BÉZIER CURVES

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Interpolation or... approximation!

Previous curve design methods based on **interpolation**

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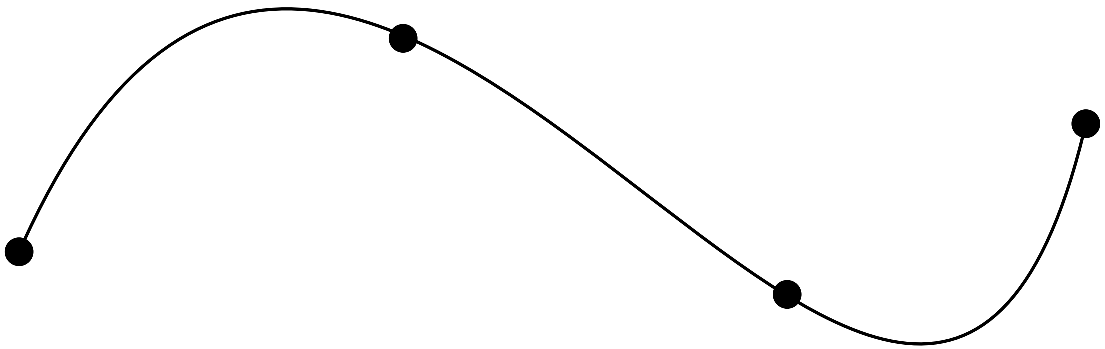

Interpolating curve

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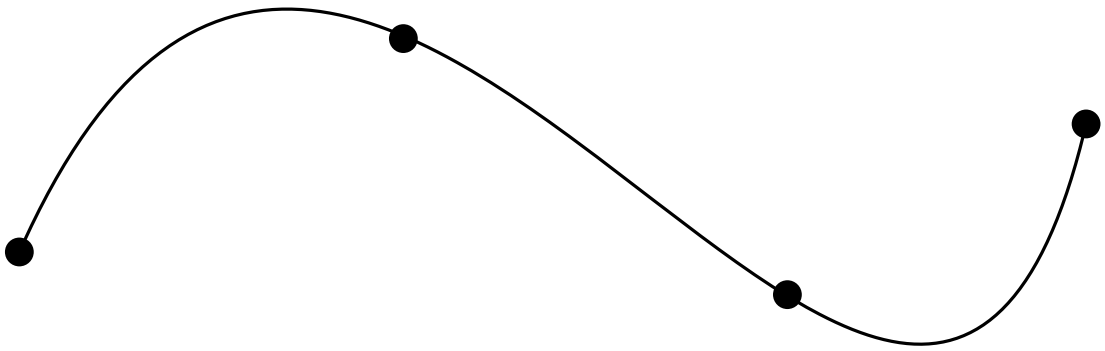
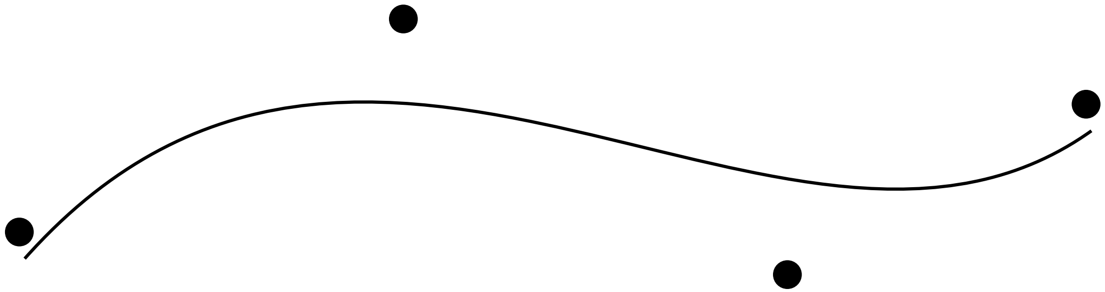


Interpolating curve

Curve passes exactly through given points



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Curve passes near the given points

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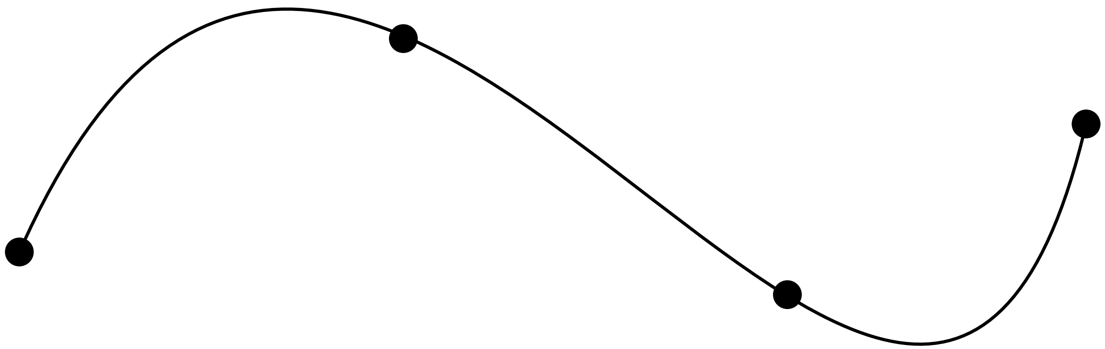

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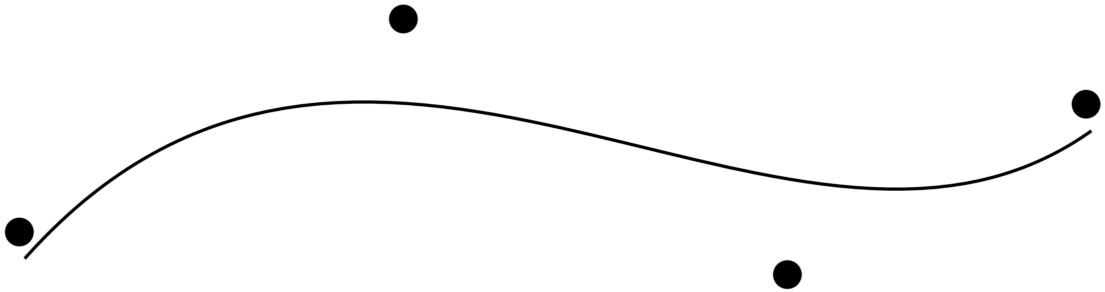


Approximating curve

Curve passes near the given points

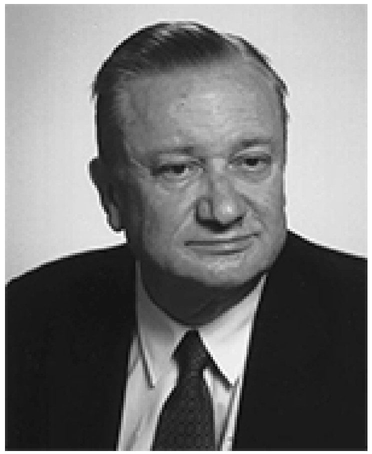
What's wrong with interpolation? Curve change when moving points can be unpredictable

Approximating curves can provide better "shape control"

Bézier curves

Named after Pierre Bézier (1910-1999)

- Worked on automatizing the process of designing cars
- Paul de Casteljau (1930-2022) developed similar methods for Citröen, but were published later



INTRODUCTION TO BÉZIER CURVES

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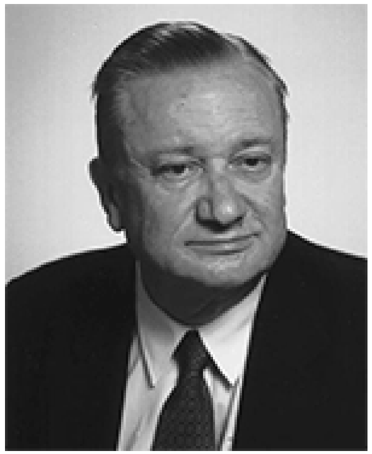
Bézier curve

- Parametric $(P(t))$
- Polynomial
- Based on *control points*



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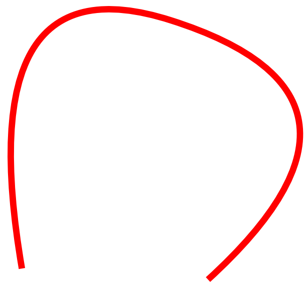

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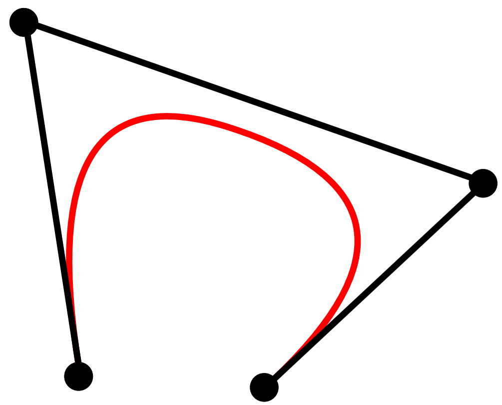
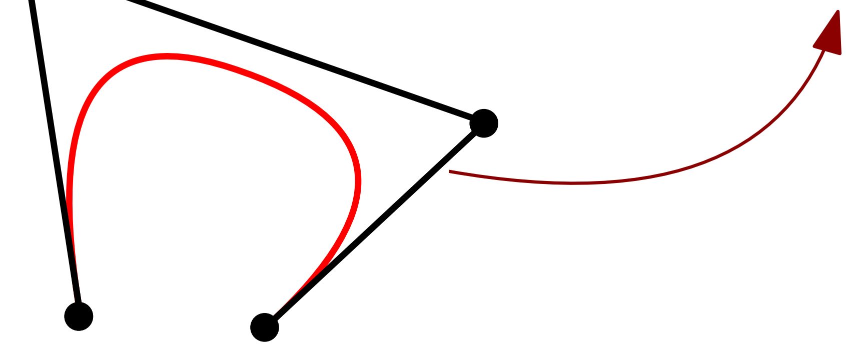


Some examples of Bézier curves

Control polygonal line, made of control points.
Each control point exerts a pull on the curve, pulling it towards itself



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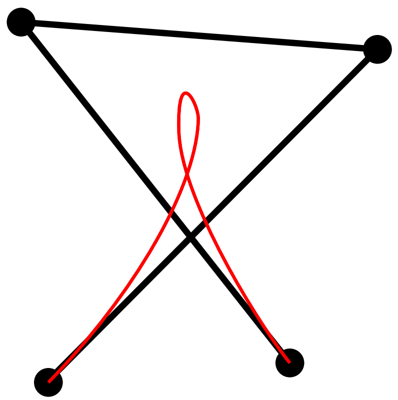

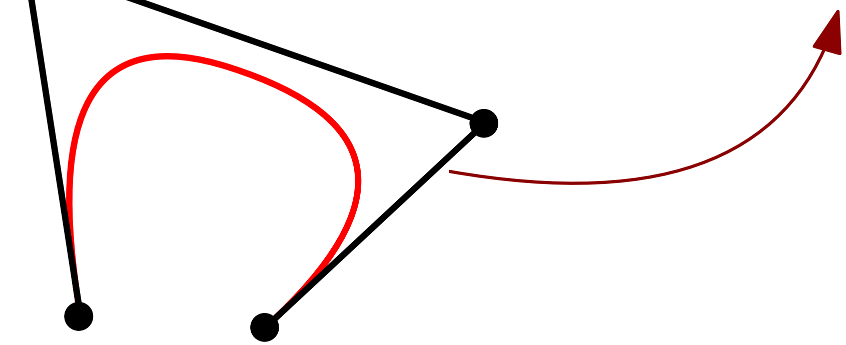

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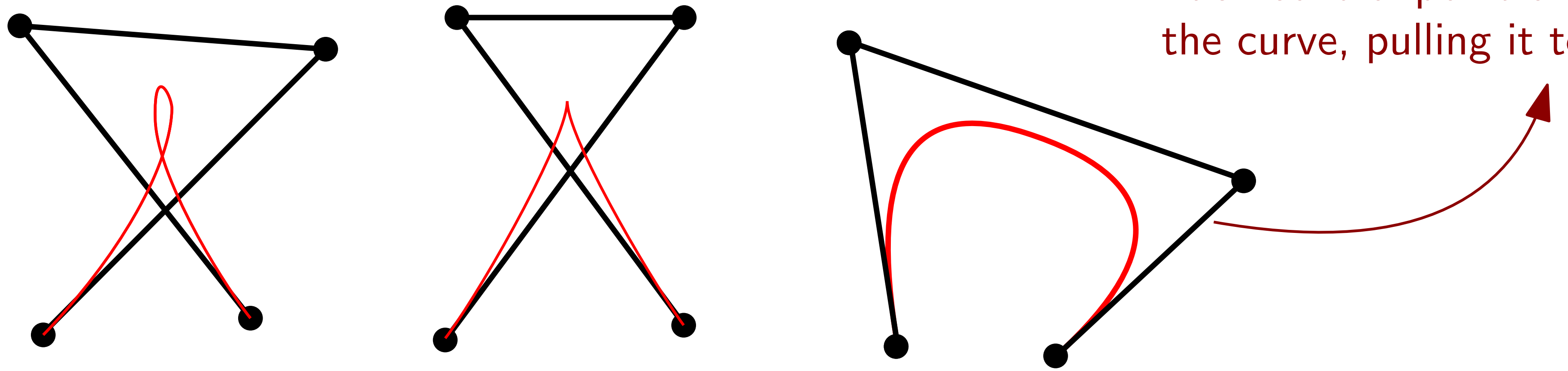

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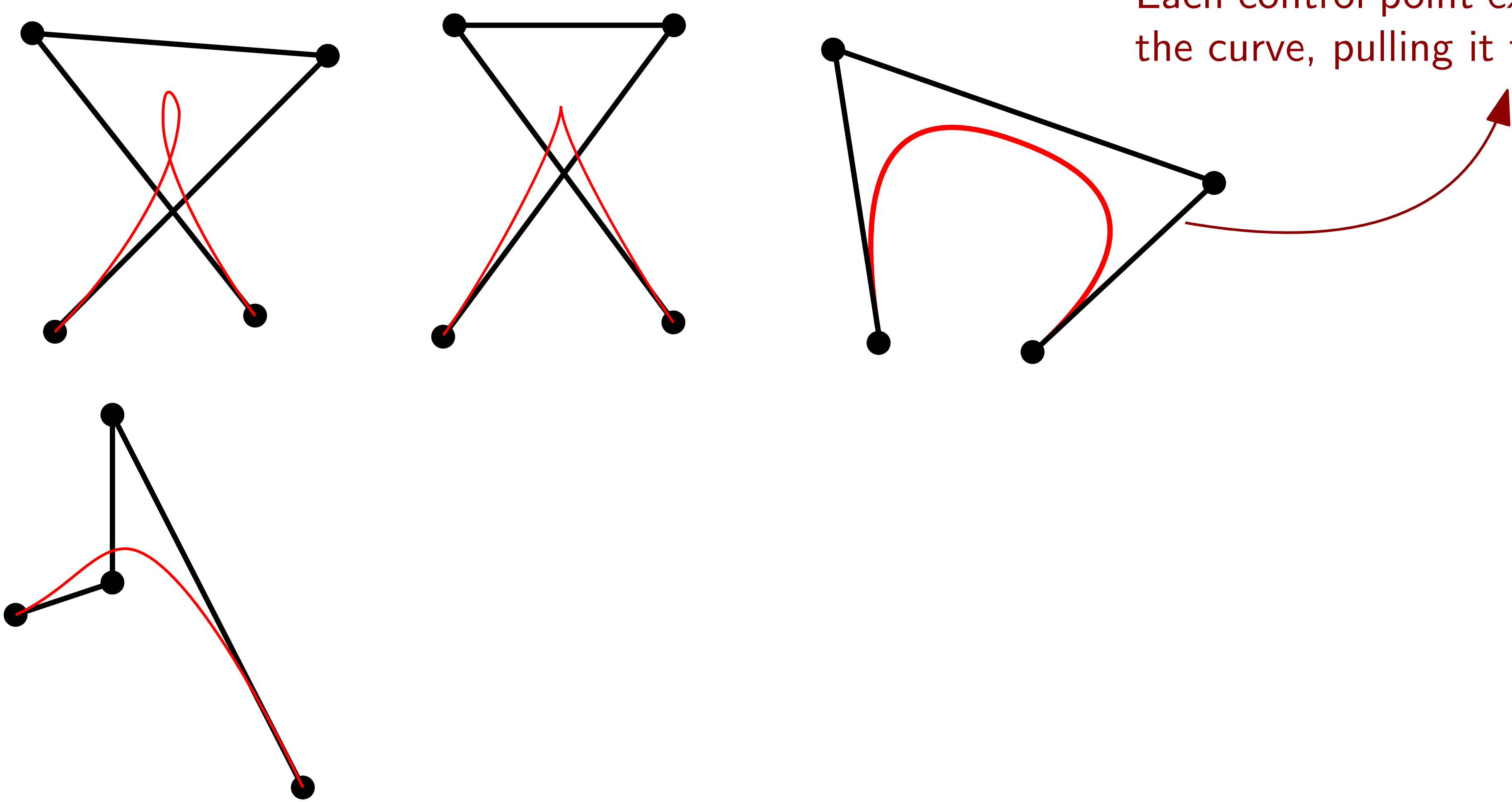

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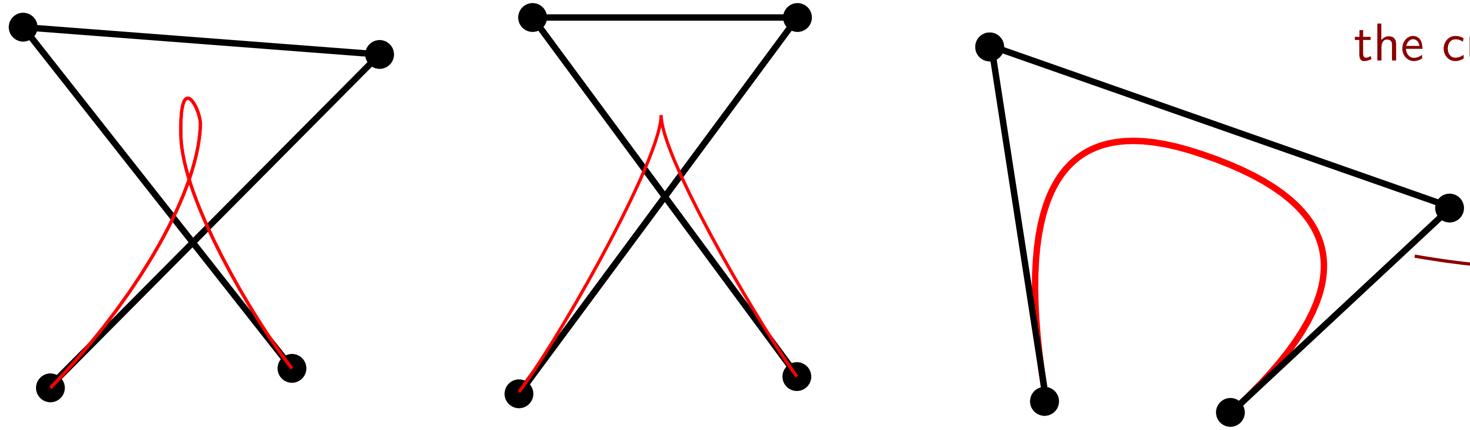

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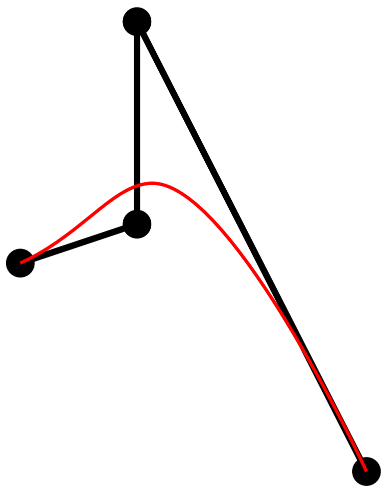

Each curve here is a polynomial, of degree....

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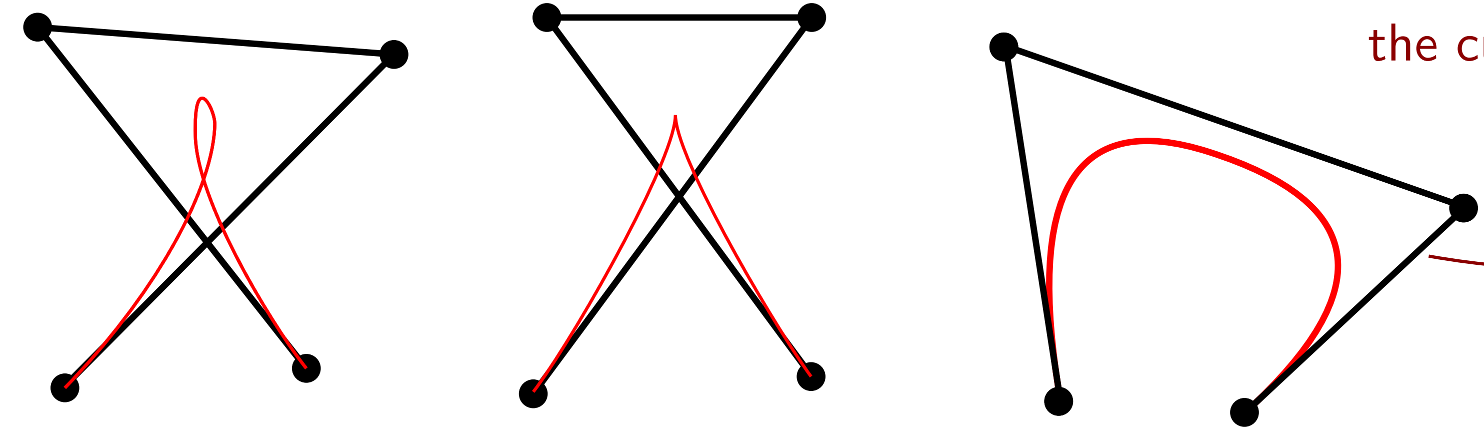

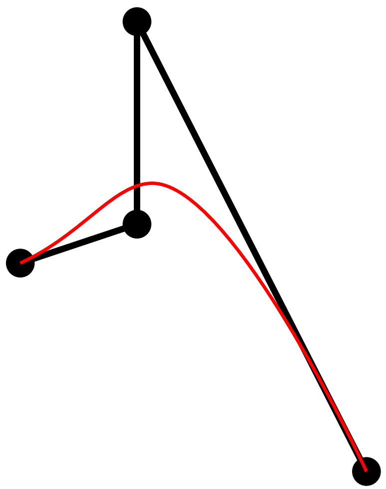

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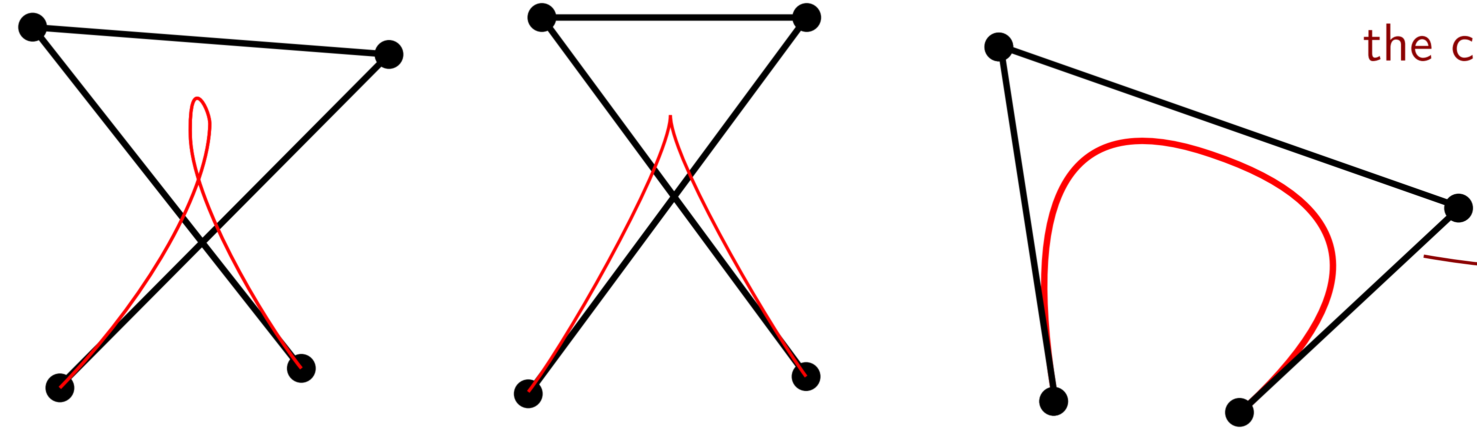

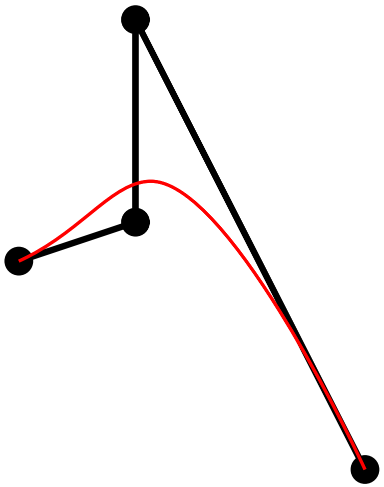

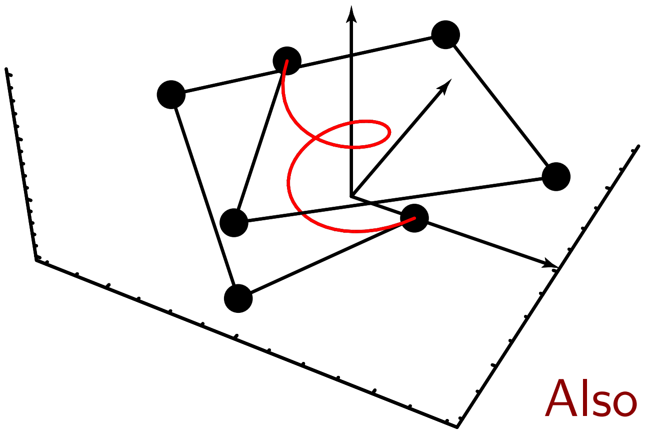

Also works in 3D!

What is a Bézier curve?

General form

$$P(t) = \sum_{i=0}^{n} P_i f_i(t) \qquad t \in [0, 1]$$

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- The basis functions must be symmetric with respect to t and $(1-t)$ (so reversing the parameter and the order of control points gives the same curve)
- Control point weights are barycentric (=affine sum): shape independent from coordinate system. That is: $P(t)$ is an affine combination of control points, so curve is affine invariant

Basis functions

The family a functions used are **Bernstein polynomials**

$$f_i(t) = B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

recall that
$0 \le i \le n$,
$\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $0! = 1$
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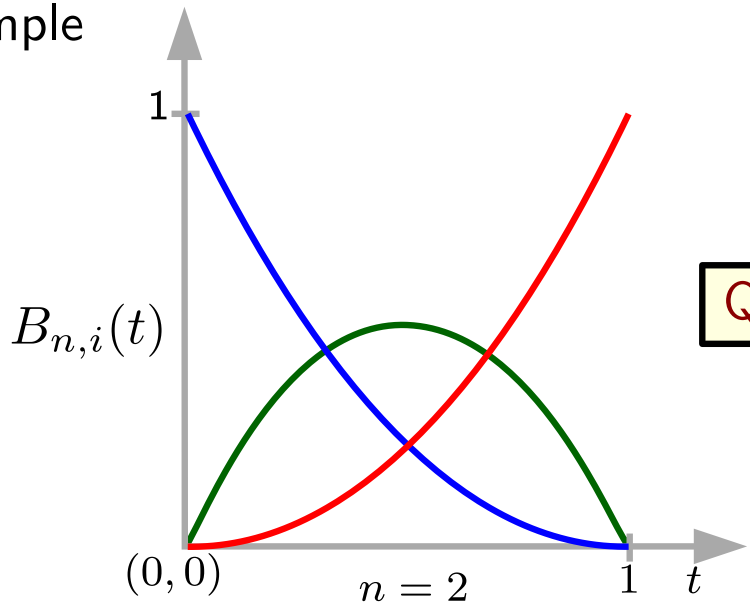

Question: Which basis function is which?

