Research digression:

Borsuk's question

A purely geometric question

Rodrigo Silveira
Universitat Politècnica de Catalunya
DAG - FME - UPC

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Karol Borsuk
(Warsaw, 1905-1982)

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Animal Husbandry
(one of the first board games with 12-side dice)

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(a.k.a. Borsuk's conjecture, 1933)

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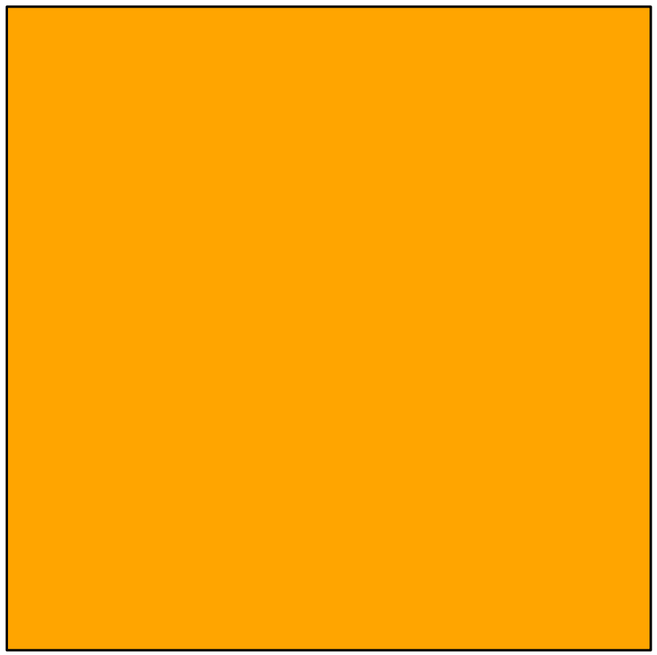

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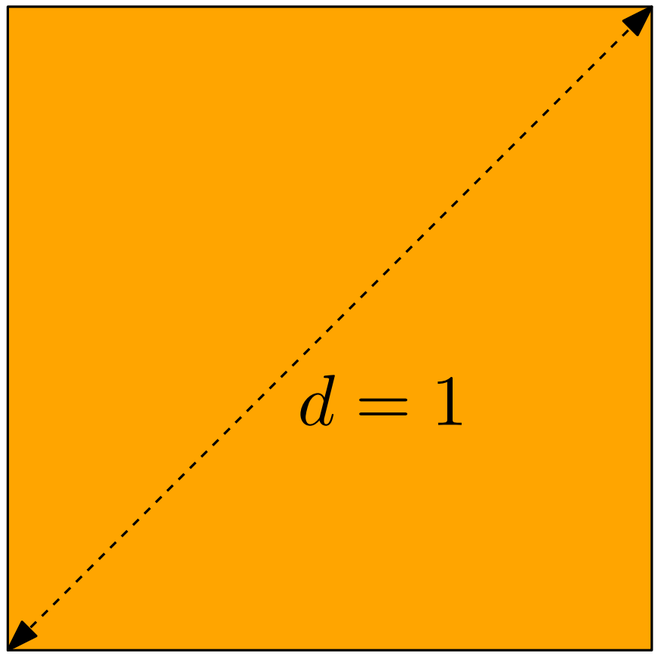

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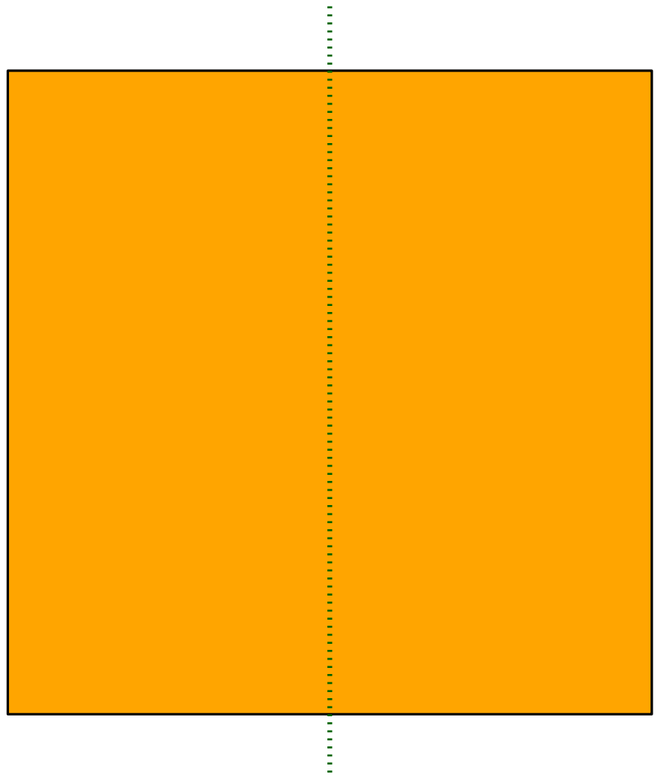

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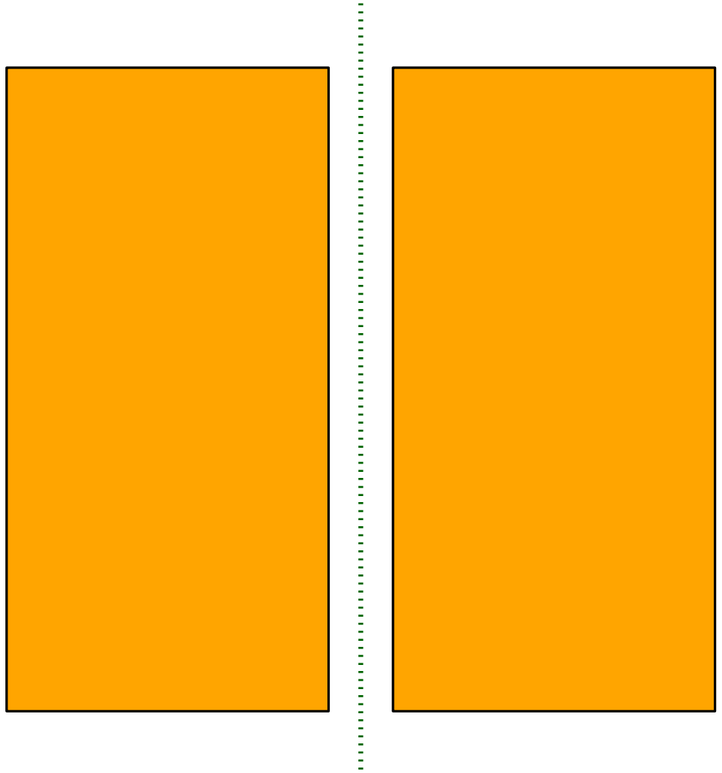

