SURFACES

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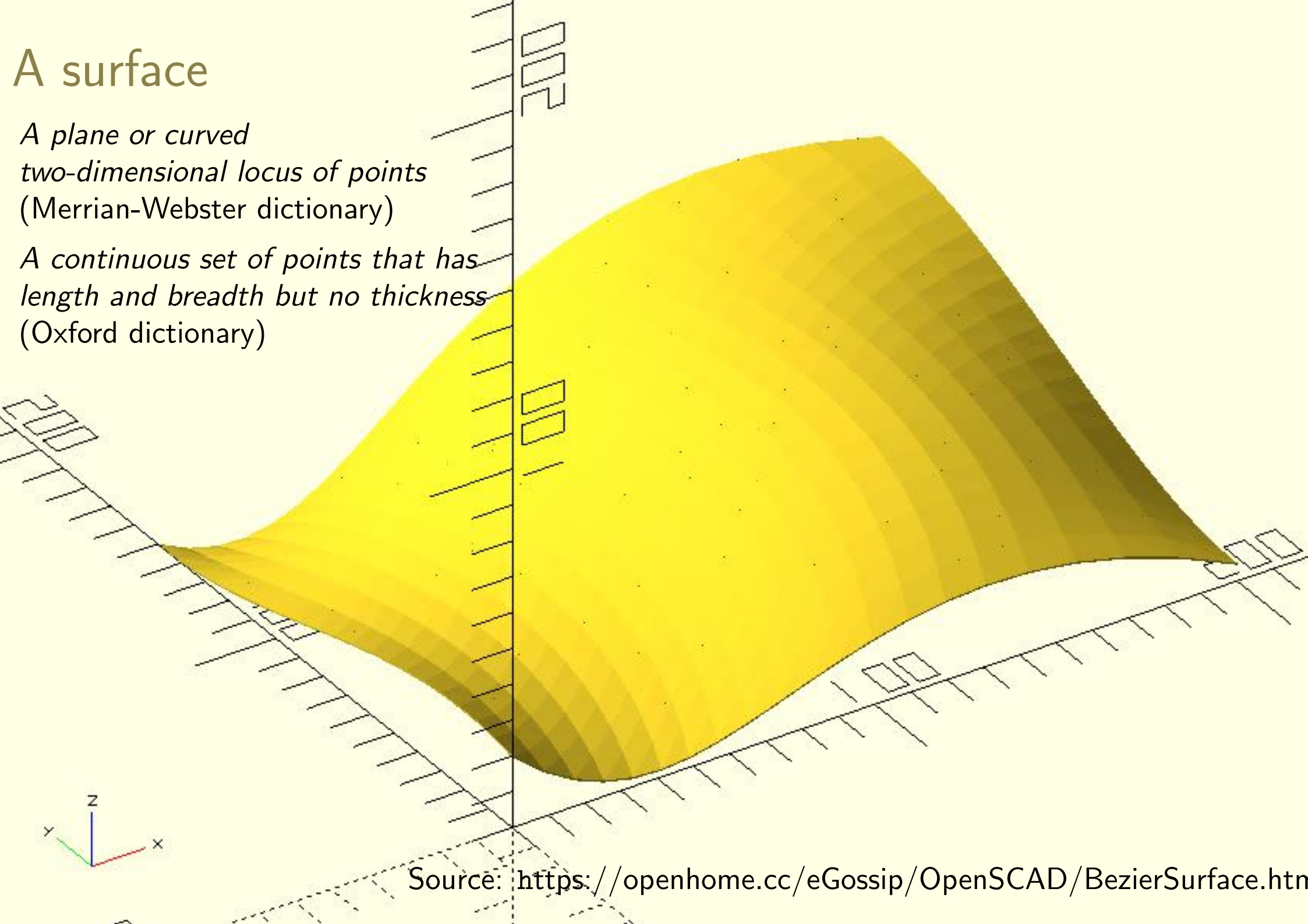
A surface

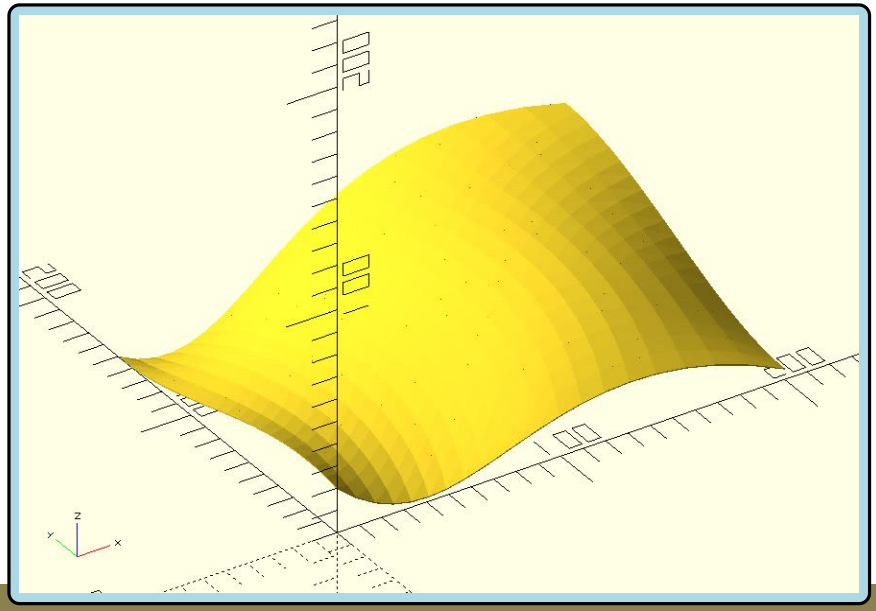
A plane or curved two-dimensional locus of points
(Merrian-Webster dictionary)

A continuous set of points that has length and breadth but no thickness
(Oxford dictionary)



Source: https://openhome.cc/eGossip/OpenSCAD/BezierSurface.htm

PARAMETRIZING SURFACES



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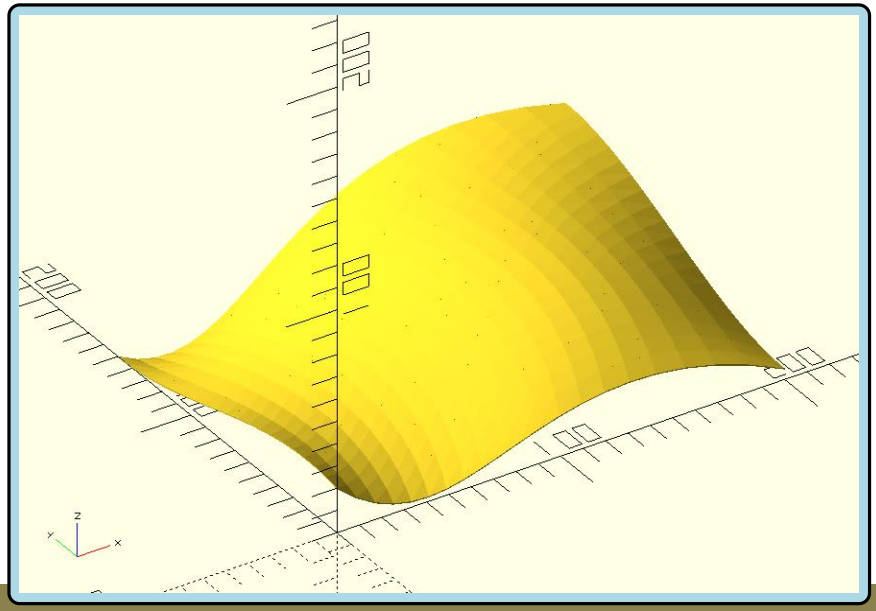
1) Explicit equation



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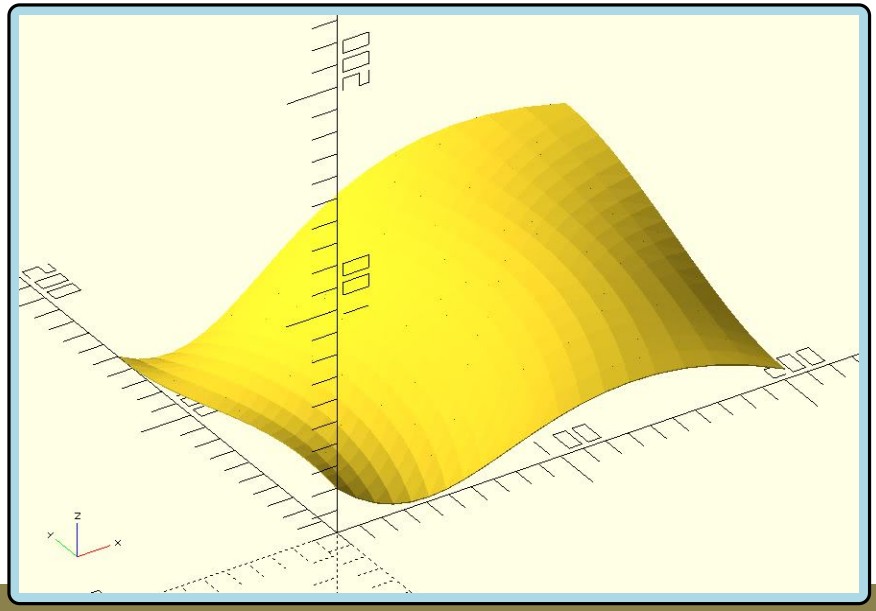

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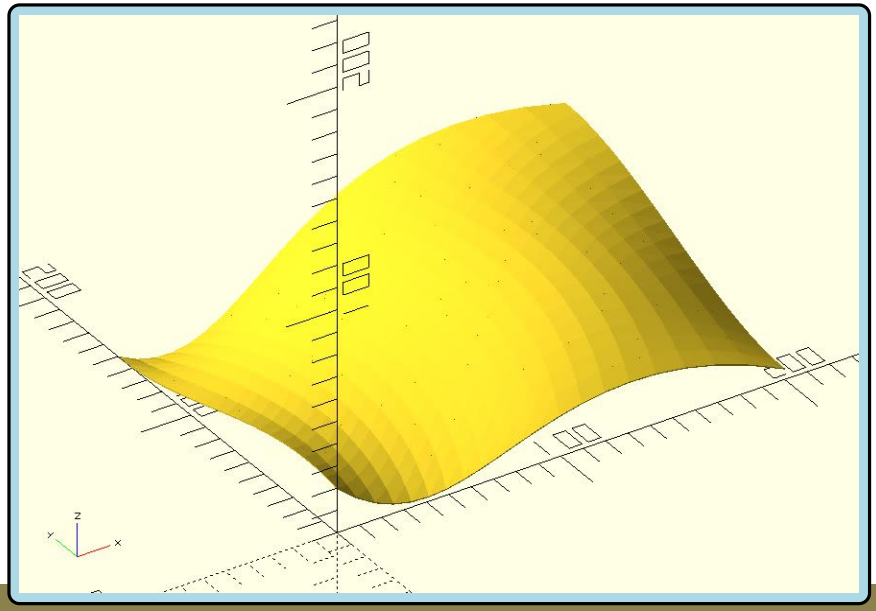

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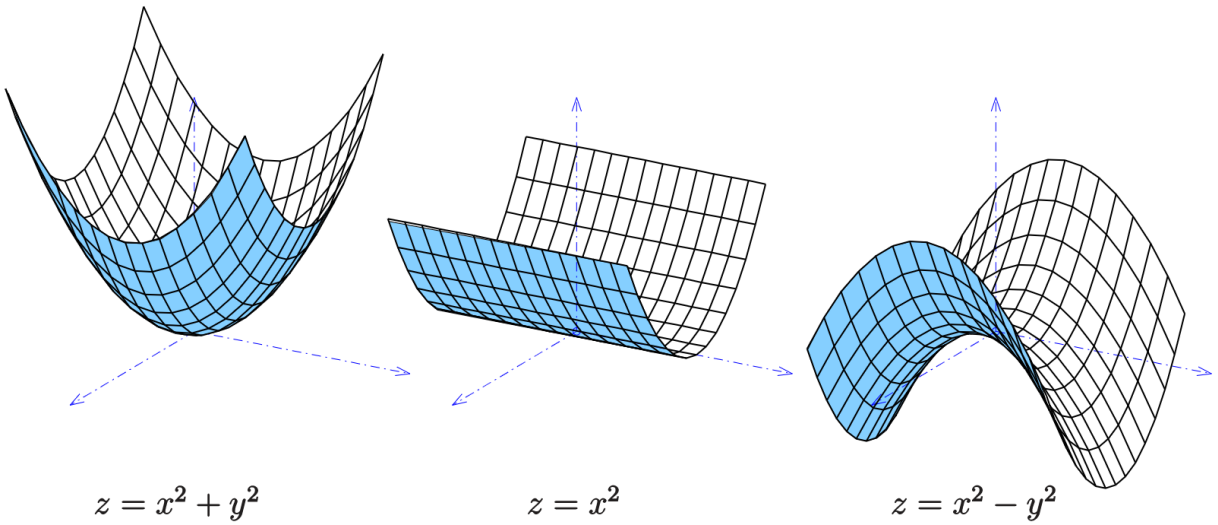
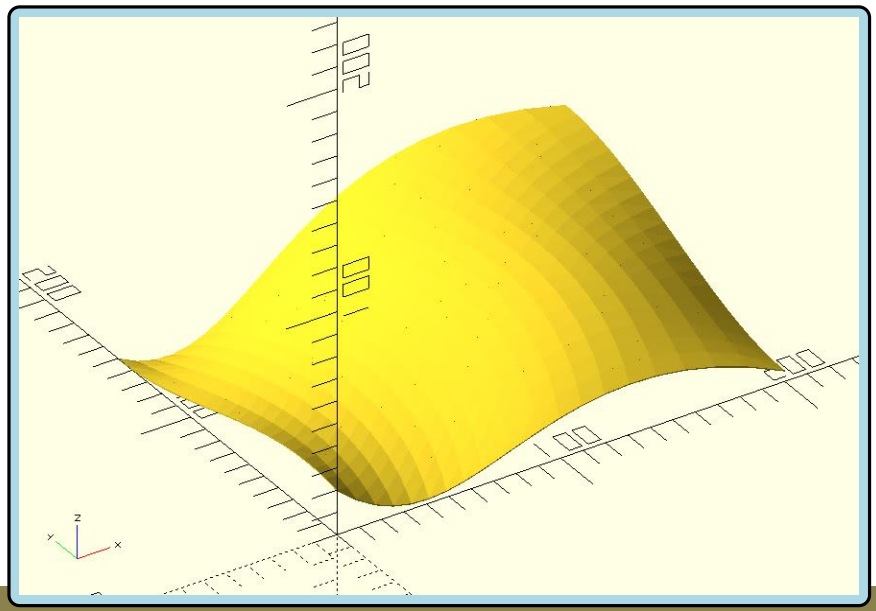


[From Wikimedia commons - by Ag2gaeh - Own work]



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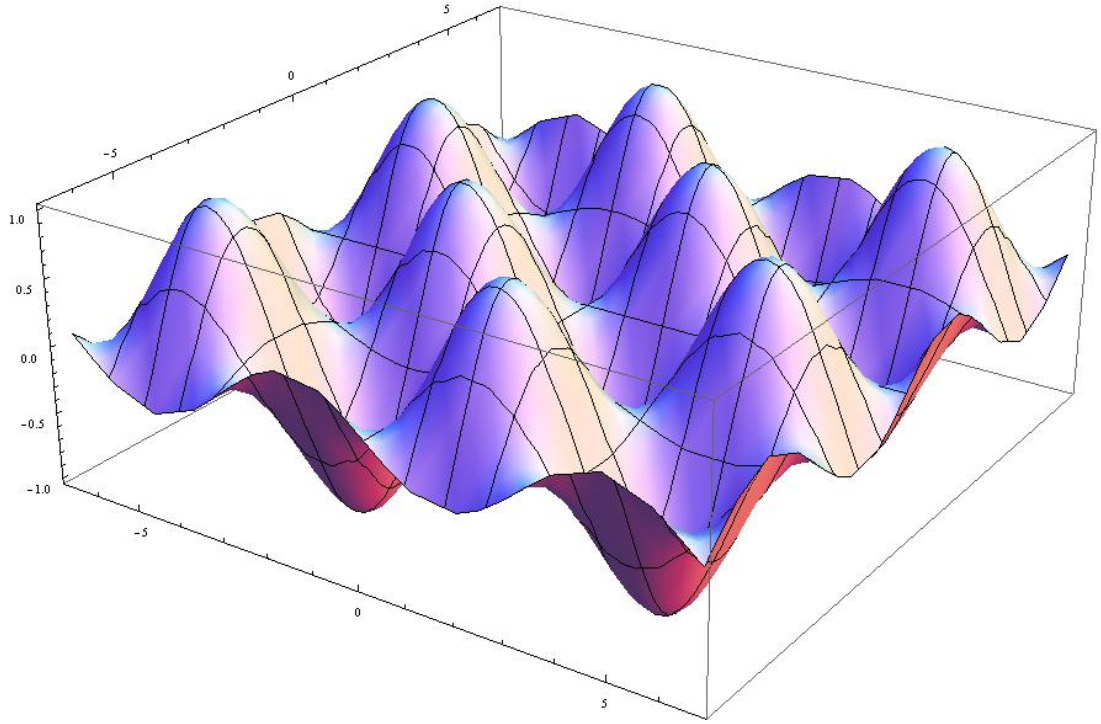
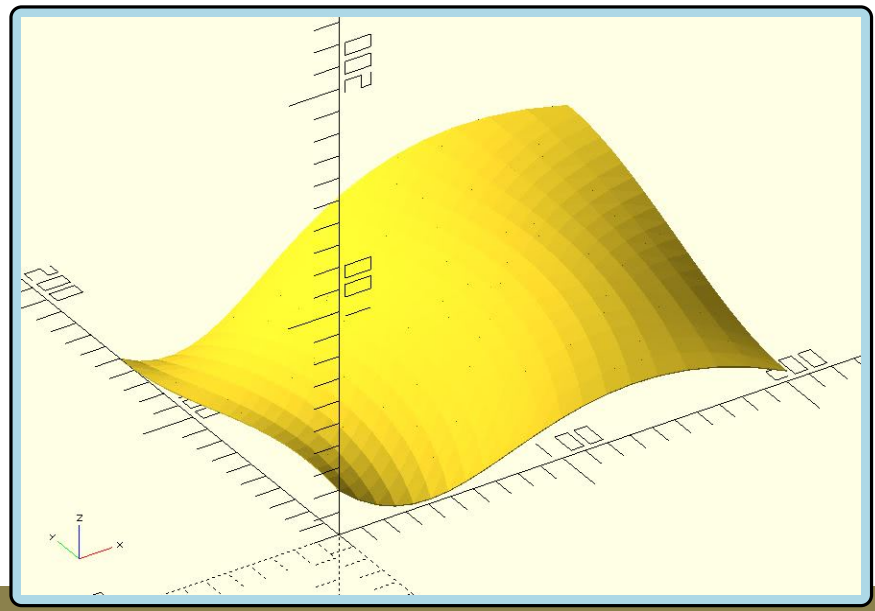
Example: $z = \sin x \sin y$