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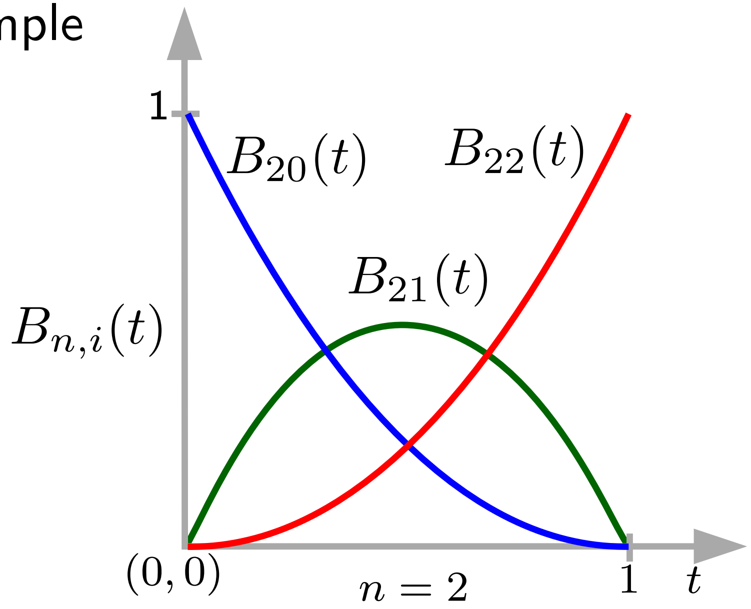

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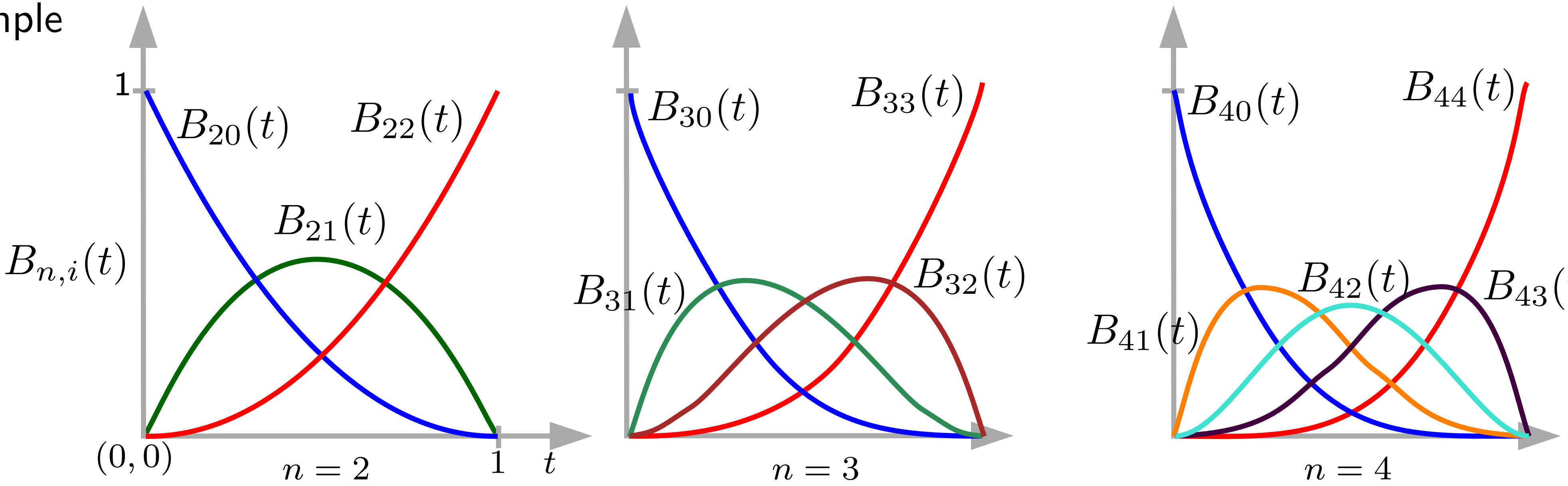

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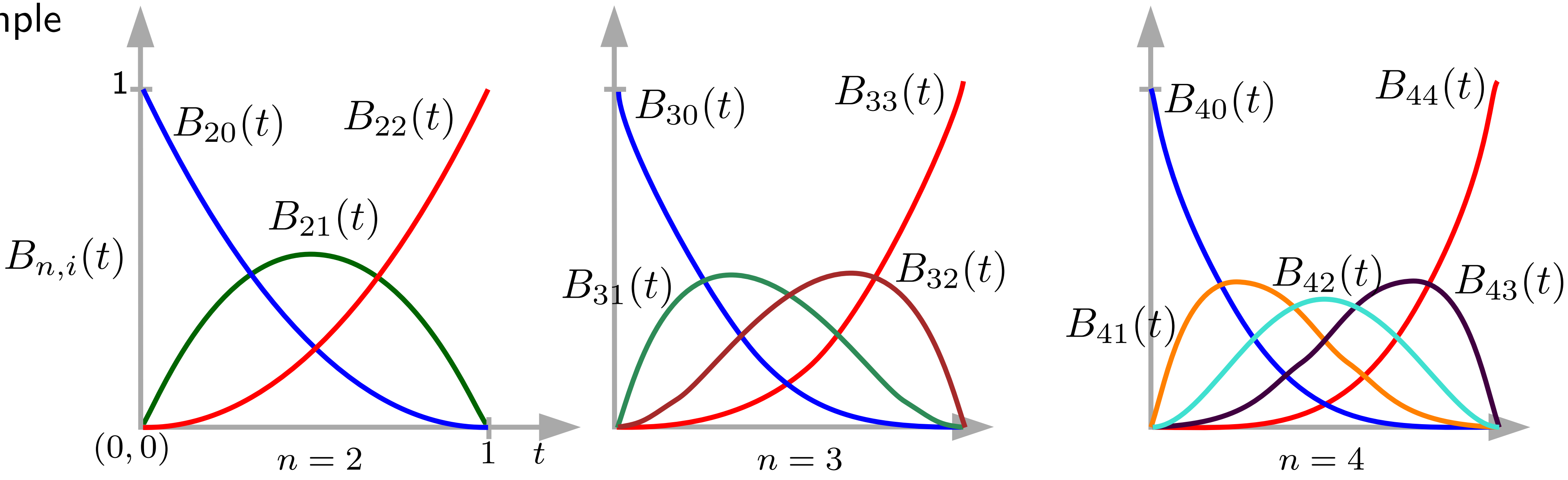

The Bézier curve becomes $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \qquad t \in [0,1]$

Example: degree-2 Bézier curve

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For $n = 2$, we have $B_{n,i}(t) = \binom{2}{i} t^i (1-t)^{2-i}$, for $0 \leq i \leq 2$

So, for $n = 2$, these are the three Bernstein polynomials:

- $B_{2,0}(t) = \binom{2}{0} t^0 (1-t)^{2-0} = (1-t)^2$
- $B_{2,1}(t) = \binom{2}{1} t^1 (1-t)^{2-1} = 2t(1-t)$
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So the quadratic Bézier curve is

$$P(t) = (1-t)^2 P_0 + 2t(1-t) P_1 + t^2 P_2$$

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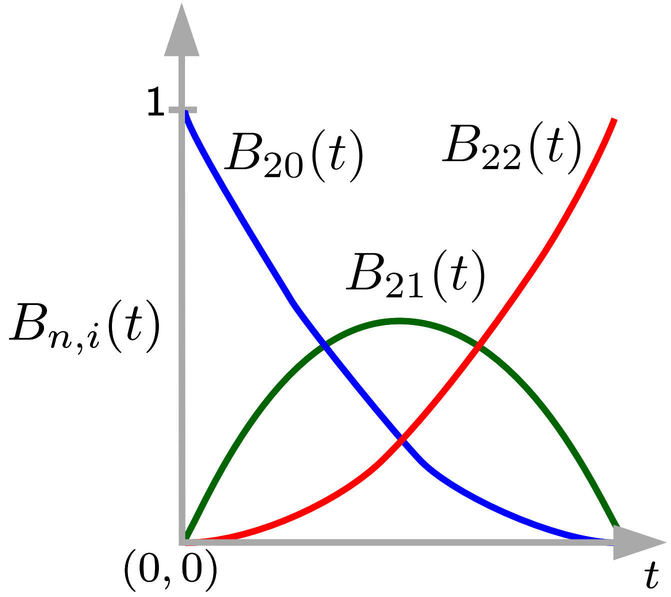
Example?

Question: Does this curve satisfy the properties in the previous slide?

Properties of Bézier curves

1. Endpoint interpolation
2. Symmetry
3. Affine invariance
4. Invariance under affine parameter transformations
5. Convex hull property
6. Pseudolocal control
7. Variation-diminishing property



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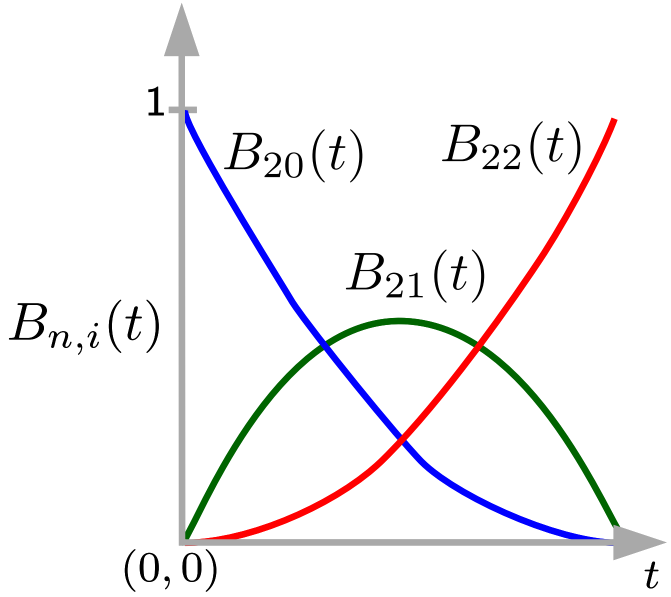

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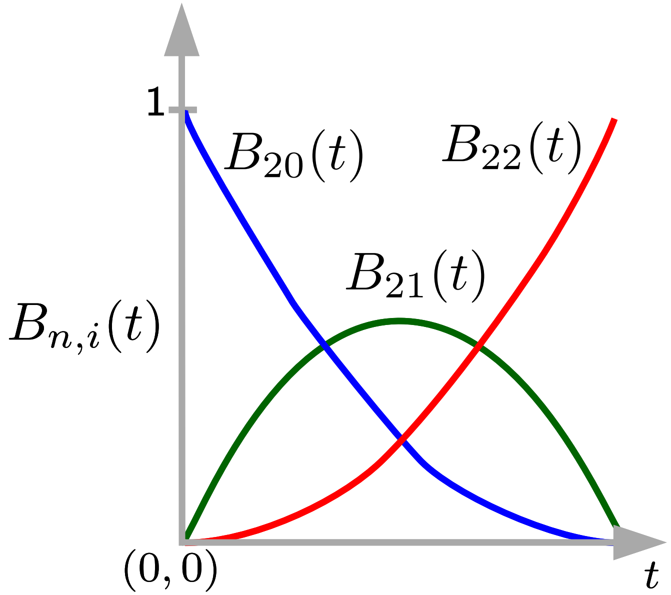
Properties of Bézier curves

3. Affine invariance

Applying an affine transformation to the curve is the same as applying the transformation to the control points

More precisely: $f(P(t)) = \sum_{i=0}^{n} f(P_i) B_{n,i}(t)$, for any affine map f, i.e., $f(v) = Av + W$



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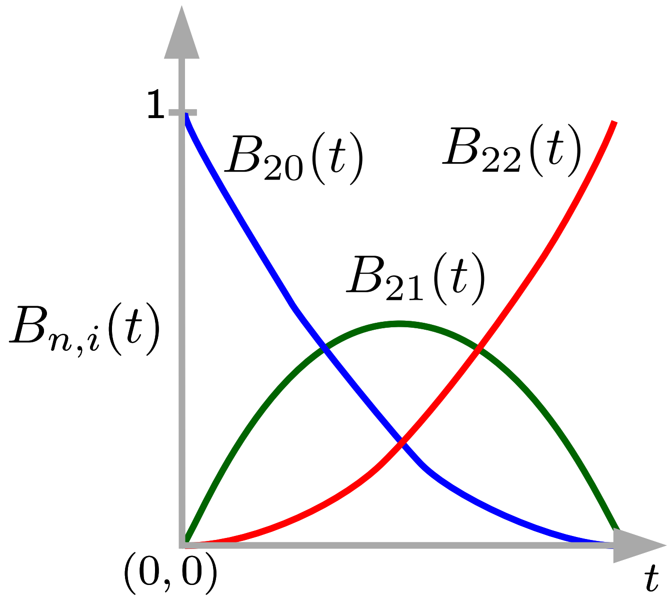
We can prove this by showing that $\sum_{i=0}^{n} B_{n,i}(t) = 1$

This follows from the binomial theorem:
$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$, with $a = t$ and $b = (1-t)$

Therefore Bézier curves are invariant under affine transformations!



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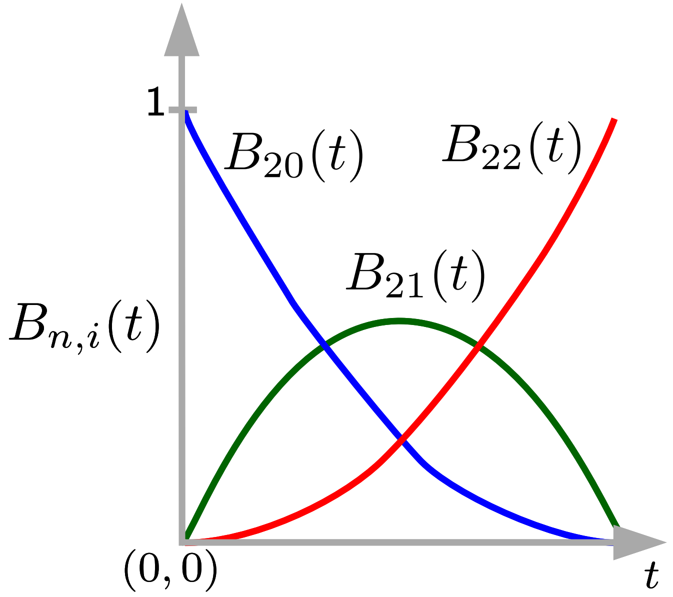

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4. Invariance under affine parameter transformations

If $C(t) = \sum_{i=0}^{n} P_i B_{n,i}(t)$ for $t \in [0,1]$, and $D(t) = \sum_{i=0}^{n} P_i B_{n,i}(\frac{u-a}{b-a})$ for $u \in [a,b]$, then curves C and D are equal

Practical consecuence: it is easy to have a curve defined over $[a,b]$ instead of $[0,1]$

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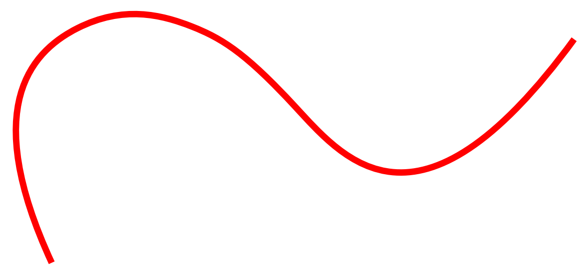
5. Convex hull property

The curve lies inside the convex hull of the control points



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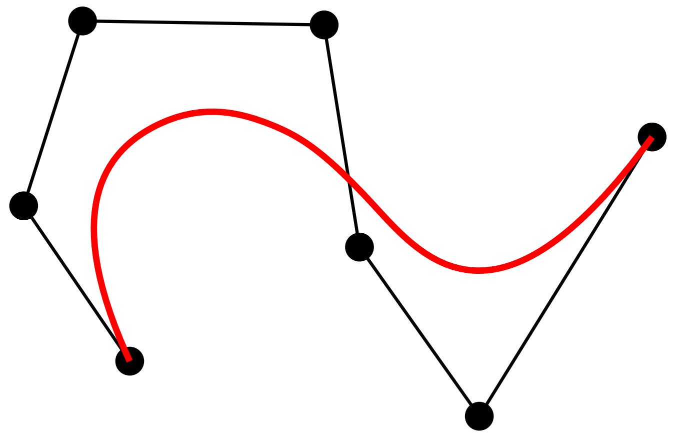

Properties of Bézier curves

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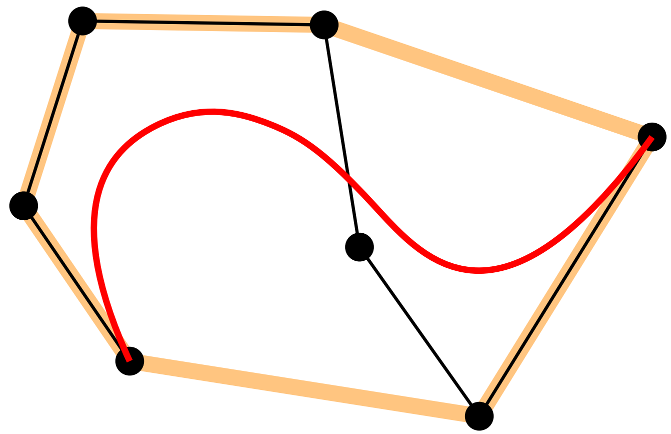

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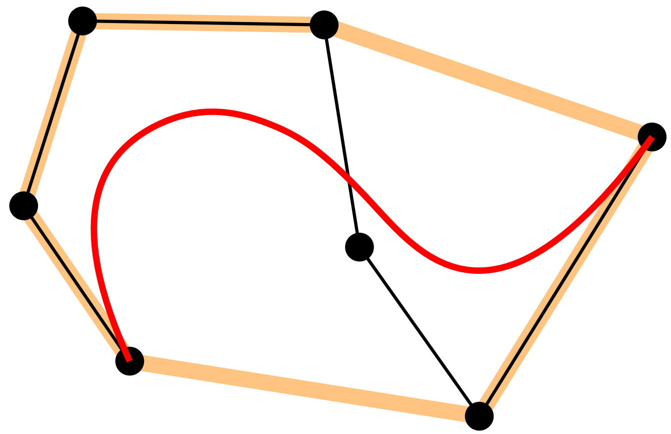

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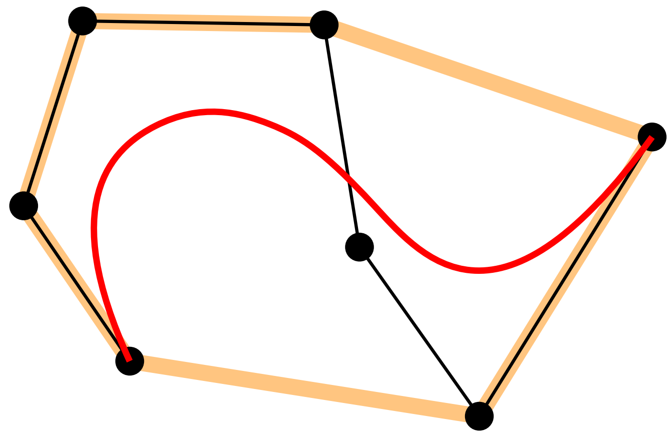

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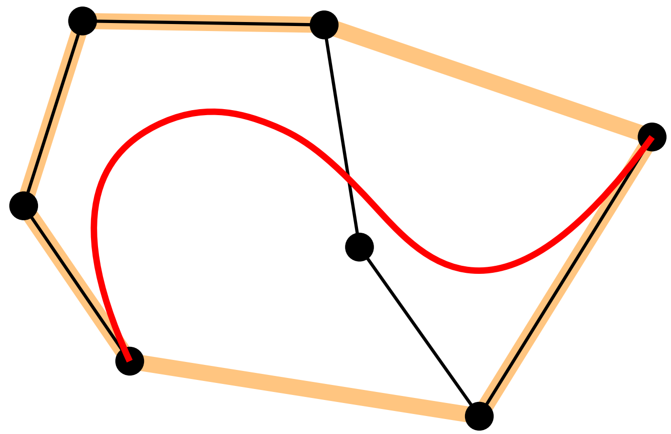

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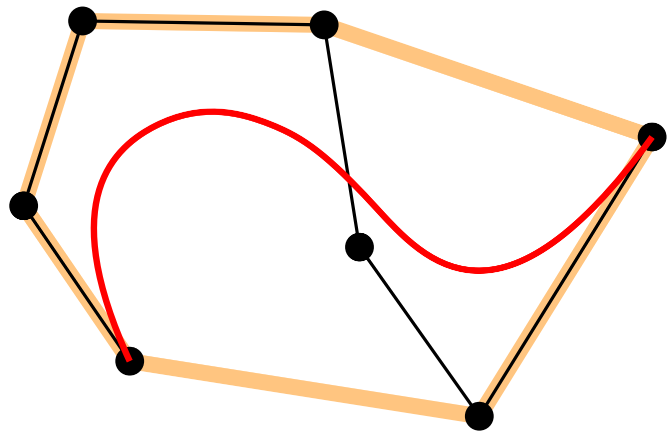

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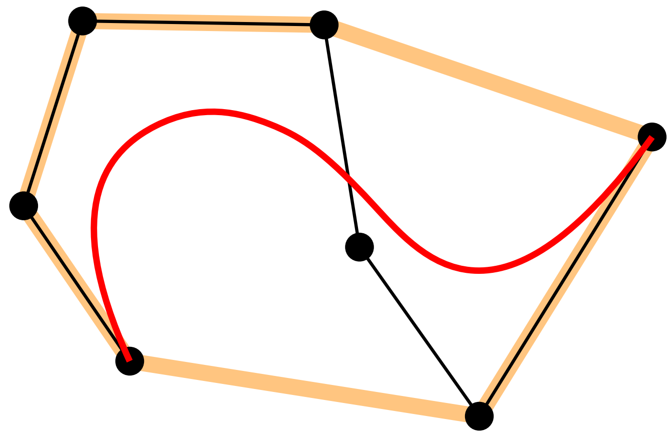
Point weights add up to one and are in $[0, 1]$, thus $P(t)$ is a **convex combination** of the control points.

The convex hull of the a set of points S is **exactly** the set of all convex combinations of points in S, thus all points in the curve belong to the convex hull.

Properties of Bézier curves



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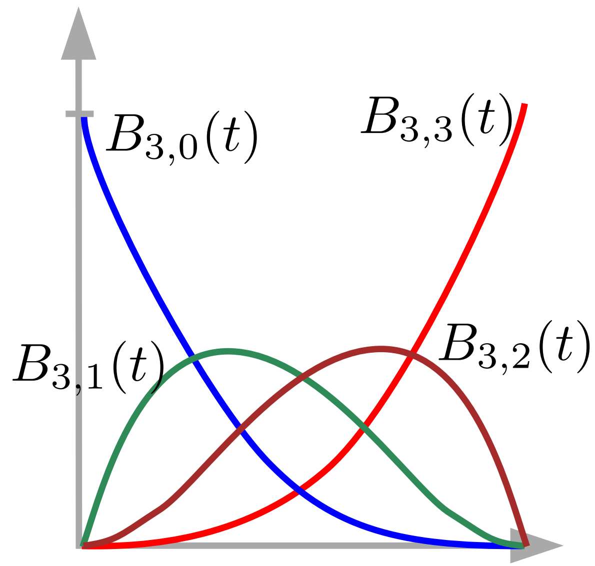
Question: What does this say about collinear control points?

Properties of Bézier curves

6. “Pseudolocal” control

Question: When does a control point influence the curve most?



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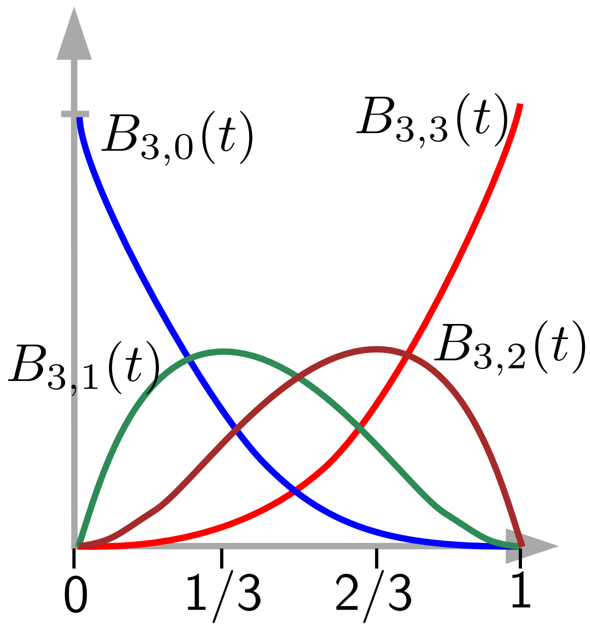
The ith Bernstein polynomial has only one maximum, at $t = i/n$.



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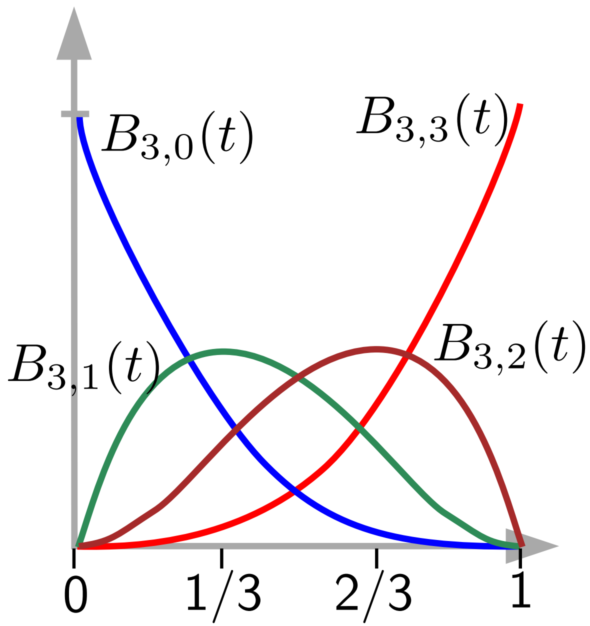

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Consequence: if we move only one control point, P_i, the curve is mostly affected around $t = i/n$. This makes the effect of the change more or less predictable.

However, note that the change still affects the whole curve (so it is **global control**—not local).

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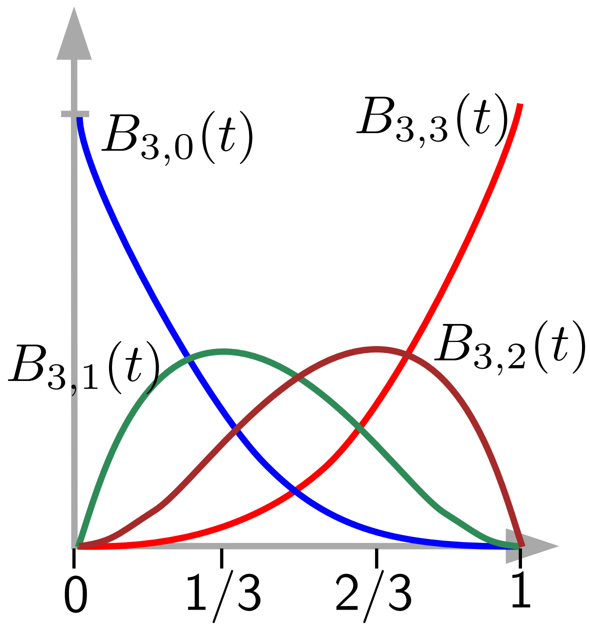


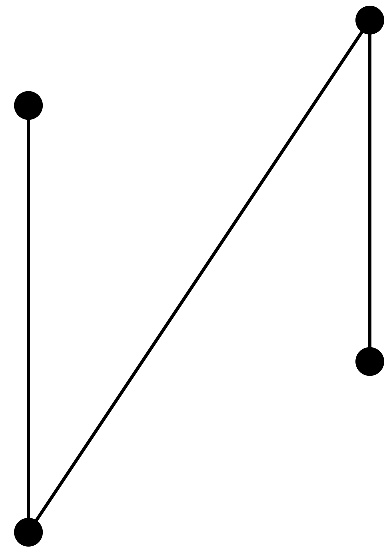
local maxima of $B_{n,i}$s

Question: What happens to $P(t)$ if P_k is moved by a vector (α, β)?