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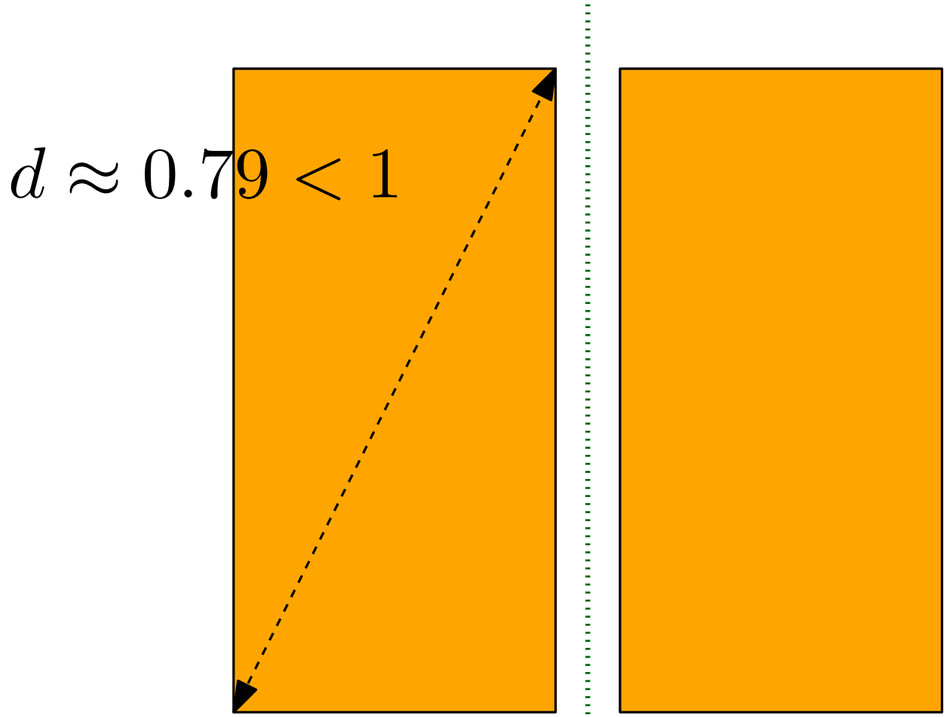

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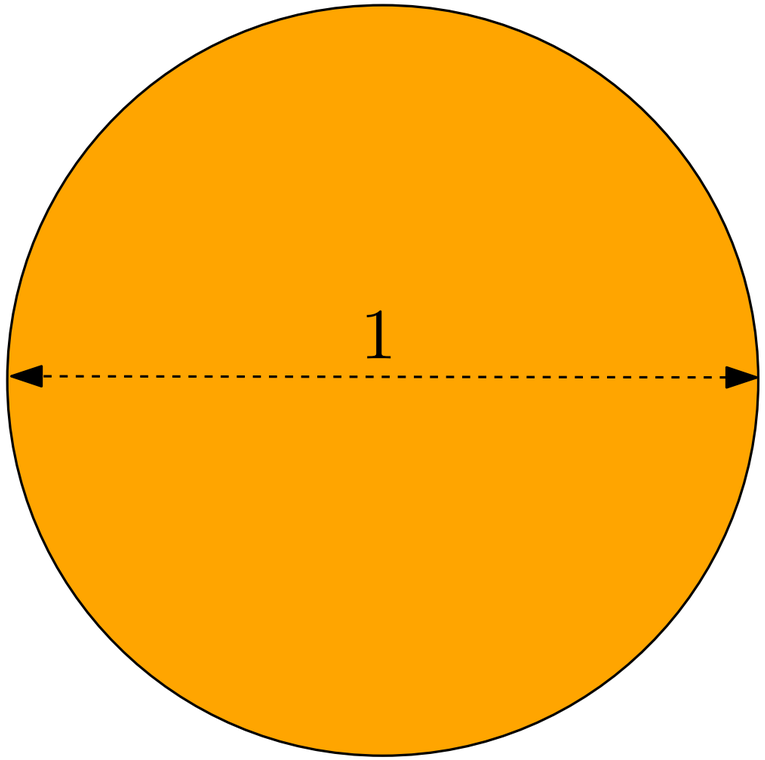

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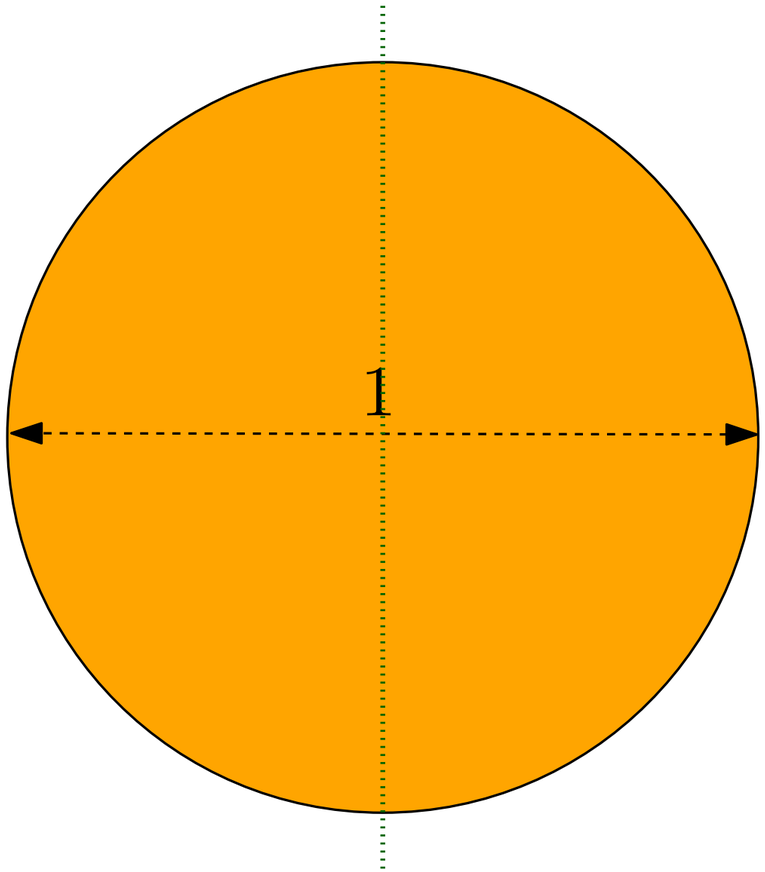

