SPLINE INTERPOLATION

Rodrigo Silveira

Curve and Surface Design
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

What can you do with Hermite curves? Combine them!

- A single Hermite curve is not so useful in practice, but it gets more interesting if you can put together several of them!

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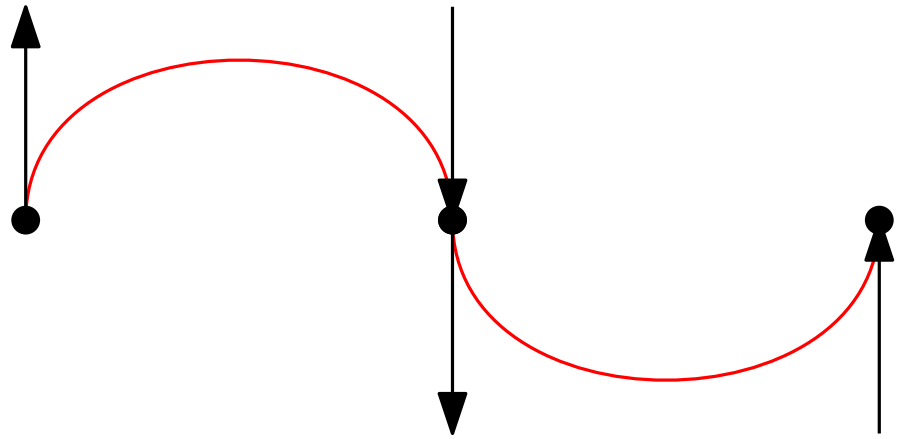

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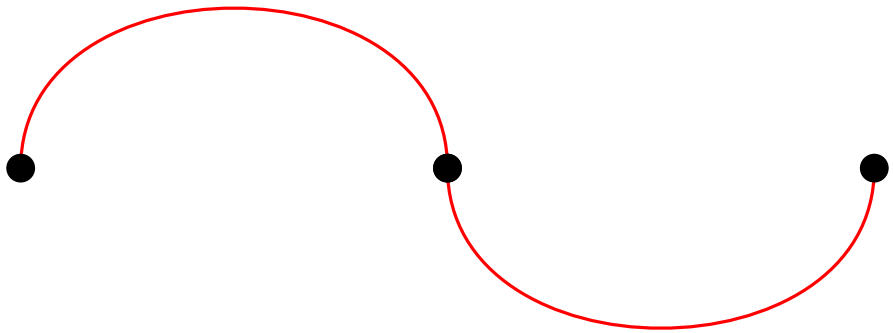

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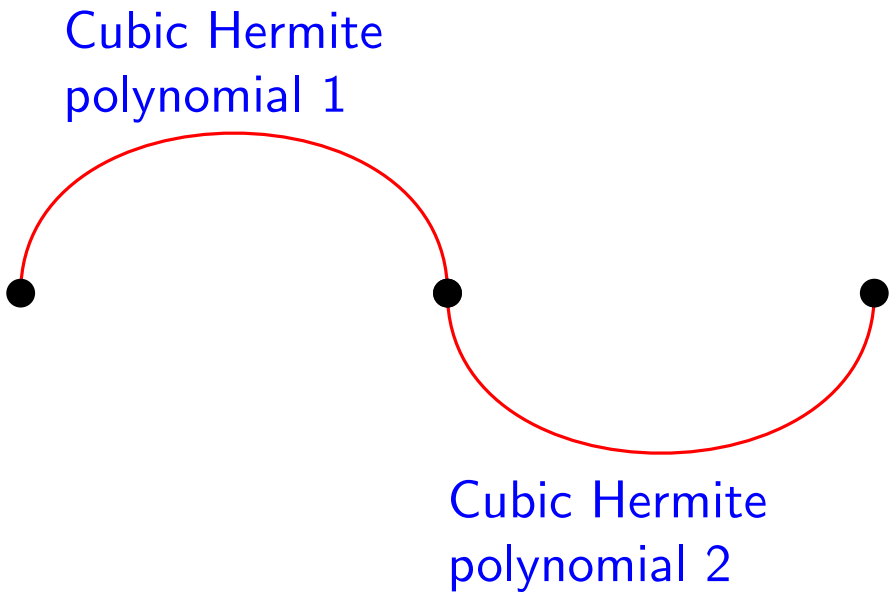