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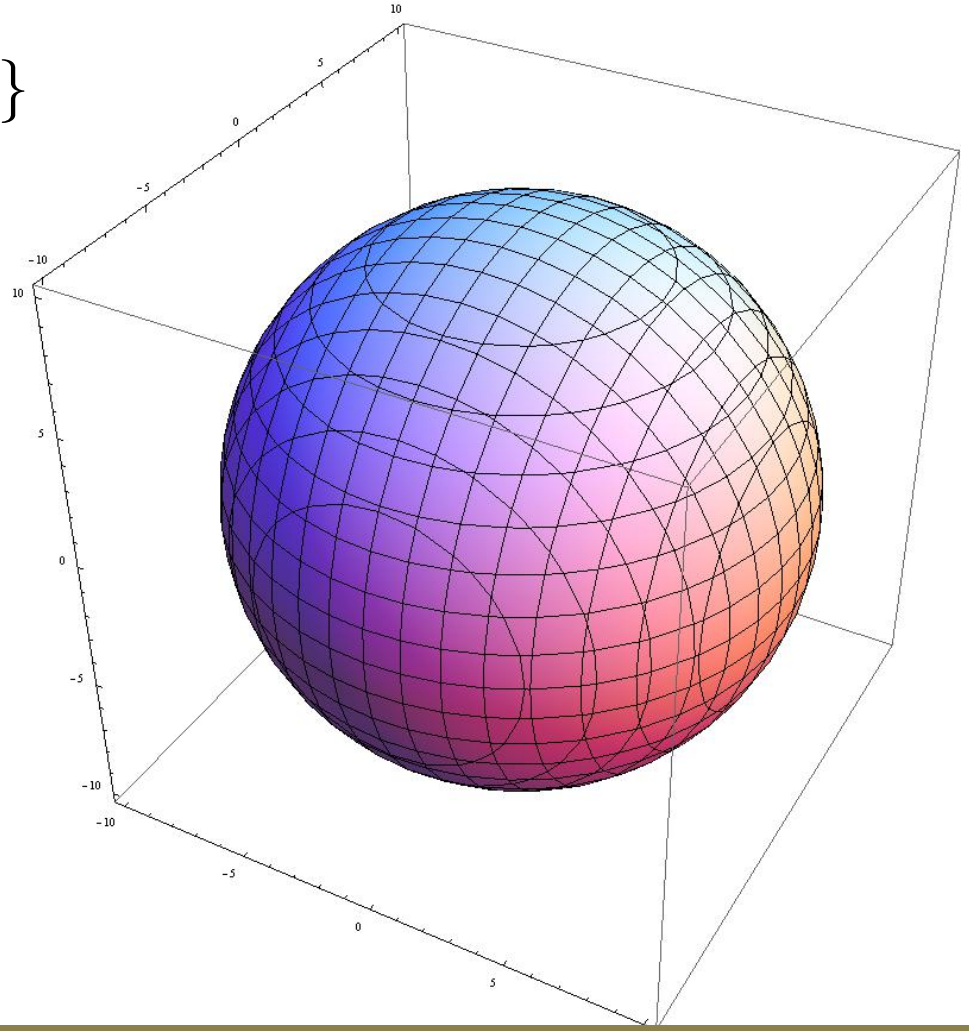

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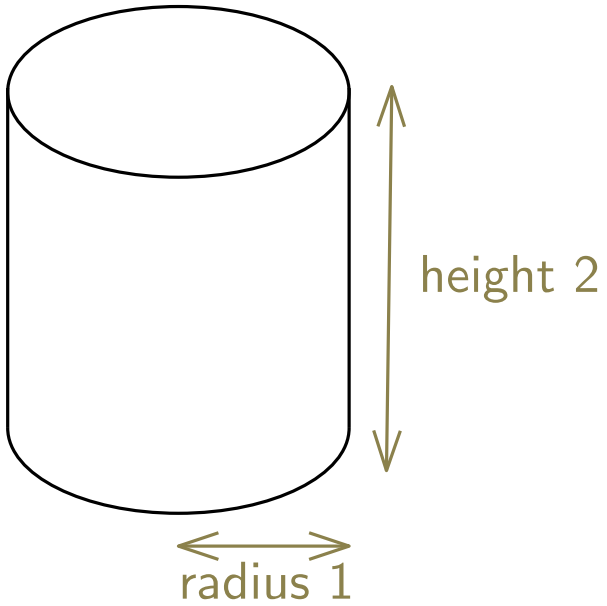

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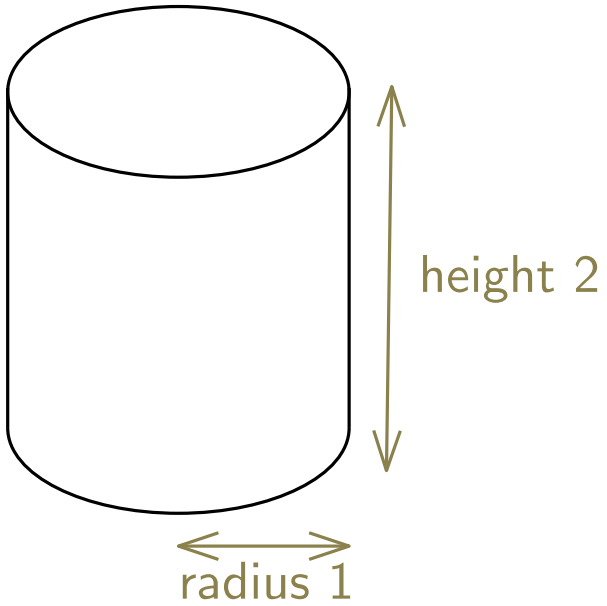

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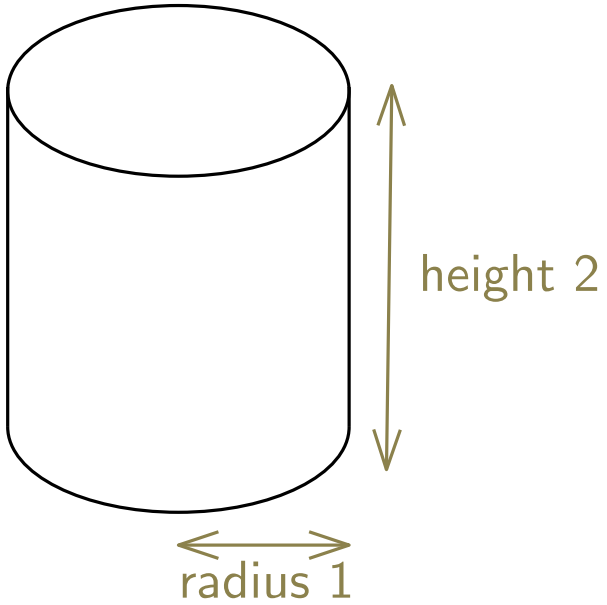

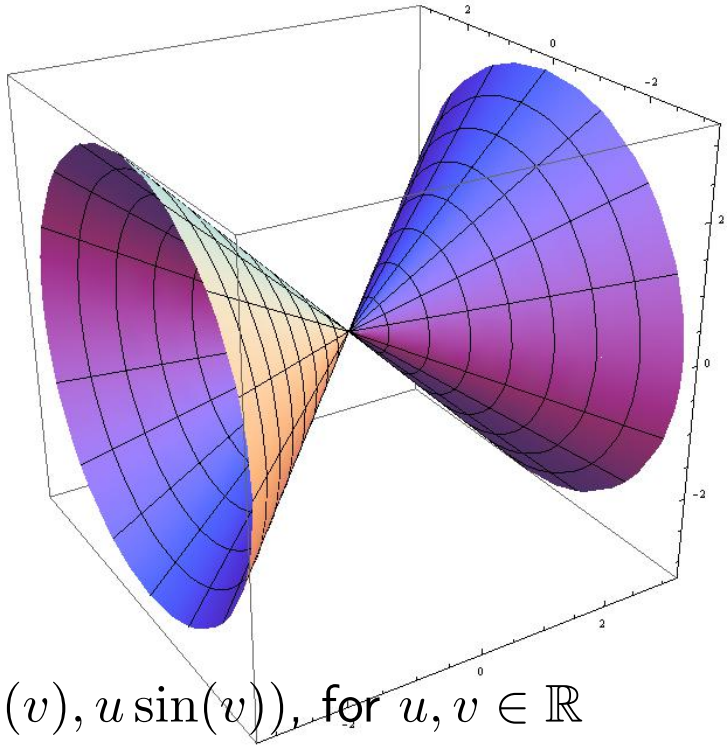

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→ the choice for surface design for CAD and graphics

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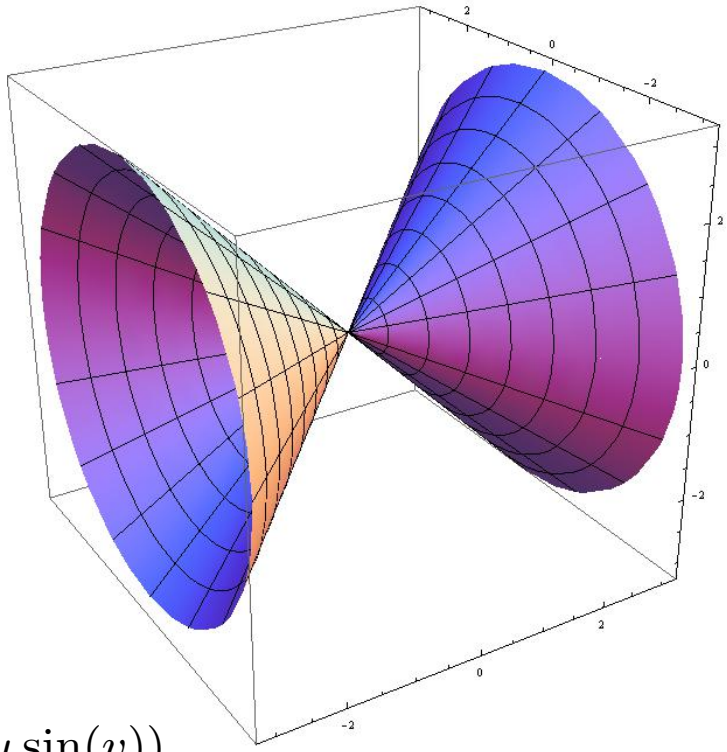

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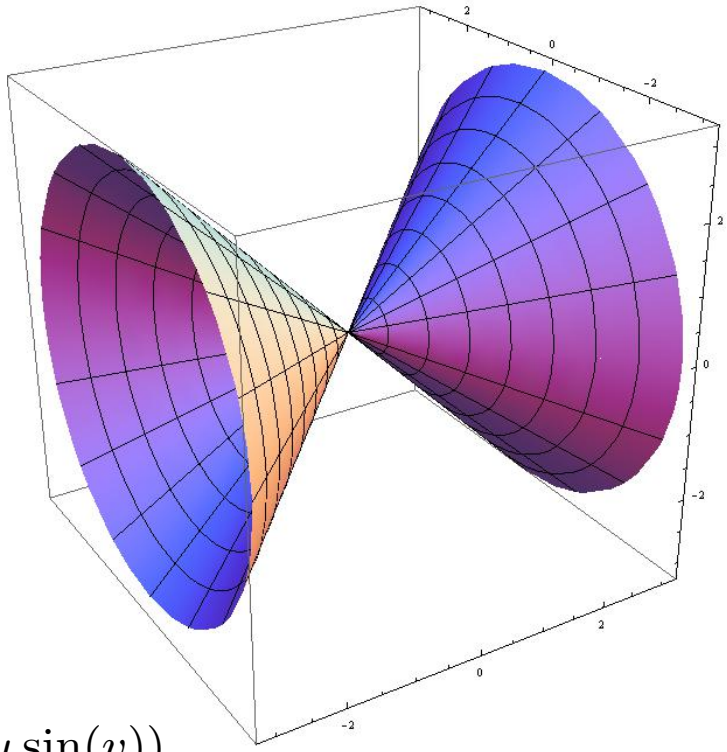

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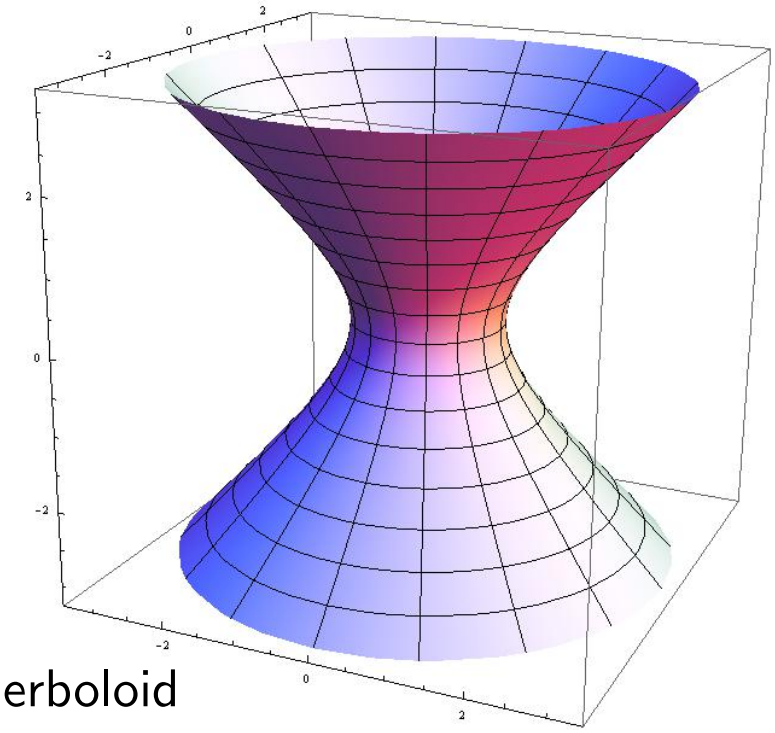

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Source: `http://math.arizona.edu/~models`

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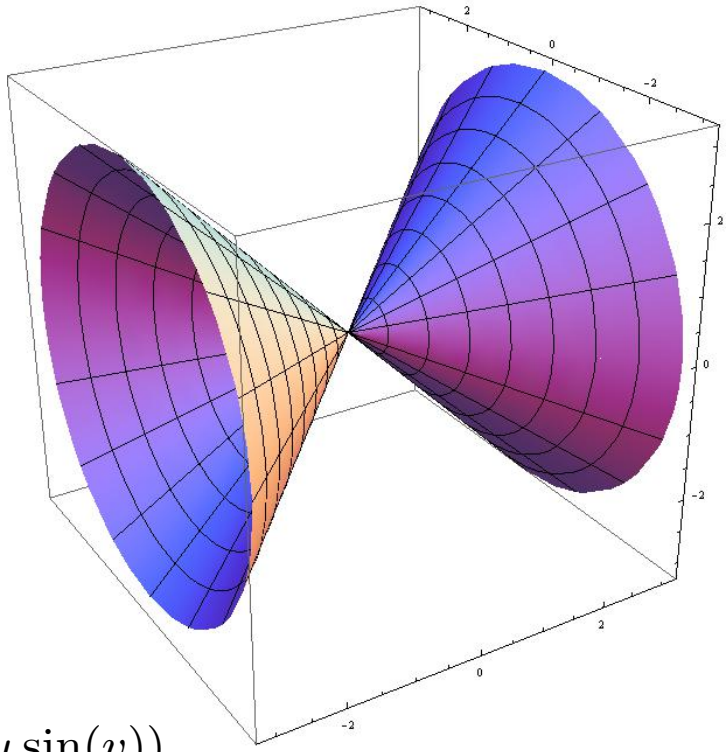

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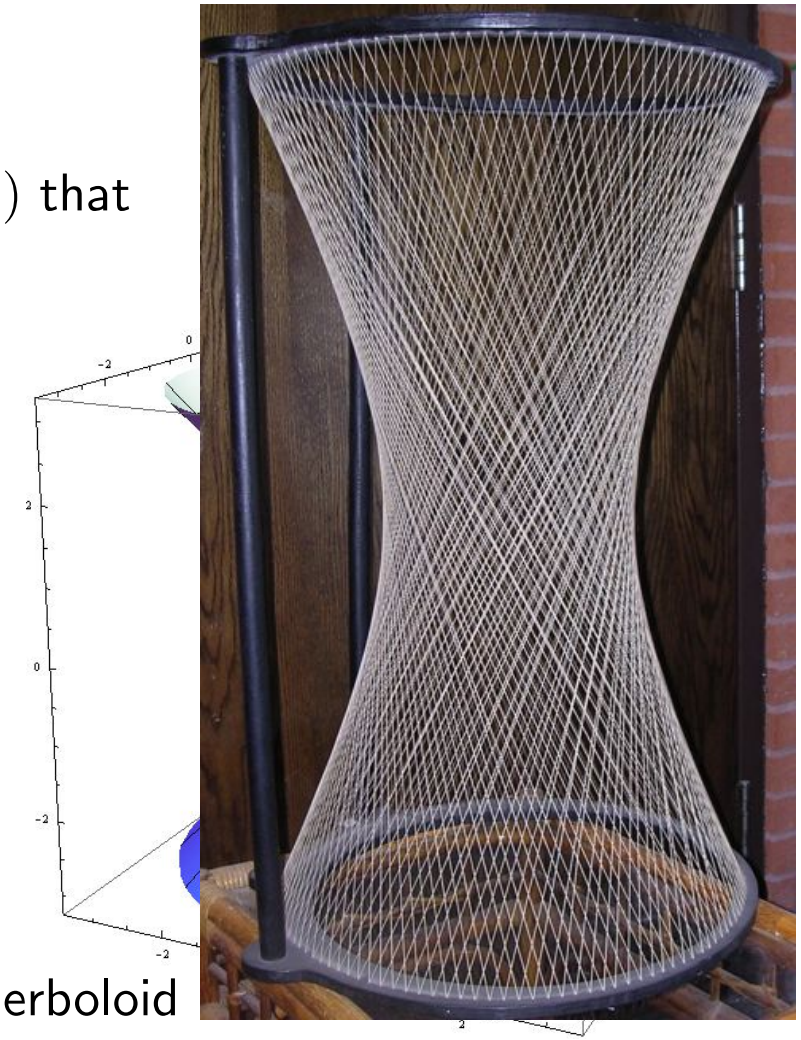

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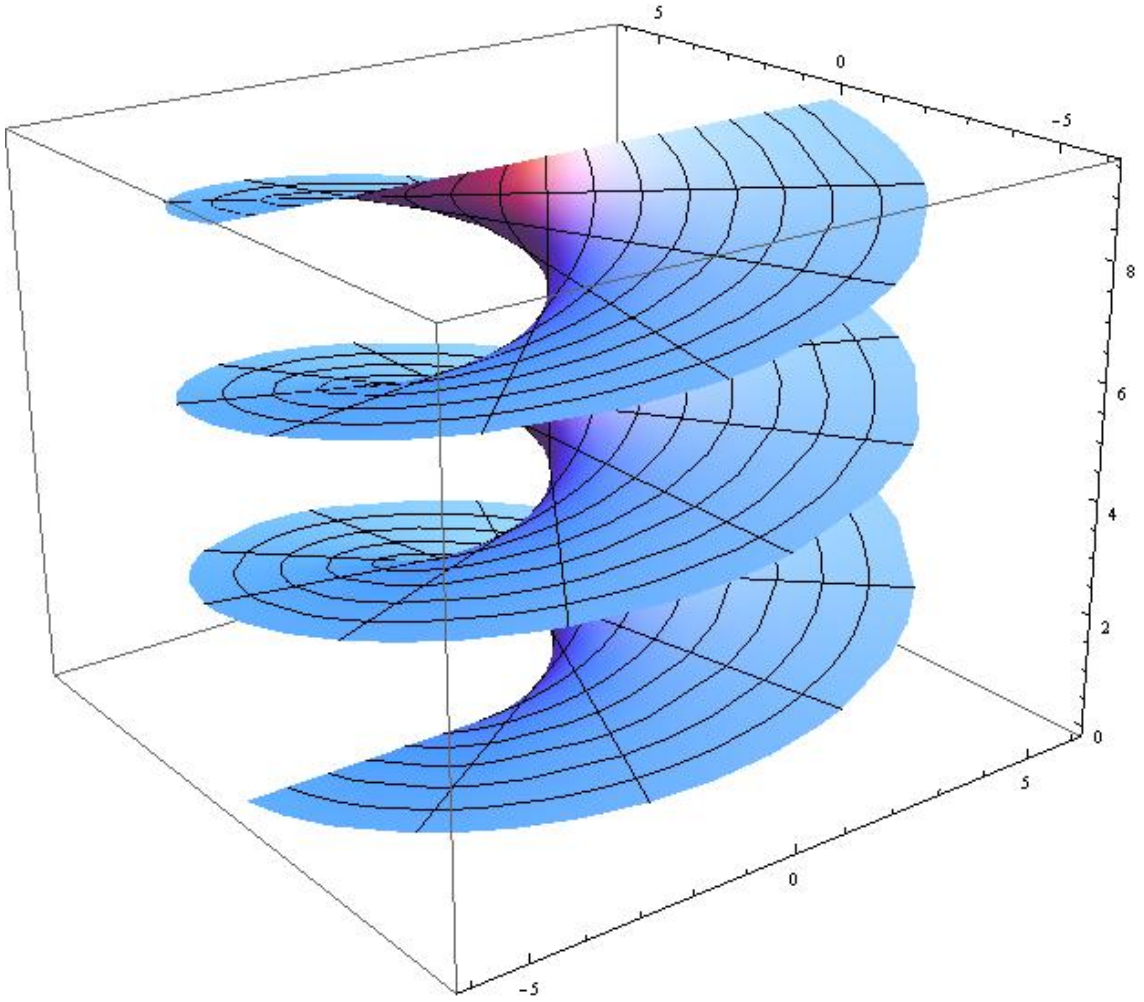