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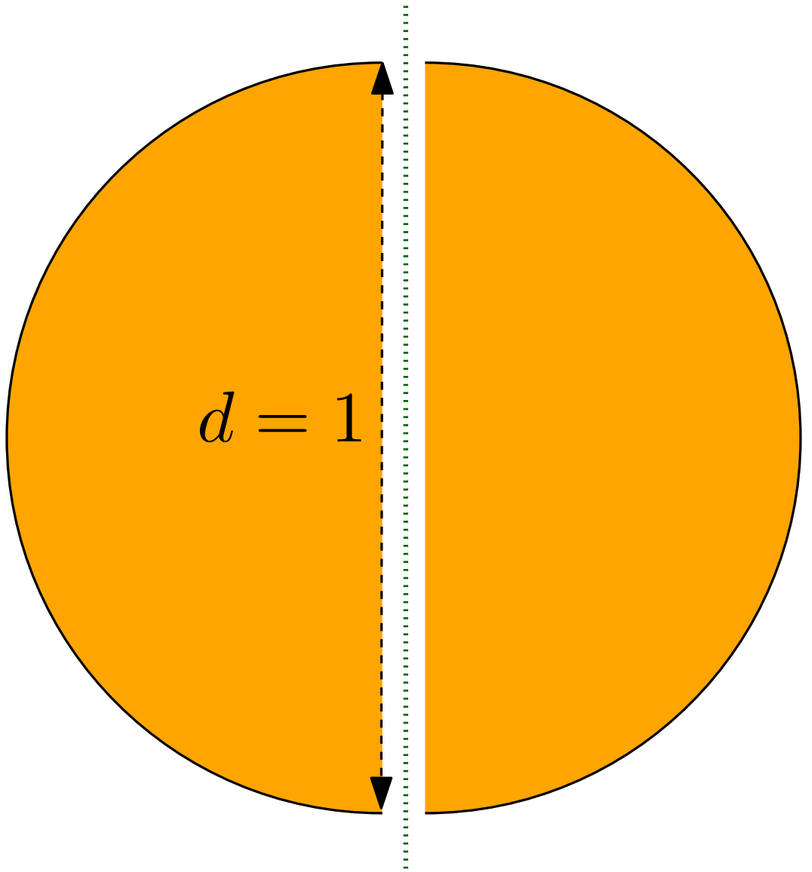

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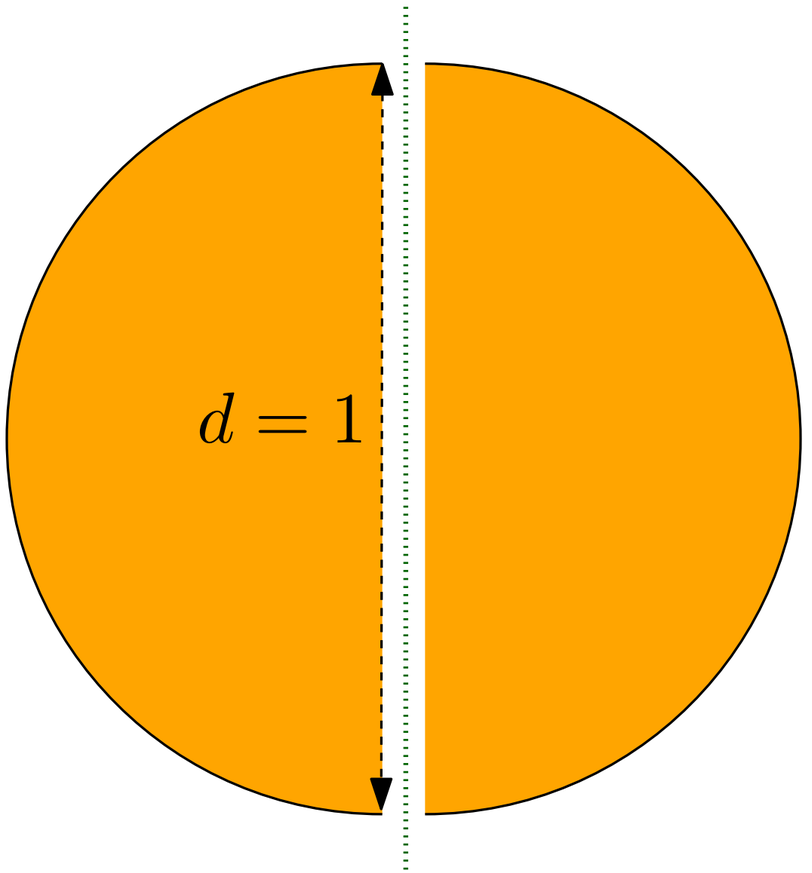

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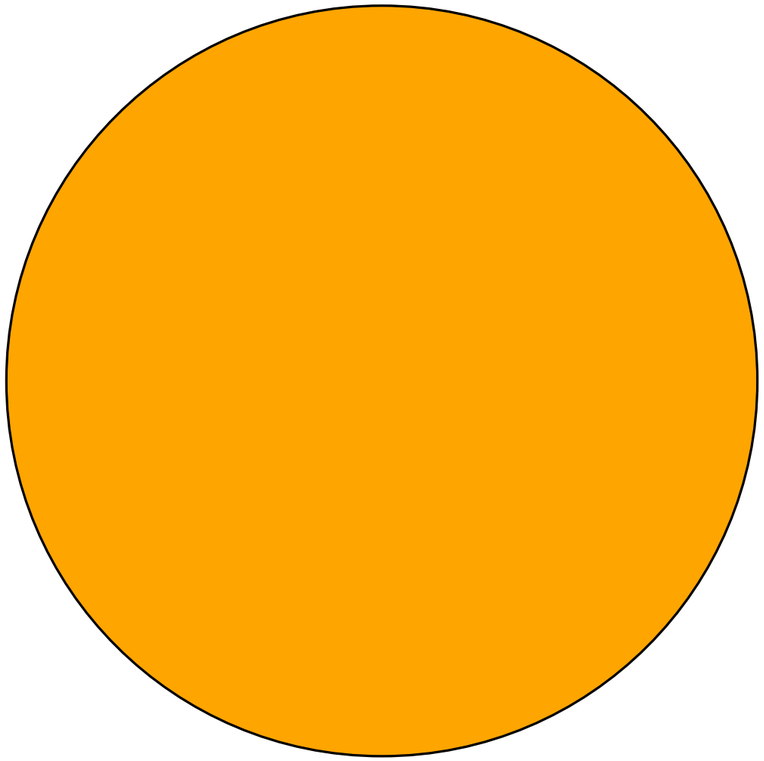

