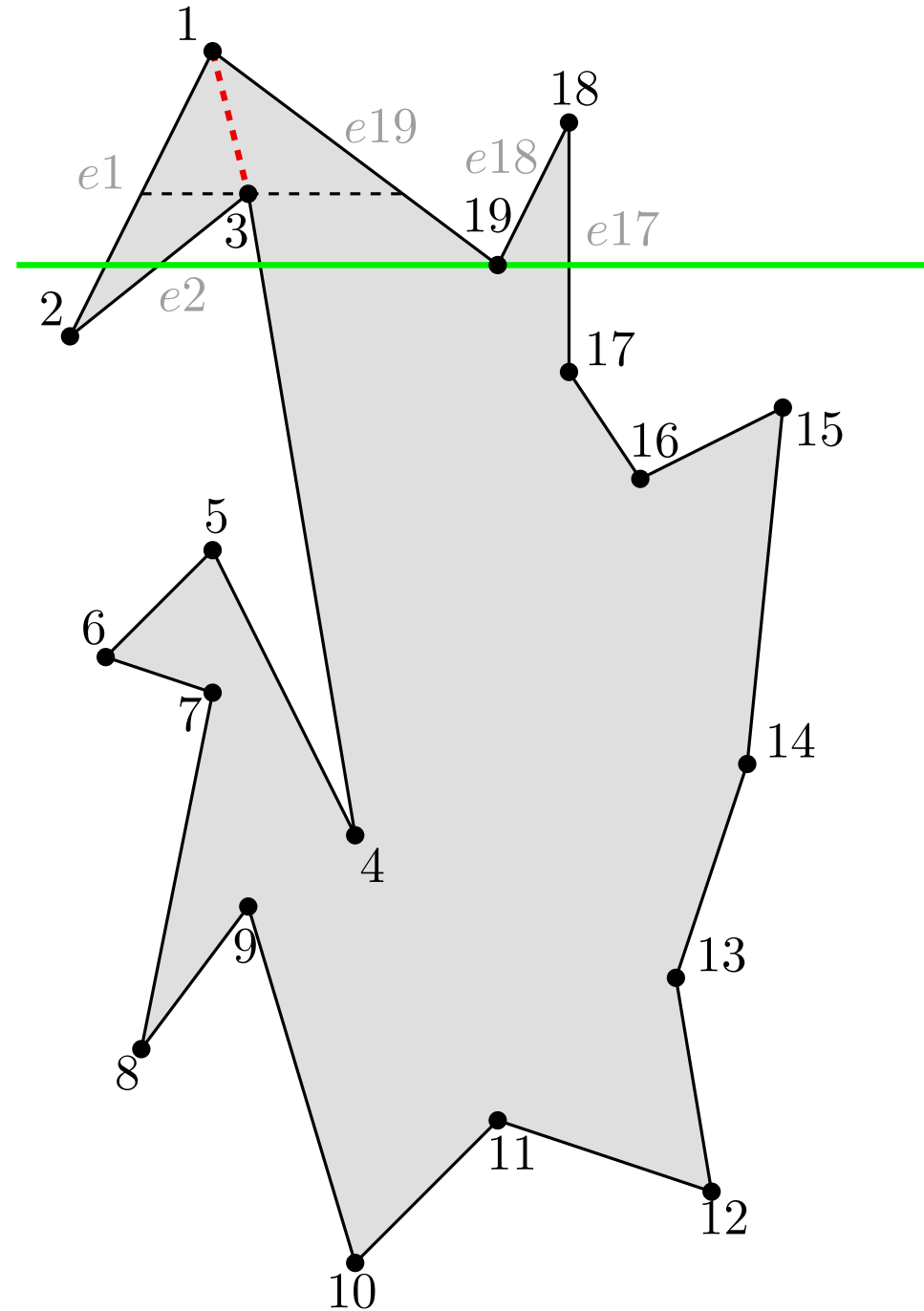
Monotone partition

vertex

19

sweep line

v_3	v_3	v_{18}
$e_1, e_2, e_3, \cancel{e_{19}}, \cancel{e_{18}}, e_{17}$		



TRIANGULATING POLYGONS

Monotone partition

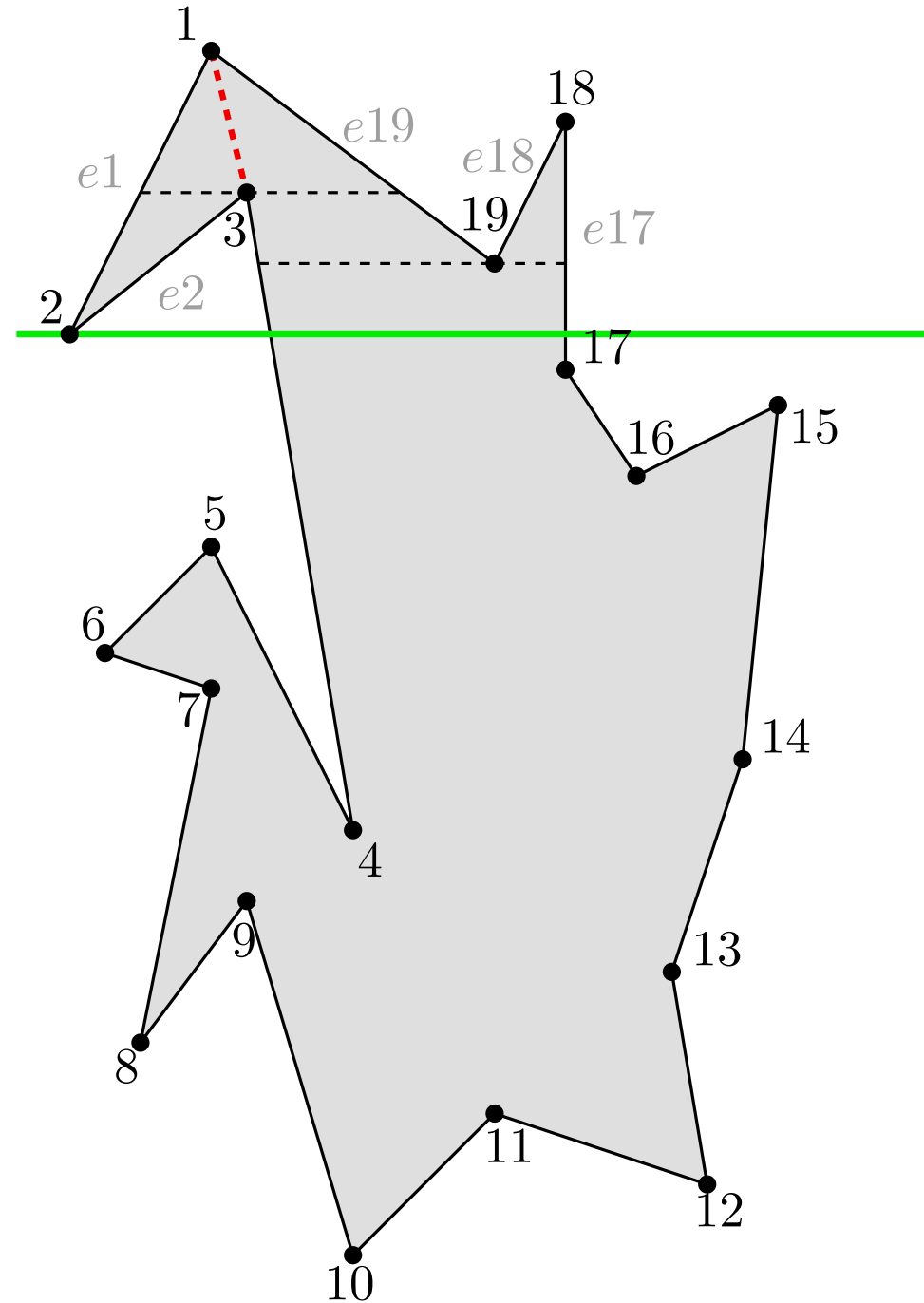
vertex

2

sweep line

v_3 v_{19}

~~e_1, e_2~~ , e_3, e_{17}

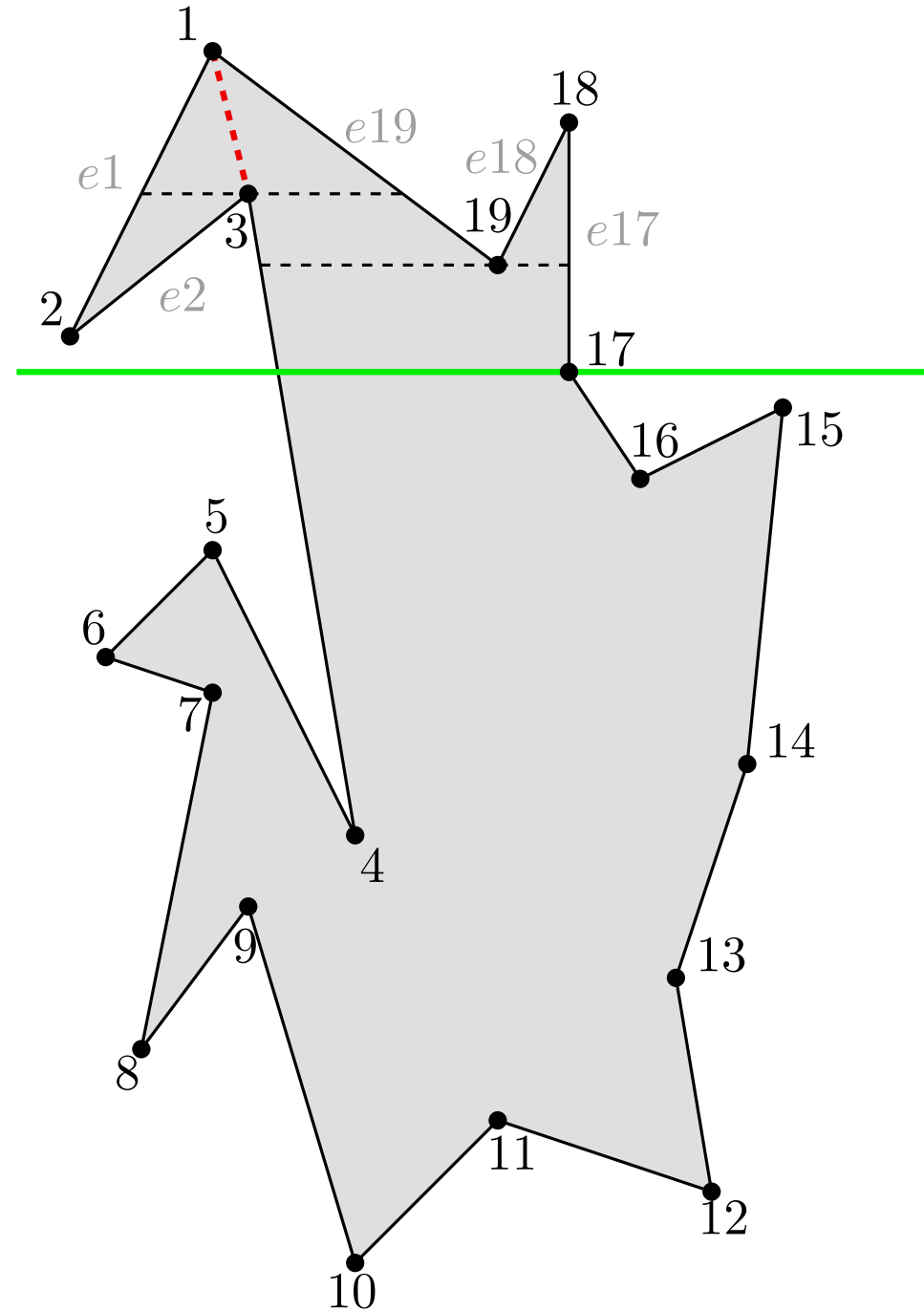


Monotone partition

17

 ~~v_{19}~~ ~~$e3, e17$~~

e16



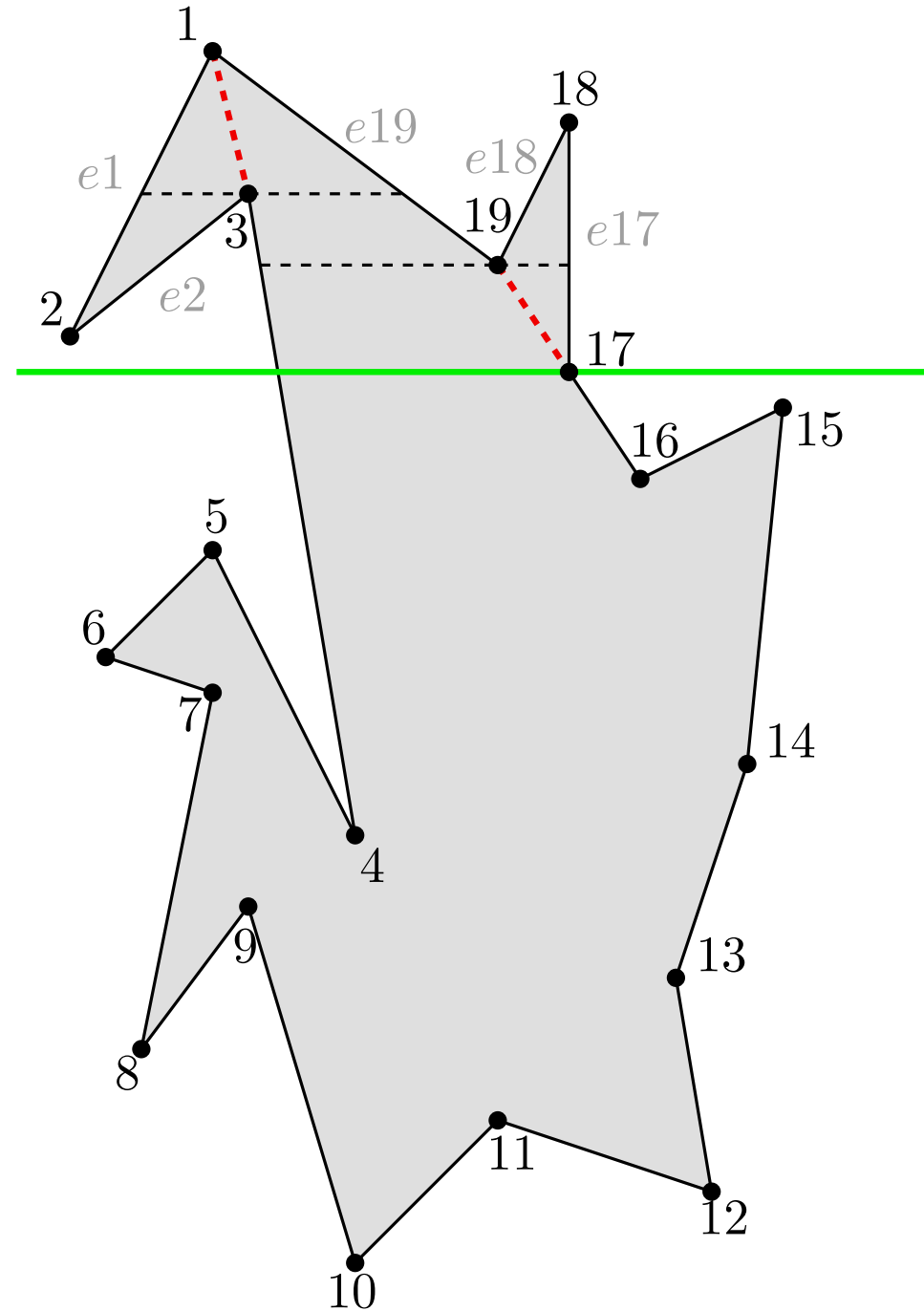
Monotone partition

17

 ~~v_{19}~~
$$e3, \cancel{e17}$$

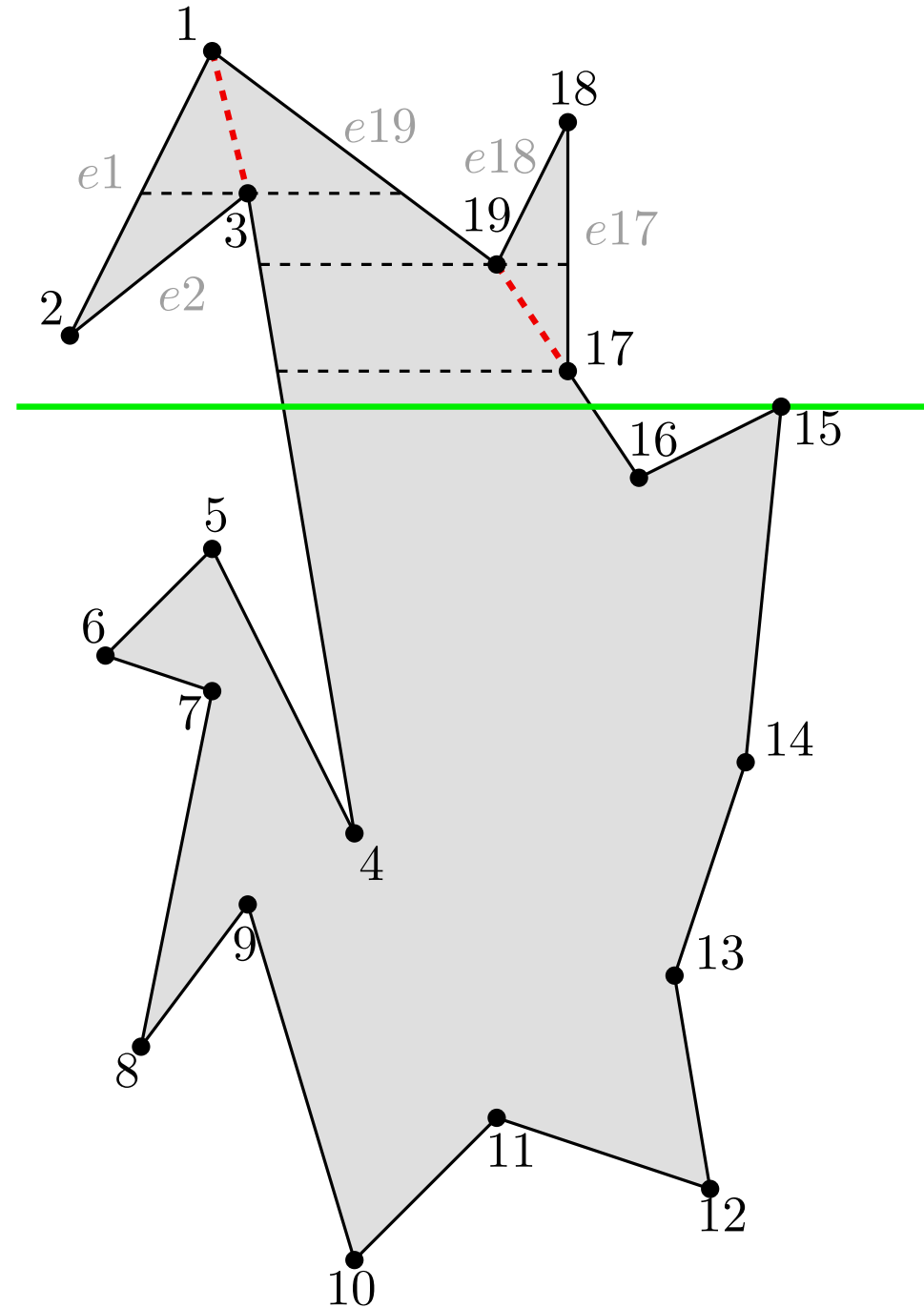
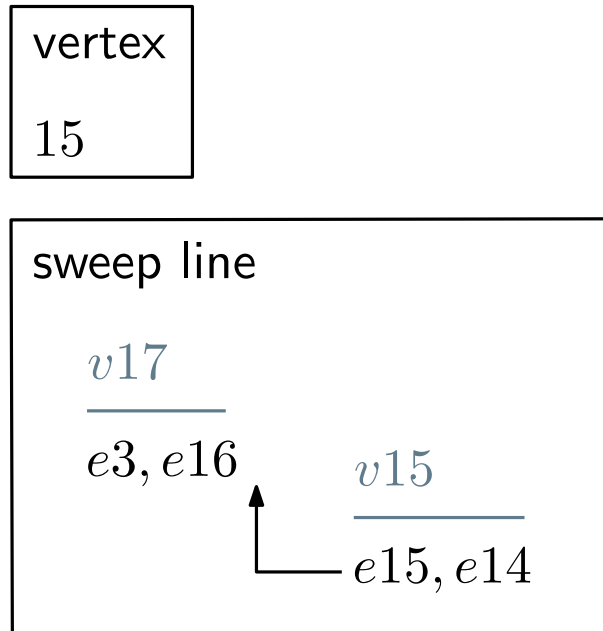
e16

Diagonal $v_{19} - v_{17}$



TRIANGULATING POLYGONS

Monotone partition



TRIANGULATING POLYGONS

Monotone partition

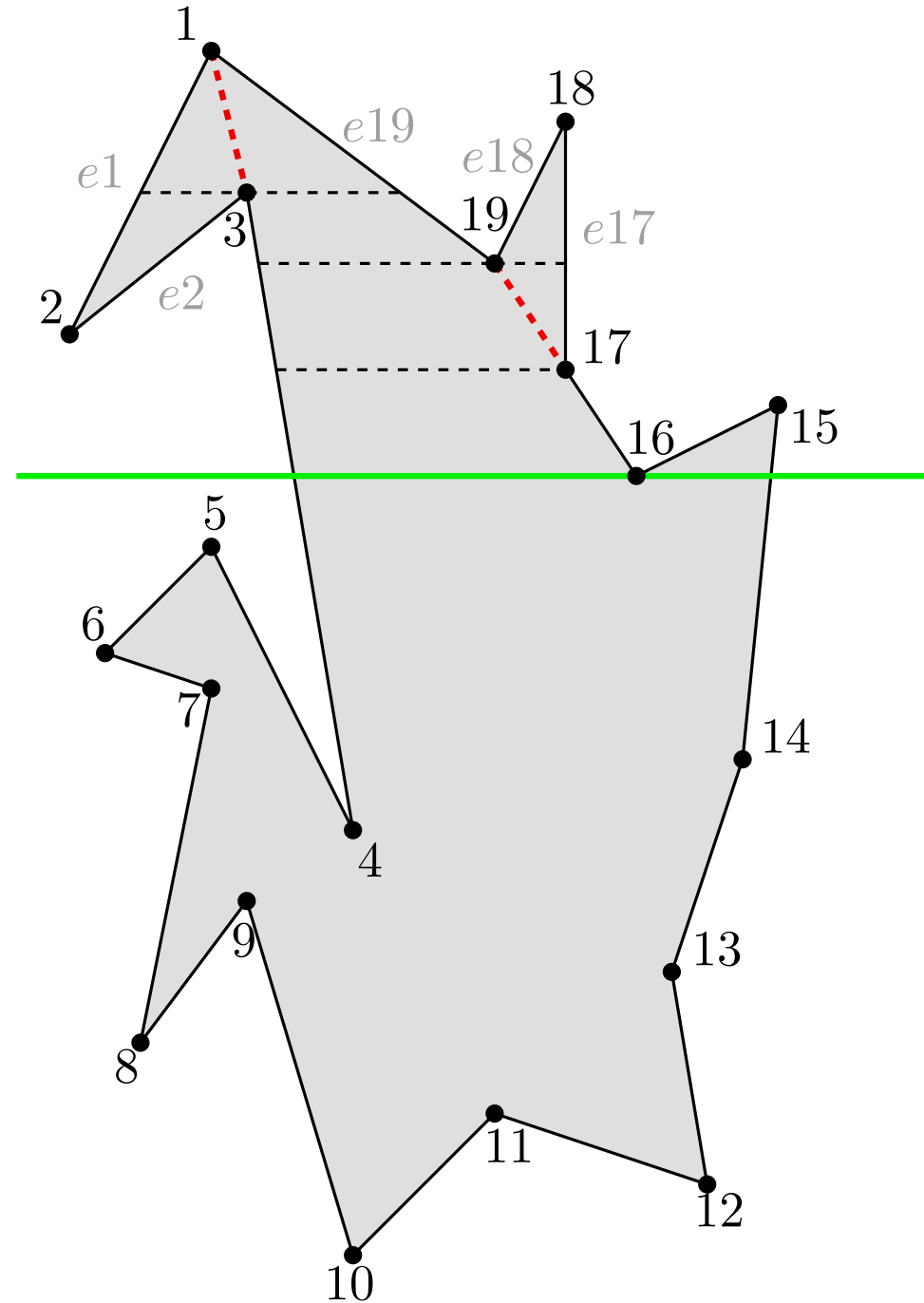
vertex

16

sweep line

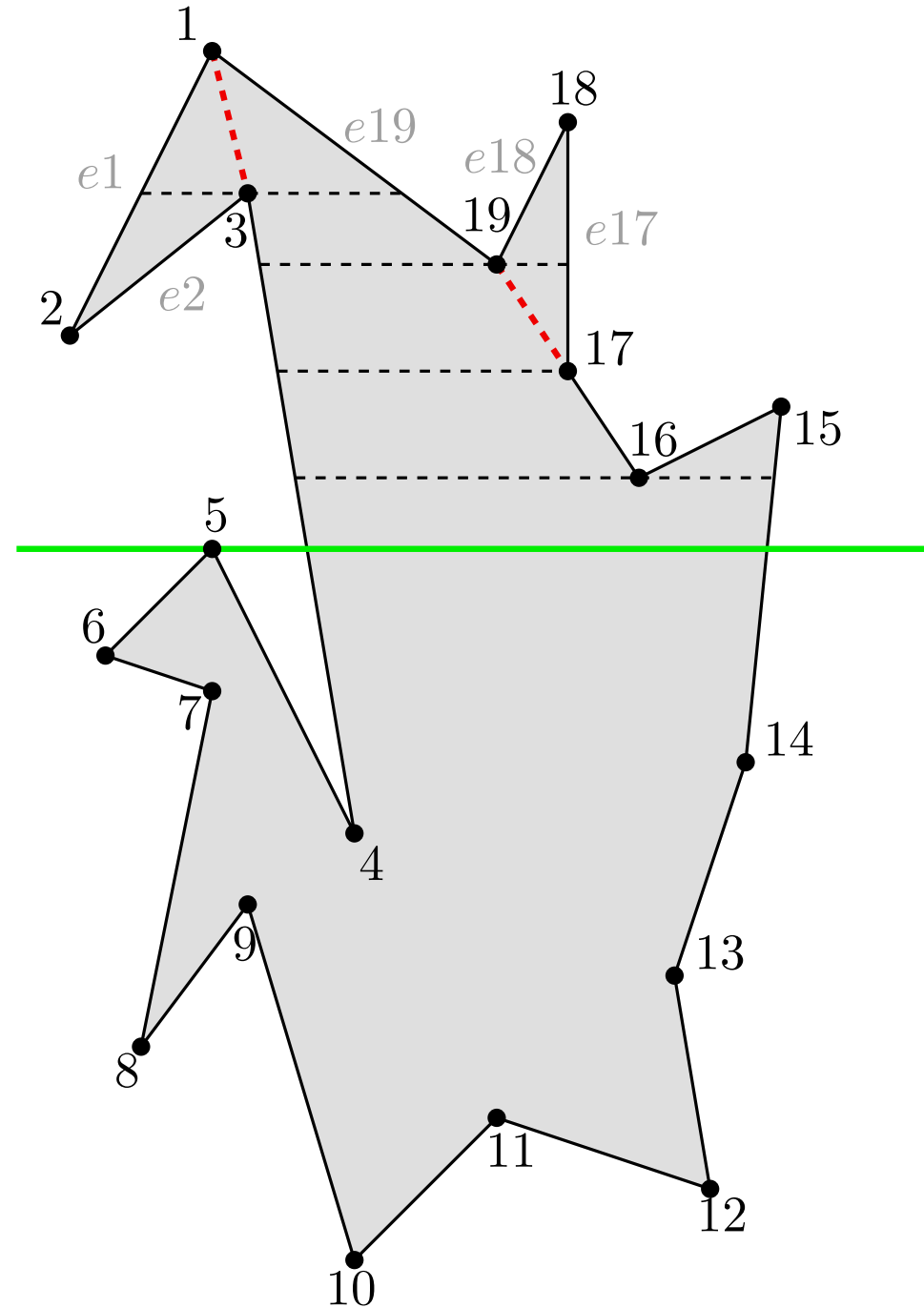
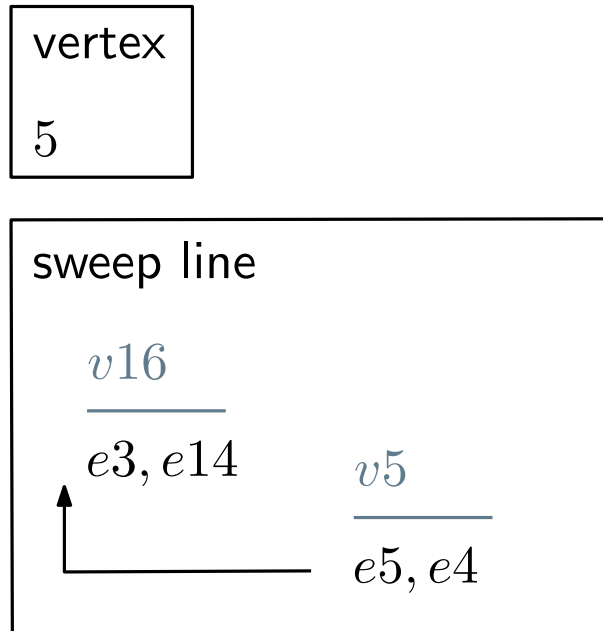
~~v_{17}~~ ~~v_{15}~~