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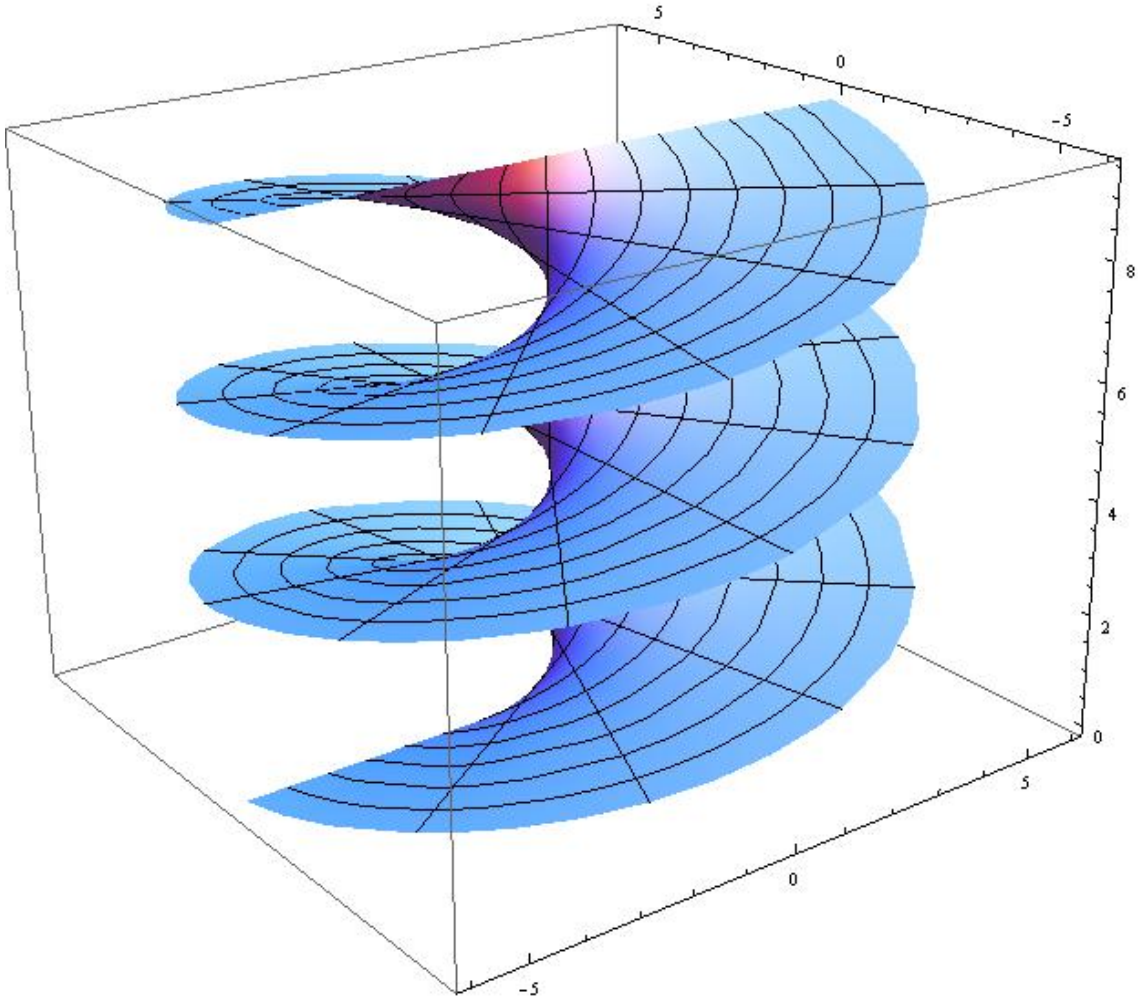

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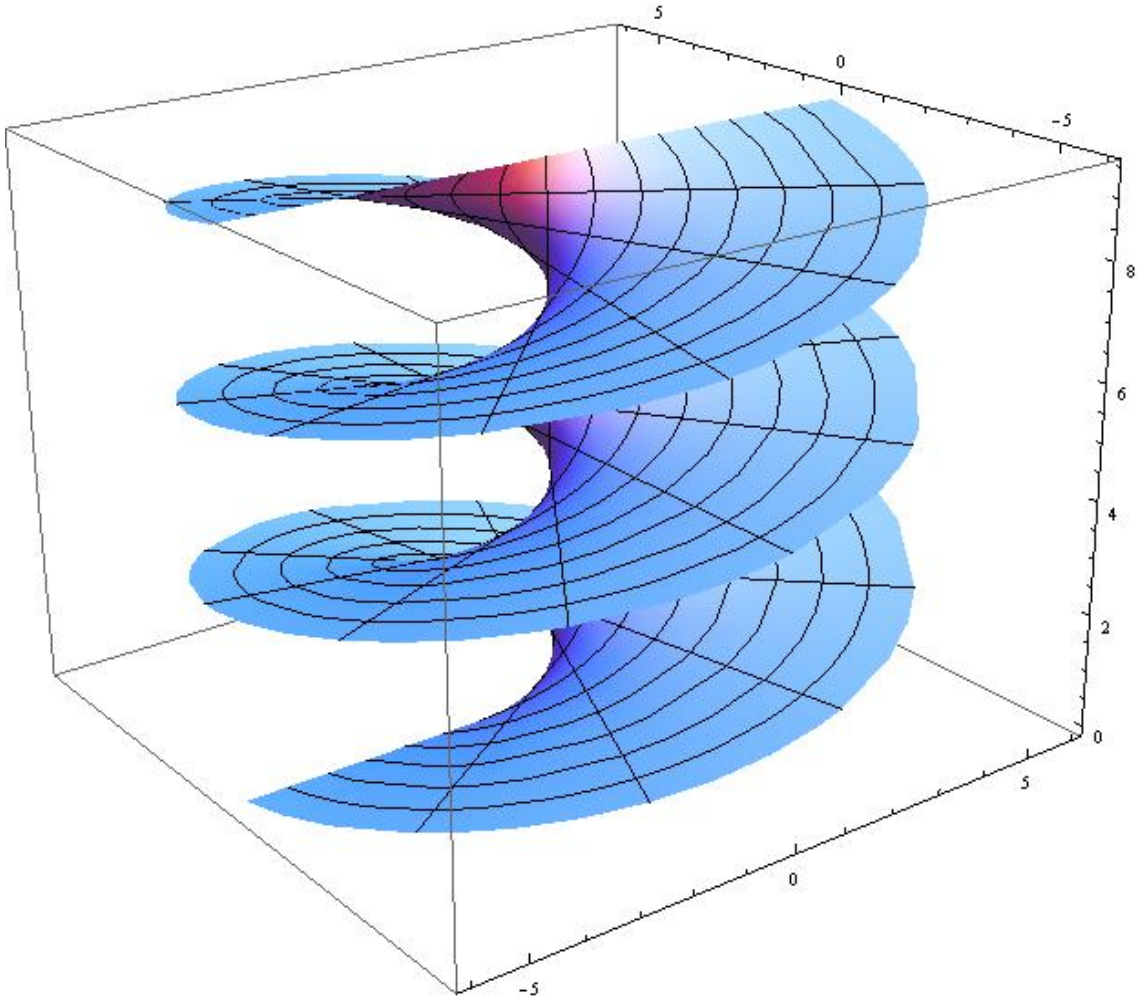

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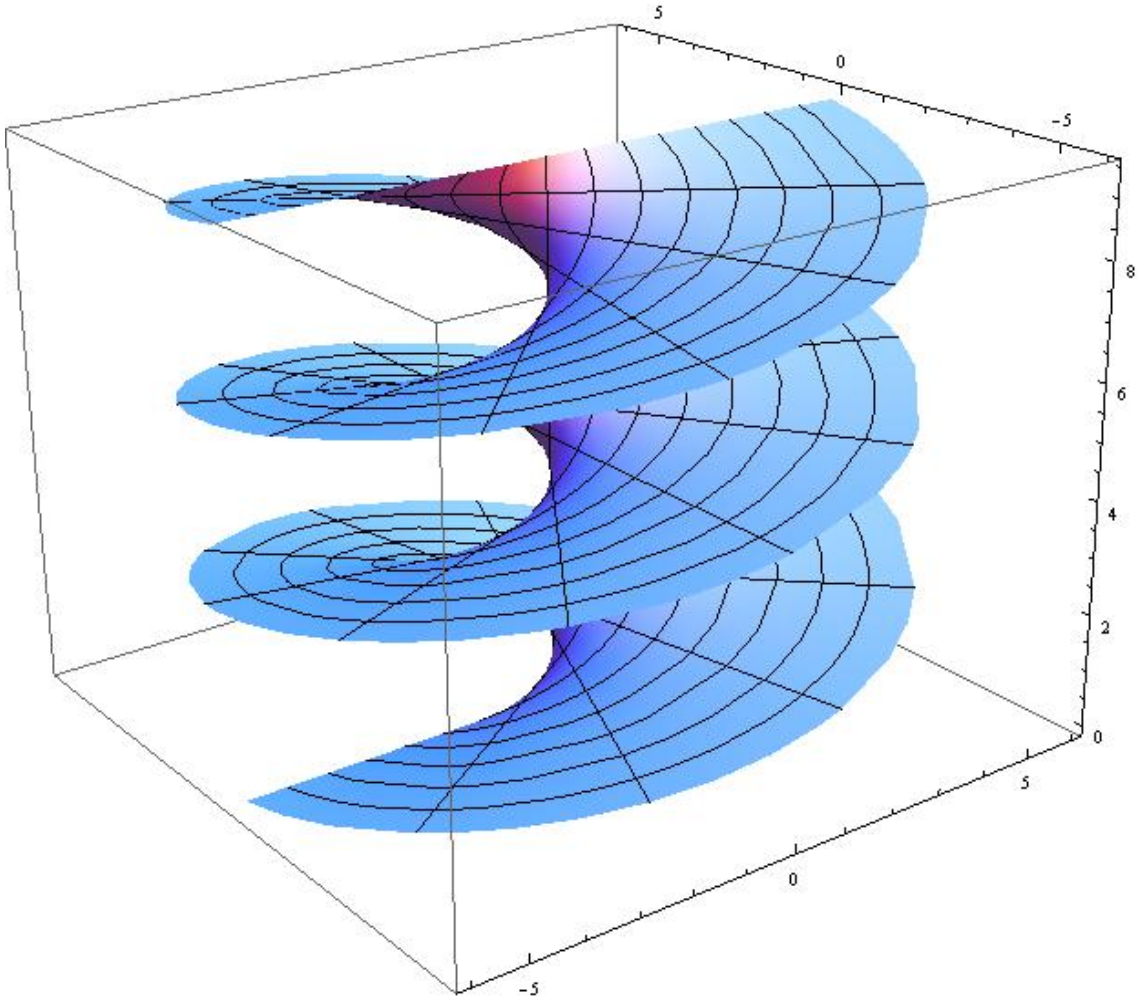

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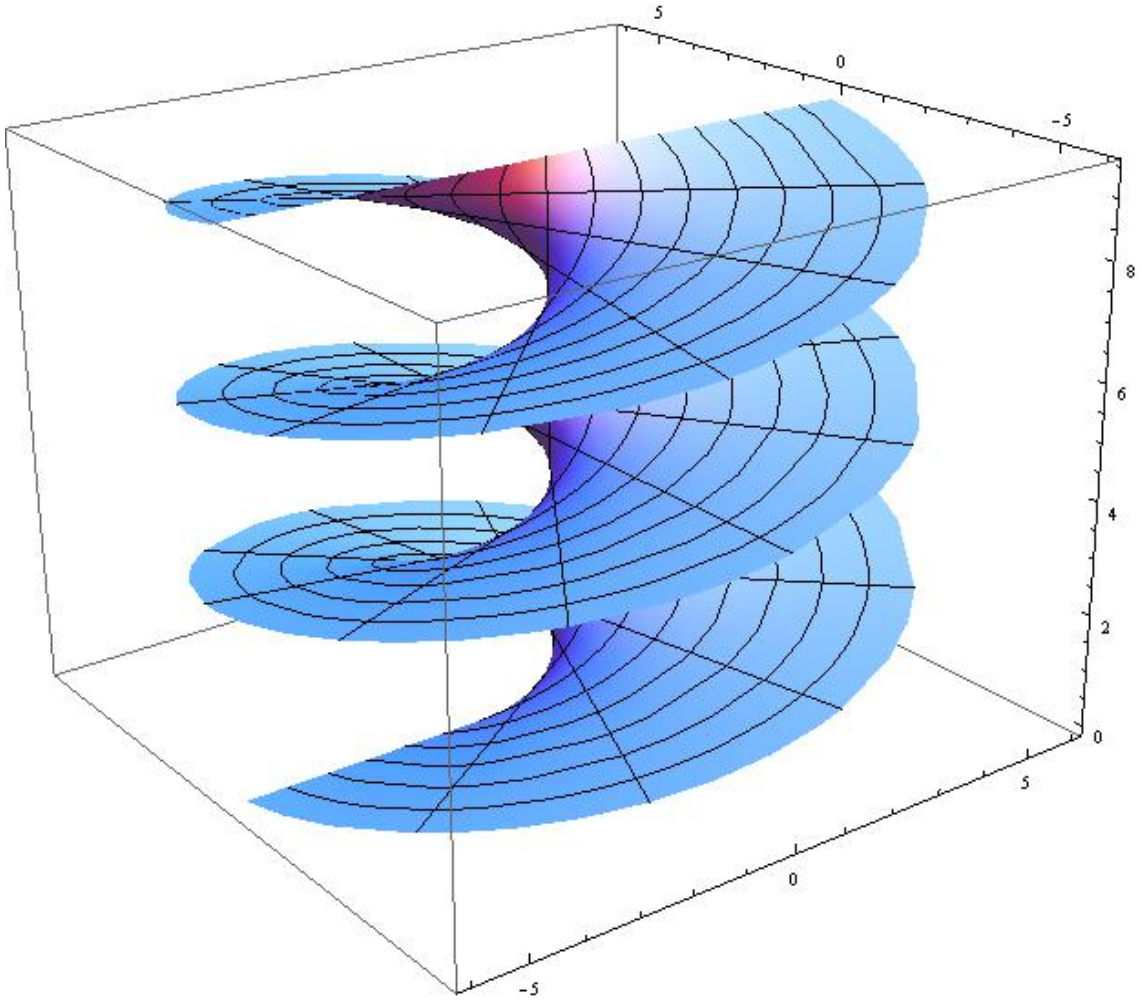

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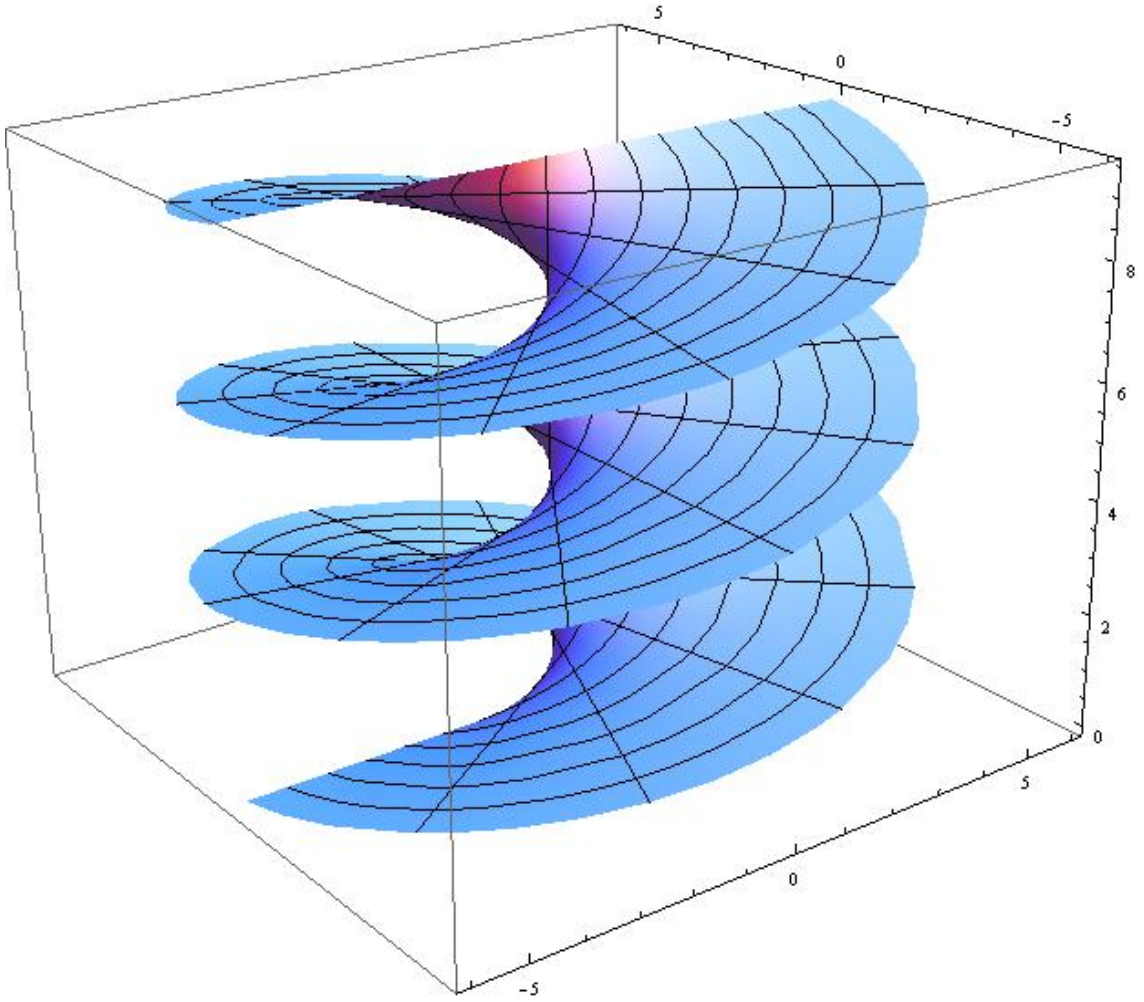

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Thus: $S(t, \lambda) = (1 - \lambda)Q(t) + \lambda P(t)$
$= (1 - \lambda)(0, 0, bt) + \lambda(a\cos t, a\sin t, bt)$
$= (a\lambda\cos t, a\lambda\sin t, bt)$

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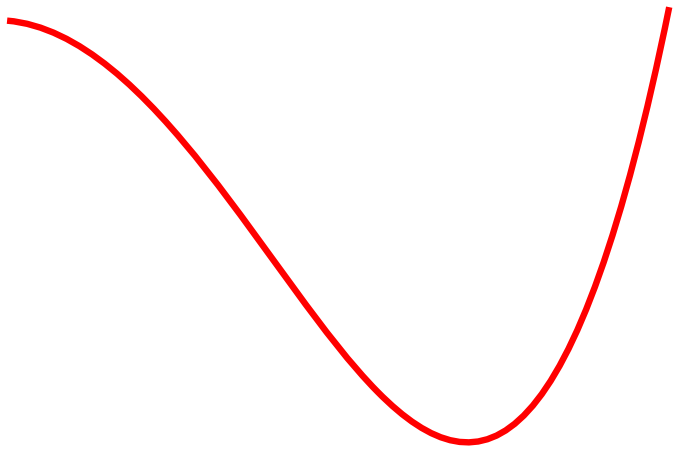