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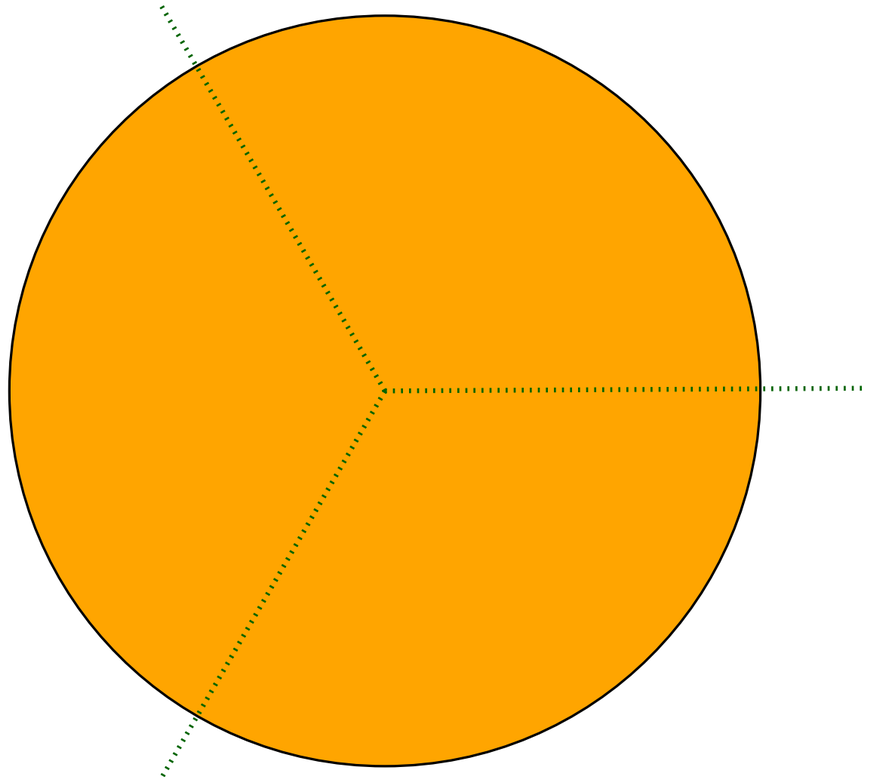

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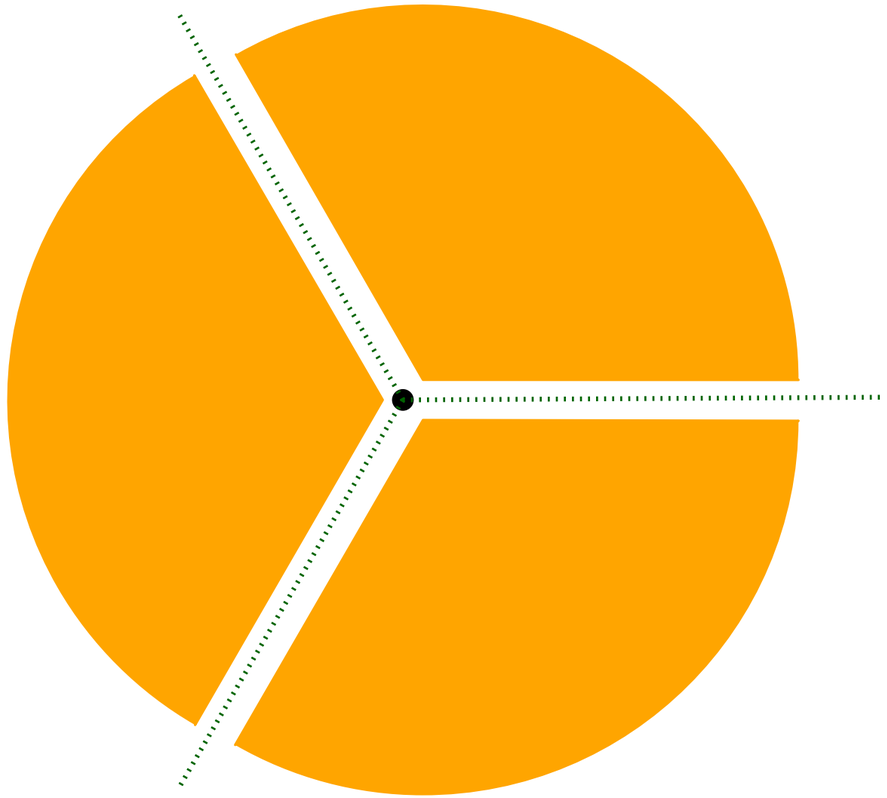

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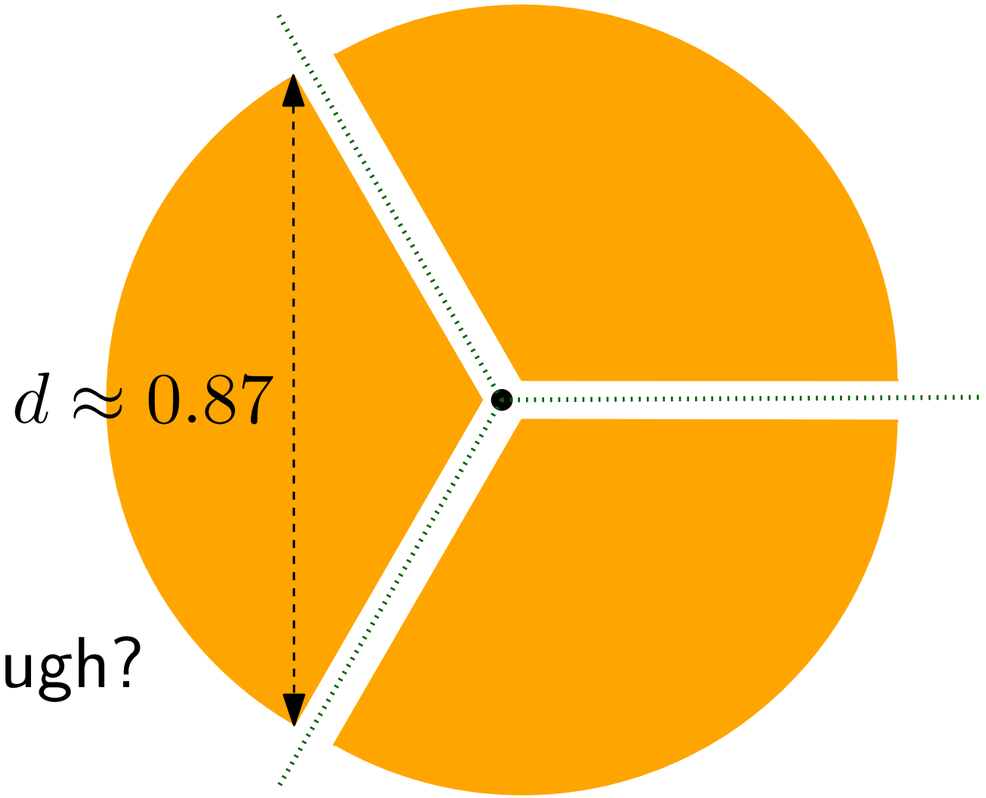

