BEZIER SURFACES

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TECHNIQUE: CARTESIAN PRODUCT

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A simple way to combine curves into surfaces

See [Salomon, 1.9]

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Consider two parametric curves $Q(u) = \sum_{i=1}^{n} f_i(u)Q_i$ and $R(w) = \sum_{i=1}^{m} g_i(u)R_i$, where the Q_i and R_i are (control) points

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$P(u, w) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(u)g_j(w)P_{ij}$ where each P_{ij} is a point (or vector)

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Example: bilinear surfaces

See [Salomon, 2.3]

$Q(u) = (1-u)P_0 + uP_1$, for $0 \leq u \leq 1$, and
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Then $P(u, w) = (1-u)(1-w)P_{00} + (1-u)wP_{01} + u(1-w)P_{10} + uwP_{11}$
for $P_{00}, P_{01}, P_{10}, P_{11}$ points, e.g., in $\mathbb{R}^3$

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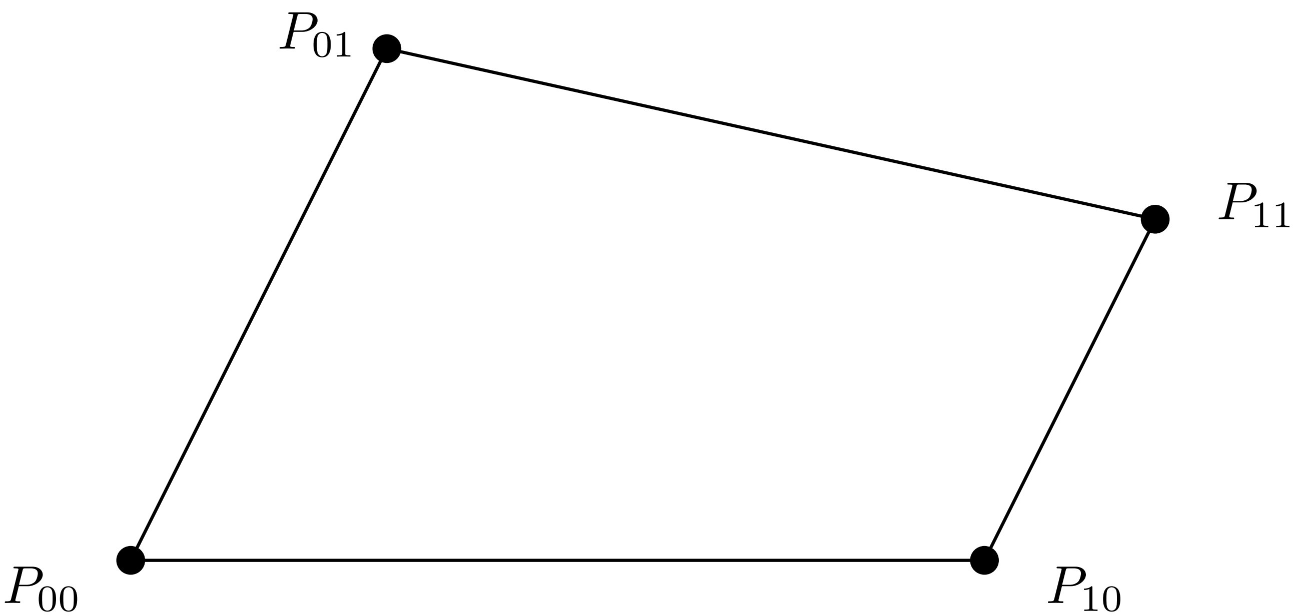

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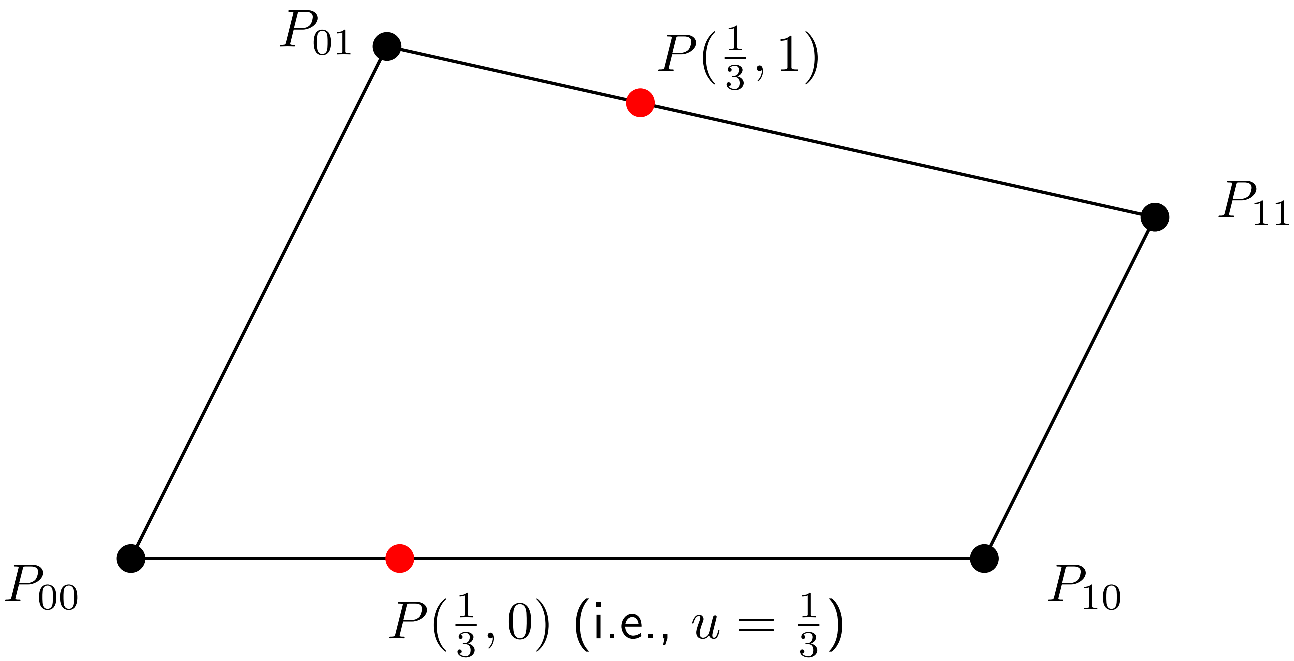

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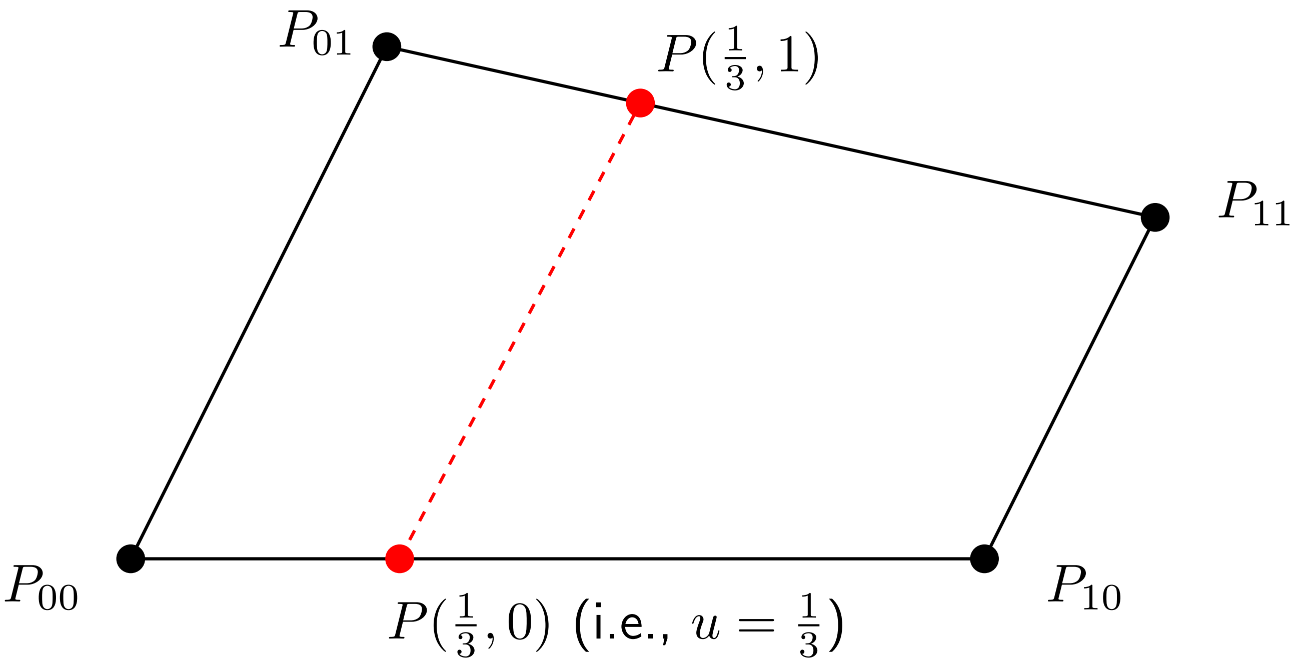

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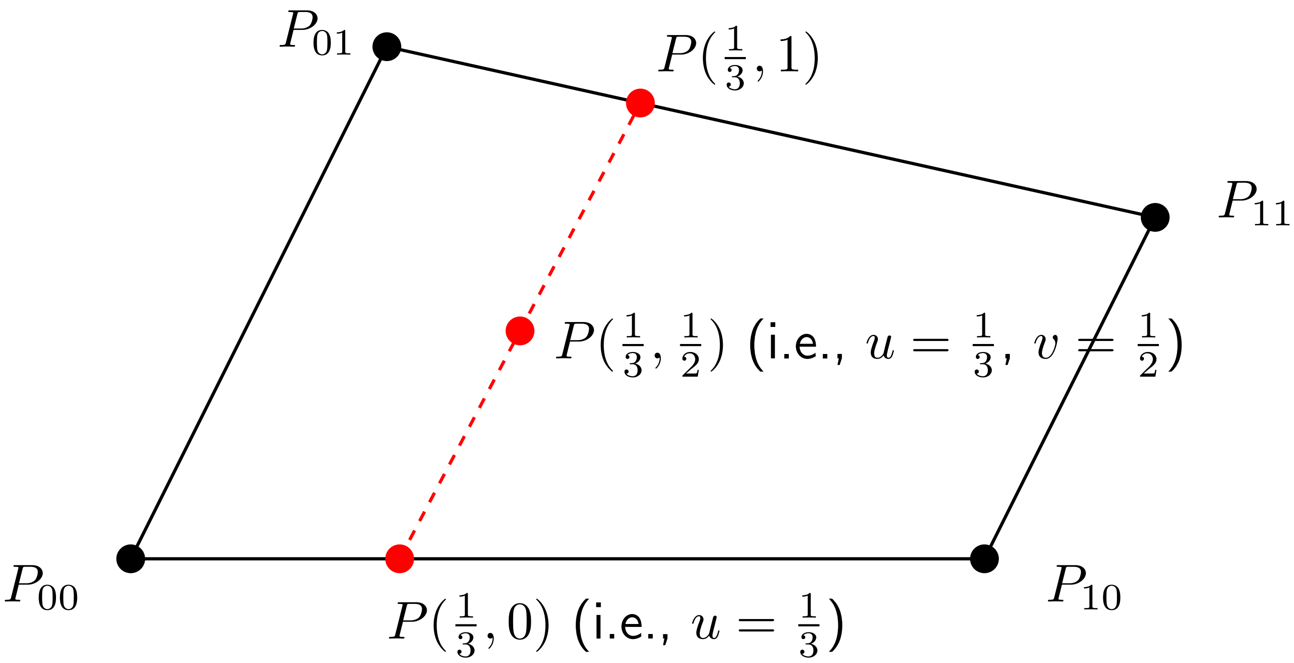

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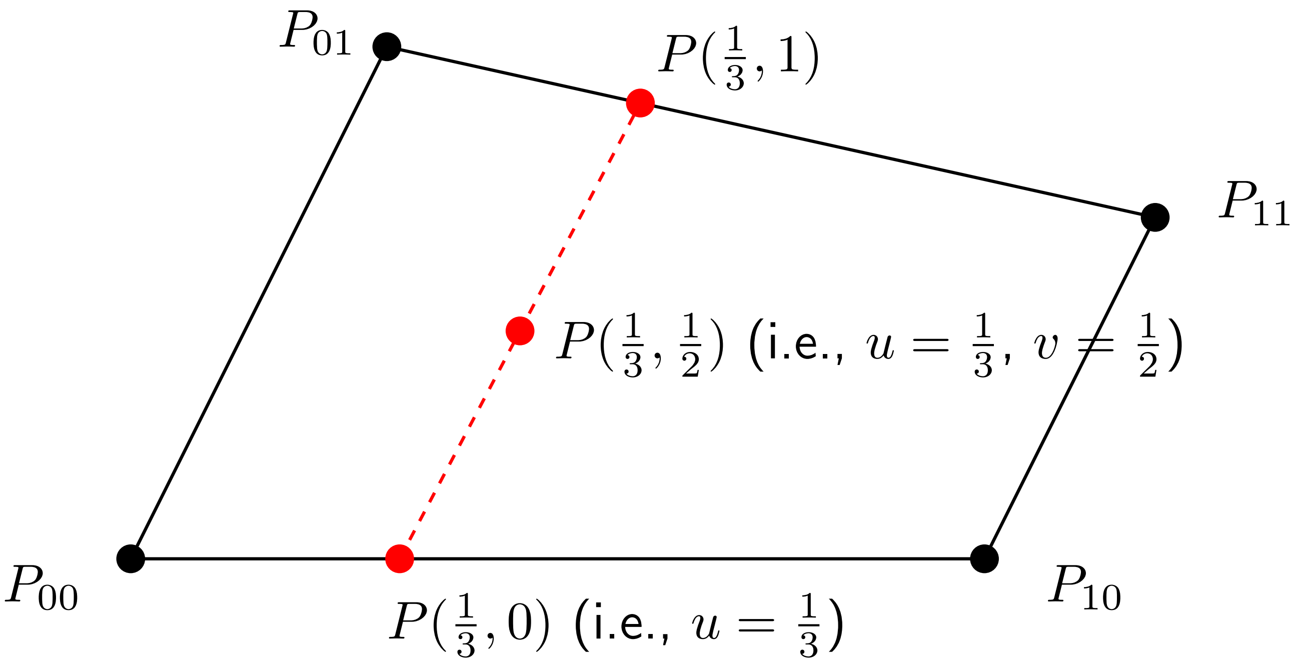