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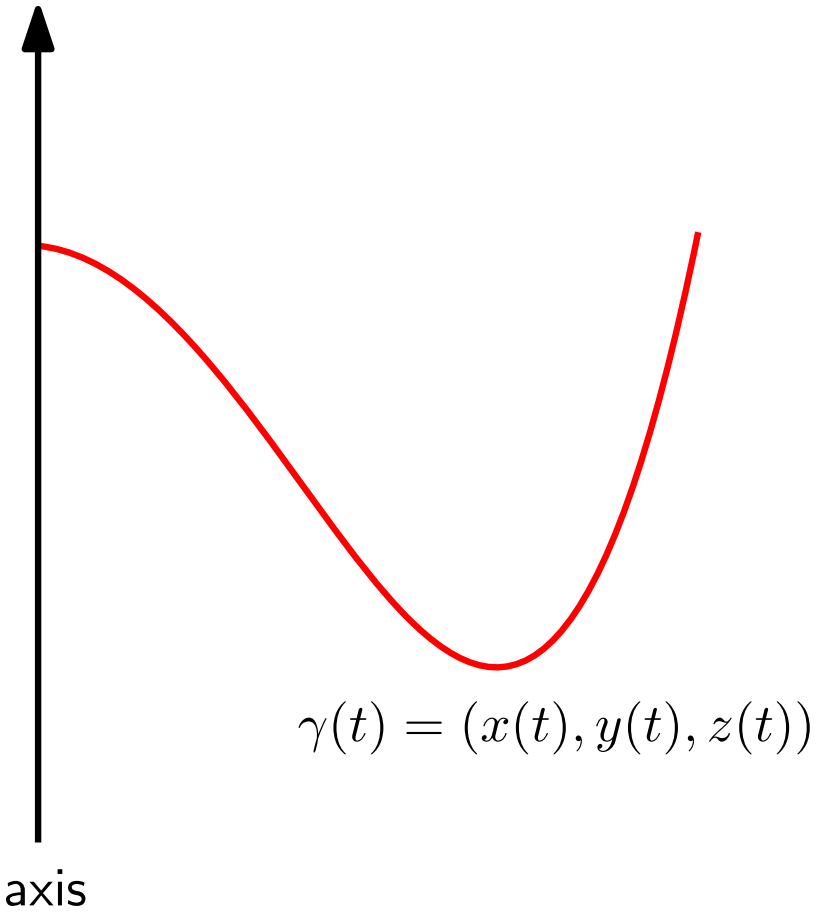

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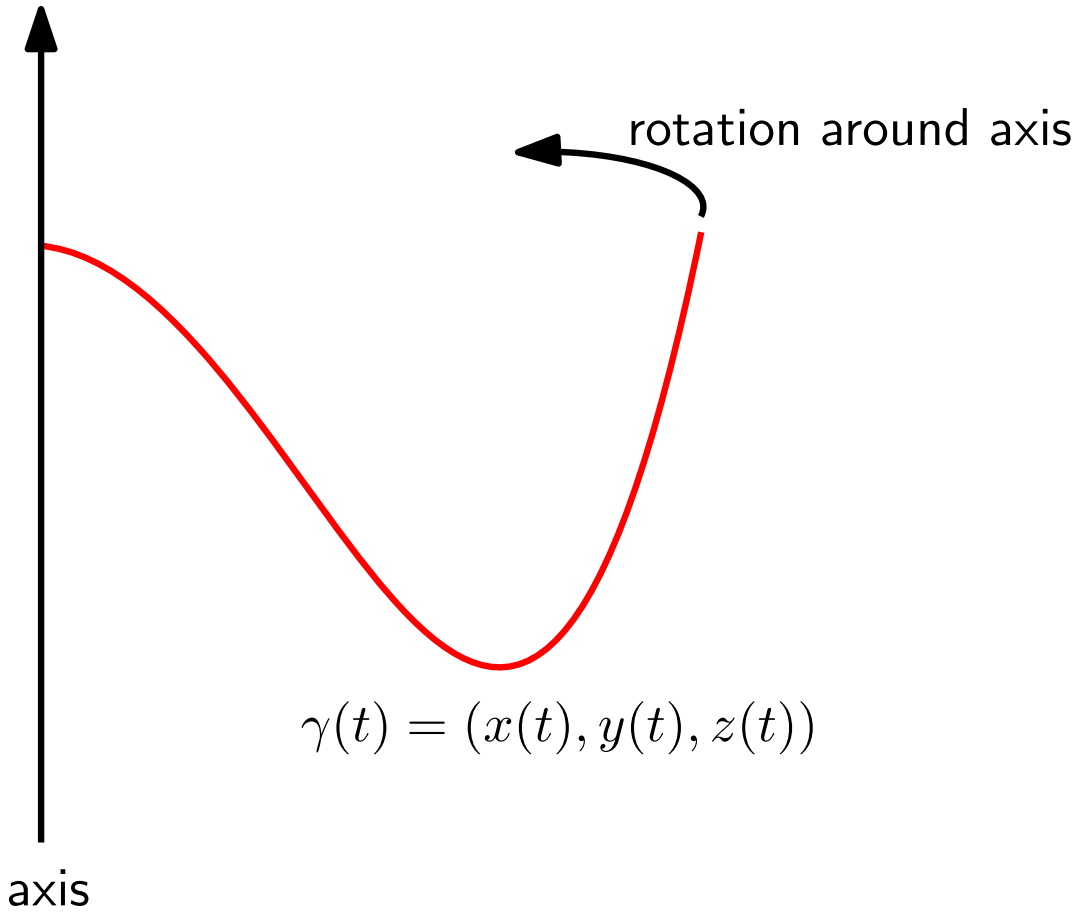

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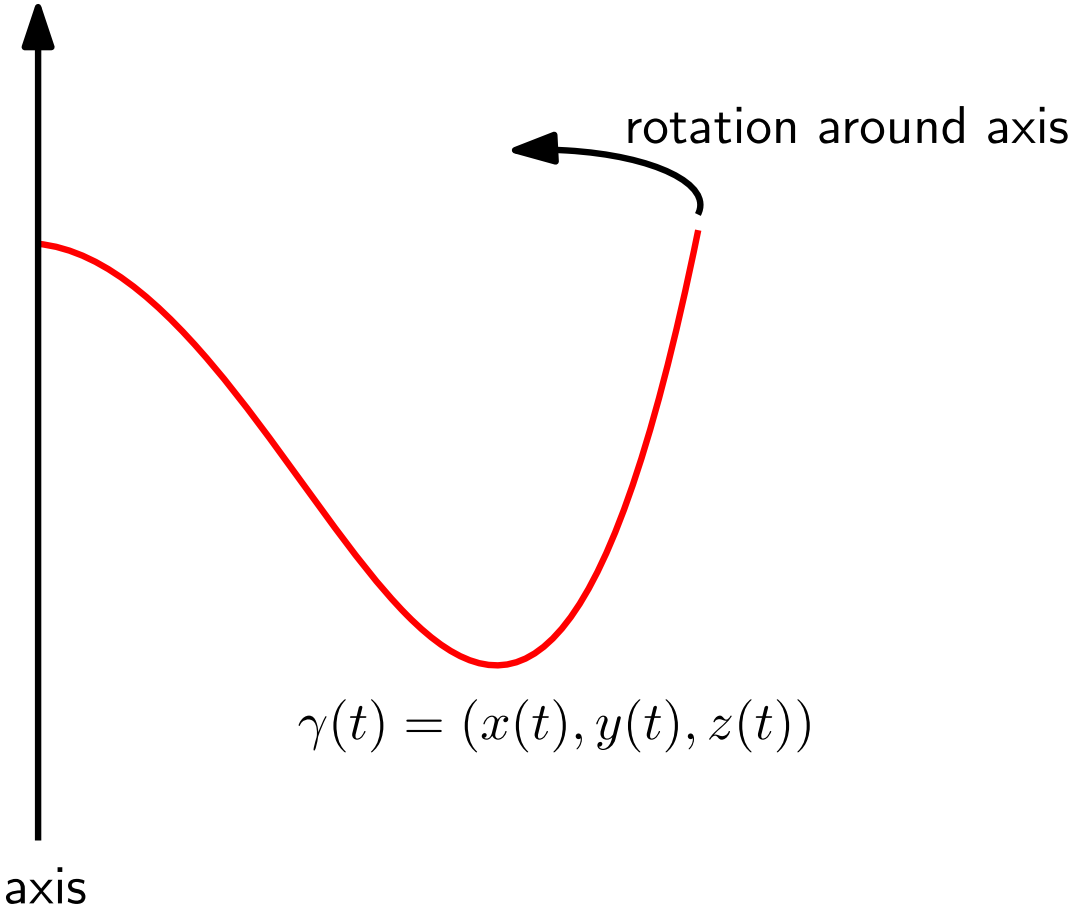

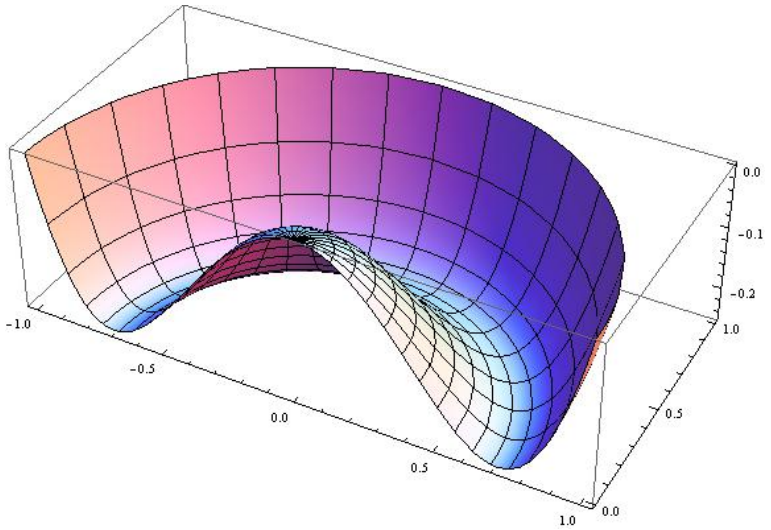

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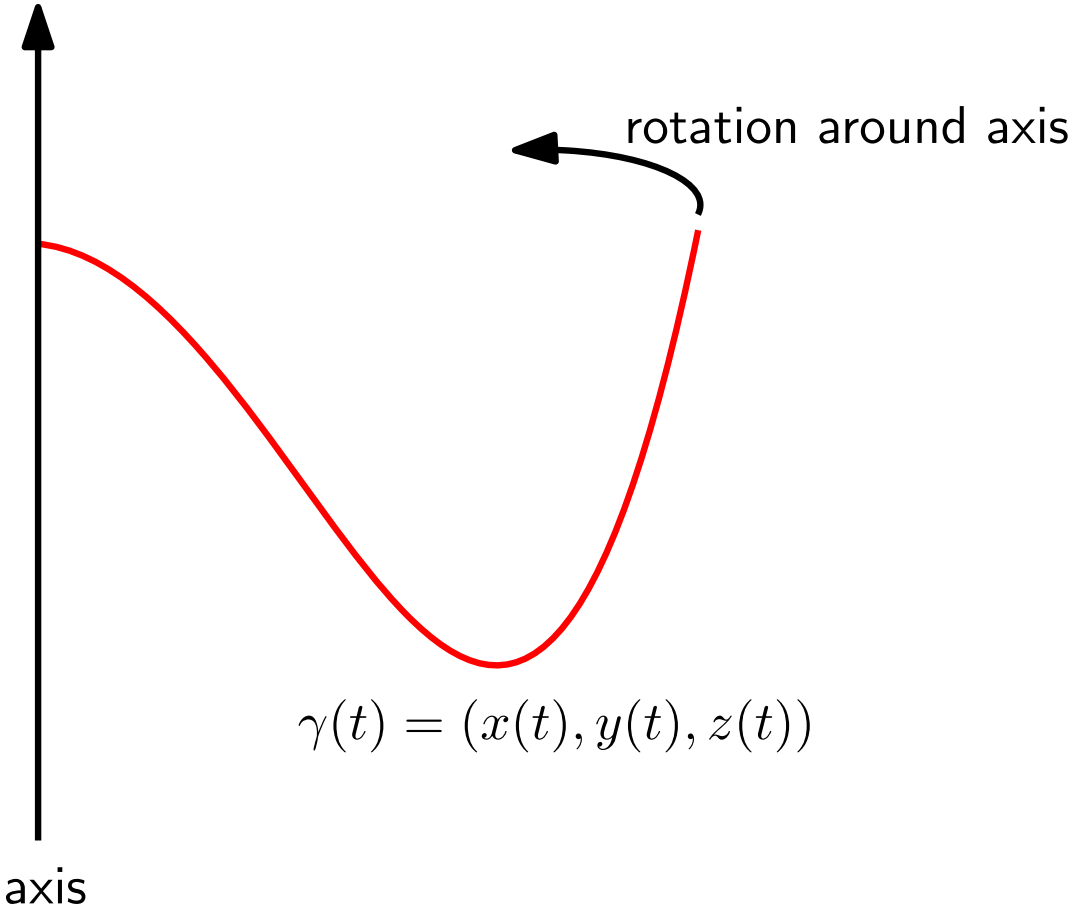

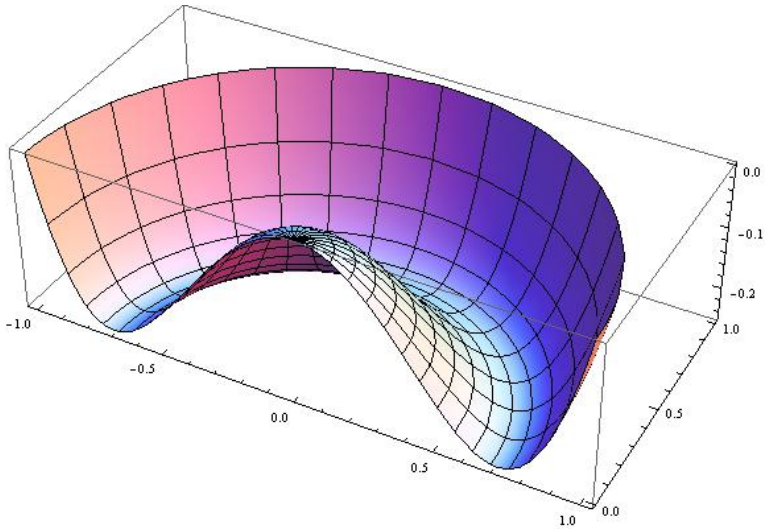

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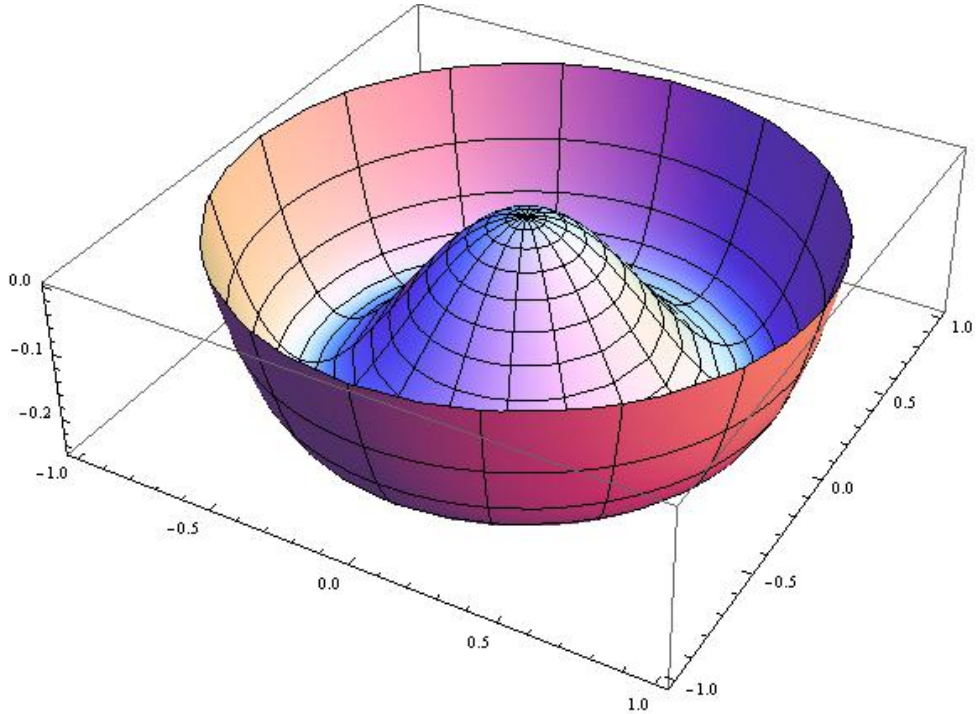

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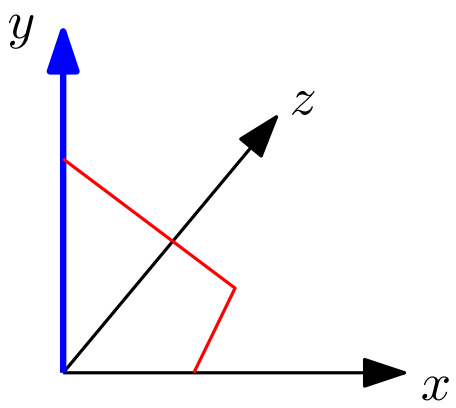

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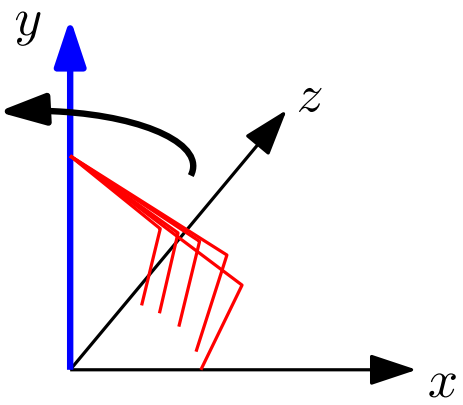

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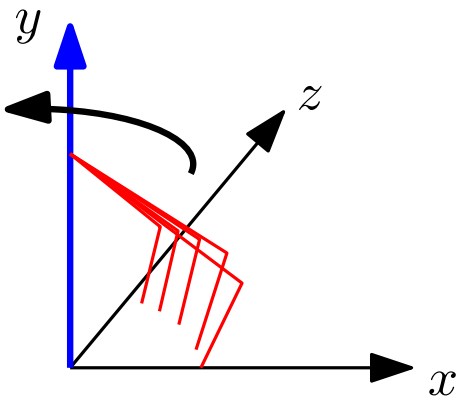

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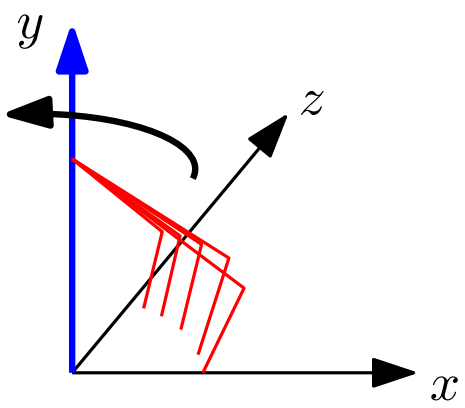

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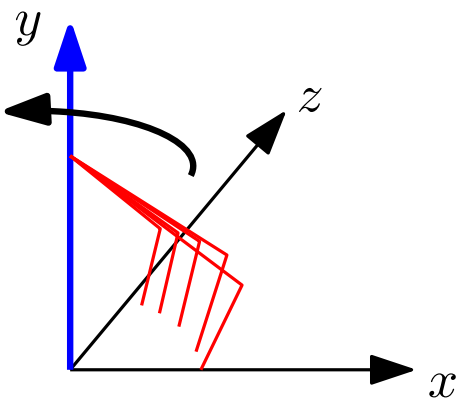

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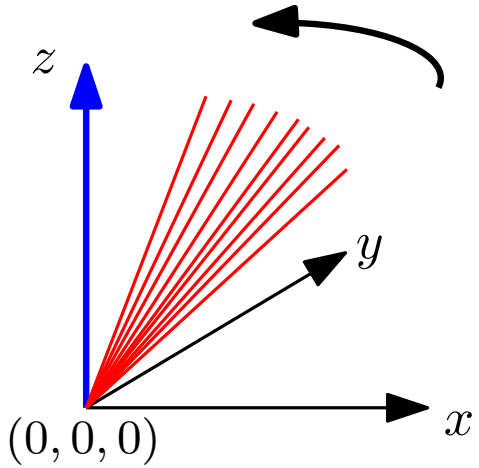