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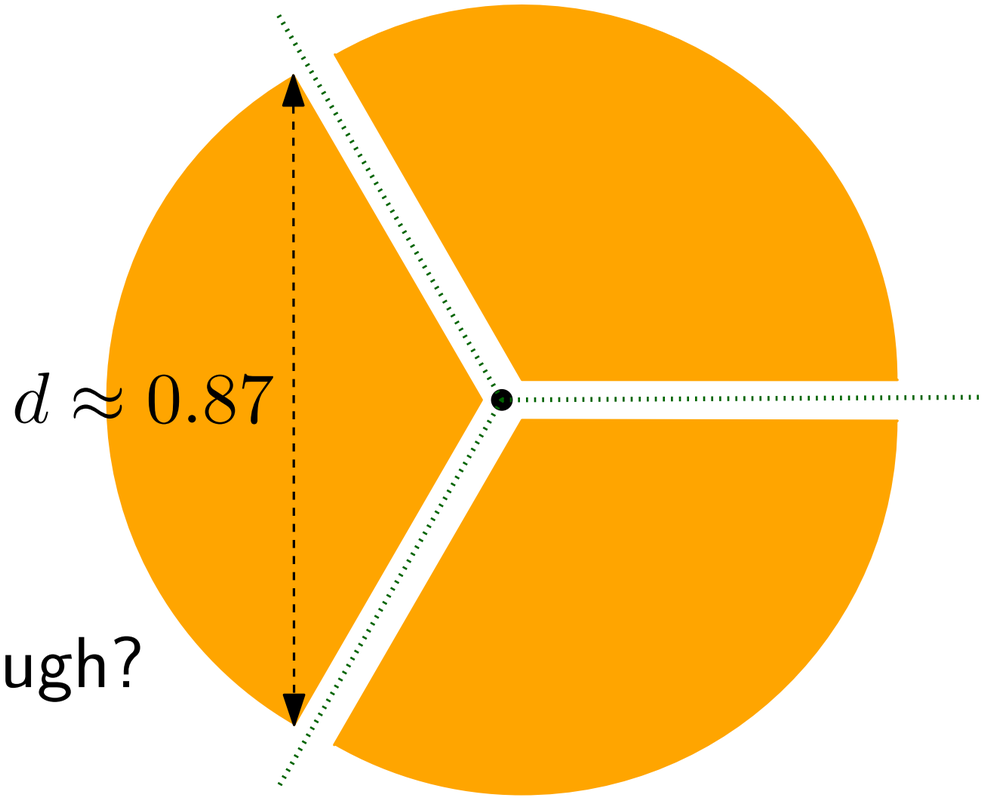

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Answer: **Yes!**

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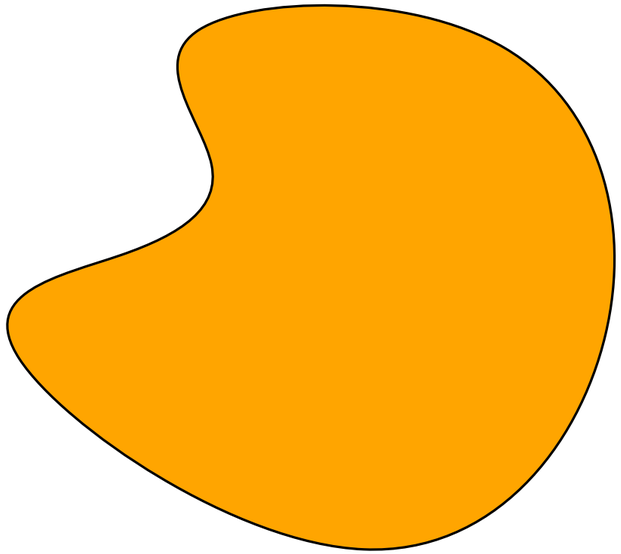

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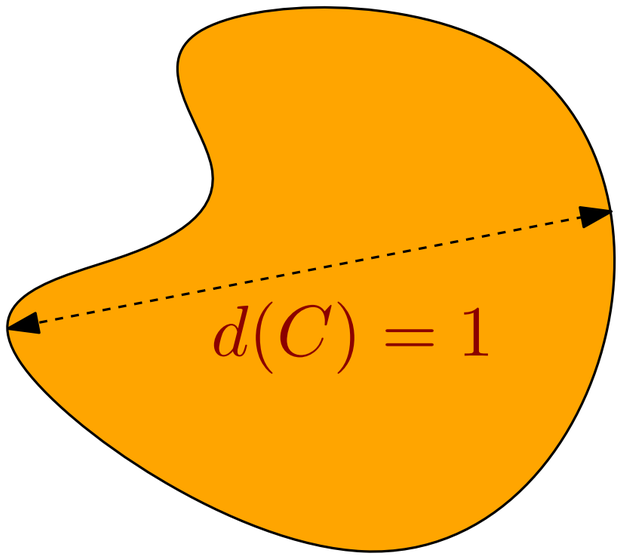

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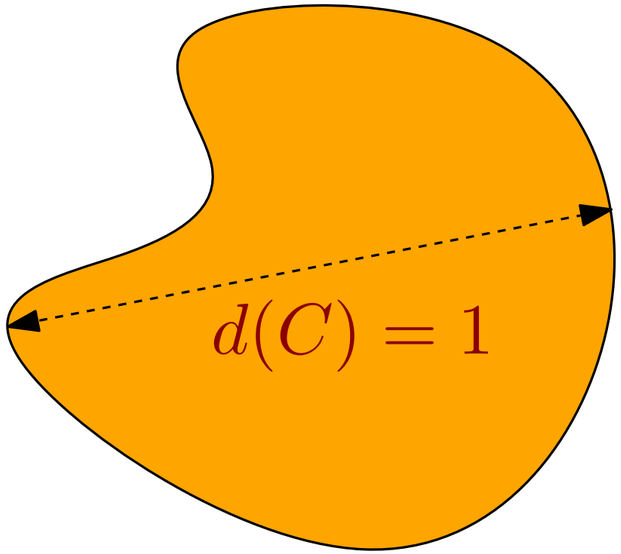