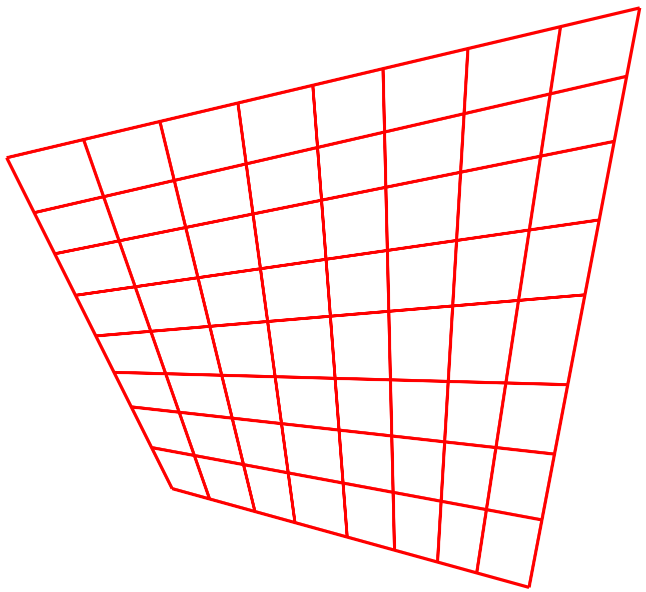

Bilinear surfaces can be highly curved

Try it: https://www.desmos.com/3d/qylacc01wq

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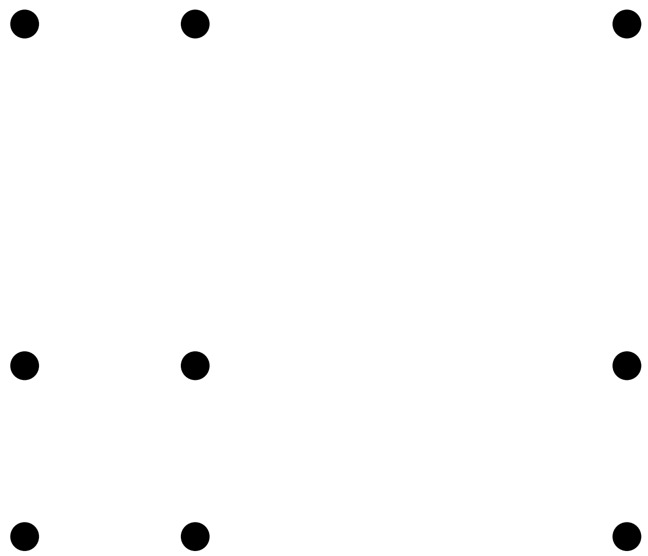

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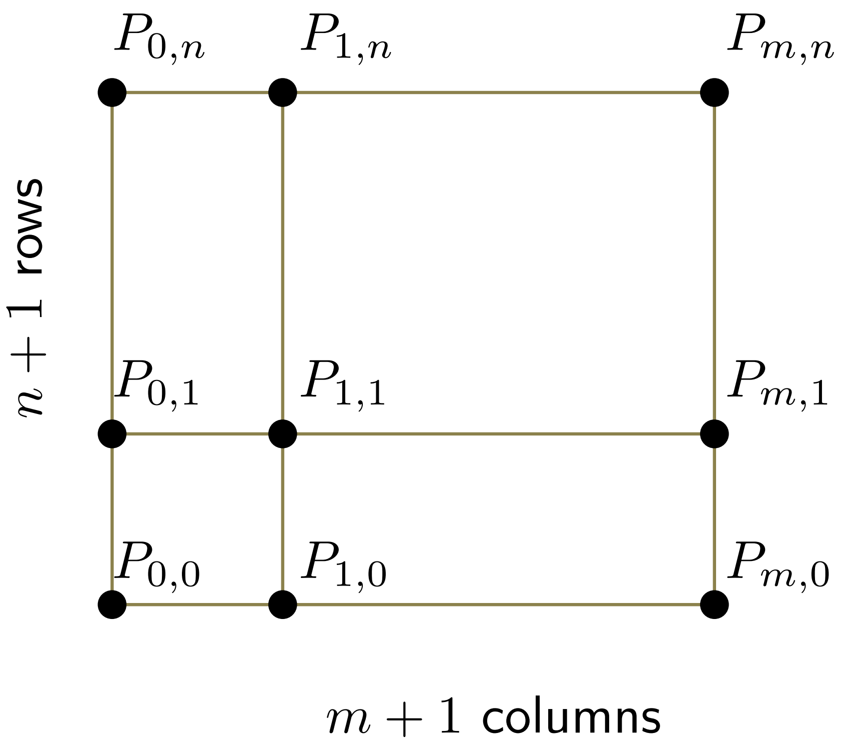

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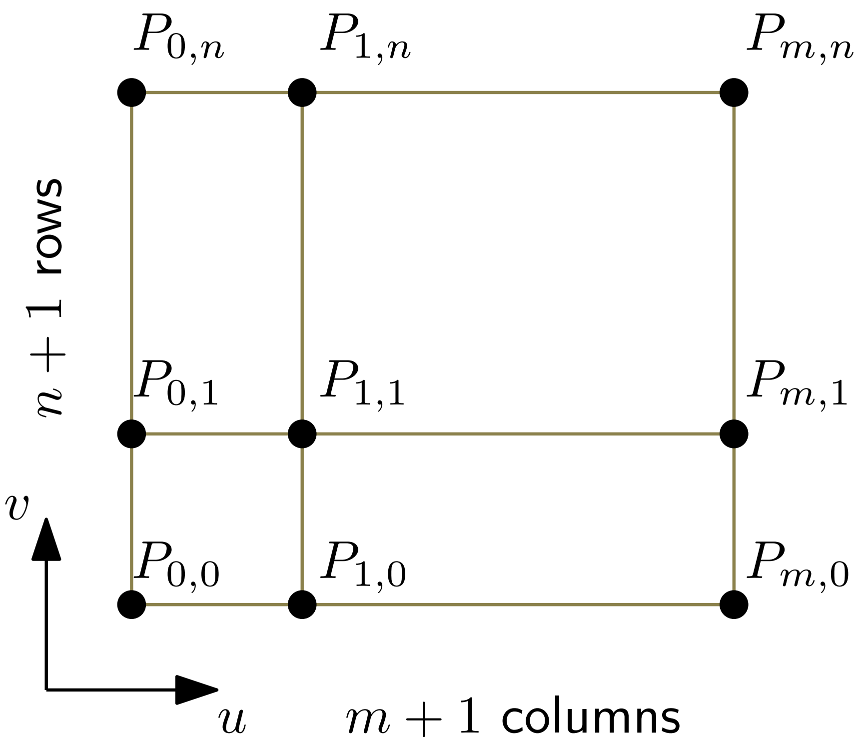

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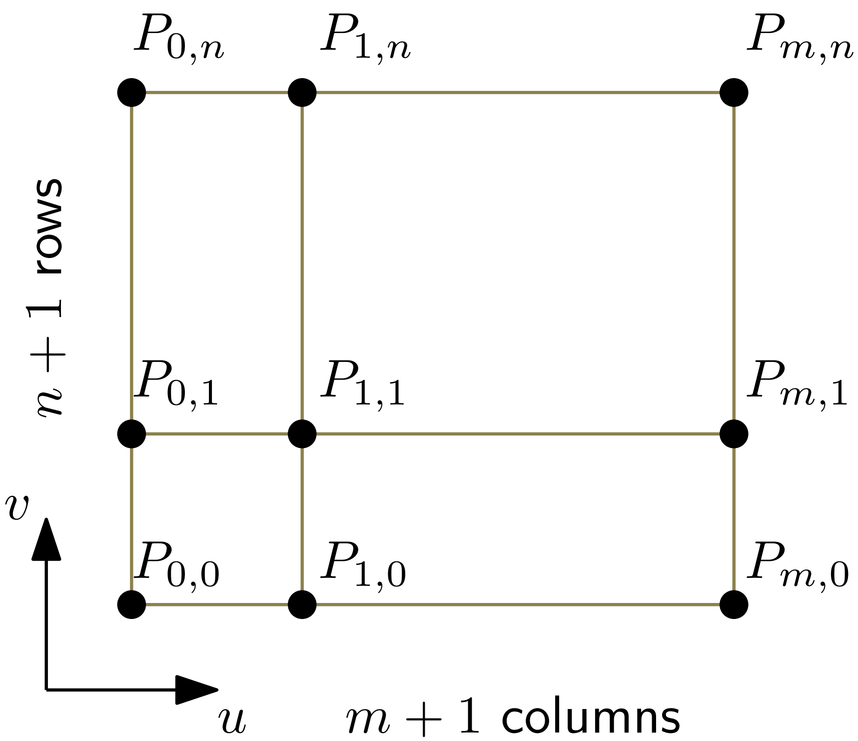

$$S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) P_{i,j}$$

$$0 \leq u, v \leq 1$$

where the terms $B_{m,i}$ and $B_{n,j}$ are the Bernstein polynomials, same as in Bézier curves

Idea: use grid of control points

Consider a grid of $(m+1) \times (n+1)$ control points arranged in a rectangular grid

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In matrix form:

$$S(u,v) = (B_{m,0}(u), B_{m,1}(u), \ldots, B_{m,m}(u)) \begin{pmatrix} P_{0,0} & P_{0,1} & \ldots & P_{0,n} \\ P_{1,0} & P_{1,1} & \ldots & P_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m,0} & P_{m,1} & \ldots & P_{m,n} \end{pmatrix} \begin{pmatrix} B_{n,0}(v) \\ B_{n,1}(v) \\ \vdots \\ B_{n,n}(v) \end{pmatrix}$$

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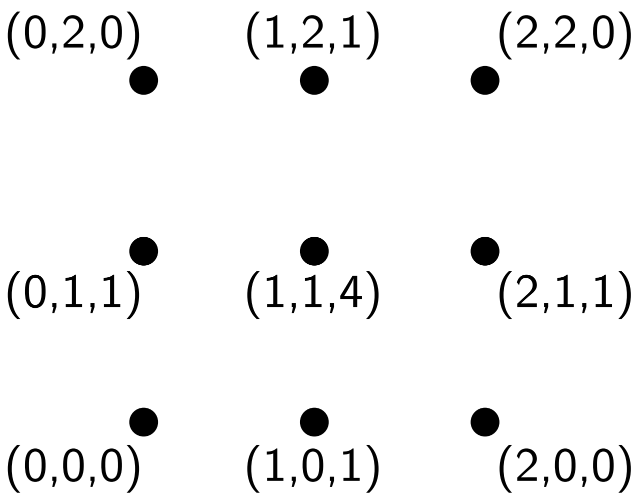
Note that each entry is a 3D point!

Example



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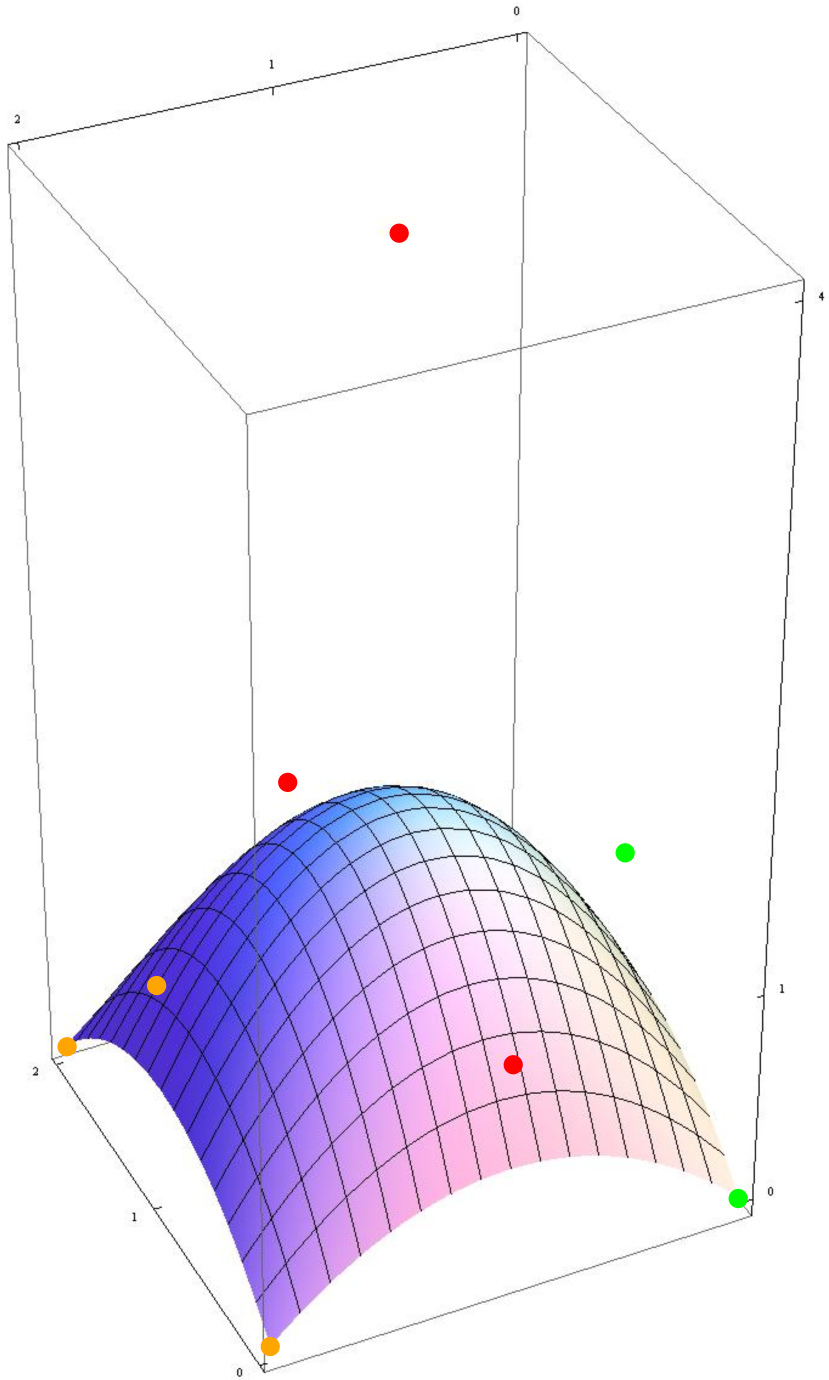
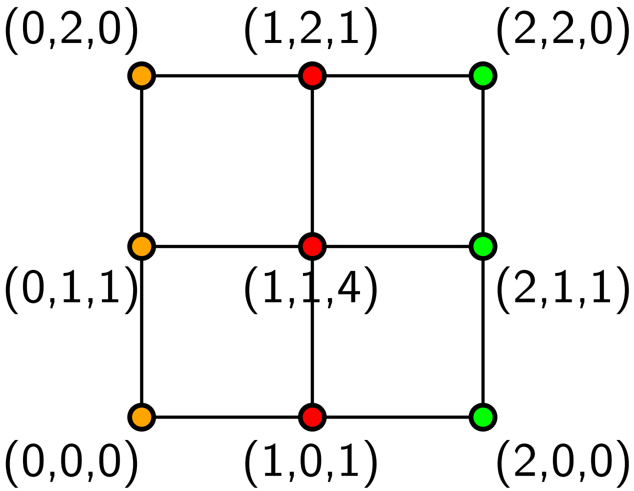
Example