Properties of Bézier curves

7. Variation-diminishing property

The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygonal line



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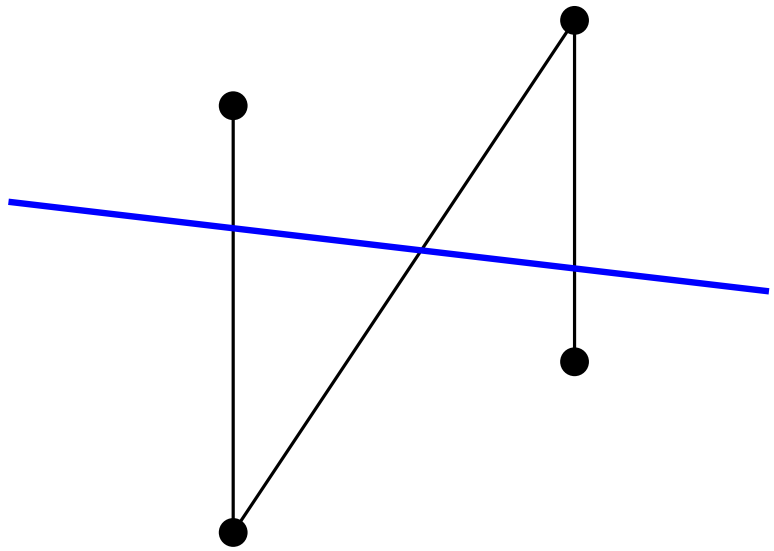

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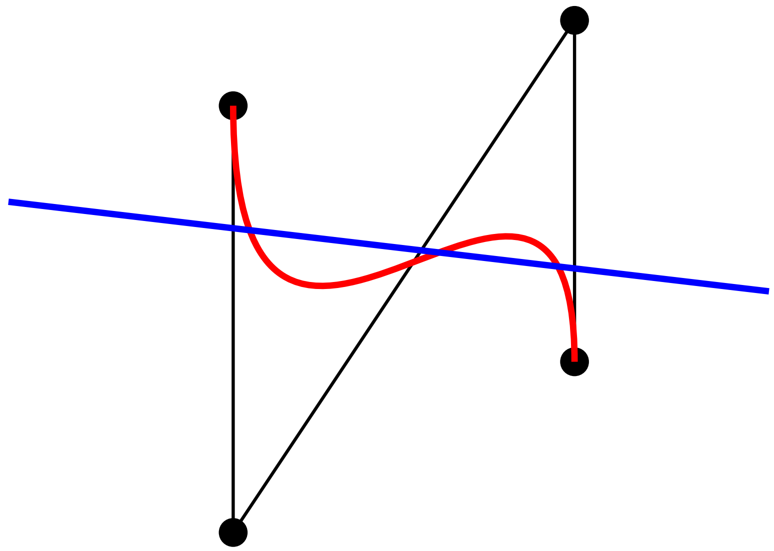

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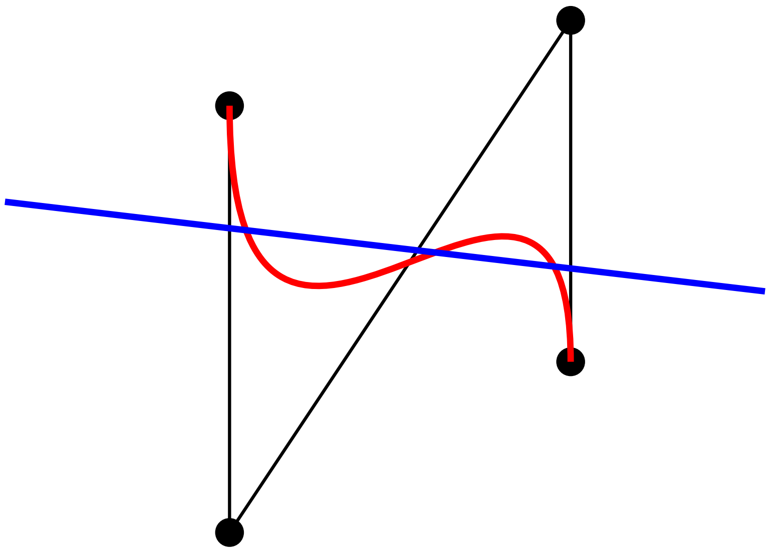

Proof? Later, after looking at **degree elevation**

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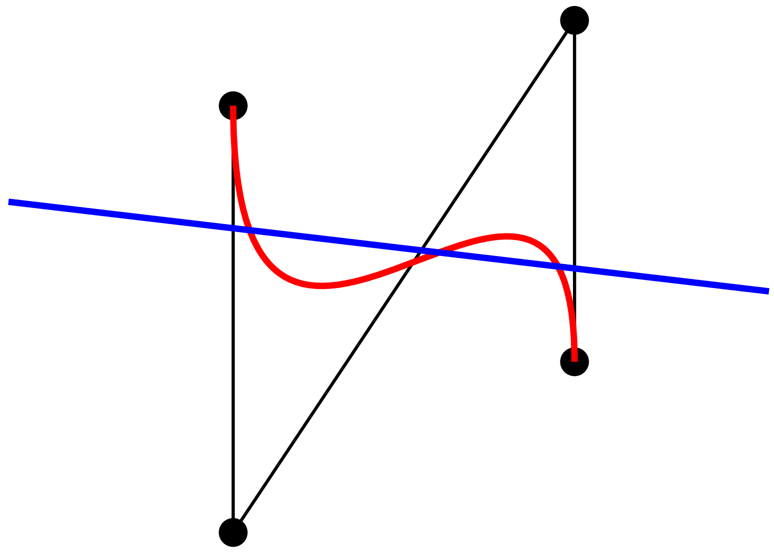

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Question: What does it imply for control points in convex position?

Connecting two curves

- In practice, one should avoid high-degree Bézier curves
- Better use many low-degree curves (they give local control)
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COMPOSITE BÉZIER CURVES

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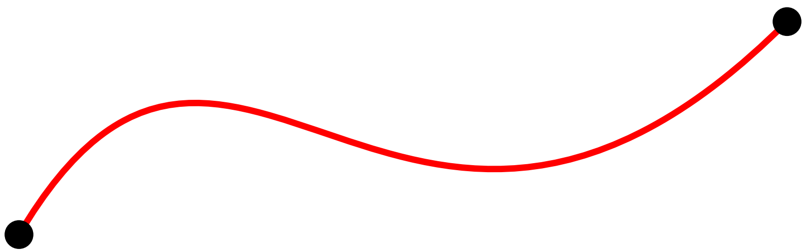

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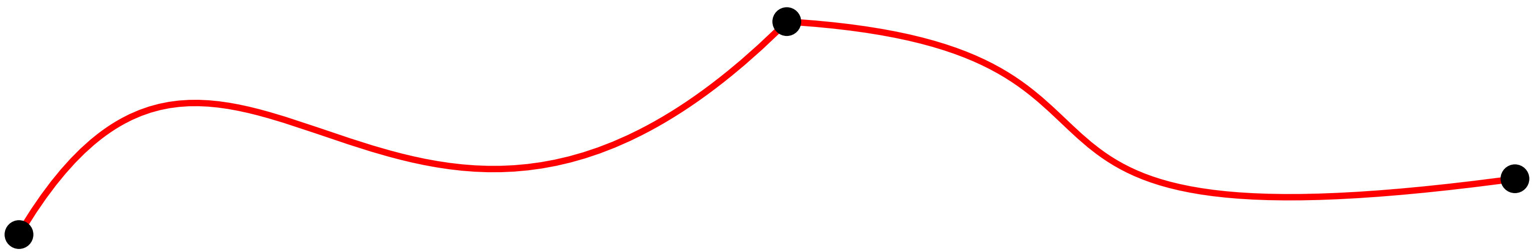

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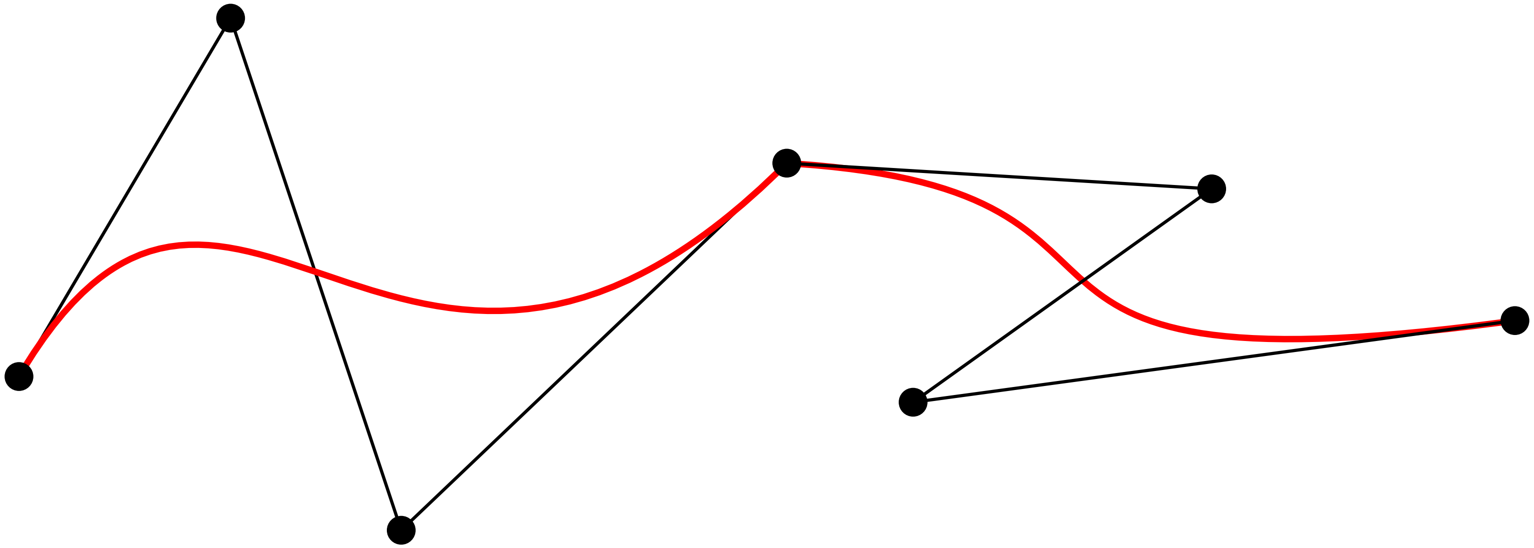

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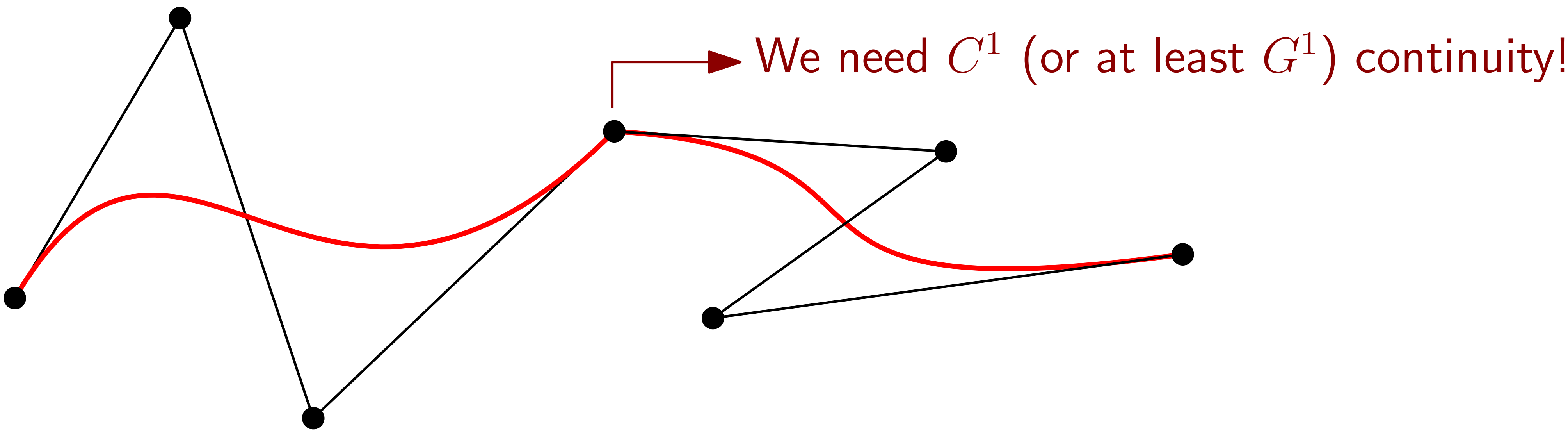

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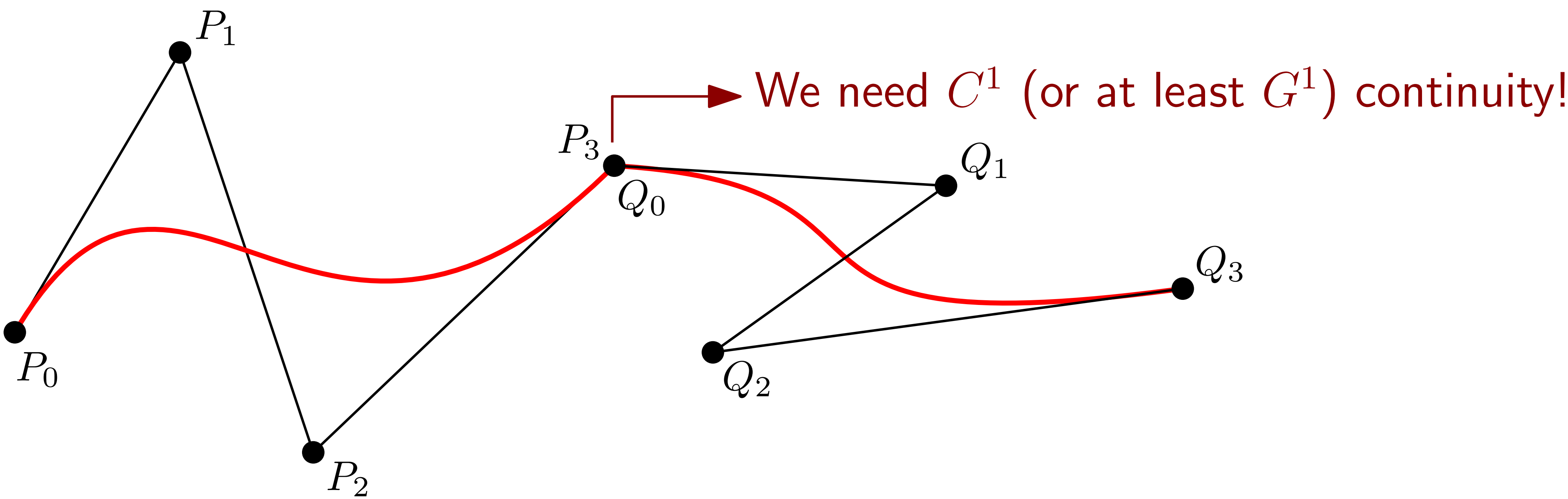

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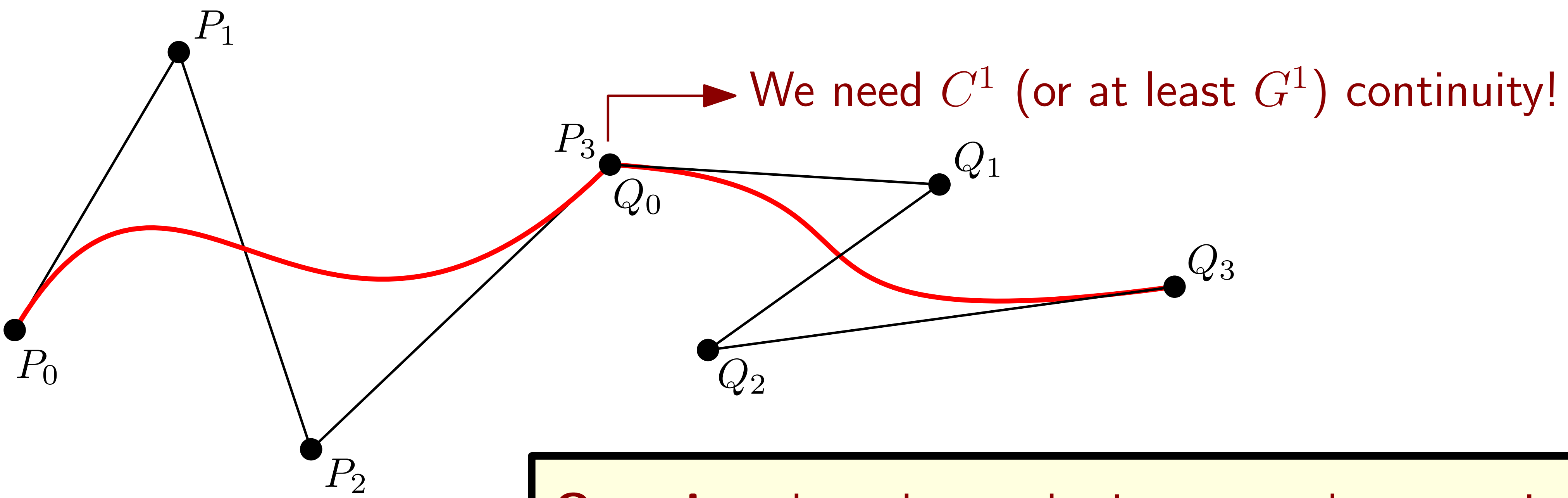
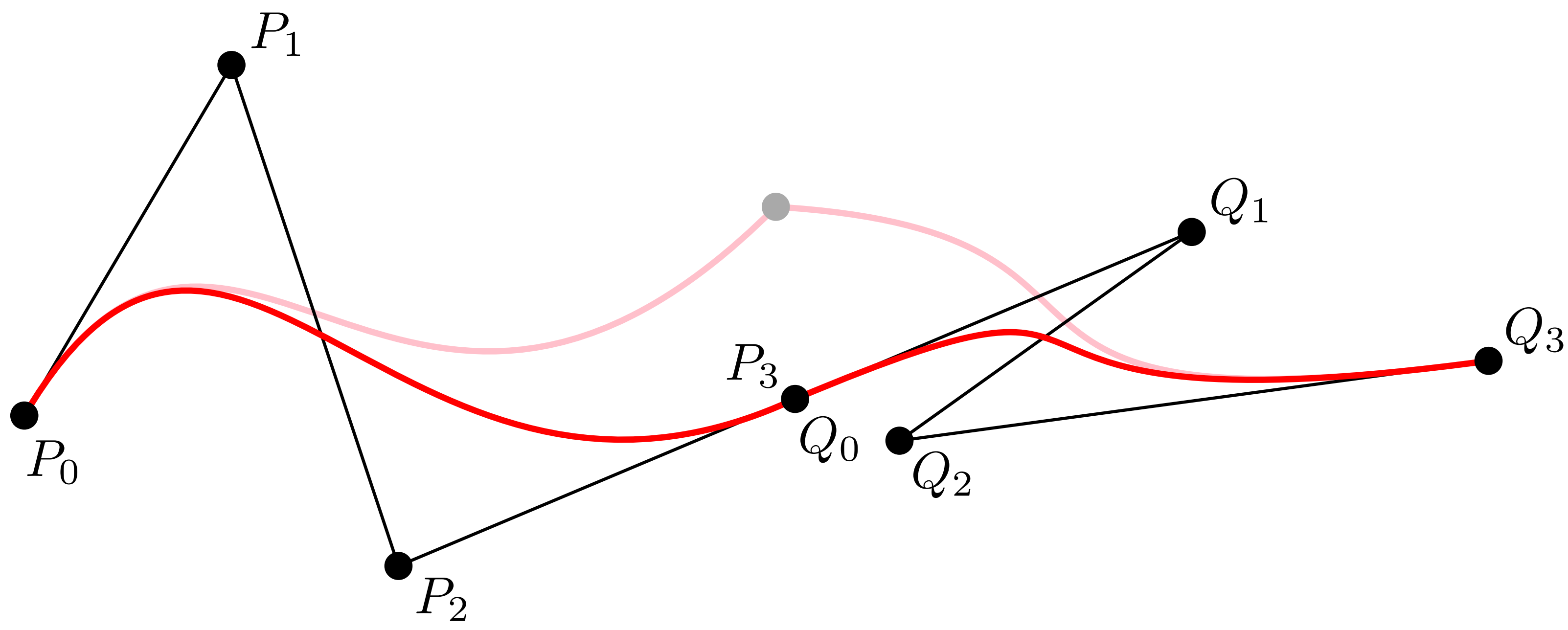


> **Question:** how do we obtain a smooth connection?

Connecting two curves



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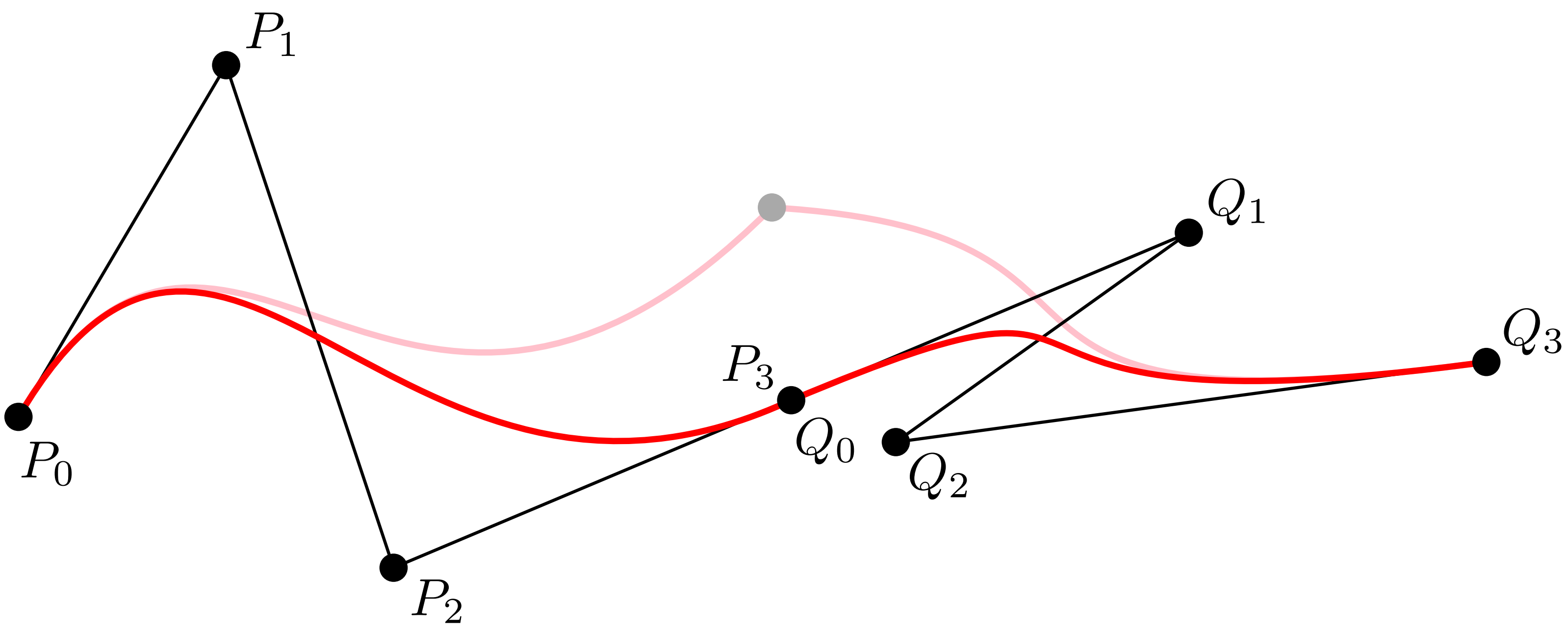

In general, for a curve P with $(n+1)$ control points and Q with $(m+1)$, the C^1-continuity condition is

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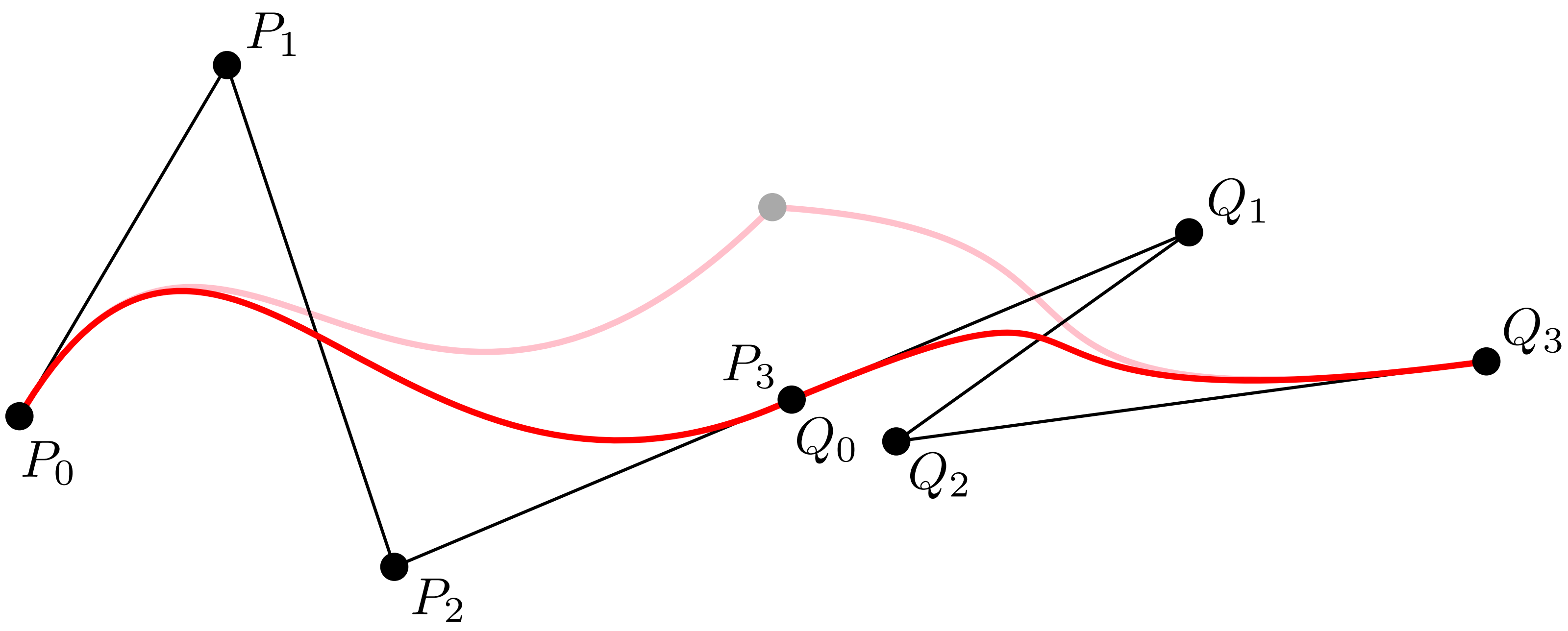

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Question: how can you obtain higher-degree continuity?