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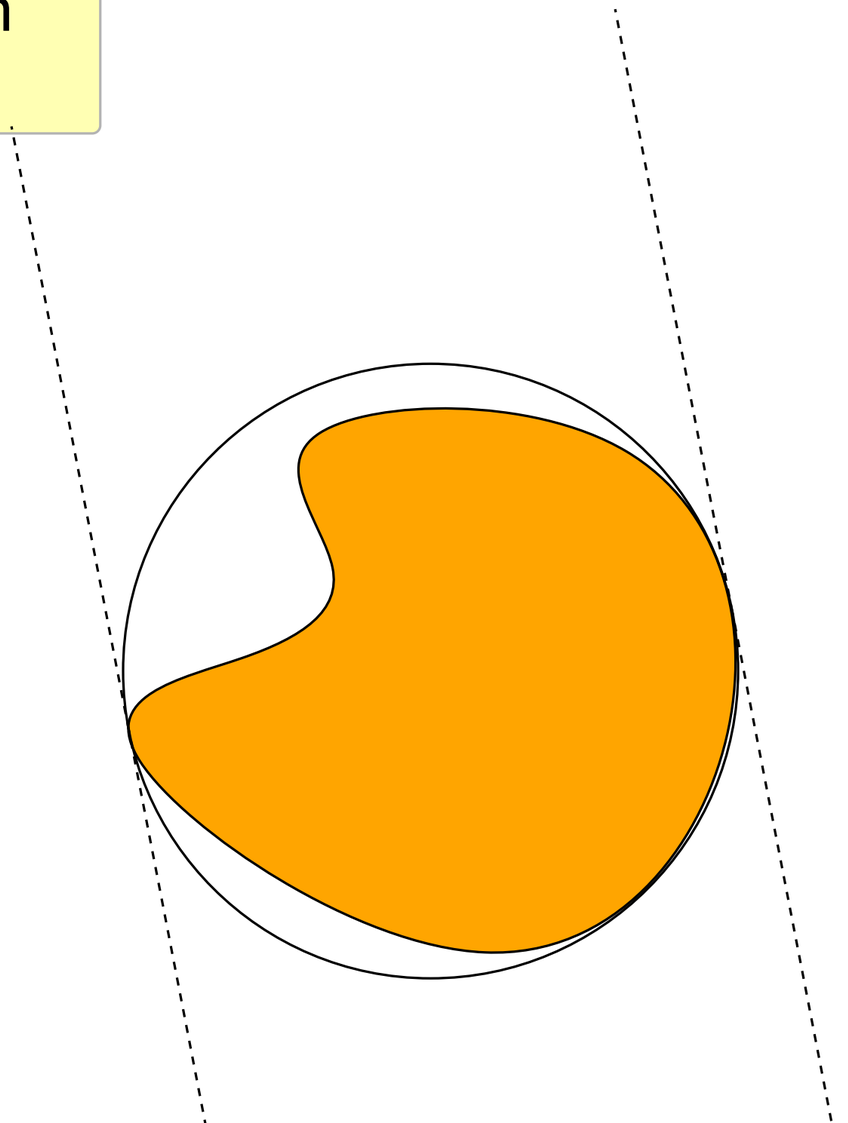

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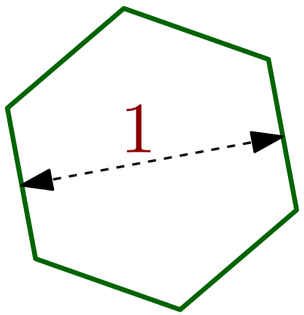

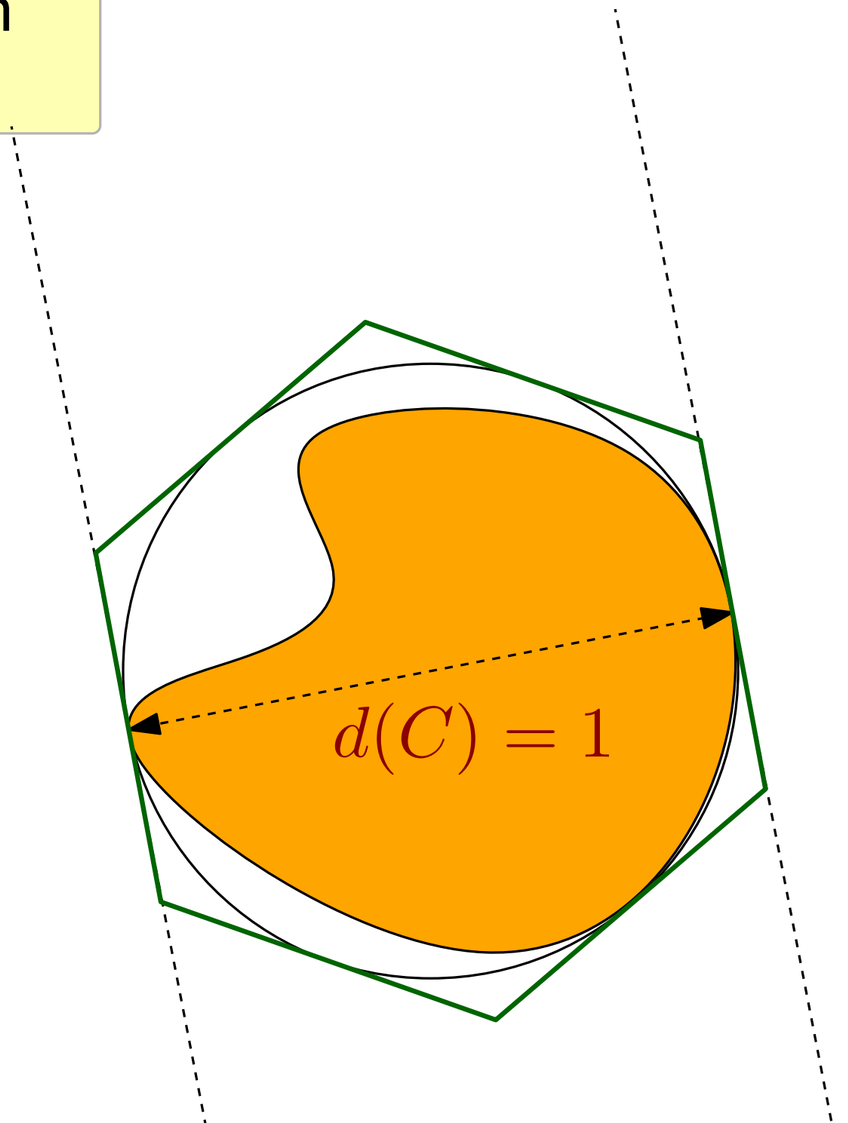

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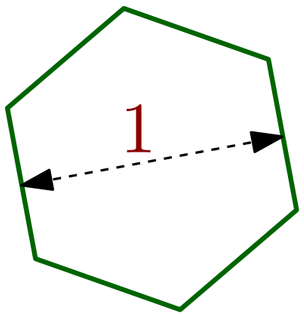