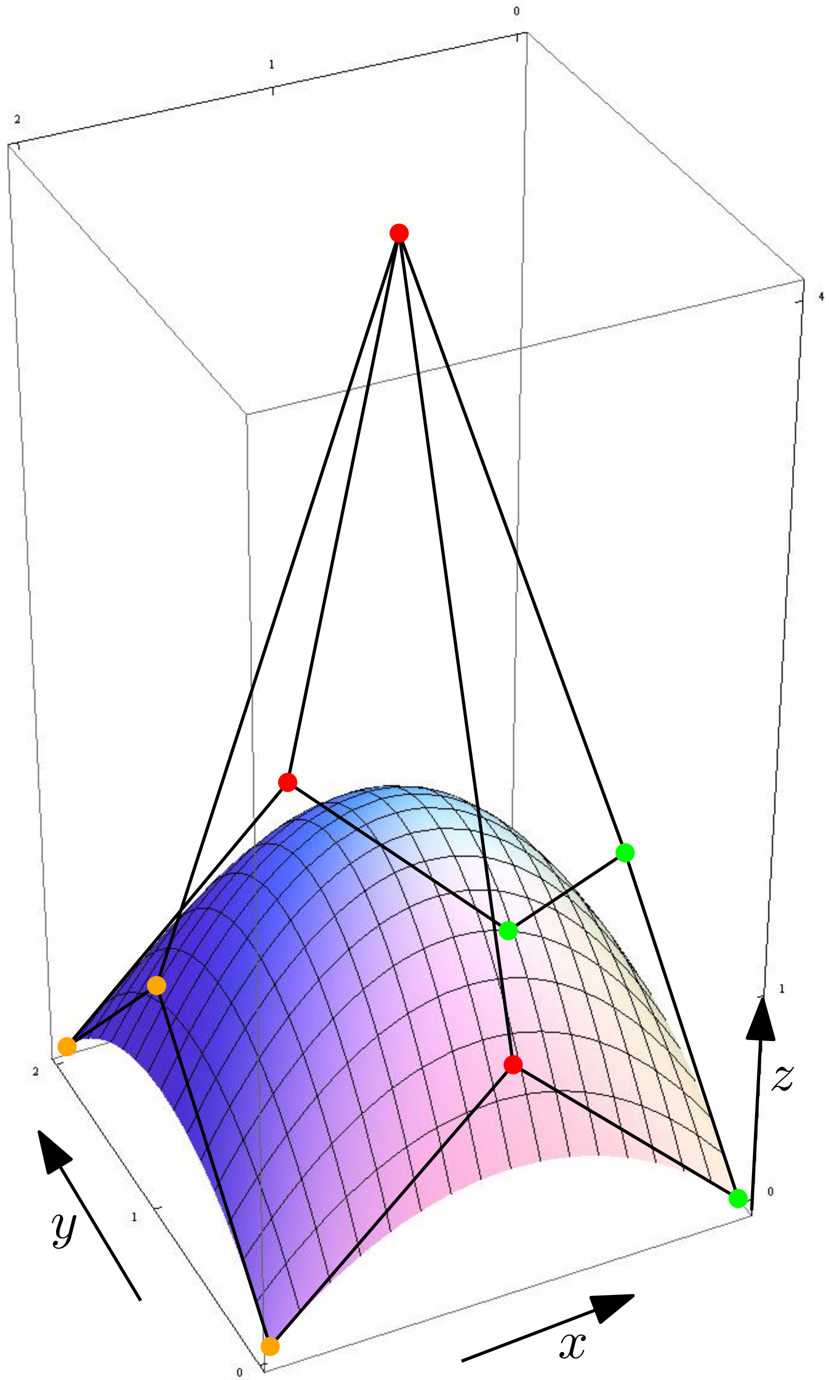
(0,2,0)	(1,2,1)	(2,2,0)
(0,1,1)	(1,1,4)	(2,1,1)
(0,0,0)	(1,0,1)	(2,0,0)



Biquadratic Bézier surface patch [Salomon, Fig 6.20]

Example





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Properties of Bézier surface (on rectangular grid)

$0 \le u, v \le 1$

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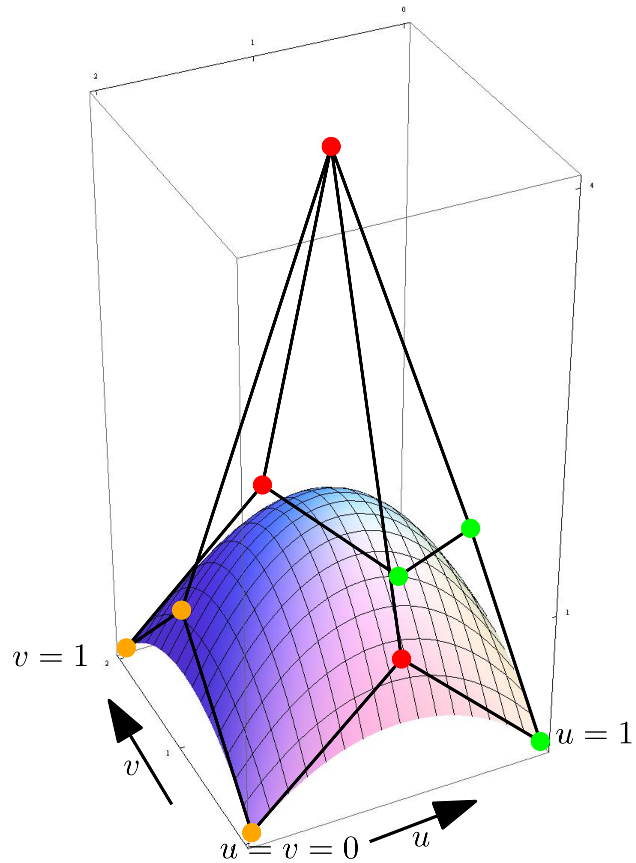

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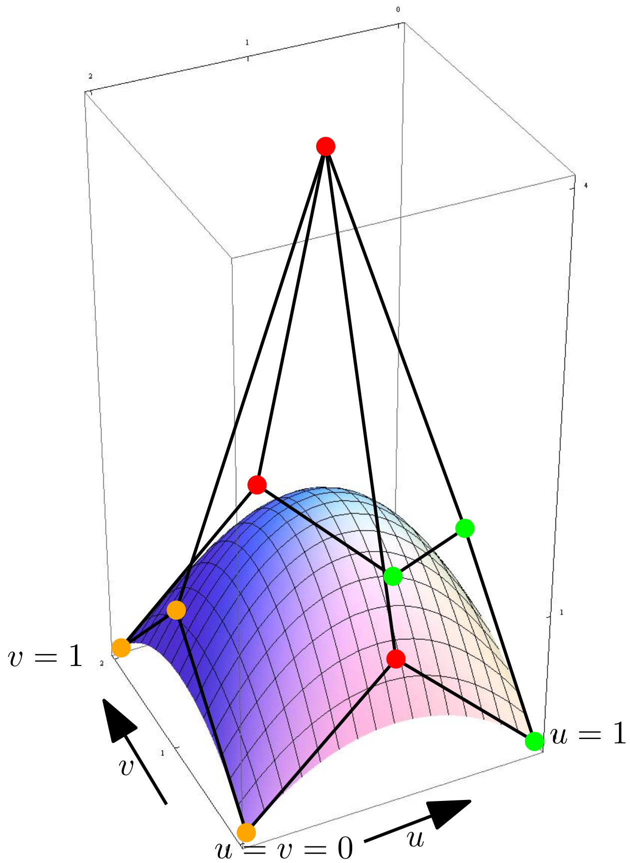

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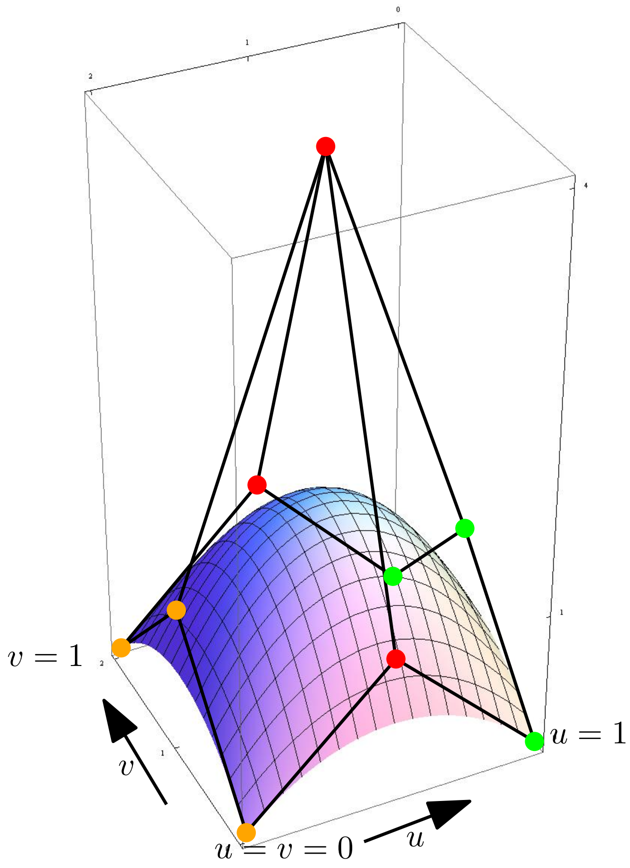
- Boundary curves



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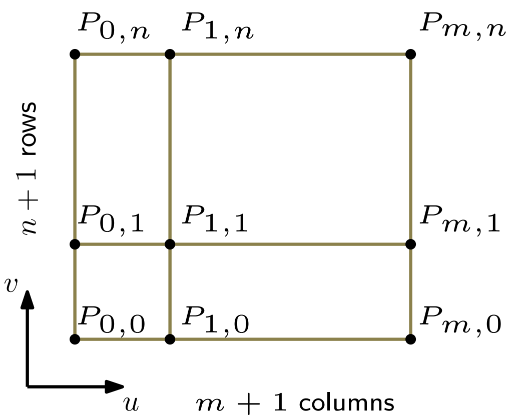

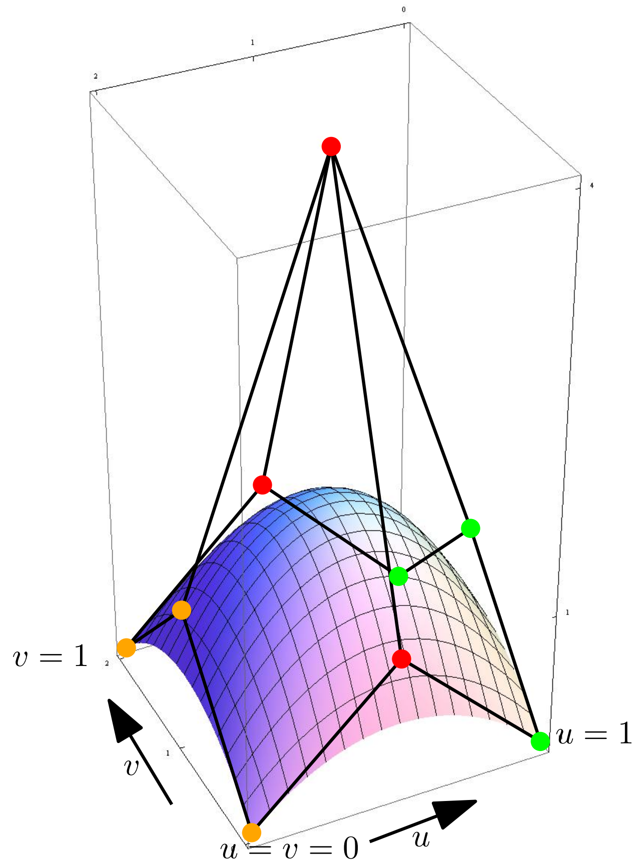

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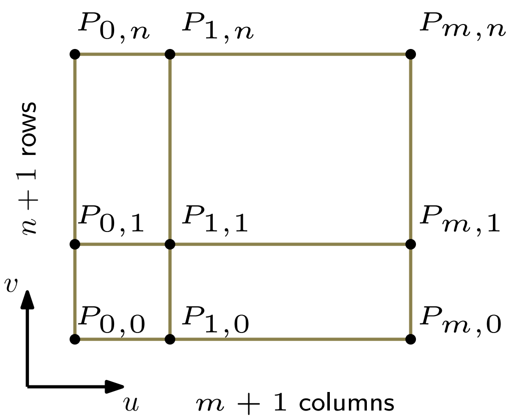

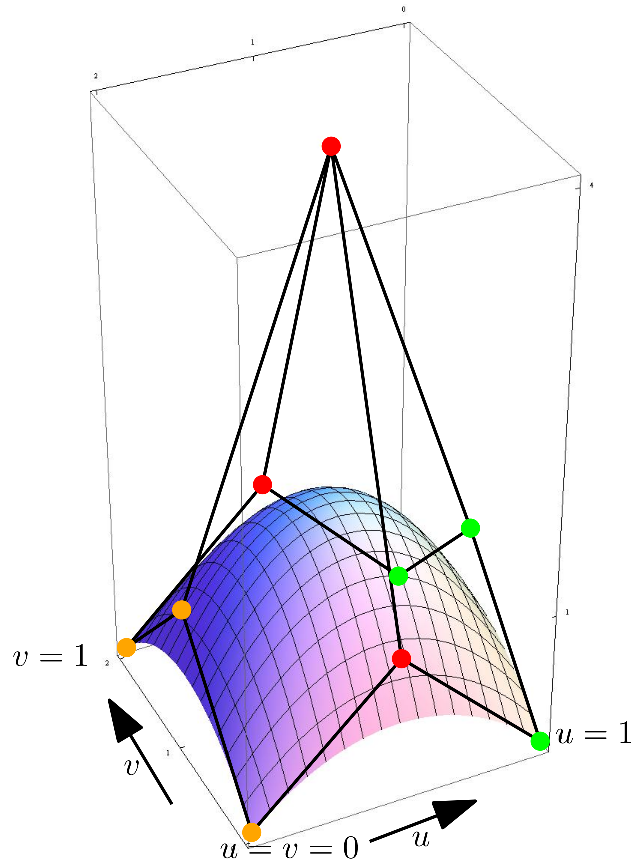

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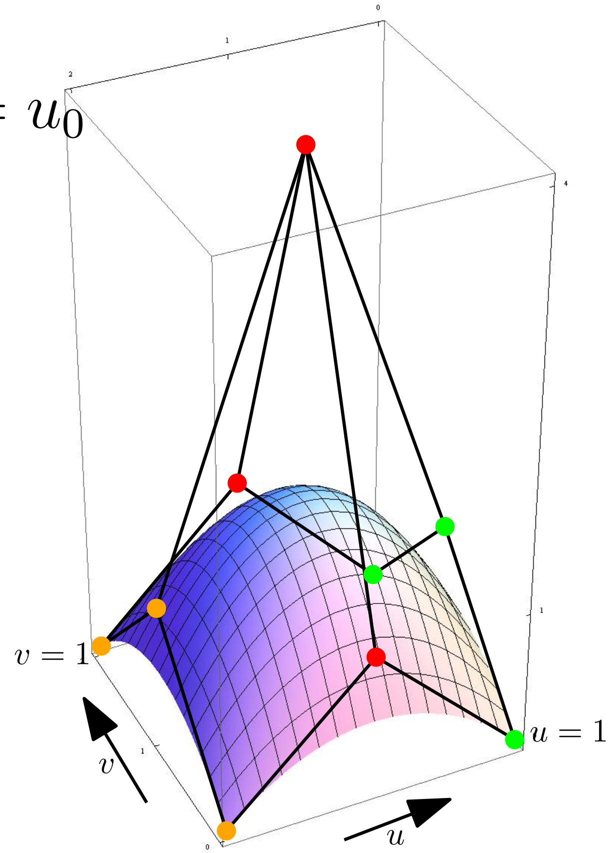

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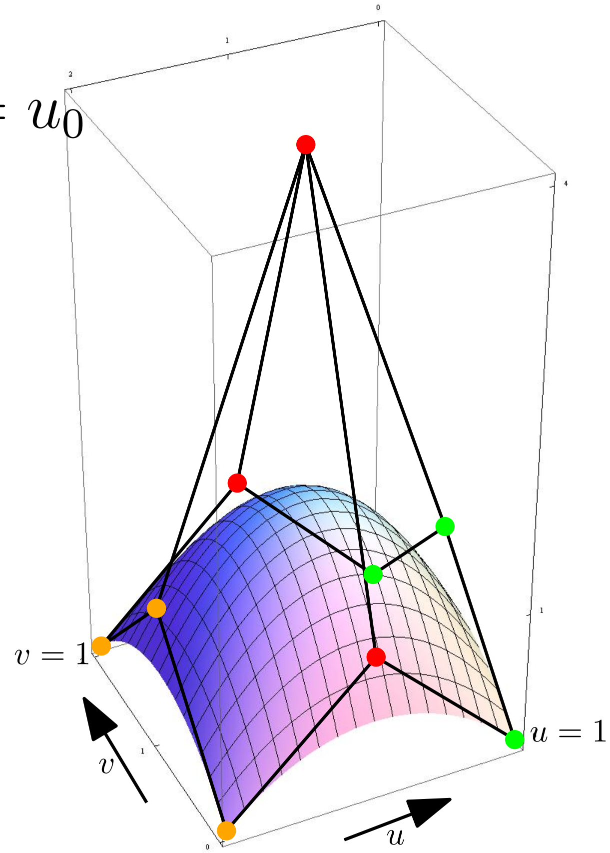