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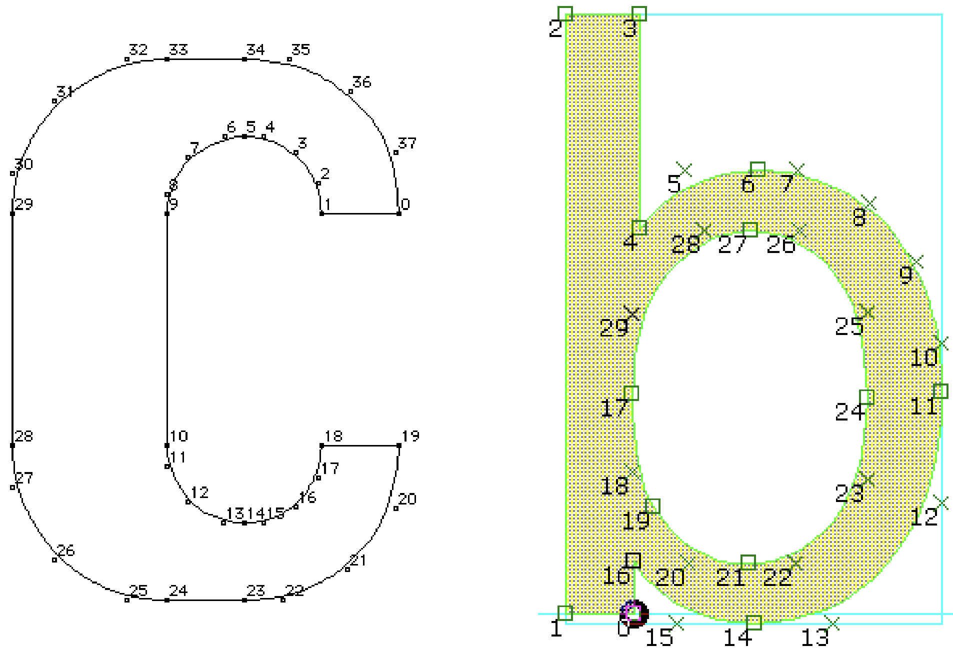

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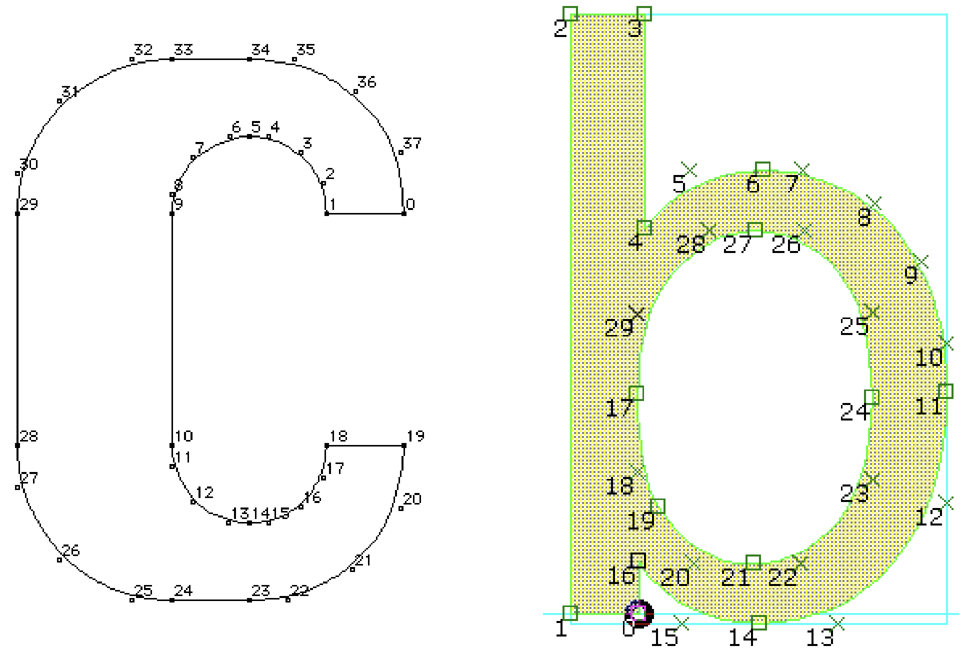

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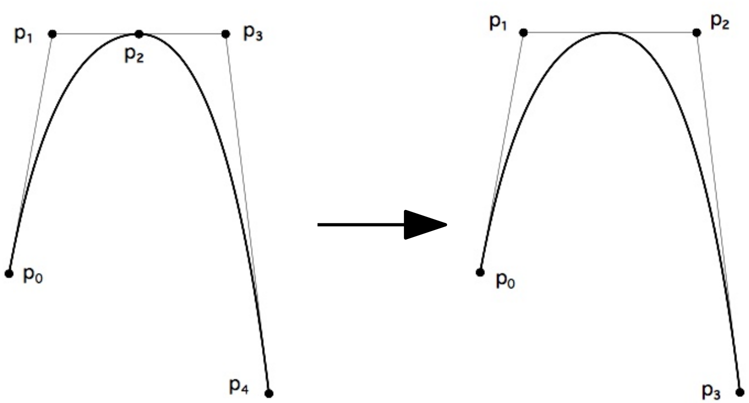
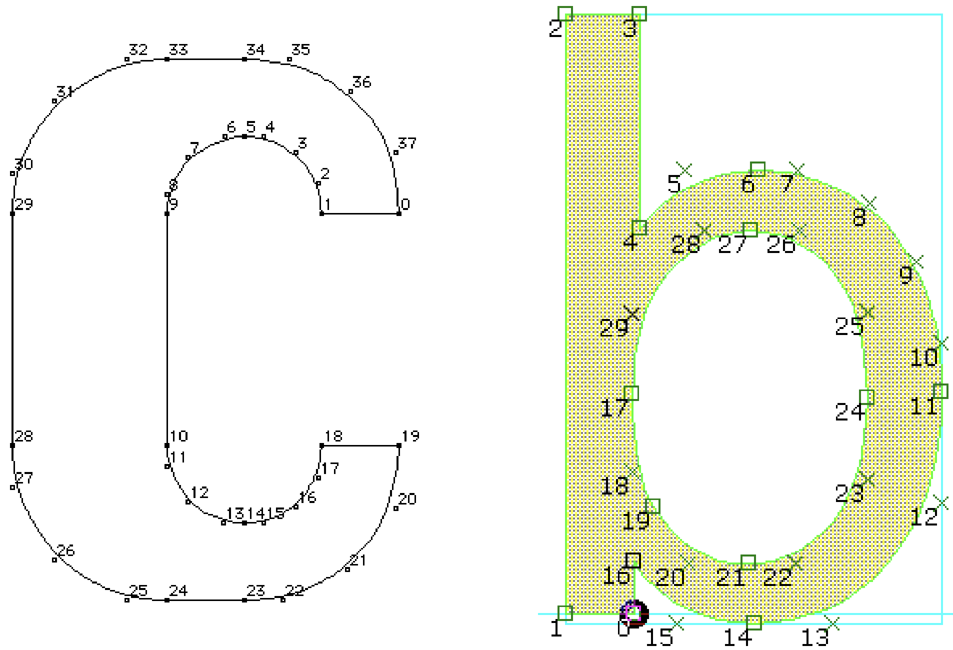




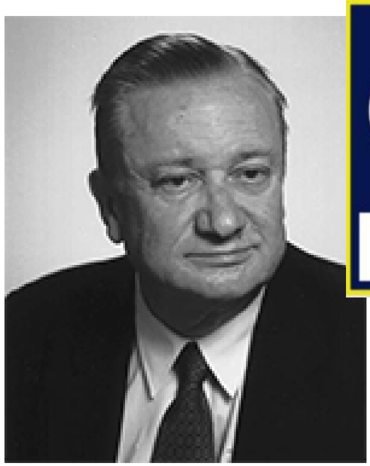
Glyphs of two characters in a true-type font

Difference between points on-curve and off-curve (only off-curve points are stored)

BÉZIER CURVES AS LINEAR INTERPOLATION

An alternative approach to Bézier curves

De Casteljau (Citroën) followed a different approach based on **repeated linear interpolation**



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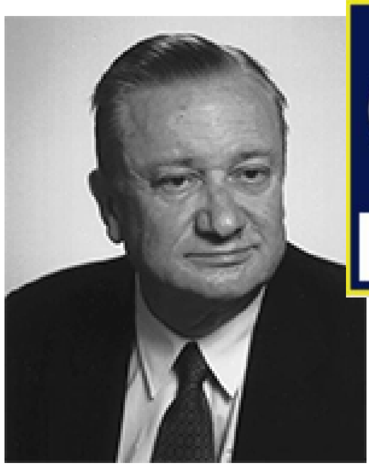

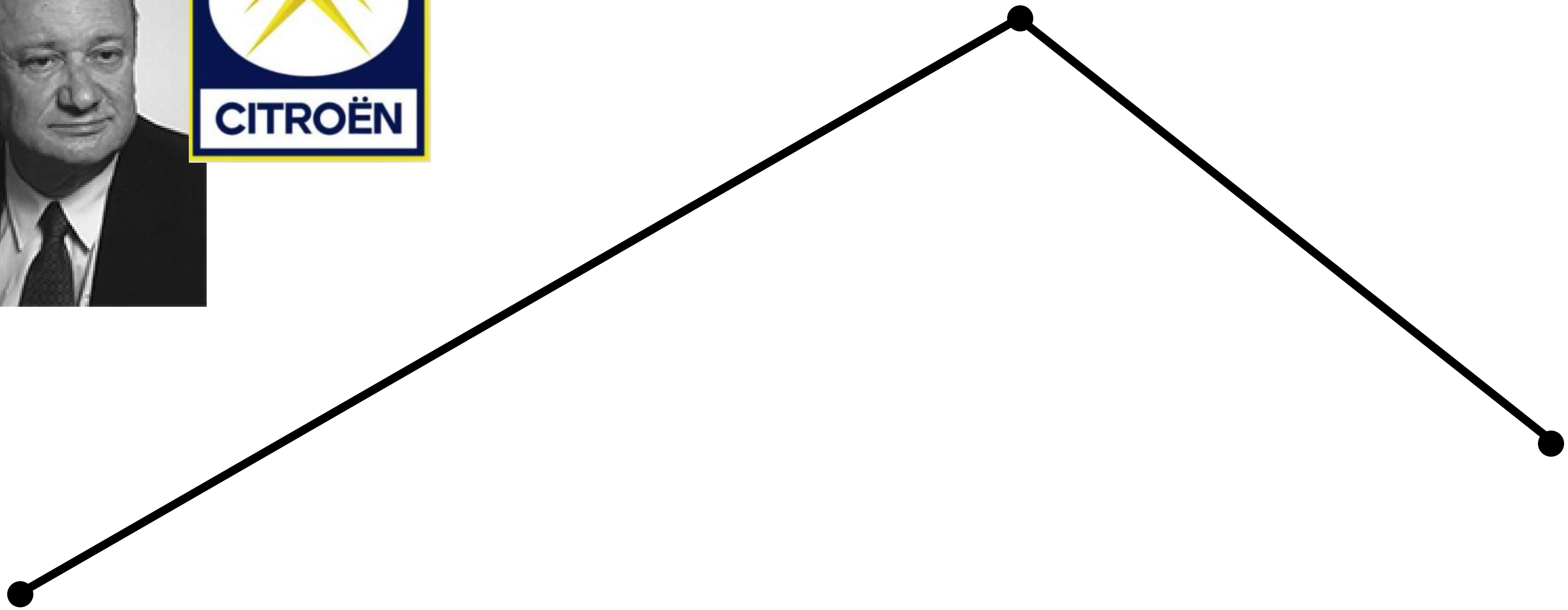

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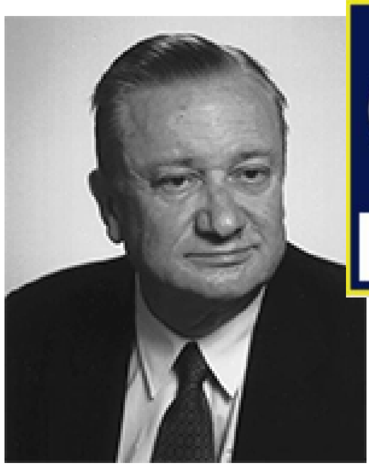

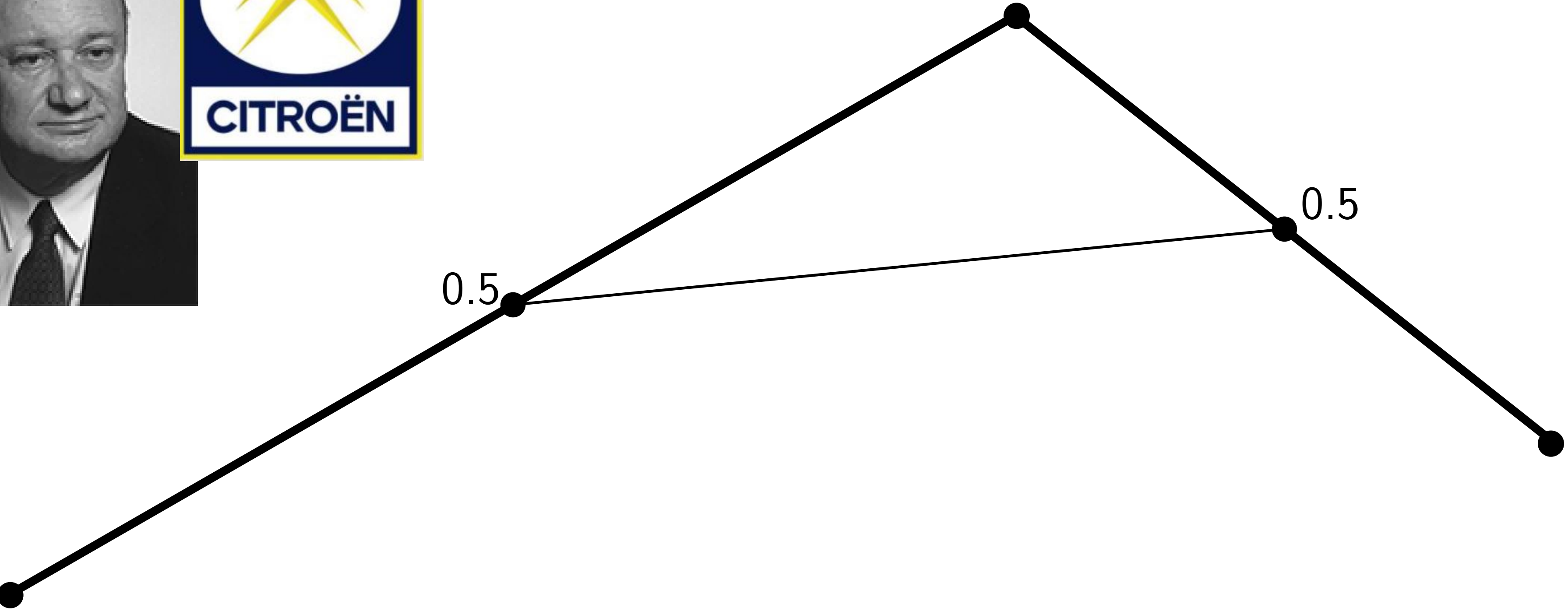

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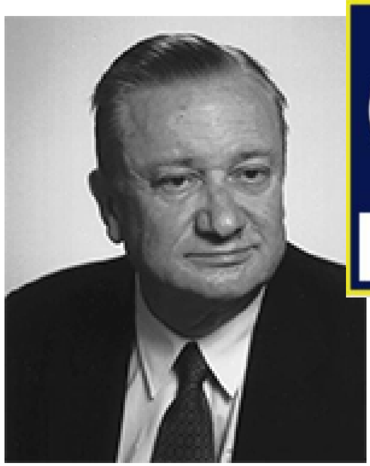

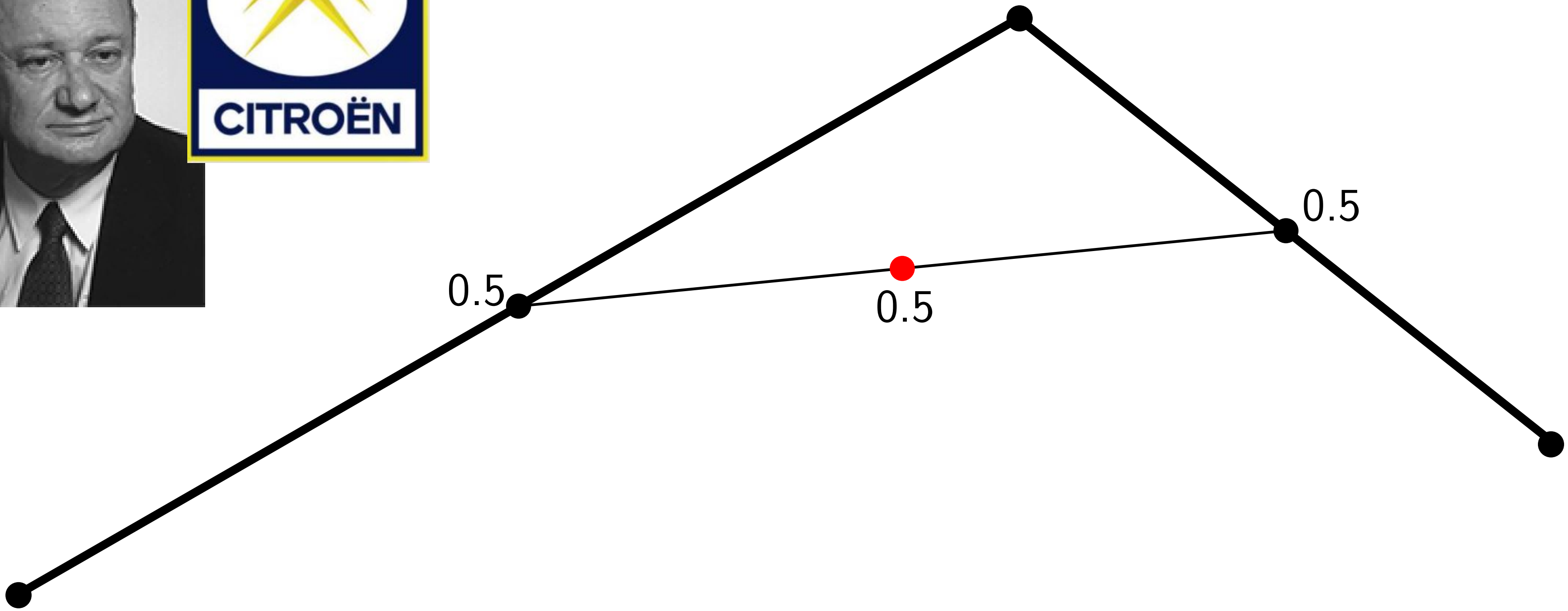

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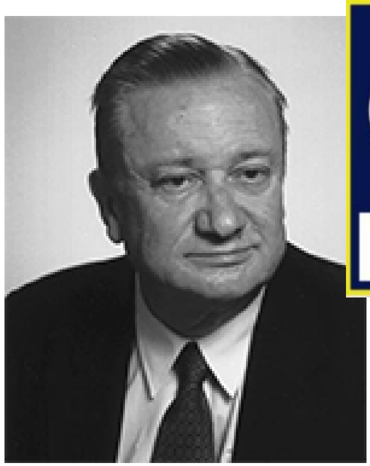

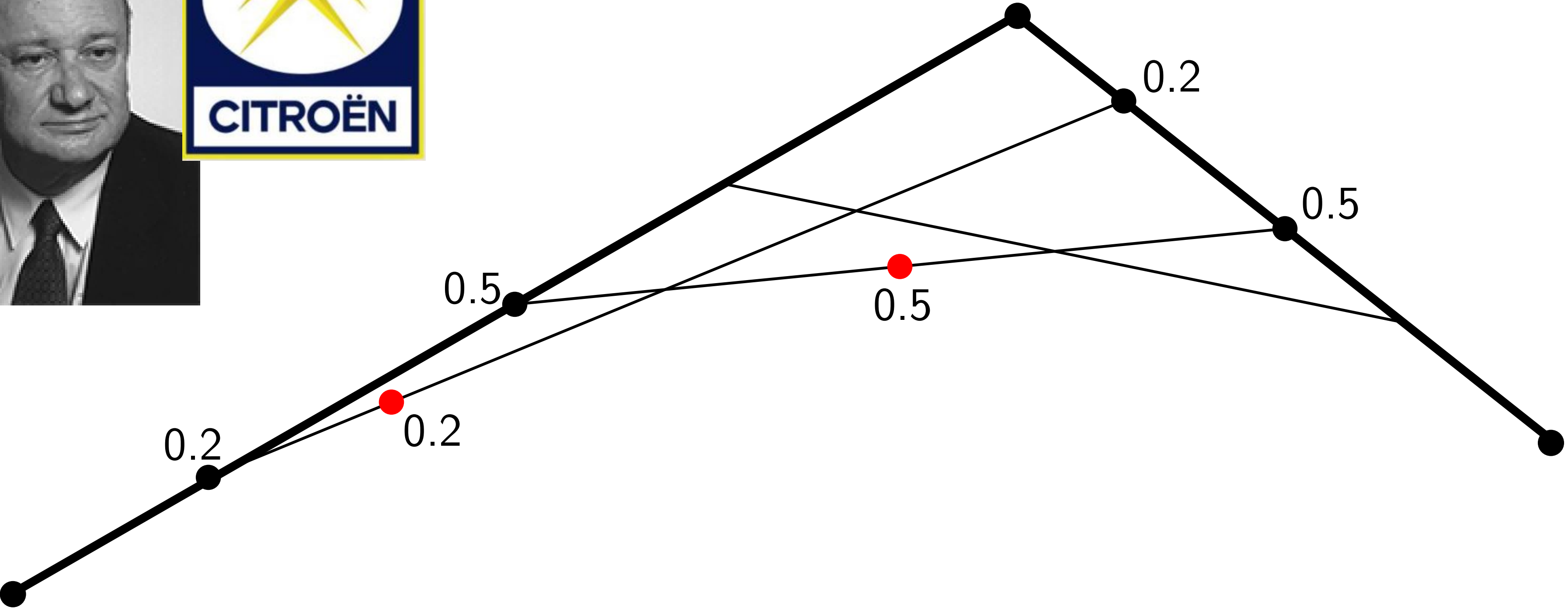

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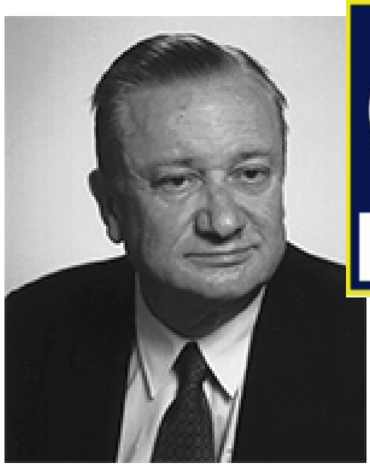

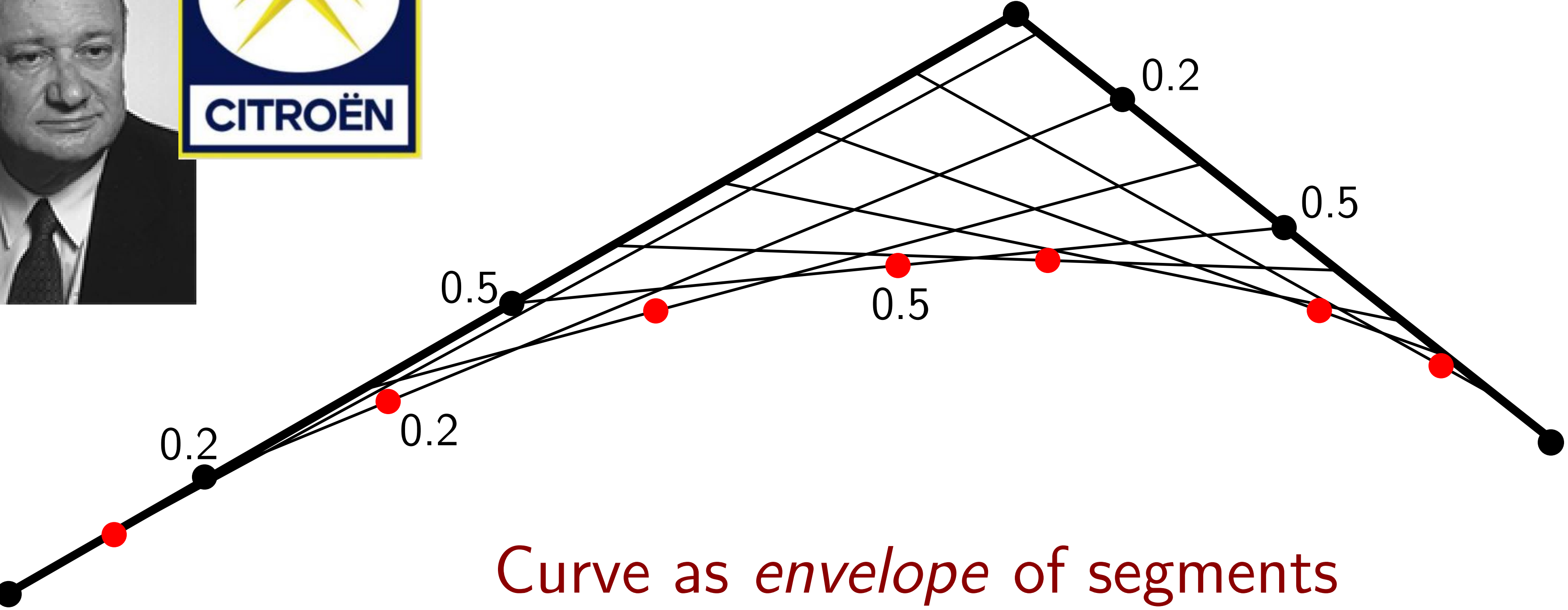

Curve as *envelope* of segments

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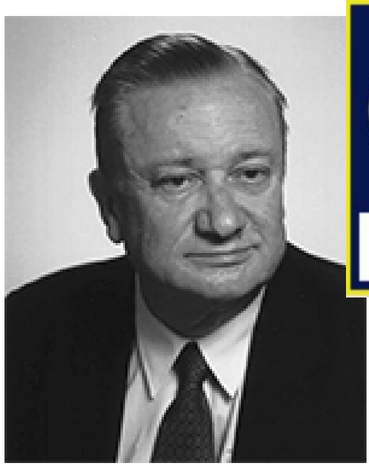

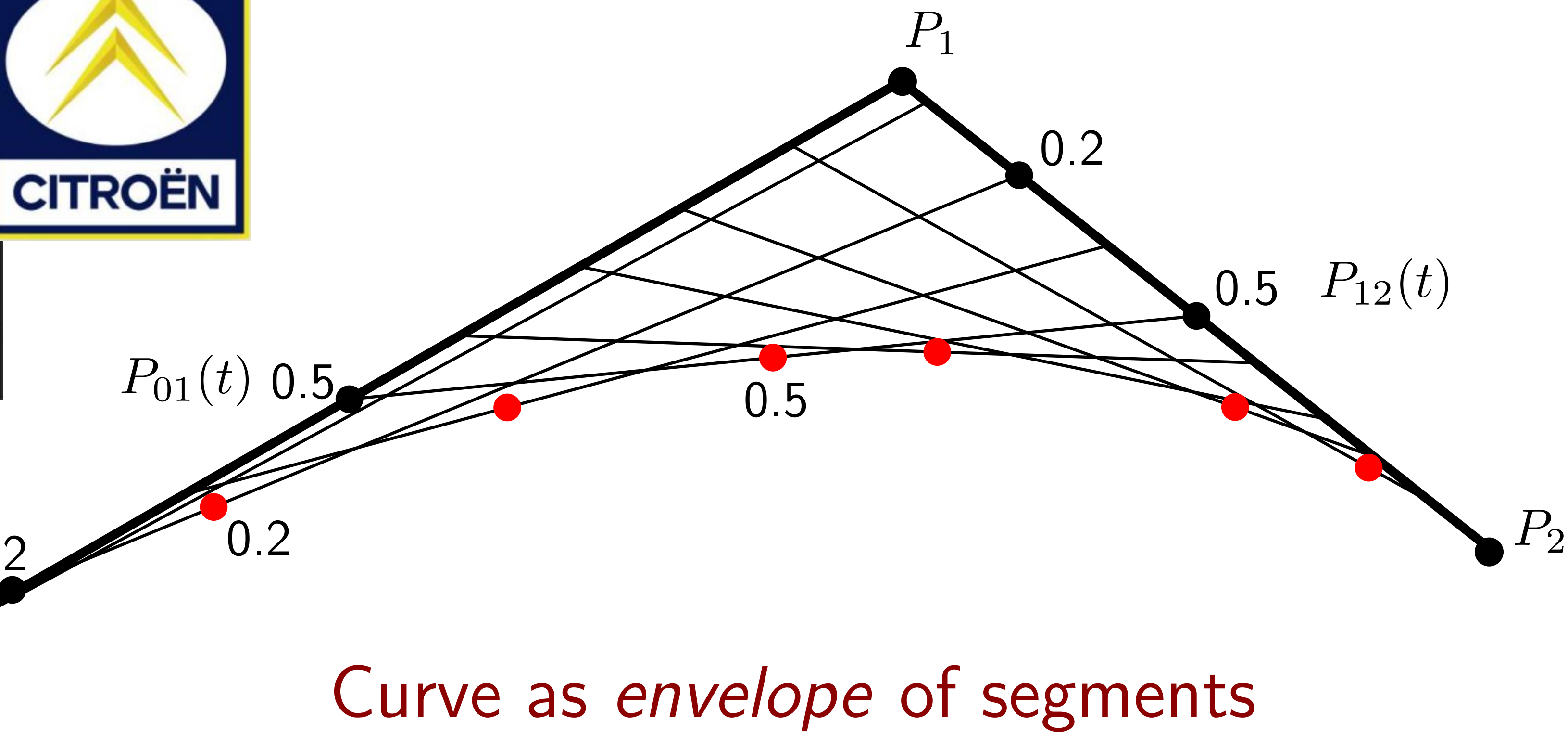

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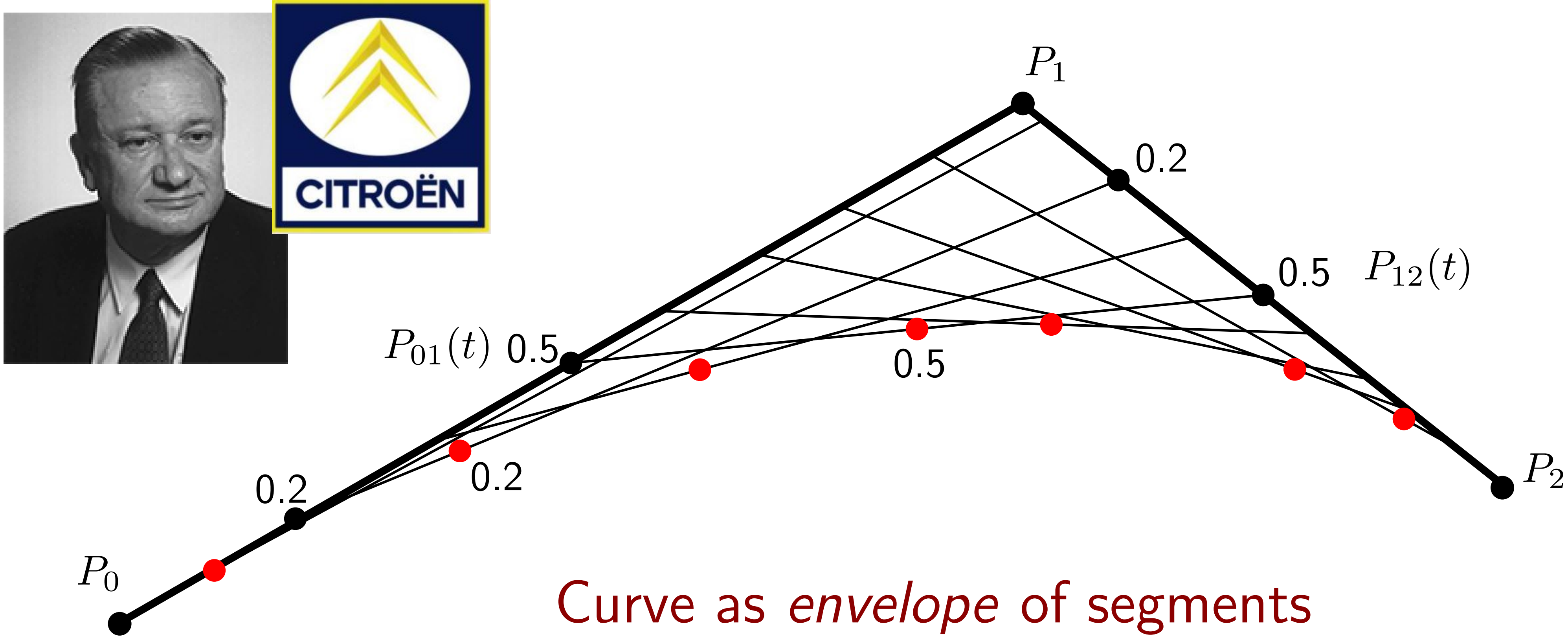

Curve as *envelope* of segments

> **Question:** What is the expression of this envelope, as a function of t?