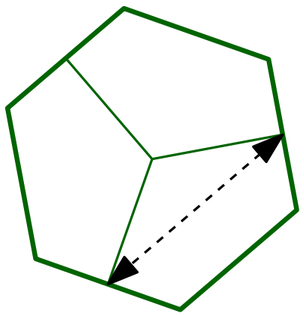

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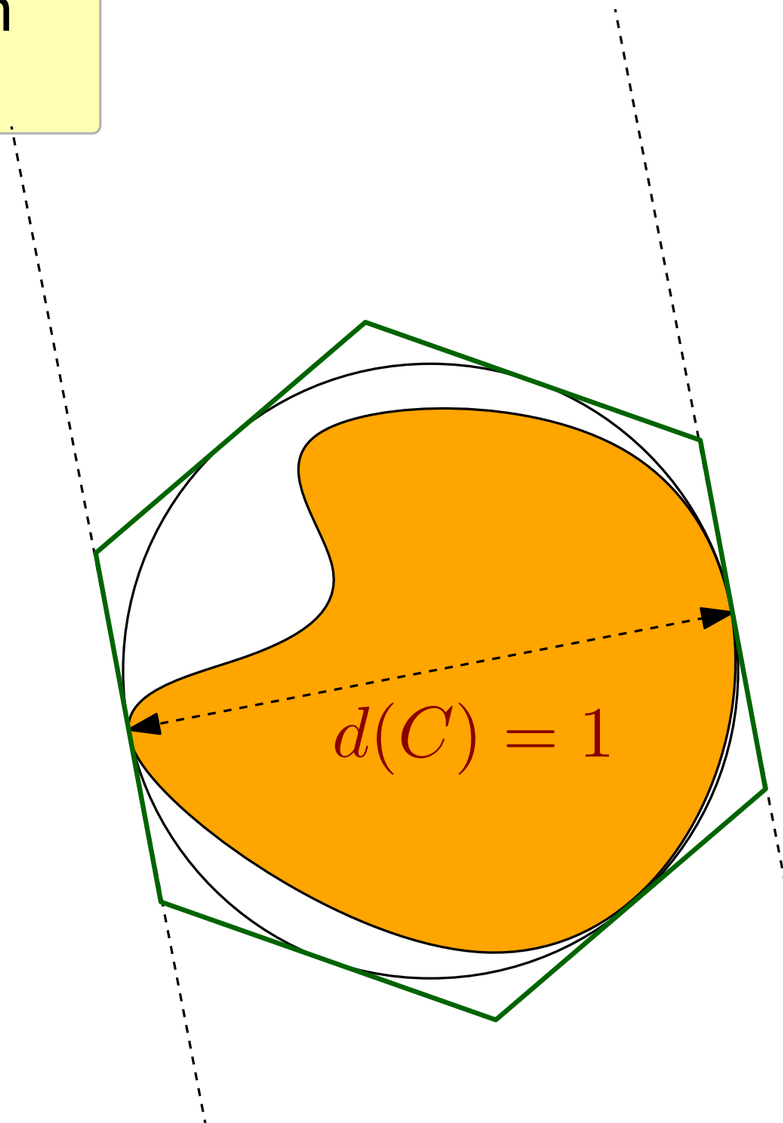

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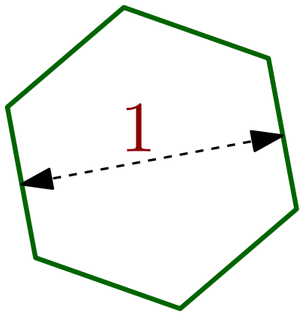

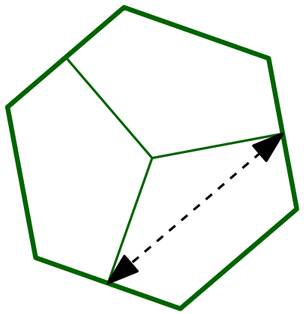

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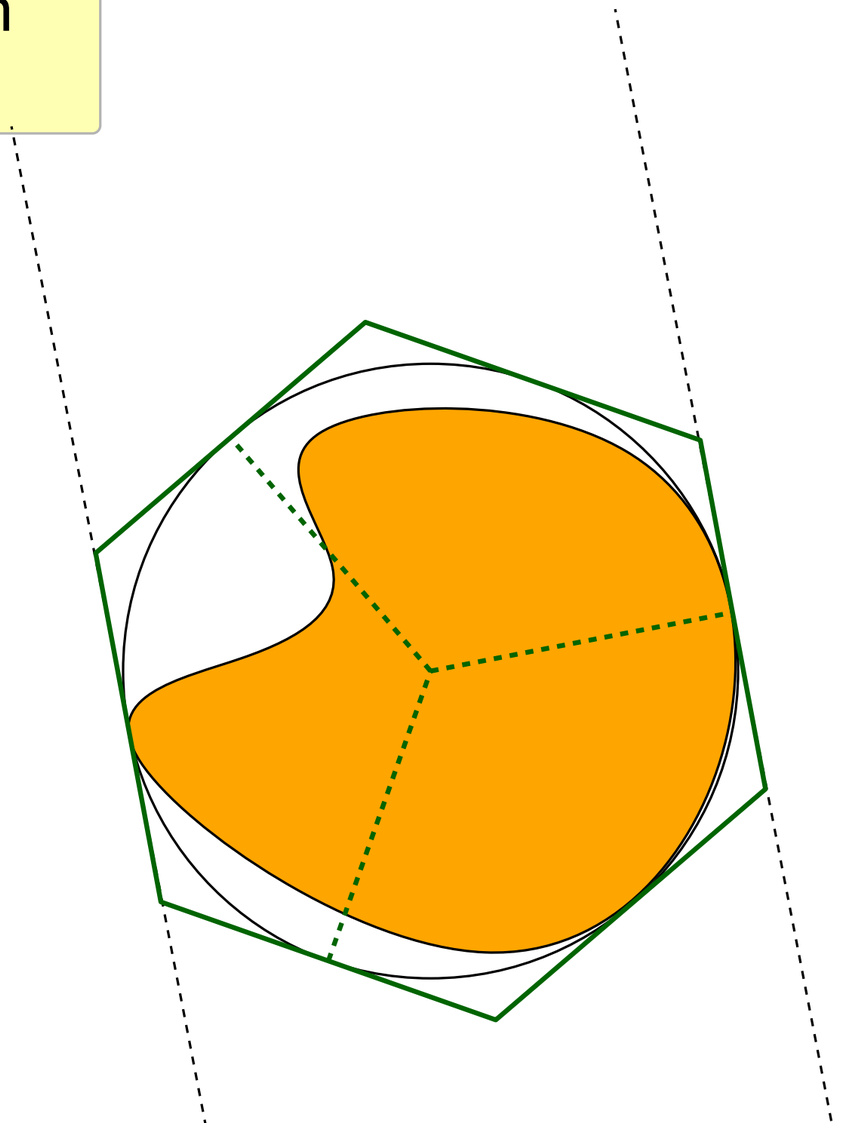

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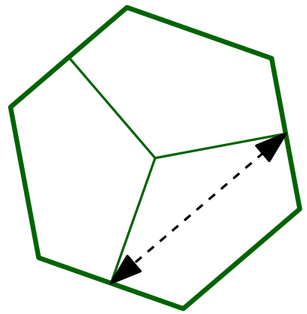