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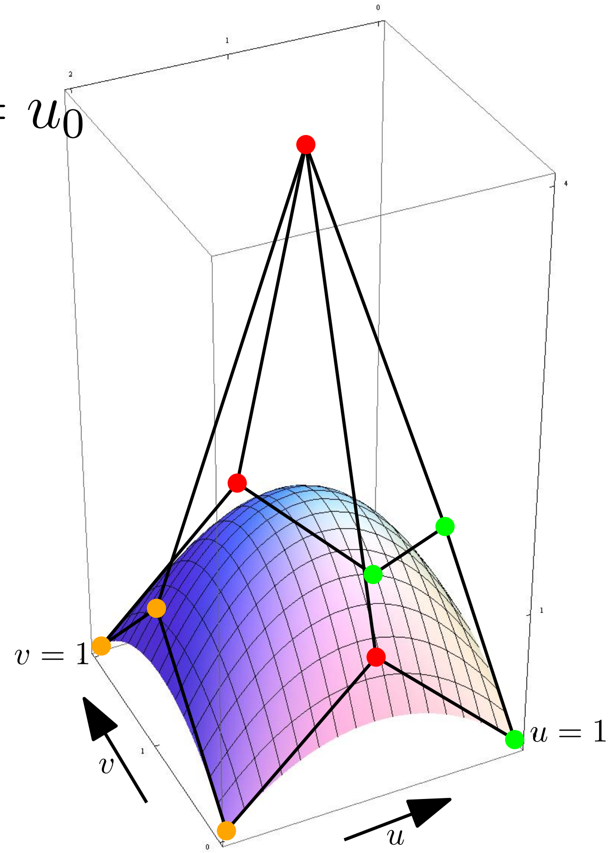
new control point Q_j



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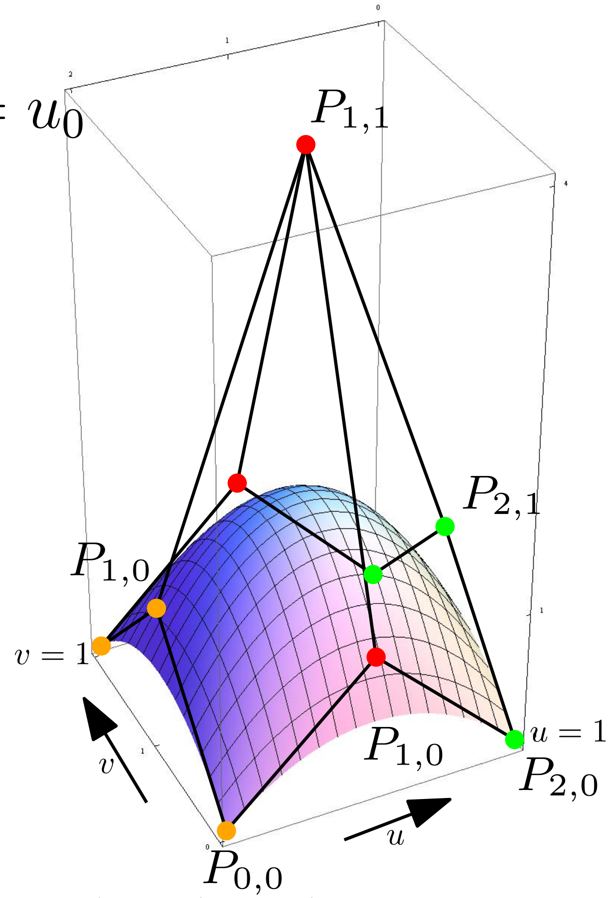

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Example on the right, for $u = 0.75$, $n = m = 2$

$$S(0.75, v) = \sum_{j=0}^{2} B_{n,j}(v) \left(\sum_{i=0}^{2} B_{m,i}(u_0) P_{i,j}\right)$$
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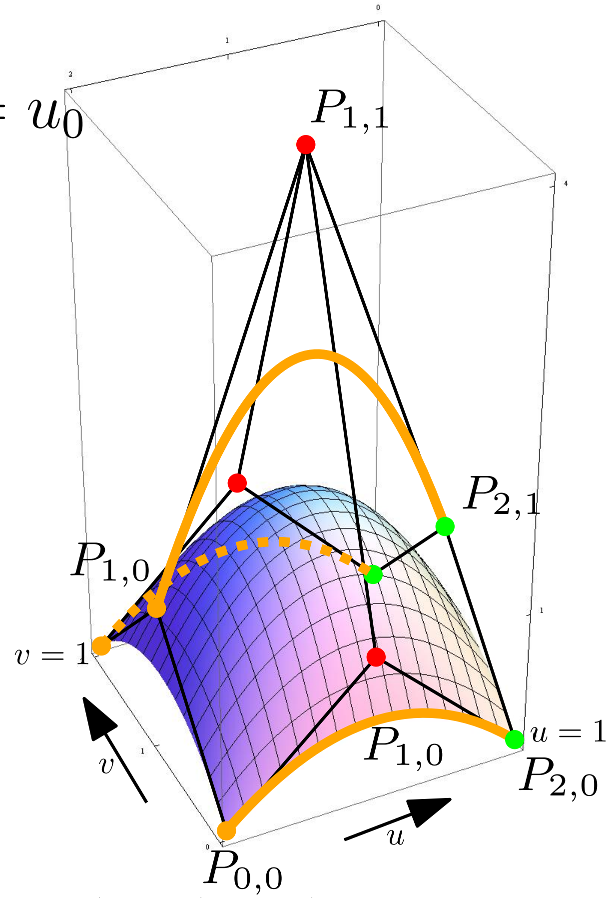

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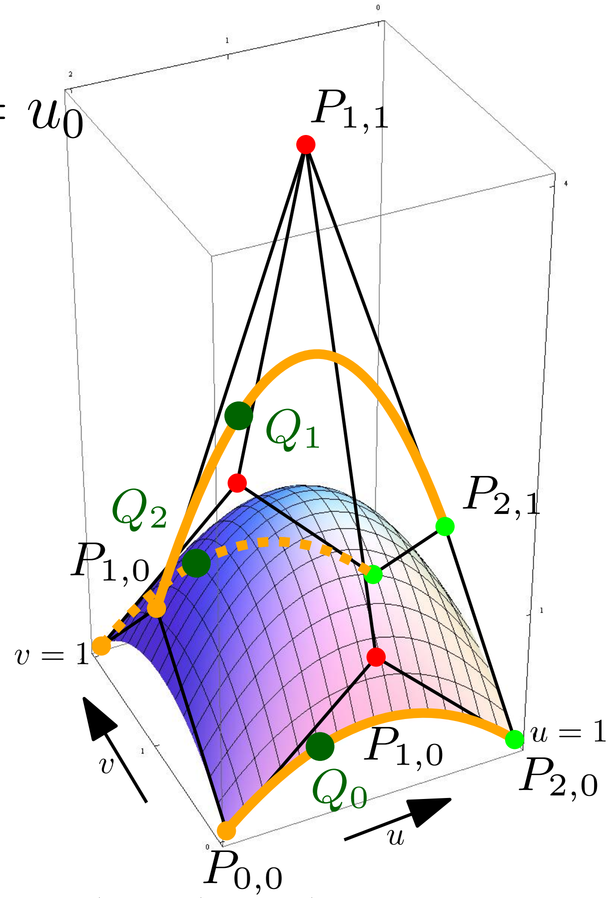


Example on the right, for $u = 0.75$, $n = m = 2$

$$S(0.75, v) = \sum_{j=0}^{2} B_{n,j}(v) \left(\sum_{i=0}^{2} B_{m,i}(u_0) P_{i,j}\right)$$
$$S(0.75, v) = \sum_{j=0}^{2} B_{n,j}(v) \underbrace{\left(B_{m,0}(0.75) P_{0,j} + B_{m,1}(0.75) P_{1,j} + B_{m,2}(0.75) P_{2,j}\right)}_{\text{new control points, for } j = 0,1,2 \text{ (i.e., } Q_0, Q_1, Q_2)}$$

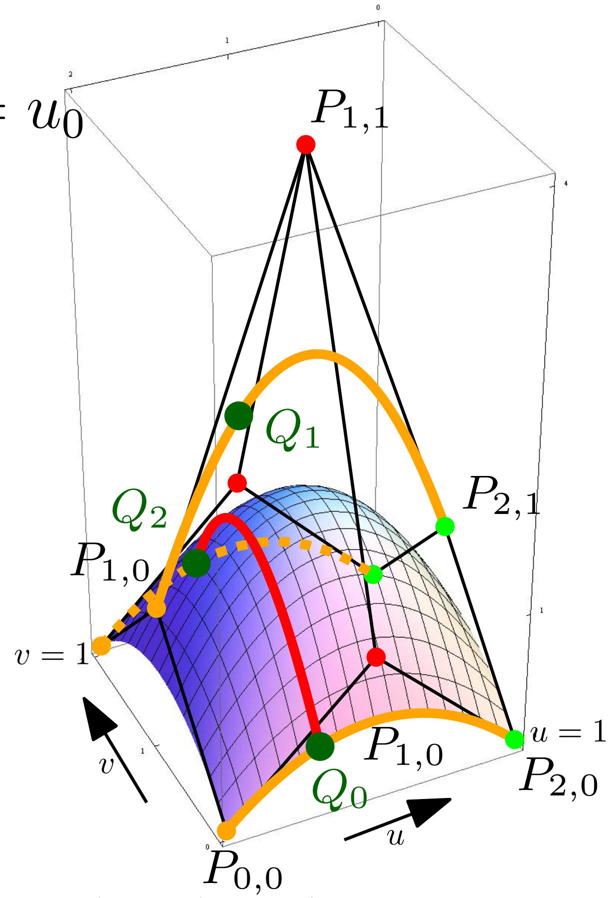
Properties of Bézier surface (on rectangular grid)

- Uniparametric curves are Bézier curves (i.e., for any fixed u or fixed v)

Consider the curve obtained for a fix parameter, e.g., $u = u_0$

Then we have:

$$S(u_0, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u_0) B_{n,j}(v) P_{i,j}$$
$$= \sum_{j=0}^{n} B_{n,j}(v) \underbrace{\left(\sum_{i=0}^{m} B_{m,i}(u_0) P_{i,j}\right)}_{\text{new control point } Q_j}$$



Example on the right, for $u = 0.75$, $n = m = 2$

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Properties of Bézier surface (on rectangular grid)

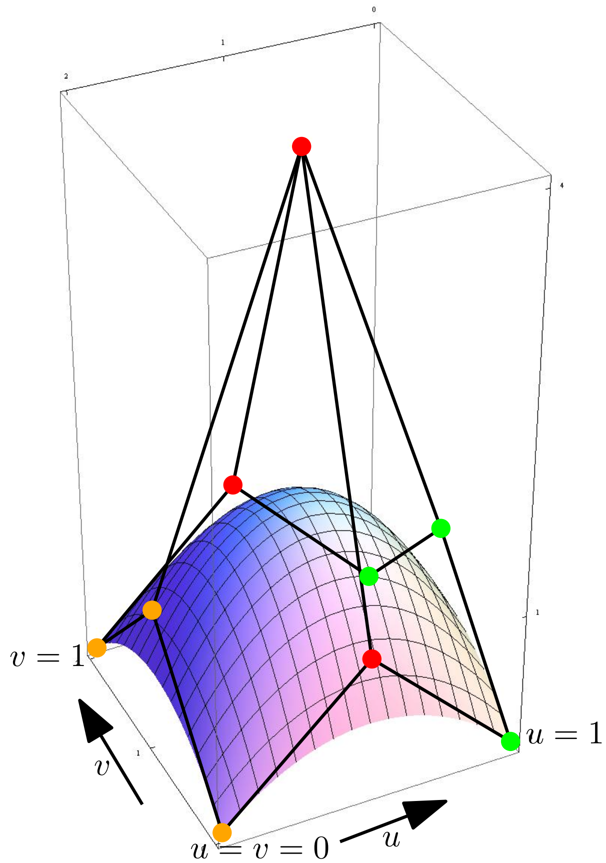
$$S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) P_{i,j}$$

- Uniparametric curves (i.e., fixed u or fixed v)

  All uniparametric curves are Bézier curves

- Affine invariance

  $\sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) = 1$ for all (u, v)



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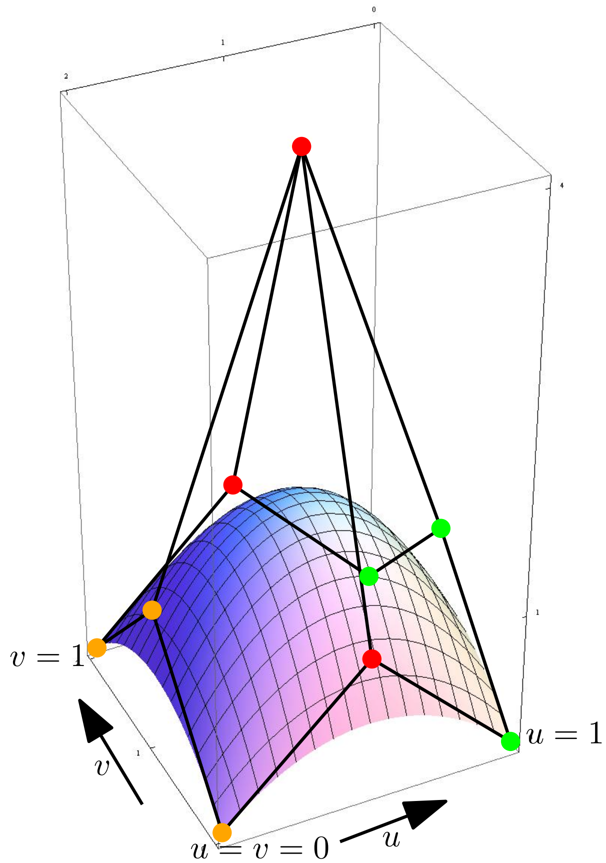

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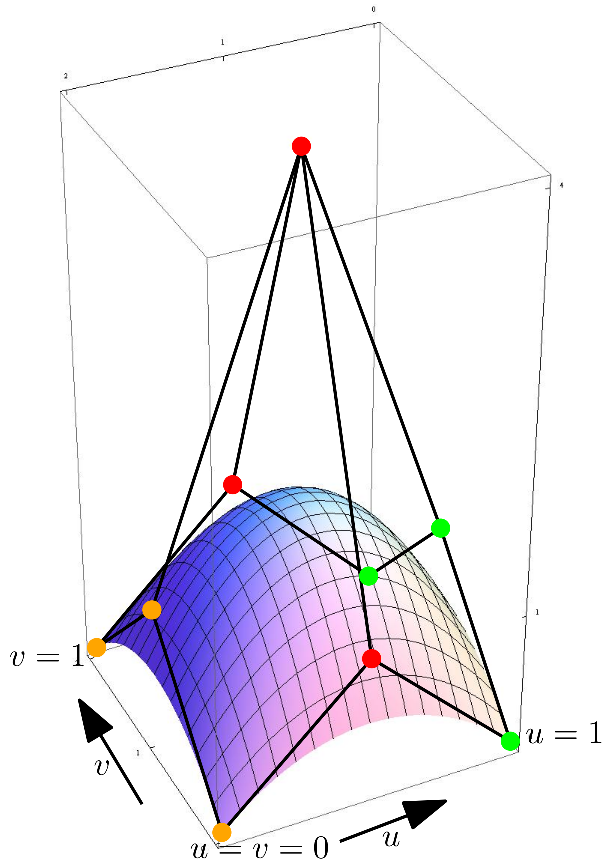
  All uniparametric curves are Bézier curves