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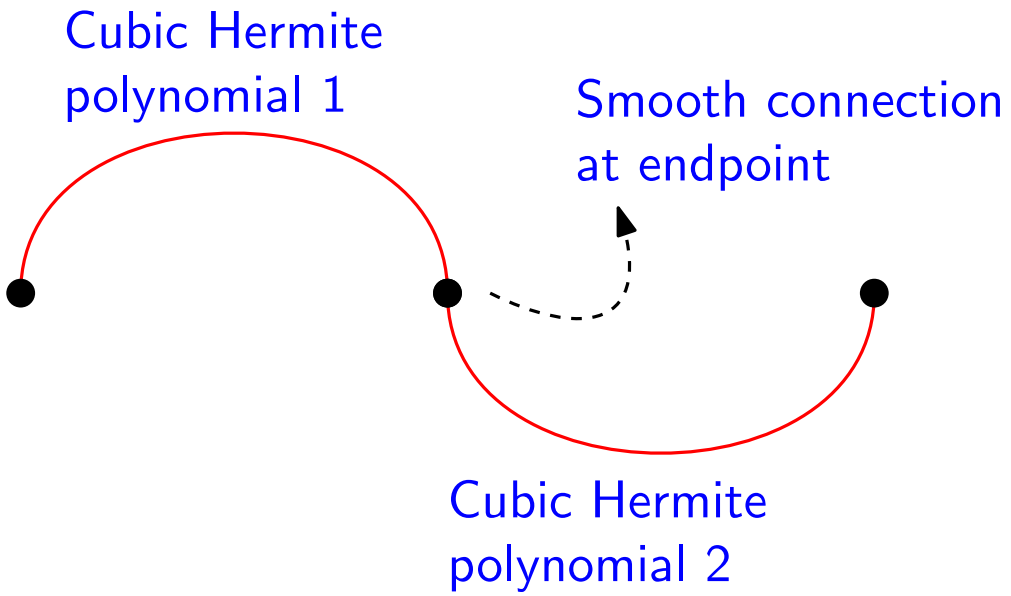

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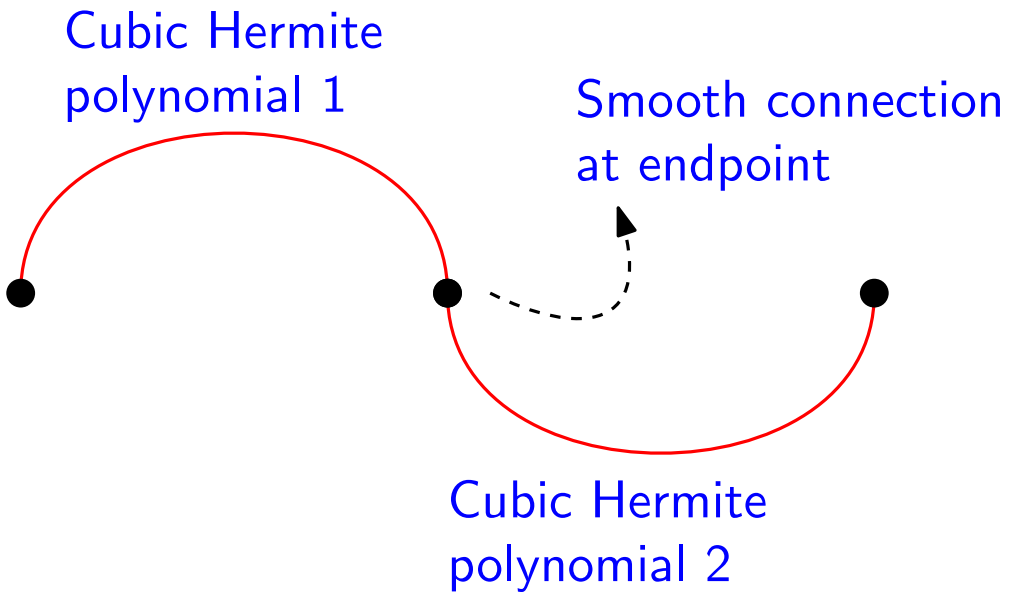