$e_3, e_{16}, e_{15}, e_{14}$



TRIANGULATING POLYGONS

Monotone partition



TRIANGULATING POLYGONS

Monotone partition

vertex

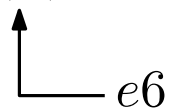
6

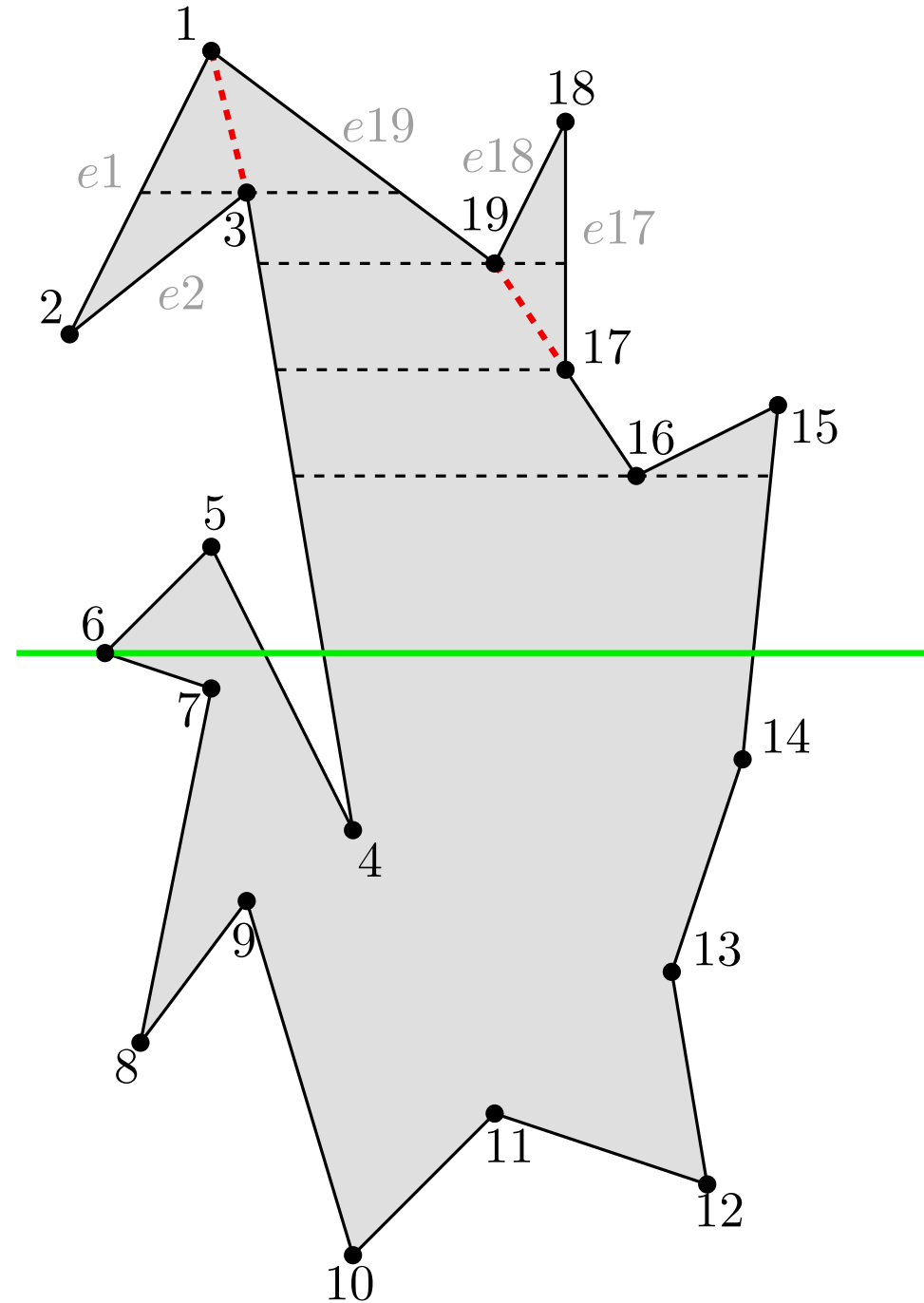
sweep line

~~v5~~

v16

~~e5, e4, e3, e14~~

 $e6$



TRIANGULATING POLYGONS

Monotone partition

vertex

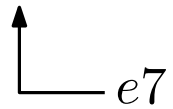
7

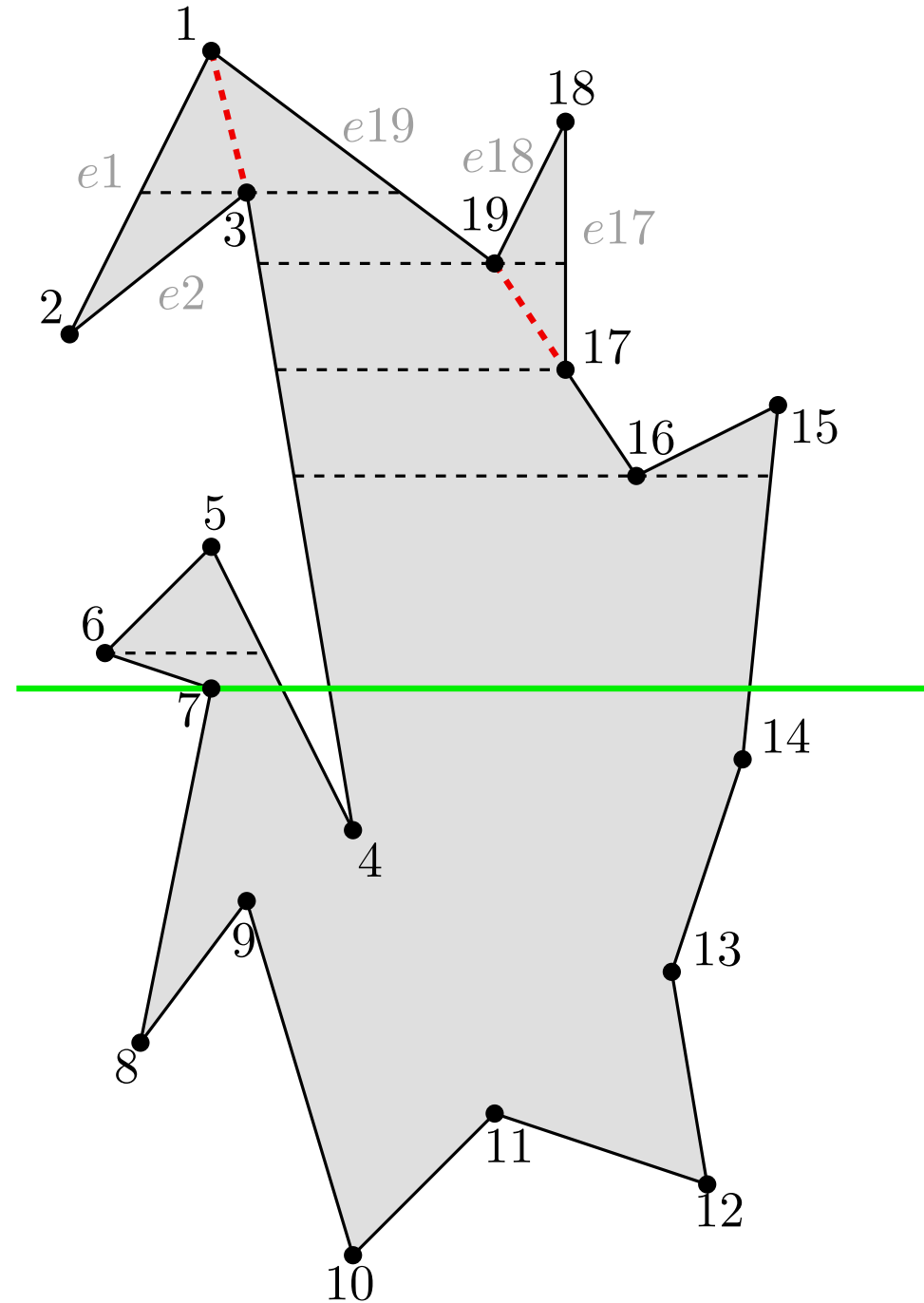
sweep line

~~v6~~

v16

~~e6, e4, e3, e14~~

 $e7$



TRIANGULATING POLYGONS

Monotone partition

vertex

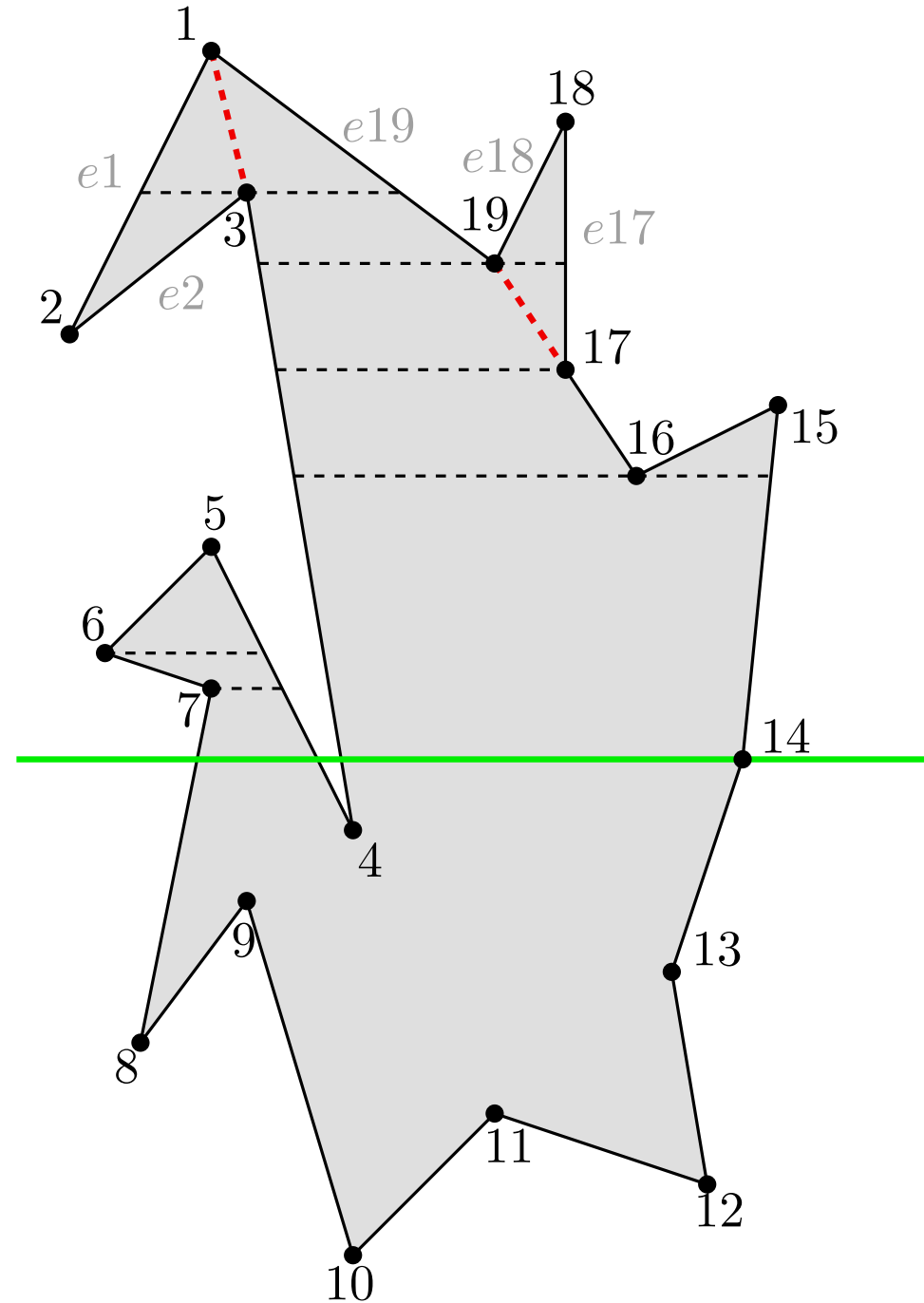
14

sweep line

v_7 ~~v_6~~

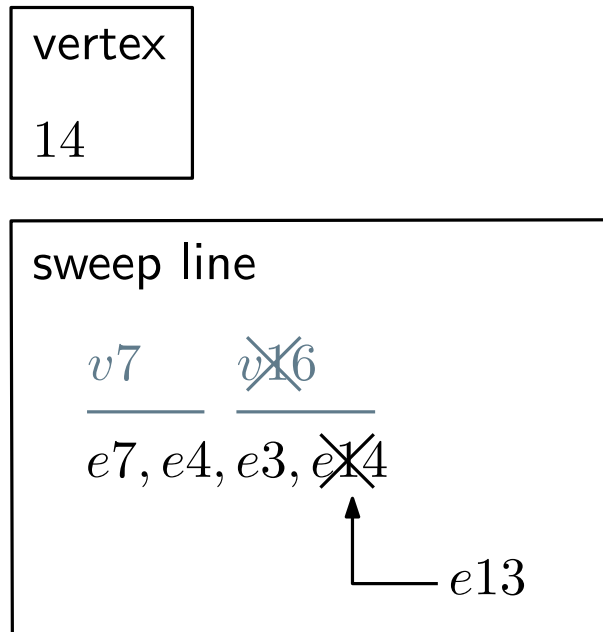
$e_7, e_4, e_3, \cancel{e_4}$

$\uparrow$
 e_{13}

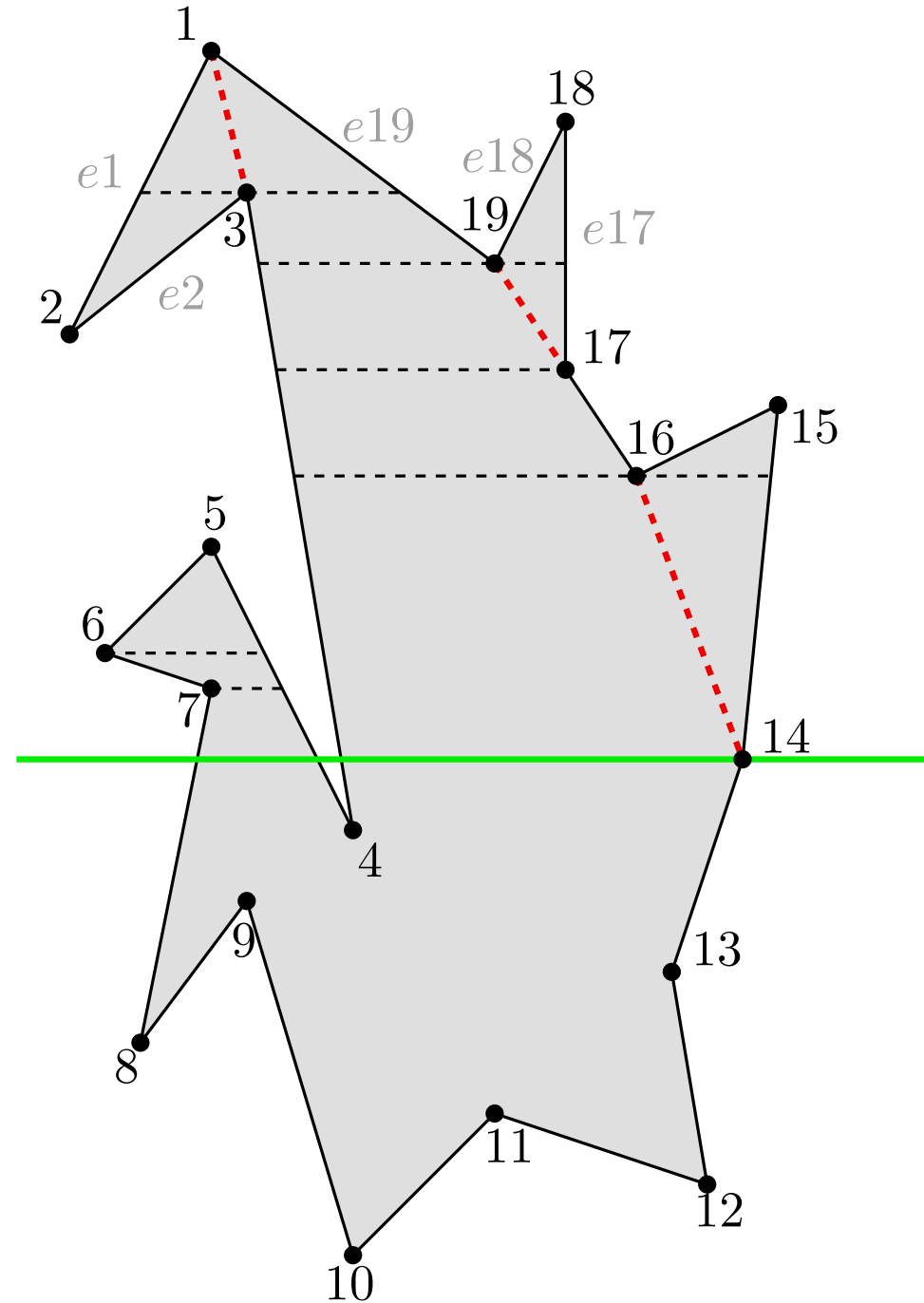


TRIANGULATING POLYGONS

Monotone partition



Diagonal $v16 - v14$



TRIANGULATING POLYGONS

Monotone partition

vertex

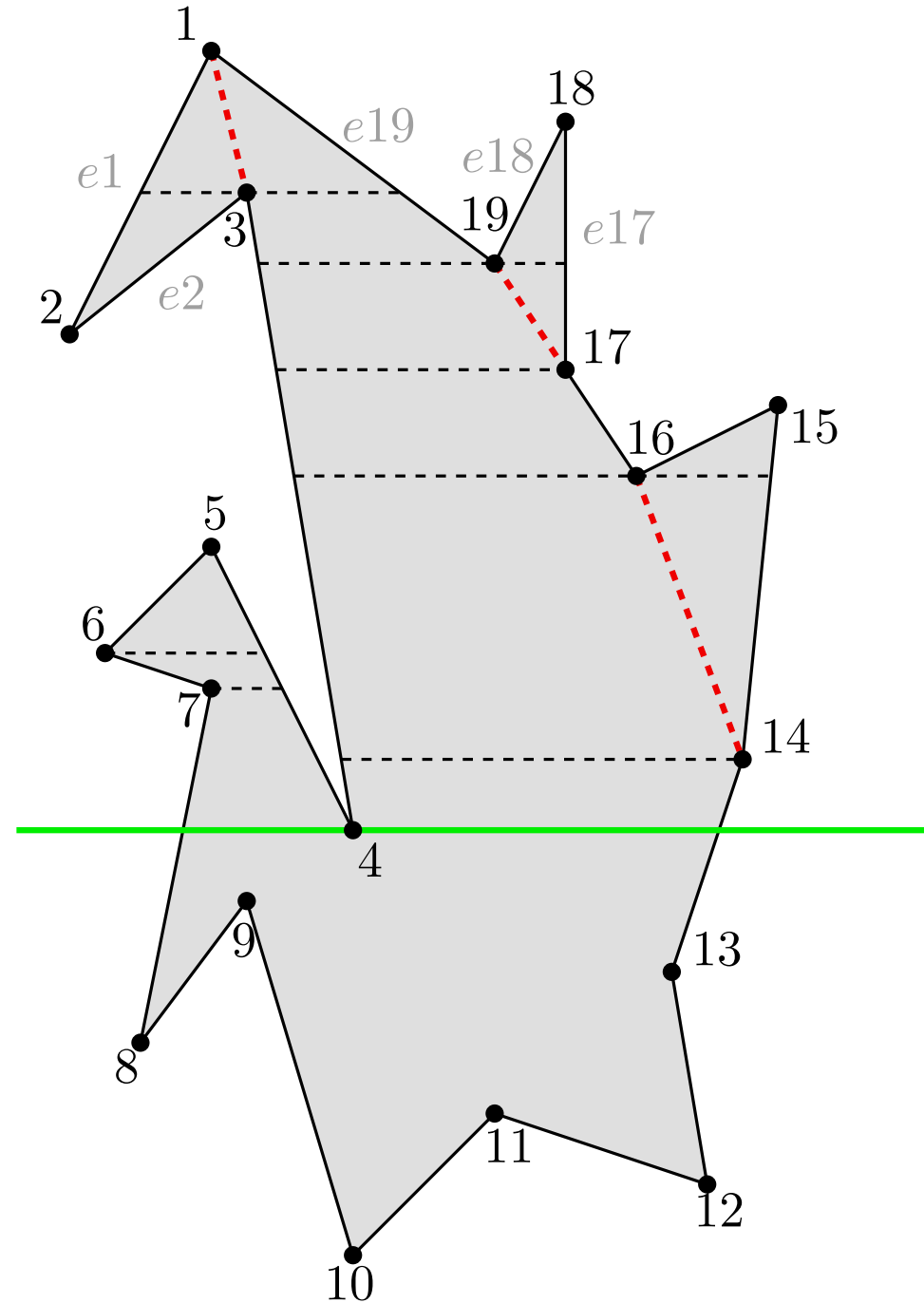
4

sweep line

~~v_7~~

~~v_{14}~~

$e_7, \cancel{e_4}, \cancel{e_3}, e_{13}$



TRIANGULATING POLYGONS

Monotone partition

vertex

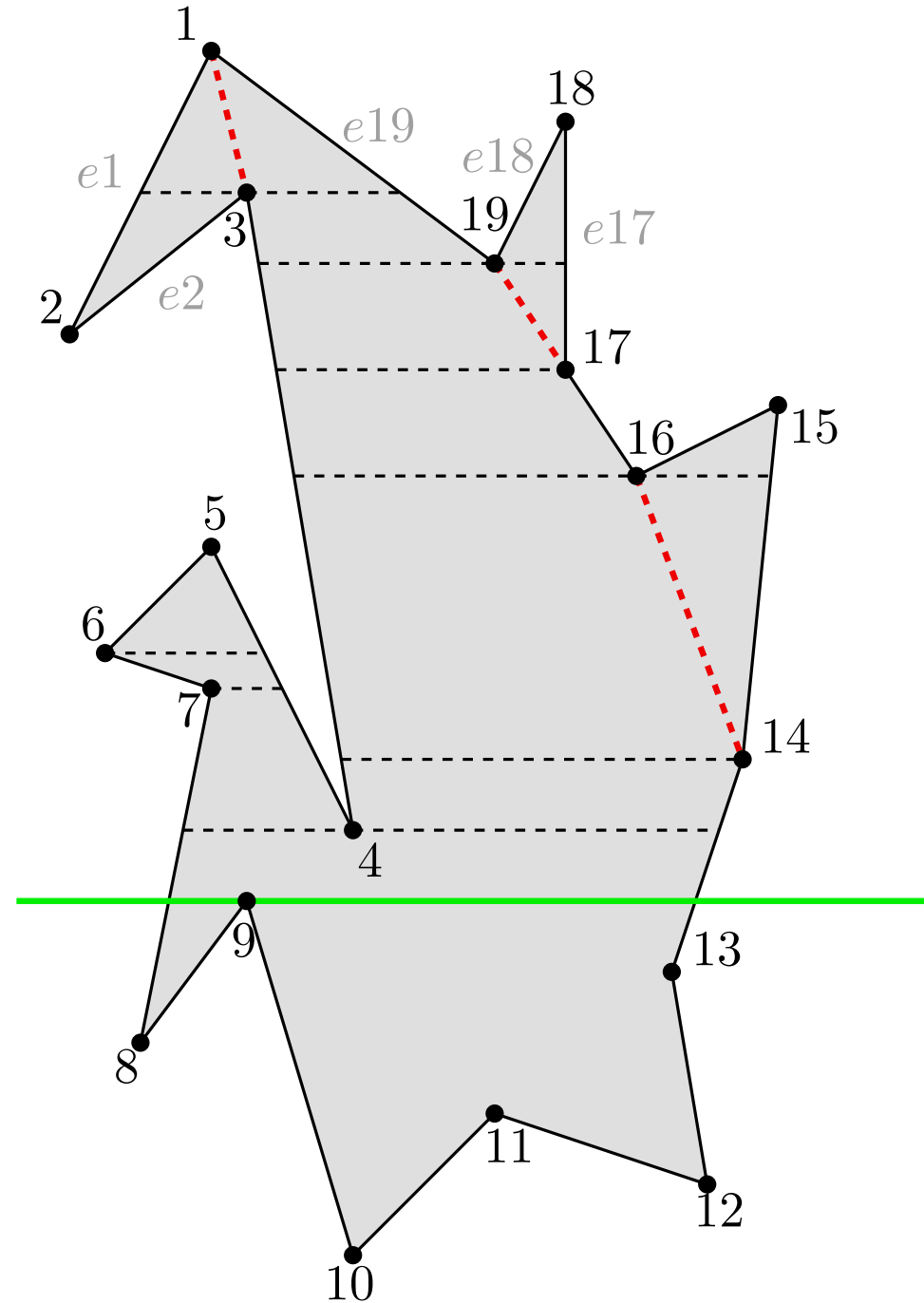
9

sweep line

~~v4~~

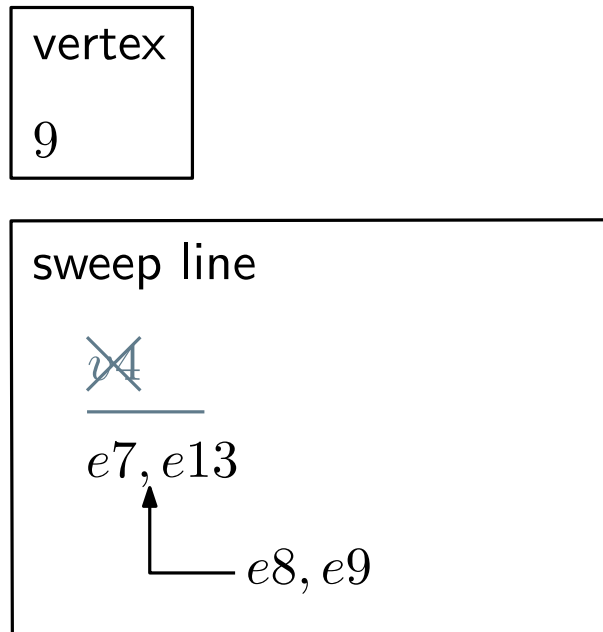
$e7, e13$

$e8, e9$

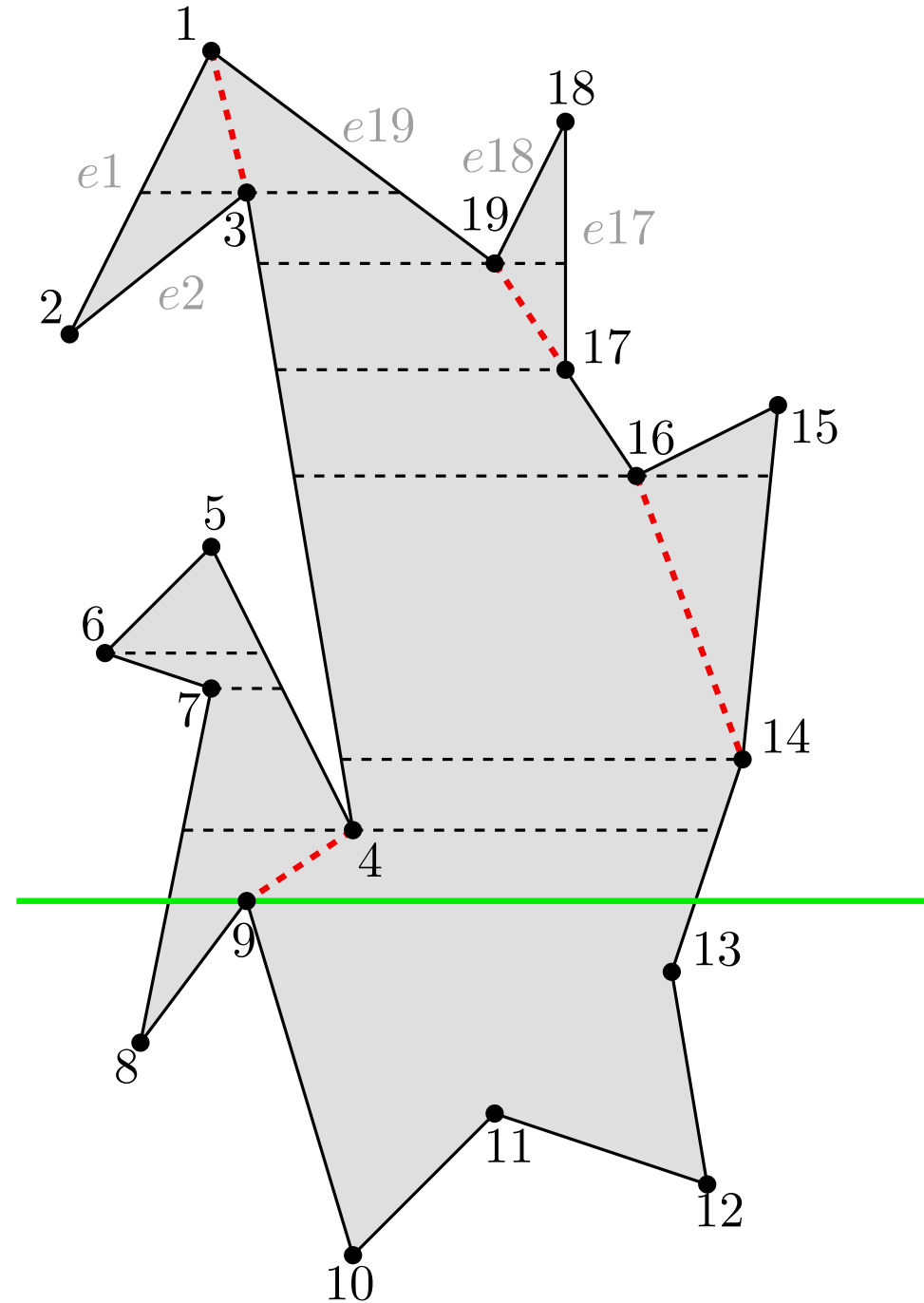


TRIANGULATING POLYGONS

Monotone partition



Diagonal $v_4 - v_9$



TRIANGULATING POLYGONS

Monotone partition

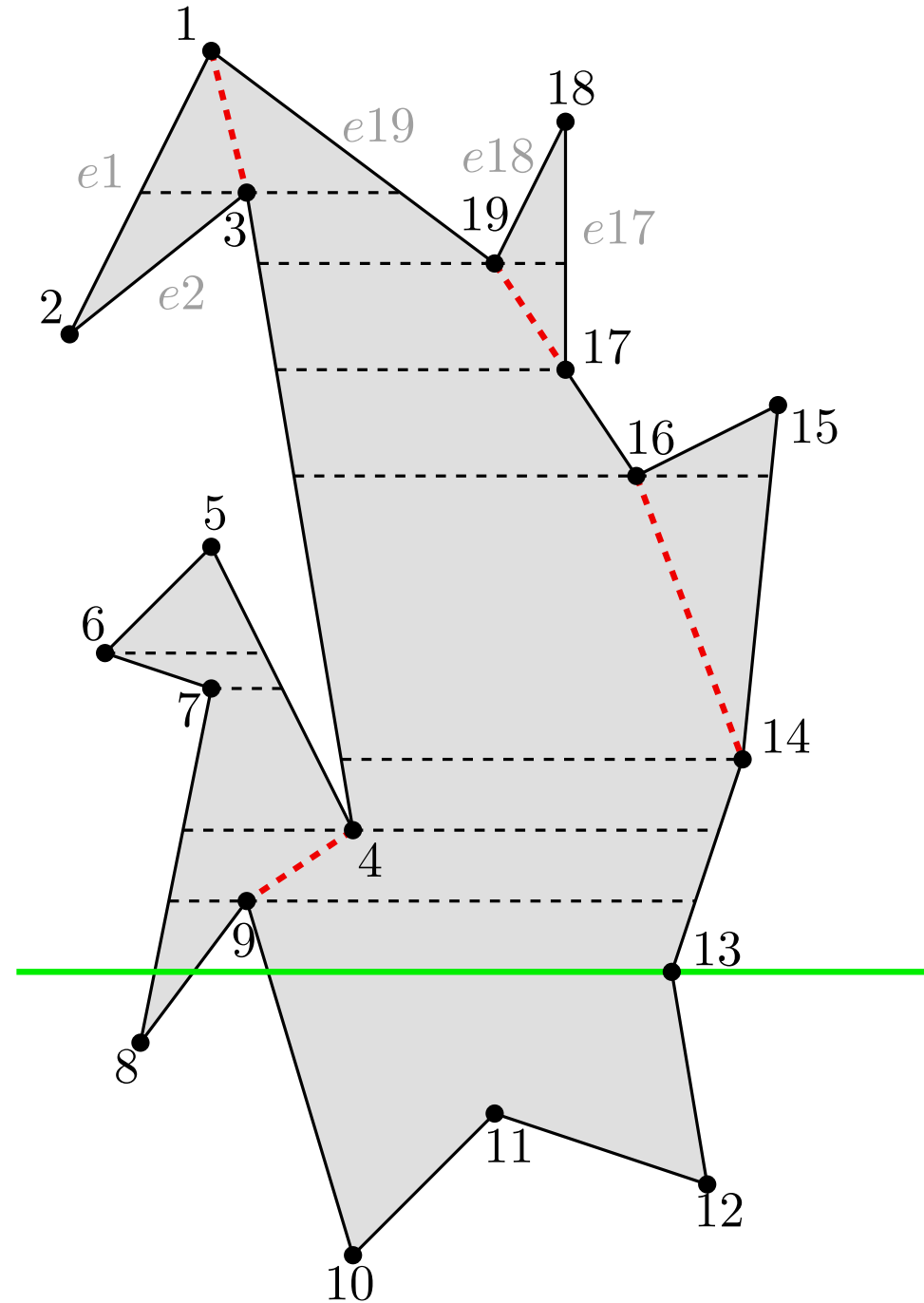
vertex

13

sweep line

$v9$ ~~$v9$~~
 $e7, e8, \cancel{e9}, e13$

↑
 $e12$



TRIANGULATING POLYGONS

Monotone partition

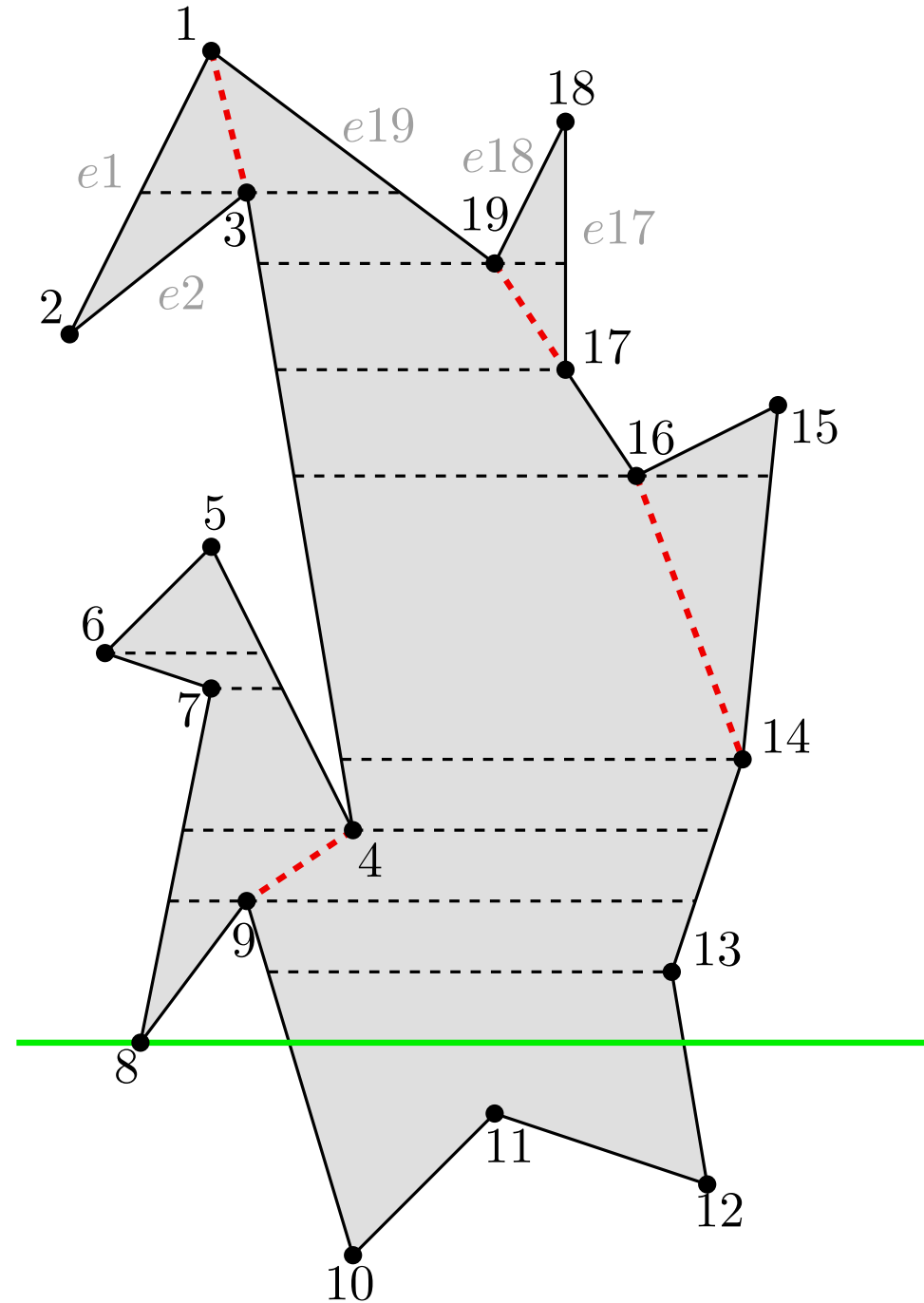
vertex

8

sweep line

v_9 v_{13}

~~e_7, e_8, e_9, e_{12}~~

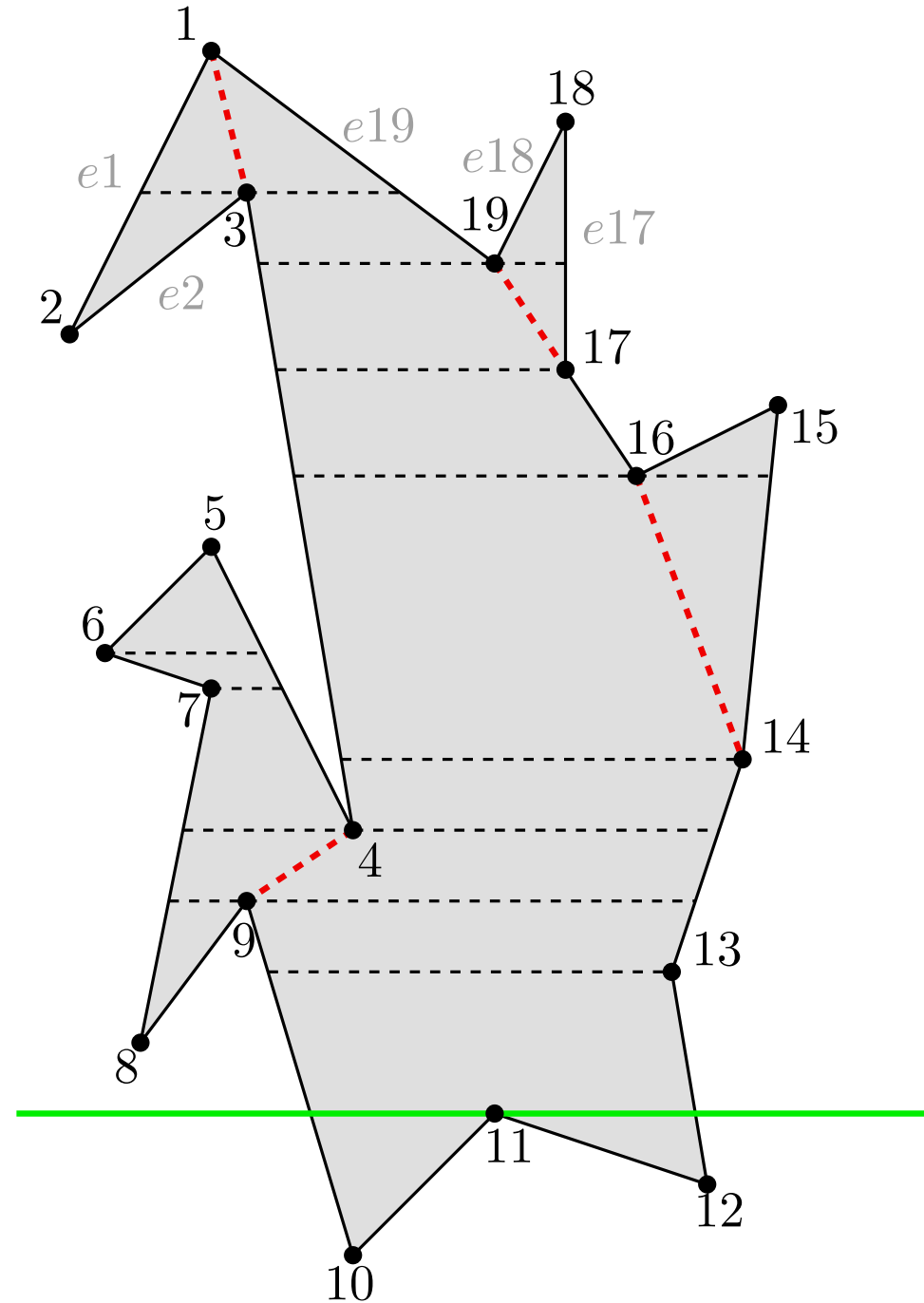


TRIANGULATING POLYGONS

Monotone partition

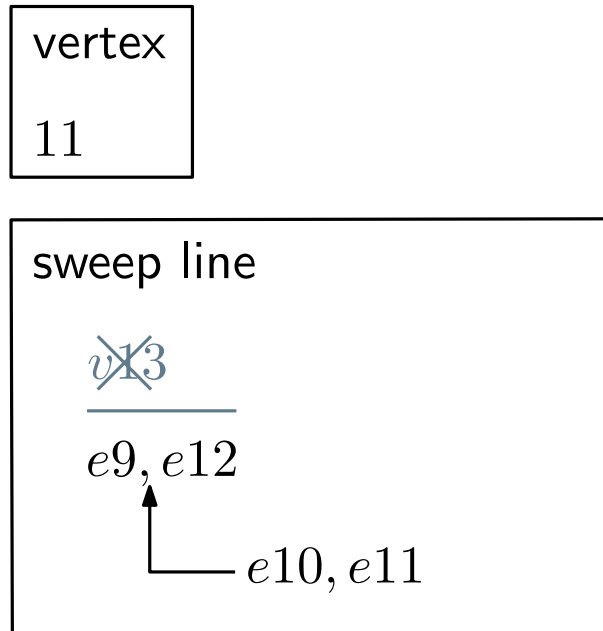
vertex
11

sweep line
 ~~$v \times 3$~~
 $e9, e12$
 $e10, e11$

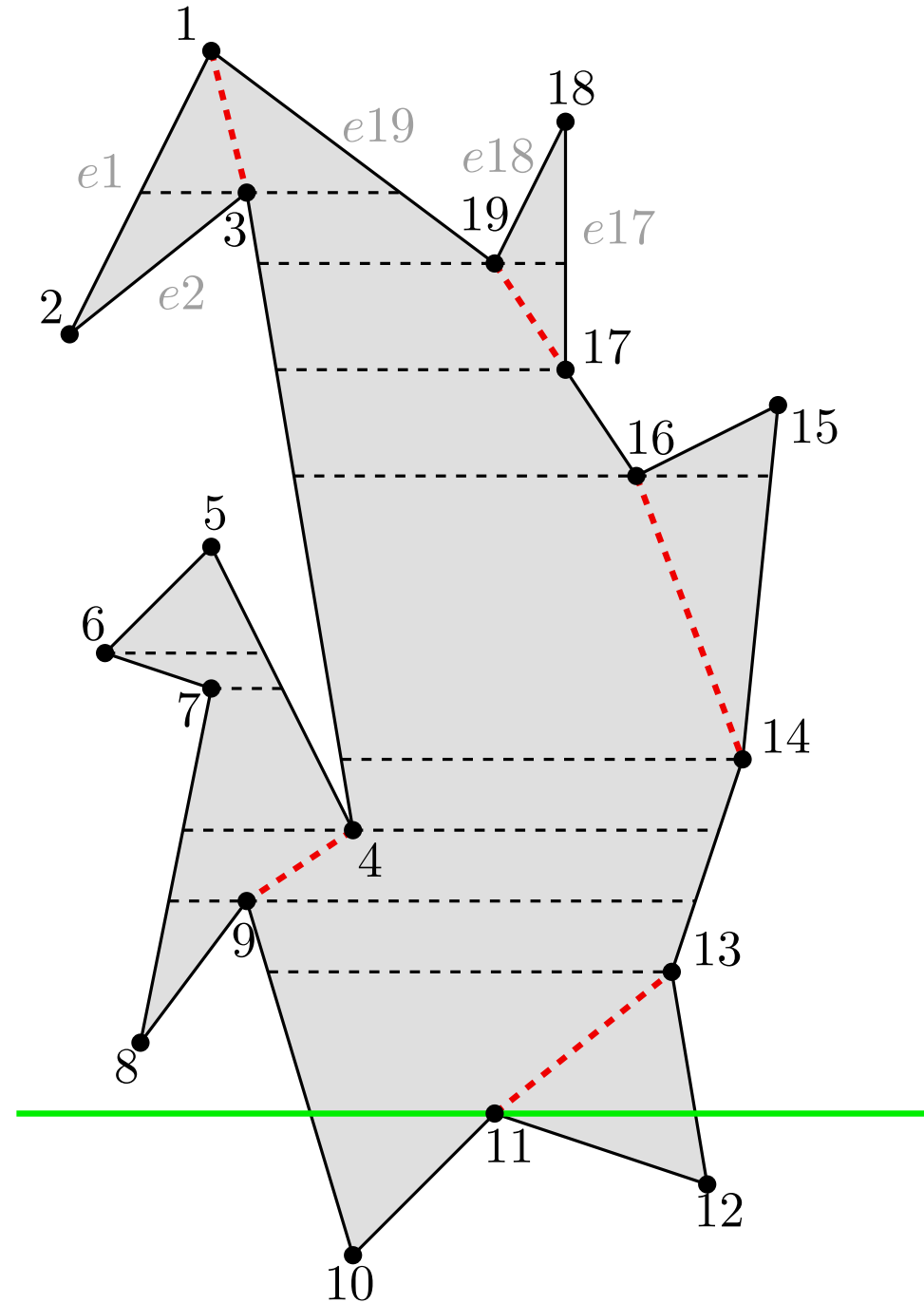


TRIANGULATING POLYGONS

Monotone partition



Diagonal $v_{13} - v_{11}$



TRIANGULATING POLYGONS

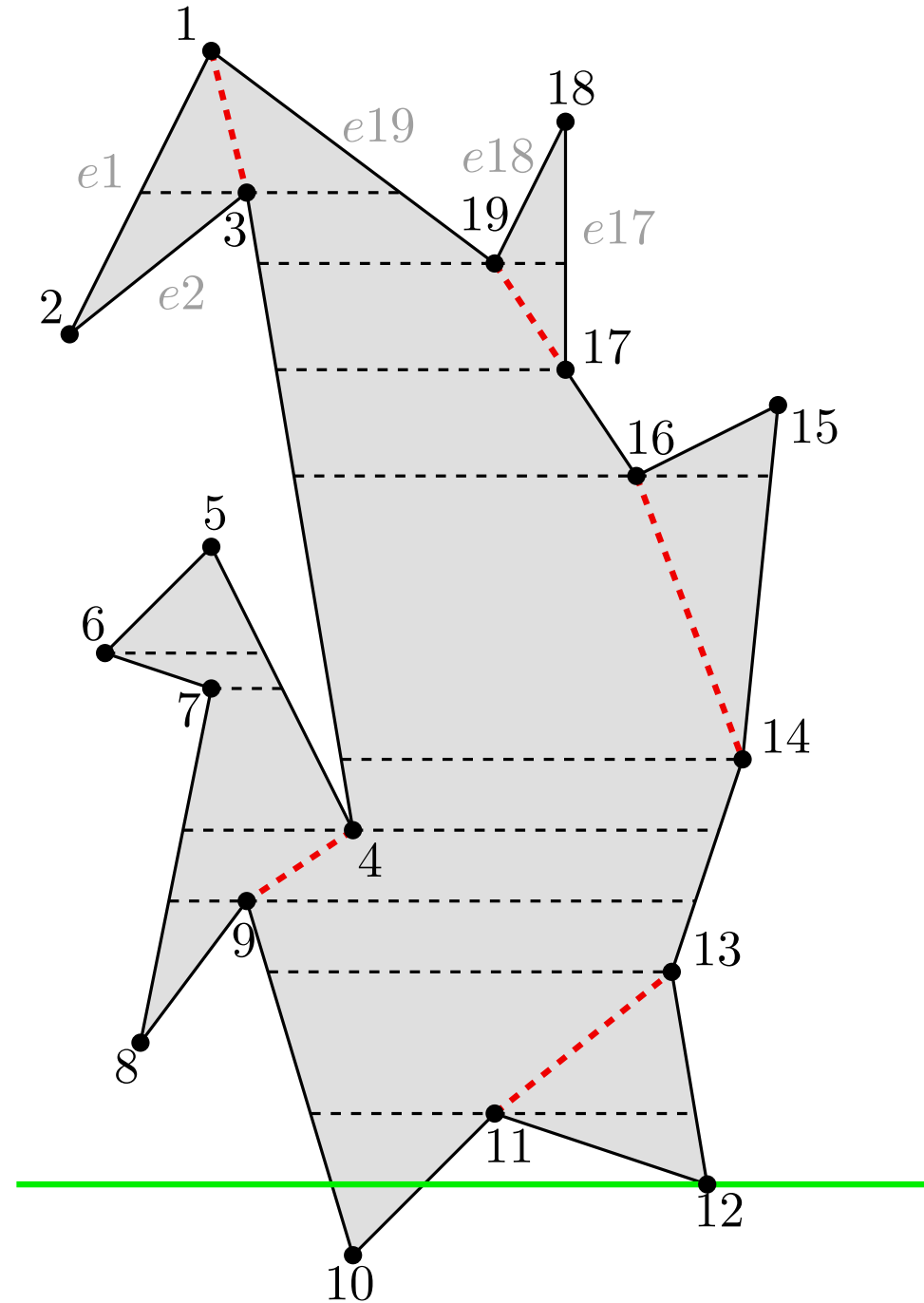
Monotone partition

vertex

12

sweep line

$\frac{v_{11}}{e_9, e_{10}, e_{11}, e_{12}}$



TRIANGULATING POLYGONS

Monotone partition

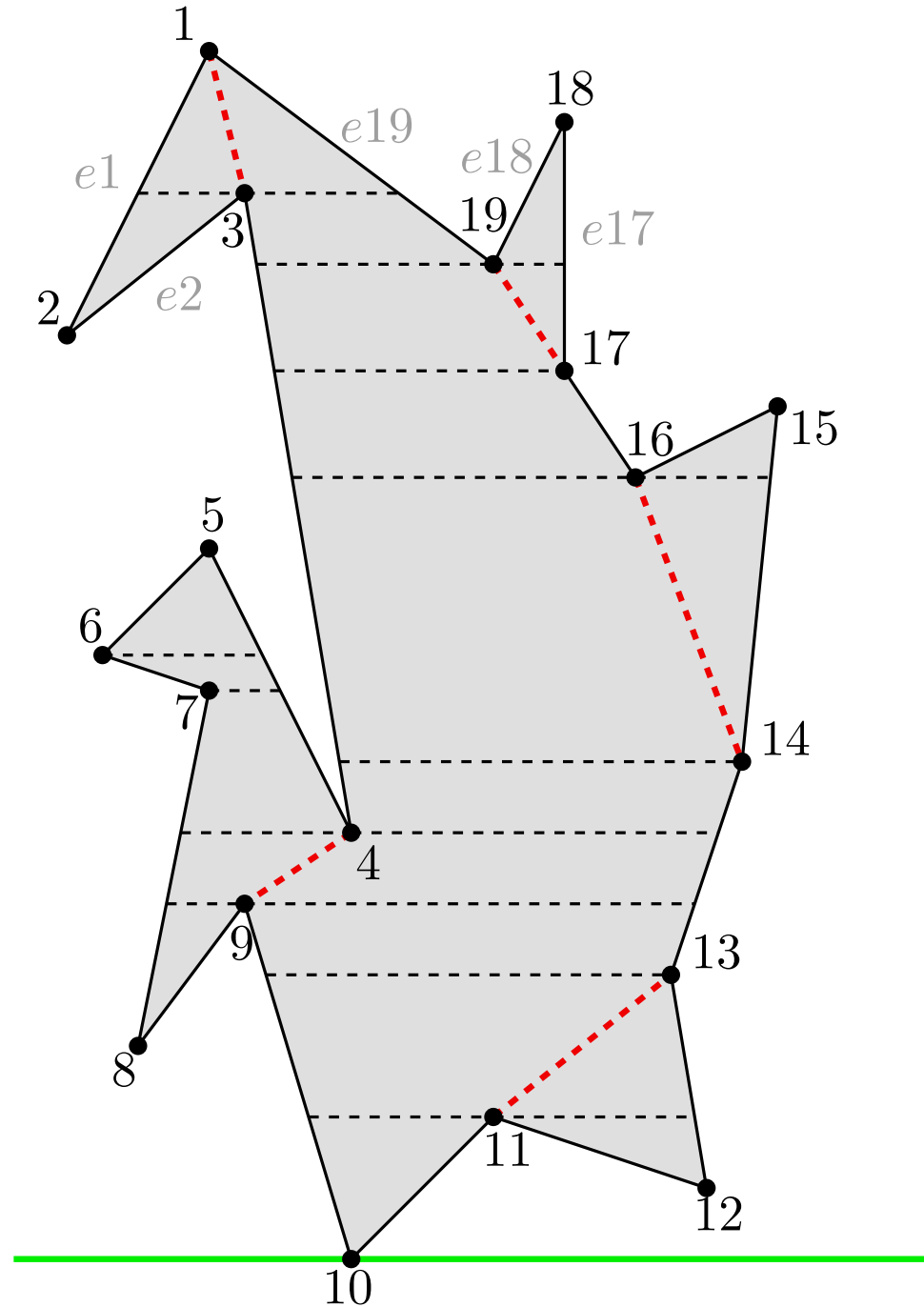
vertex

10

sweep line

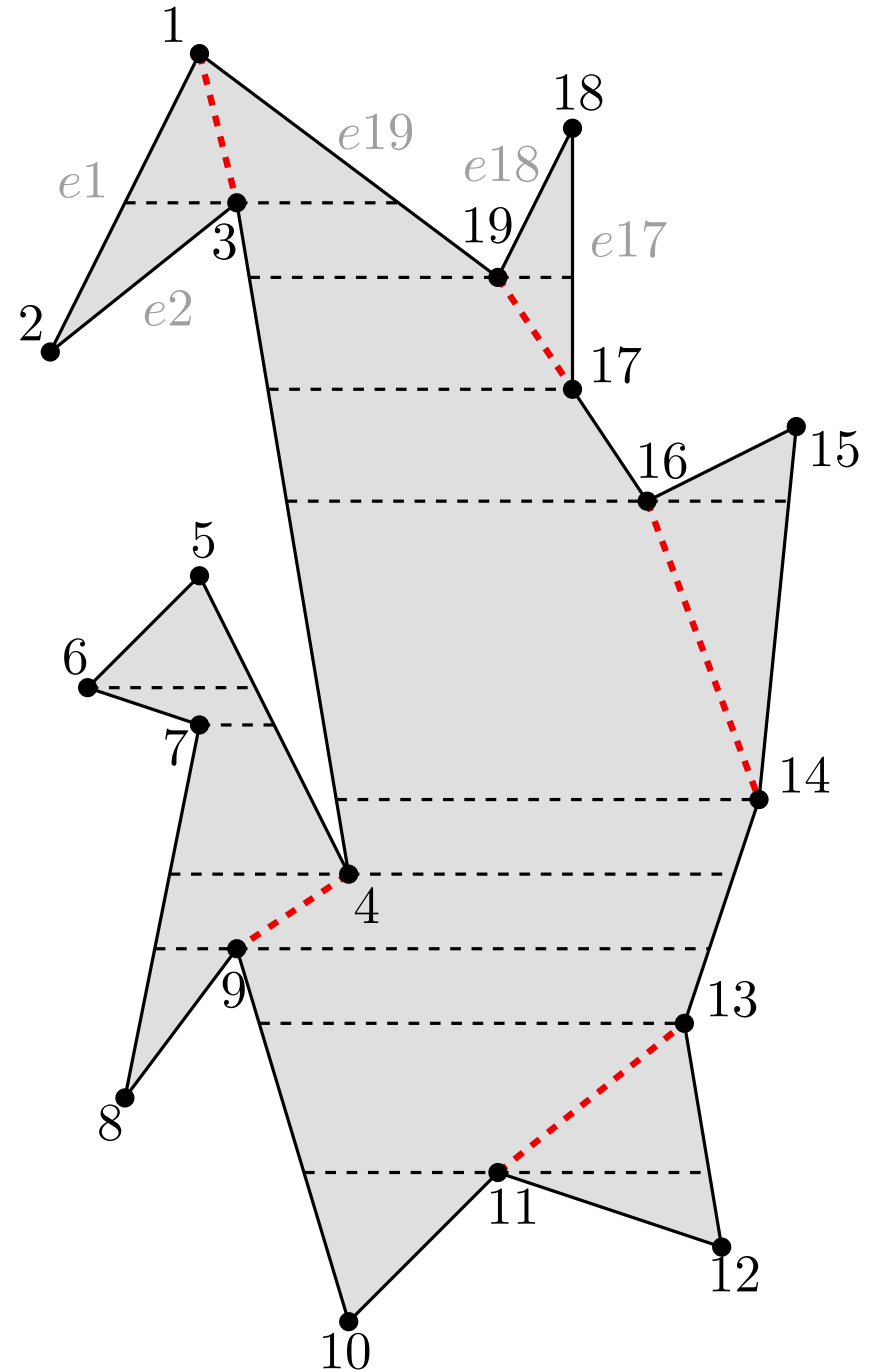
v_{11}

~~e_9, e_{10}~~



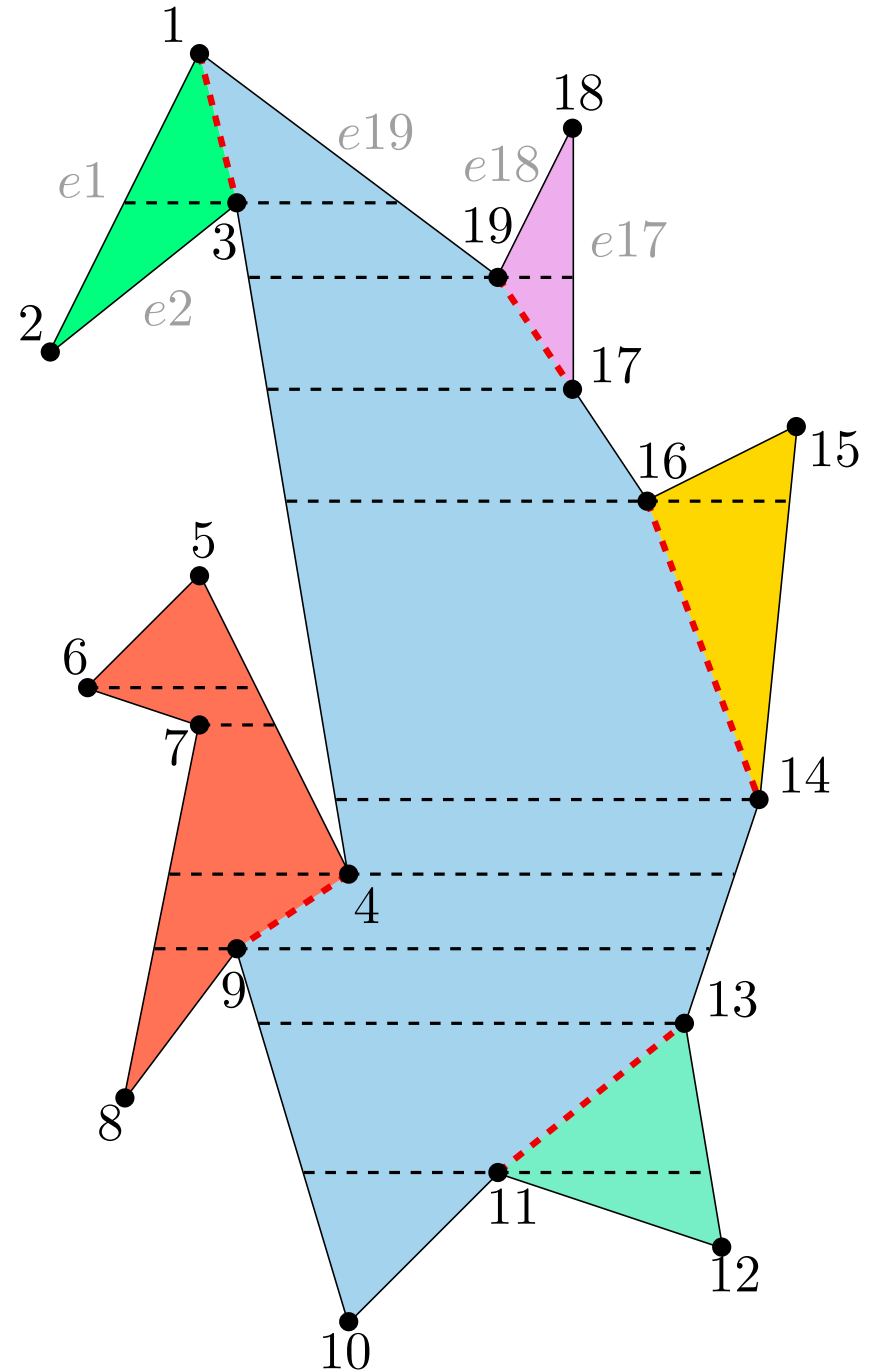
TRIANGULATING POLYGONS

Monotone partition



TRIANGULATING POLYGONS

Monotone partition



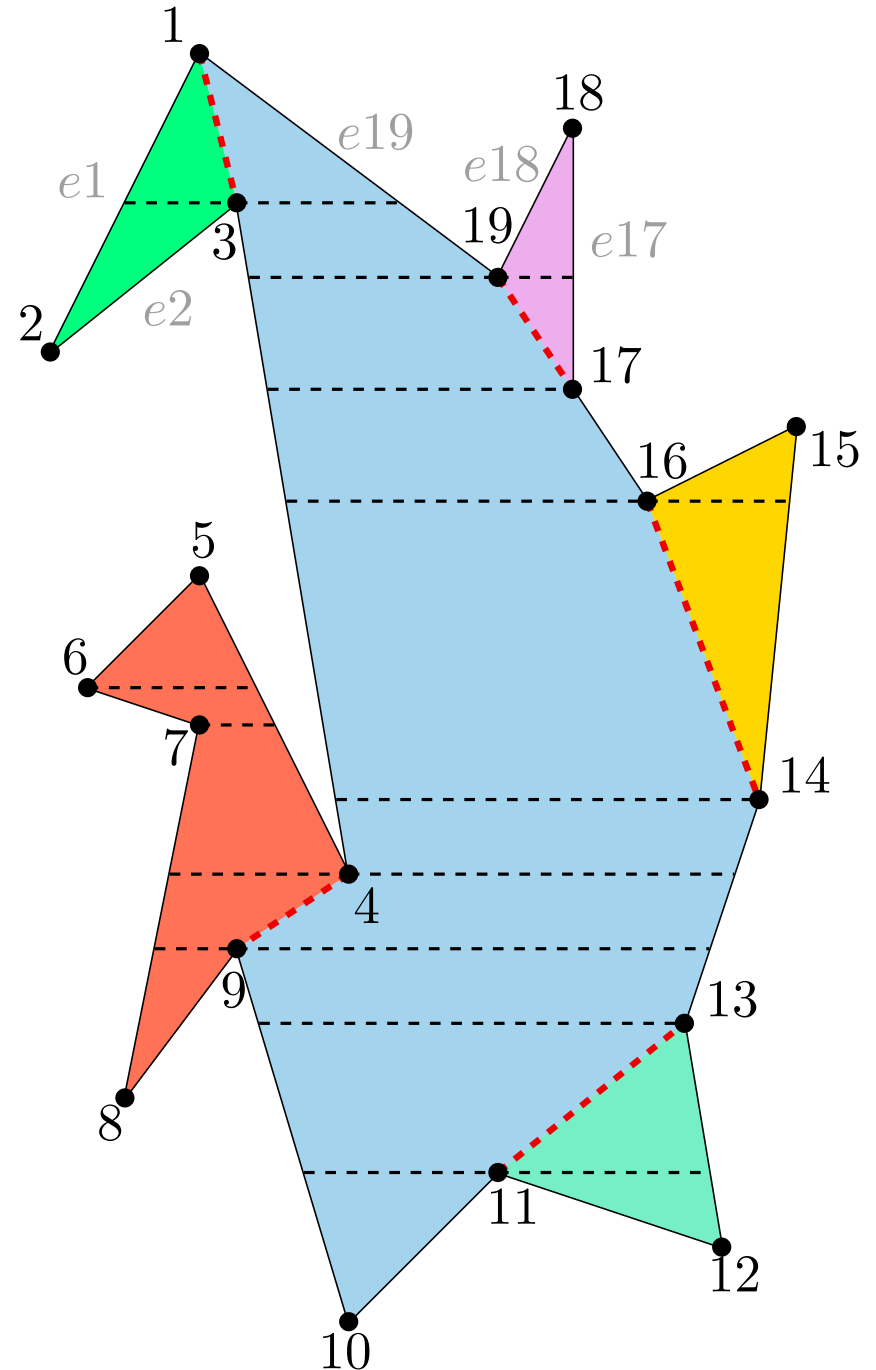
TRIANGULATING POLYGONS

Monotone partition

Running time:

- Sorting the vertices in the event queue: $O(n \log n)$ time.
- On each event, update sweep line: replace, insert or delete vertices or edges in $O(\log n)$ time each.
- There are n events.

The algorithm runs in $O(n \log n)$ time.



TRIANGULATING POLYGONS

Summarizing