De Casteljau's algorithm

For 3 points P_0, P_1, P_2, and $0 \leq t \leq 1$, we have:

$$P_{01}(t) = (1-t)P_0 + tP_1$$
$$P_{12}(t) = (1-t)P_1 + tP_2$$
$$P(t) = P_{012}(t) = (1-t)P_{01}(t) + tP_{12}(t)$$

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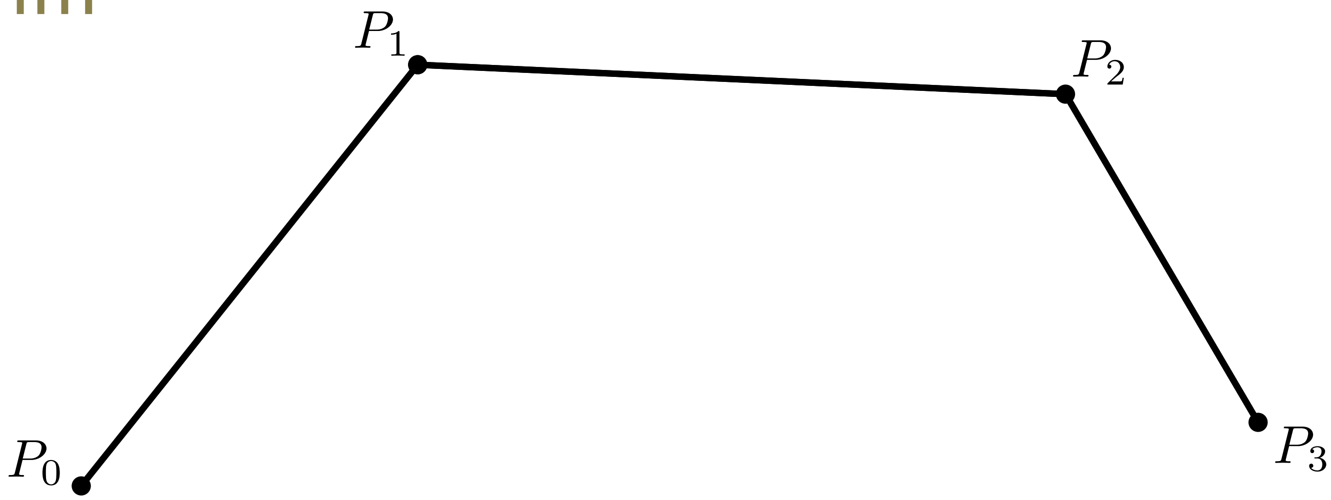
Known as the *Recursive or geometric construction method*

The final curve is given by $P(t) = P_{0\ldots n}(t)$

De Casteljau's algorithm

Example for $n = 3$ and $t = 1/2$

$P(t) = P_{0123}(t)$



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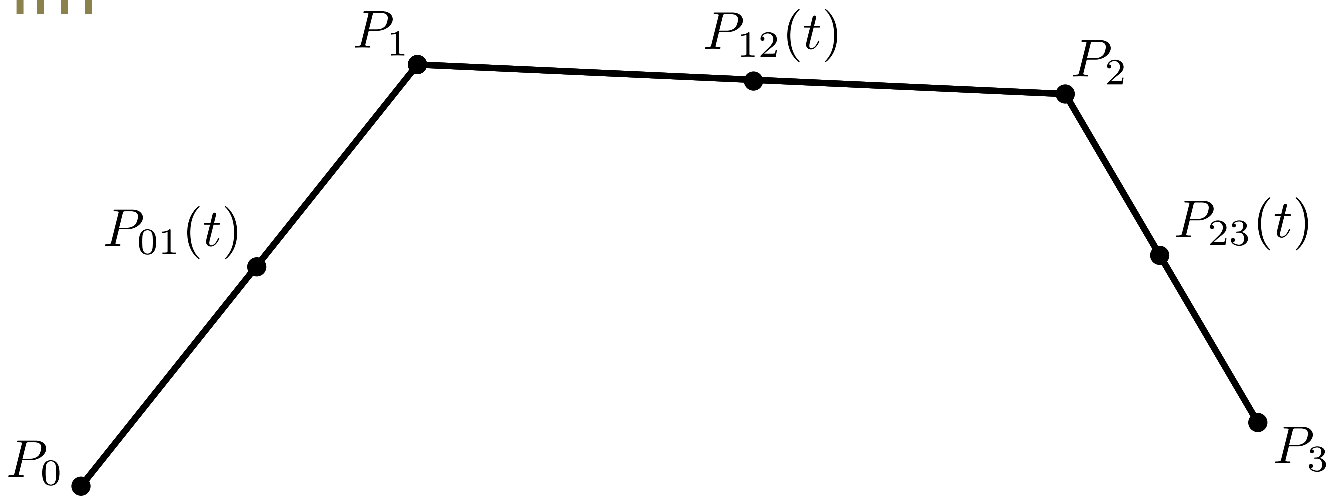

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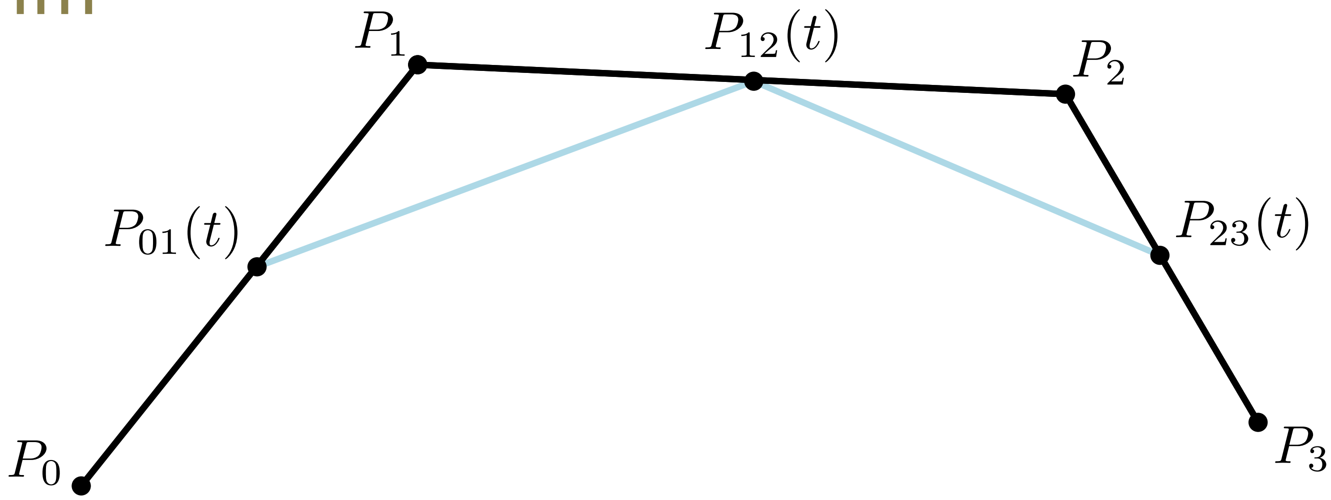

BÉZIER CURVES AS LINEAR INTERPOLATION

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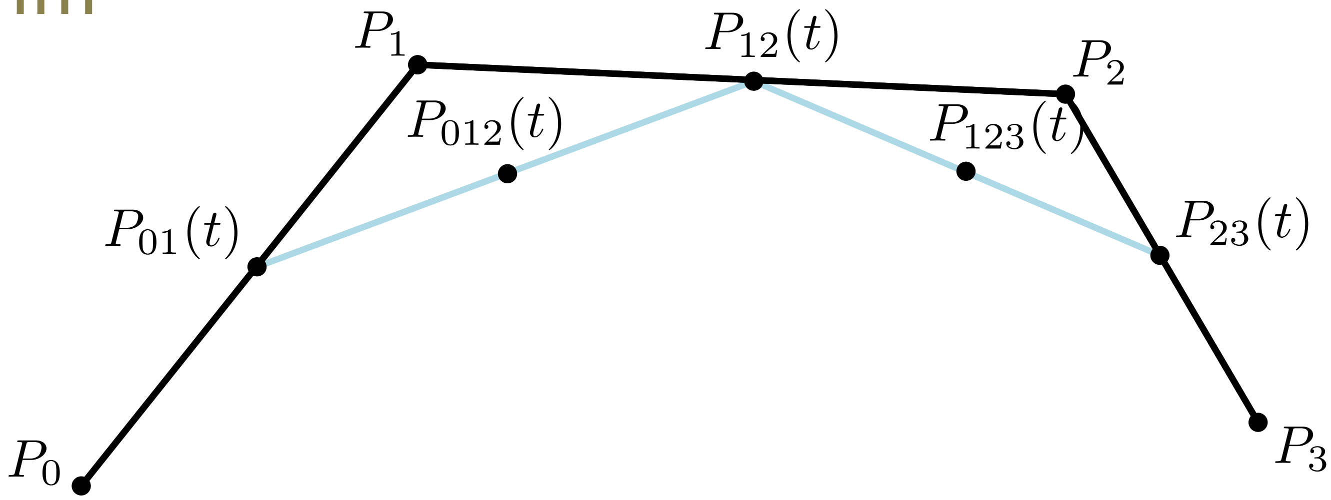

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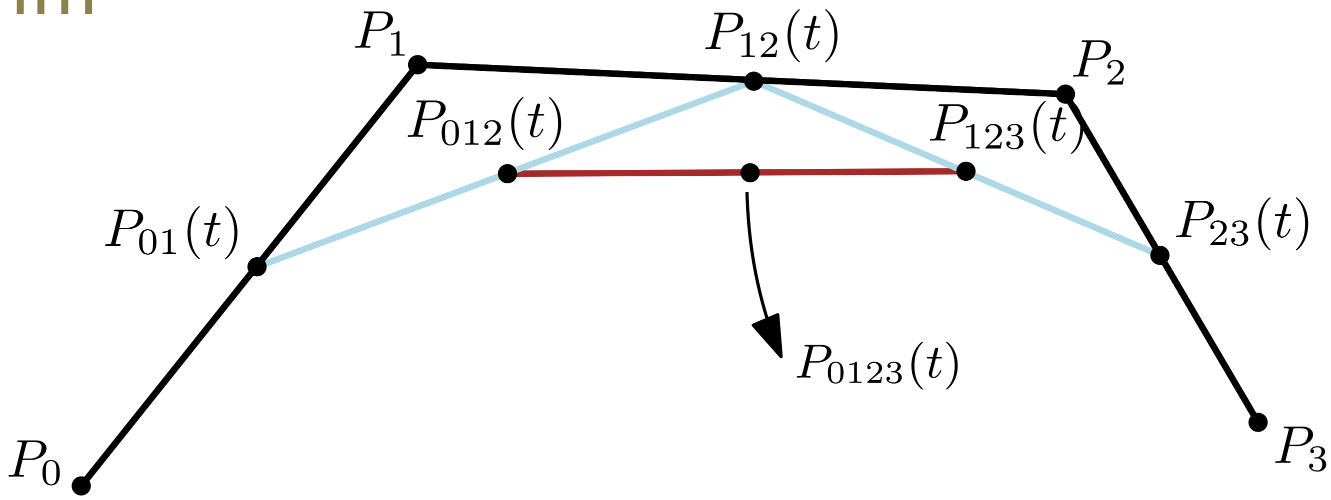

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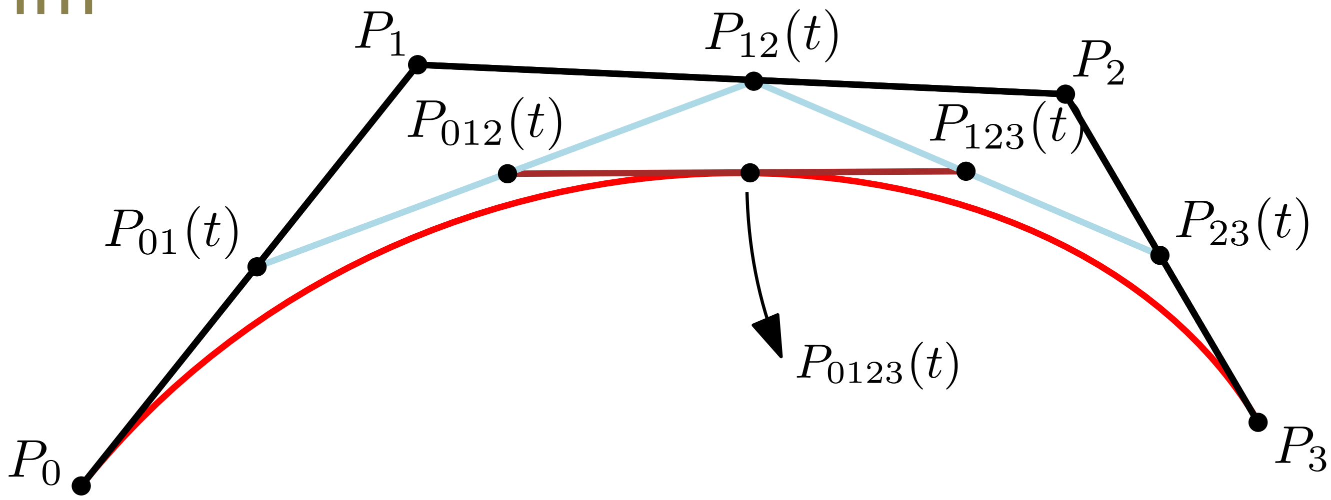

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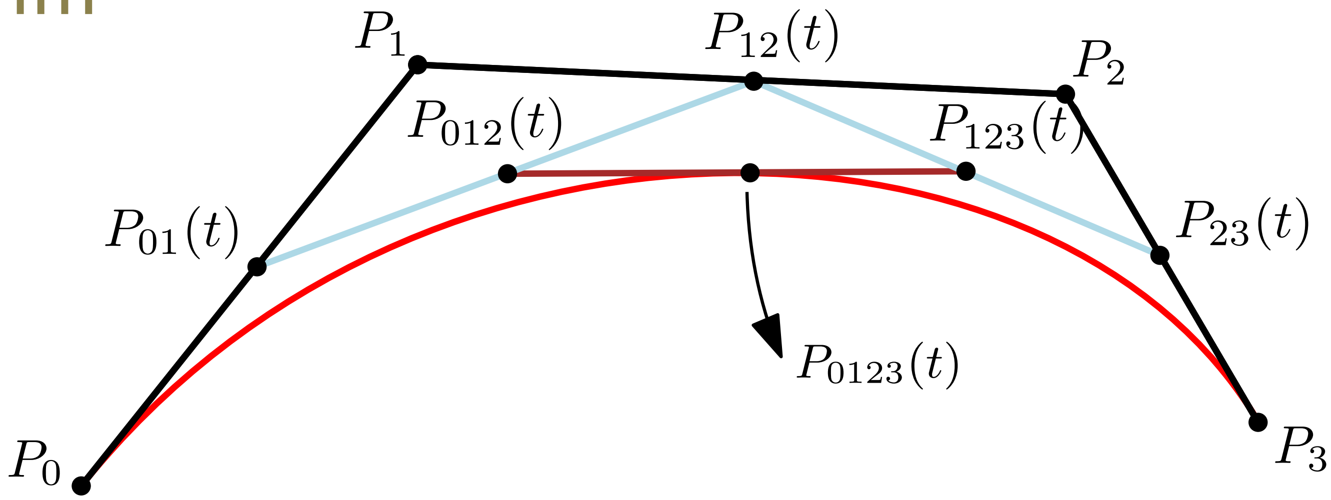

See `https://javascript.info/bezier-curve` for several animations

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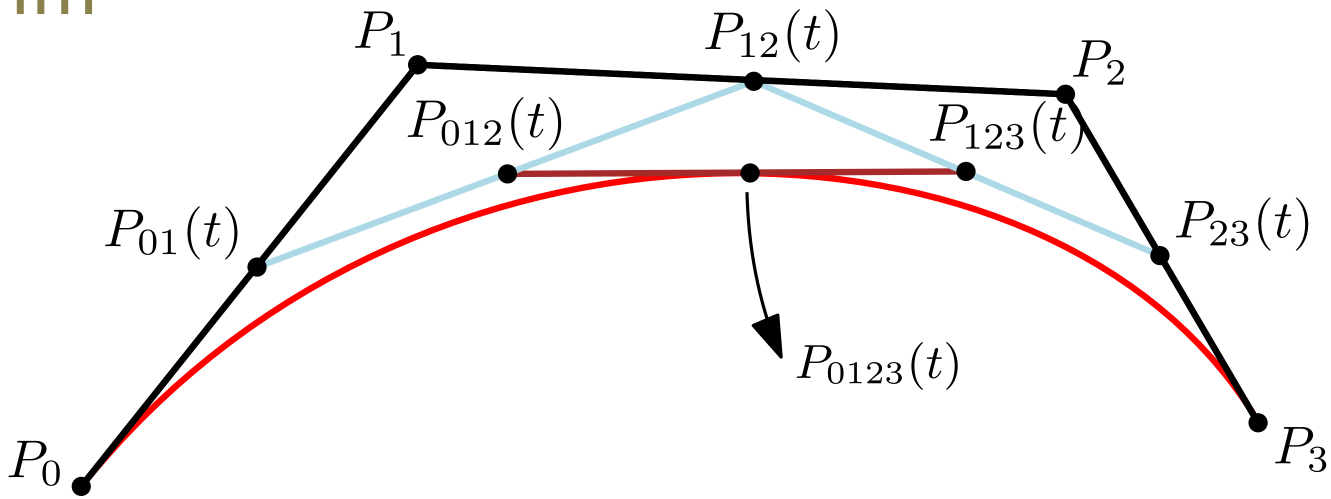

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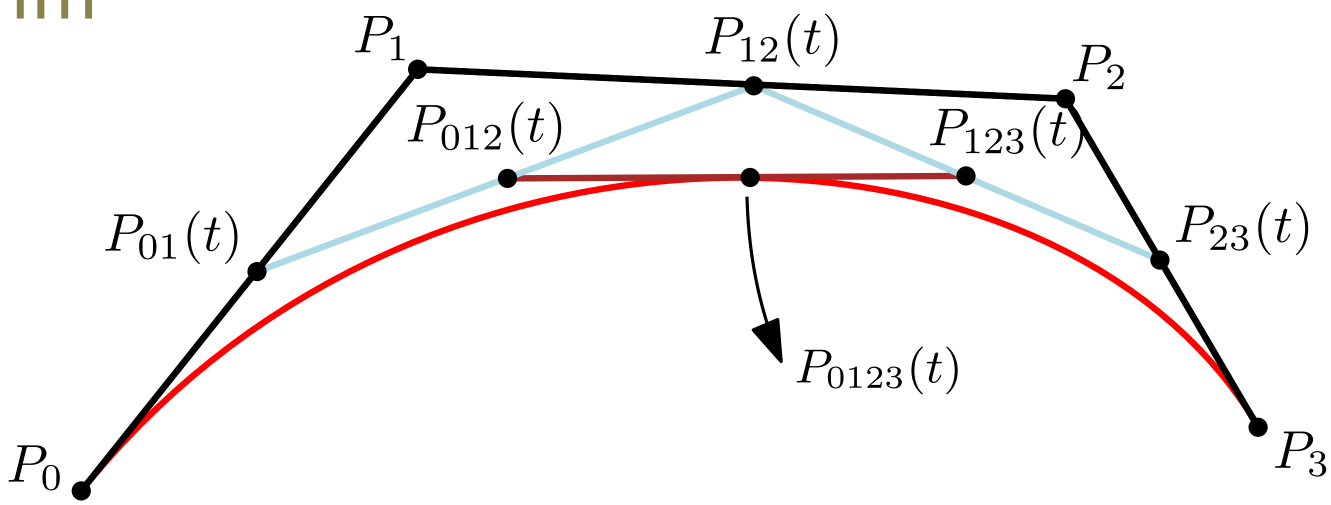

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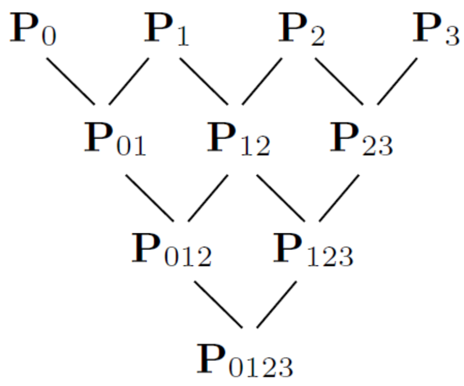
Step	Points constructed	#points
1	$\mathbf{P}_{01}\ \mathbf{P}_{12}\ \mathbf{P}_{23}\ldots\mathbf{P}_{n-1,n}$	n
2	$\mathbf{P}_{012}\ \mathbf{P}_{123}\ \mathbf{P}_{234}\ldots\mathbf{P}_{n-2,n-1,n}$	$n-1$
3	$\mathbf{P}_{0123}\ \mathbf{P}_{1234}\ \mathbf{P}_{2345}\ldots\mathbf{P}_{n-3,n-2,n-1,n}$	$n-2$
$\vdots$	$\vdots$	$\vdots$
n	$\mathbf{P}_{0123\ldots n}$	



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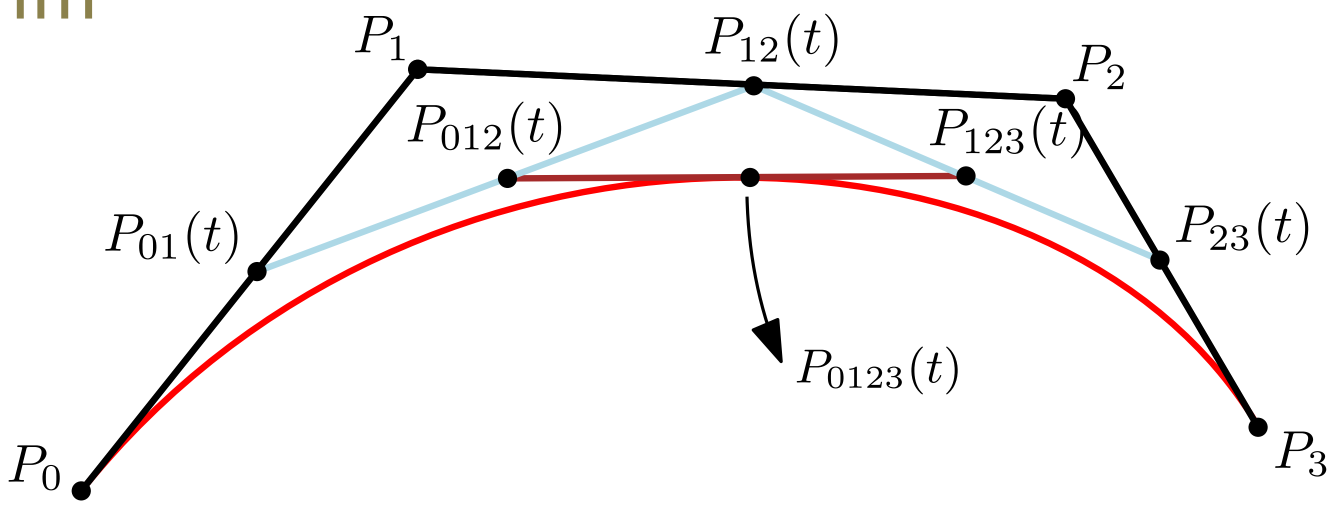

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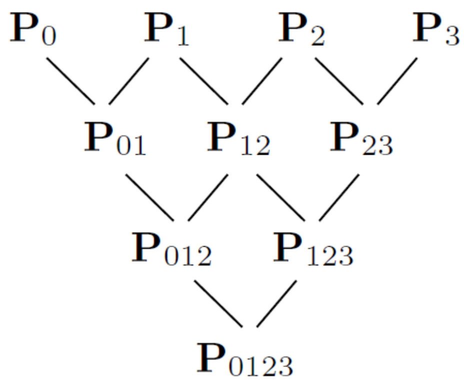

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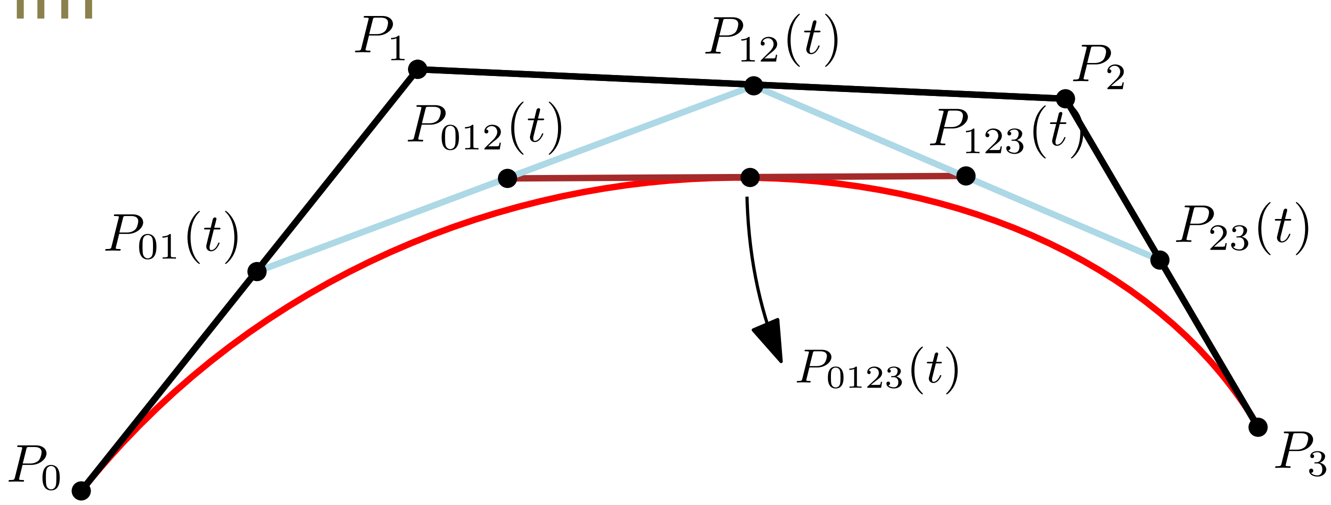

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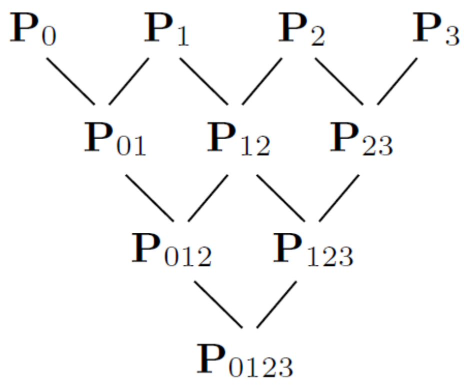

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$$n + (n-1) + (n-2) + \cdots + 2 + 1 = n(n+1)/2$$

De Casteljau's algorithm

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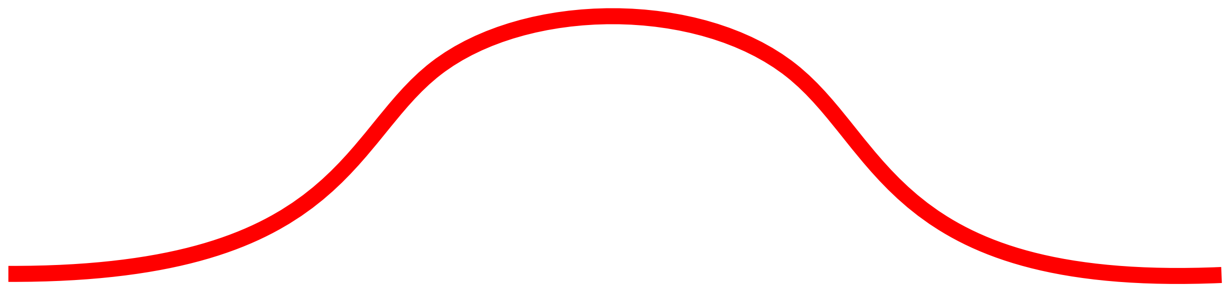

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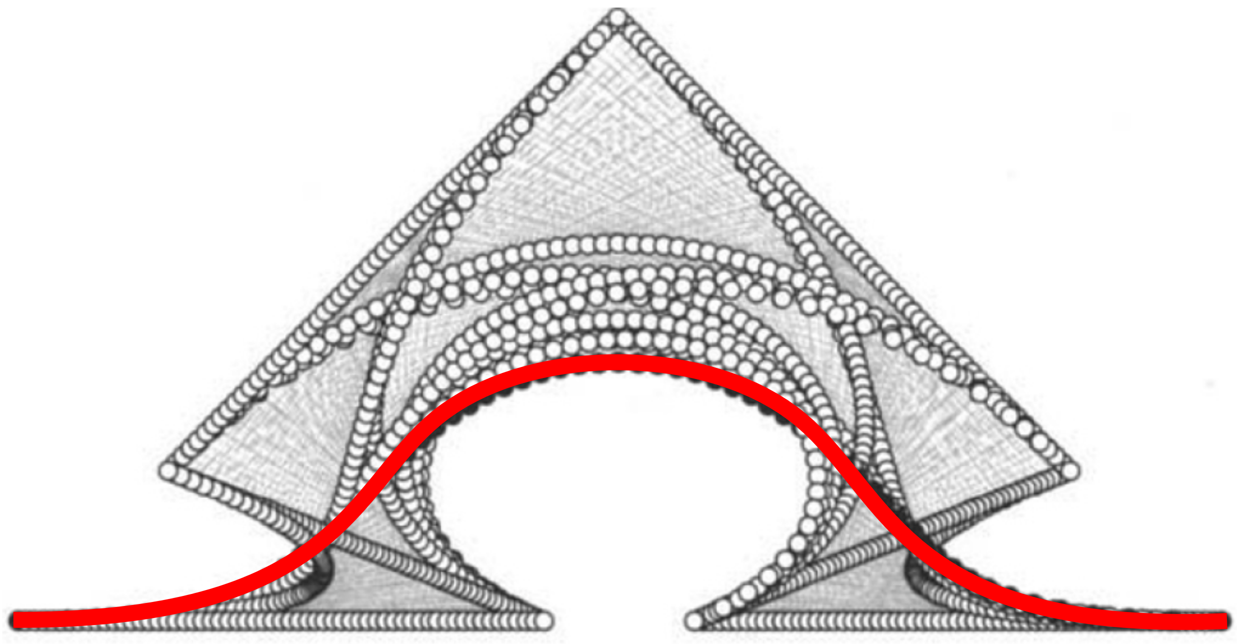

Figure 4.3 The de Casteljau algorithm: 60 points are computed on a degree six curve; all intermedediate points $\mathbf{b}_i^r$ are shown.