- Affine invariance

  $\sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) = 1$ for all (u, v)

- Convex hull property

- No variation diminishing property



Smooth connection of rectangular Bézier patches

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- Same idea as for curves

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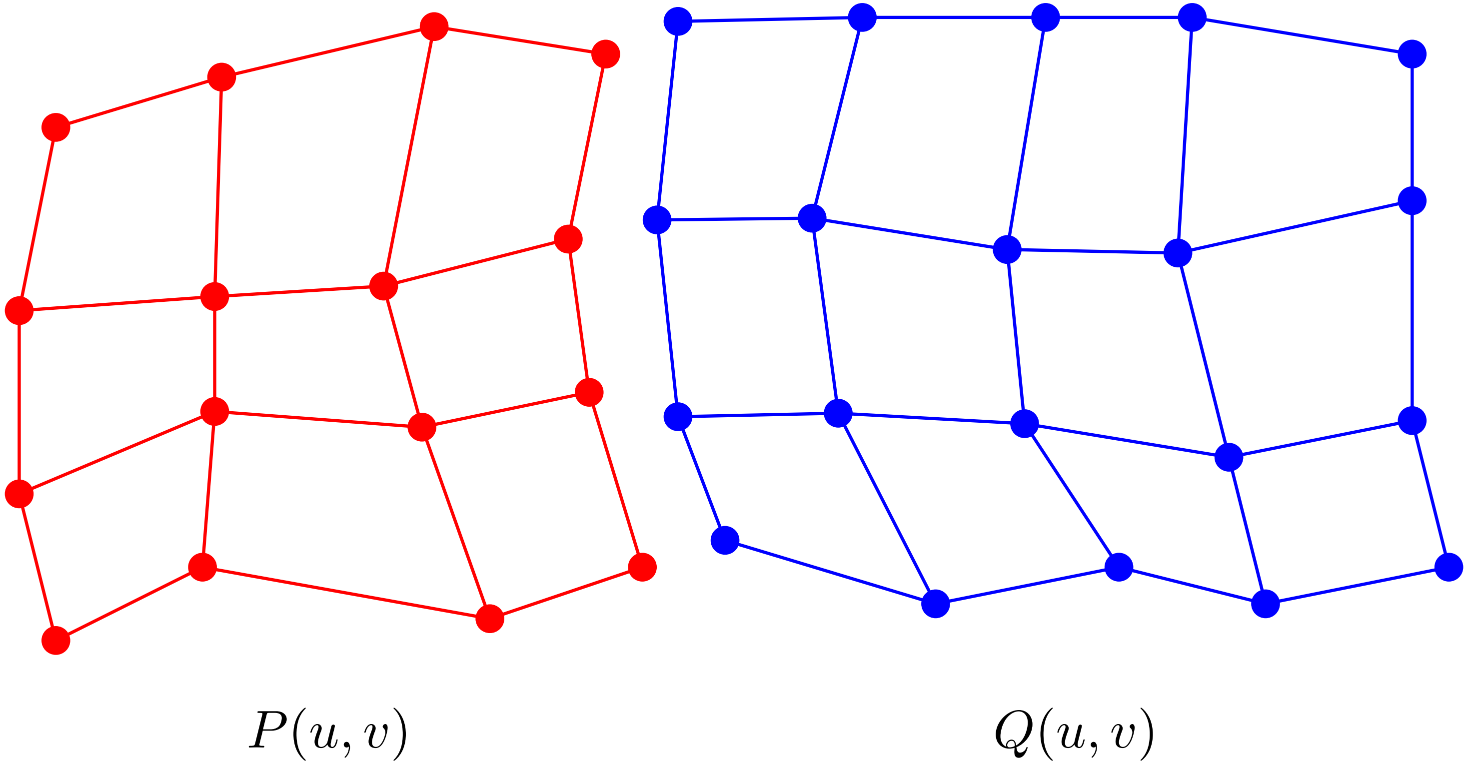
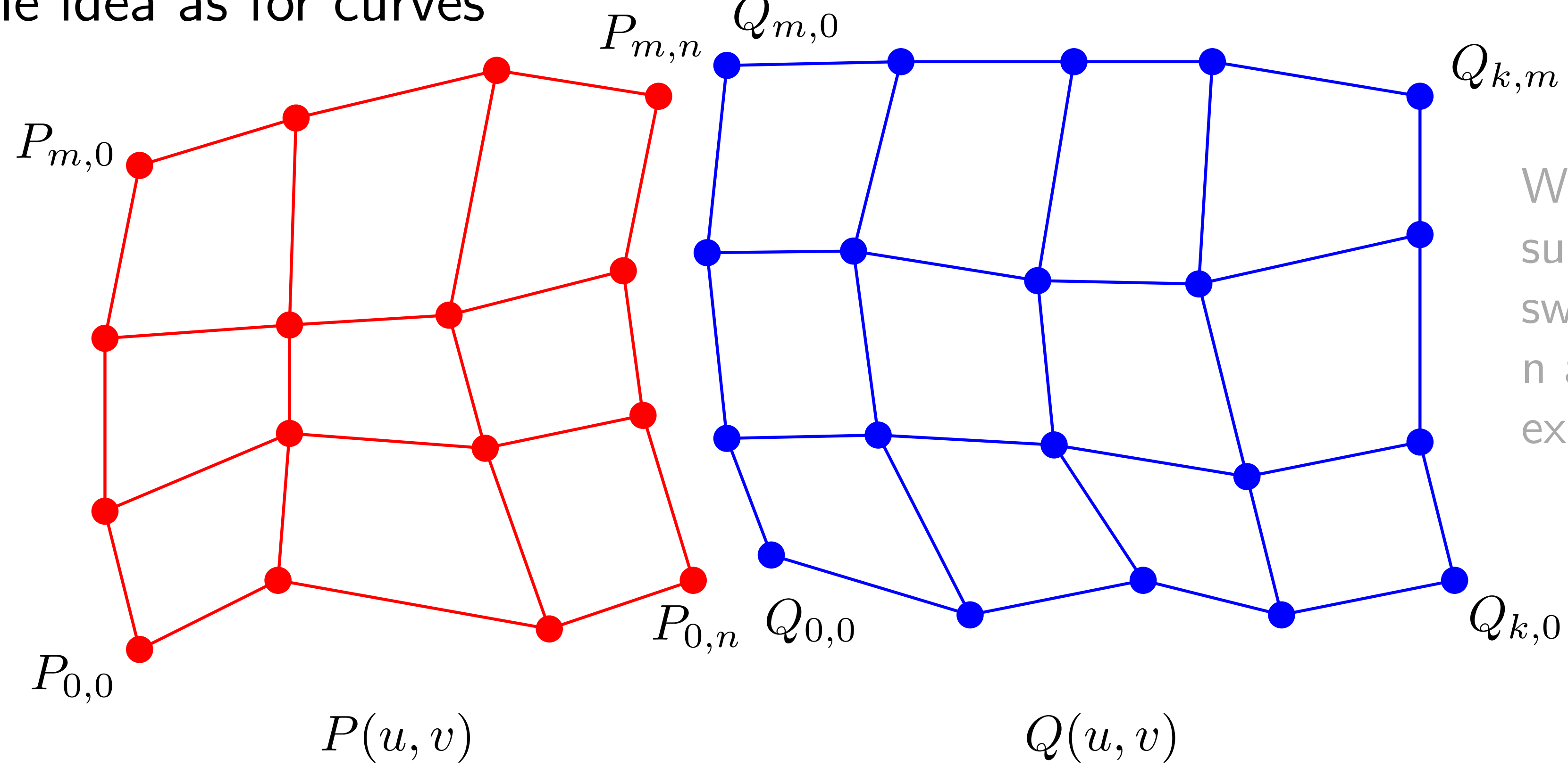


Smooth connection of rectangular Bézier patches

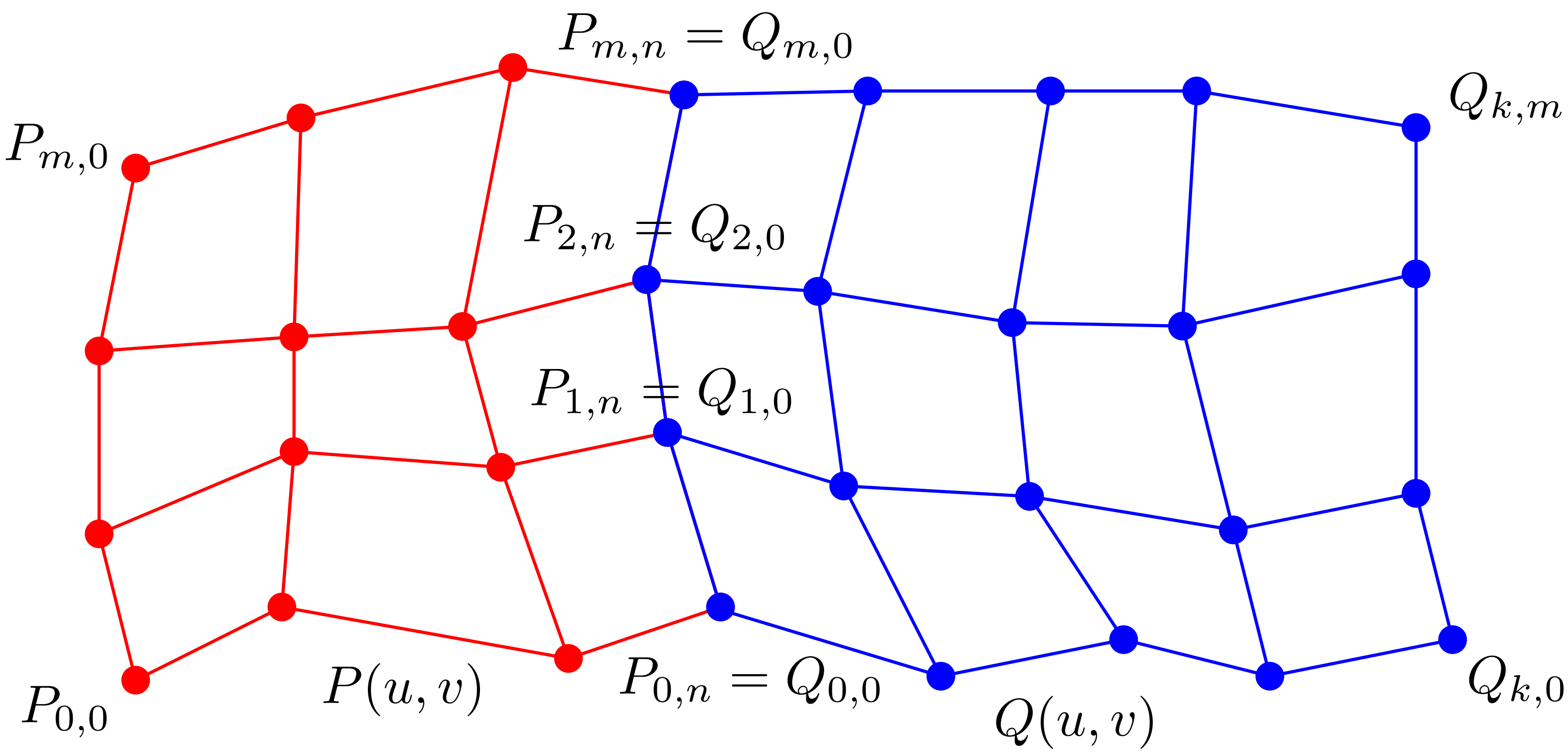
- Same idea as for curves



Warning: subindices swapped, and n and m exchanged

Smooth connection of rectangular Bézier patches

- Same idea as for curves



Smooth connection of rectangular Bézier patches

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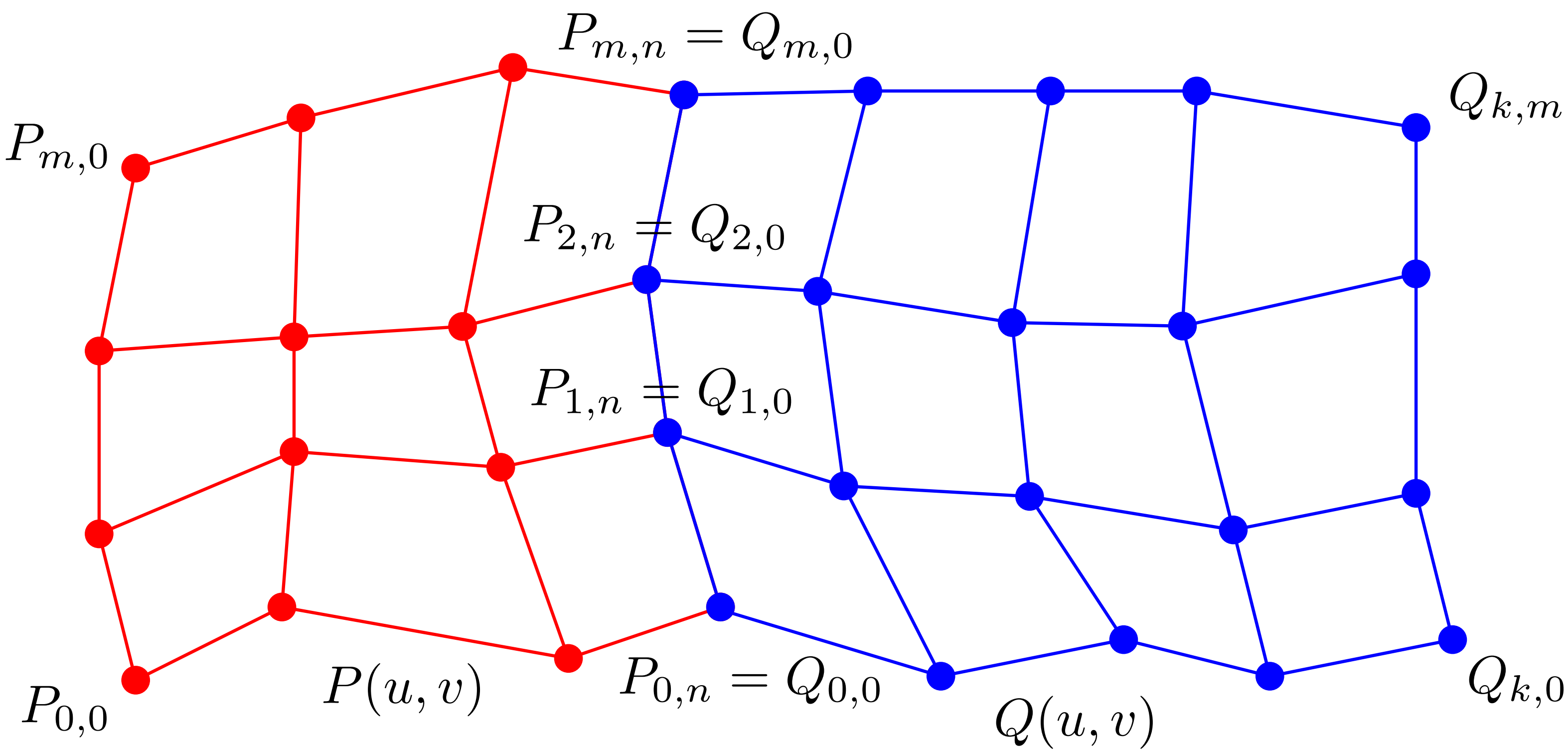
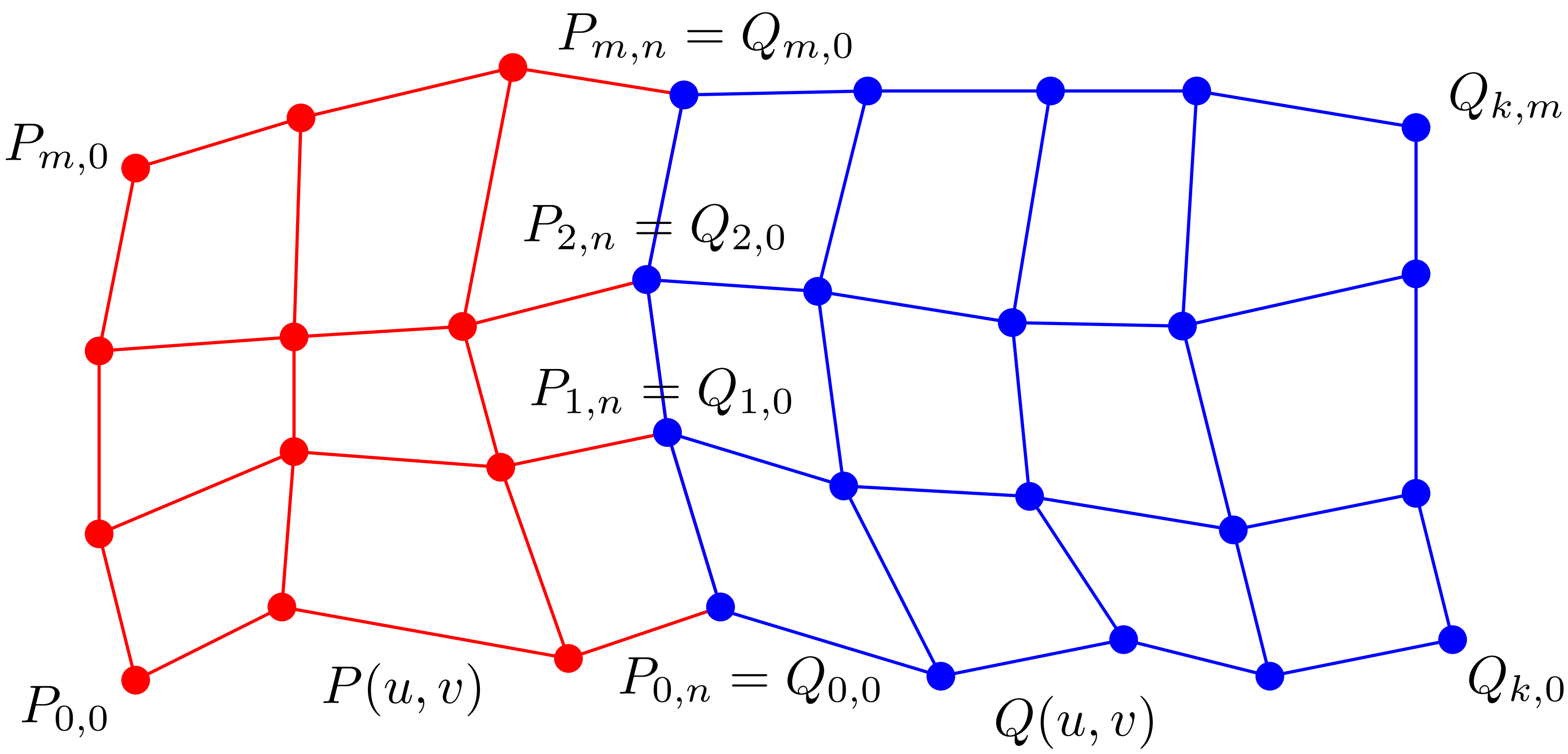