What is a cone? Let's start from the *implicit equation*

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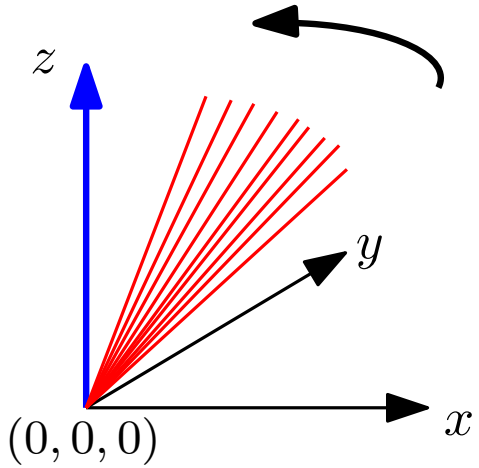

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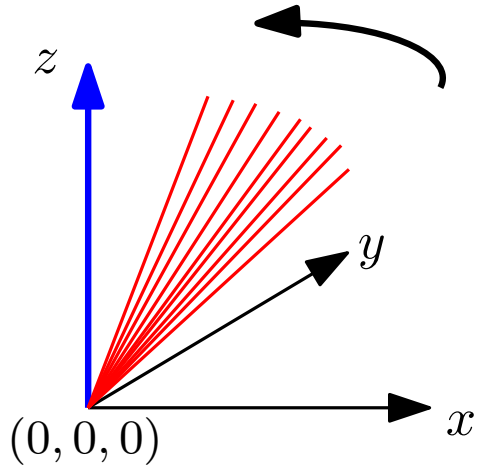

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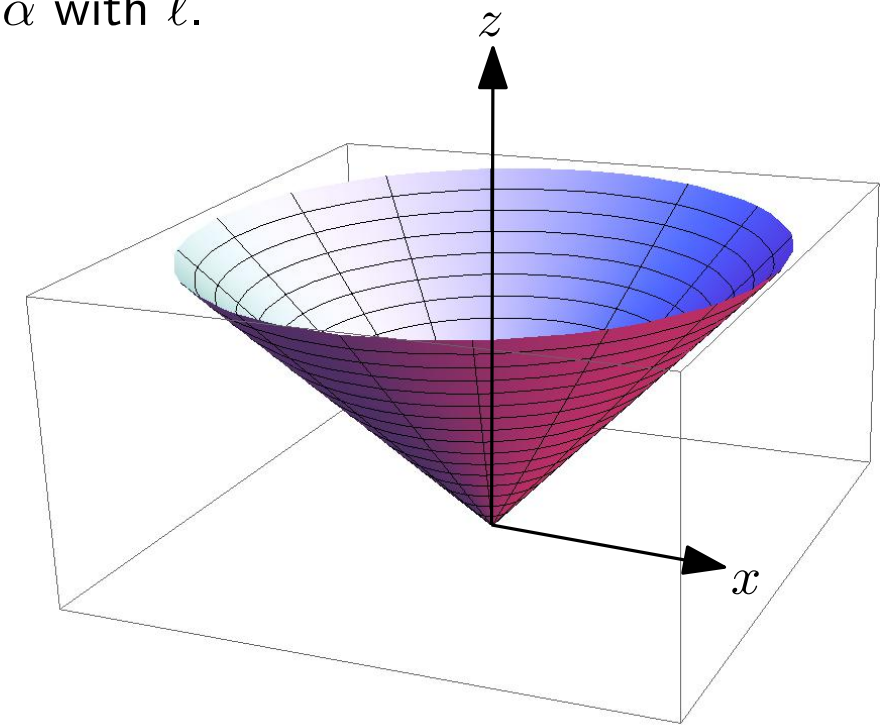

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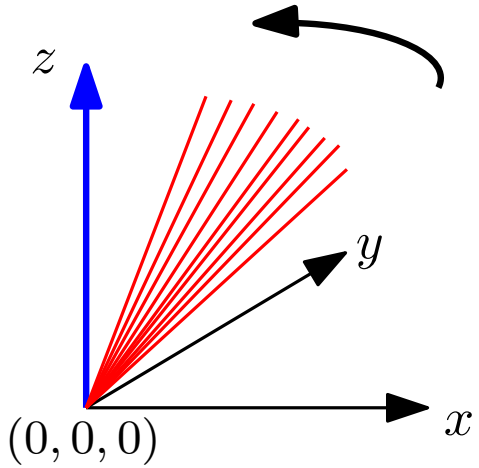

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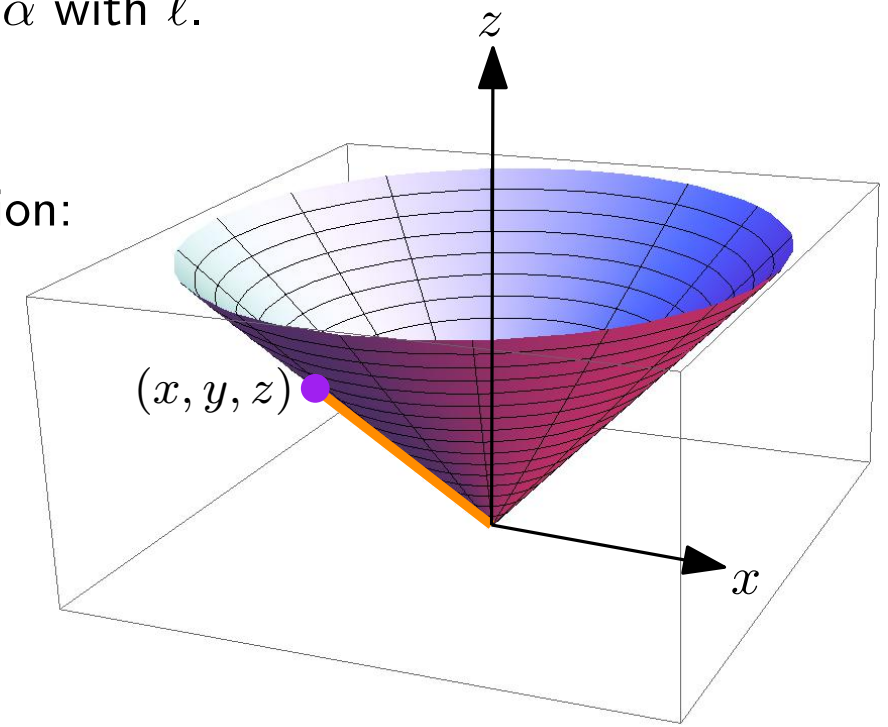

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EXAMPLES OF SURFACES

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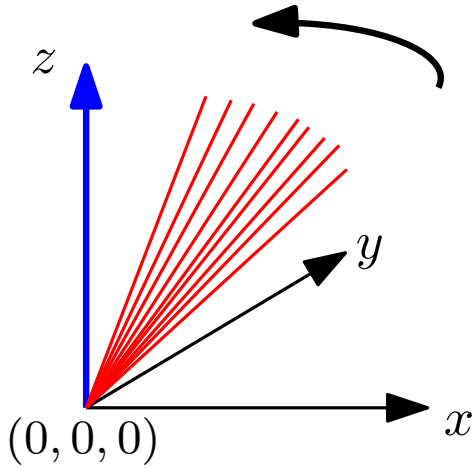

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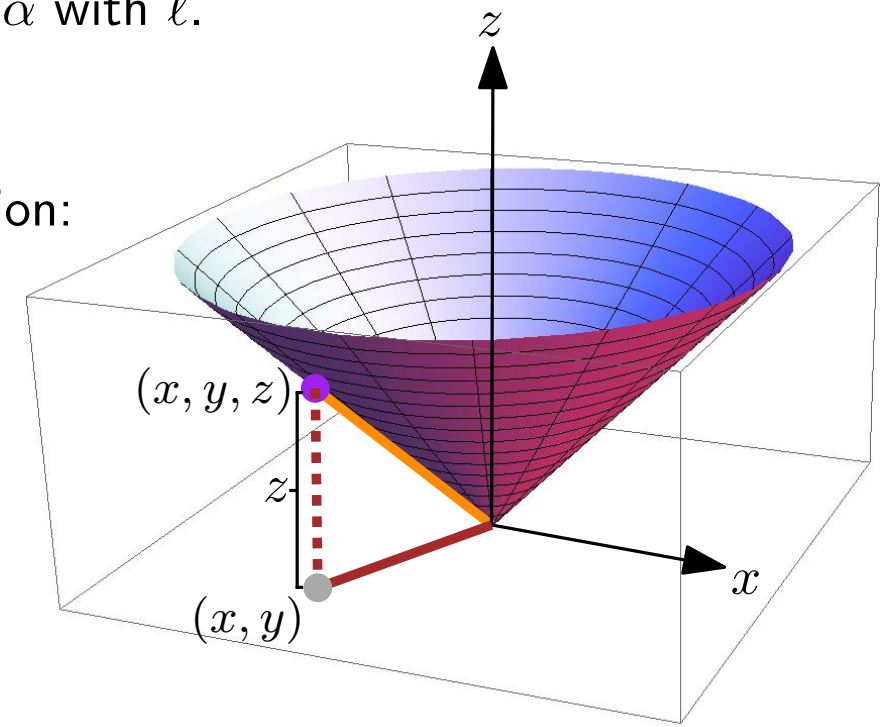

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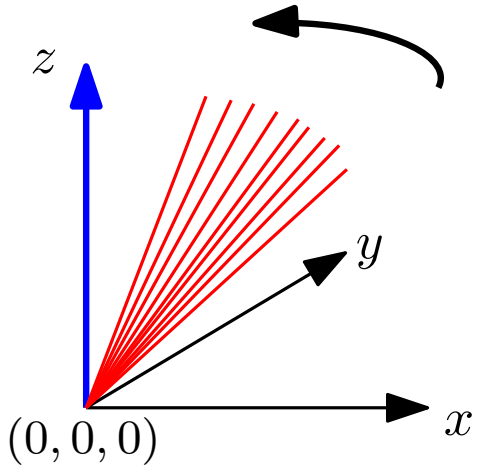

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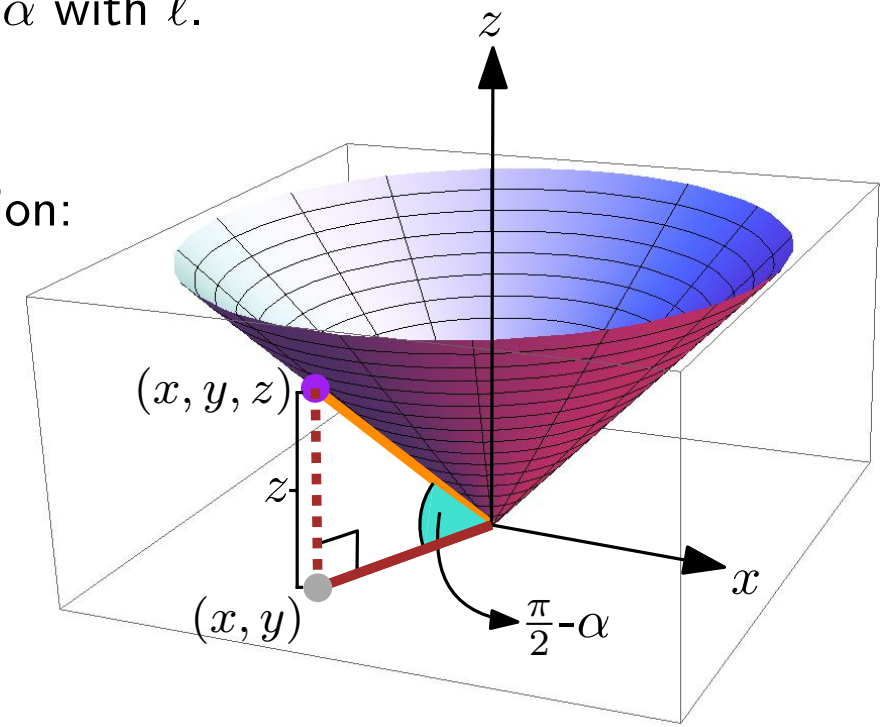

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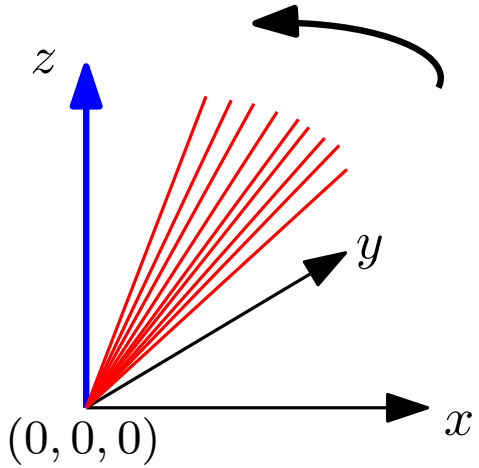

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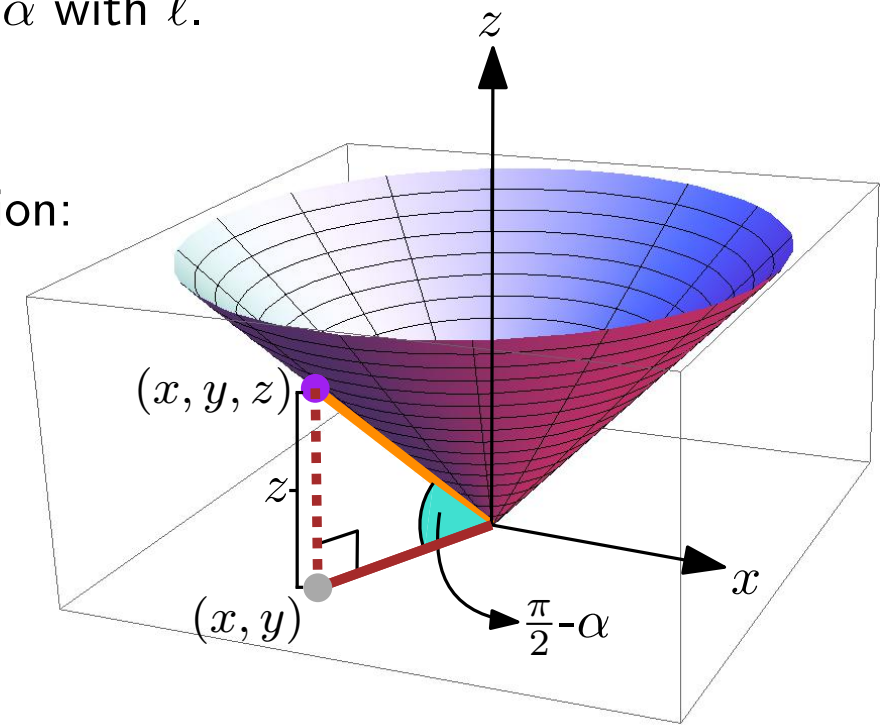

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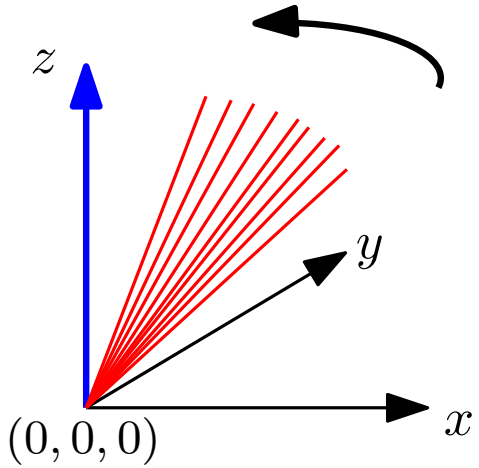

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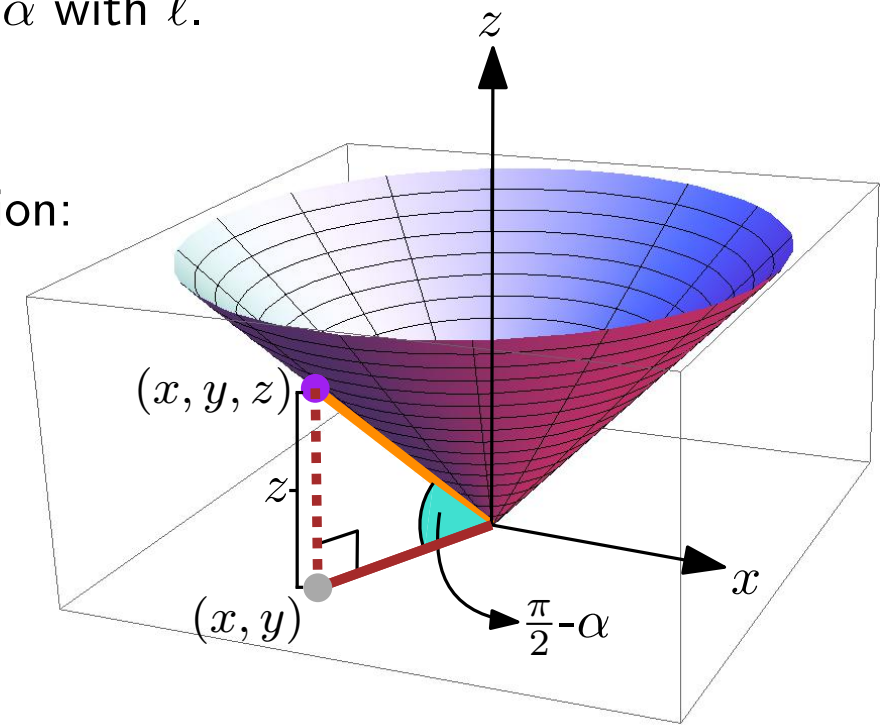

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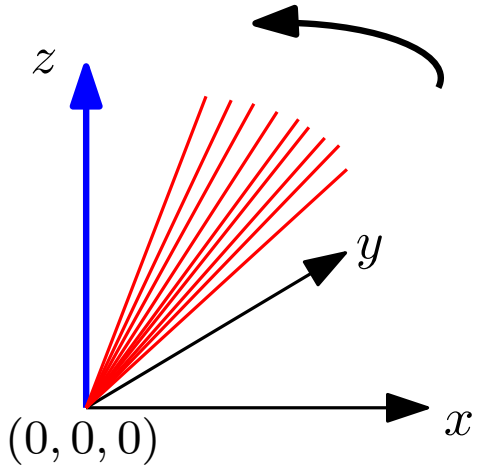

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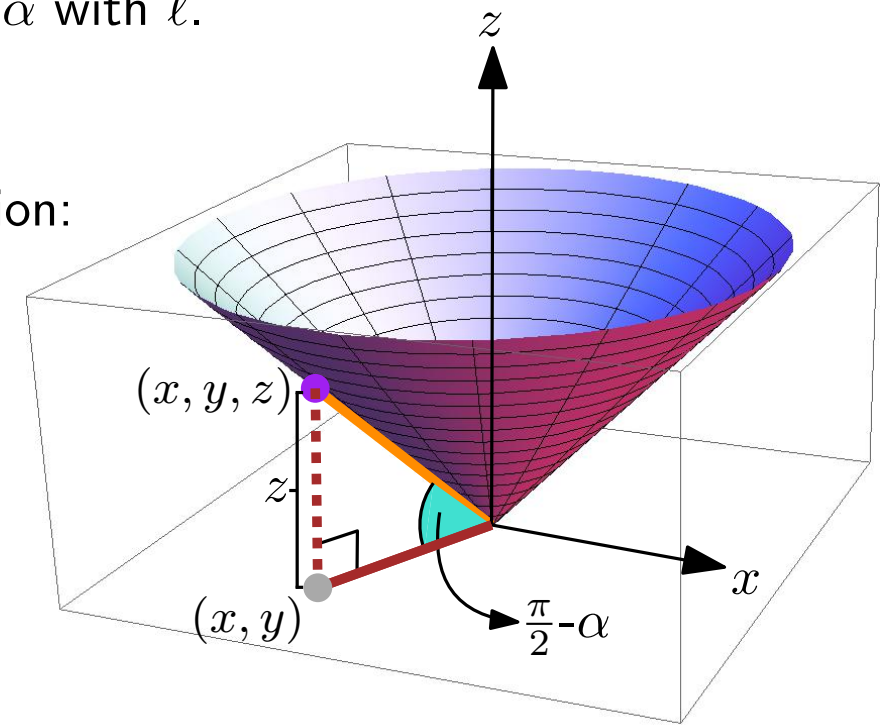