Running time of polygon triangulation:

- $O(n^2)$ by subtracting ears
- $O(n^2)$ by inserting diagonals
- $O(n \log n)$ by:
 1. Decomposing the polygon into monotone subpolygons in $O(n \log n)$ time
 2. Triangulating each monotone subpolygon in $O(n)$ time

TRIANGULATING POLYGONS

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Is it possible to triangulate a polygon in $o(n \log n)$ time?

TRIANGULATING POLYGONS

Summarizing

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 1. Decomposing the polygon into monotone subpolygons in $O(n \log n)$ time
 2. Triangulating each monotone subpolygon in $O(n)$ time

Is it possible to triangulate a polygon in $o(n \log n)$ time?

Yes.

There exists an algorithm to triangulate an n -vertex simple polygon in $O(n)$ time, but it is too complicated and, in practice, it is not used:

B. Chazelle, Triangulating a Simple Polygon in Linear Time,
Discrete and Computational Geometry, 6:485–524, 1991.

STORING THE POLYGON TRIANGULATION

Storing the polygon triangulation

Possible options, advantages and disadvantages

Storing the polygon triangulation

Possible options, advantages and disadvantages

Storing the list of all the diagonals of the triangulation

Storing the polygon triangulation

Possible options, advantages and disadvantages

Storing the list of all the diagonals of the triangulation

Advantage: small memory usage.

Disadvantage: it suffices to draw the triangulation, but it does not contain the proximity information. For example, finding the triangles incident to a given diagonal, or finding the neighbors of a given triangle are expensive computations.

Storing the polygon triangulation

Possible options, advantages and disadvantages

Storing the list of all the diagonals of the triangulation

Advantage: small memory usage.

Disadvantage: it suffices to draw the triangulation, but it does not contain the proximity information. For example, finding the triangles incident to a given diagonal, or finding the neighbors of a given triangle are expensive computations.

For each triangle, storing the sorted list of its vertices and edges, as well as the sorted list of its neighbors.

Storing the polygon triangulation

Possible options, advantages and disadvantages

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Storing the polygon triangulation

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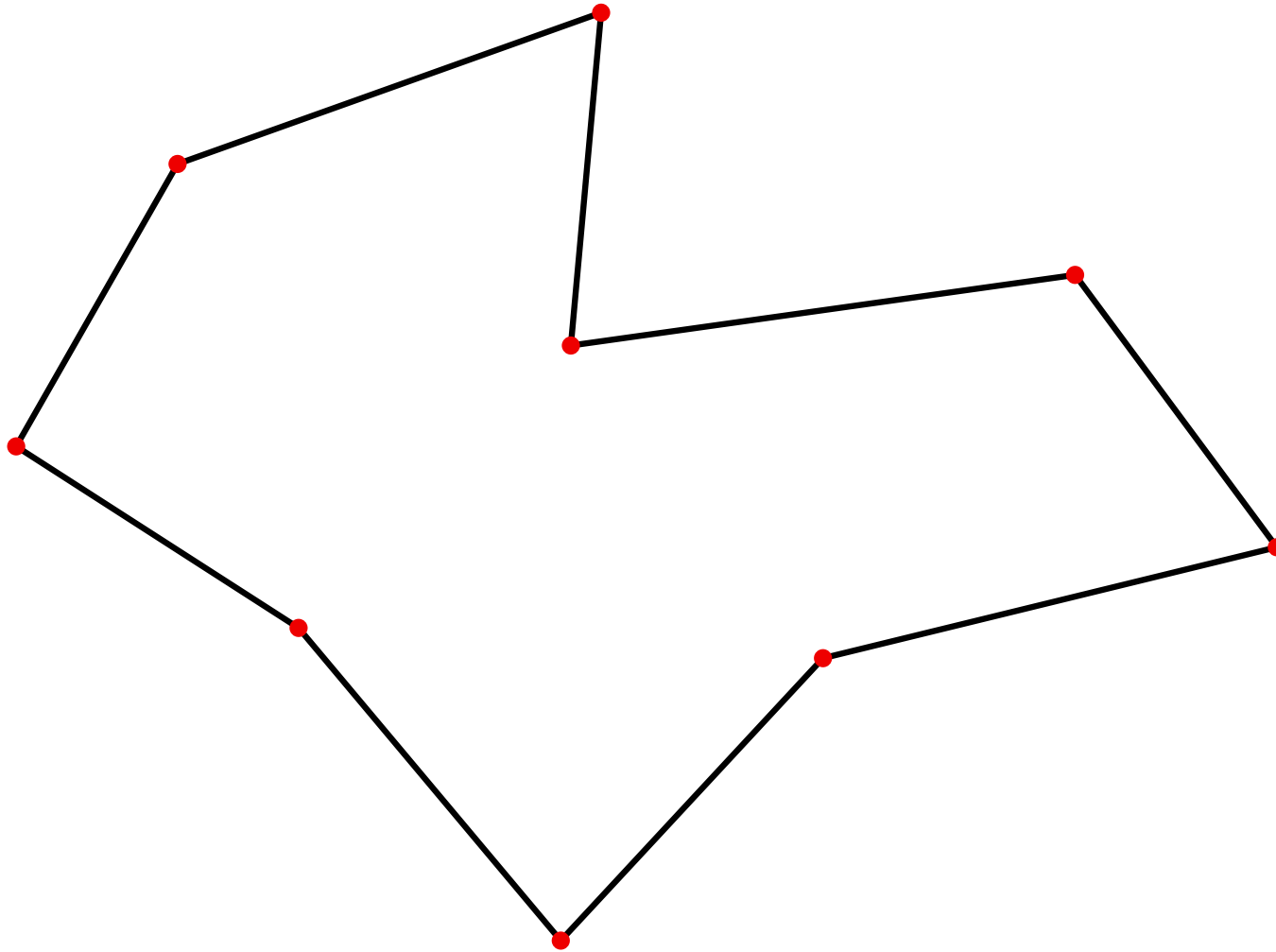
Disadvantage: the stored data is redundant and it uses more space than required.

The data structure which is most frequently used to store a triangulation is the DCEL (doubly connected edge list).

The DCEL is also used to store plane partitions, polyhedra, meshes, etc.