Figure from book by Farin (page 47)

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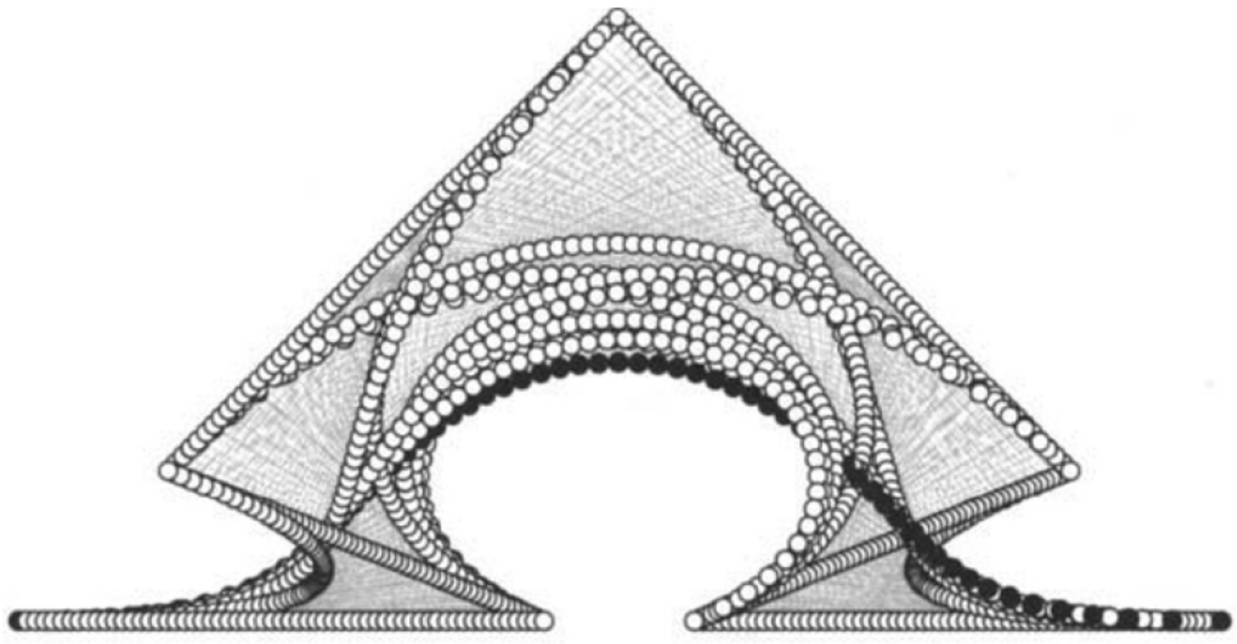

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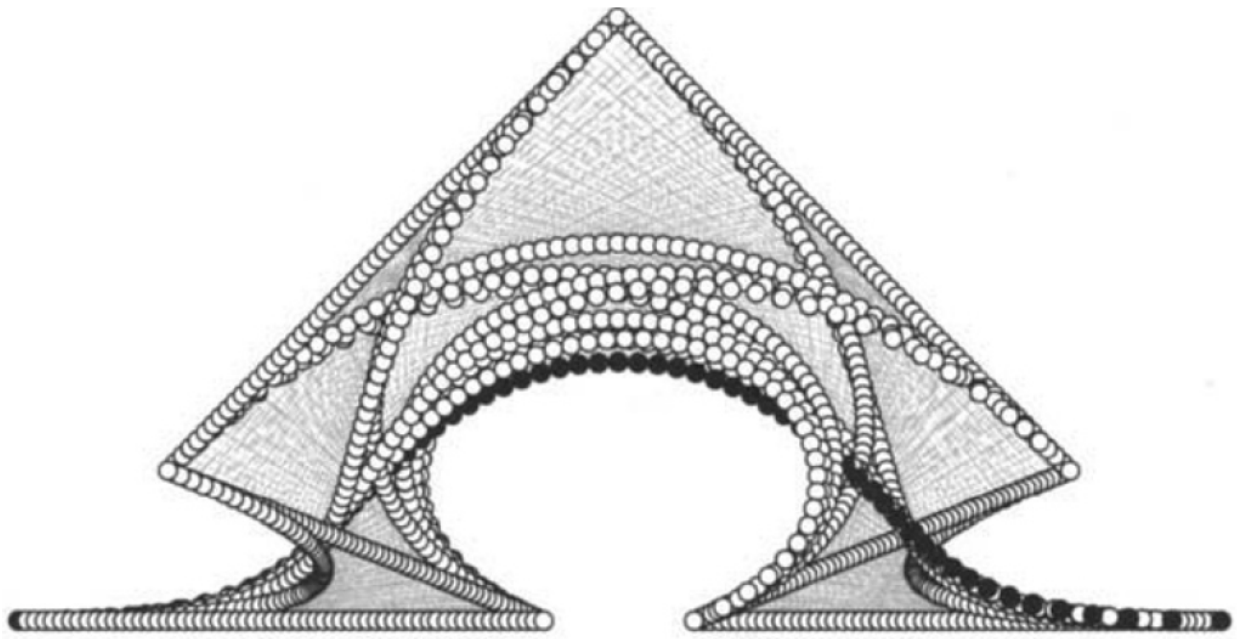

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Question for later: Is the computation based on Bernstein polynomials faster?

Using De Casteljau's to subdivide a curve

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(We need this when we need more flexibility to design the curve)

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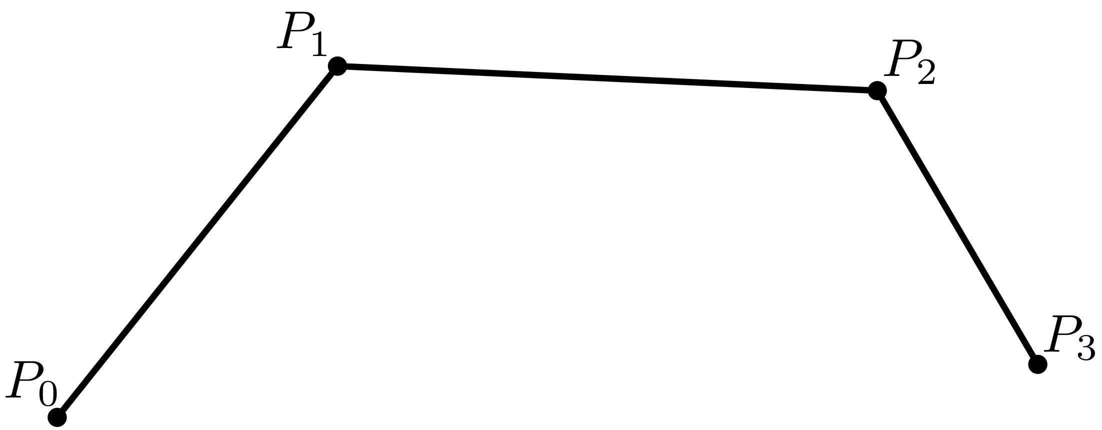

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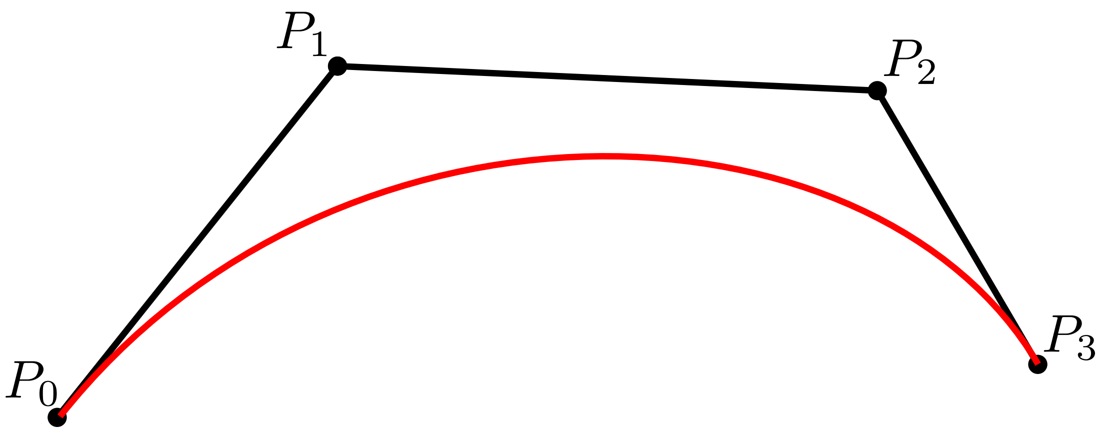

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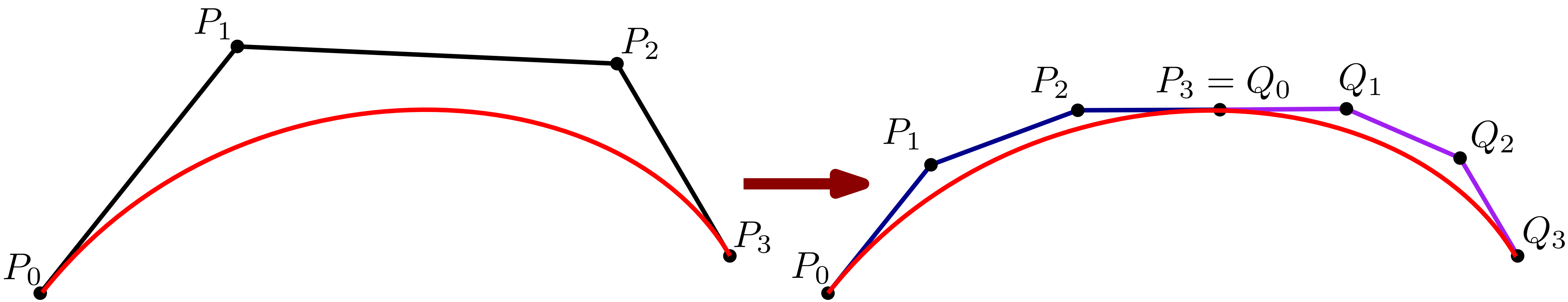

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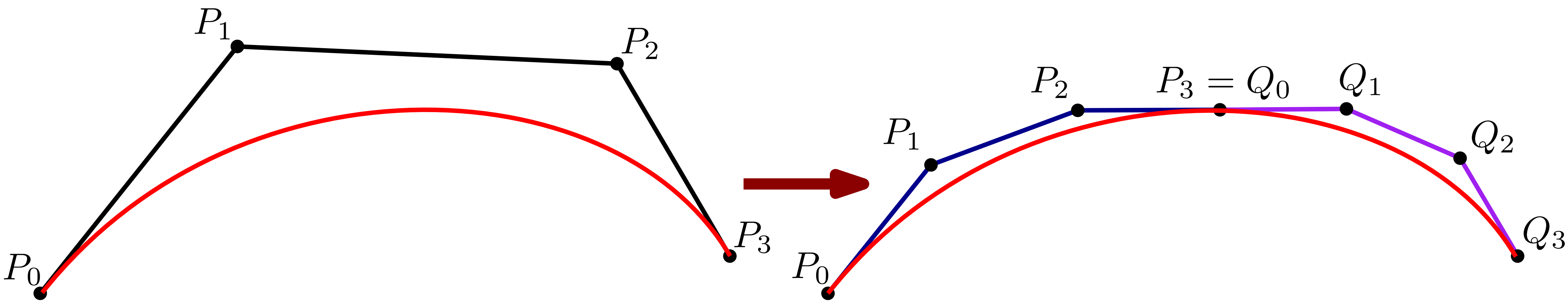

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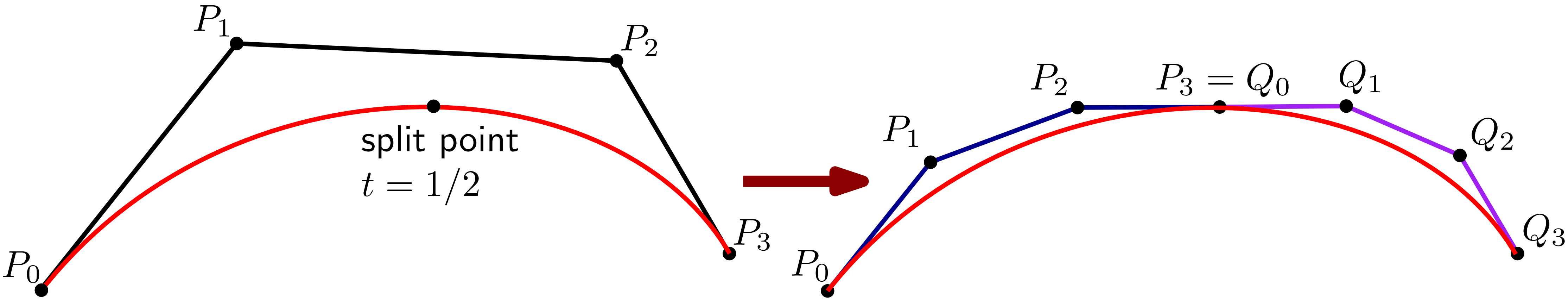

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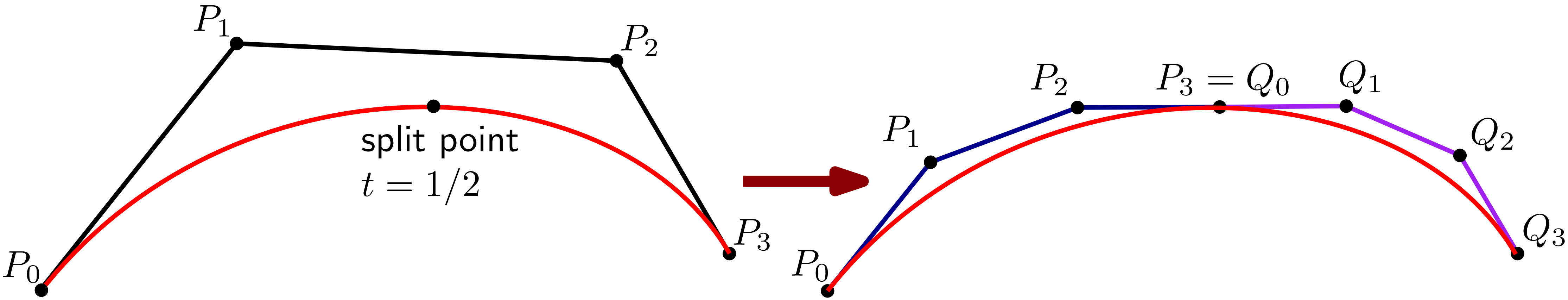

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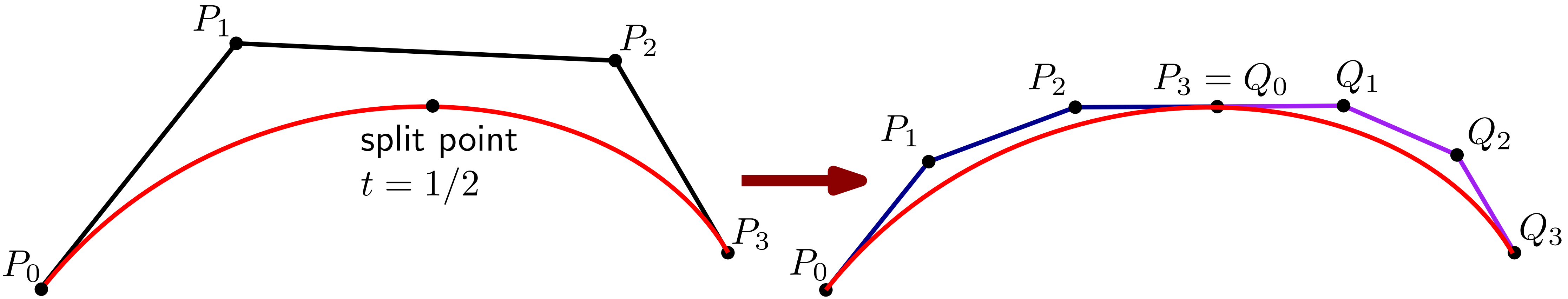

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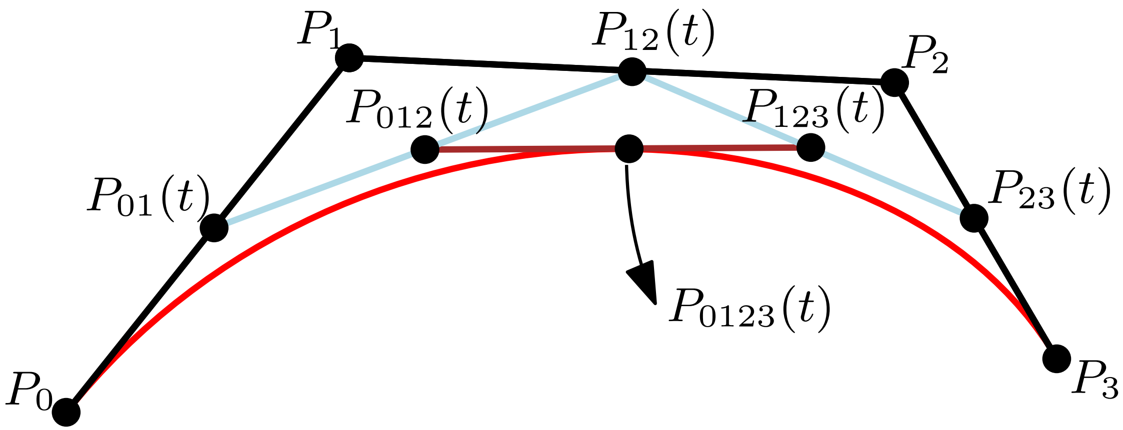

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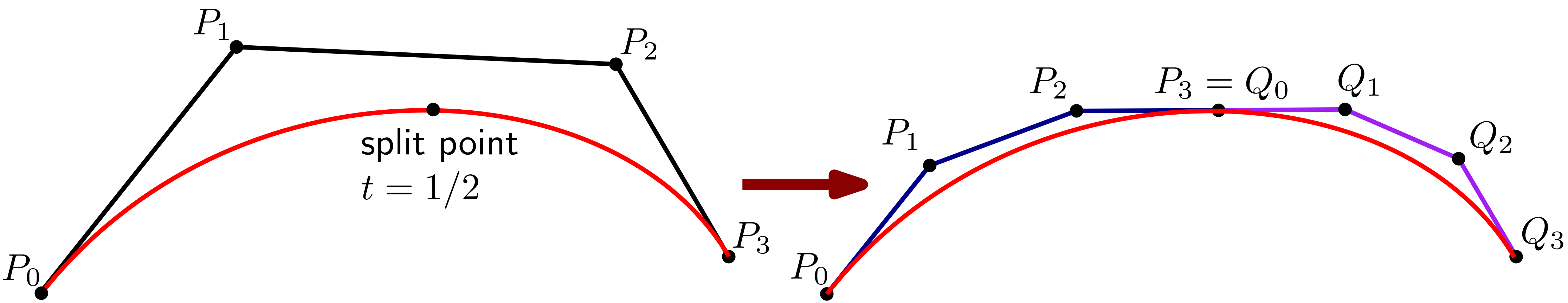

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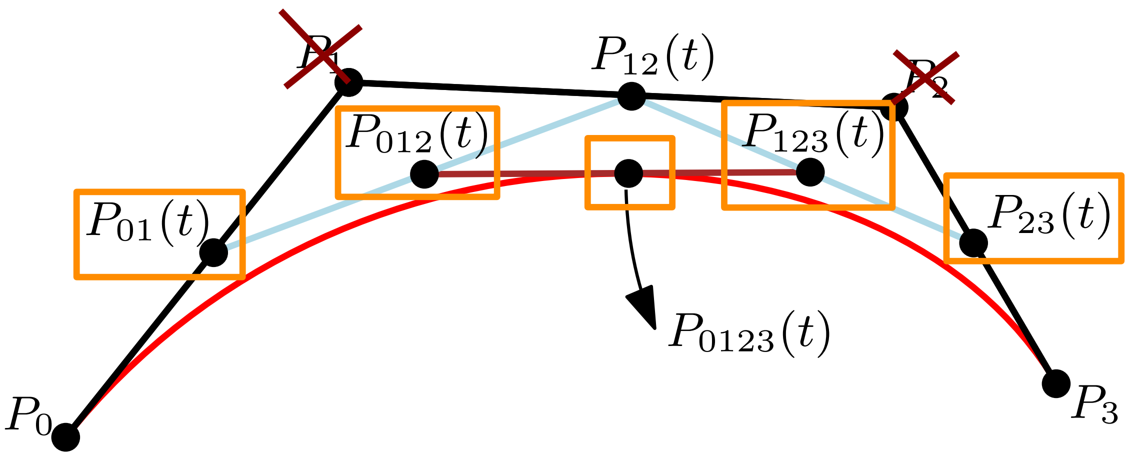

In general, the subdivision is done by:

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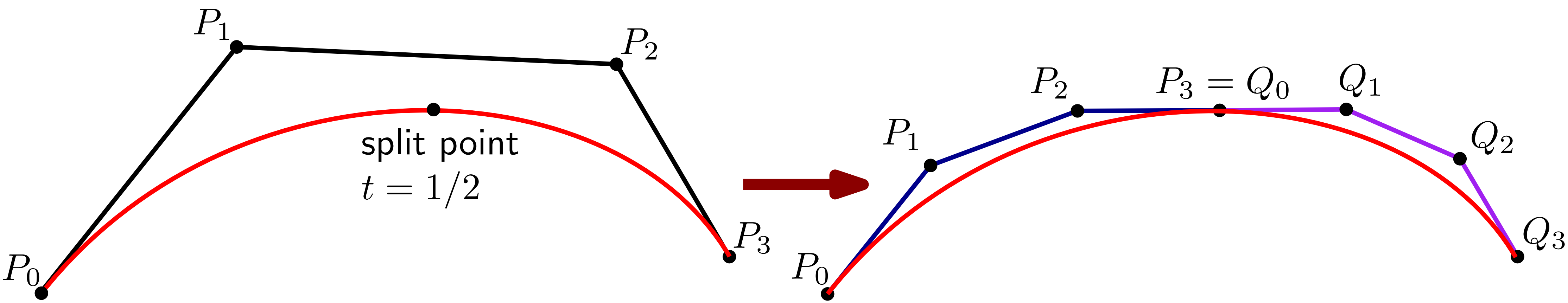

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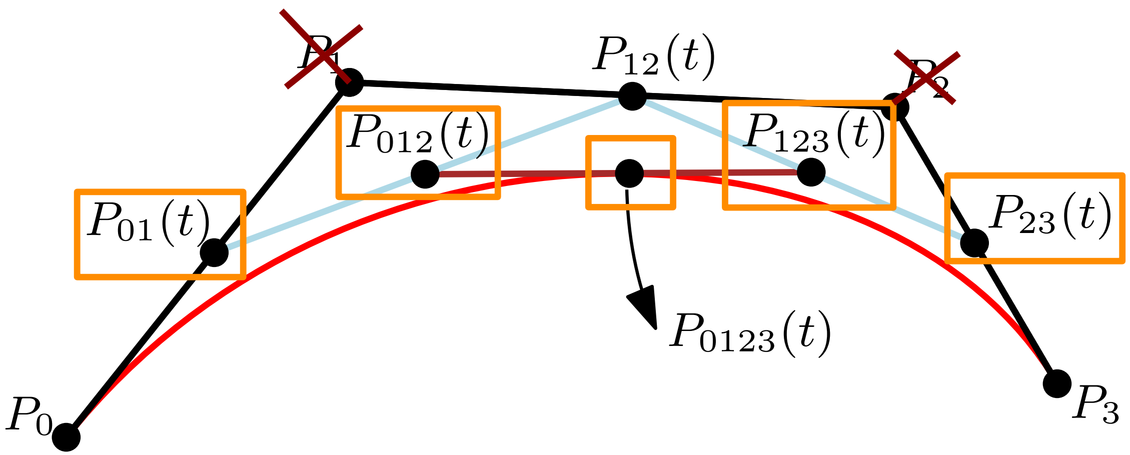

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This method is also useful for **clipping**

Computation of a Bézier curve

Recall definition

$B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

$P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t)$

recall that $0 \leq i \leq n$, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, and $0! = 1$

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→ This can also be stored in a table, and reused for other points

Even faster: forward differences

Idea: find a method to "jump" from one point in $P(t)$ to the next one ,$P(t+\Delta)$, using only a few computations for each jump

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If dP would exist, then we could do:

$$\begin{aligned} P(0) &= P_0 \\ P(0+\Delta) &= P(0) + dP = P_0 + dP \\ P(2\Delta) &= P(\Delta) + dP = P_0 + 2dP \\ P(i \cdot \Delta) &= P((i-1)\Delta) + dP = P_0 + i \cdot dP \end{aligned}$$

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Infinite series

But becomes finite if $P(t)$ has constant degree!

Forward differences for cubic Bézier curve

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$P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$
or, equivalently,
$P(t) = at^3 + bt^2 + ct + d$
where:
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Now it is easy to differentiate:

$P'(t) = 3at^2 + 2bt + c$ $\quad P''(t) = 6at + 2b$ $\quad P'''(t) = 6a$

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or, equivalently,

$P(t) = at^3 + bt^2 + ct + d$

where:

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Forward differences for cubic Bézier curve

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Problem: dP depends on t!