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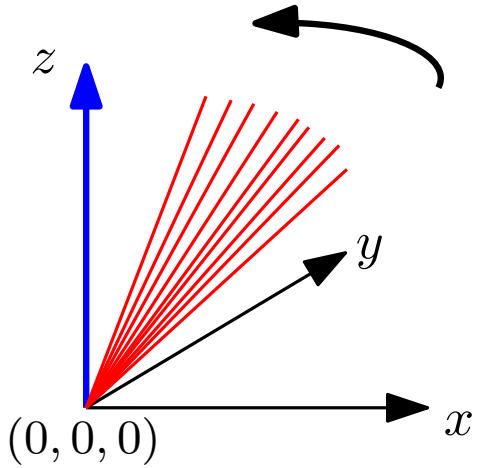

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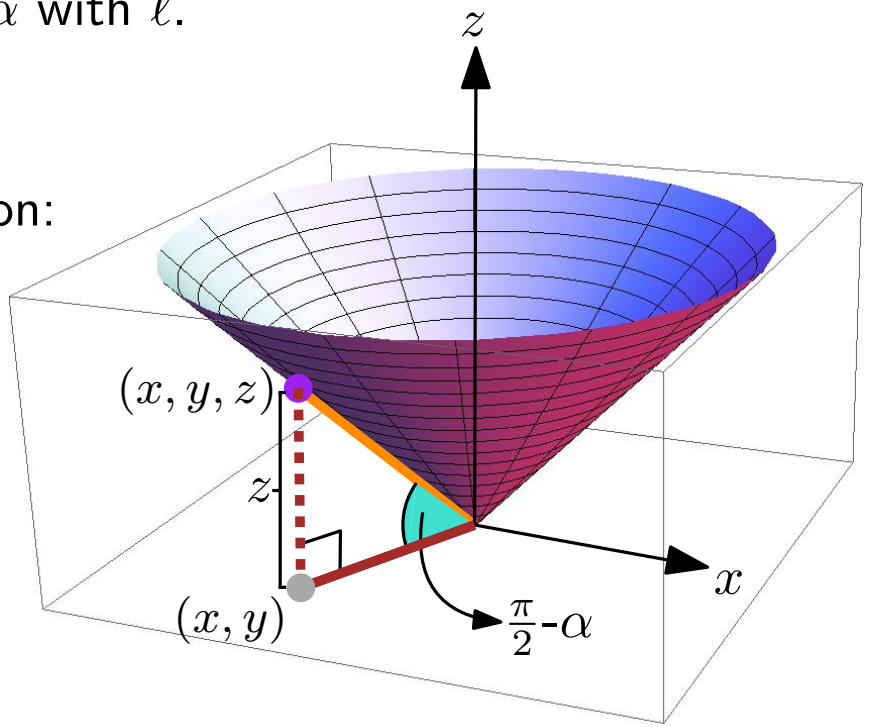

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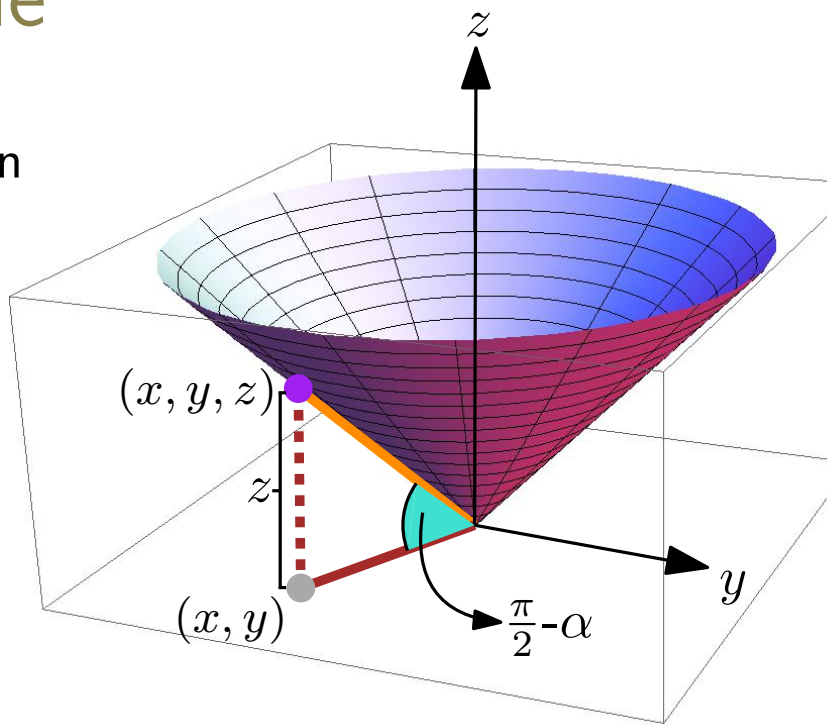

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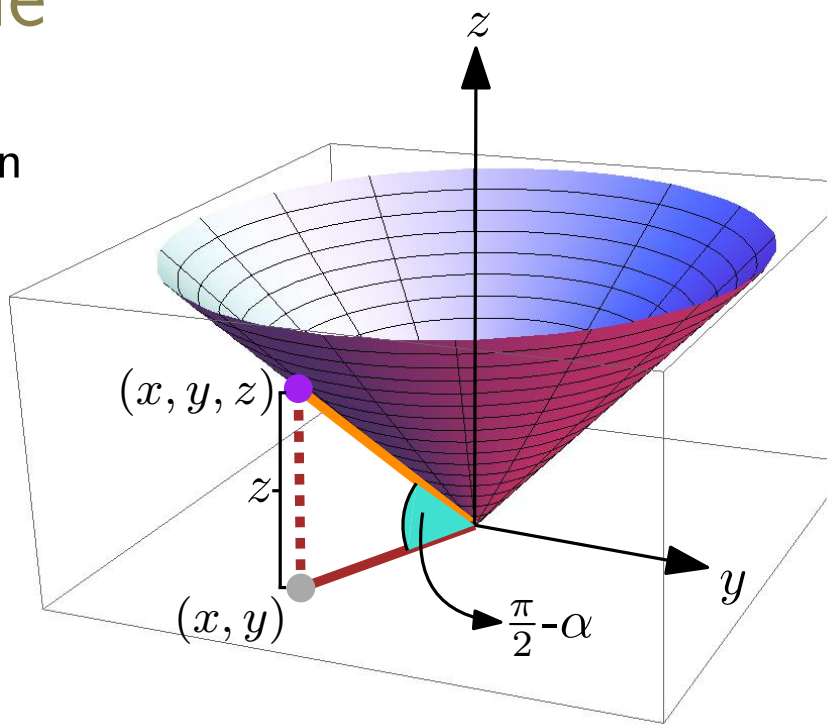

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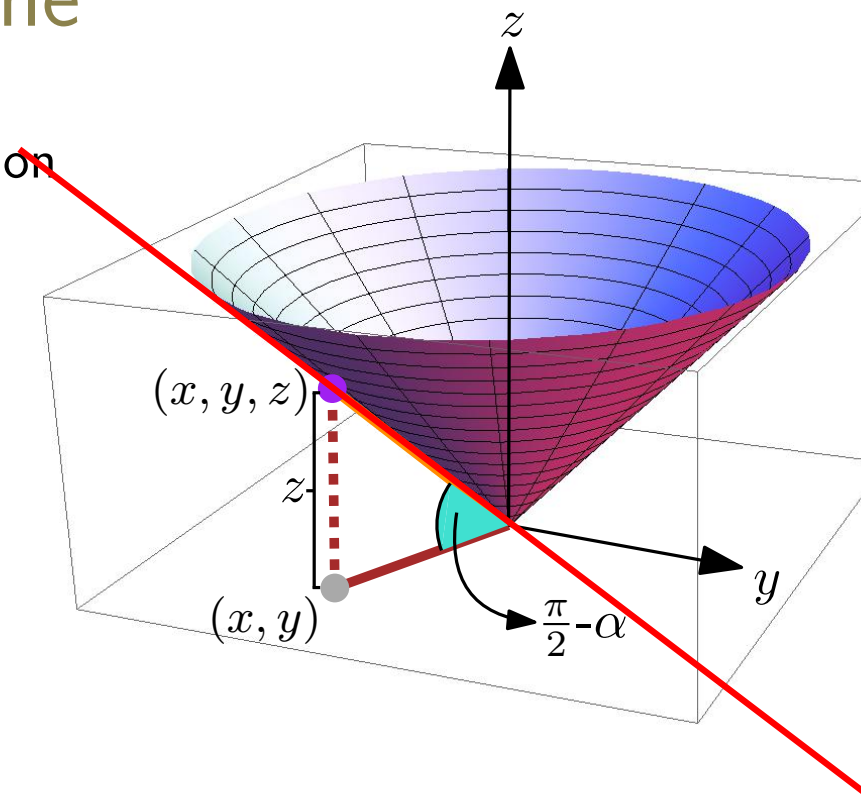

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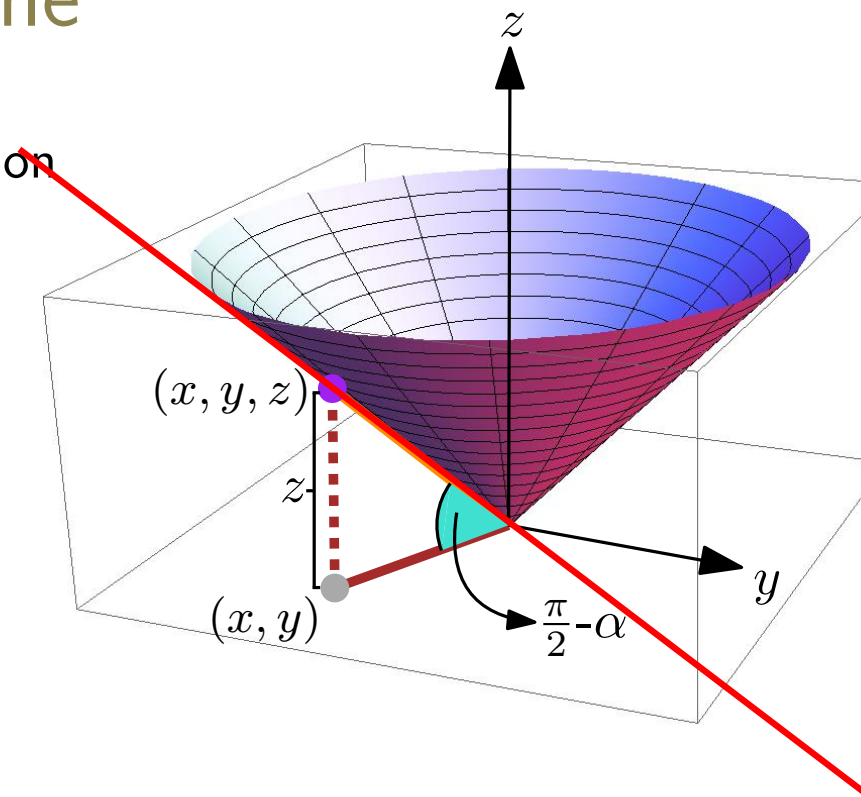

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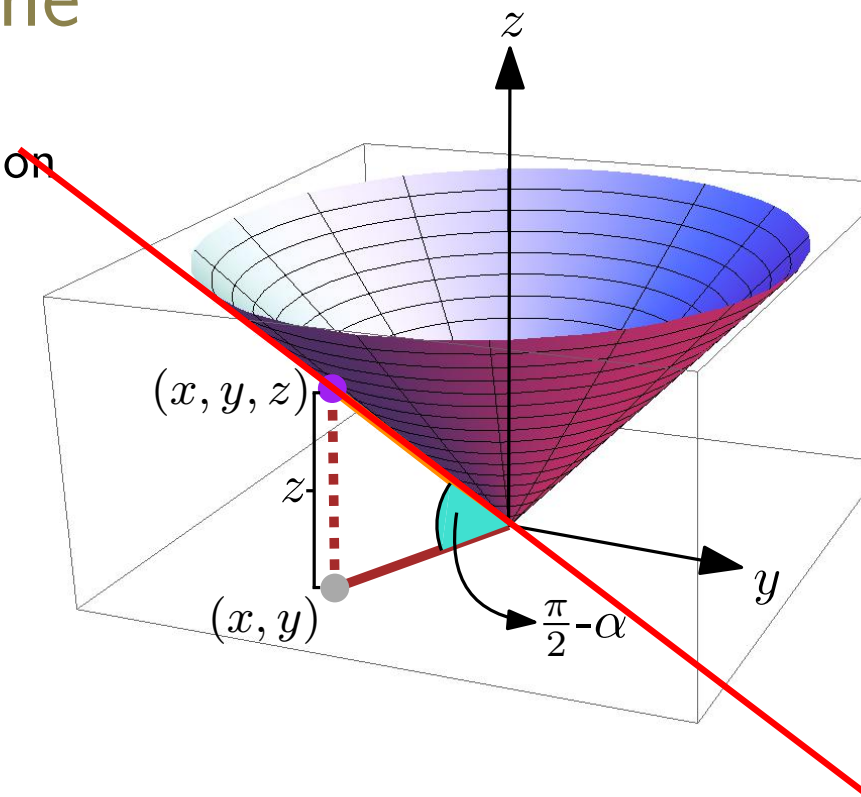

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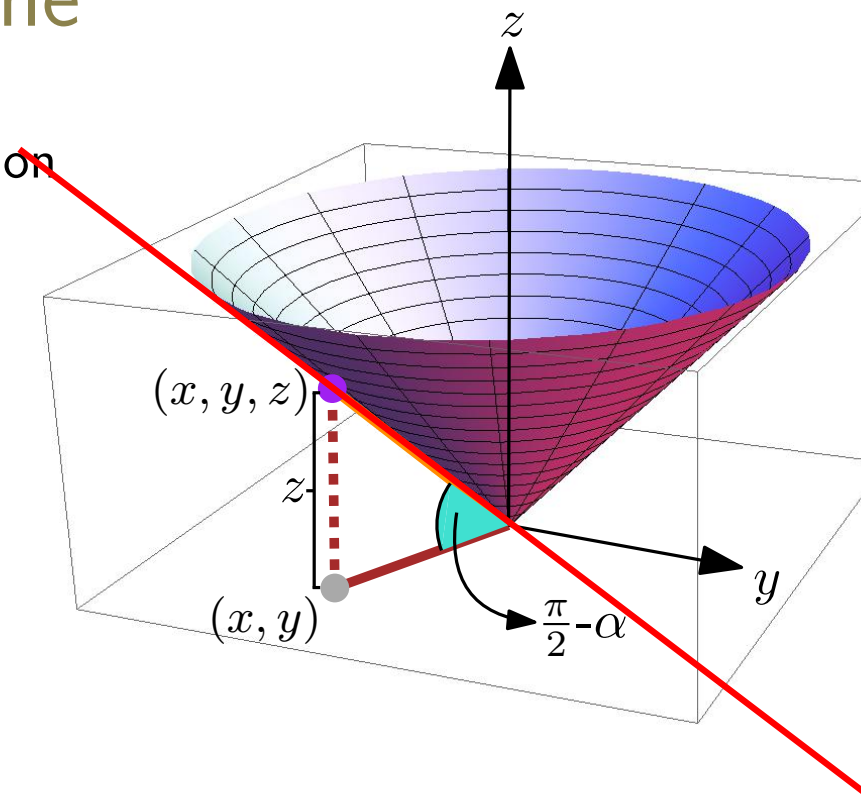

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EXAMPLES OF SURFACES

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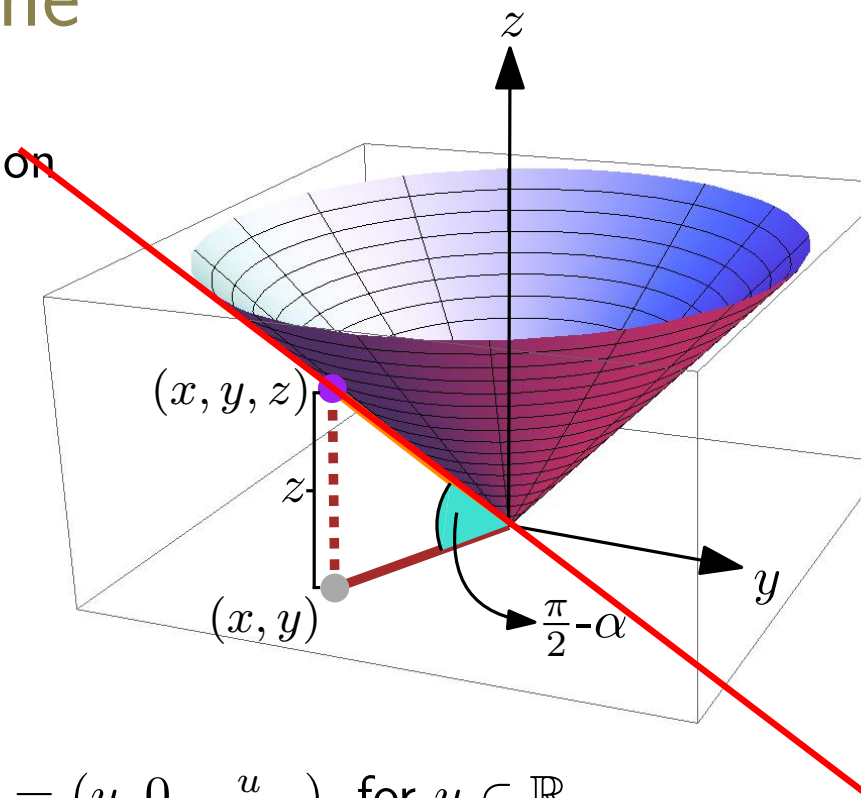

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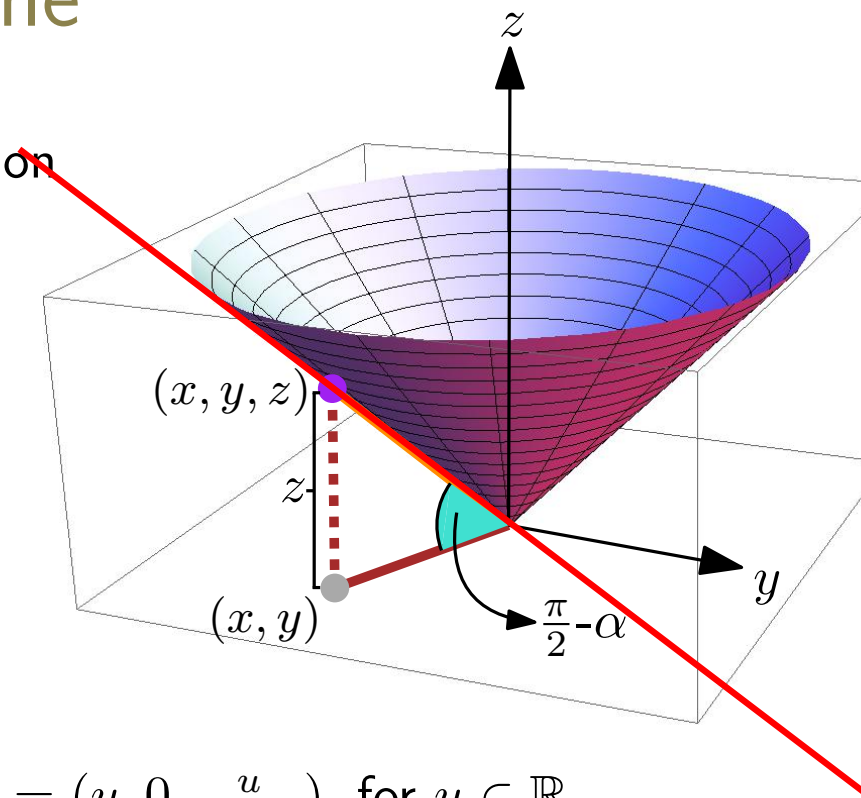