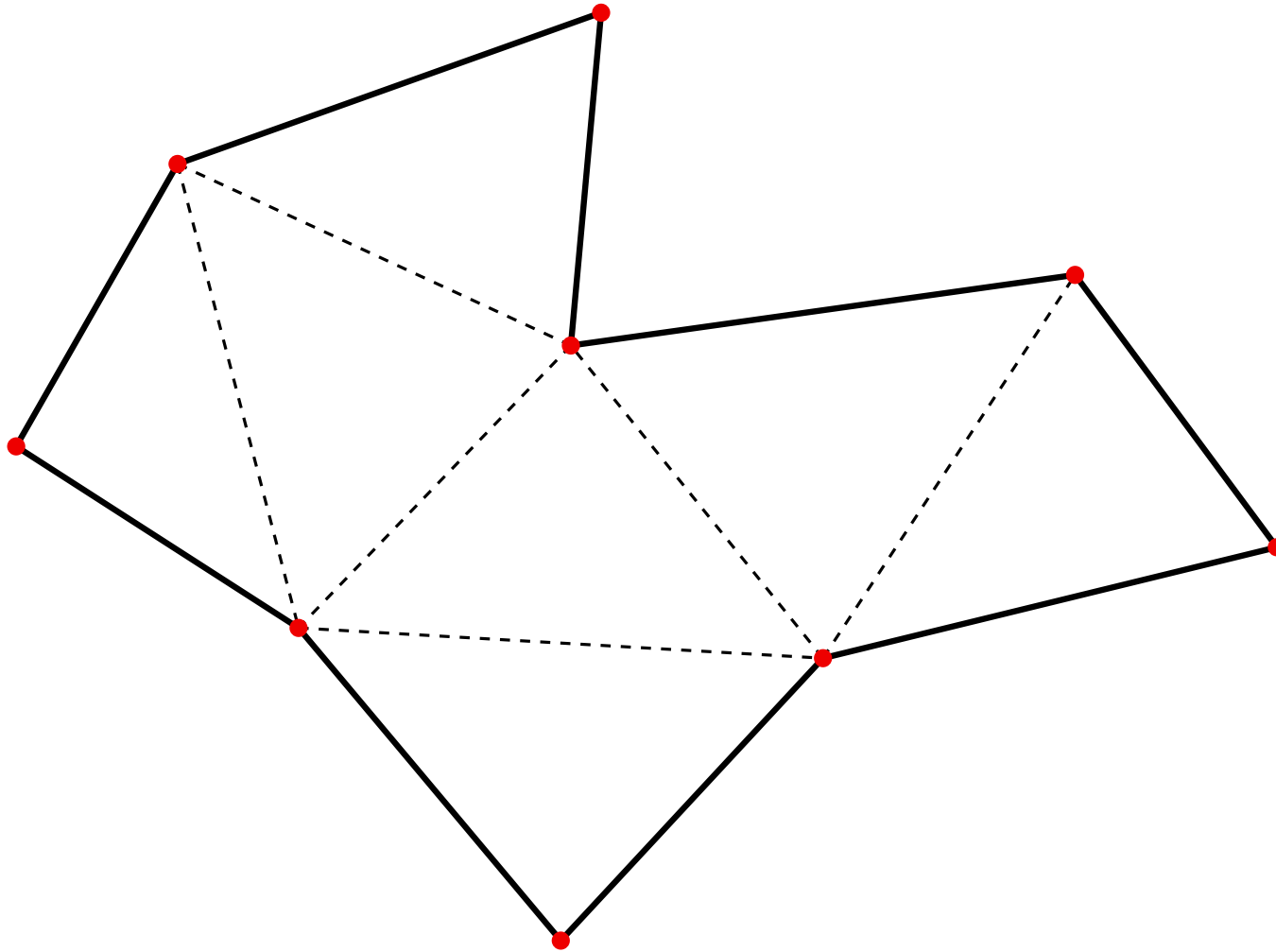
Storing the polygon triangulation

DCEL



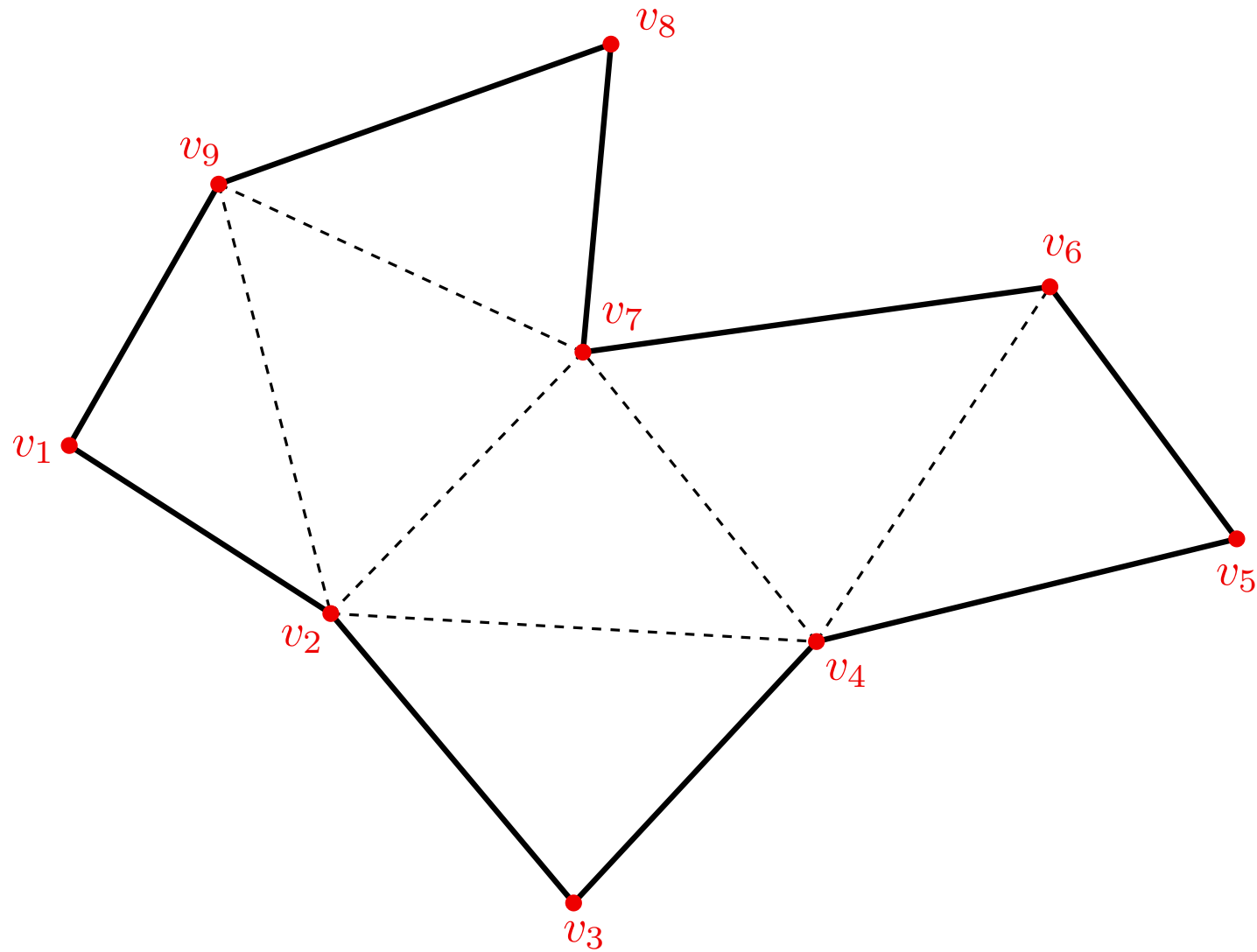
Storing the polygon triangulation

DCEL



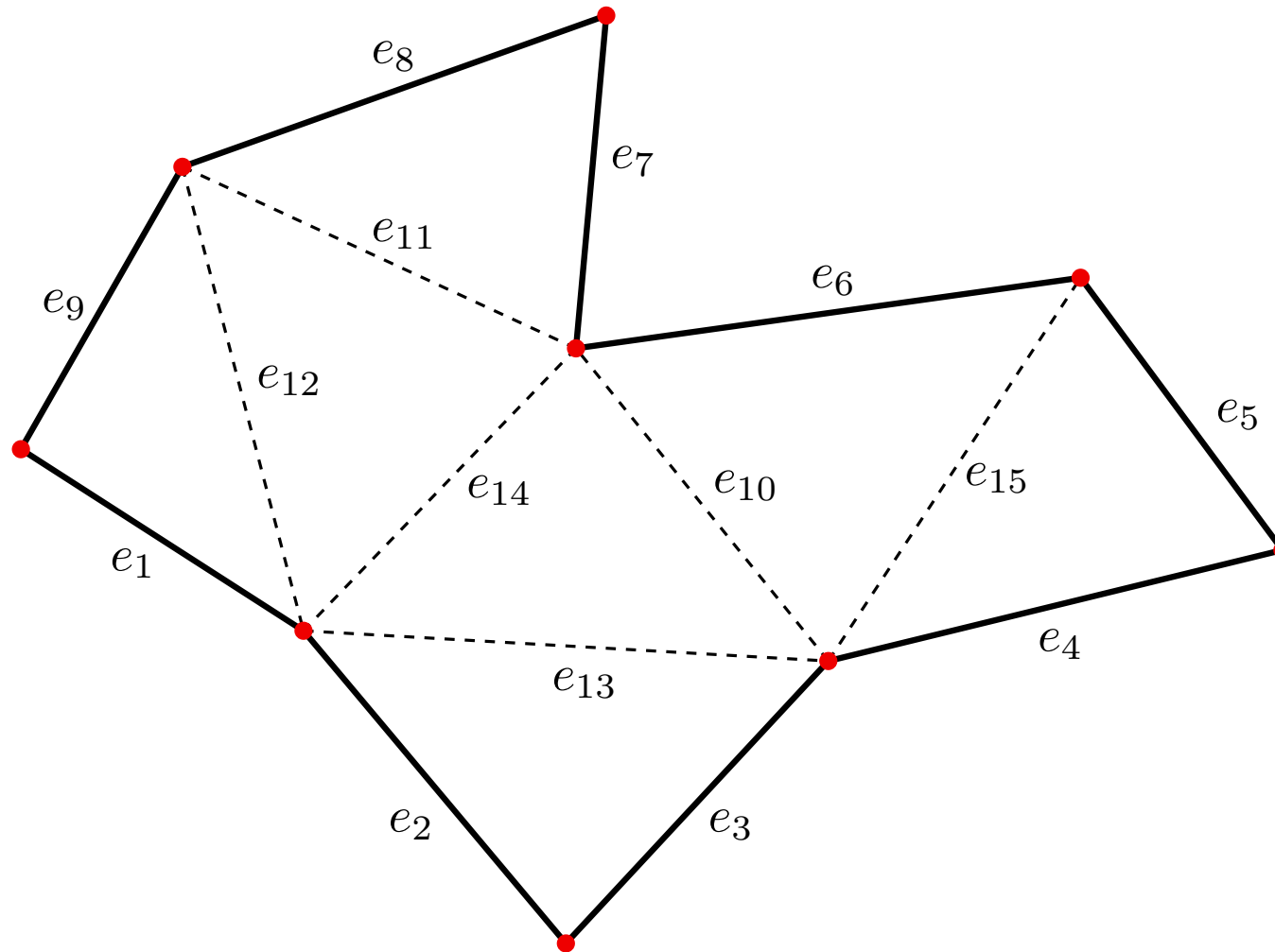
Storing the polygon triangulation

DCEL



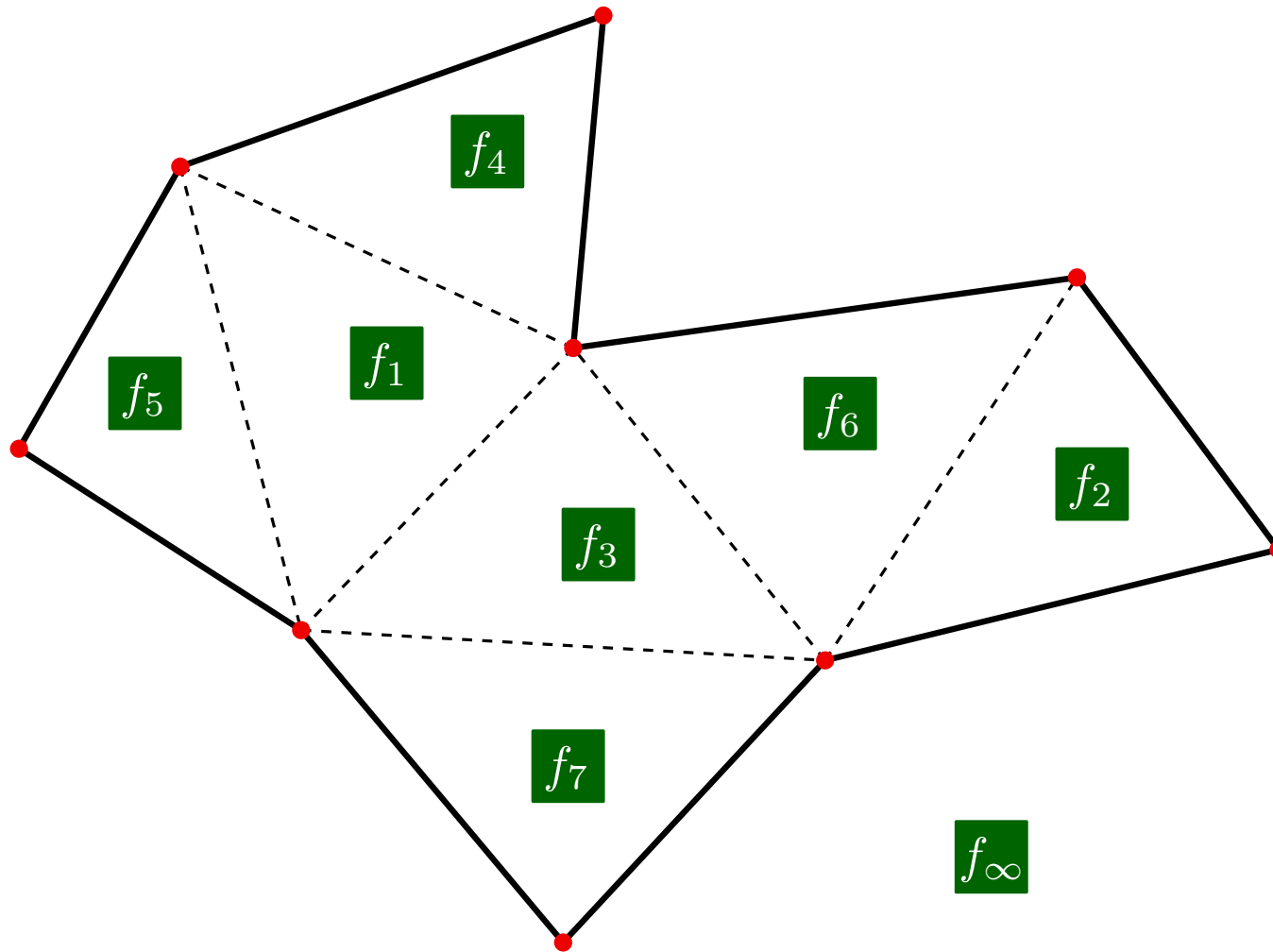
Storing the polygon triangulation

DCEL



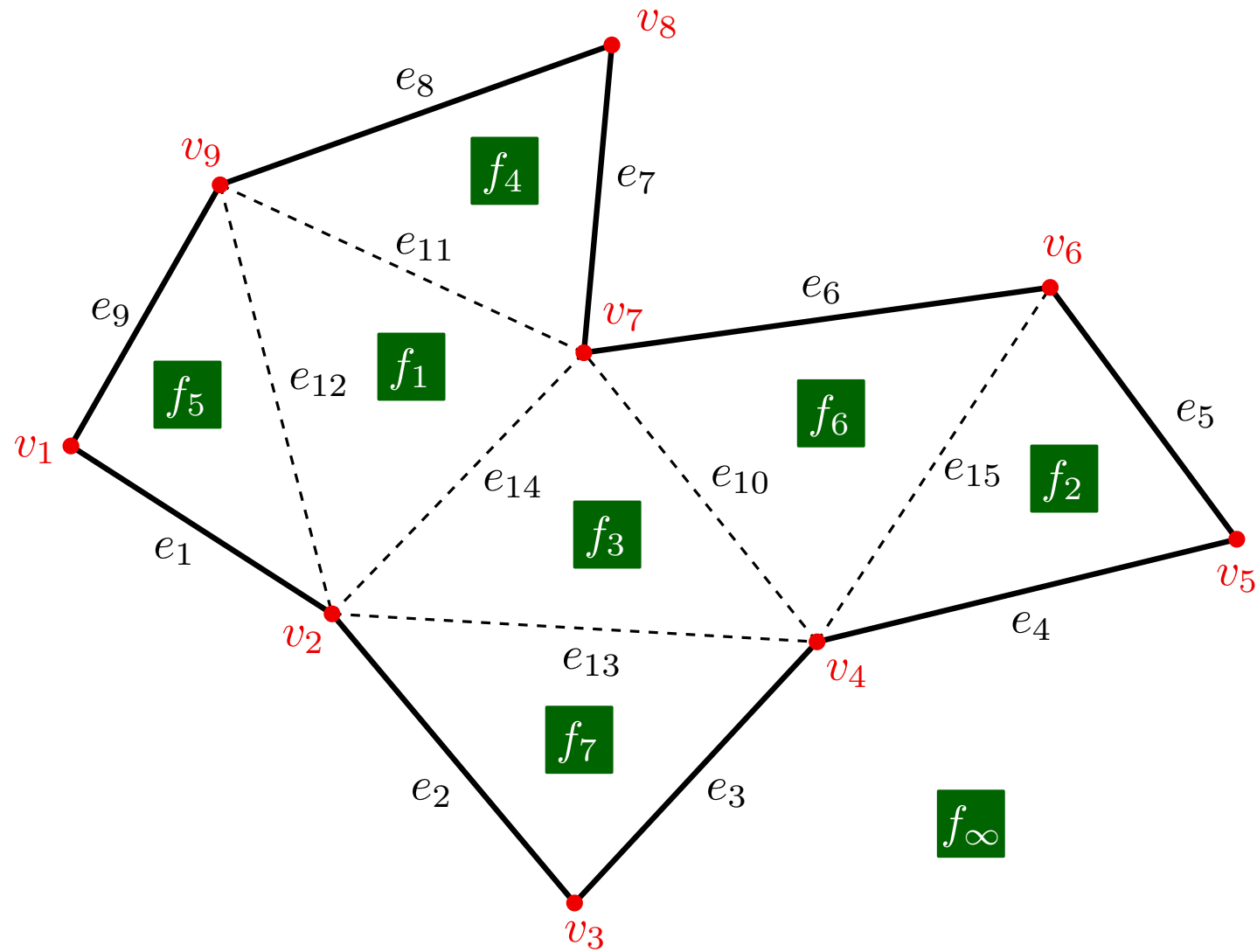
Storing the polygon triangulation

DCEL

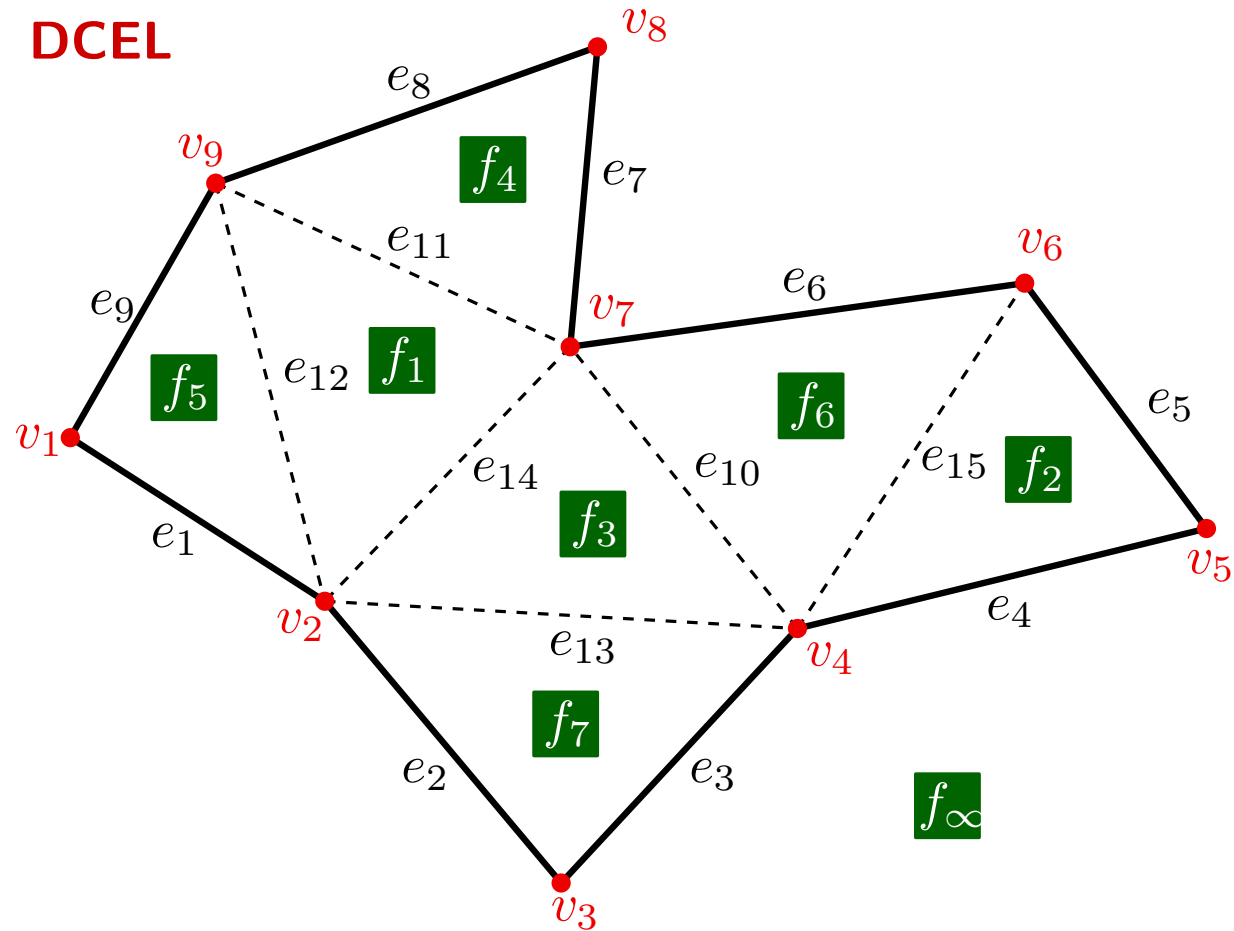


Storing the polygon triangulation

DCEL



Storing the polygon triangulation

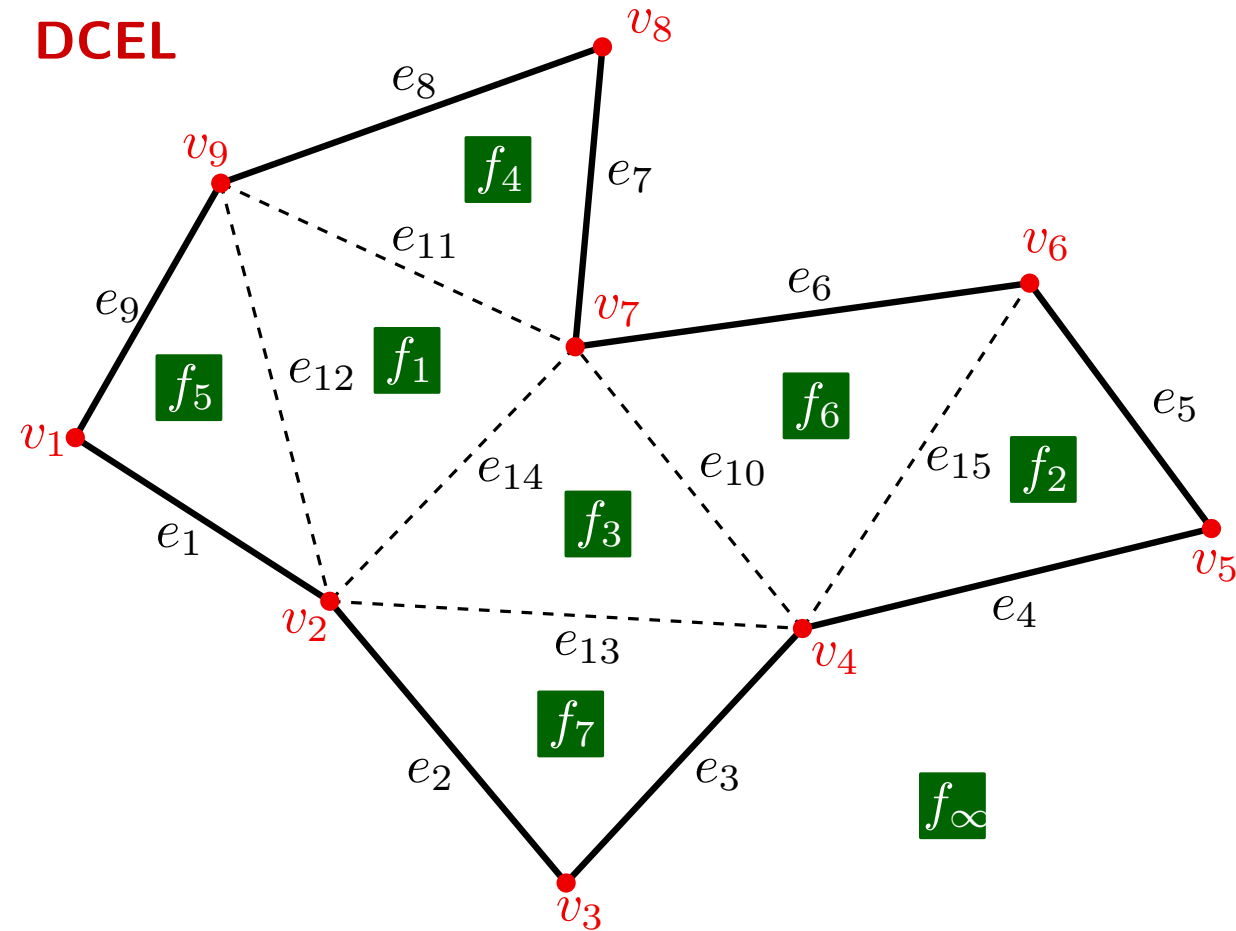


Storing the polygon triangulation

Table of vertices

v	x	y	e
1	x_1	y_1	1
2	x_2	y_2	1
3	x_3	y_3	2
4	x_4	y_4	10
5	x_5	y_5	4
6	x_6	y_6	6
7	x_7	y_7	10
8	x_8	y_8	8
9	x_9	y_9	9

DCEL

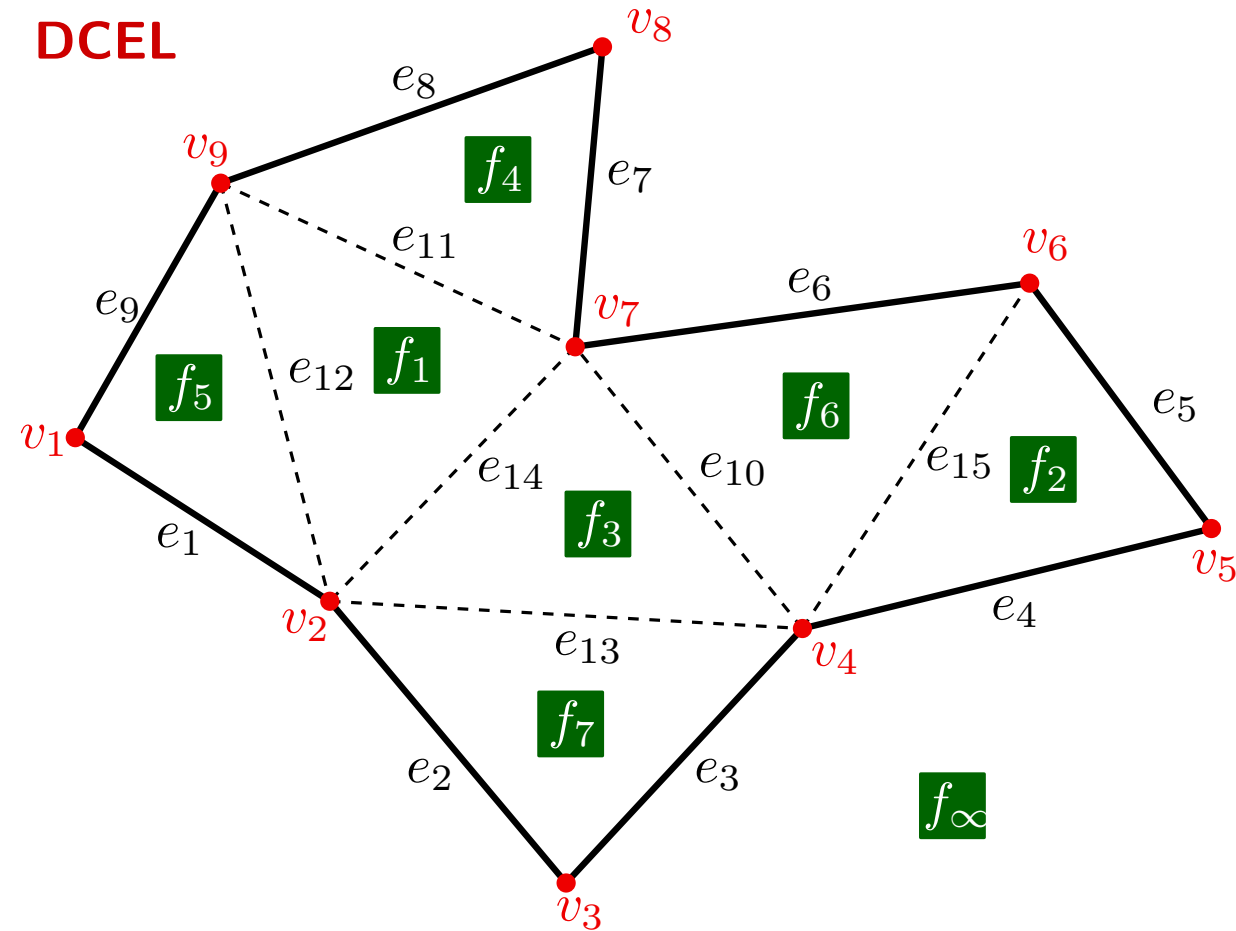


Storing the polygon triangulation

Table of faces

f	e
1	11
2	4
3	10
4	11
5	1
6	6
7	2
∞	9

DCEL

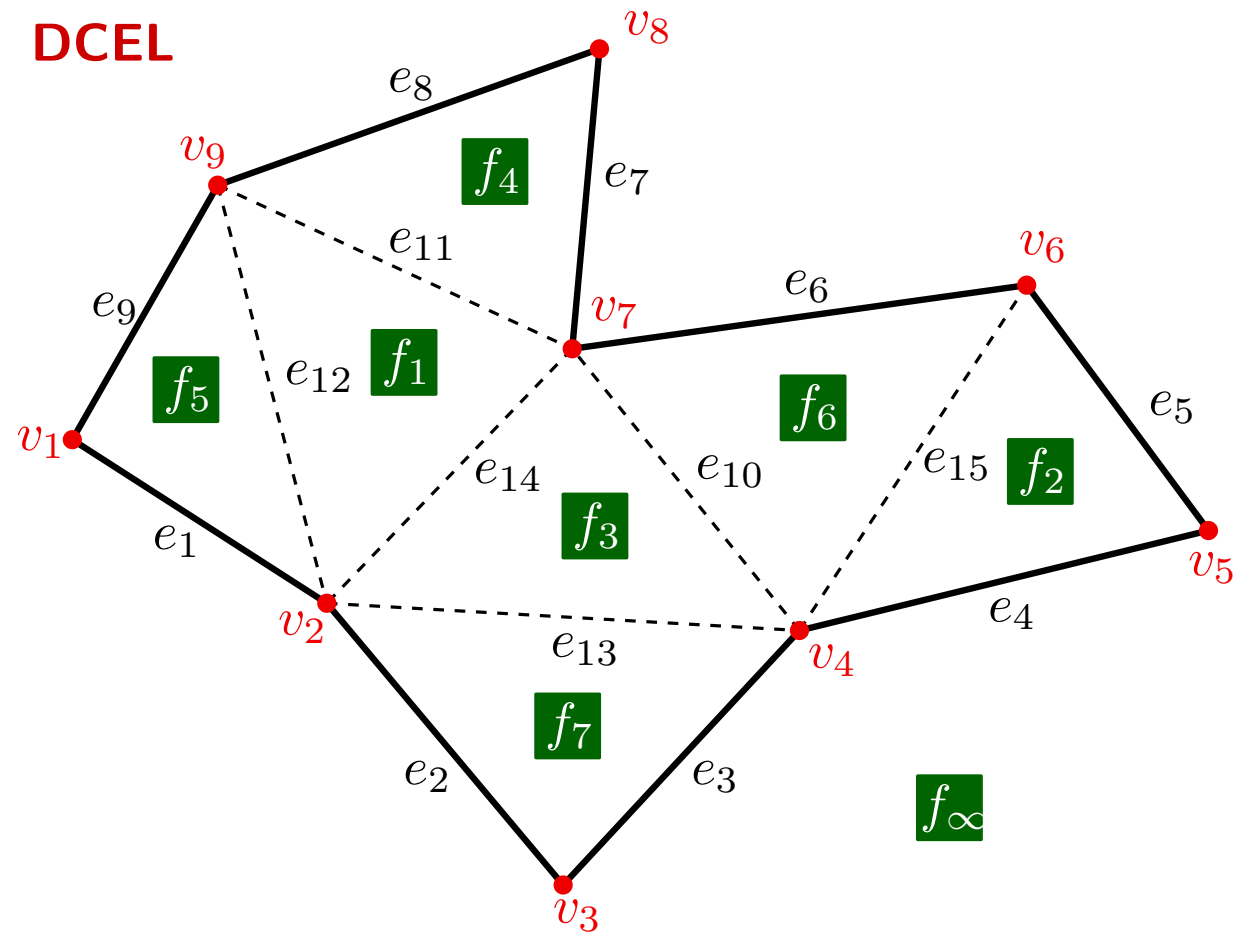


Storing the polygon triangulation

DCEL

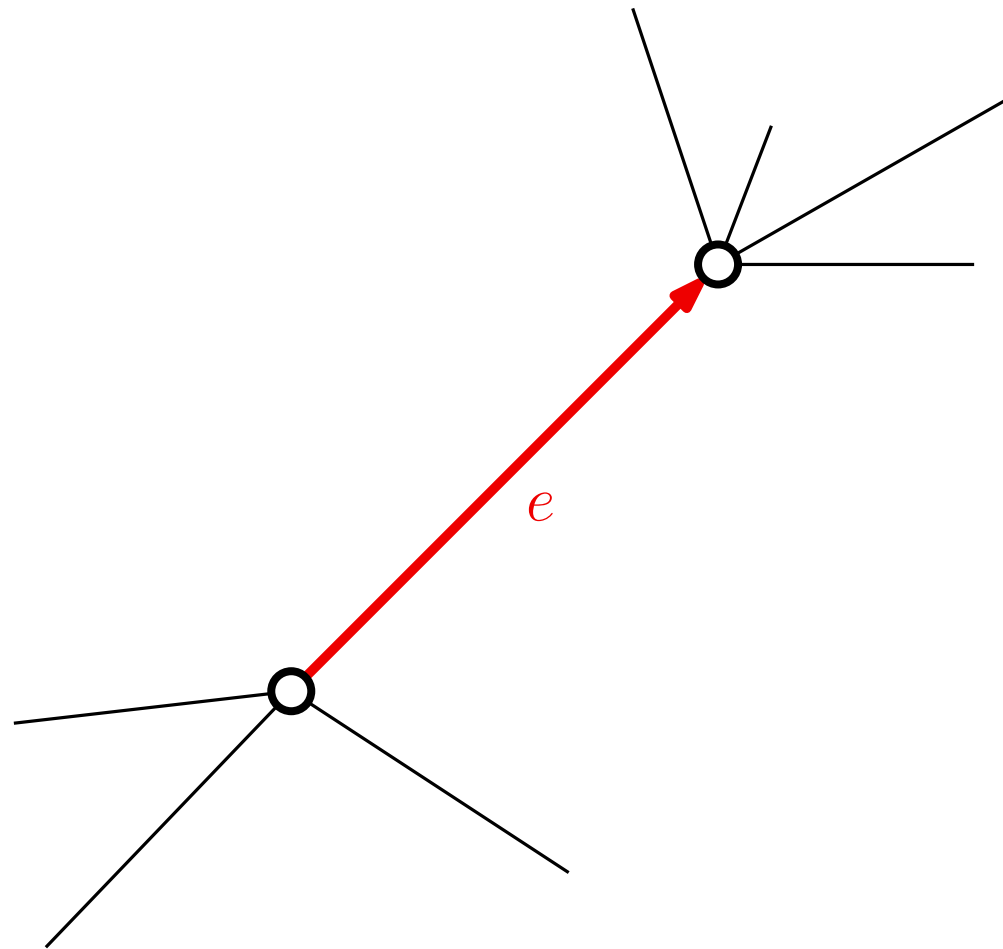
e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	2
2	2	3	7	∞	13	3
3	4	3	∞	7	4	2
4	4	5	2	∞	15	5
5	5	6	2	∞	4	6
6	6	7	6	∞	15	7
7	7	8	4	∞	11	8
8	8	9	4	∞	7	9
9	9	1	5	∞	12	1
10	4	7	3	6	13	6
11	9	7	4	1	8	14
12	2	9	5	1	1	11
13	2	4	3	7	14	3
14	2	7	1	3	12	10
15	4	6	6	2	10	5