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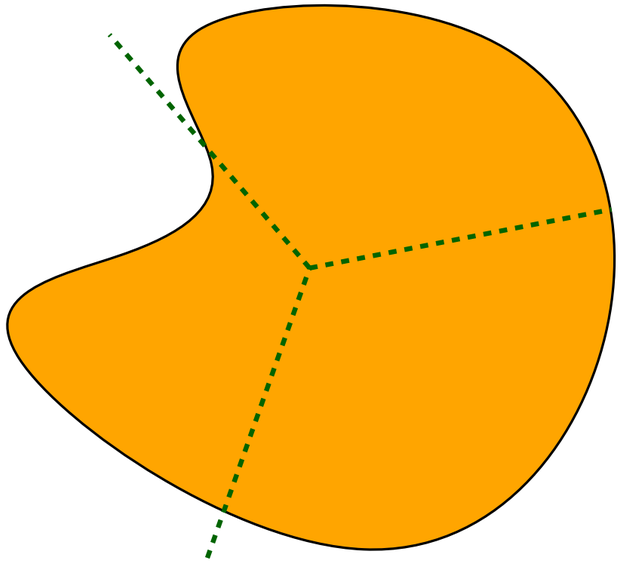

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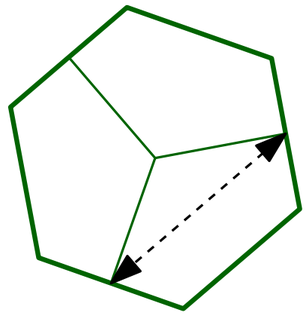

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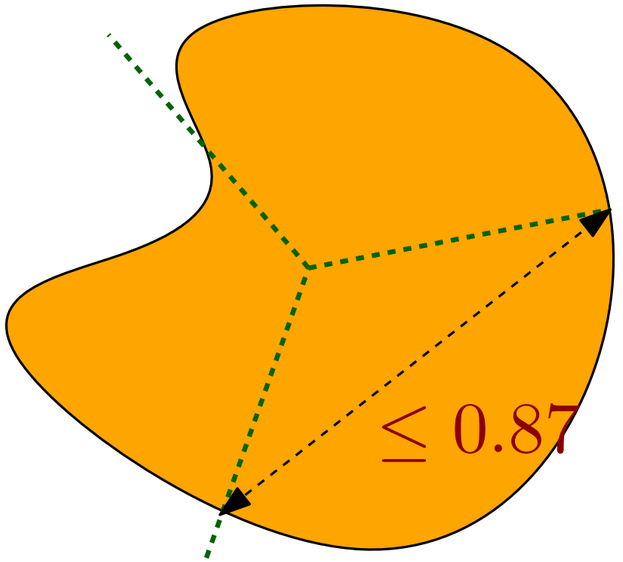

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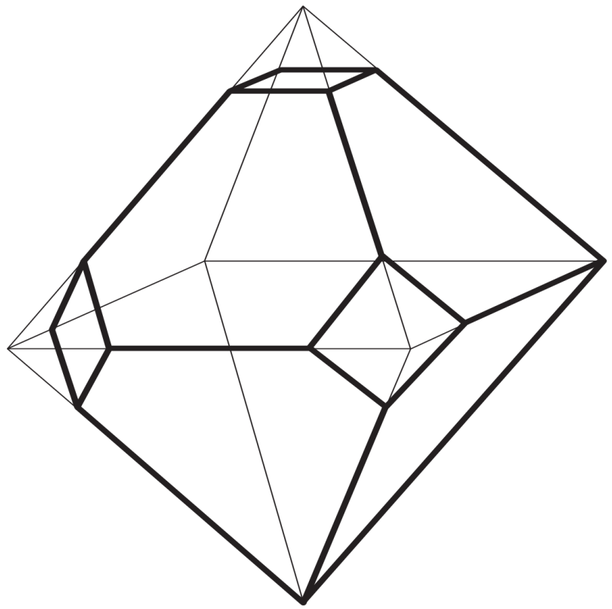

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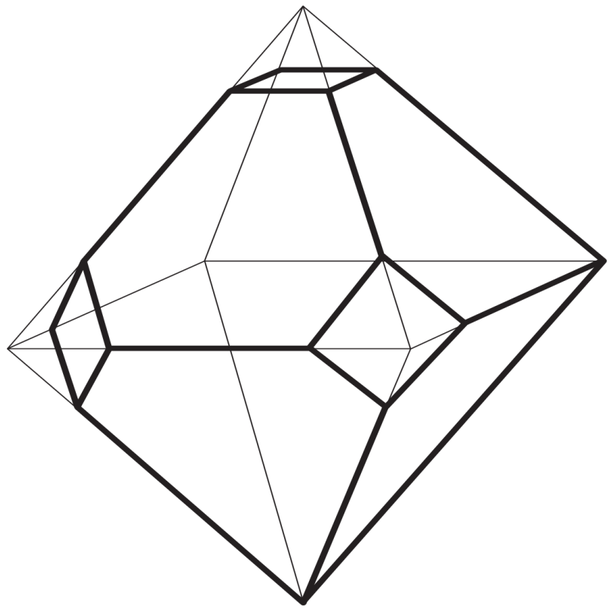

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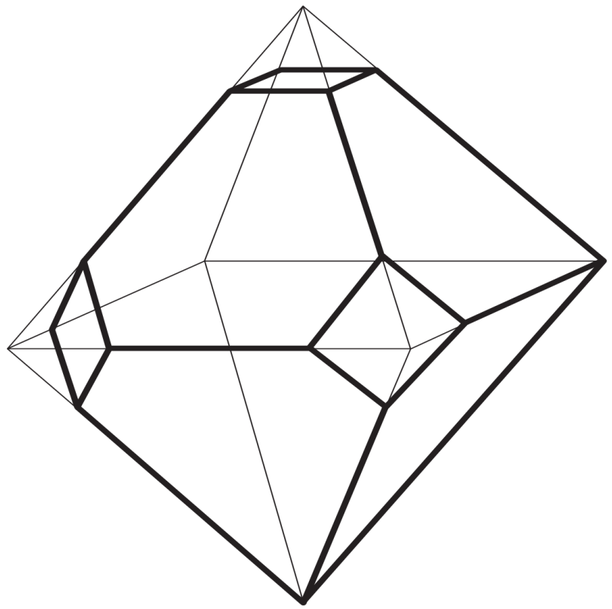

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- Since then, the minimum d has been improved: $d = 946$ (1994), $d = 561$ (1997), $d = 298$ (2003), $d = 65$ (2014), $d = 64$ (2014).

Any question?

Want to learn more?

- J. Kahn and G. Kalai, A counterexample to Borsuk's conjecture, Bull. Amer. Math. Soc. (N.S.), 29 (1993), 60–62. https://arxiv.org/abs/math/9307229

- Zong, C. Borsuk's partition conjecture. Jpn. J. Math. 16, 185–201 (2021). https://doi.org/10.1007/s11537-021-2007-7