- Continuity (C^0-cont) $\quad P_{i,n} = Q_{i,0} \quad \forall i = 0, \ldots, m$

Smooth connection of rectangular Bézier patches

- Same idea as for curves



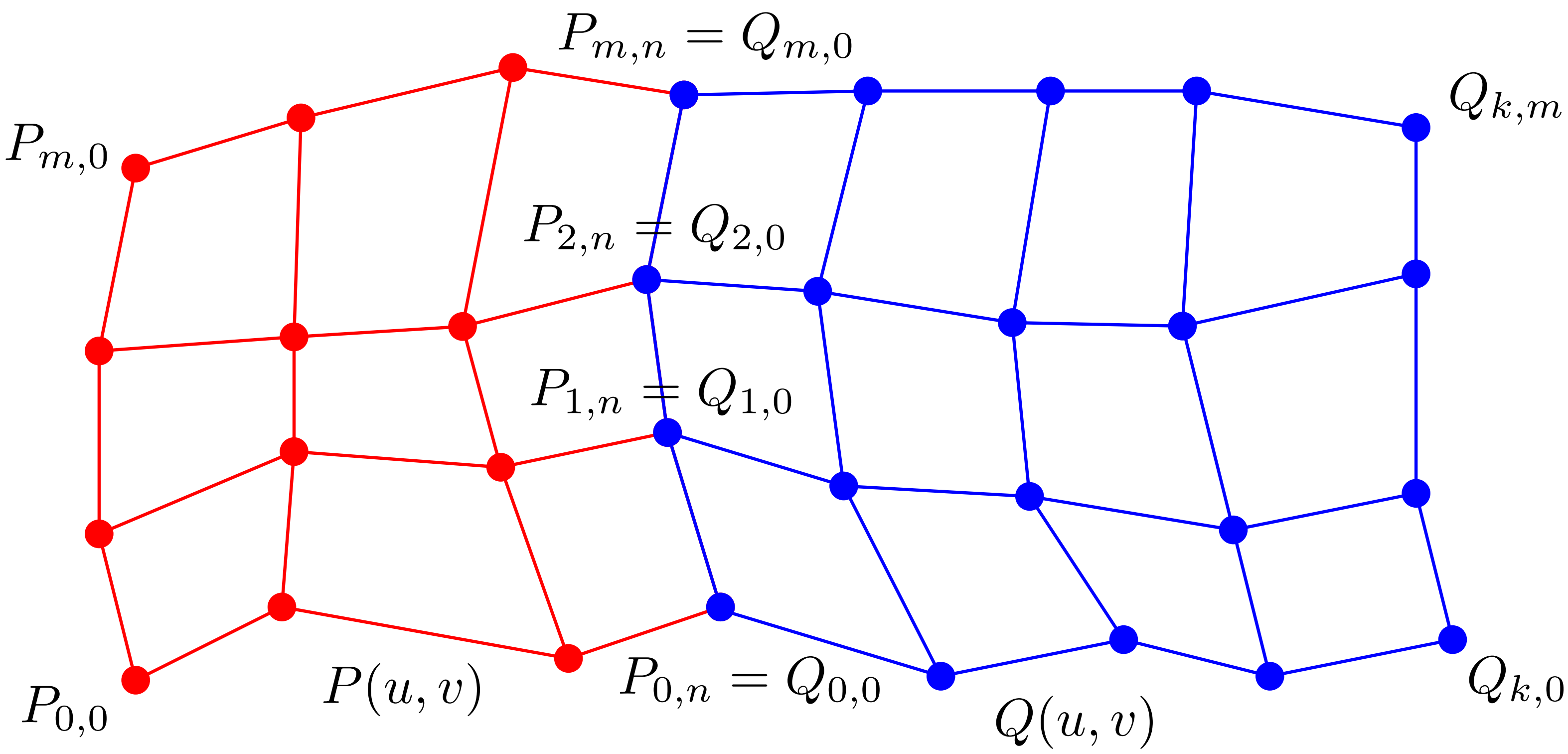
- Continuity (C^0-cont)

$$P_{i,n} = Q_{i,0} \quad \forall i = 0, \ldots, m$$

- Smoothness (C^1-cont)

Smooth connection of rectangular Bézier patches

- Same idea as for curves



- Continuity (C^0-cont) $\qquad P_{i,n} = Q_{i,0} \quad \forall i = 0, \ldots, m$
- Smoothness (C^1-cont) $\qquad \left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0}$

CONNECTING BEZIER SURFACES

Smoothness condition (C^1-continuity)

$$\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0}$$

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$$\frac{\partial P(u,v)}{\partial u}\bigg|_{u=1} = \sum_{i=0}^{m} \binom{m}{i} n v^i (1-v)^{m-i} P_{i,n} - \sum_{i=0}^{m} \binom{m}{i} n v^i (1-v)^{m-i} P_{i,n-1}$$

Smoothness condition (C^1-continuity)

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$$= n \sum_{i=0}^{m} \binom{m}{i} v^i (1-v)^{m-i} (P_{i,n} - P_{i,n-1})$$

Smoothness condition (C^1-continuity)

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$$\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \sum_{i=0}^{m}\binom{m}{i}nv^i(1-v)^{m-i}P_{i,n} - \sum_{i=0}^{m}\binom{m}{i}nv^i(1-v)^{m-i}P_{i,n-1}$$

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Analogously,

$$\left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0} = k\sum_{i=0}^{m}\binom{m}{i}v^i(1-v)^{m-i}(Q_{i,1} - Q_{i,0})$$

Smoothness condition (C^1-continuity)

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$$P(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) P_{i,j} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} u^i (1-u)^{m-i} \binom{n}{j} v^j (1-v)^{n-j} P_{i,j}$$

$$\begin{aligned}\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} &= \sum_{i=0}^{m} \binom{m}{i} n v^i (1-v)^{m-i} P_{i,n} - \sum_{i=0}^{m} \binom{m}{i} n v^i (1-v)^{m-i} P_{i,n-1} \\ &= n \sum_{i=0}^{m} \binom{m}{i} v^i (1-v)^{m-i} (P_{i,n} - P_{i,n-1})\end{aligned}$$

Analogously,

$$\left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0} = k \sum_{i=0}^{m} \binom{m}{i} v^i (1-v)^{m-i} (Q_{i,1} - Q_{i,0})$$

Therefore, the condition for C^1-continuity is: $n(P_{i,n} - P_{i,n-1}) = k(Q_{i,1} - Q_{i,0}) \;\; \forall i = 0, \ldots, m$

Smoothness condition (C^1-continuity)

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$$\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \sum_{i=0}^{m}\binom{m}{i}nv^i(1-v)^{m-i}P_{i,n} - \sum_{i=0}^{m}\binom{m}{i}nv^i(1-v)^{m-i}P_{i,n-1}$$

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Analogously,

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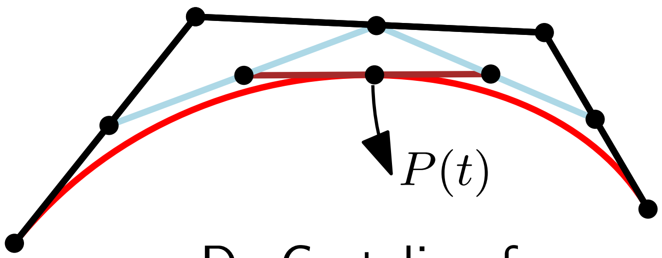
Therefore, the condition for C^1-continuity is: $\boxed{n(P_{i,n} - P_{i,n-1}) = k(Q_{i,1} - Q_{i,0}) \quad \forall i = 0, \ldots, m}$

If we just want G^1-cont, it is enough with $P_{i,n} - P_{i,n-1} = \alpha(Q_{i,1} - Q_{i,0})$, for some $\alpha \neq 0 \in \mathbb{R}$

Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves

Applying De Casteljau to each dimension

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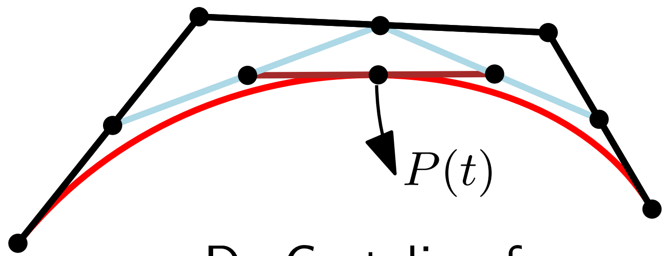

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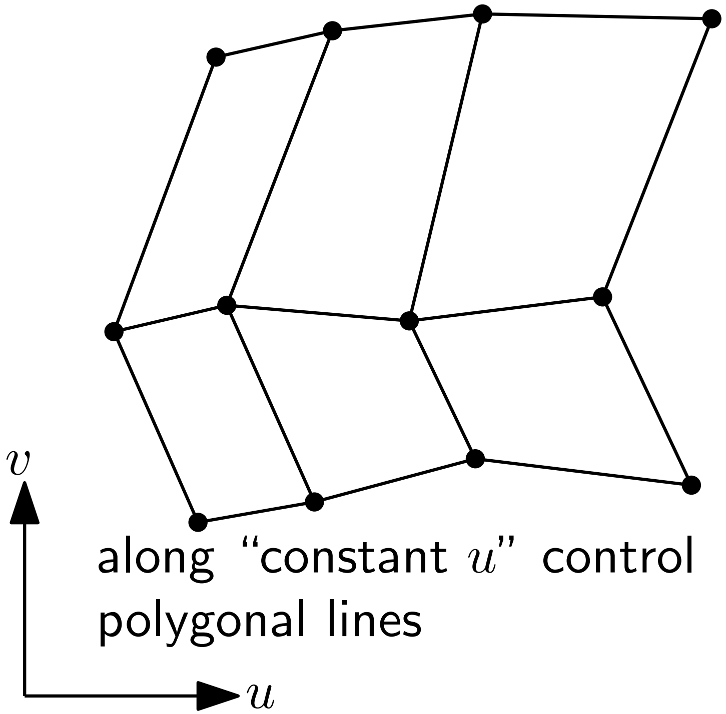

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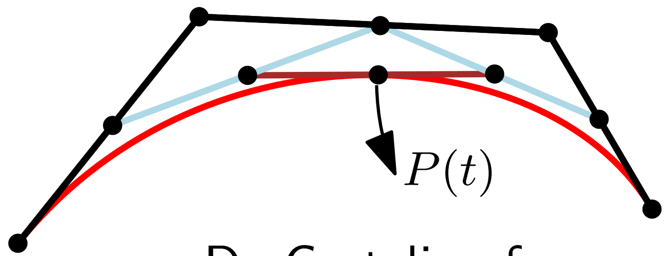