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- More precisely: at each data point (except for the first and last), two polynomials connect, and their derivatives up to the $(k-1)$th have the same values

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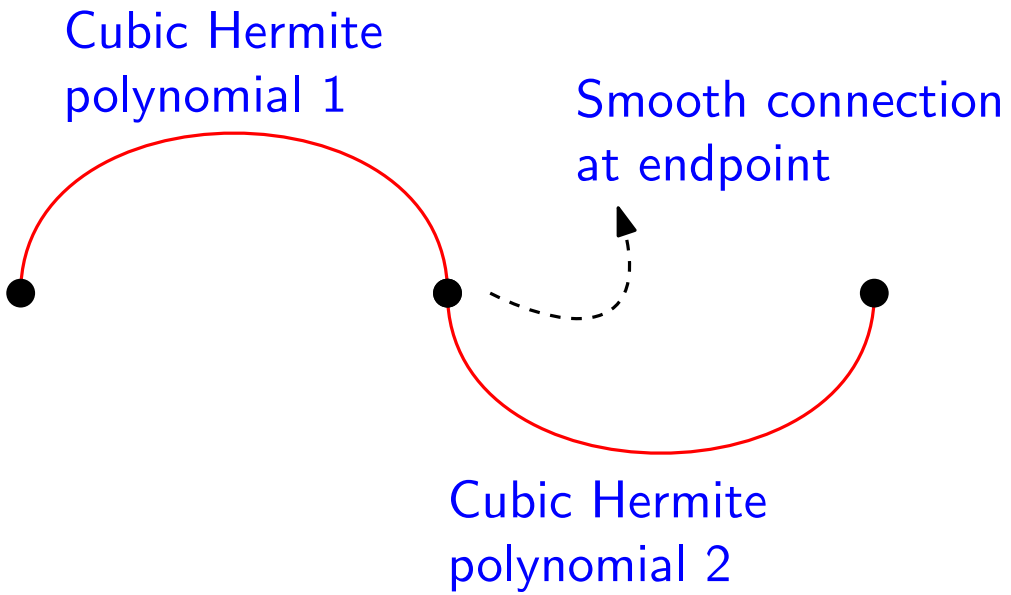

- **Spline**: set of polynomial curves of degree k that are smoothly connected at data points
- More precisely: at each data point (except for the first and last), two polynomials connect, and their derivatives up to the $(k-1)$th have the same values
- Splines allow keeping the degree low, which in turn allows avoiding the wiggling of high-degree polynomials, as well as the complexity of their computation

Cubic splines

- Input: n points $P_1, \ldots, P_n$ and 2 tangent vectors $\overrightarrow{v_1}$, $\overrightarrow{v_n}$ (other possibilities exist, see later)
- Output: A curve consisting of $n-1$ cubic Hermite curves that are smoothly connected (up to 2nd derivative) at the intermediate points $P_2, \ldots, P_{n-1}$