DCEL



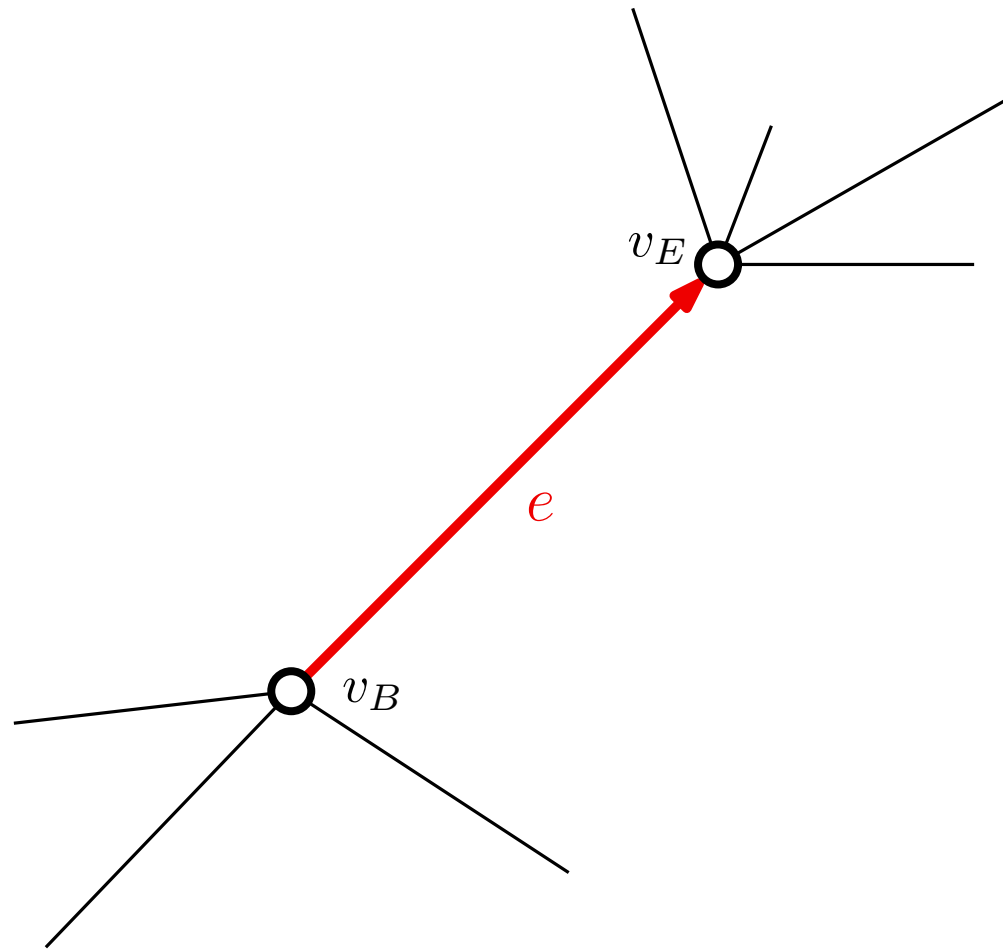
Storing the polygon triangulation

DCEL



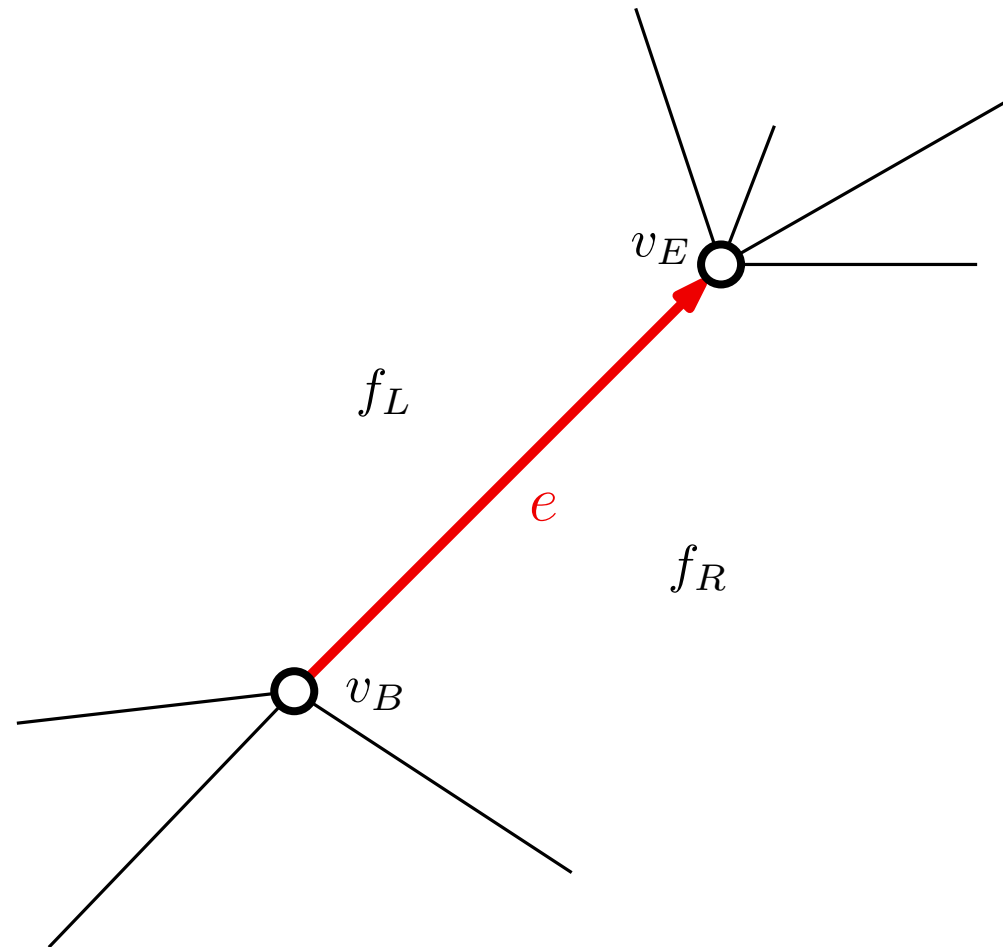
Storing the polygon triangulation

DCEL



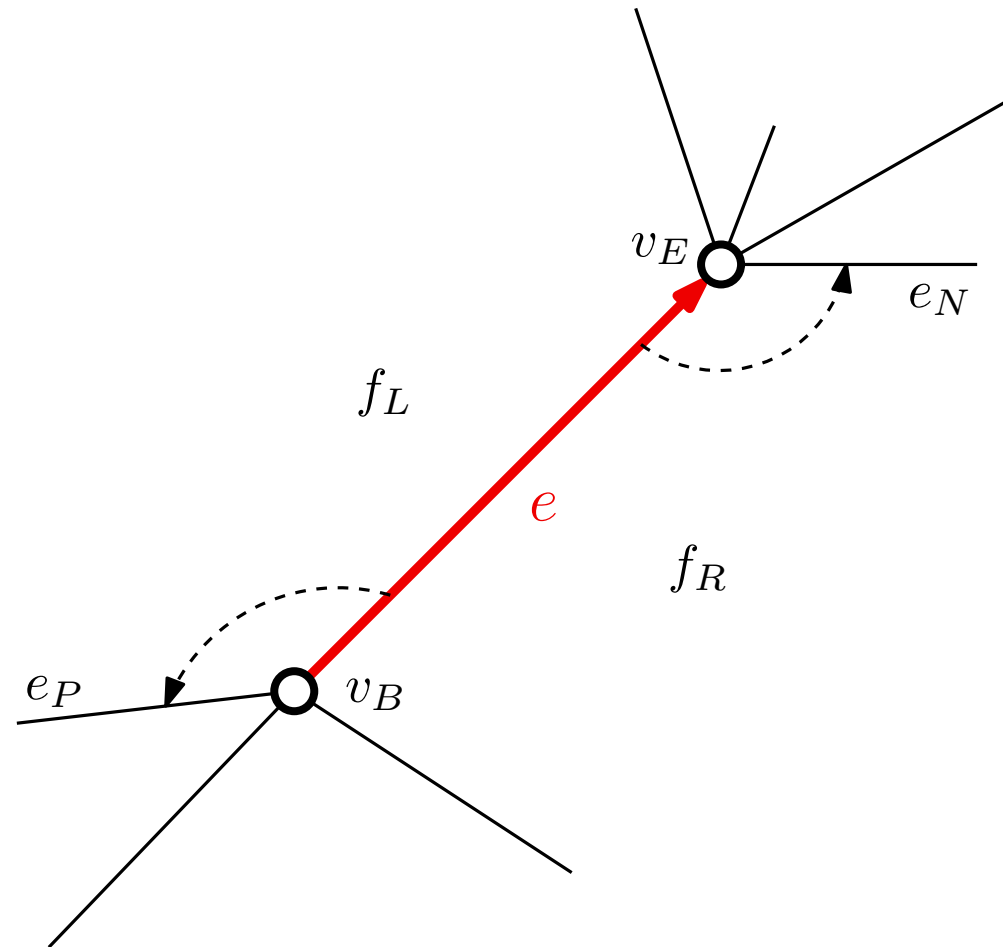
Storing the polygon triangulation

DCEL



Storing the polygon triangulation

DCEL

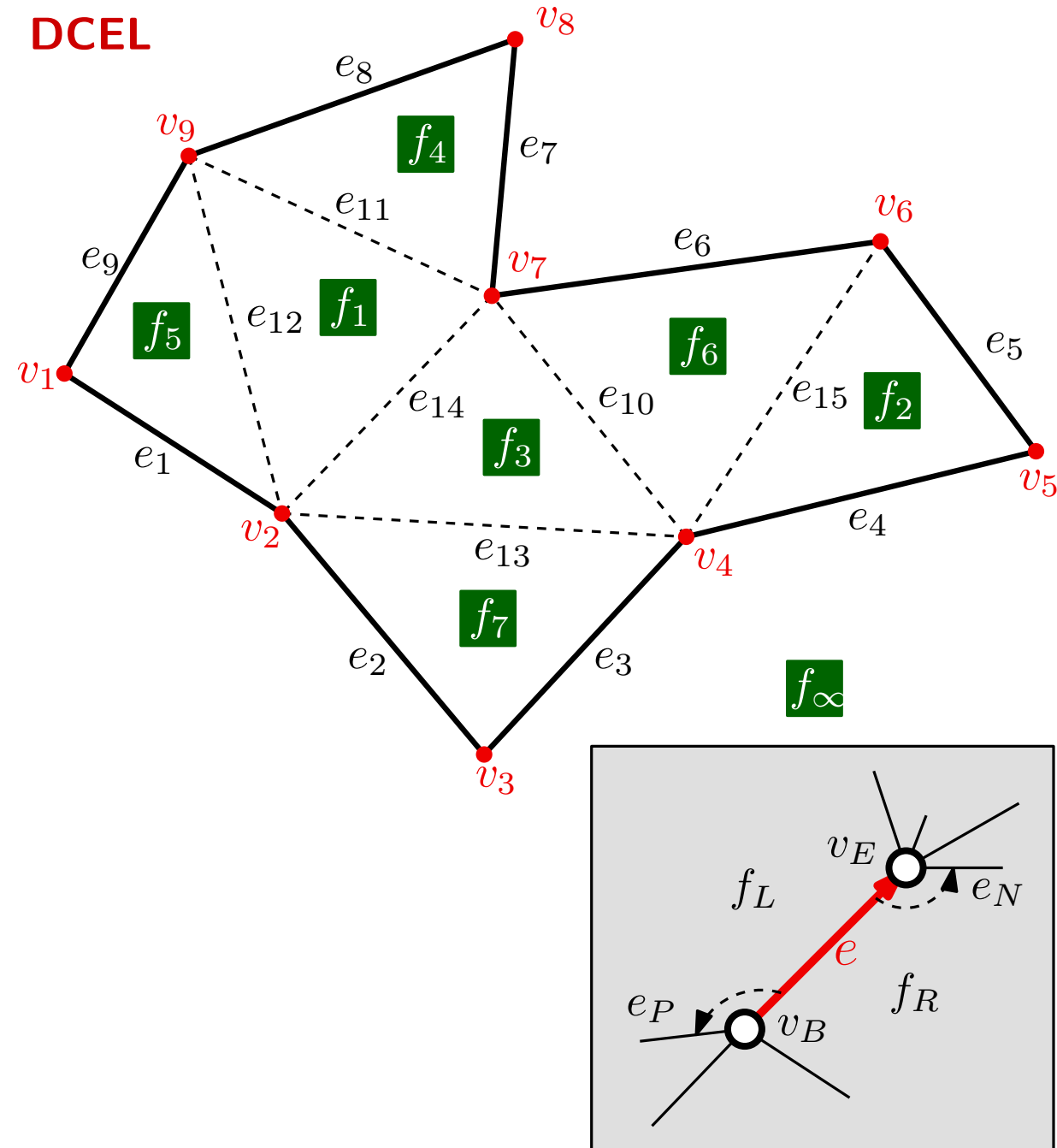


Storing the polygon triangulation

DCEL

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	2
2	2	3	7	∞	13	3
3	4	3	∞	7	4	2
4	4	5	2	∞	15	5
5	5	6	2	∞	4	6
6	6	7	6	∞	15	7
7	7	8	4	∞	11	8
8	8	9	4	∞	7	9
9	9	1	5	∞	12	1
10	4	7	3	6	13	6
11	9	7	4	1	8	14
12	2	9	5	1	1	11
13	2	4	3	7	14	3
14	2	7	1	3	12	10
15	4	6	6	2	10	5

DCEL

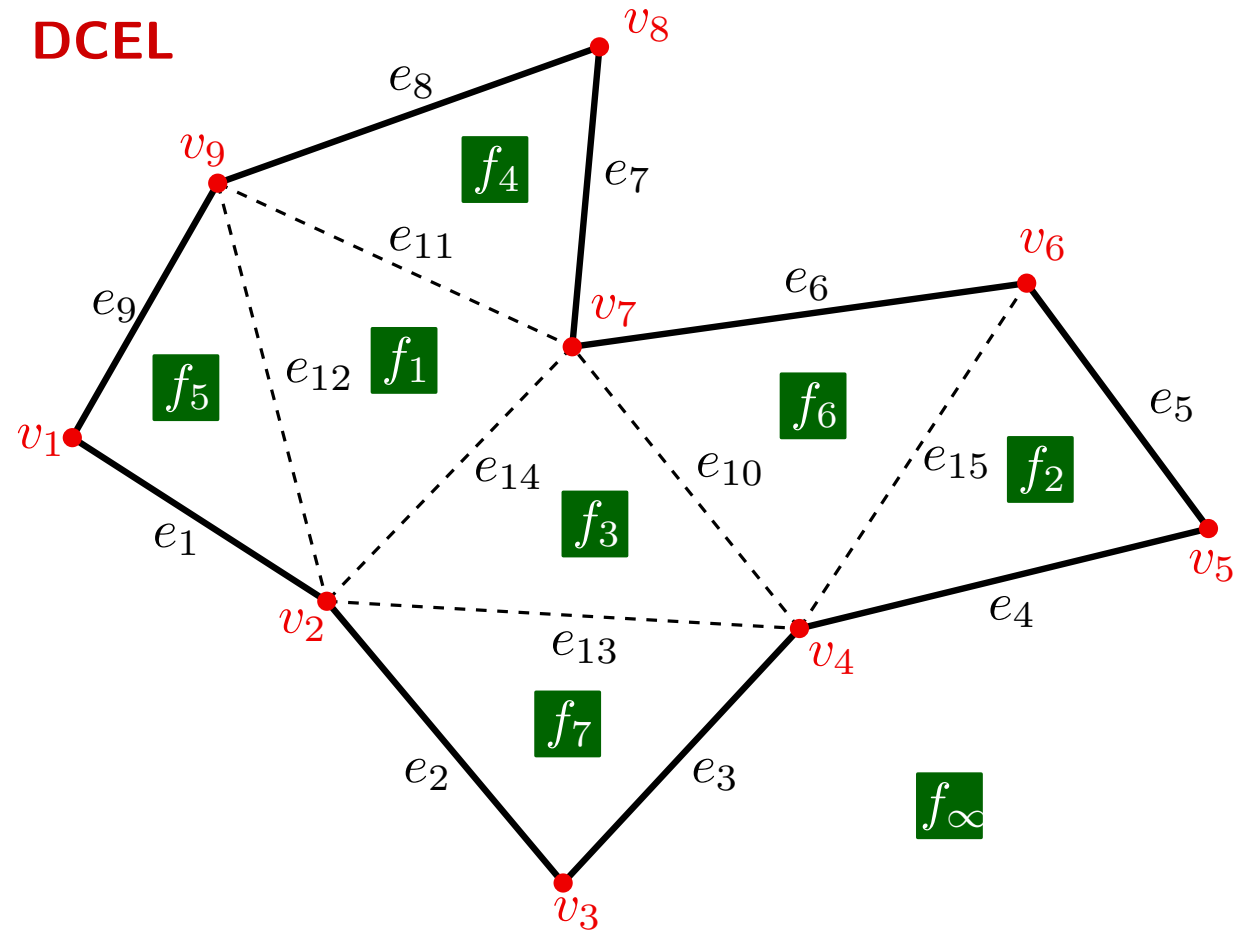


Storing the polygon triangulation

Storage space

- For each face:
1 pointer
- For each vertex:
2 coordinates + 1 pointer
- For each edge:
6 pointers

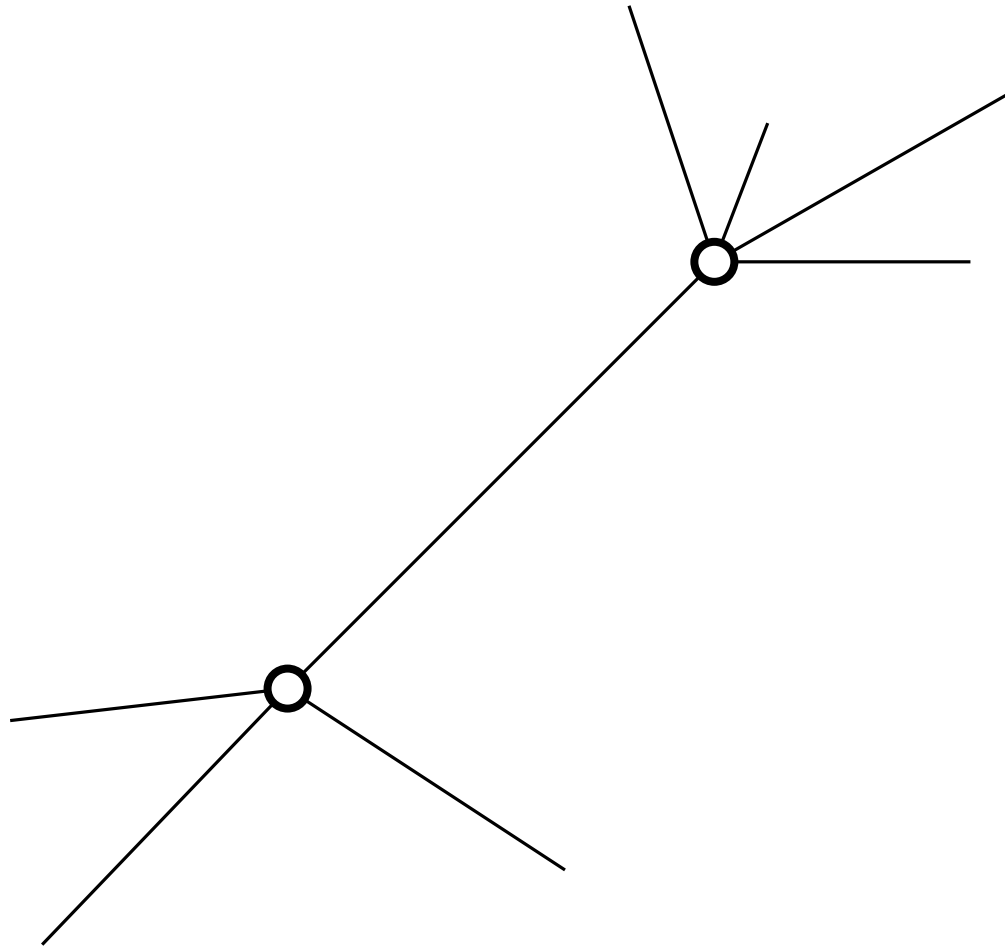
In total, the storage space is $O(n)$.



Storing the polygon triangulation

DCEL

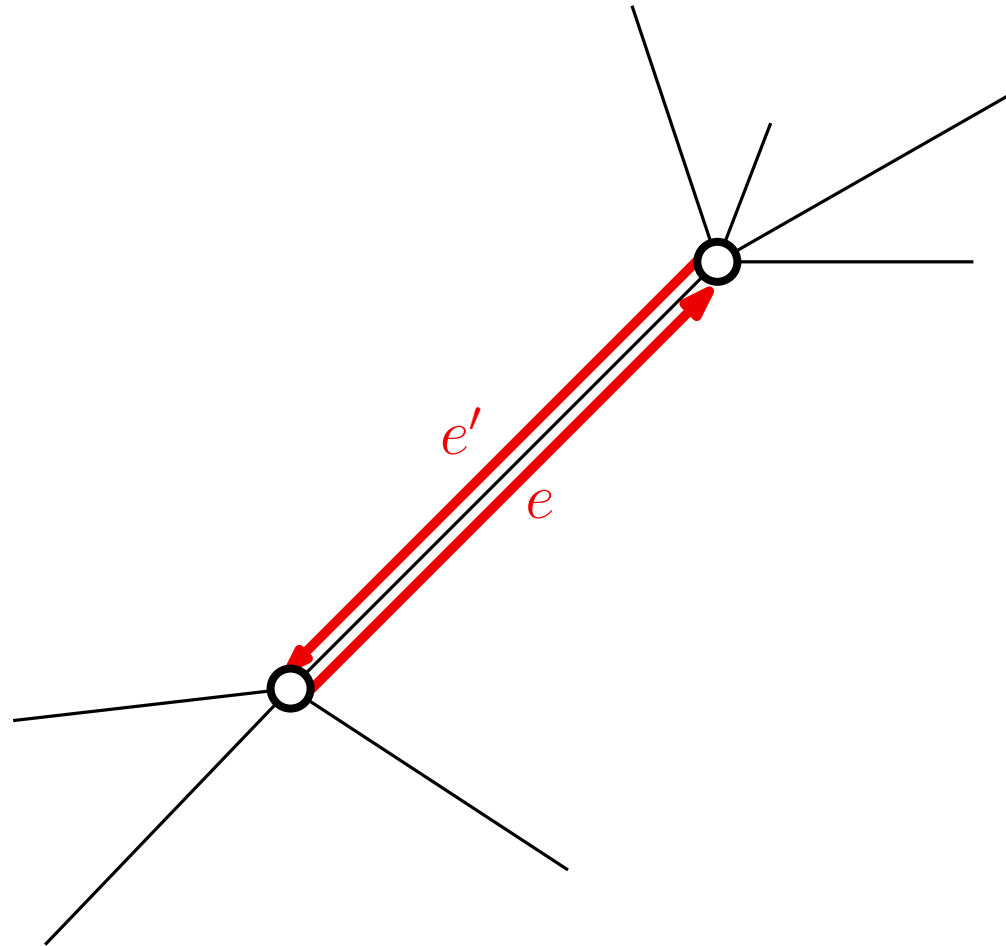
There are other DCEL variants, as for example:



Storing the polygon triangulation

DCEL

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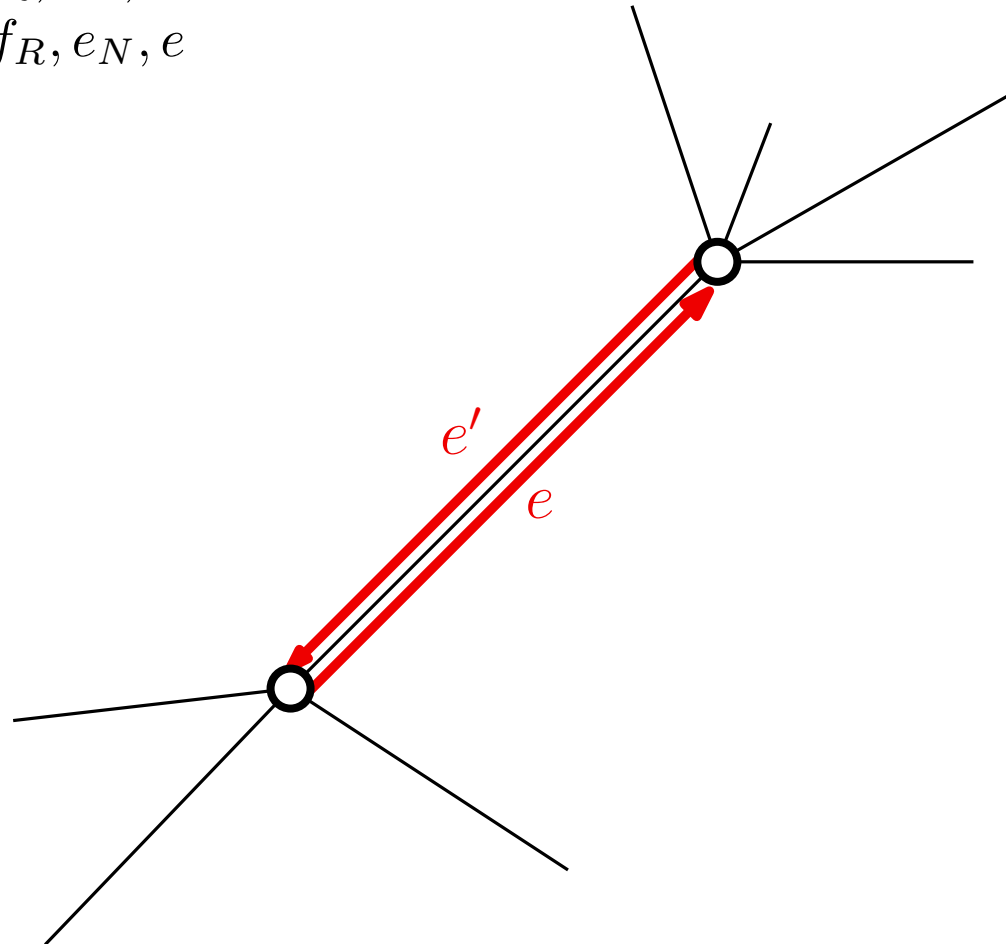


Storing the polygon triangulation

DCEL

There are other DCEL variants, as for example:

$$\begin{aligned} e &\longrightarrow v_B, f_R, e_N, e' \\ e' &\longrightarrow v_B, f_R, e_N, e \end{aligned}$$

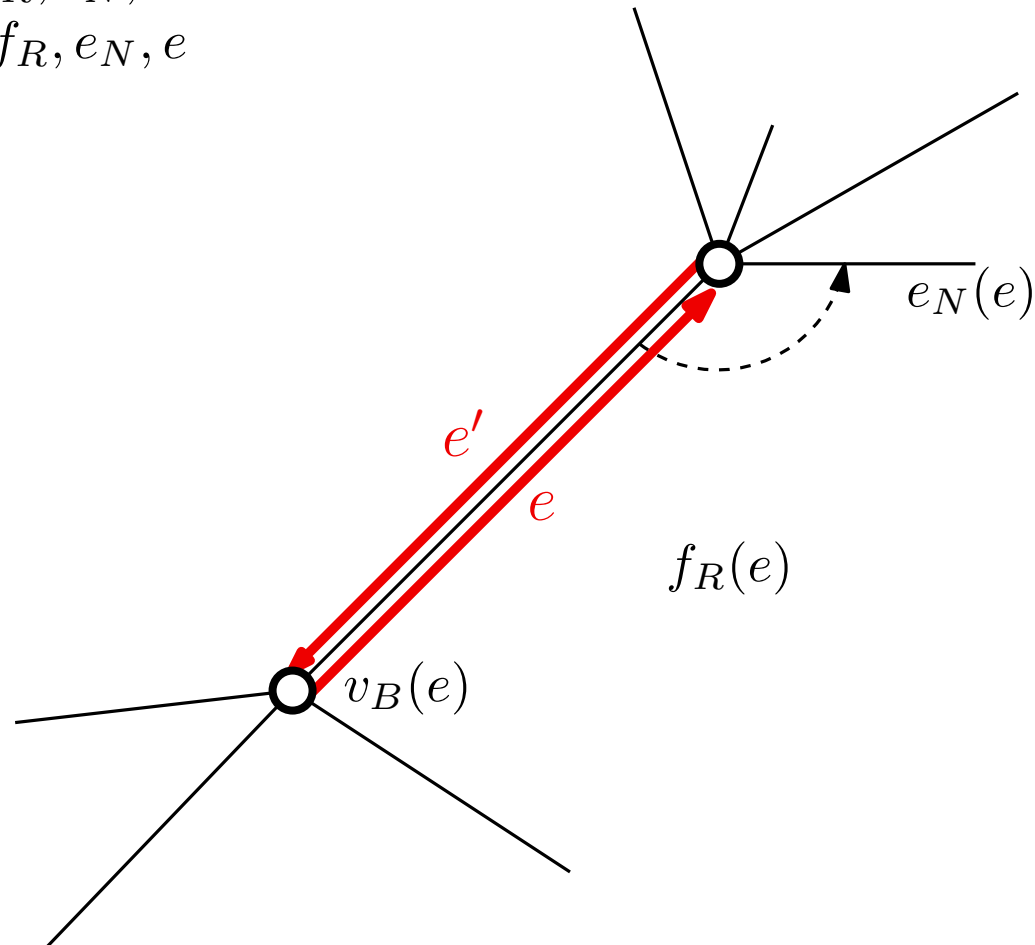


Storing the polygon triangulation

DCEL

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$$\begin{aligned} e &\longrightarrow v_B, f_R, e_N, e' \\ e' &\longrightarrow v_B, f_R, e_N, e \end{aligned}$$