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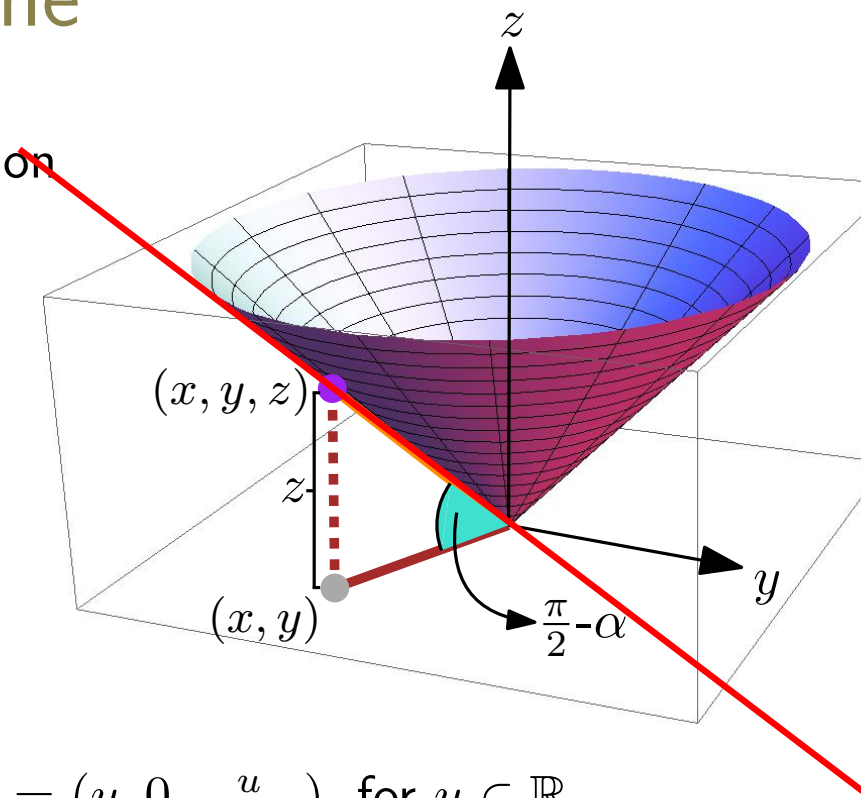

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parametric equation of the cone

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Normal vector, tangent plane

Let S be a surface parametrized as
$S(u, v) = (x(u, v), y(u, v), z(u, v))$ for (u, v) in some domain

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If the functions x, y, z have partial derivatives, we consider $\begin{cases} \frac{\partial S}{\partial u} = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) \\ \frac{\partial S}{\partial v} = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) \end{cases}$

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If the functions x, y, z have partial derivatives, we consider $\begin{cases} \frac{\partial S}{\partial u} = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) \\ \frac{\partial S}{\partial v} = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) \end{cases}$

A point P is called *regular* if $\frac{\partial S}{\partial u}(P)$ and $\frac{\partial S}{\partial v}(P)$:

- exist
- are continuous at P
- their cross product is not zero

If one of these conditions does not hold for P, it is called a *singular* point

A surface is called *regular* if all its points are regular

Normal vector, tangent plane

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Tangent plane

The plane tangent to S at P is given by the plane defined by P and $\vec{N}$

Normal vector, tangent plane

$\frac{\partial S}{\partial u}(P)$ and $\frac{\partial S}{\partial v}(P)$ are tangent vectors in the u and v directions

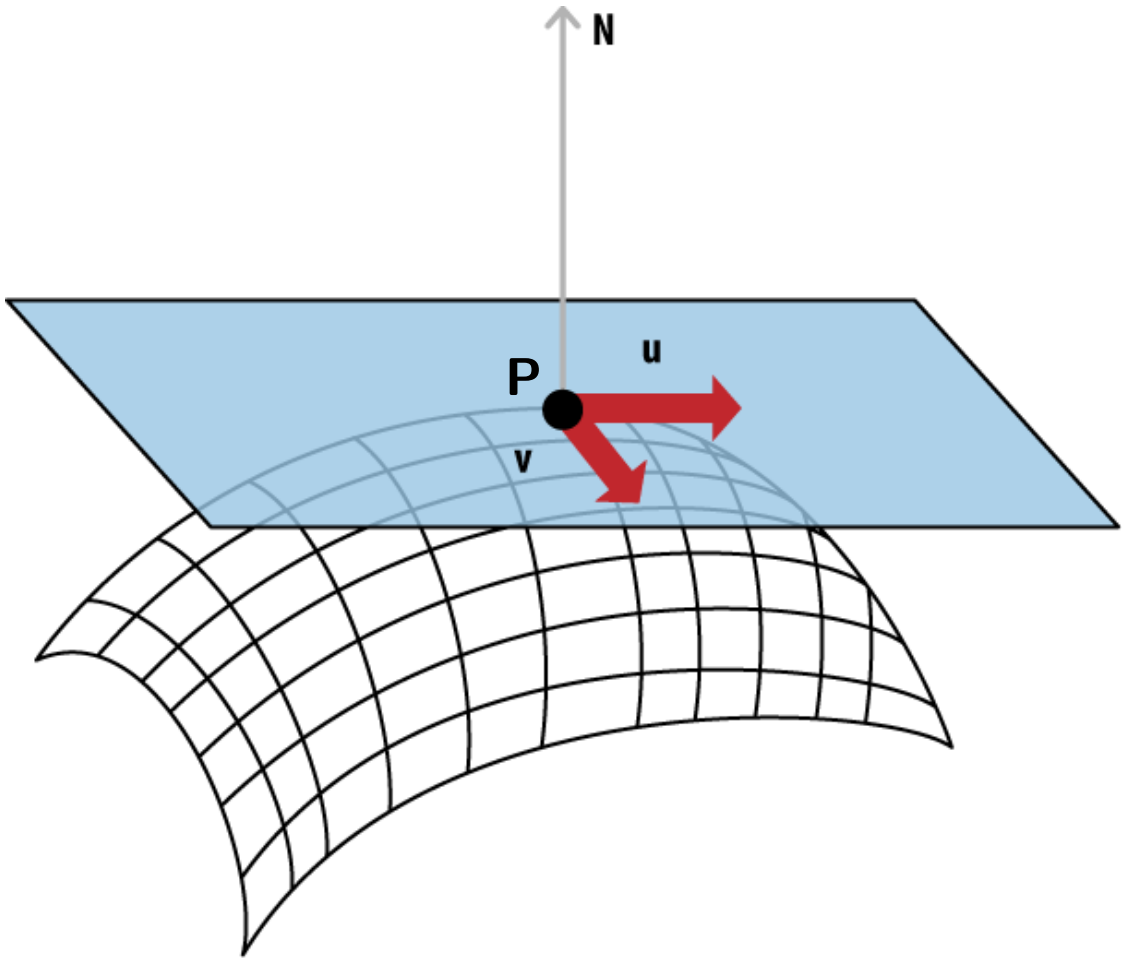

Figure from oreilley.com

Normal vector, tangent plane

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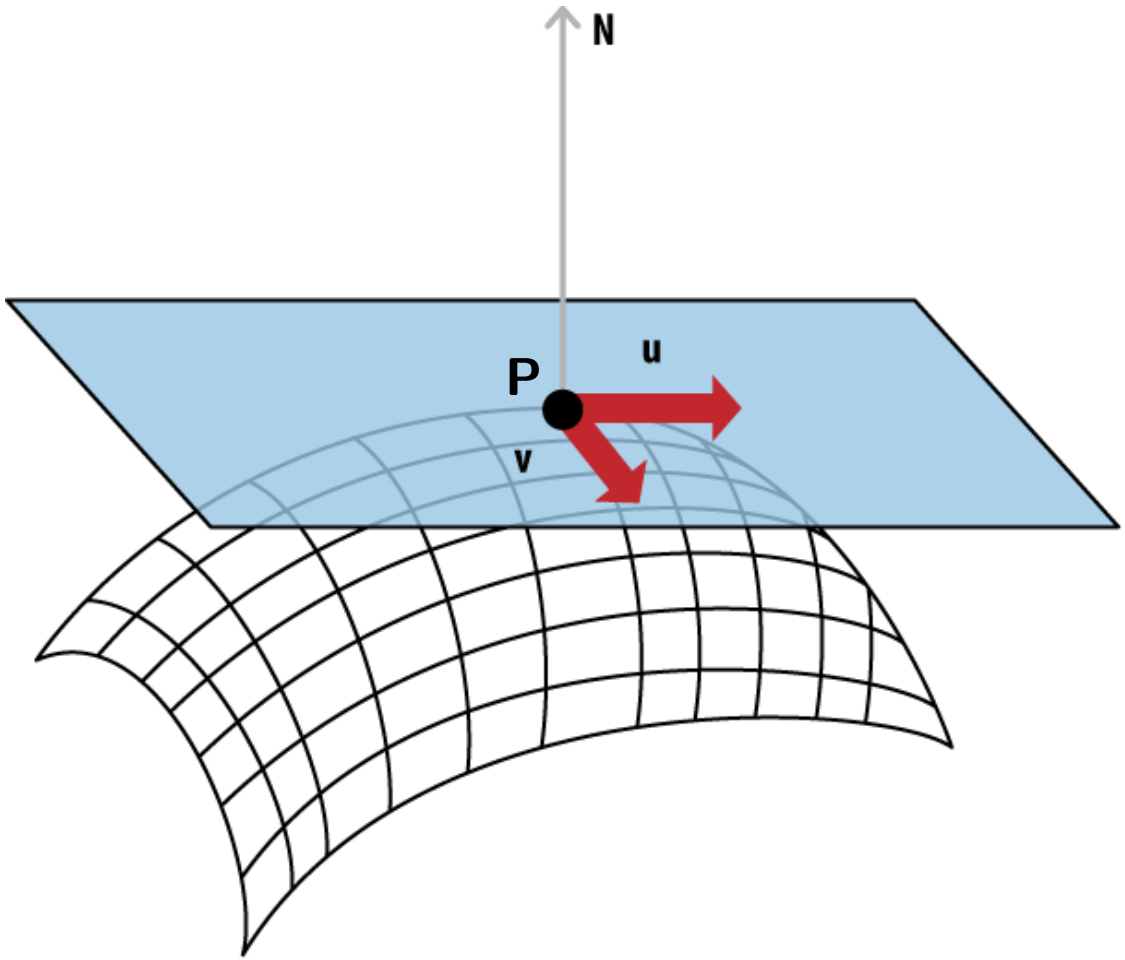

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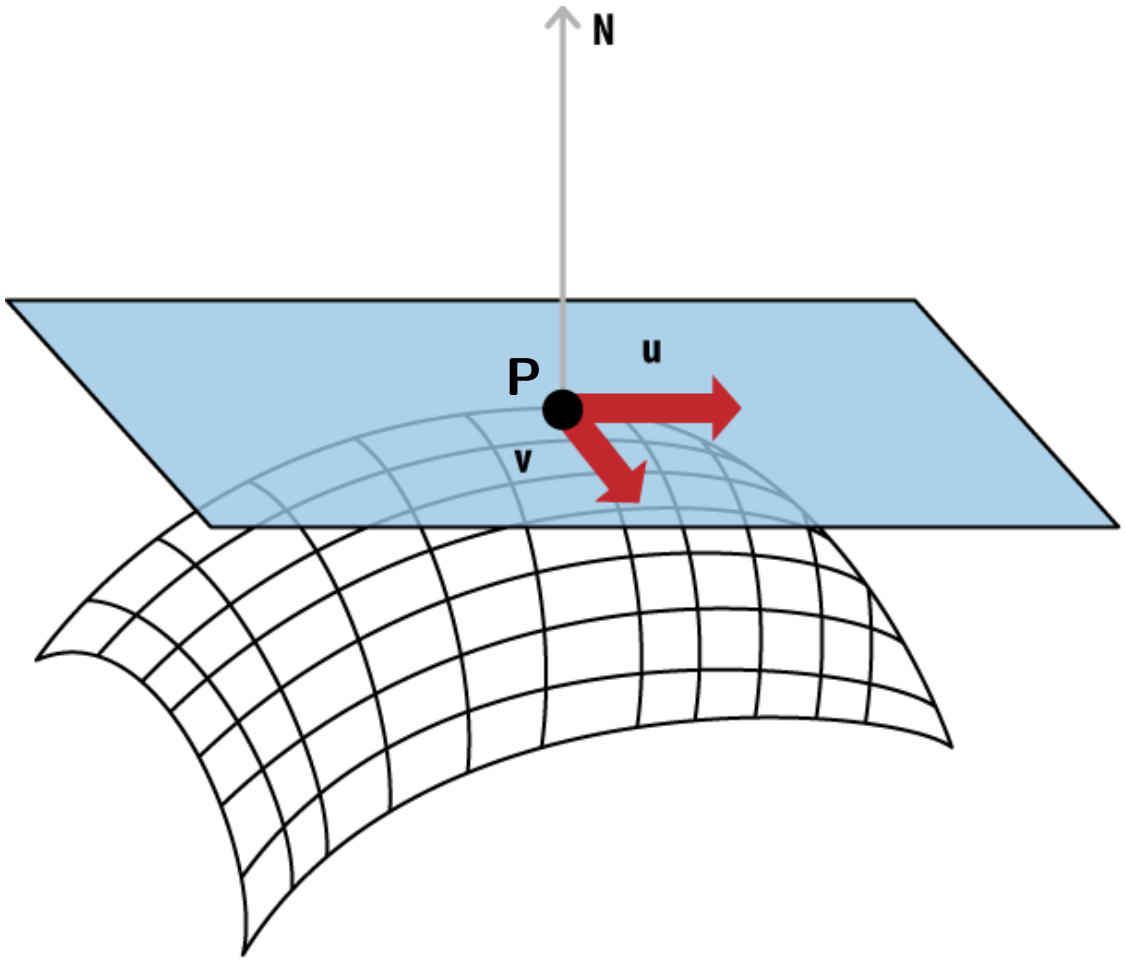
Exceptions:

- Surface does not have a tangent plane at P
- Parametrization is irregular at P



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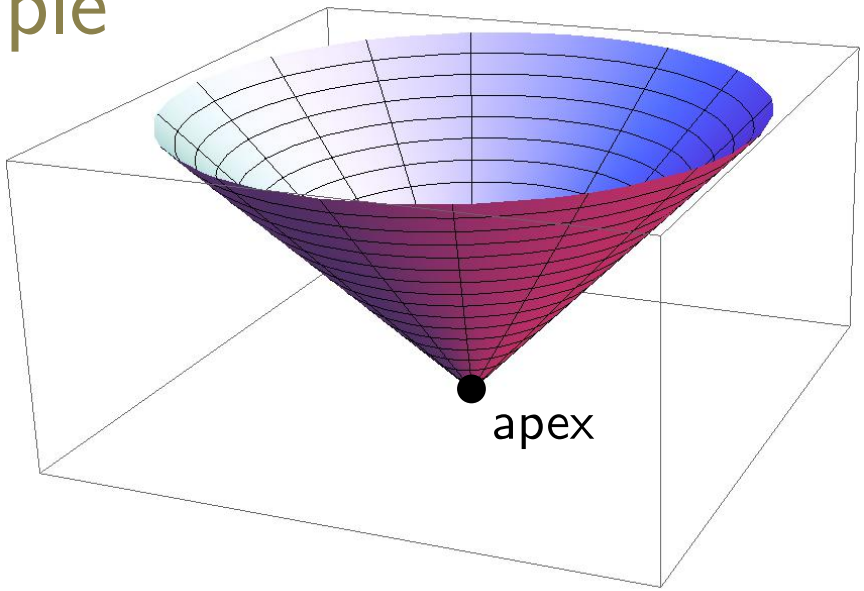
Normal vector, tangent plane: example

The apex of a cone is a **singular point**



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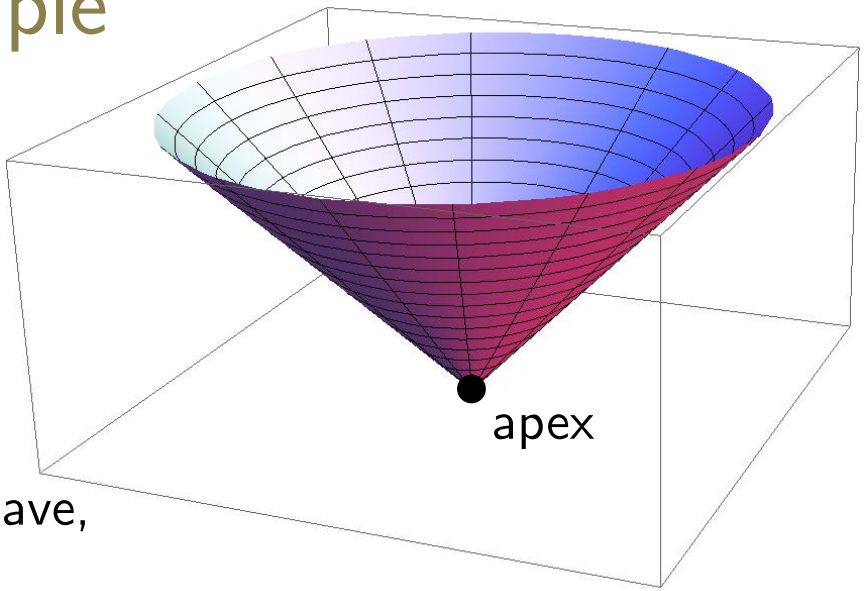

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Consider the cone $x^2 + y^2 = z^2$, which can be parametrized as:

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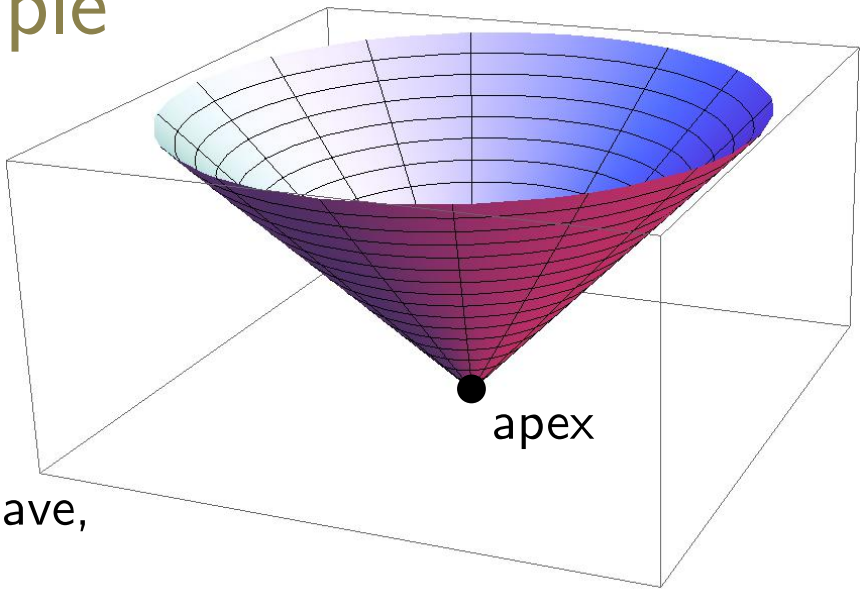

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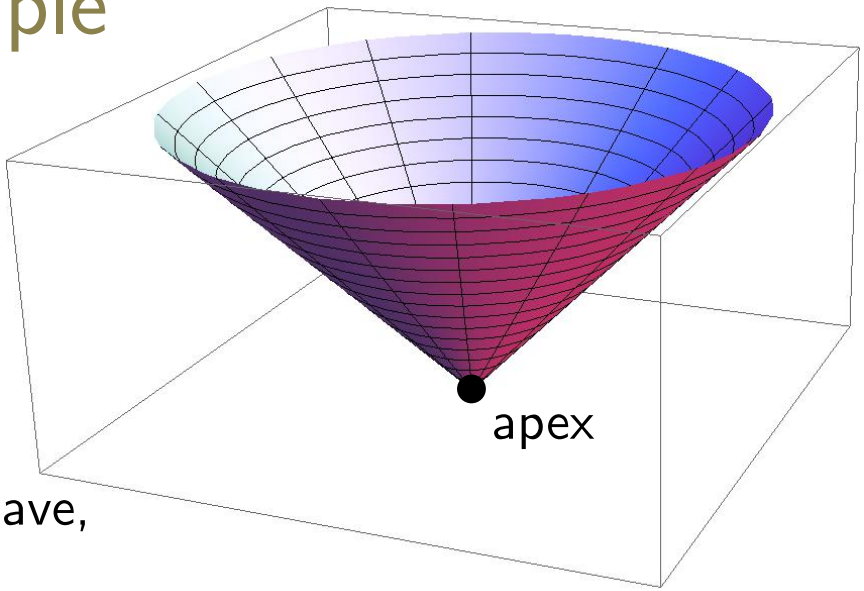