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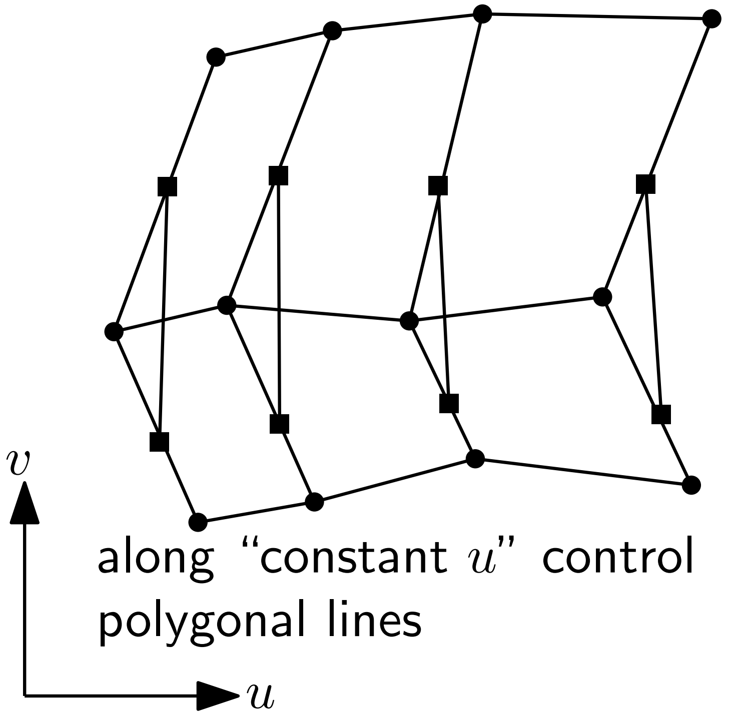
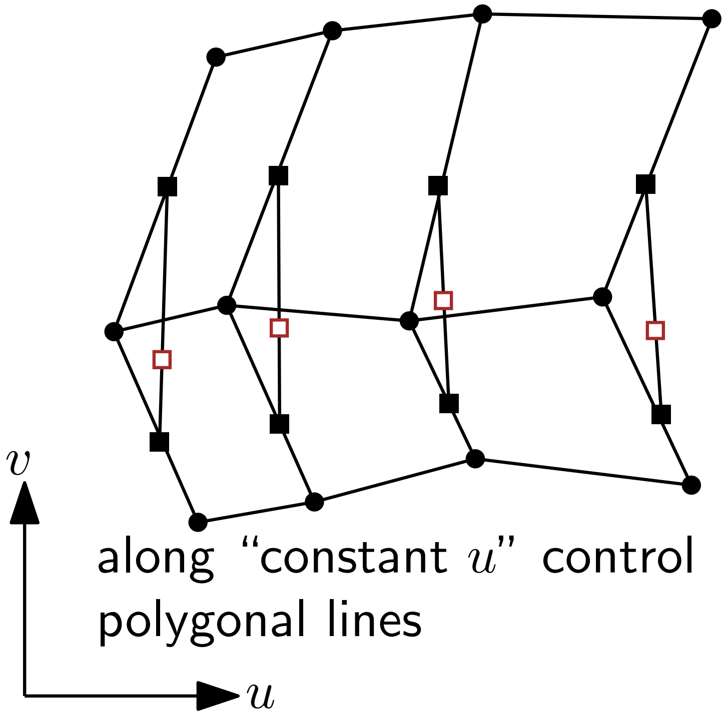


along "constant u" control polygonal lines

Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)



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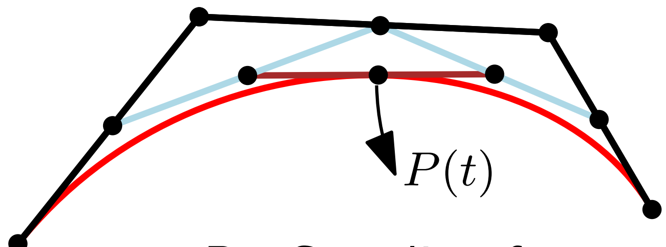

De Casteljau for curves

Applying De Casteljau to each dimension

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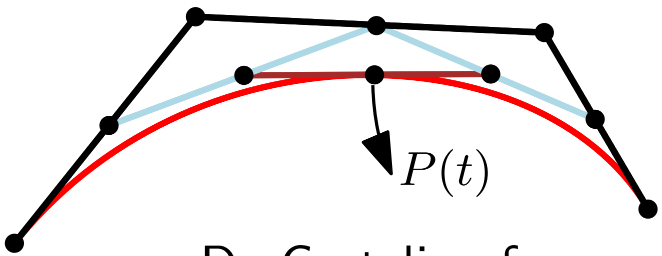

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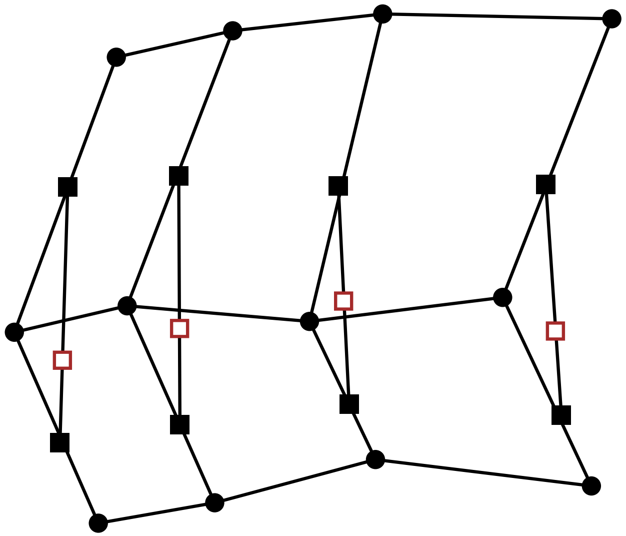

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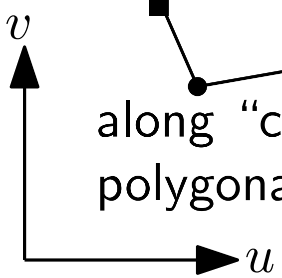
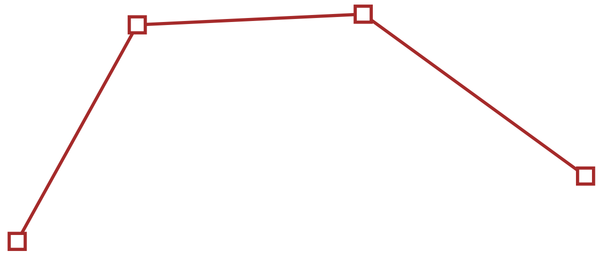
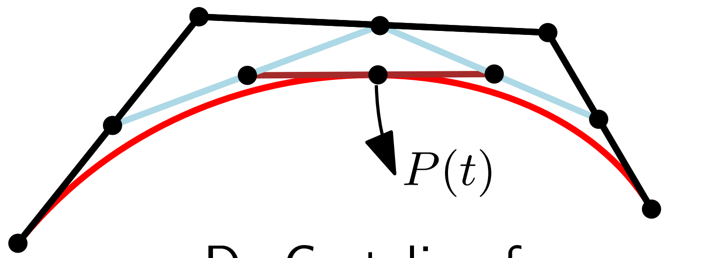




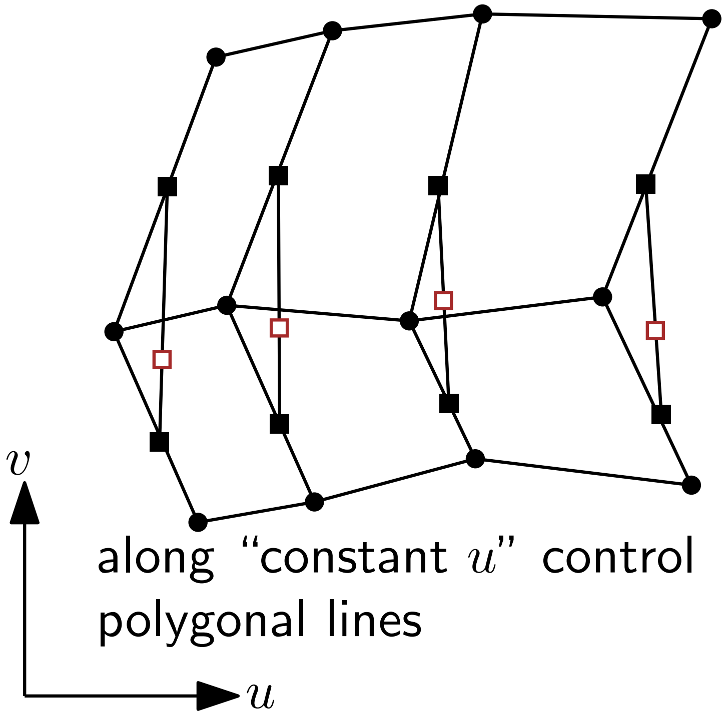
along one "constant v" polygon

Applying De Casteljau to each dimension

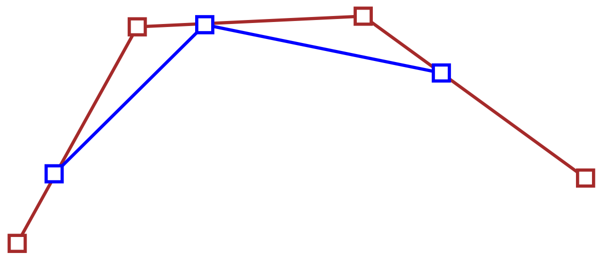
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves

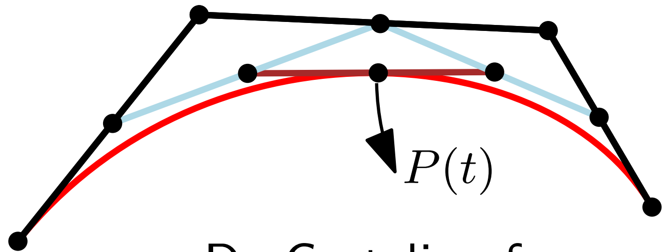


along "constant u" control polygonal lines



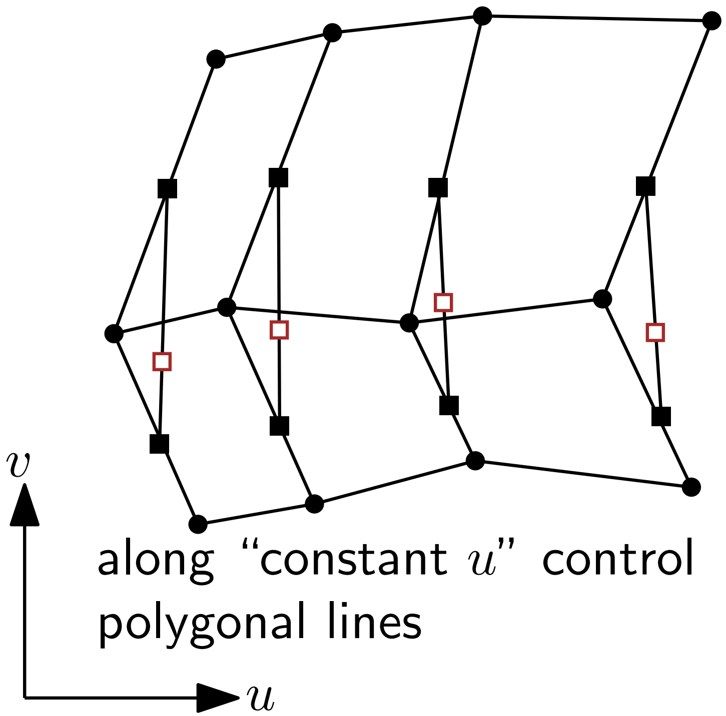
along one "constant v" polygon

Applying De Casteljau to each dimension

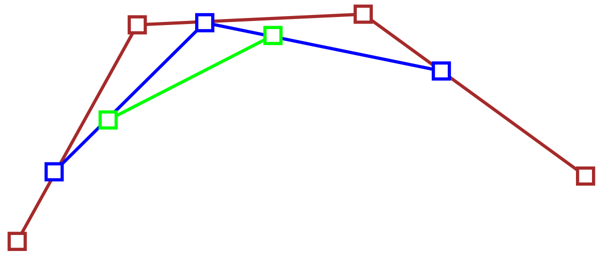


De Casteljau for curves

- For surfaces: apply it in two phases (along u, and along v)



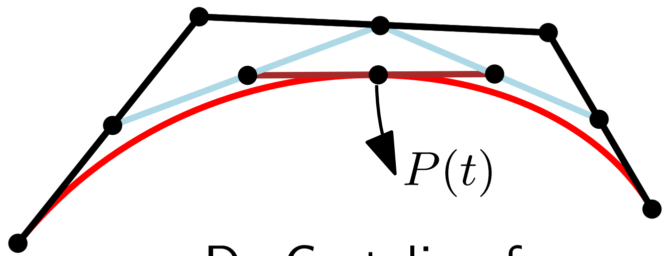
along "constant u" control polygonal lines



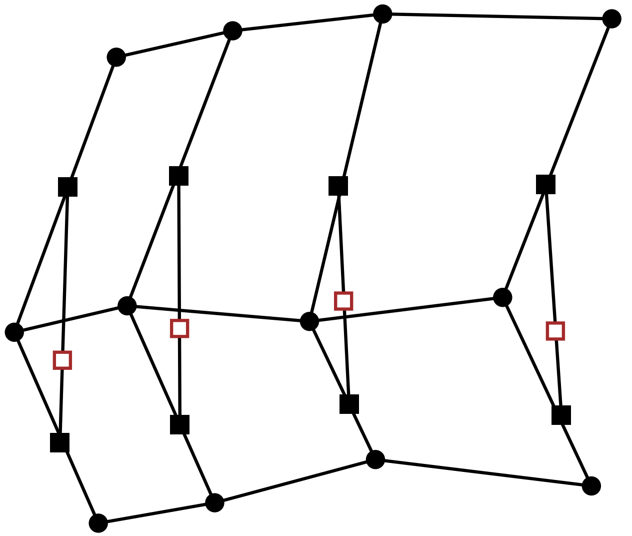
along one "constant v" polygon

Applying De Casteljau to each dimension

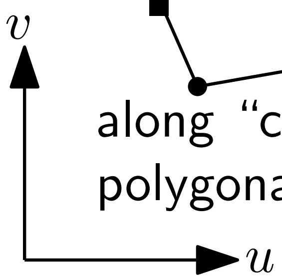
- For surfaces: apply it in two phases (along u, and along v)

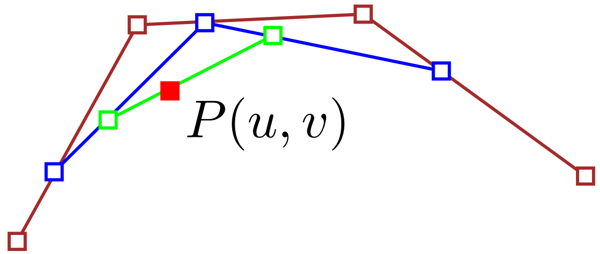


De Casteljau for curves



along "constant u" control polygonal lines

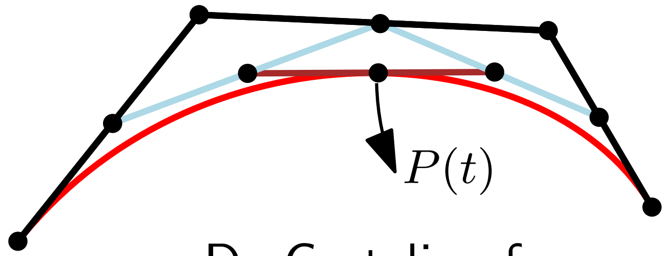




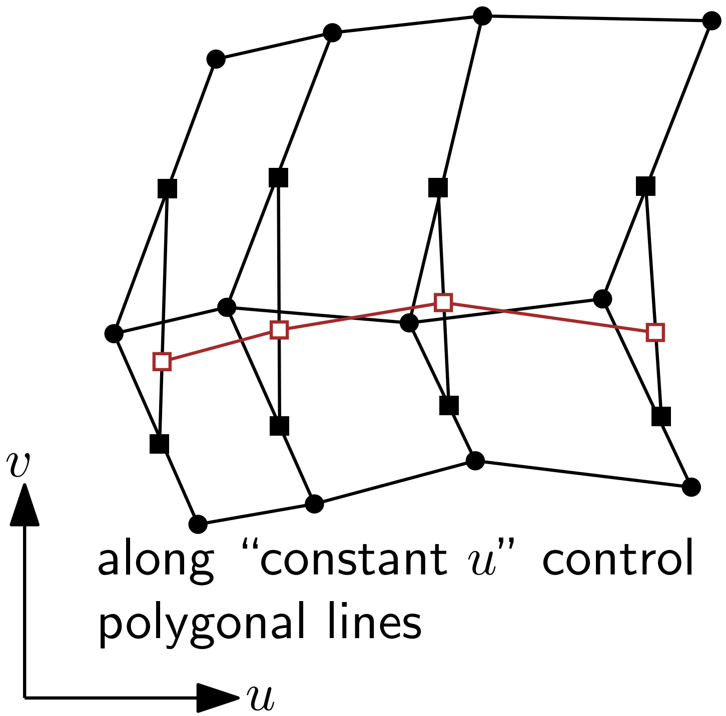
along one "constant v" polygon

Applying De Casteljau to each dimension

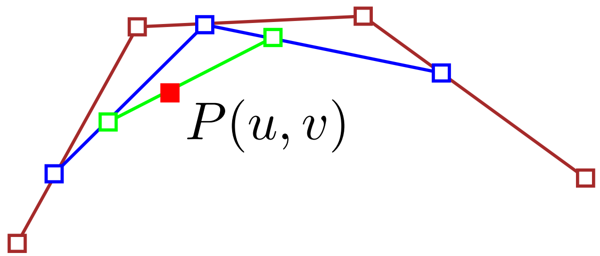
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves