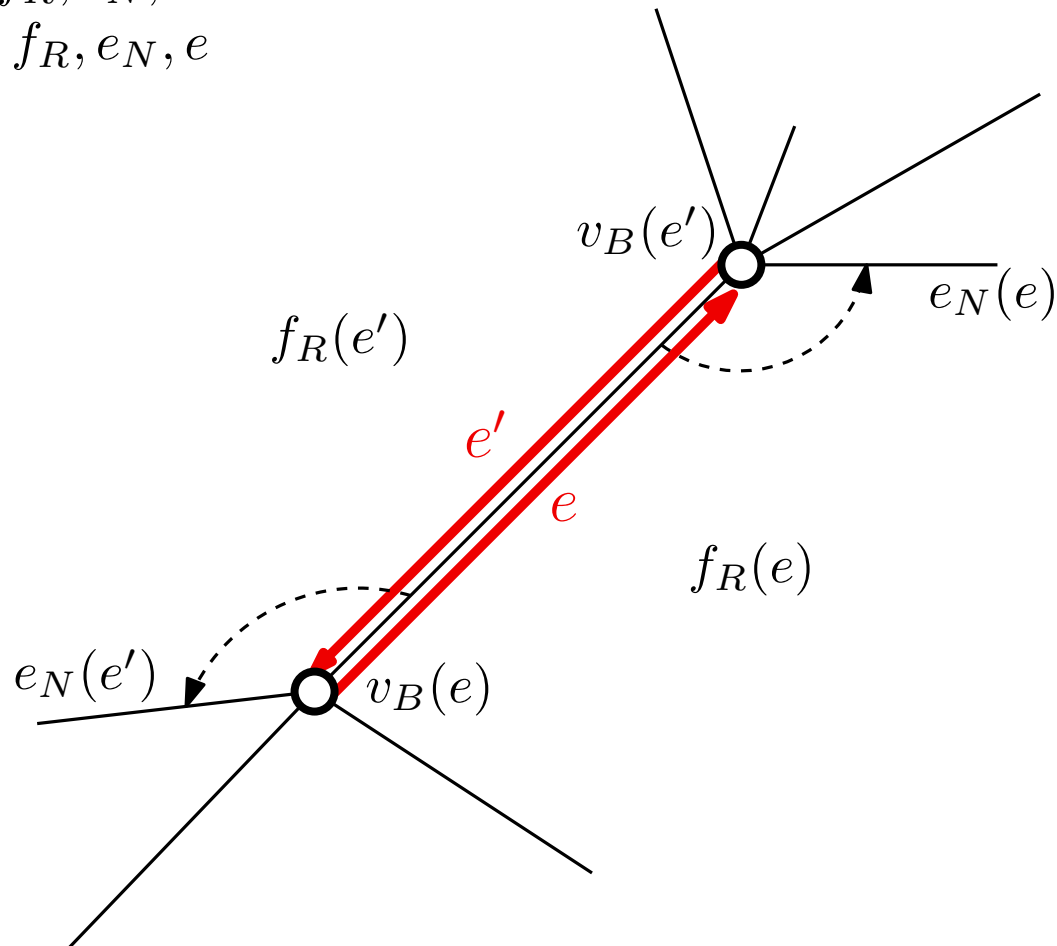


Storing the polygon triangulation

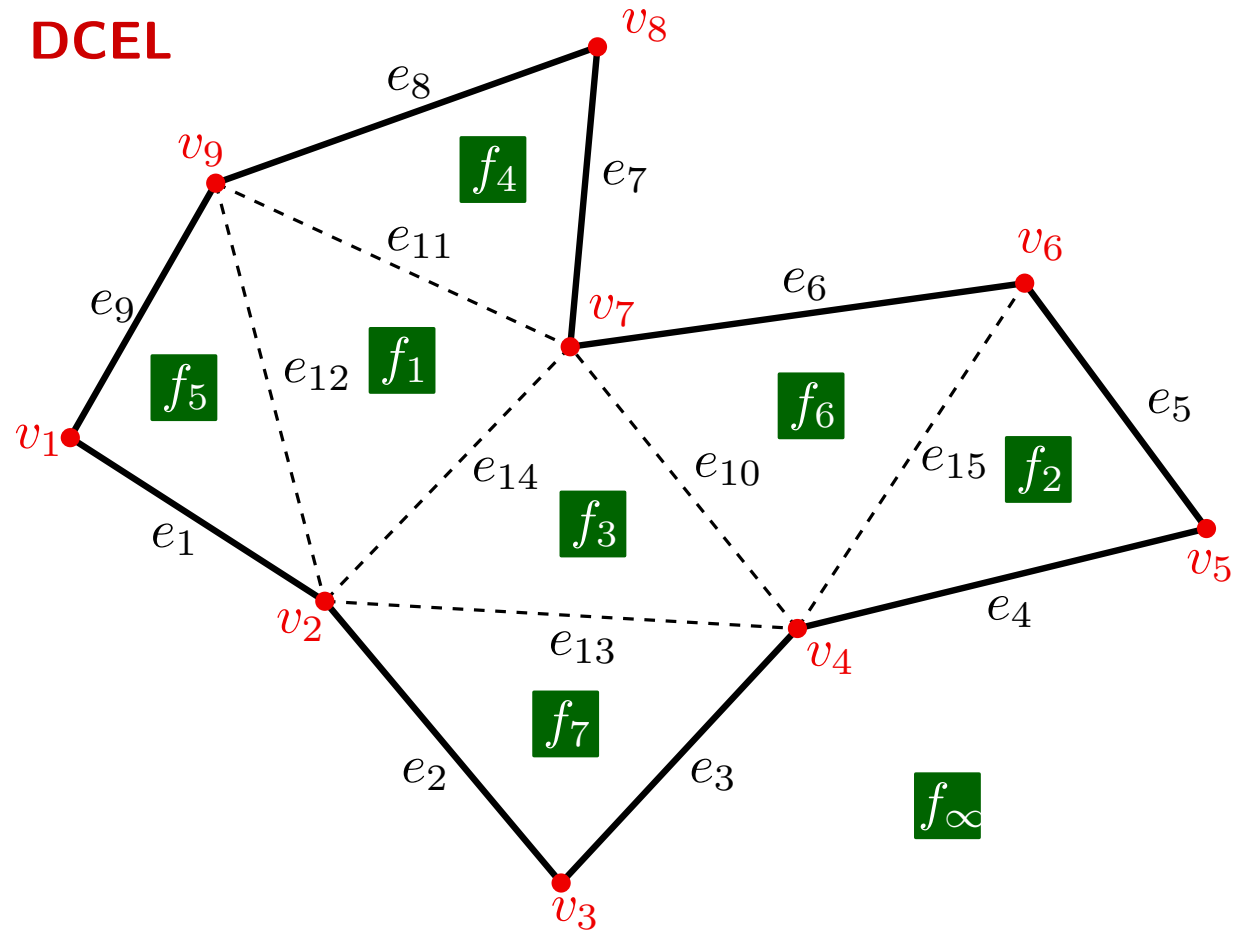
DCEL

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$$\begin{aligned} e &\longrightarrow v_B, f_R, e_N, e' \\ e' &\longrightarrow v_B, f_R, e_N, e \end{aligned}$$



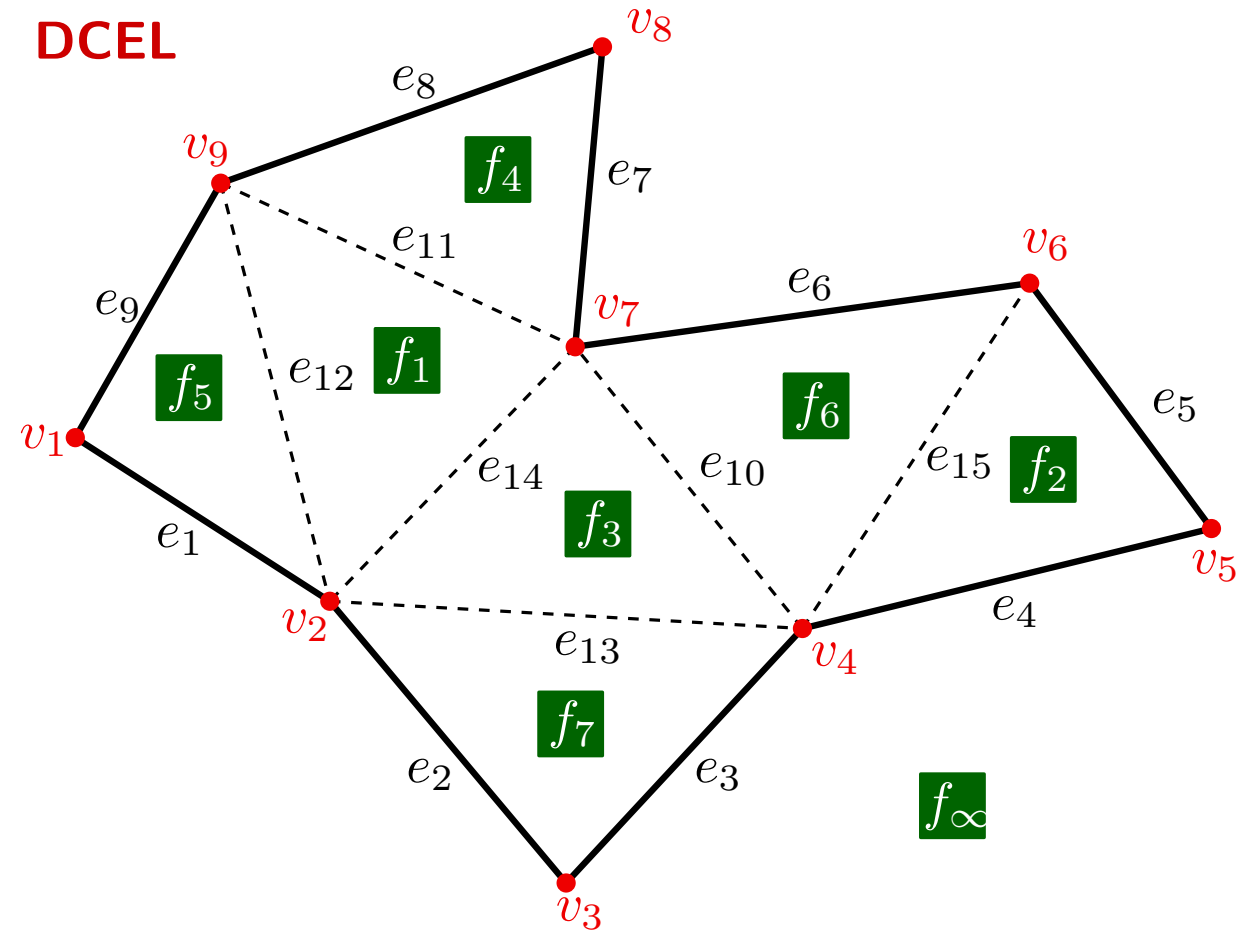
Storing the polygon triangulation



Storing the polygon triangulation

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	2
2	2	3	7	∞	13	3
3	4	3	∞	7	4	2
4	4	5	2	∞	15	5
5	5	6	2	∞	4	6
6	6	7	6	∞	15	7
7	7	8	4	∞	11	8
8	8	9	4	∞	7	9
9	9	1	5	∞	12	1
10	4	7	3	6	13	6
11	9	7	4	1	8	14
12	2	9	5	1	1	11
13	2	4	3	7	14	3
14	2	7	1	3	12	10
15	4	6	6	2	10	5

DCEL

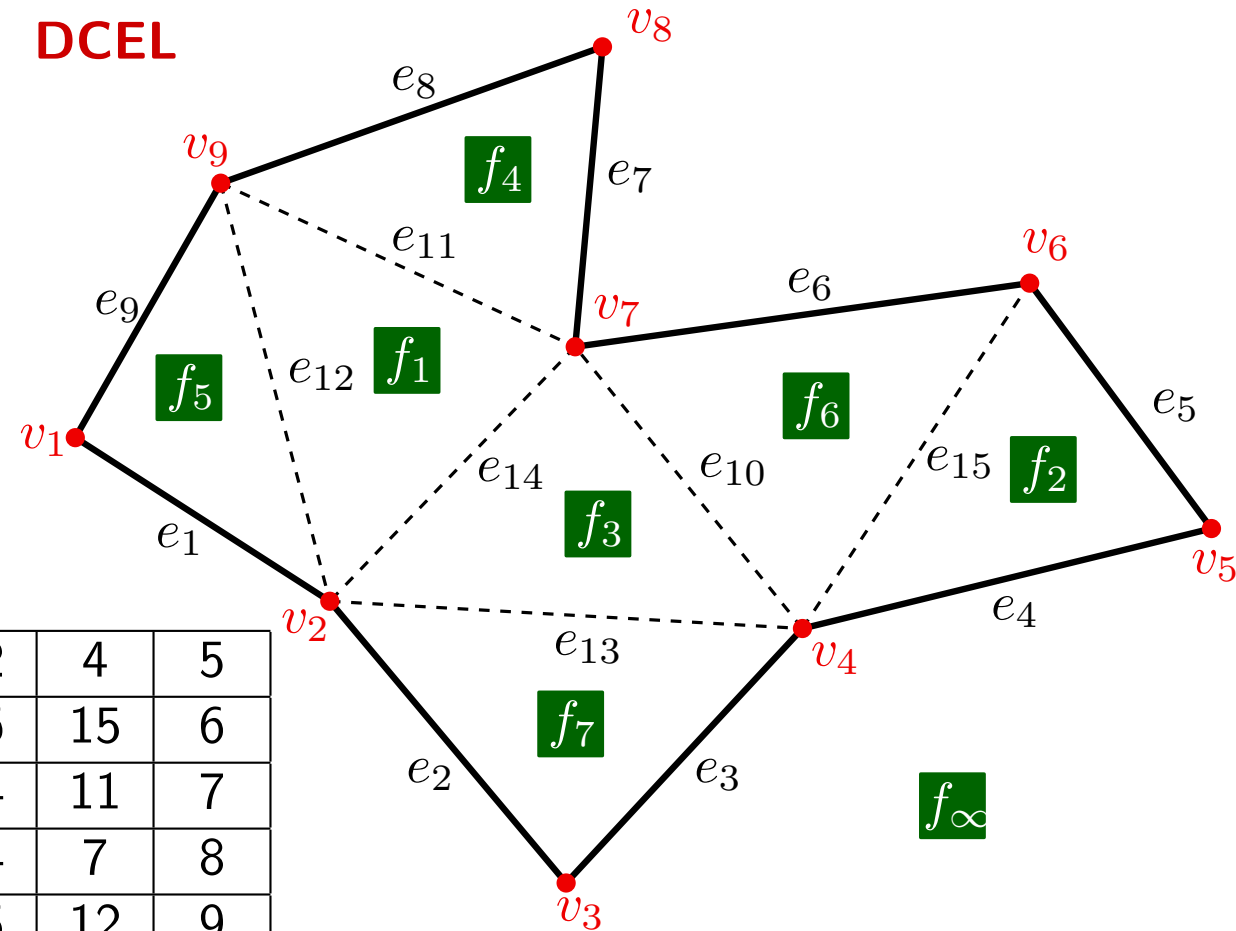


Storing the polygon triangulation

e	v_B	f_R	e_N	e'
1	1	∞	2	1'
2	2	∞	3	2'
3	4	7	2	3'
4	4	∞	5	4'
5	5	∞	6	5'
6	6	∞	7	6'
7	7	∞	8	7'
8	8	∞	9	8'
9	9	∞	1	9'
10	4	6	6	10'
11	9	1	14	11'
12	2	1	11	12'
13	2	7	3	13'
14	2	3	10	14'
15	4	2	5	15'
1'	2	5	9	1
2'	3	7	13	2
3'	3	∞	4	3
4'	5	2	15	4

5'	6	2	4	5
6'	7	6	15	6
7'	8	4	11	7
8'	9	4	7	8
9'	1	5	12	9
10'	7	3	13	10
11'	7	4	8	11
12'	9	5	1	12
13'	4	3	14	13
14'	7	1	12	14
15'	6	6	10	15

DCEL



Storing the polygon triangulation

How to build the DCEL



Storing the polygon triangulation

How to build the DCEL

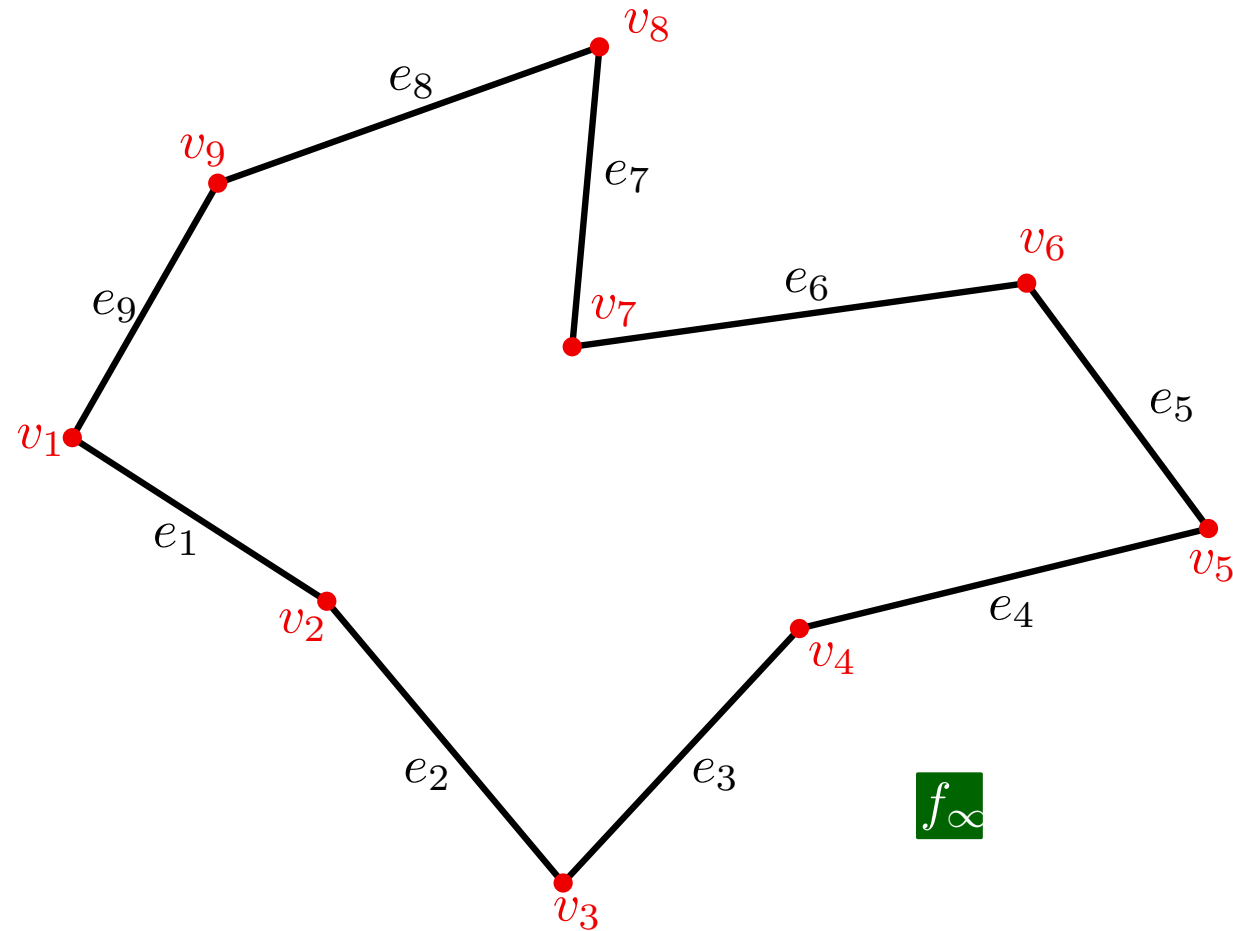
Algorithm 1: subtracting ears



Storing the polygon triangulation

How to build the DCEL

Algorithm 1: subtracting ears

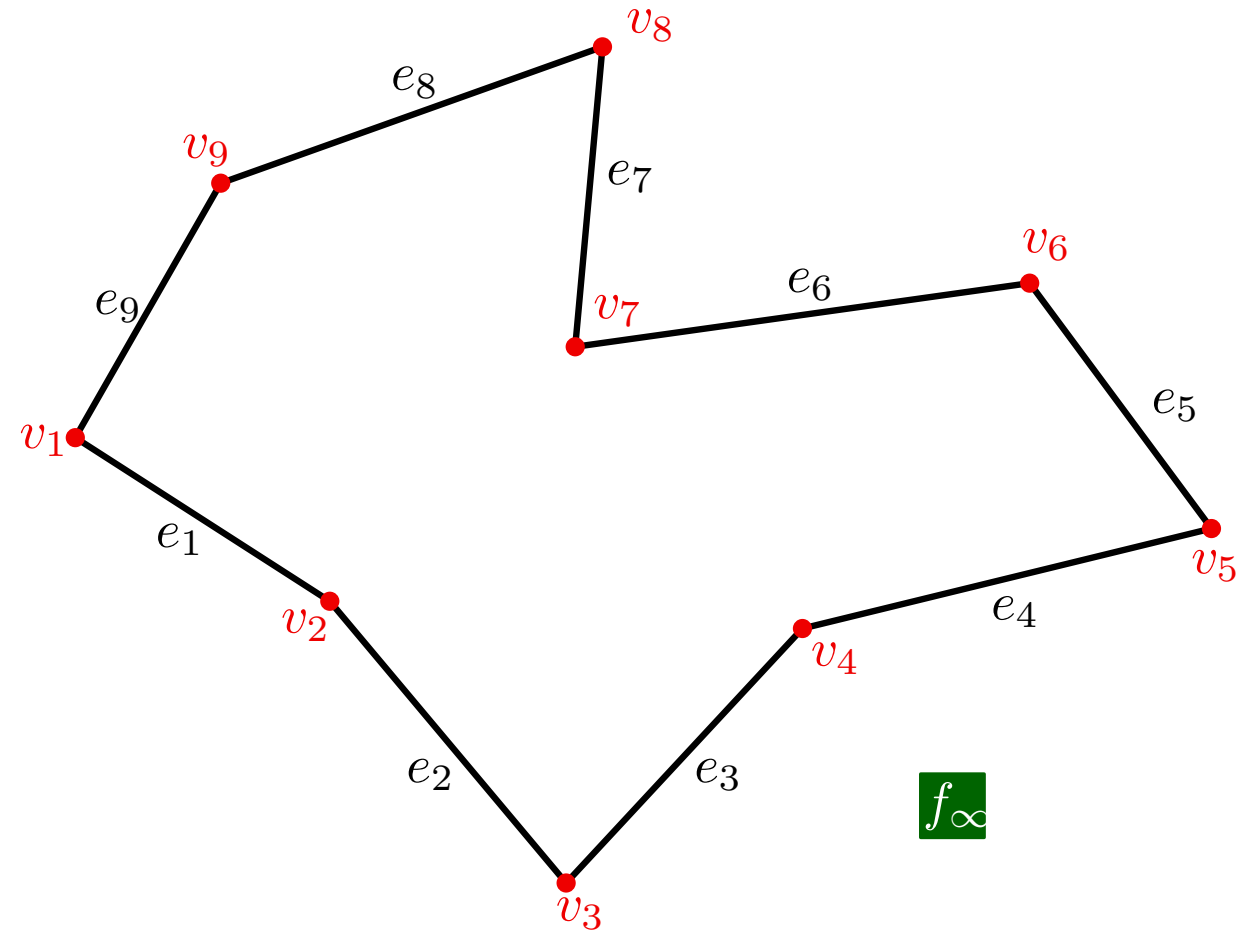


Storing the polygon triangulation

How to build the DCEL

Algorithm 1: subtracting ears

Initialize



Storing the polygon triangulation

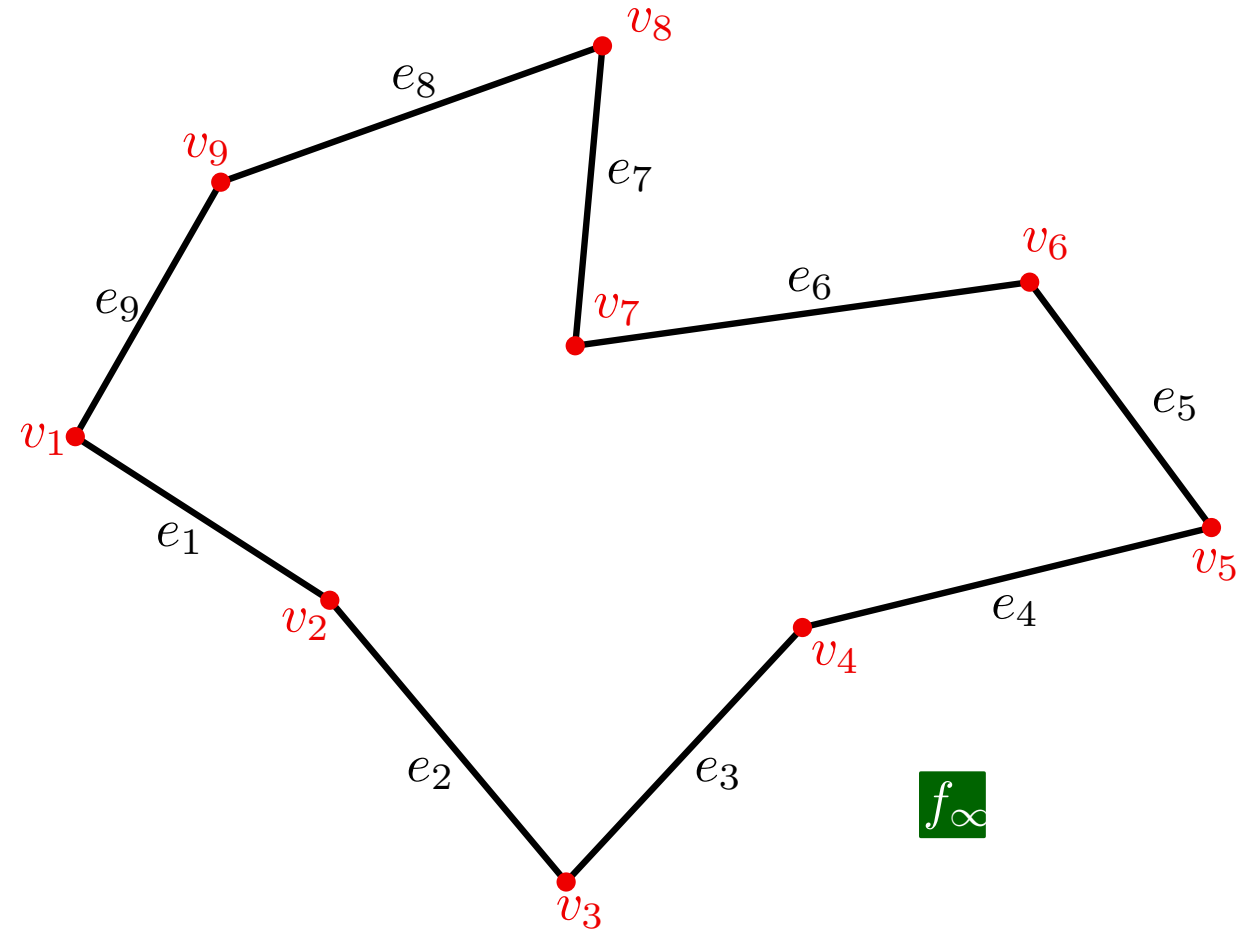
How to build the DCEL

Algorithm 1: subtracting ears

Initialize

Table of vertices

v	x	y	e
1	x_1	y_1	1
2	x_2	y_2	2
3	x_3	y_3	3
4	x_4	y_4	4
5	x_5	y_5	5
6	x_6	y_6	6
7	x_7	y_7	7
8	x_8	y_8	8
9	x_9	y_9	9



Storing the polygon triangulation

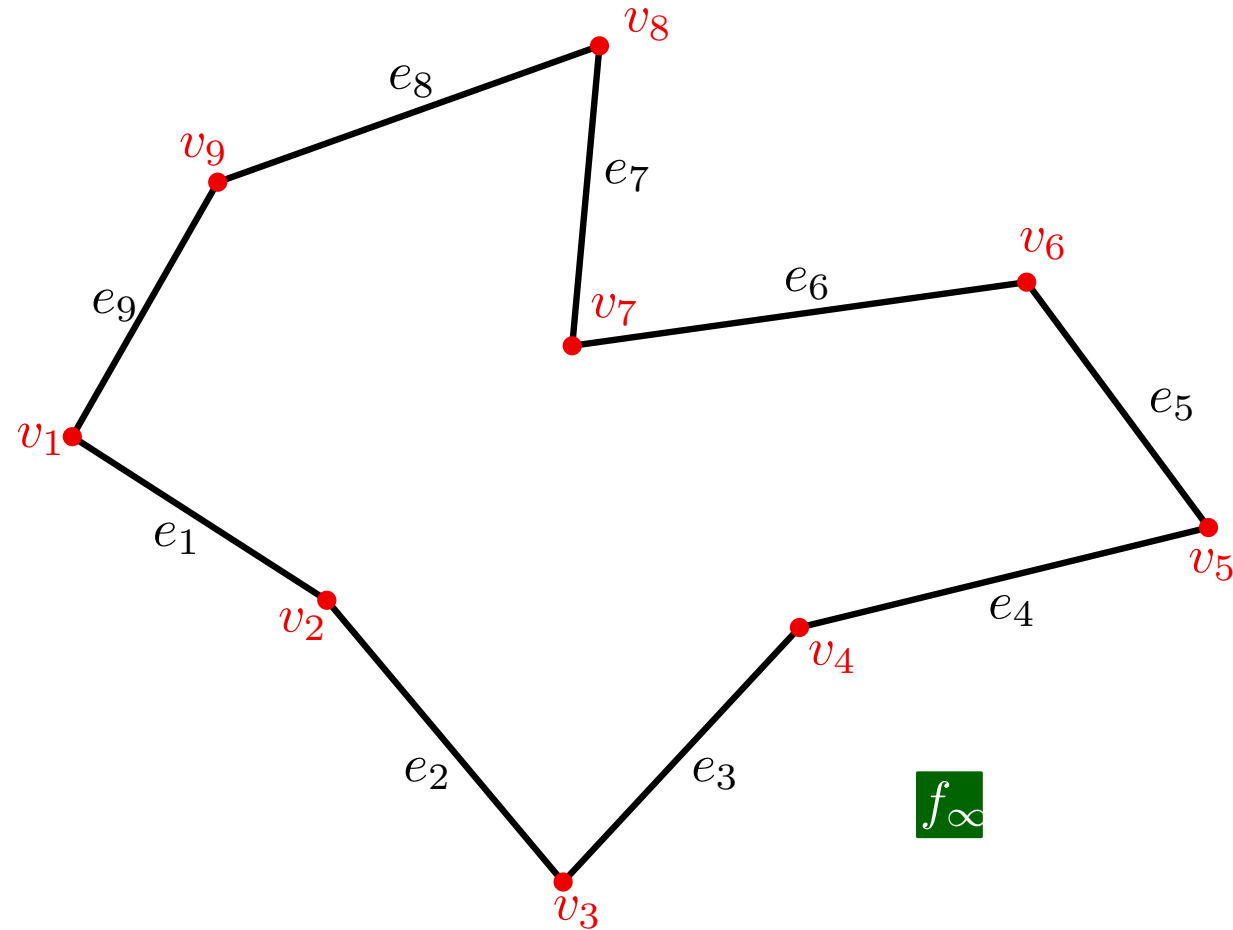
How to build the DCEL

Algorithm 1: subtracting ears

Initialize

Table of faces

f	e
∞	9



Storing the polygon triangulation

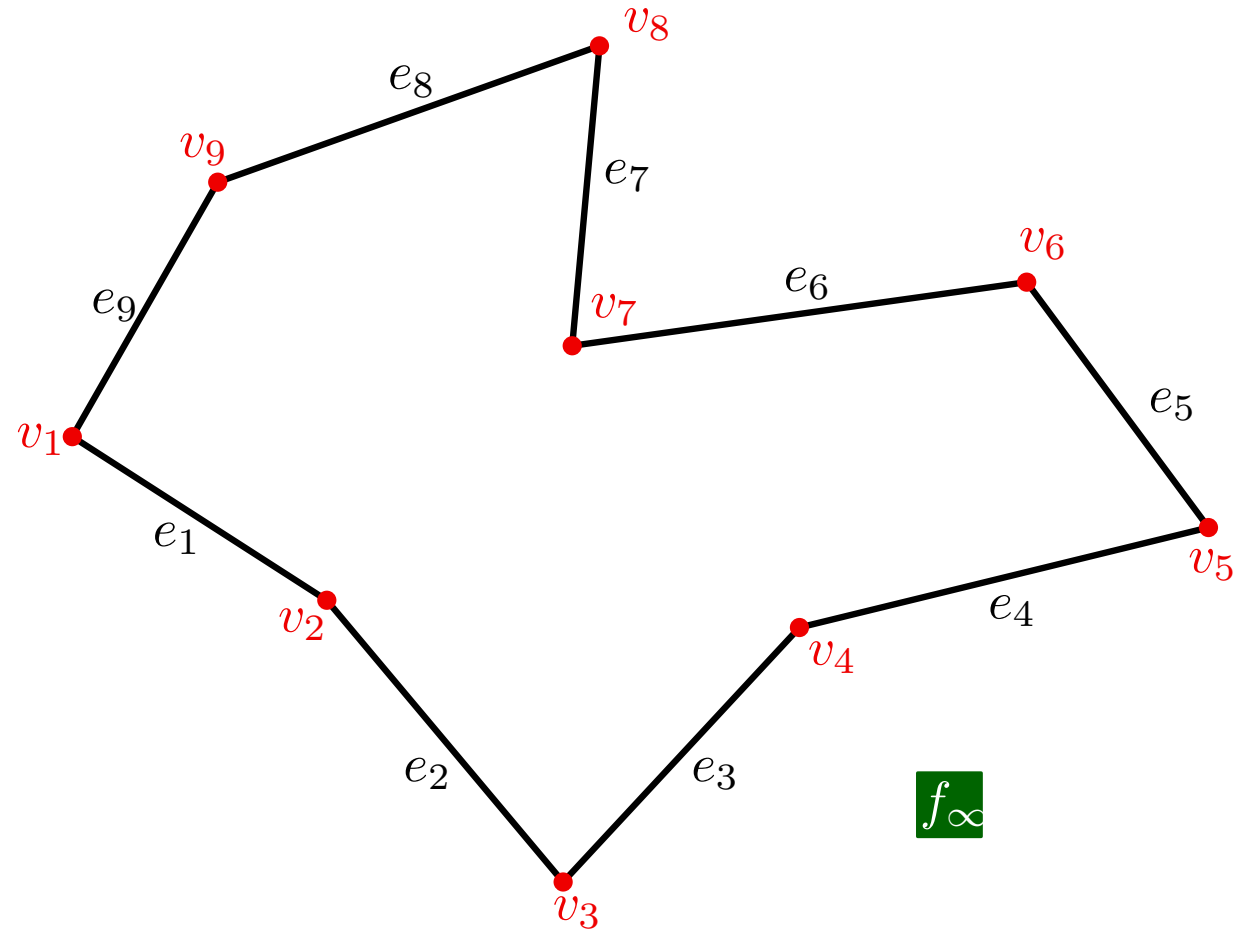
How to build the DCEL

Algorithm 1: subtracting ears

Initialize

DCEL

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2		∞		2
2	2	3		∞		3
3	3	4		∞		4
4	4	5		∞		5
5	5	6		∞		6
6	6	7		∞		7
7	7	8		∞		8
8	8	9		∞		9
9	9	1		∞		1