Forward differences for cubic Bézier curve

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We combine the three increments (dP, ddP and $dddP$) in an algorithms as follows:

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```
procedure FastCubicBézierSketch
    Compute dP, ddP and dddP for t=0
    P ← P_0
    for t = 0 to 1 step Δ do
        Draw point P
        P ← P + dP
        dP ← dP + ddP
        ddP ← ddP + dddP
```

Forward differences for cubic Bézier curve

Final code, trying to reuse computations as much as possible

procedure FastCubicBézier
    $Q_1 \leftarrow 3\Delta$
    $Q_2 \leftarrow Q_1 \cdot \Delta$ $\triangleright\ 3\Delta^2$
    $Q_3 \leftarrow \Delta^3$
    $Q_4 \leftarrow 2Q_2$ $\triangleright\ 6\Delta^2$
    $Q_5 \leftarrow 6Q_3$ $\triangleright\ 6\Delta^3$
    $Q_6 \leftarrow P_0 - 2P_1 + P_2$
    $Q_7 \leftarrow 3(P_1 - P_2) - P_0 + P_3$ $\triangleright\ a$
    $P \leftarrow P_0$
    $dP \leftarrow (P_1 - P_0)Q_1 + Q_6 \cdot Q_2 + Q_7 \cdot Q_3$
    $ddP \leftarrow Q_6 \cdot Q_4 + Q_7 \cdot Q_5$
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        Draw point P
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for $t = 0$ to 1 step Δ **do**

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The reduction in # of operations is huge: Ignoring the initialization, 3 sums for each evaluation of t

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Bézier curves are often expressed in matrix form

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Question: how many sums/products to evaluate $P(t)$?

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- Recall: curve subdivision took a degree-n curve and produced two curves of degree-n ($2n + 1$ control points in total)
- Alternative: add points (increase degree) while preserving curve

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original curve		new curve
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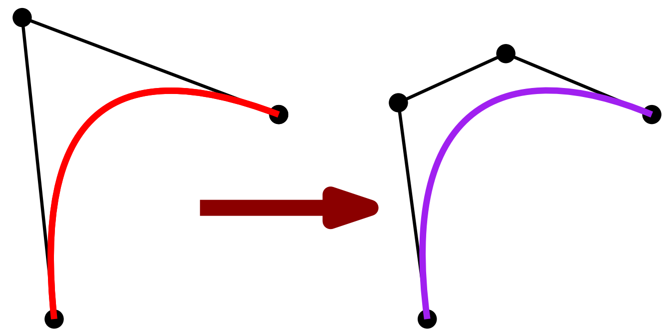

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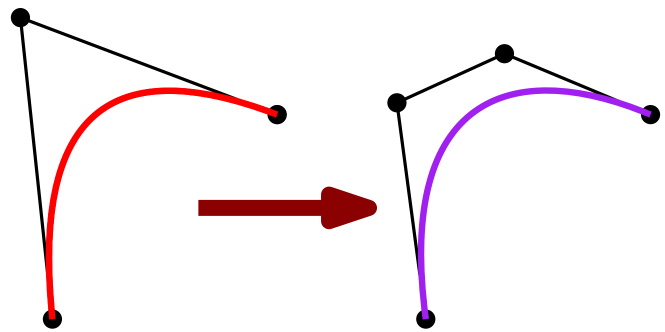

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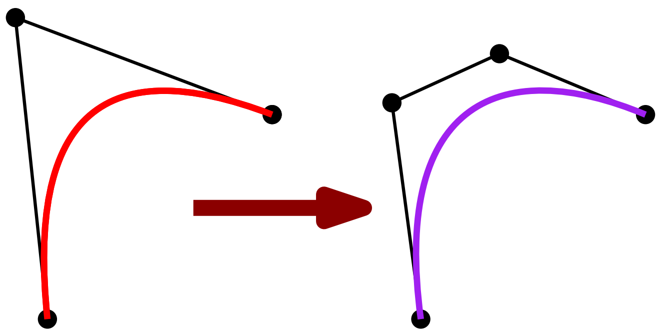

Producing control points for $P_{n+1}(t)$

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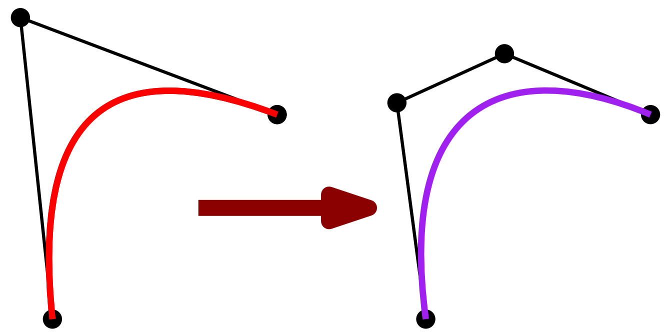

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With some basic algebraic tricks one can write $P(t)$ as an $(n+1)$-degree Bézier curve

DEGREE ELEVATION

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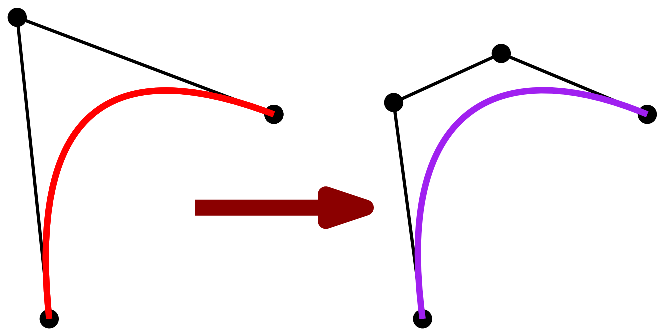

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Start from trivial identity $P(t) = (t + (1-t))P(t) = tP(t) + (1-t)P(t)$

Use that $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i$, and extract coefficientes of new $(n+2)$ control points

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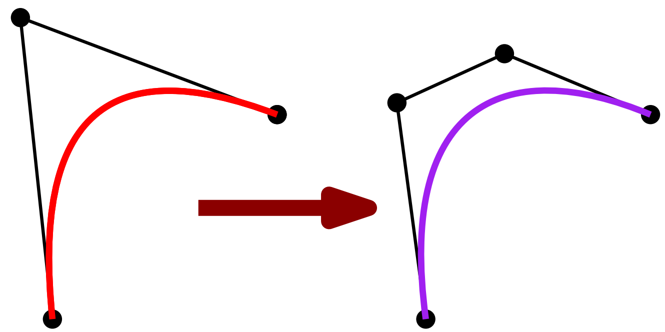

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Result:

$$P(t) = tP(t) + (1-t)P(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} t^i (1-t)^{n-i+1} \left(\frac{i}{n+1} P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i\right)$$

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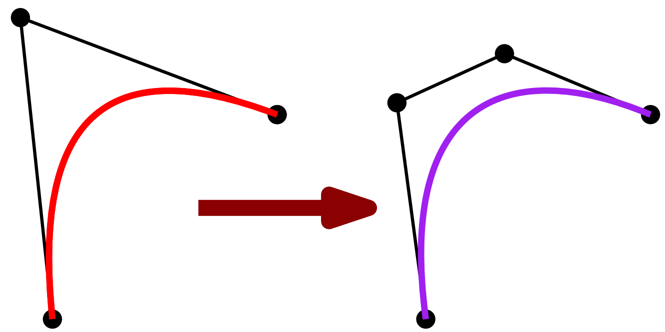

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Bézier curve of degree $(n+1)$!

$Q_i = \alpha_i P_{i-1} + (1-\alpha_i) P_i$

new control points

Note: here we assume $P_{-1} = 0$ and $P_{n+1} = 0$

Summary

The expression obtained for $P_{n+1}(t)$ is:

$$P_{n+1}(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} t^i (1-t)^{n+1-i} \left(\frac{i}{n+1} P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i \right)$$

$$P_{n+1}(t) = \sum_{i=0}^{n+1} B_{n+1,i}(t) Q_i$$

where $Q_i = \alpha_i P_{i-1} + (1-\alpha_i) P_i$, $\alpha_i = \frac{i}{n+1}$
and $Q_0 = P_0$, $Q_{n+1} = P_n$

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Example

Summary

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Example

degree-4 curve (5 control points)

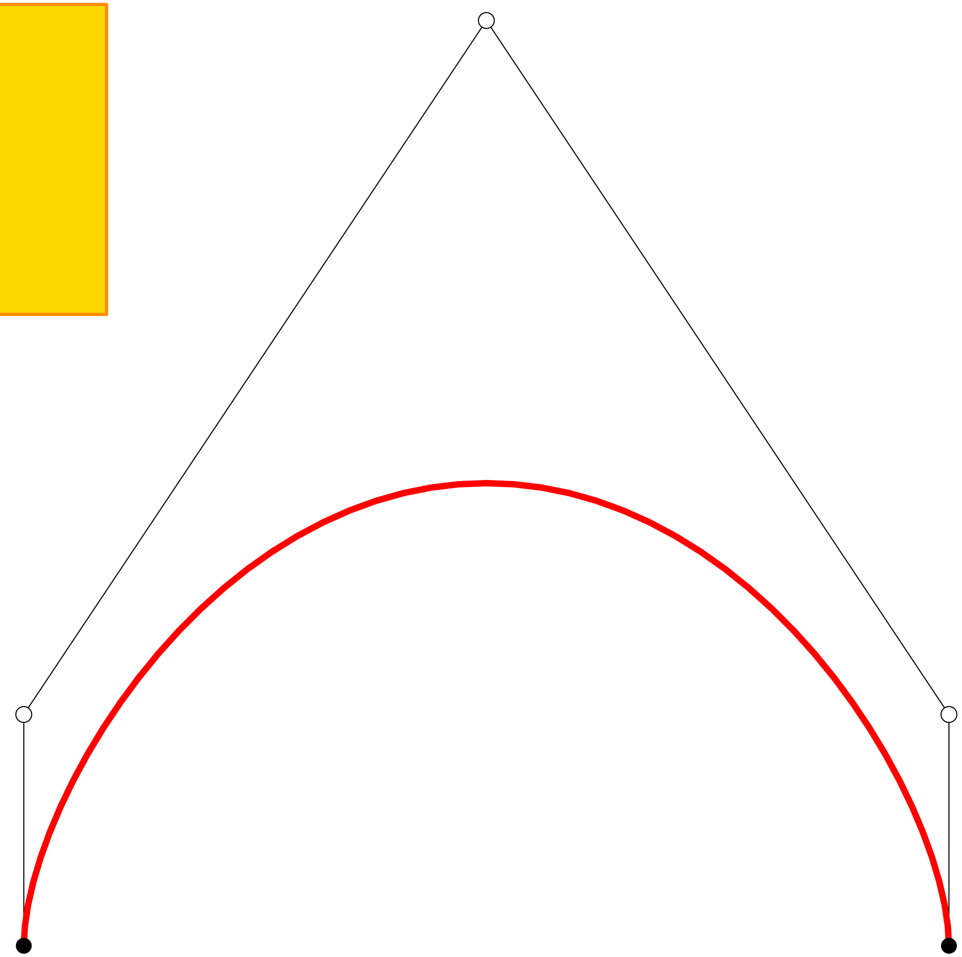

Figure from [G. Farin and D. Hansford, *The Essentials of CAGD*, AK Peters, 2000].

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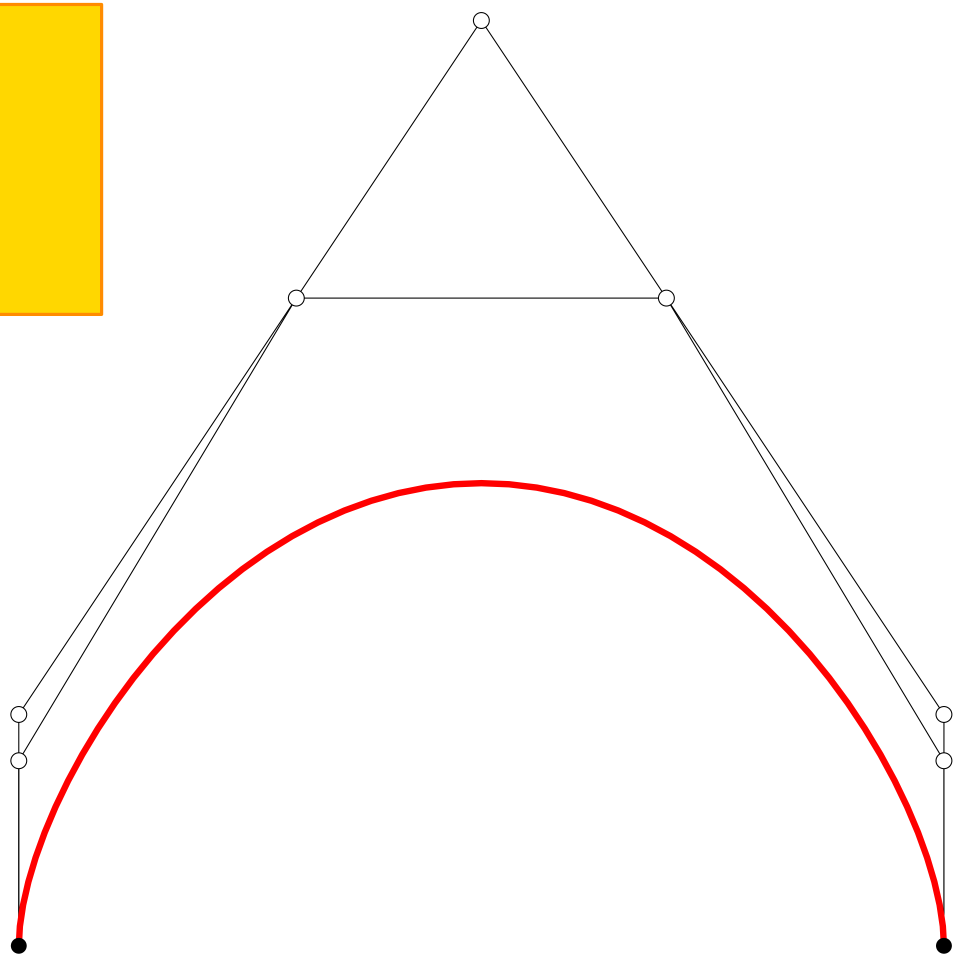

Example

degree-4 curve (5 control points)

degree-5 curve (6 control points)

Figure from [G. Farin and D. Hansford, *The Essentials of CAGD*, AK Peters, 2000].

DEGREE ELEVATION

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The expression obtained for $P_{n+1}(t)$ is:

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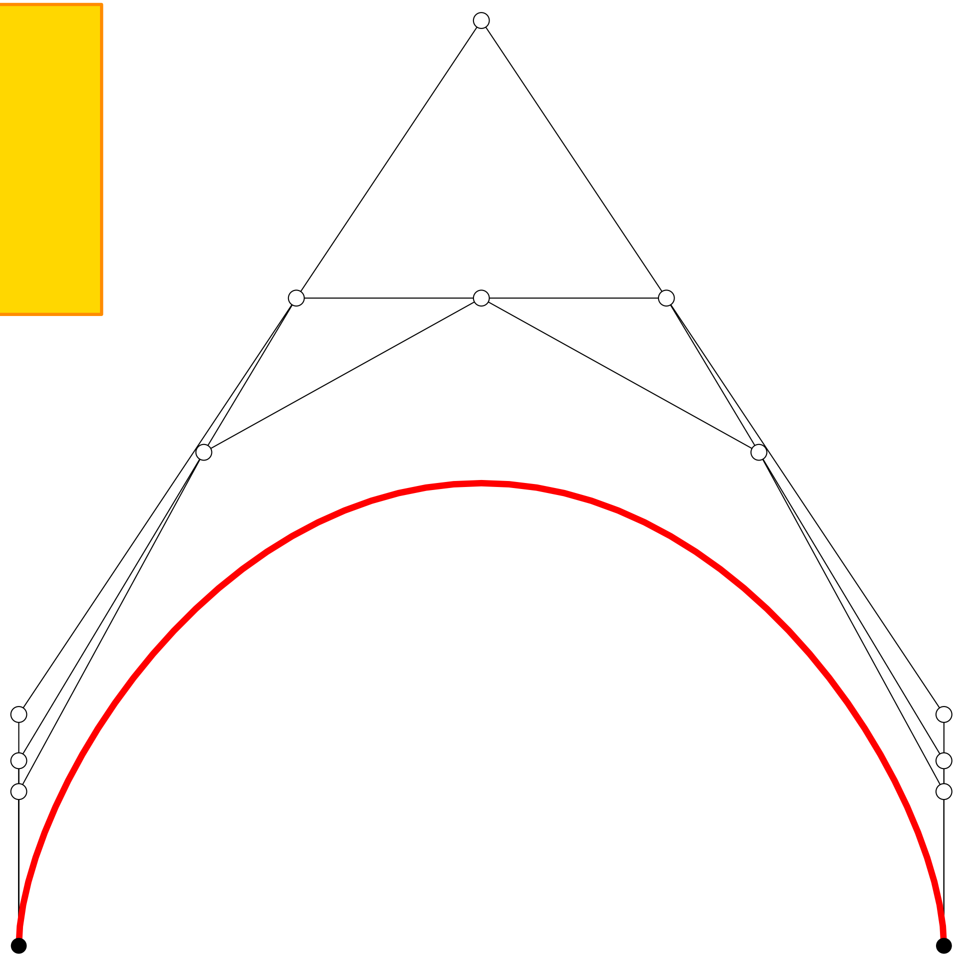

Example

degree-4 curve (5 control points)

degree-5 curve (6 control points)

degree-6 curve (7 control points)

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degree-6 curve (7 control points)

degree-n curve ($n+1$ control points)

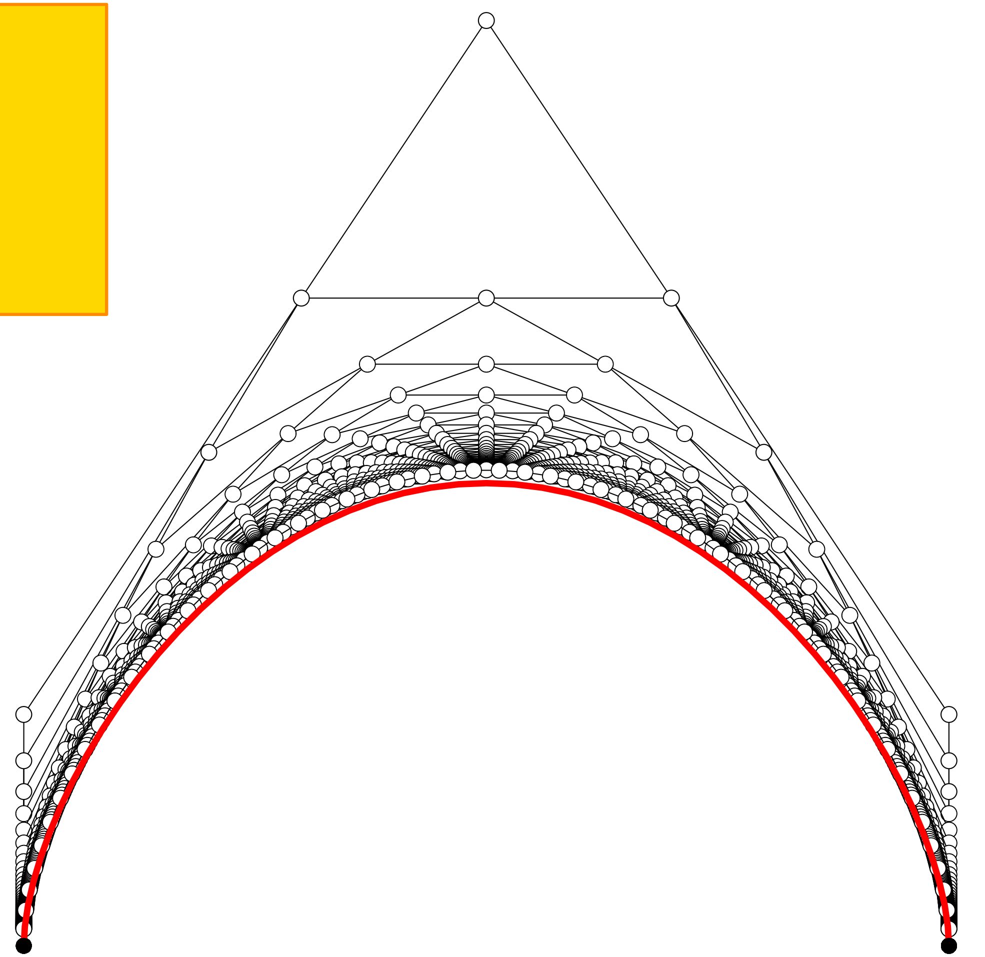