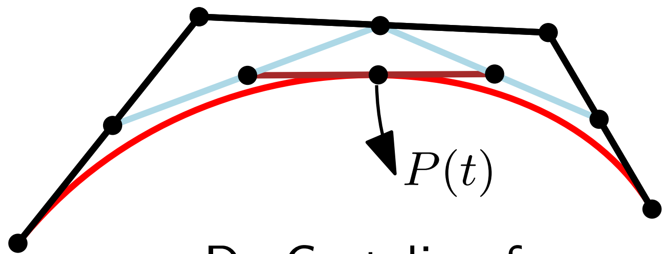
along "constant u" control polygonal lines



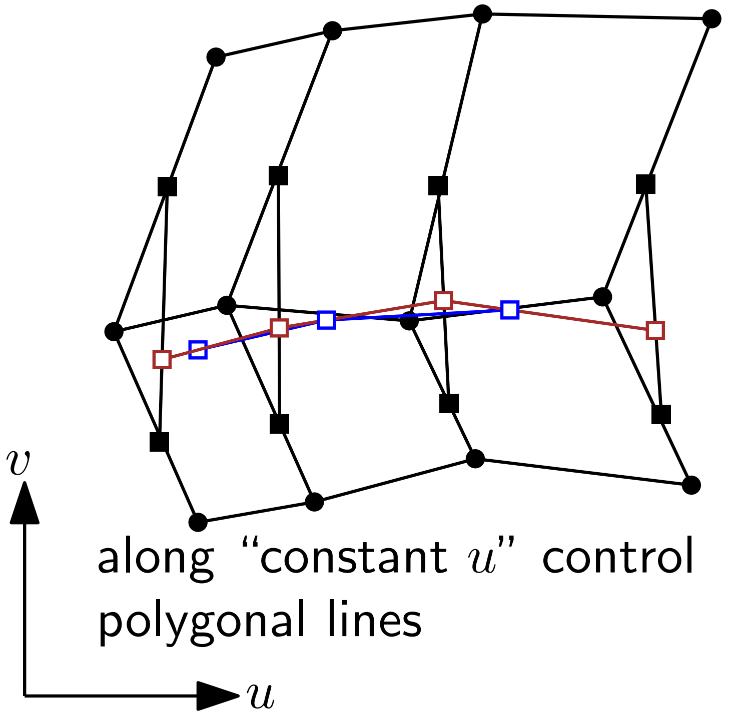
along one "constant v" polygon

Applying De Casteljau to each dimension

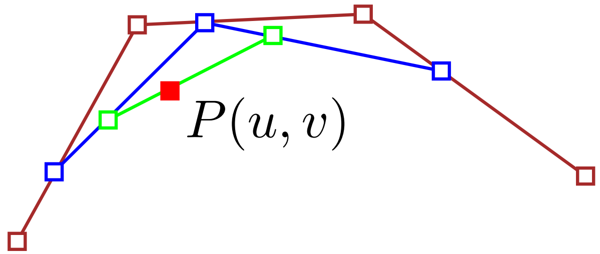
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves



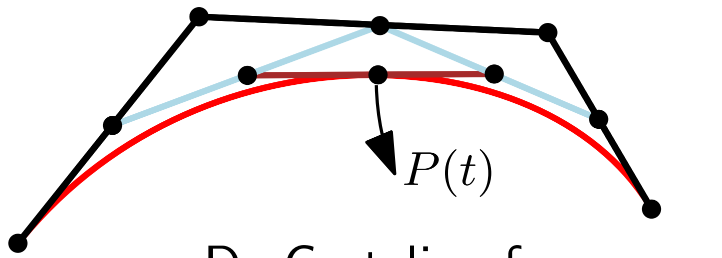
along "constant u" control polygonal lines



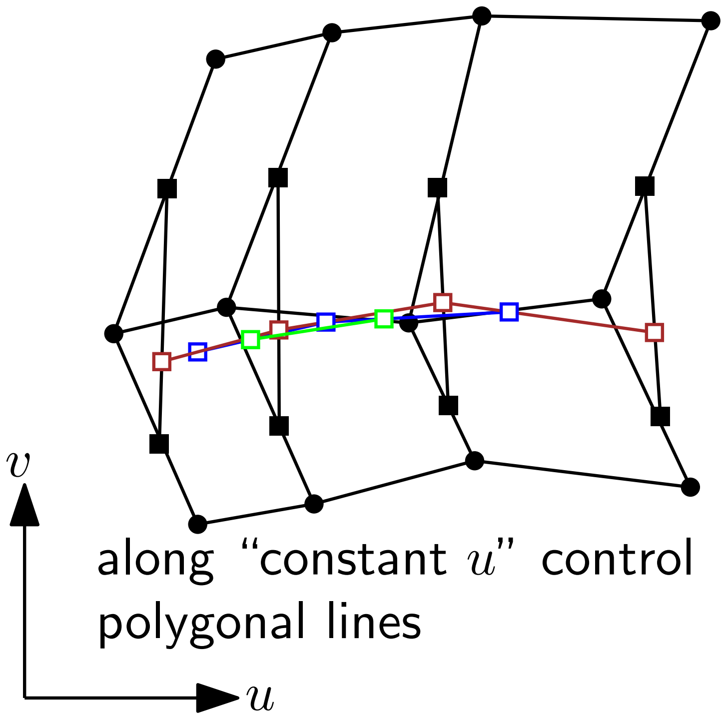
along one "constant v" polygon

Applying De Casteljau to each dimension

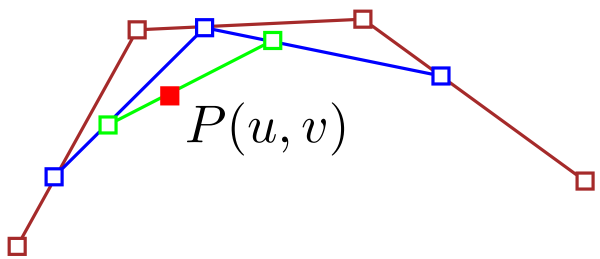
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves



along "constant u" control polygonal lines



along one "constant v" polygon

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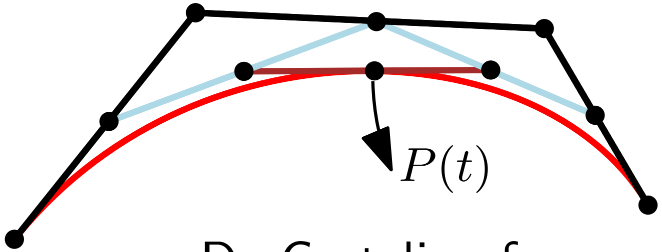
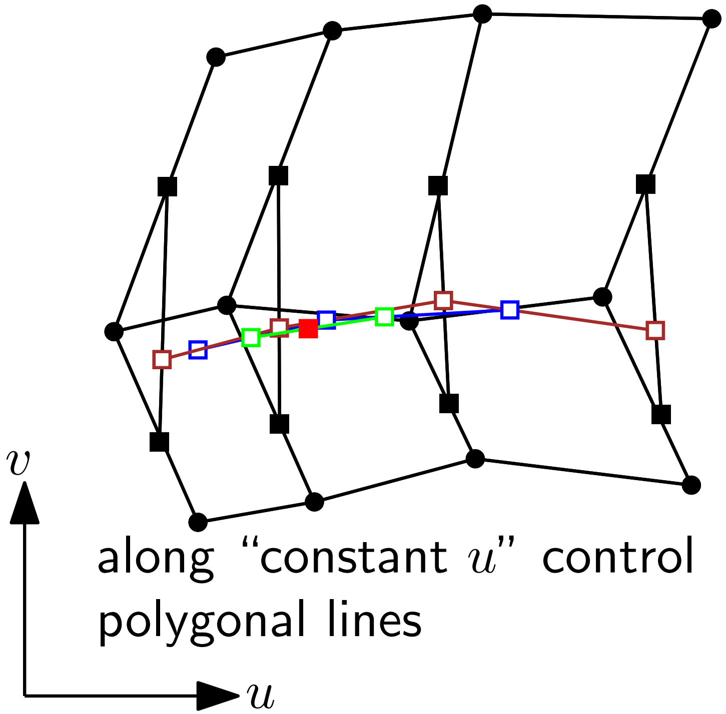
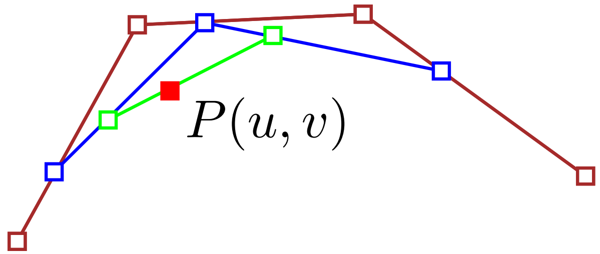


De Casteljau for curves



along "constant u" control polygonal lines



along one "constant v" polygon

- Applications of De Casteljau, e.g., to curve subdivision, also extend to surfaces

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

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Interpolating Bézier surface patch

Example from [Salomon, page 232]:

Example: We choose $m = 3$ and $n = 2$. The system of equations becomes

$$[(1-u_k)^3, 3u_k(1-u_k)^2, 3u_k^2(1-u_k), u_k^3] \begin{bmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22} \\ \mathbf{P}_{30} & \mathbf{P}_{31} & \mathbf{P}_{32} \end{bmatrix} \begin{bmatrix} (1-w_l)^2 \\ 2w_l(1-w_l) \\ w_l^2 \end{bmatrix} = \mathbf{Q}_{kl}$$

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12 equations, 12 unknowns

Rational rectangular Bézier surface patch

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Definition anologous to the one for curves

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$$P(u,w) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} B_{m,i}(u) B_{n,j}(w) P_{i,j}}{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{i,j} B_{m,i}(u) B_{n,j}(w)}$$

$0 \leq u, w \leq 1$

$w_{i,j} \in \mathbb{R}_{>0}$ for all i, j

If all weights are $w_{i,j} = 1$, it reduces to the ordinary Bézier surface

Surface patches don't need to be rectangular

Control points arranged as triangular array

$$\begin{array}{c}\mathbf{P}_{010}\\ \mathbf{P}_{001}\ \mathbf{P}_{100}\end{array}\quad \begin{array}{c}\mathbf{P}_{020}\\ \mathbf{P}_{011}\ \mathbf{P}_{110}\\ \mathbf{P}_{002}\ \mathbf{P}_{101}\ \mathbf{P}_{200}\end{array}\quad \begin{array}{c}\mathbf{P}_{030}\\ \mathbf{P}_{021}\ \mathbf{P}_{120}\\ \mathbf{P}_{012}\ \mathbf{P}_{111}\ \mathbf{P}_{210}\\ \mathbf{P}_{003}\ \mathbf{P}_{102}\ \mathbf{P}_{201}\ \mathbf{P}_{300}\end{array}\quad \begin{array}{c}\mathbf{P}_{040}\\ \mathbf{P}_{031}\ \mathbf{P}_{130}\\ \mathbf{P}_{022}\ \mathbf{P}_{121}\ \mathbf{P}_{220}\\ \mathbf{P}_{013}\ \mathbf{P}_{112}\ \mathbf{P}_{211}\ \mathbf{P}_{310}\\ \mathbf{P}_{004}\ \mathbf{P}_{103}\ \mathbf{P}_{202}\ \mathbf{P}_{301}\ \mathbf{P}_{400}\end{array}$$

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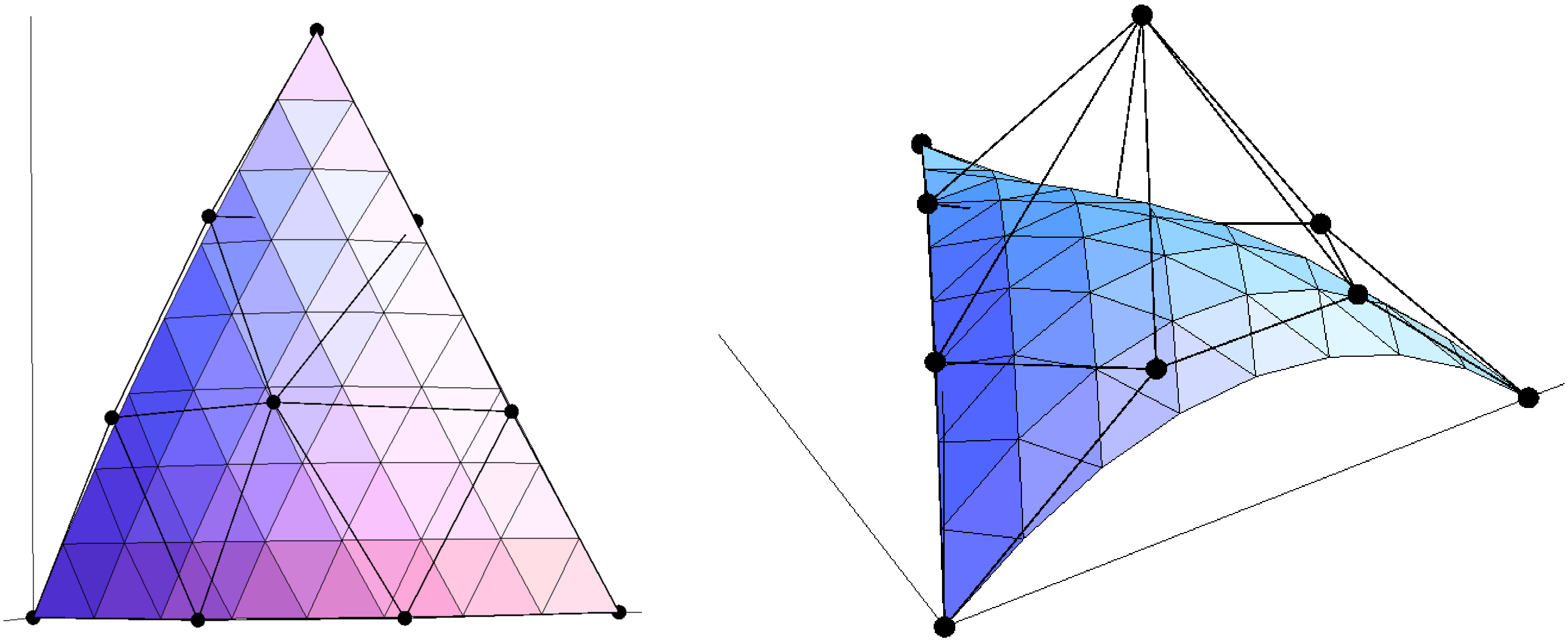
Bézier formula needs version based on three variables

$$\mathbf{P}(u,v,w) = \sum_{i+j+k=n} \mathbf{P}_{ijk} \frac{n!}{i!\,j!\,k!} u^i v^j w^k = \sum_{i+j+k=n} \mathbf{P}_{ijk} B^n_{ijk}(u,v,w)$$

with $u + v + w = 1$ (so there are only two degrees of freedom)

First Bézier surface, developed by De Casteljau in 1959!

Example



$n = 3$