Storing the polygon triangulation

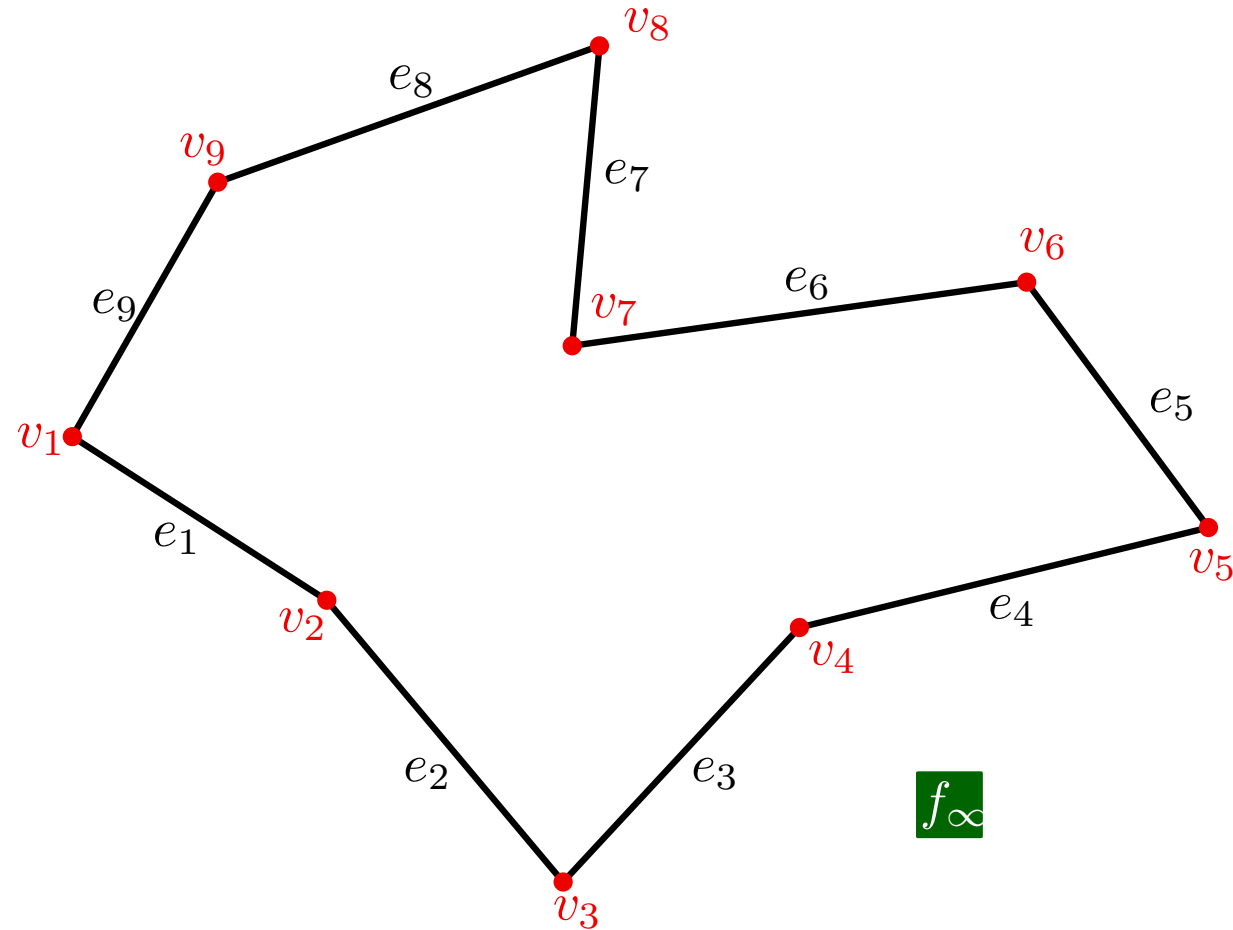
How to build the DCEL

Algorithm 1: subtracting ears

Initialize

DCEL

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2		∞		2
2	2	3		∞		3
3	4	3	∞		4	
4	4	5		∞		5
5	5	6		∞		6
6	6	7		∞		7
7	7	8		∞		8
8	8	9		∞		9
9	9	1		∞		1

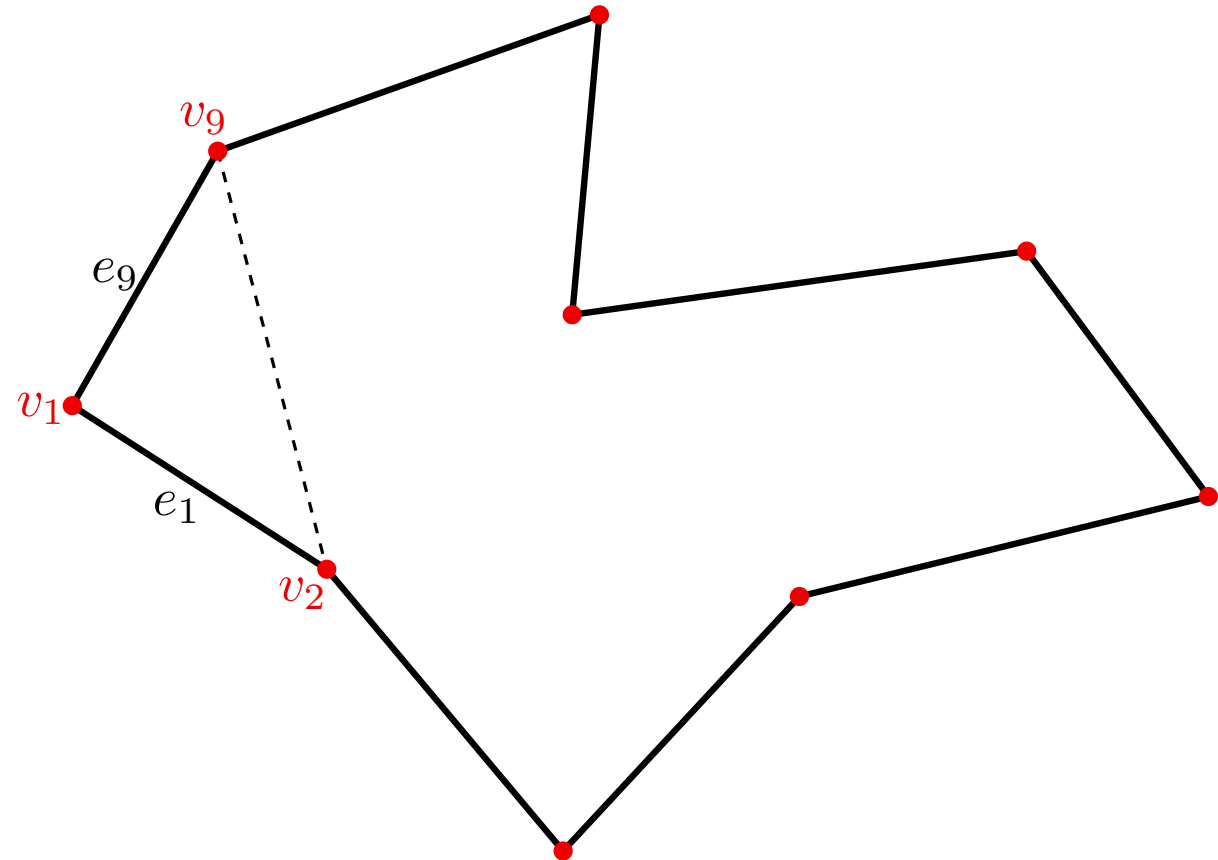


Storing the polygon triangulation

How to build the DCEL

Algorithm 1: subtracting ears

Advance

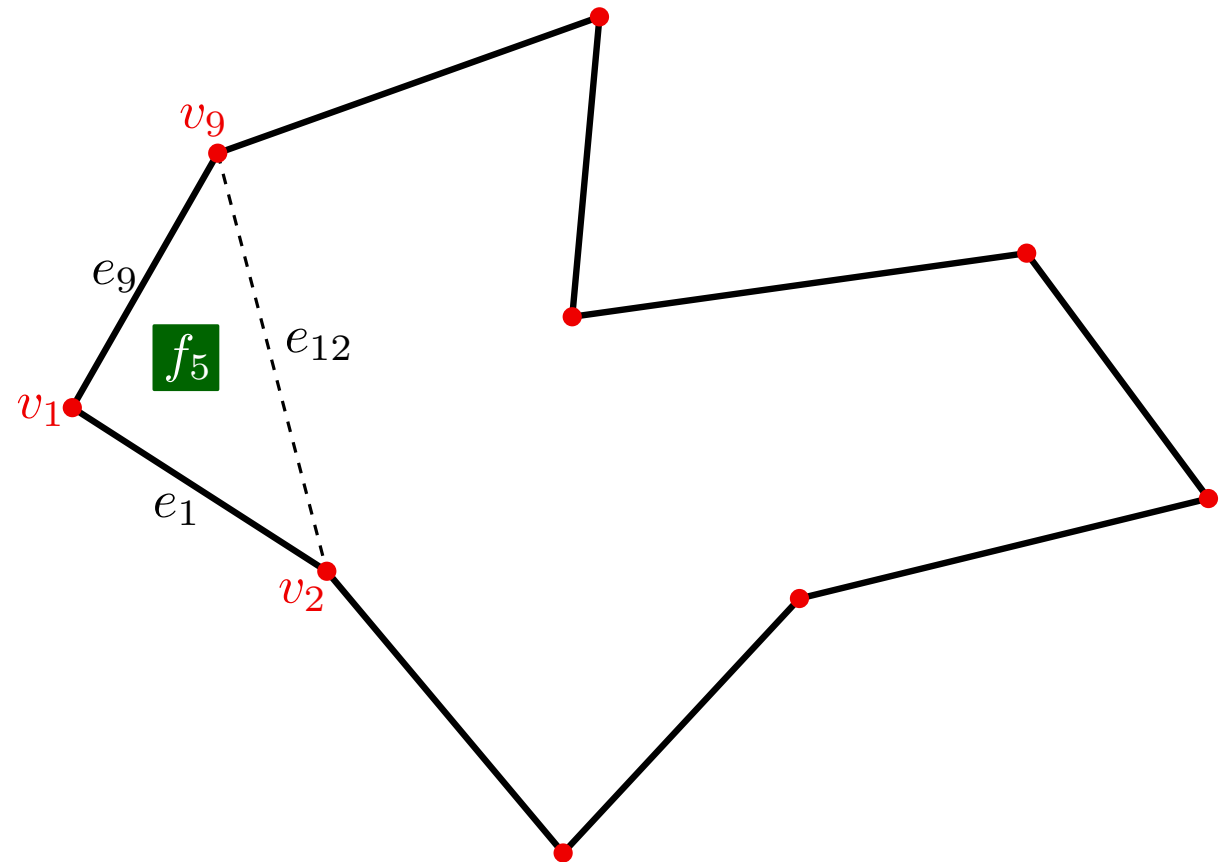


Storing the polygon triangulation

How to build the DCEL

Algorithm 1: subtracting ears

Advance



Storing the polygon triangulation

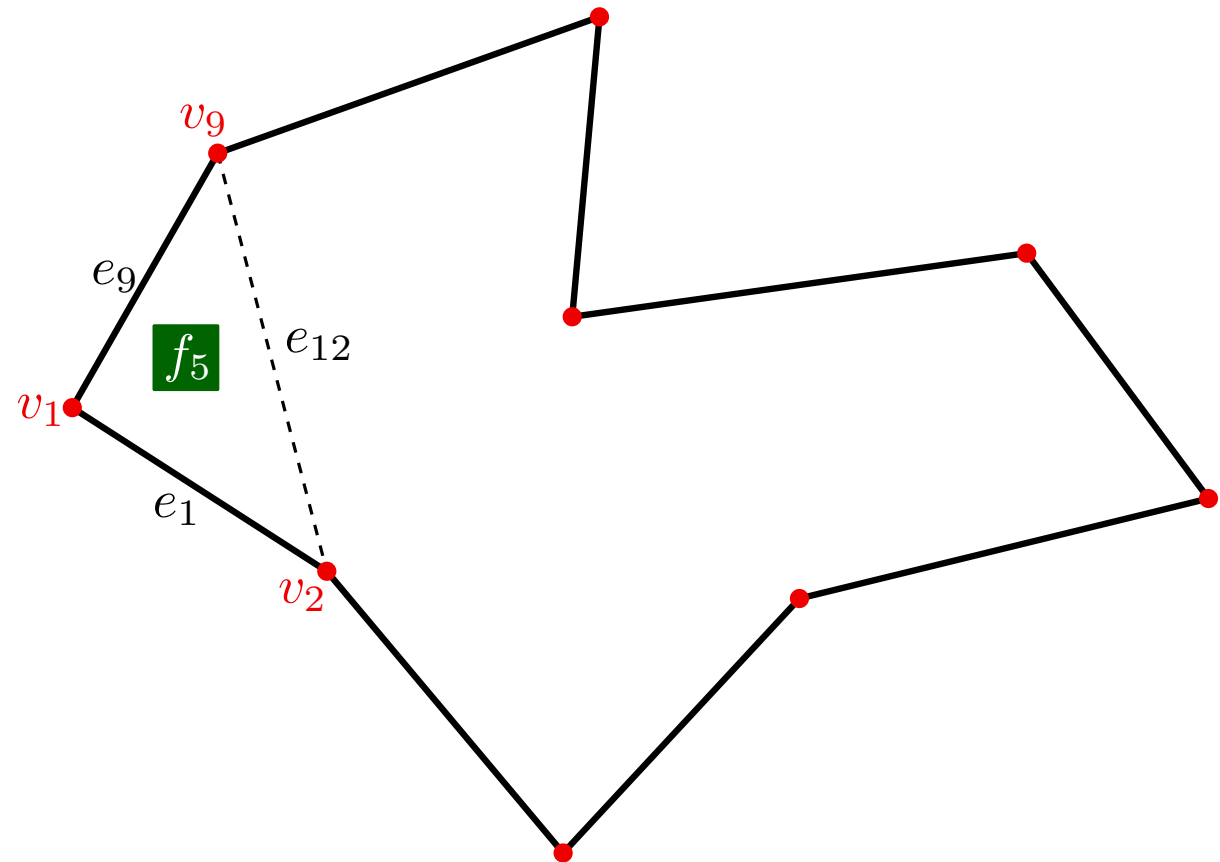
How to build the DCEL

Algorithm 1: subtracting ears

Advance

Table of faces

f	e
5	9



Storing the polygon triangulation

How to build the DCEL

Algorithm 1: subtracting ears

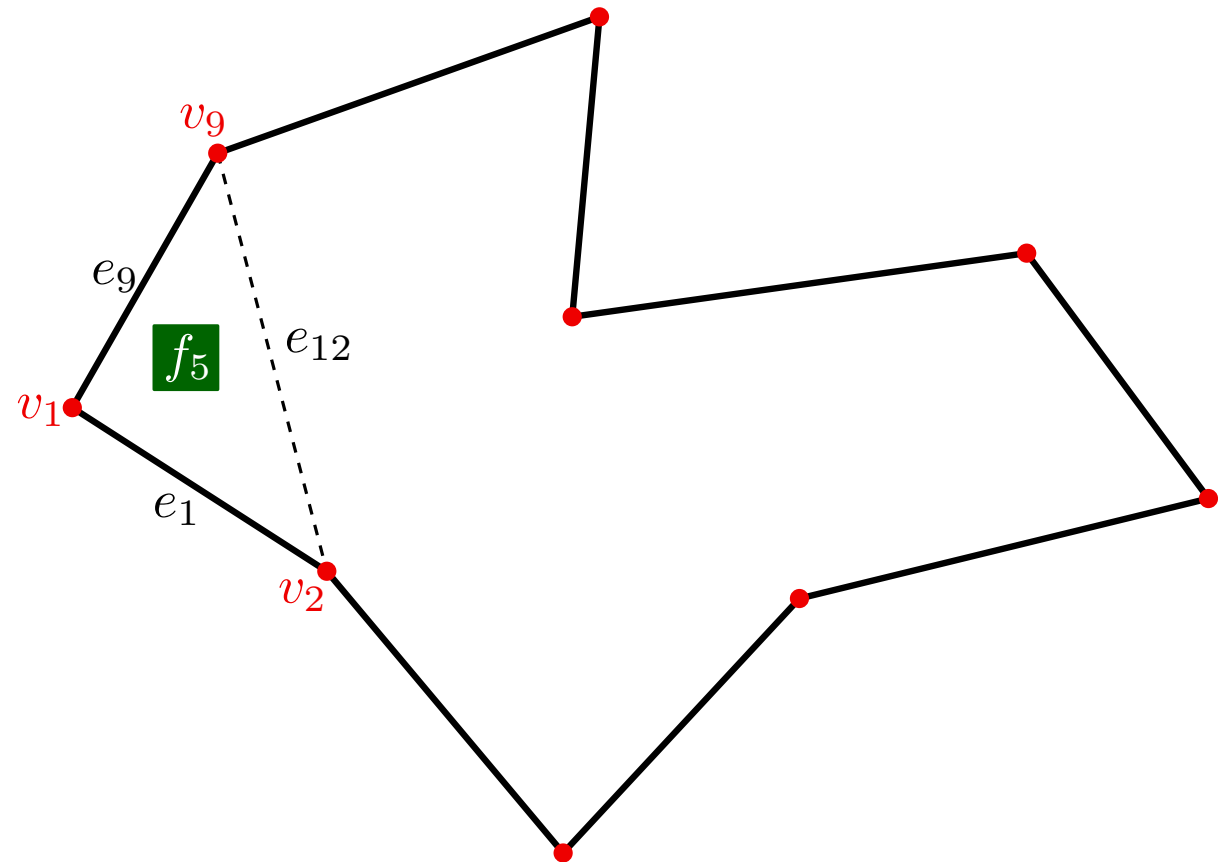
Advance

Table of faces

f	e
5	9

DCEL

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	2
9	9	1	5	∞	12	1
12	2	9	5		1	



Storing the polygon triangulation

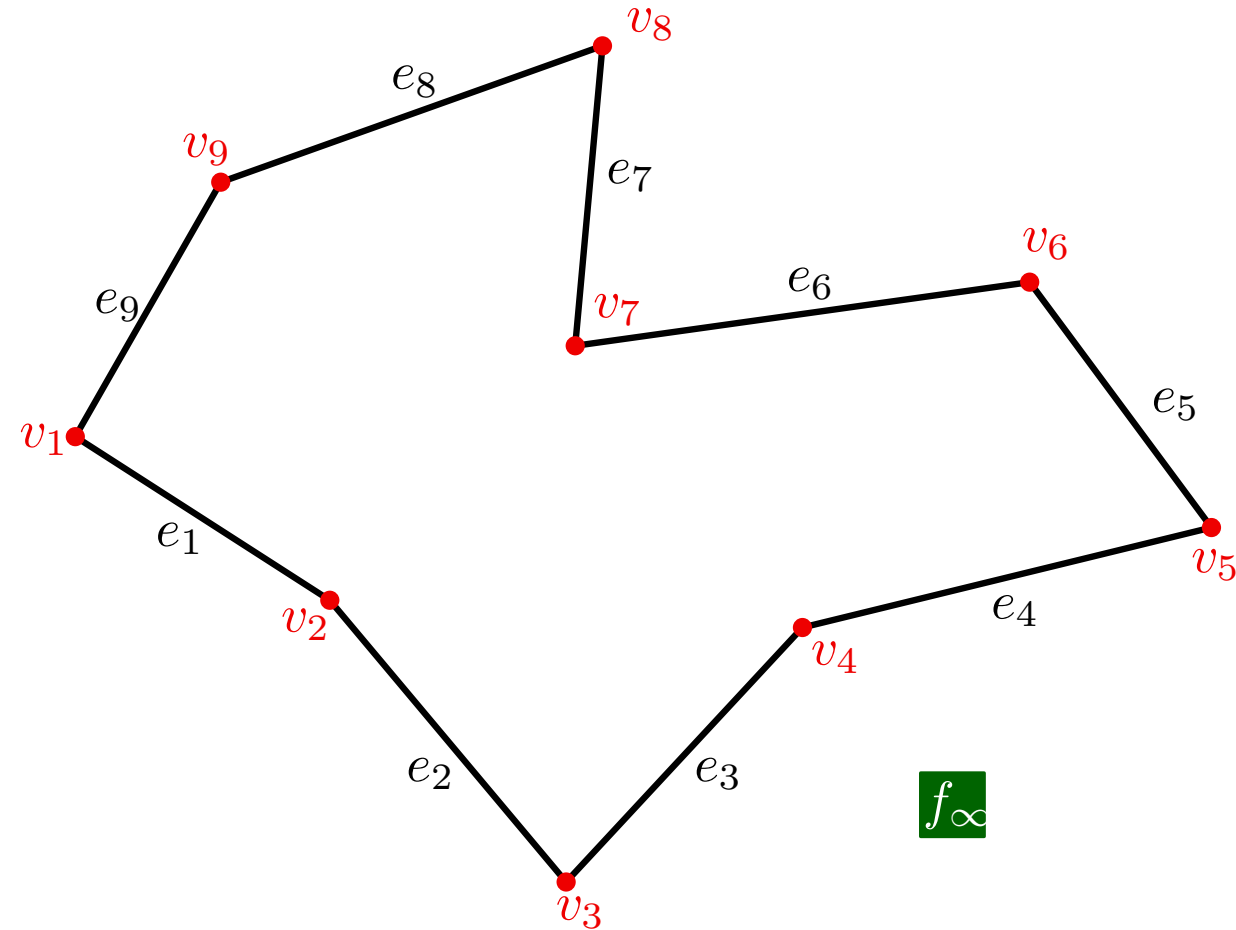
How to build the DCEL

Algorithm 2: inserting diagonals

Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

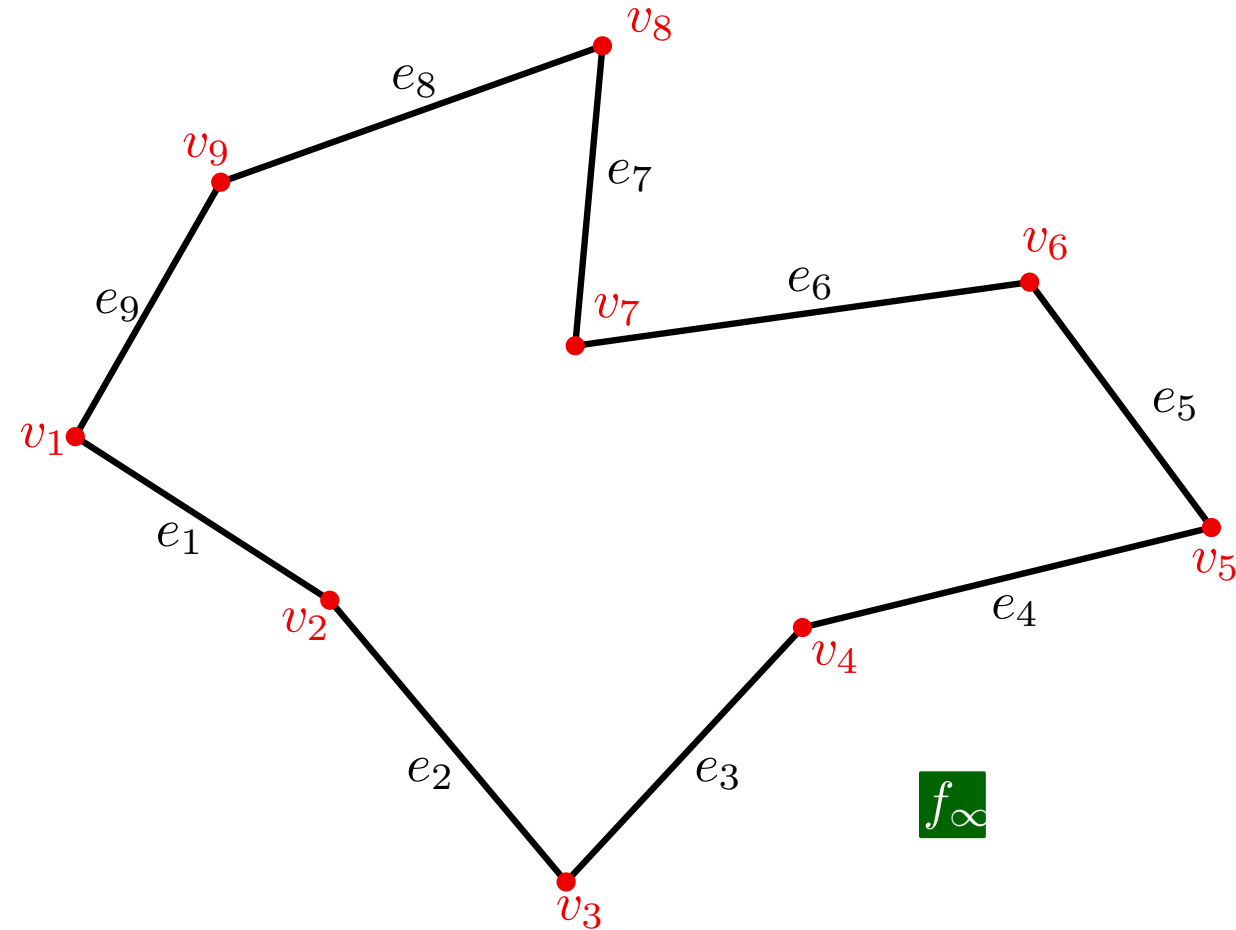


Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

Initialize



Storing the polygon triangulation

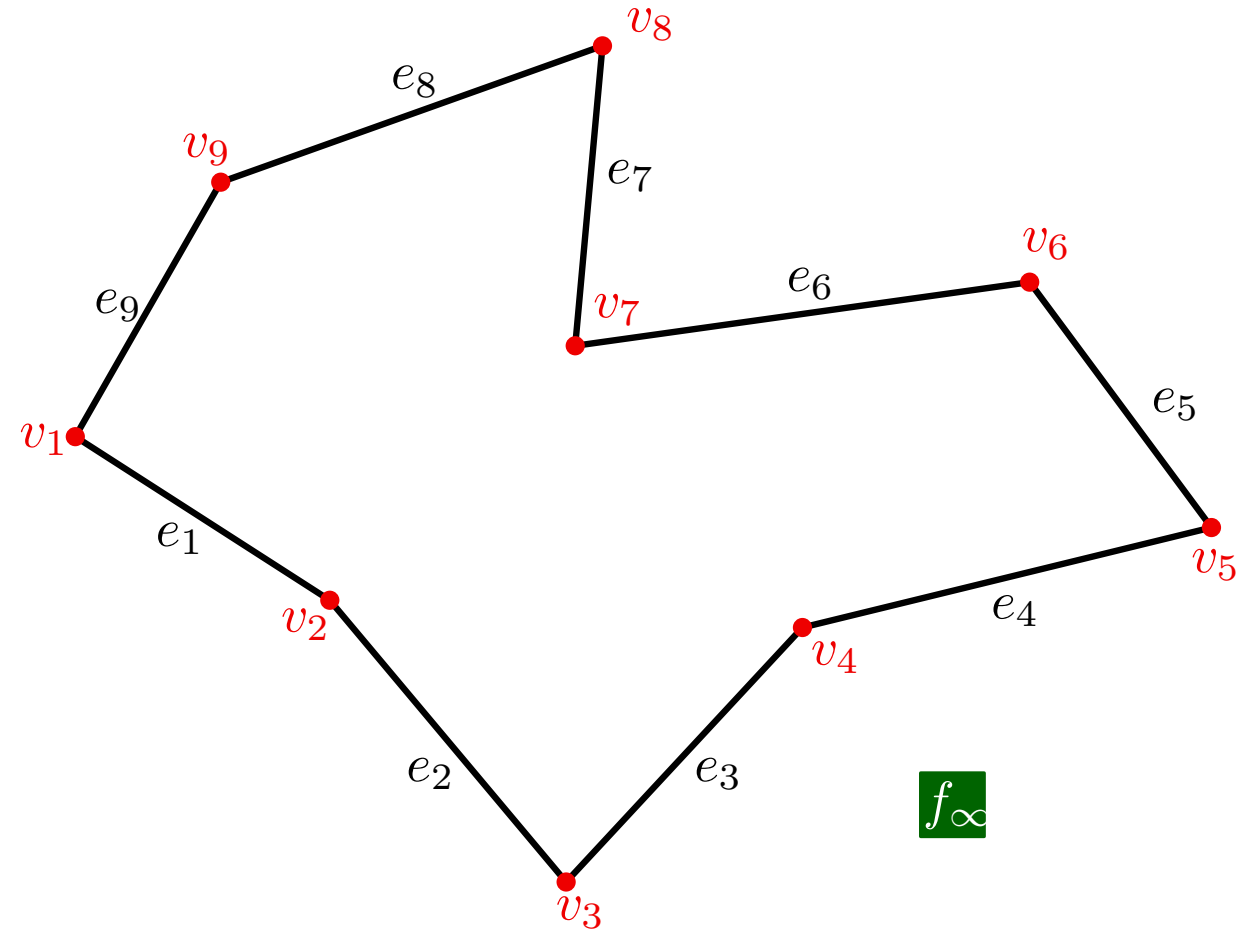
How to build the DCEL

Algorithm 2: inserting diagonals

Initialize

Table of vertices

v	x	y	e
1	x_1	y_1	1
2	x_2	y_2	2
3	x_3	y_3	3
4	x_4	y_4	4
5	x_5	y_5	5
6	x_6	y_6	6
7	x_7	y_7	7
8	x_8	y_8	8
9	x_9	y_9	9