Figure from [G. Farin and D. Hansford, *The Essentials of CAGD*, AK Peters, 2000].

DEGREE ELEVATION

Summary

Question: Can you do degree reduction?

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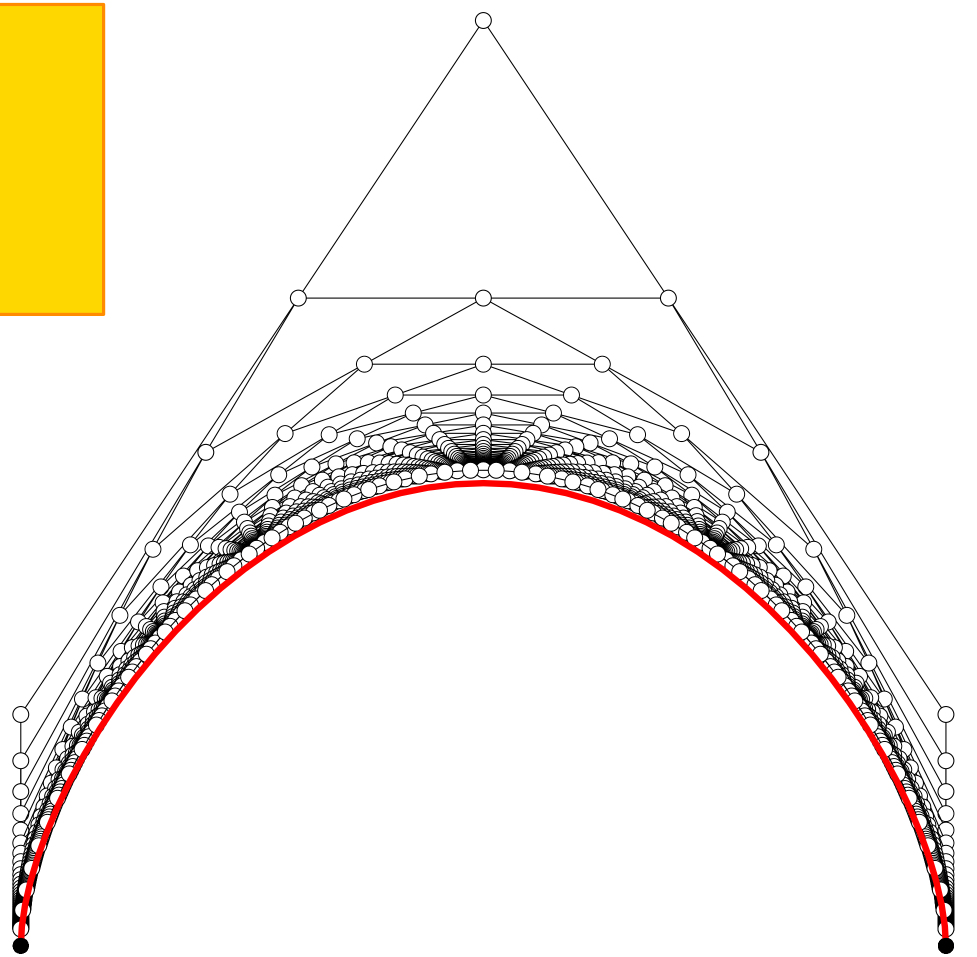

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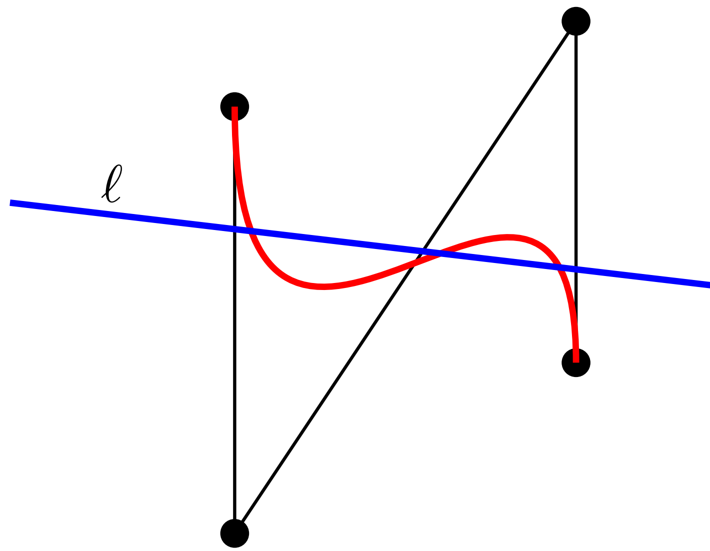
Back to the variation diminishing property

Recall the property: The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygon



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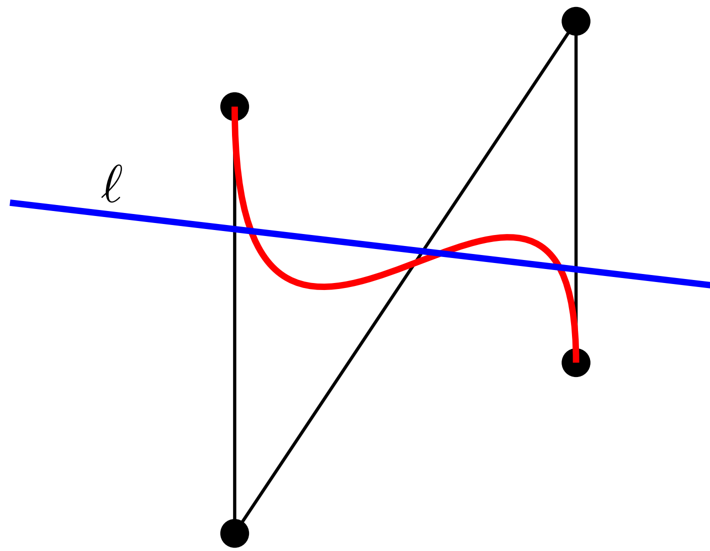

Proof sketch using degree elevation

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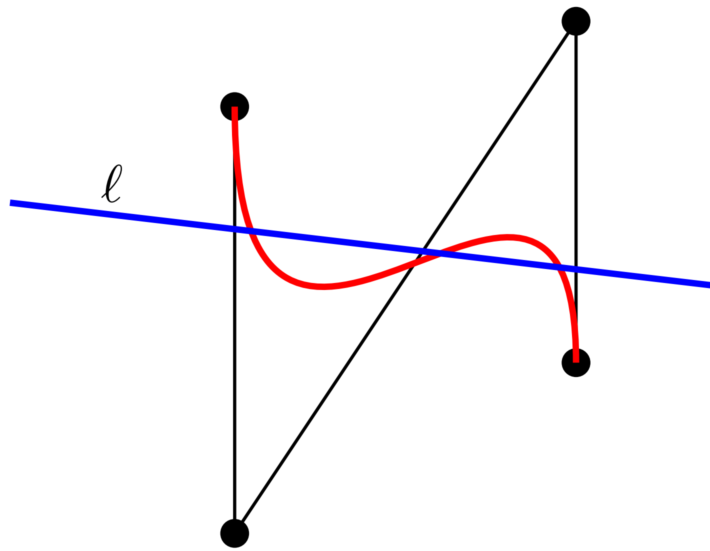

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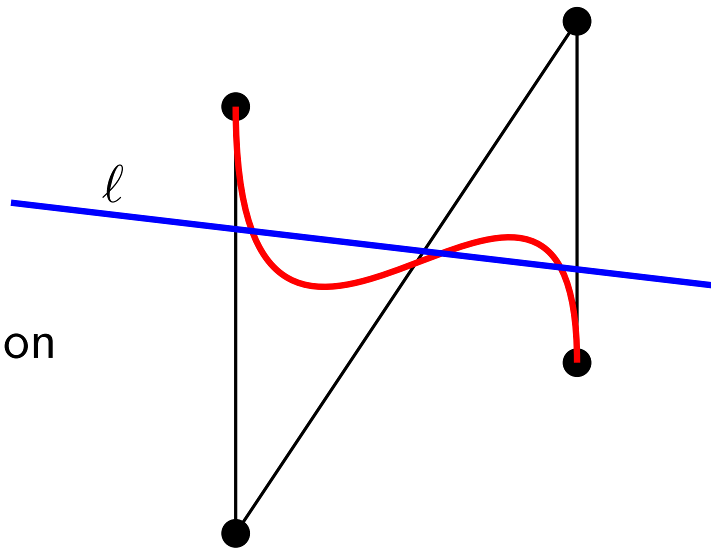

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Observation 2: degree elevation is an instance of linear interpolation

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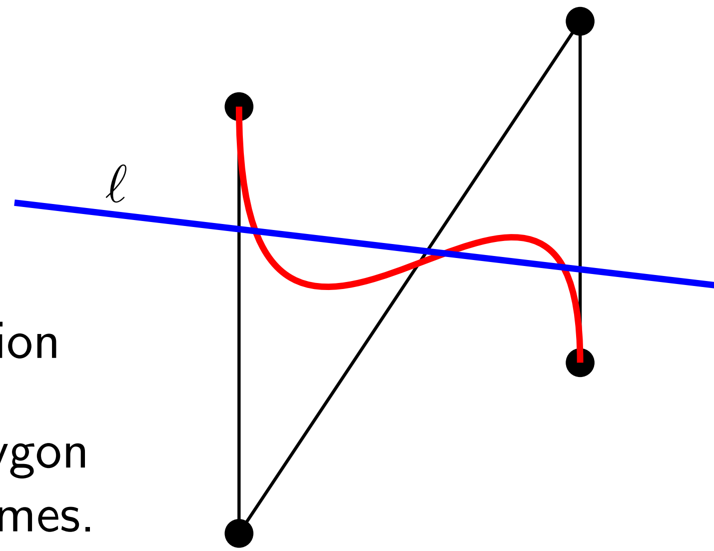

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Let R_0 be the control polygon of $P(t)$, let R_1 be the control polygon after increasing the degree by one, and R_k after increasing it k times.

Let ℓ be a given line. Then R_k has no more intersections with ℓ than R_0.

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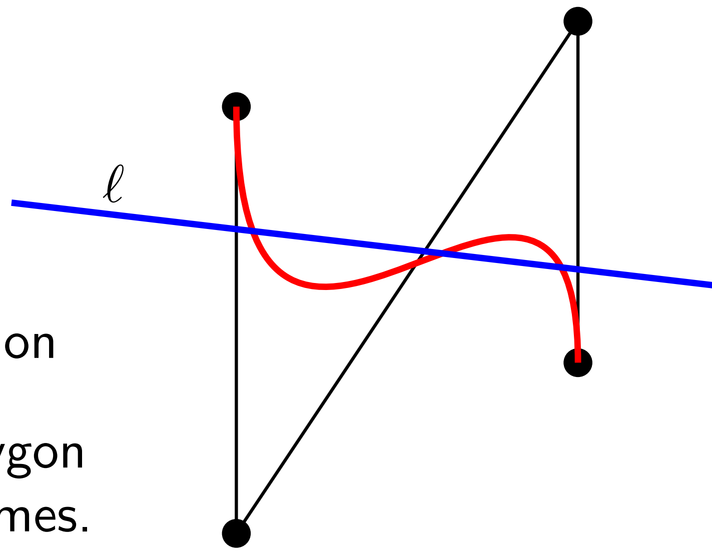

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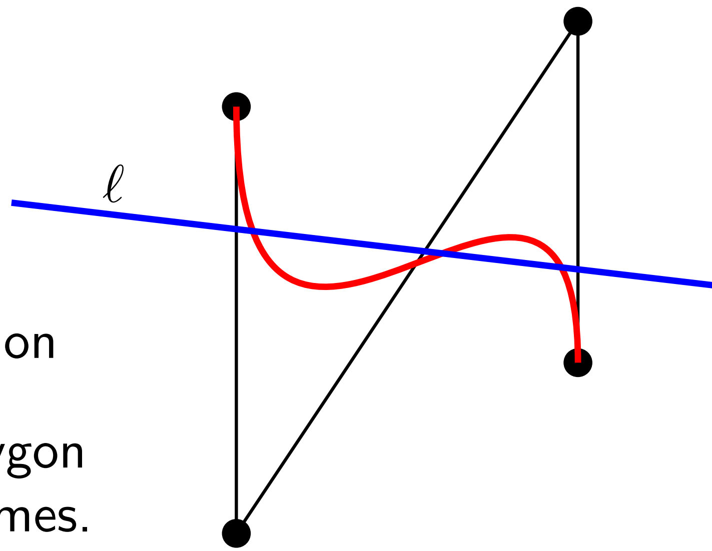

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Lemma: in the limit, as $k \to \infty$, R_k approaches the Bézier curve $P(t)$

Corollary: the Bézier curve $P(t)$ has no more intersections with ℓ than its control polygon

Goal: find Bézier curve that interpolates given points

- In general, the Bézier curve does not interpolate its control points
- There are situations in which the user may want to force the curve through some points

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INTERPOLATION WITH BÉZIER CURVES

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unknowns

We are free to choose the values of the t_is!

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For example, take $t_i = i/n = i/3$

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$$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{5}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ \frac{1}{3} & -\frac{3}{2} & 3 & -\frac{5}{6} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 3/2 \\ 2 & 3/2 \\ 3 & 0 \end{pmatrix}$$

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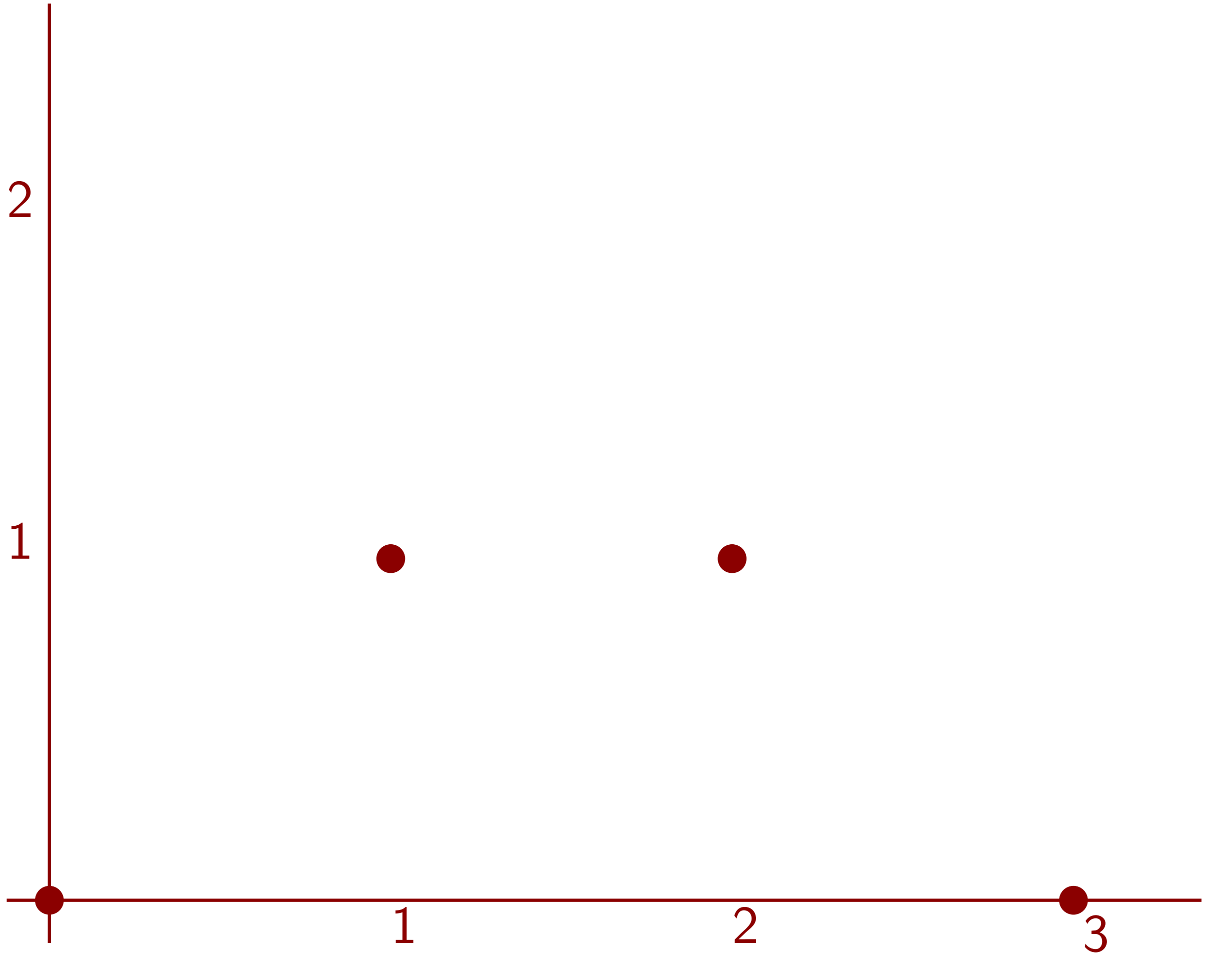

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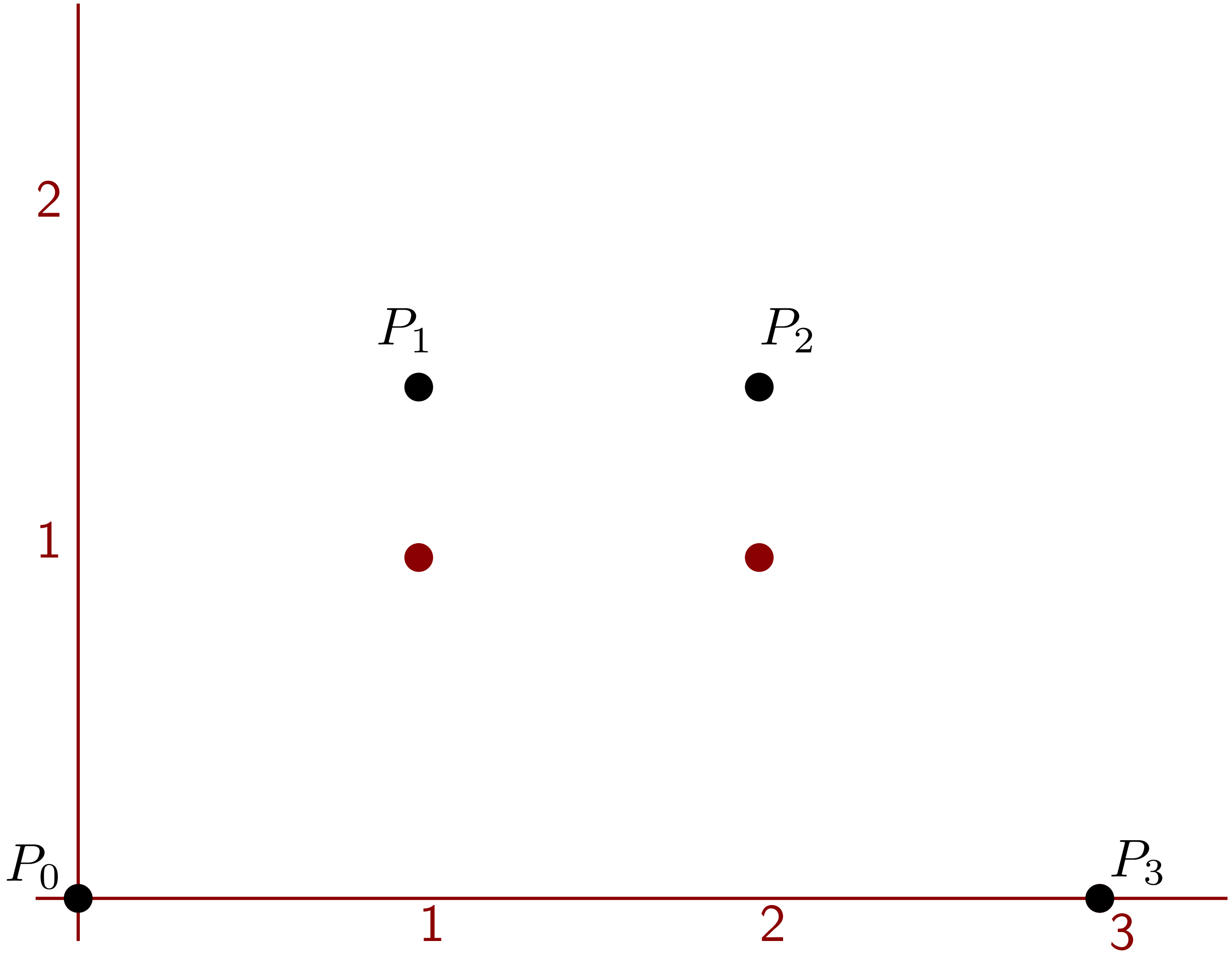

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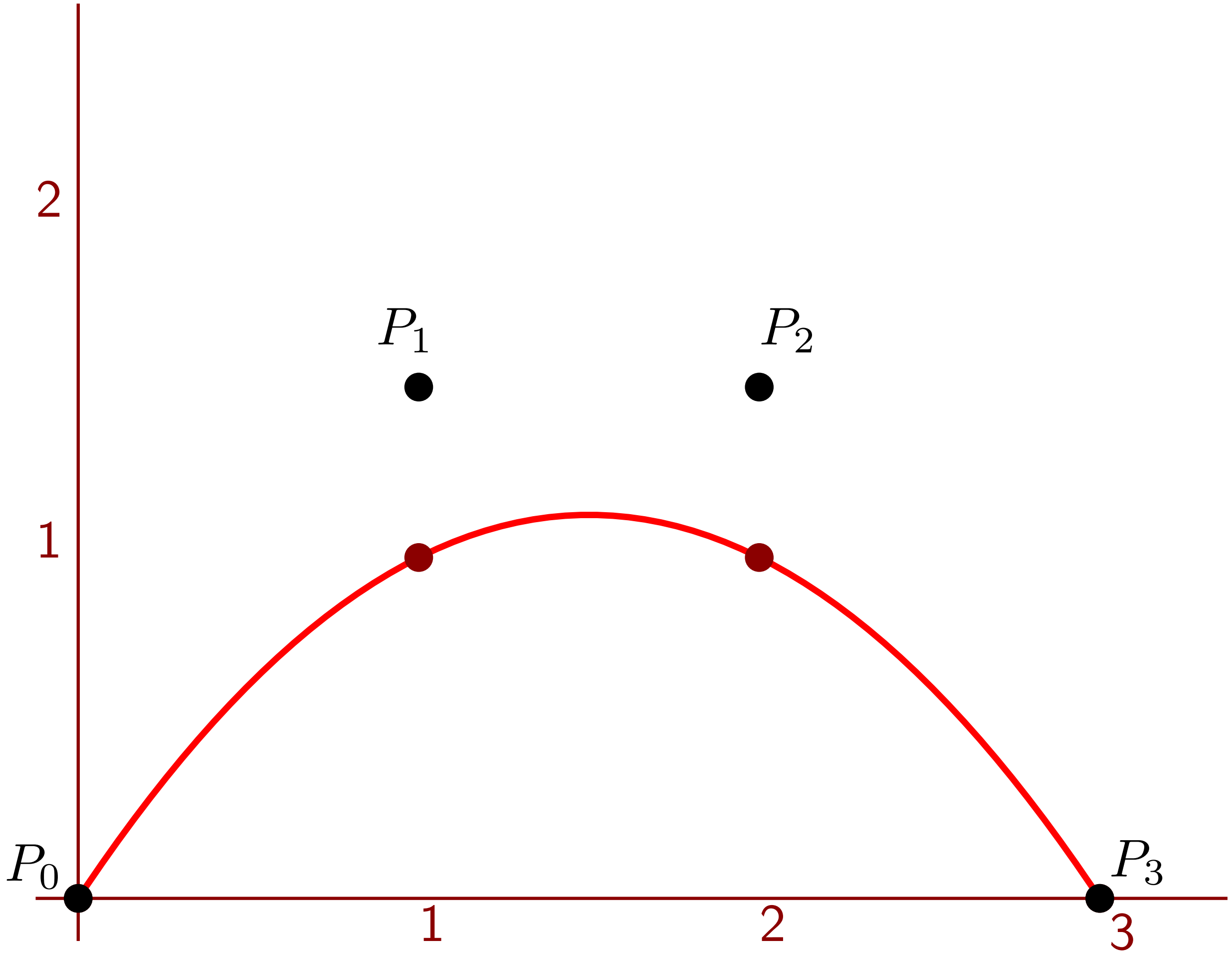

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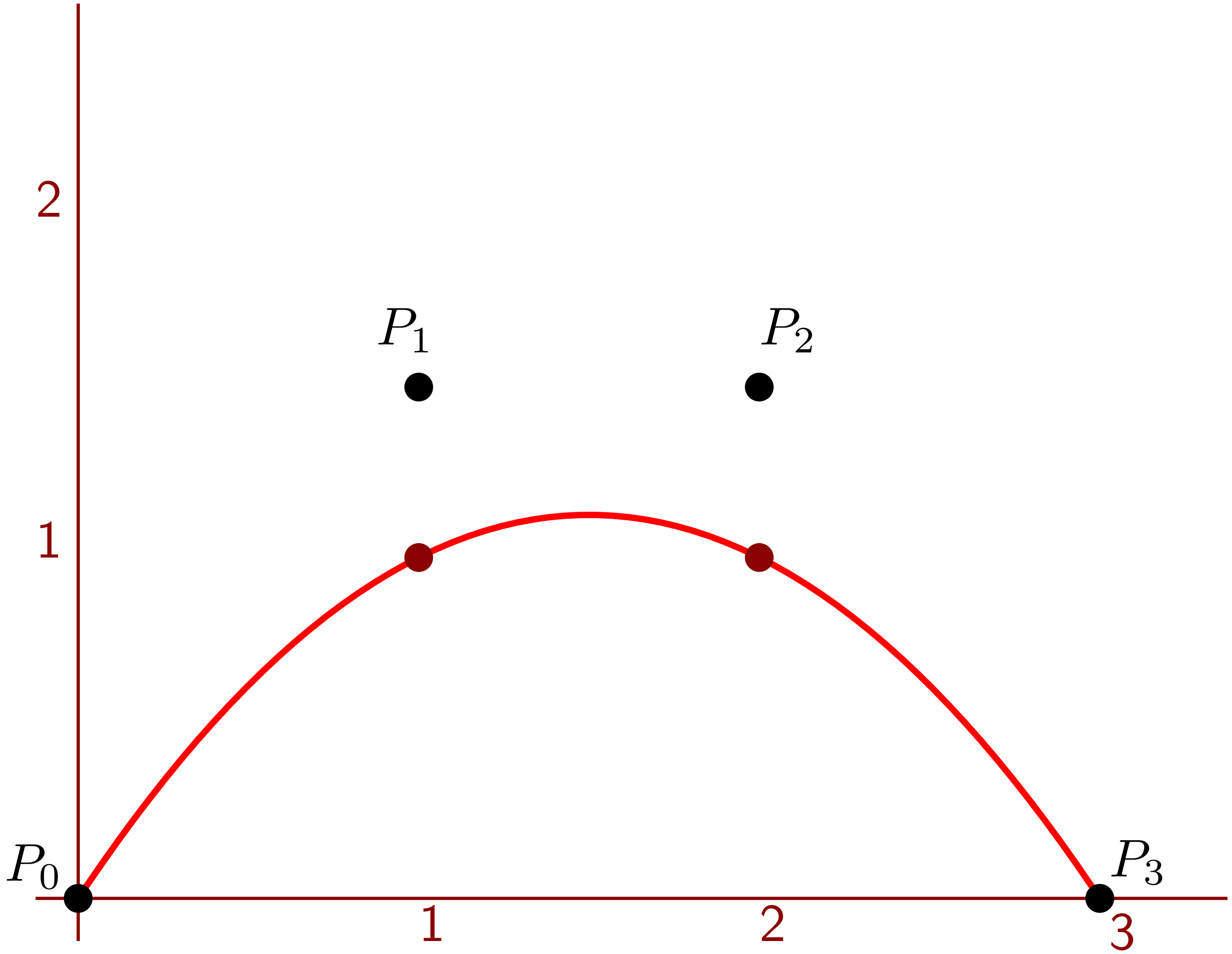
This is only one way to interpolate with Bézier curves, others are possible



Rational Bézier curves

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$$P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t)$$

Bézier curve

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Rational Bézier curve

- Weights are usually non-negative (otherwise denominator could be zero)

Advantages: why complicate things so much?

- Invariant under projections
- It can represent conic curves (impossible with Hermite or Bézier curves) (e.g., segments of circles, ellipses, hyperbolas and parabolas)

Understanding rational Bézier curves

Effect of the weights

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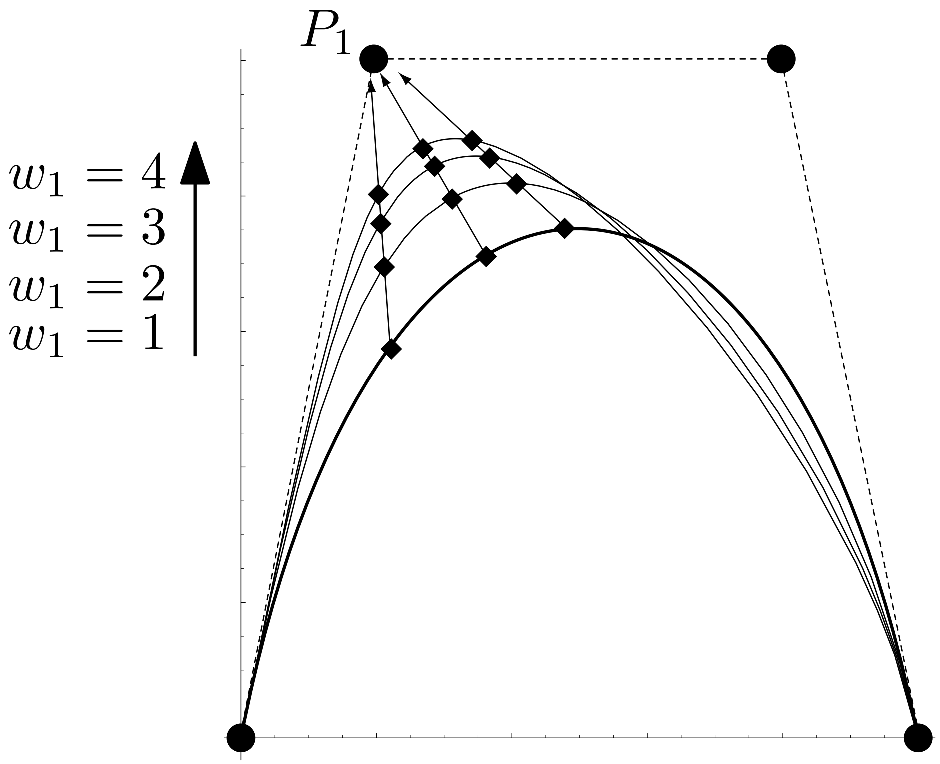
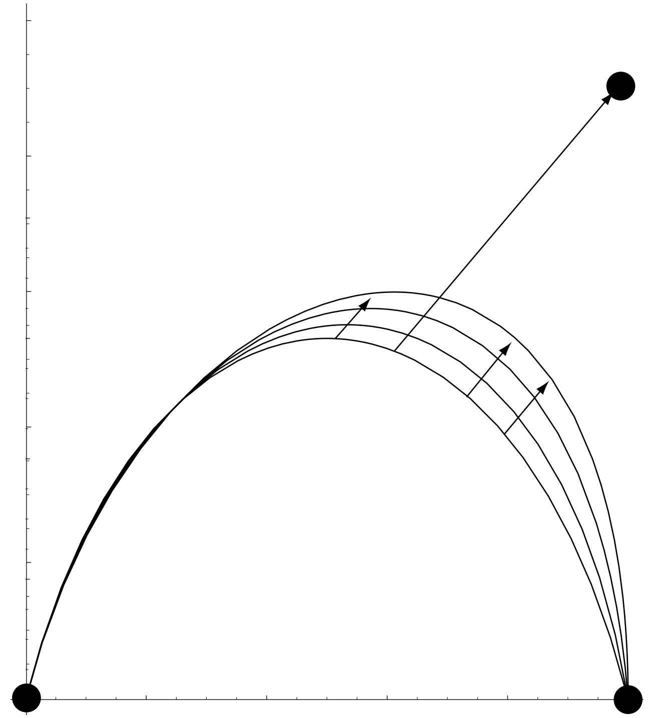
- If $w_i > 1$, the curve gets closer to P_i
- If $w_i < 1$, the curve moves away from P_i



changing weights



moving control point (weight fixed at 1)

Figure from book by Salomon (page 219)

Rational Bézier curves as curves in projective space

2D rational Bézier curve =projection of 3D nonrational Bézier curve onto 2D space!

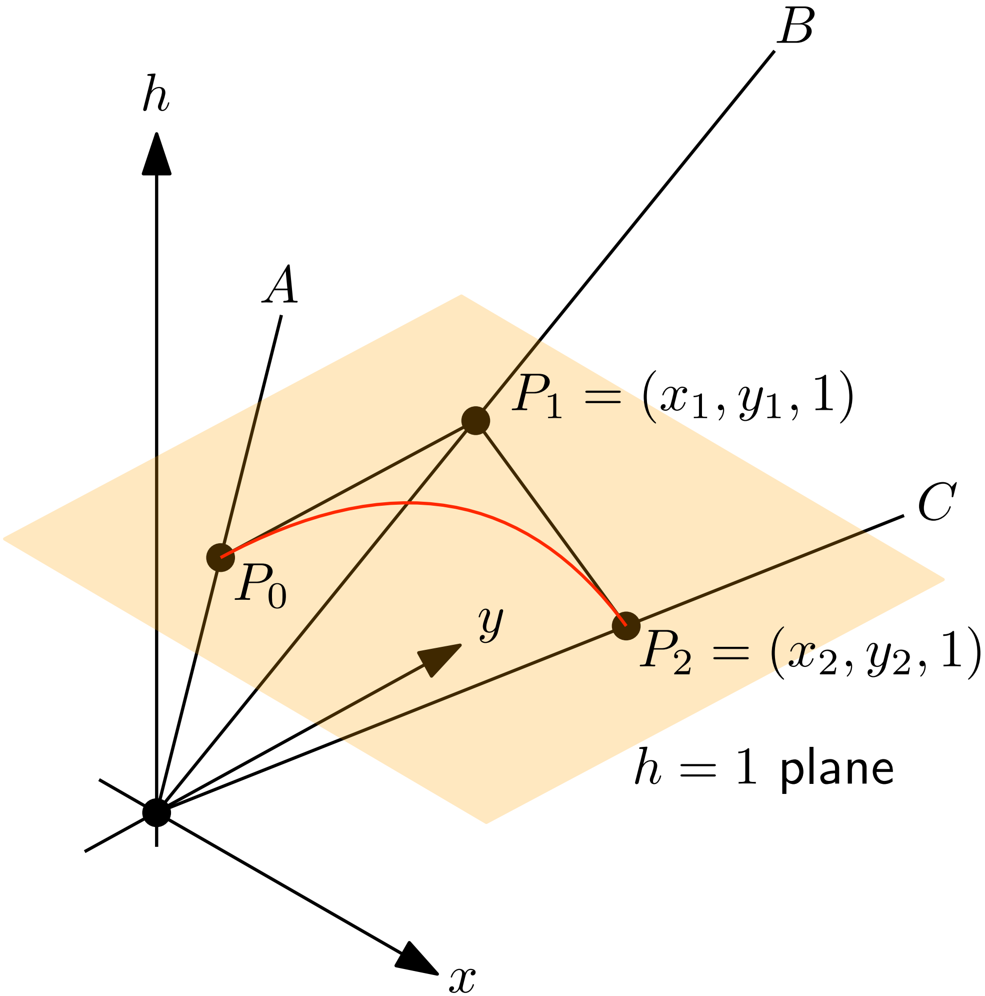


Figure adapted from book by Mortenson

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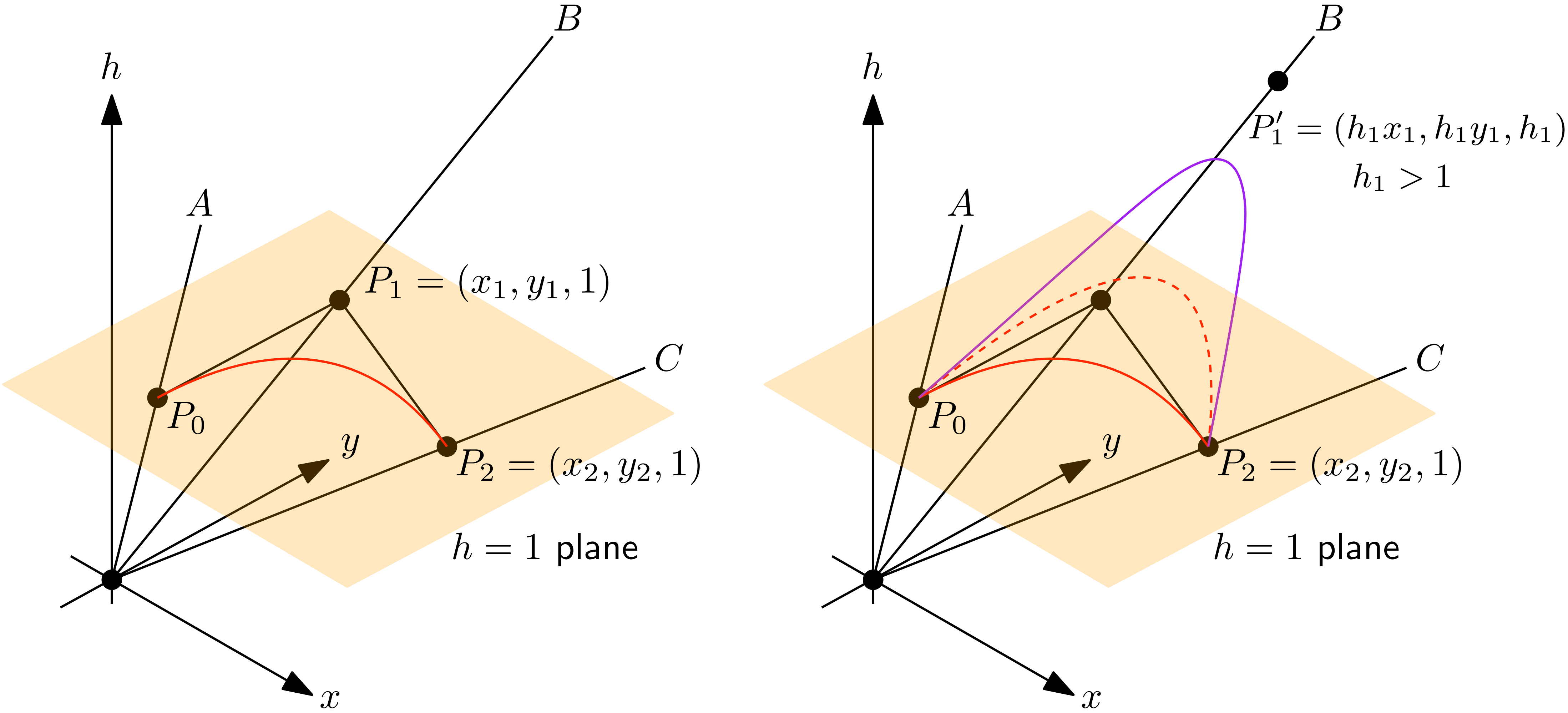

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Representing conics with rational Bézier curves

We can represent a conic curve *exactly* with a quadratic rational Bézier curve:

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Theorem Consider a conic curve $C(t)$. Then there exist weights w_0, w_1, w_2 and control points P_0, P_1, P_2 such that

$$C(t) = \frac{w_0 P_0 B_{2,0}(t) + w_1 P_1 B_{2,1}(t) + w_2 P_2 B_{2,2}(t)}{w_0 B_{2,0}(t) + w_1 B_{2,1}(t) + w_2 B_{2,2}(t)}$$

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Example

Take $w_0 = w_2 = 1$ and let $s = \frac{w_1}{1+w_1}$

- $s = 1/2$ produces a **parabolic** arc
- $s < 1/2$ produces an **elliptic** arc
- $s > 1/2$ produces a **hyperbolic** arc

for any three non-colinear control points