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The tangent vectors are as follows:

$\frac{\partial S}{\partial u} = (\cos v, \sin v, 1)$

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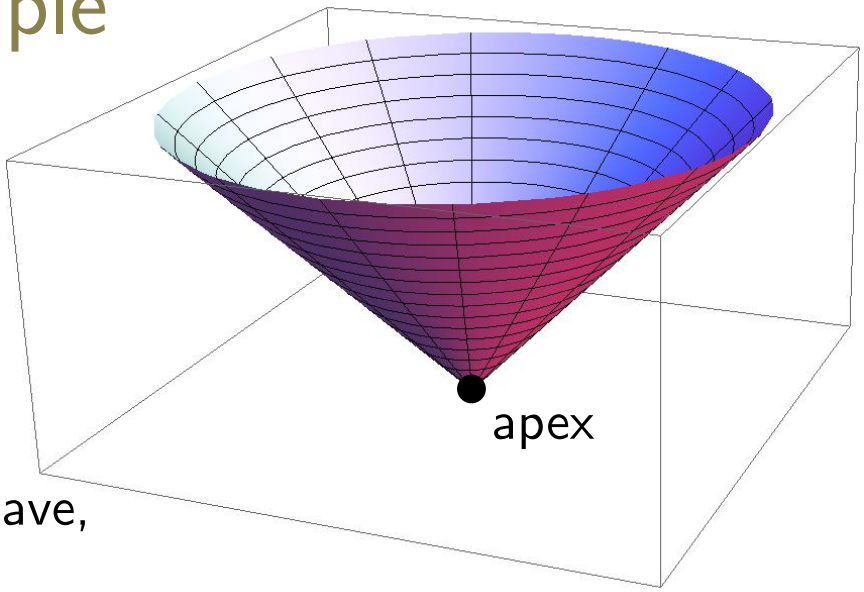

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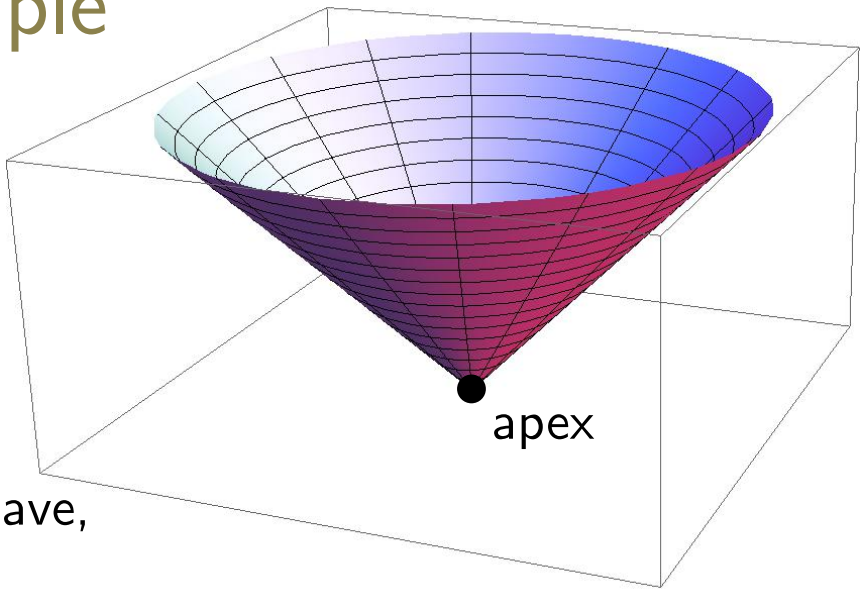

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For the apex, $P = (0,0,0)$, we get $\vec{N}(0,0) = 0$

(thus, the apex is a singular point)

Normal vector, tangent plane: example

Consider the north hemisphere of the unit sphere

Normal vector, tangent plane: example

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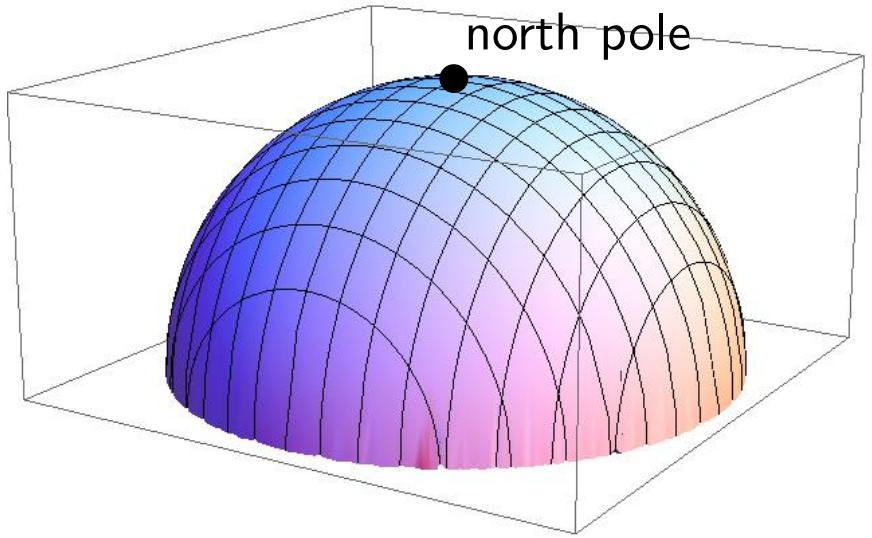

Normal vector, tangent plane: example

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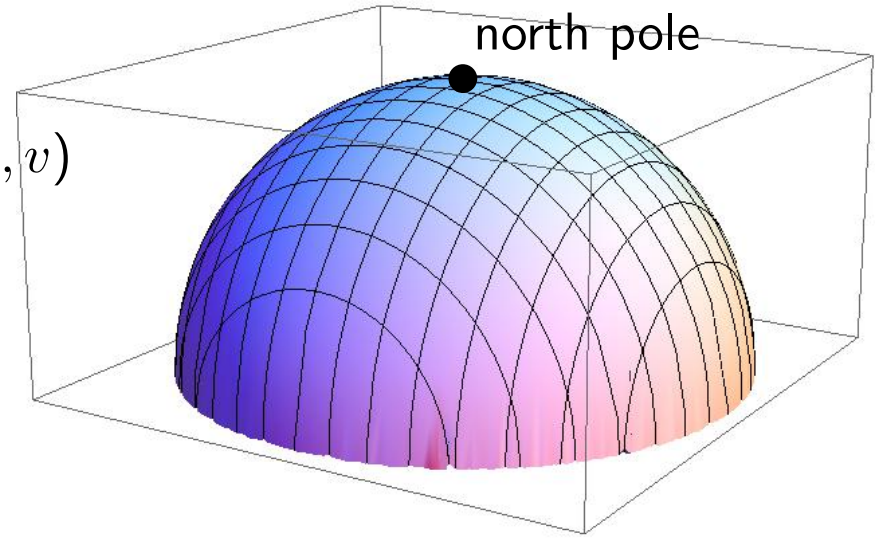

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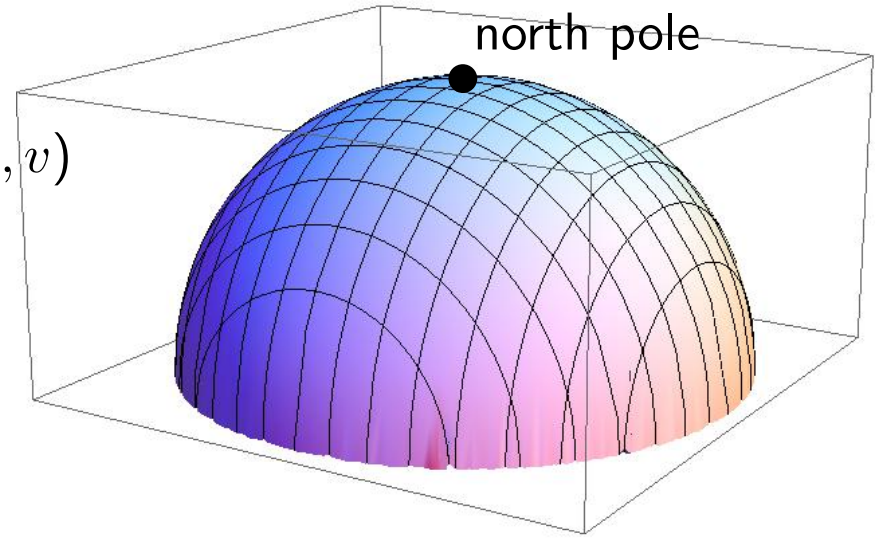

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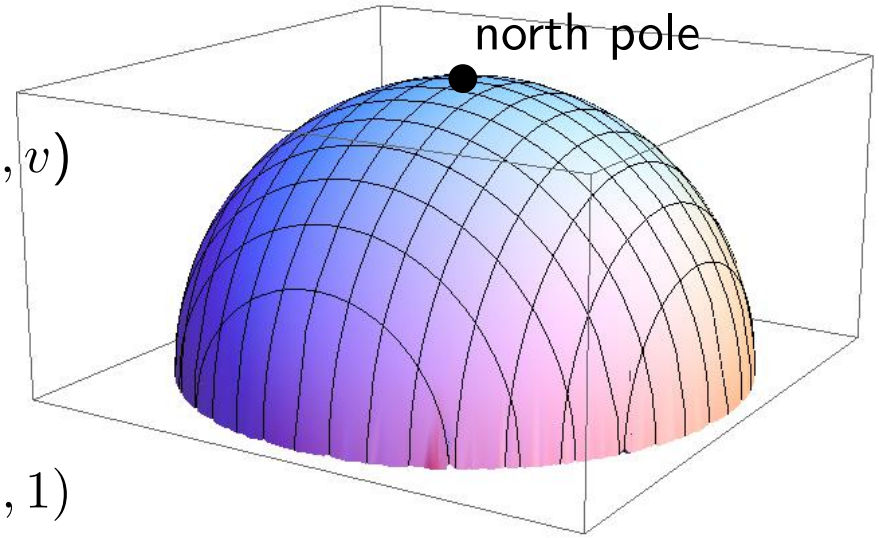

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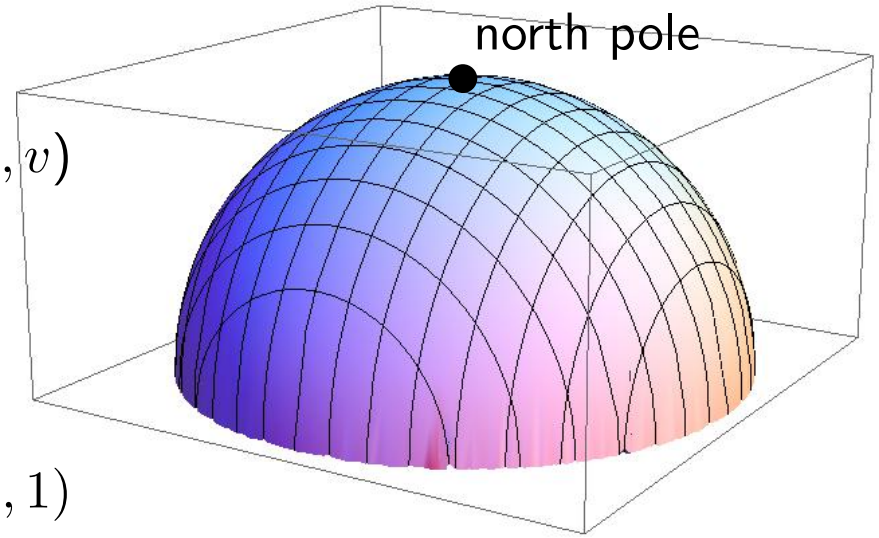

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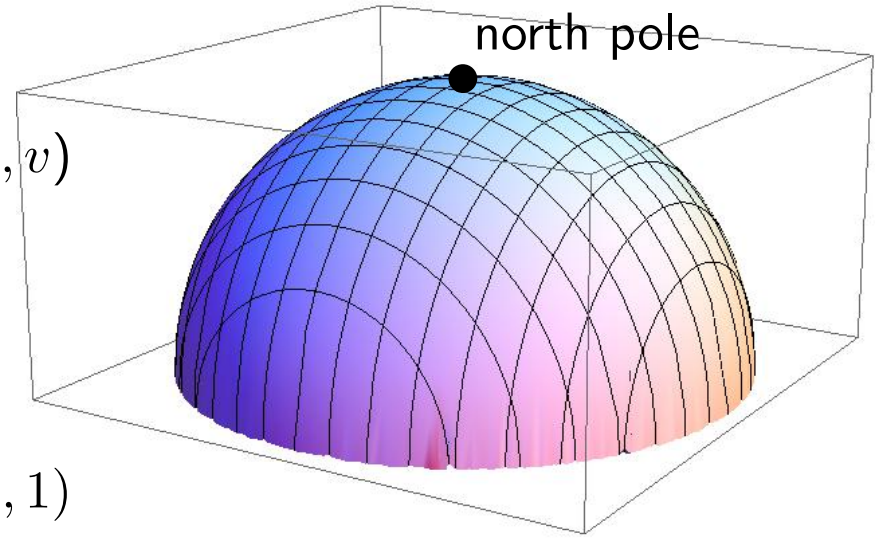

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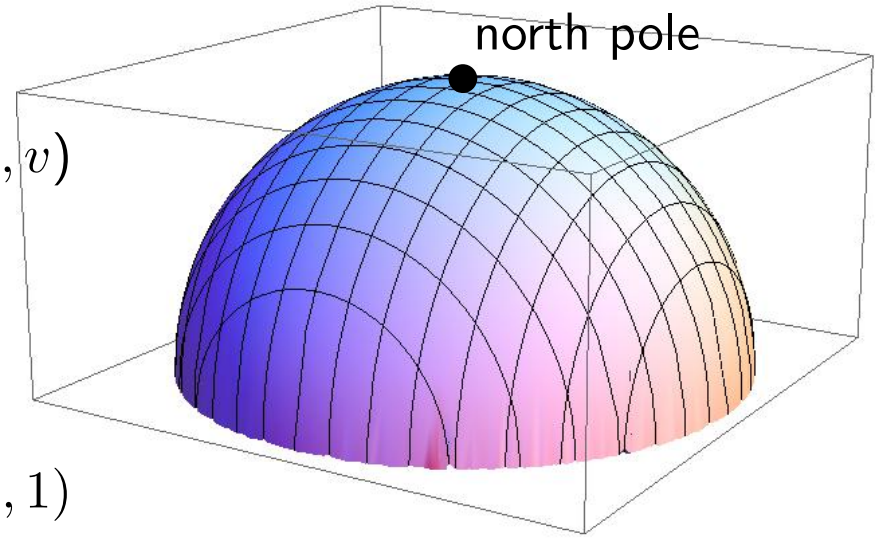

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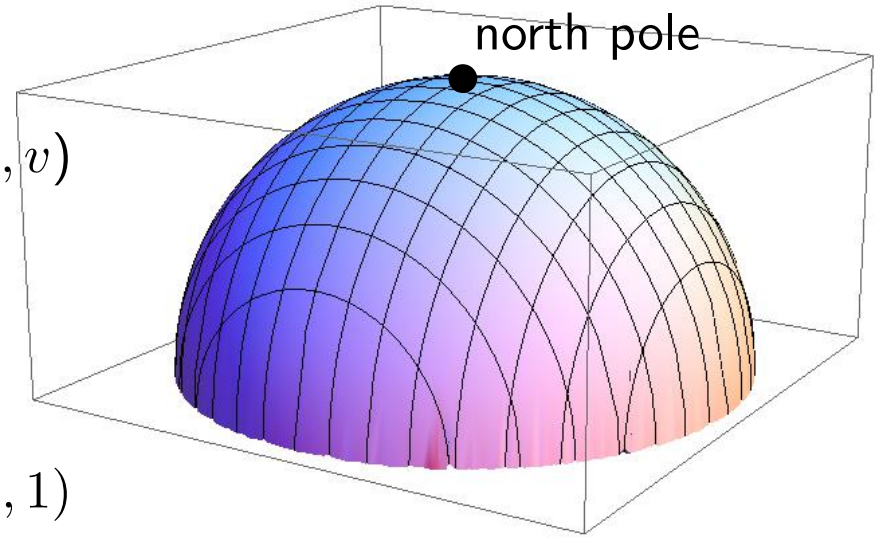

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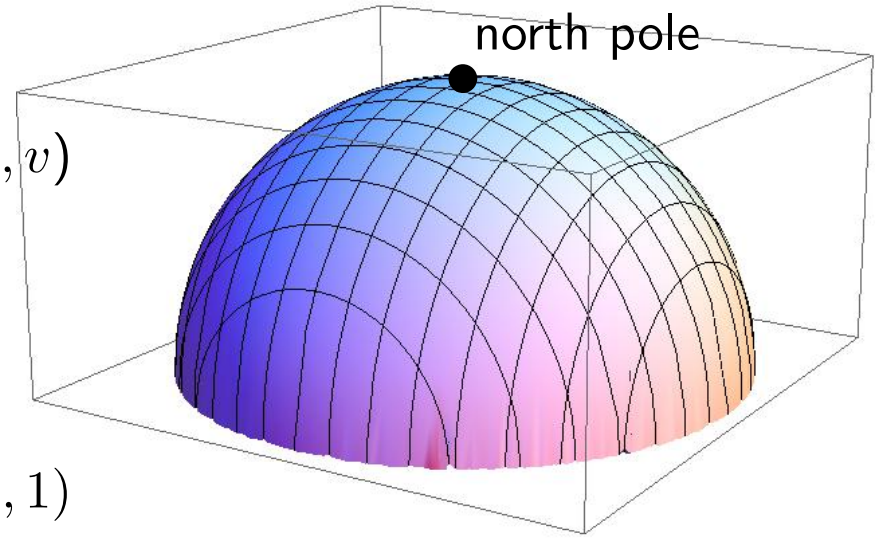

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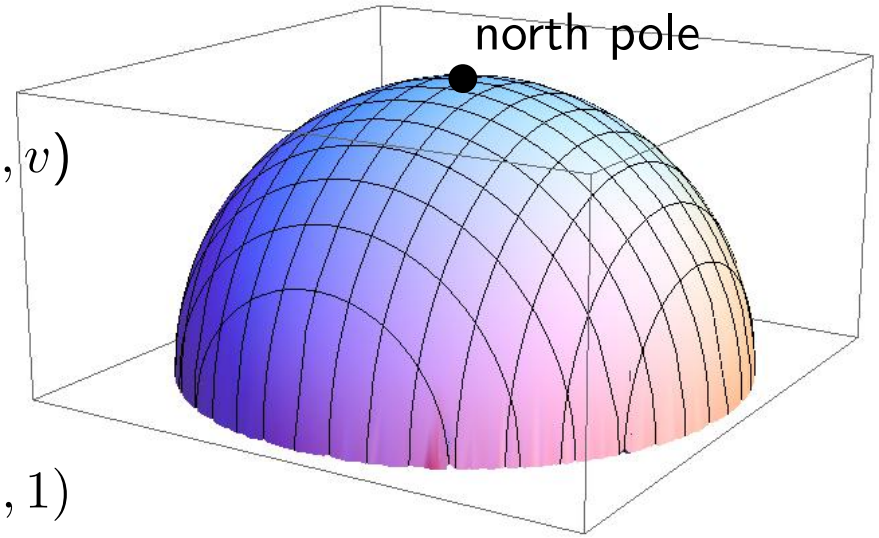

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$\rightarrow$ no issue with surface, but with parametrization