Storing the polygon triangulation

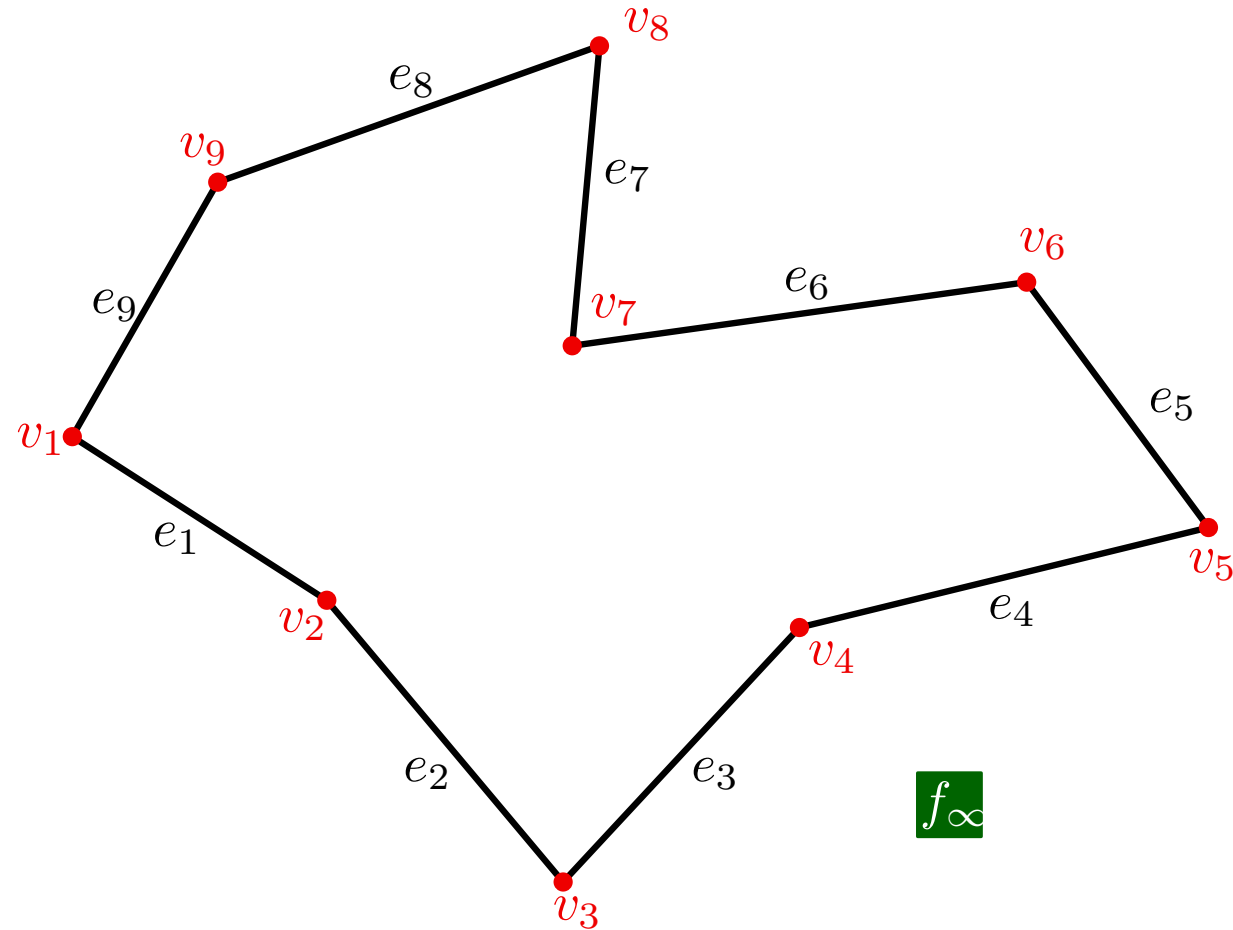
How to build the DCEL

Algorithm 2: inserting diagonals

Initialize

Table of faces

f	e
∞	9

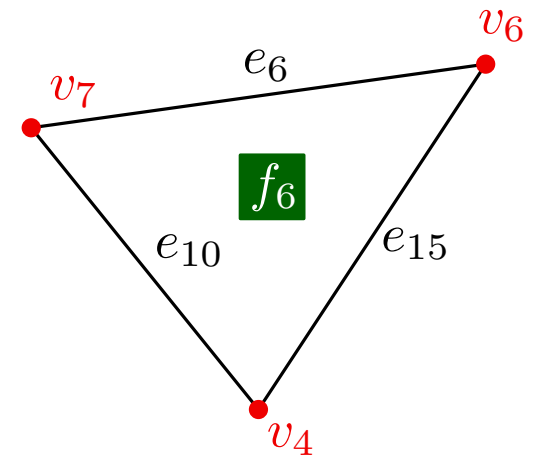


Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

Base step



Storing the polygon triangulation

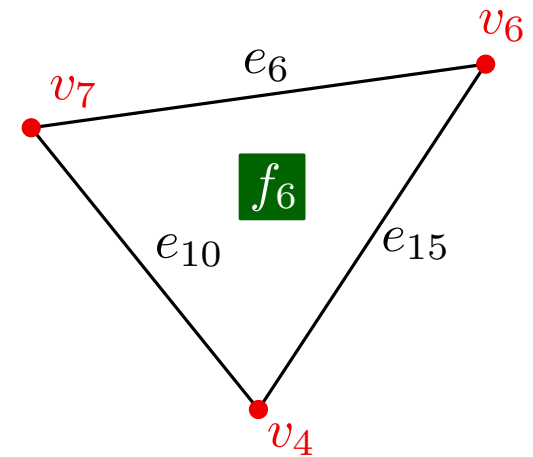
How to build the DCEL

Algorithm 2: inserting diagonals

Base step

Table of faces

f	e
6	10



Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

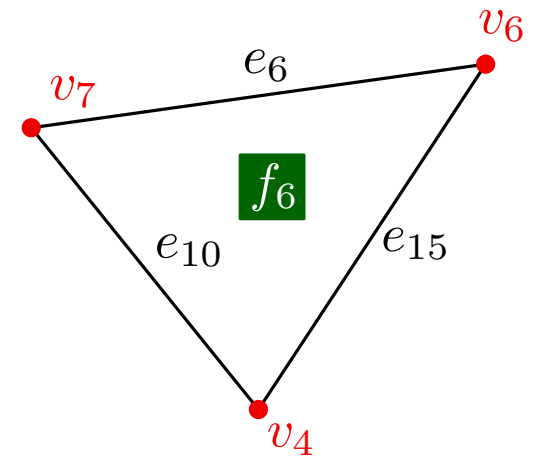
Base step

Table of faces

f	e
6	10

DCEL

e	v_B	v_E	f_L	f_R	e_P	e_N
6	6	7	6	∞	15	10
10	7	4	6	∞	6	15
15	4	6	6	∞	10	6

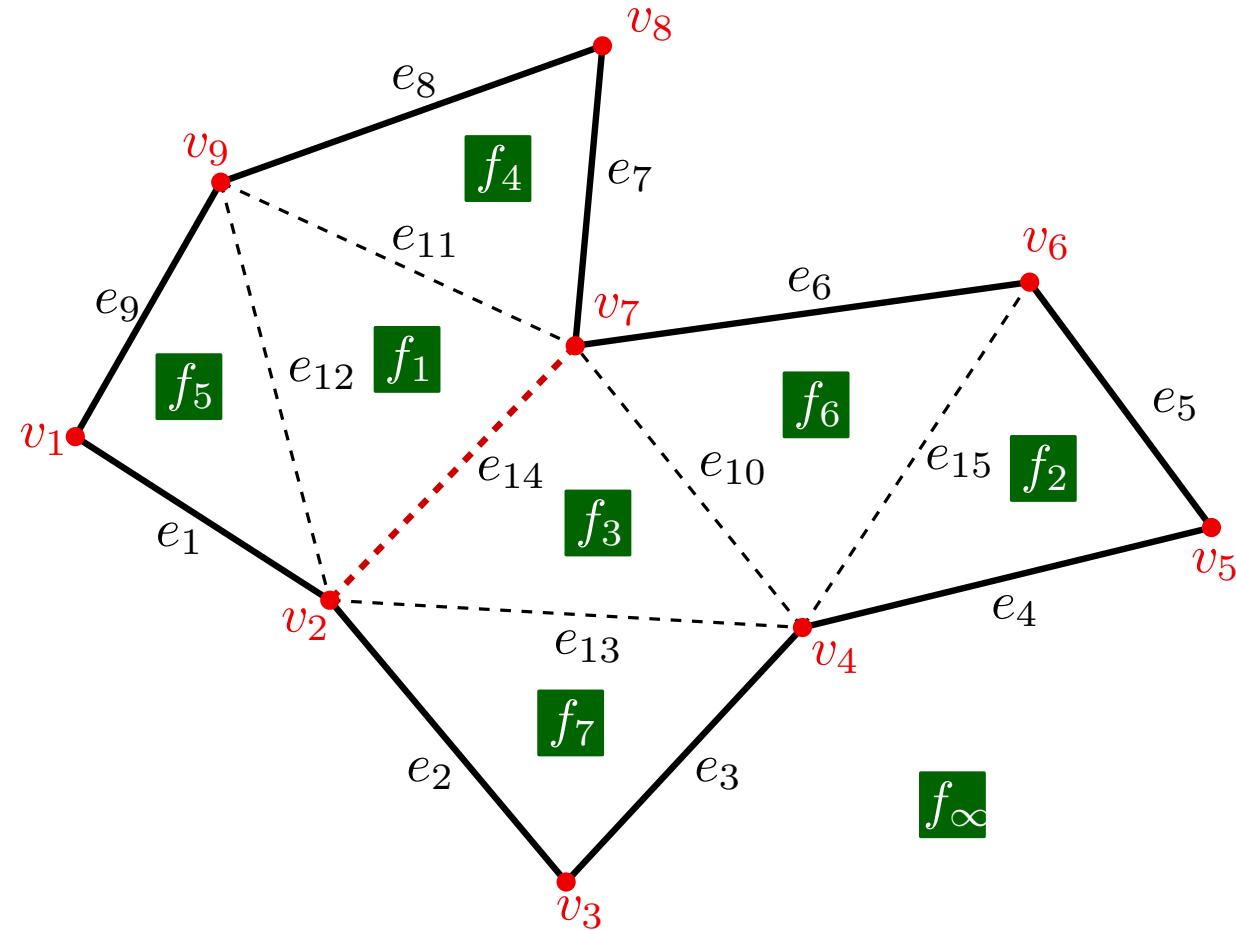


Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

Merge step



Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

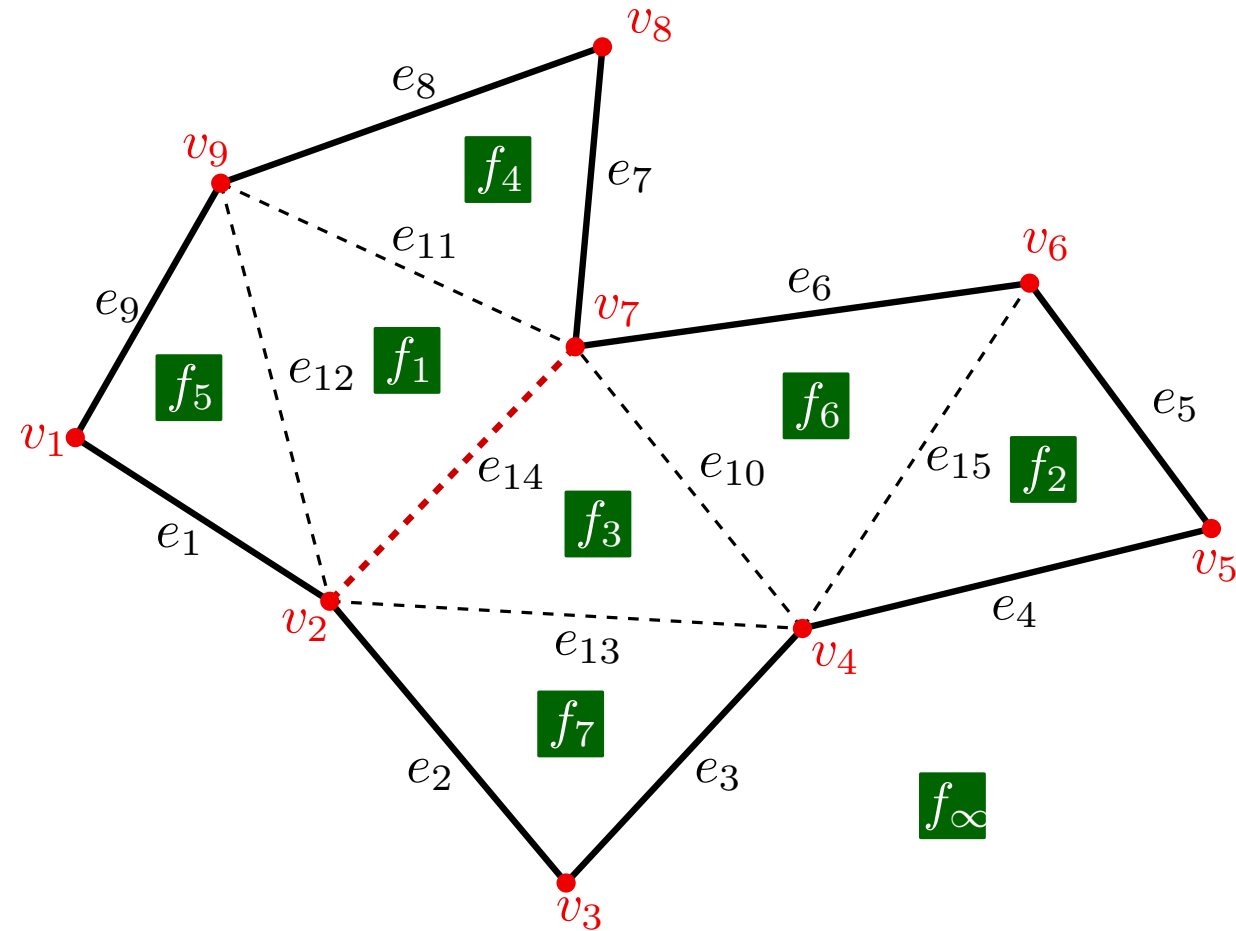
Merge step

DCEL 1

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	14
14	2	7	1	∞	12	7

DCEL 2

e	v_B	v_E	f_L	f_R	e_P	e_N
6	6	7	6	∞	15	14
14	2	7	∞	3	2	10



Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

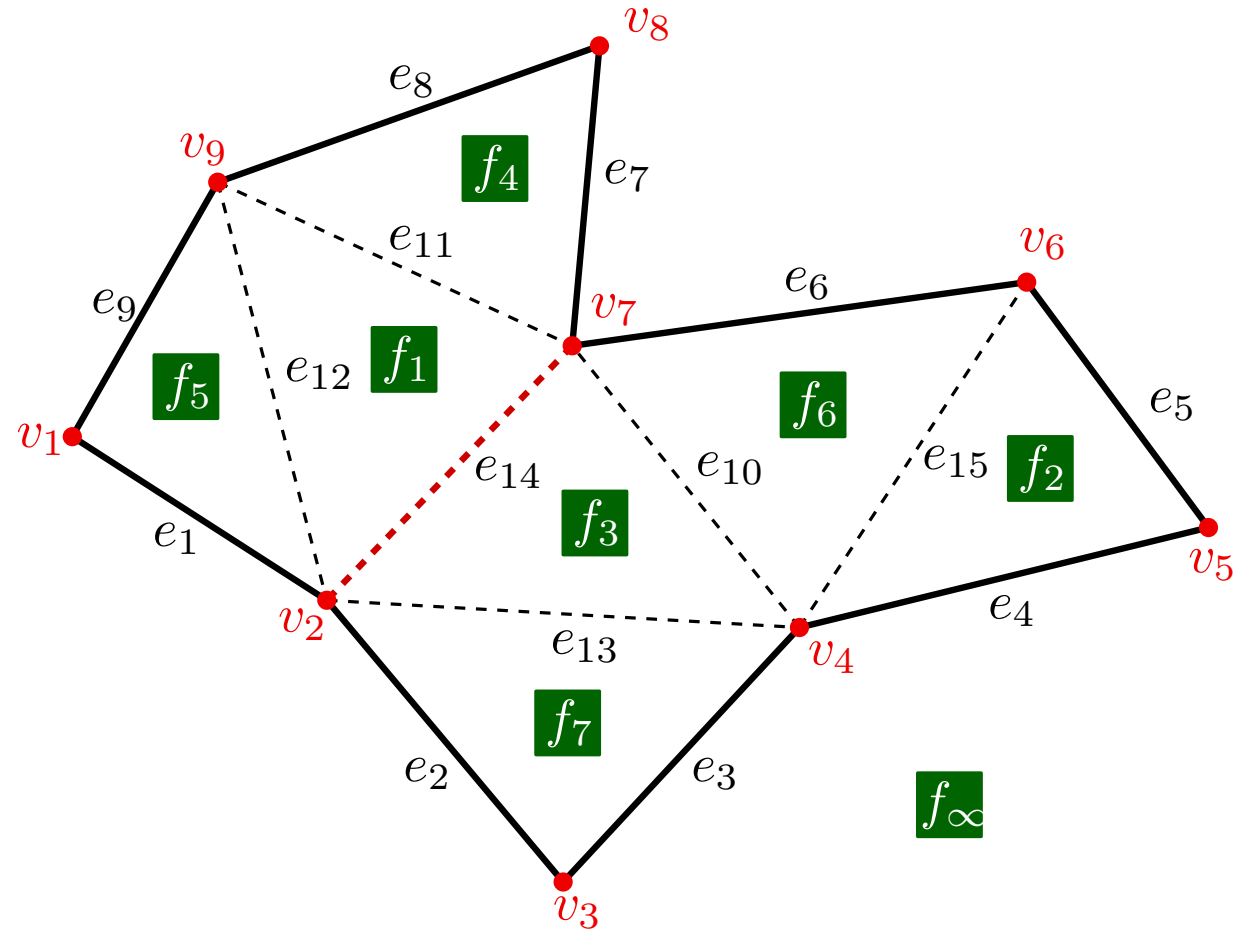
Merge step

DCEL 1

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	14
14	2	7	1	∞	12	7

DCEL 2

e	v_B	v_E	f_L	f_R	e_P	e_N
6	6	7	6	∞	15	14
14	2	7	∞	3	2	10



Storing the polygon triangulation

How to build the DCEL

Algorithm 2: inserting diagonals

Merge step

DCEL 1

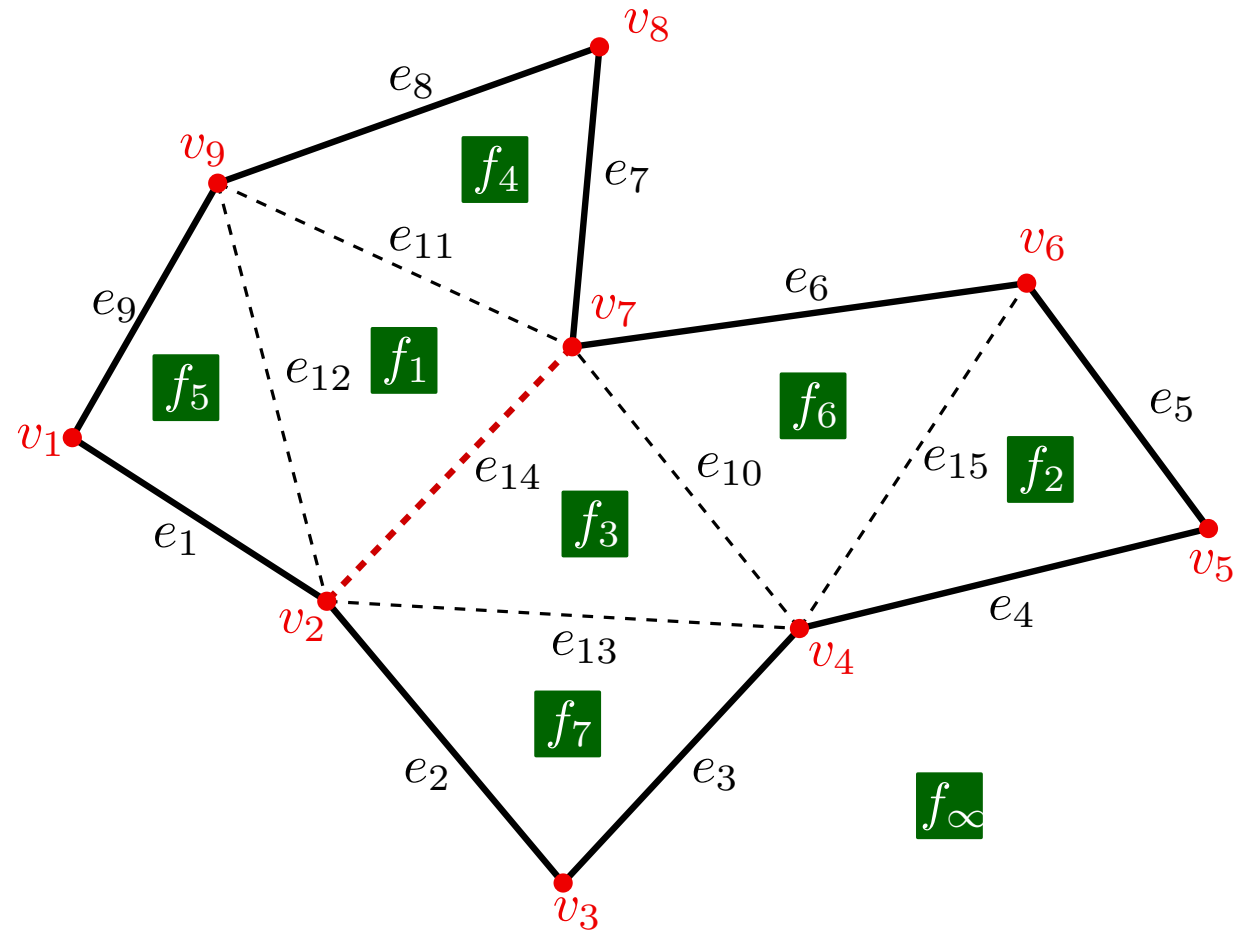
e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	14
14	2	7	1	∞	12	7

DCEL 2

e	v_B	v_E	f_L	f_R	e_P	e_N
6	6	7	6	∞	15	14
14	2	7	∞	3	2	10

Merged DCEL

e	v_B	v_E	f_L	f_R	e_P	e_N
1	1	2	5	∞	9	2
6	6	7	6	∞	15	7
14	2	7	1	3	12	10



Storing the polygon triangulation

How to build the DCEL

Algorithm 3:

Storing the polygon triangulation

How to build the DCEL

Algorithm 3:

1. Decompose into monotone polygons
2. Triangulate monotone pieces

Storing the polygon triangulation

How to build the DCEL

Algorithm 3:

1. Decompose into monotone polygons
2. Triangulate monotone pieces

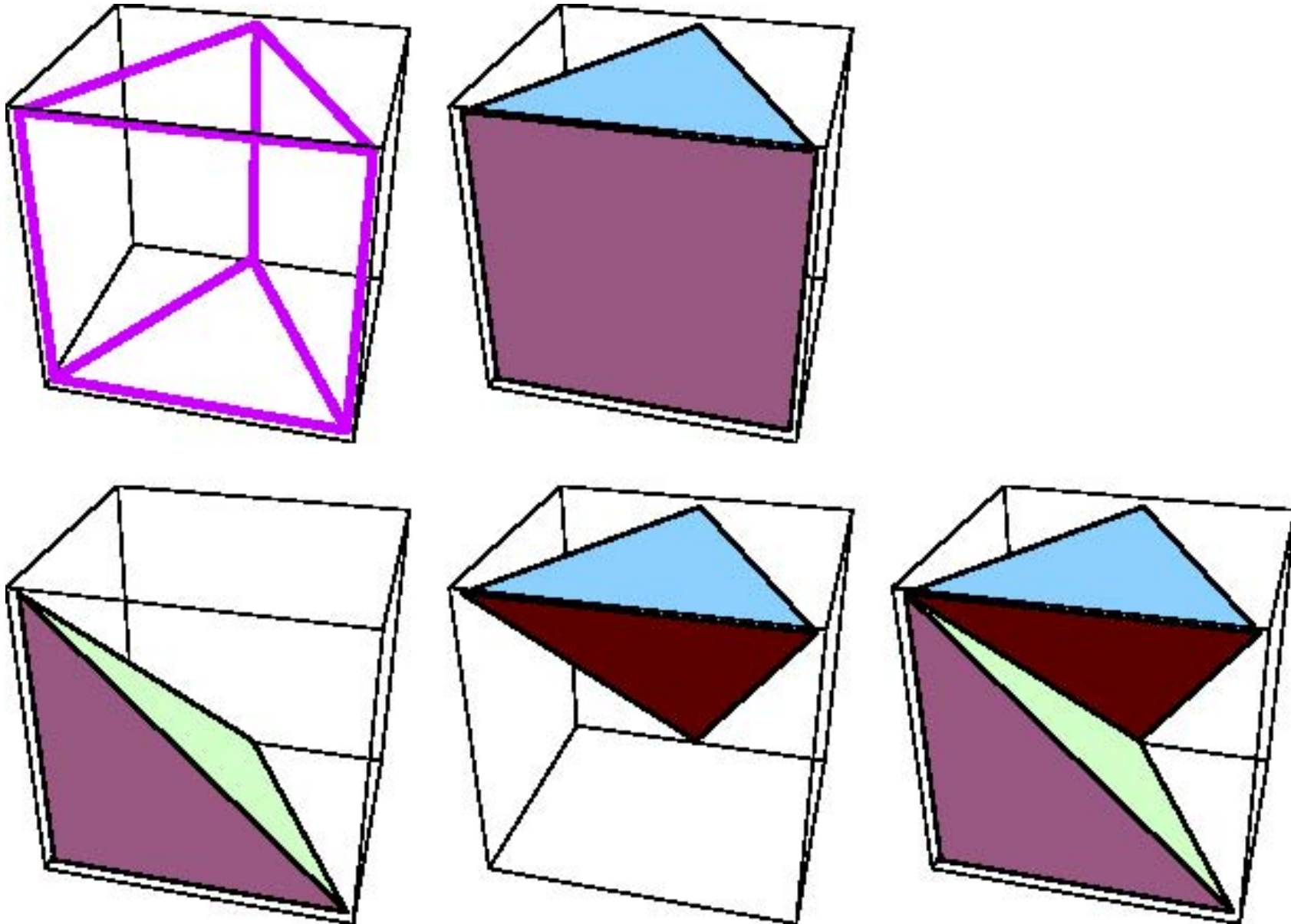
Computing the DCEL is done by combining previous strategies:

- Separating ears for triangulating each monotone subpolygon.
- Merging DCELs for putting together the monotone pieces.

WHAT HAPPENS IN 3D?

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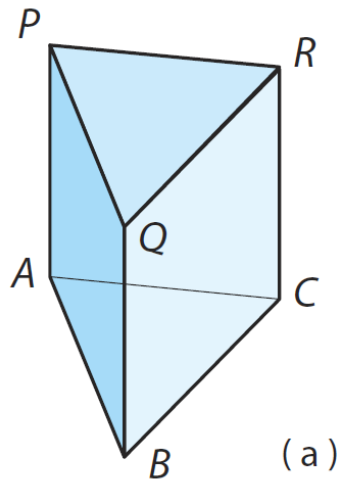
Polyhedra can be *tetrahedralized*:



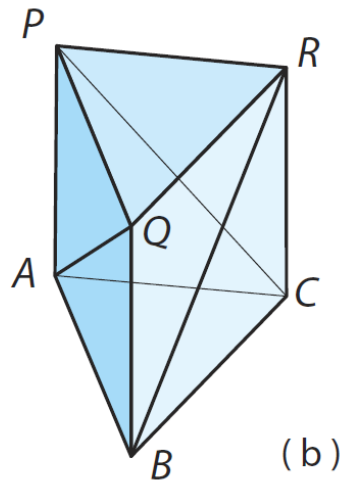
WHAT HAPPENS IN 3D?

But not always: A polyhedron that cannot be tetrahedralized:

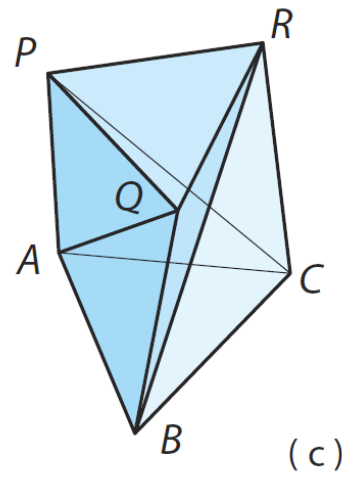
Schönhardt polyhedron



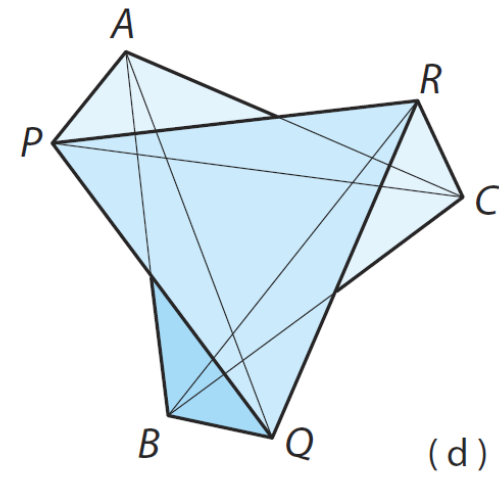
(a)



(b)



(c)



(d)

Figure from book by Devadoss and O'Rourke

Smallest polyhedron that cannot be tetrahedralized

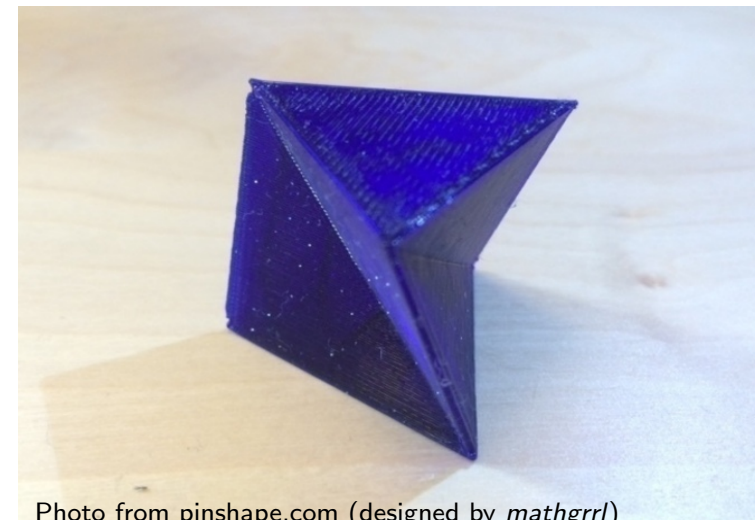
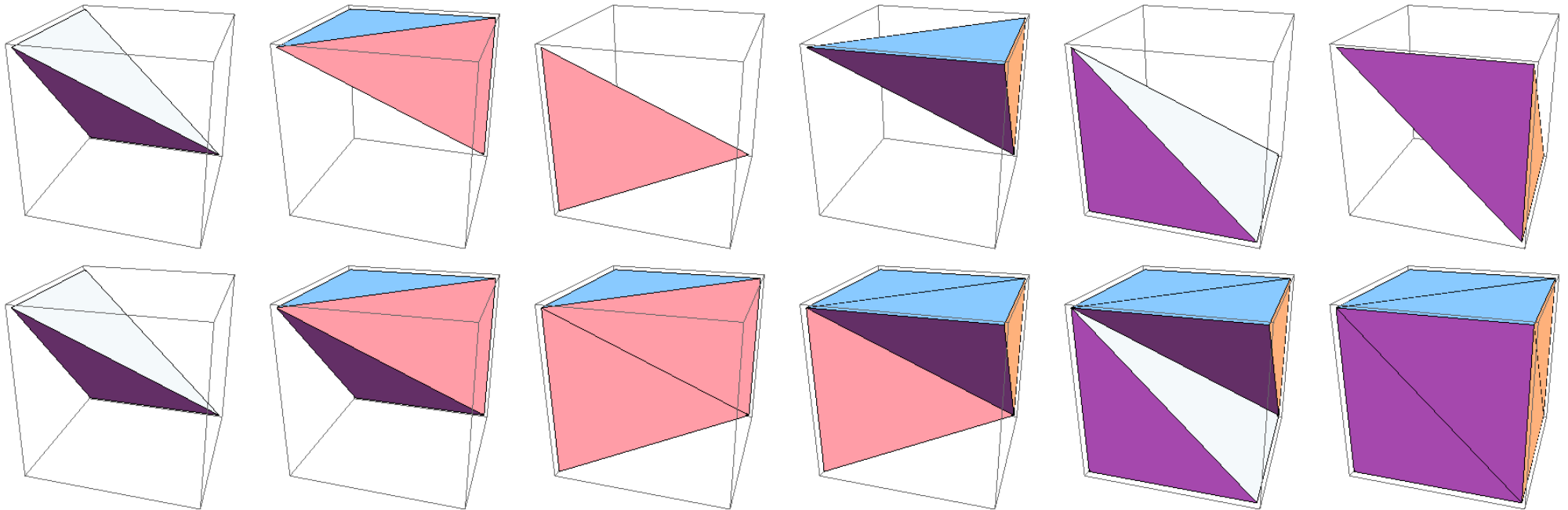


Photo from pinshape.com (designed by *mathgrrl*)

WHAT HAPPENS IN 3D?

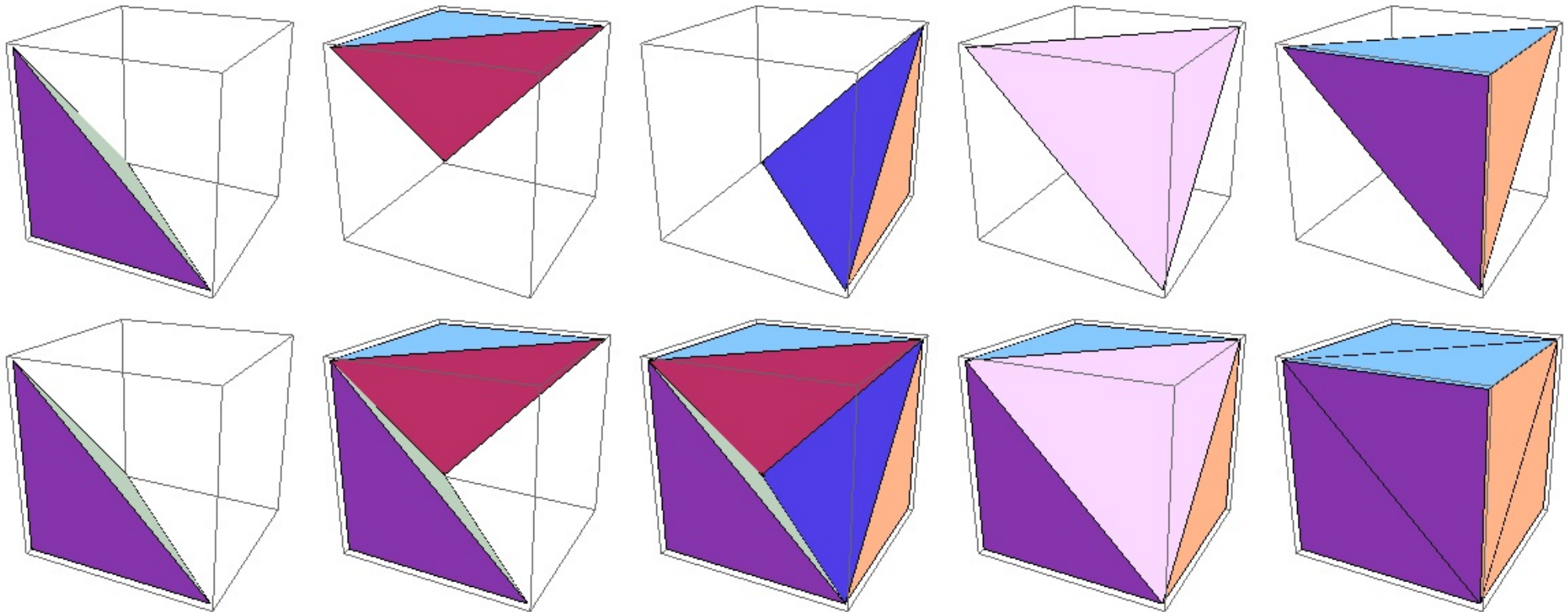
A cube can be decomposed into 6 tetrahedra...



WHAT HAPPENS IN 3D?

A cube can be decomposed into 6 tetrahedra...

but also into 5!



TRIANGULATING POLYGONS

TO LEARN MORE

- J. O'Rourke, **Computational Geometry in C (2nd ed.)**, Cambridge University Press, 1998.
- M. de Berg, O. Cheong, M. van Kreveld, M. Overmars, **Computational Geometry: Algorithms and Applications (3rd rev. ed.)**, Springer, 2008.

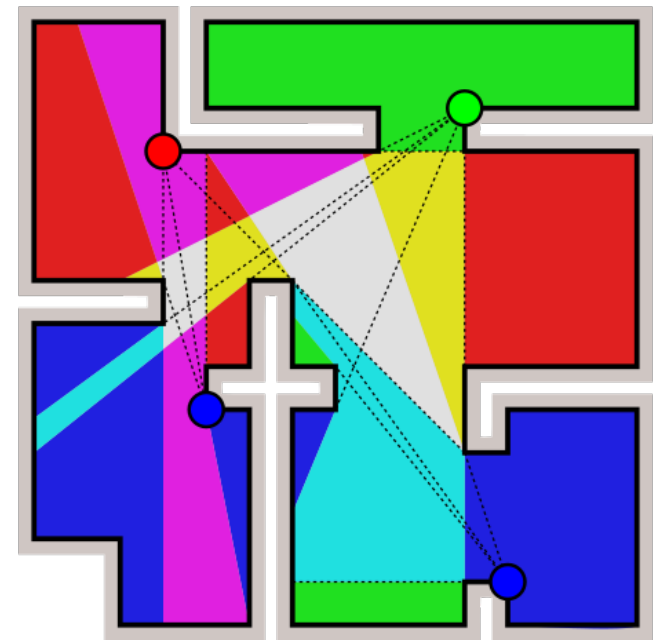
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- J. O'Rourke, **Computational Geometry in C (2nd ed.)**, Cambridge University Press, 1998.
- M. de Berg, O. Cheong, M. van Kreveld, M. Overmars, **Computational Geometry: Algorithms and Applications (3rd rev. ed.)**, Springer, 2008.

A NICE APPLICATION

The art gallery theorem



- J. O'Rourke, **Art Gallery Theorems and Algorithms**, Oxford University Press, 1987.
<http://www.science.smith.edu/~jorourke/books/ArtGalleryTheorems/art.html>