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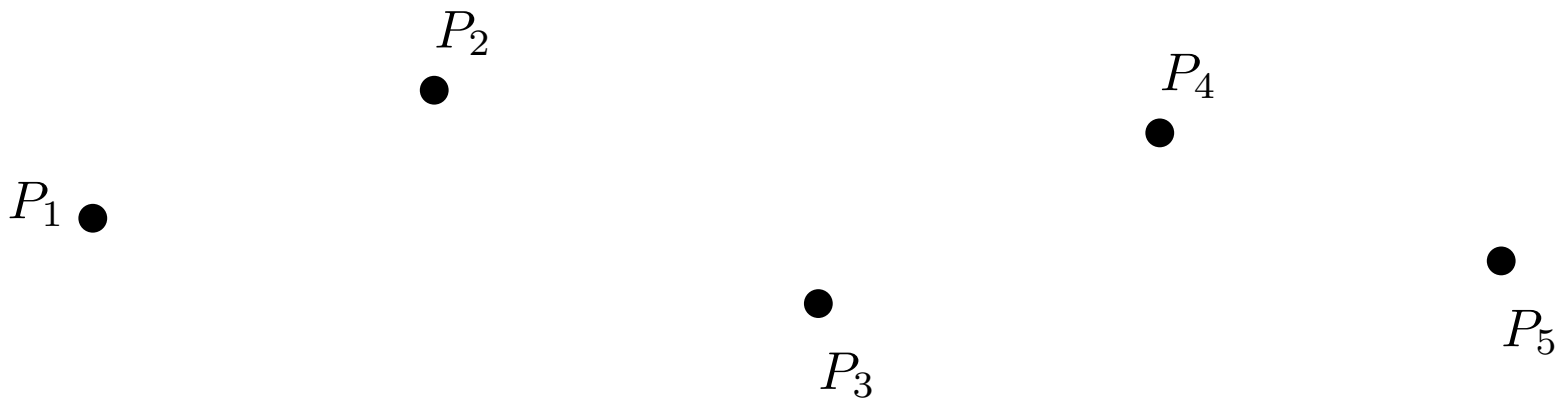
Example with $n = 5$



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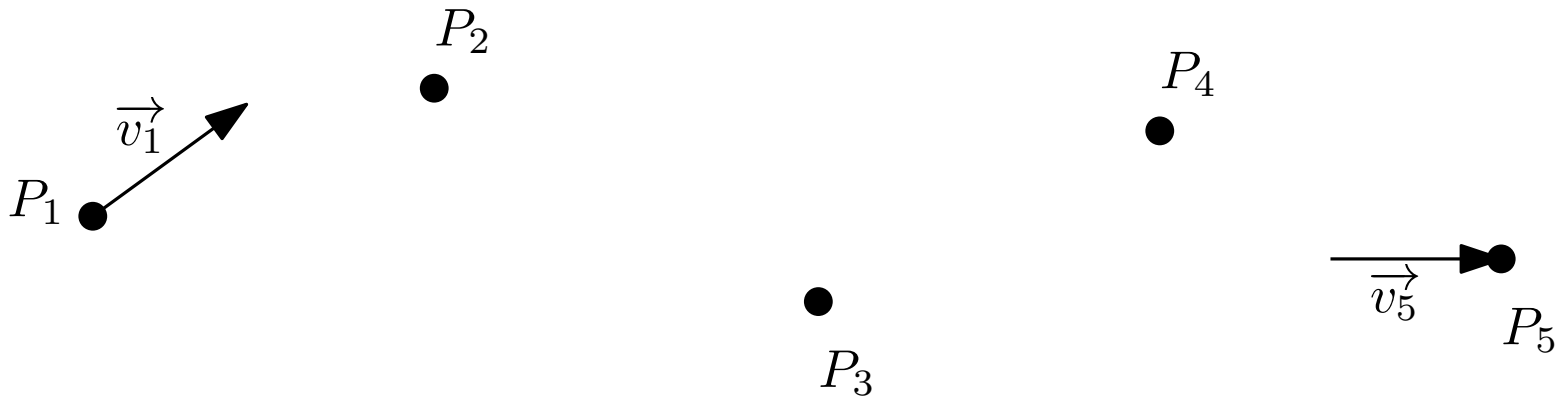

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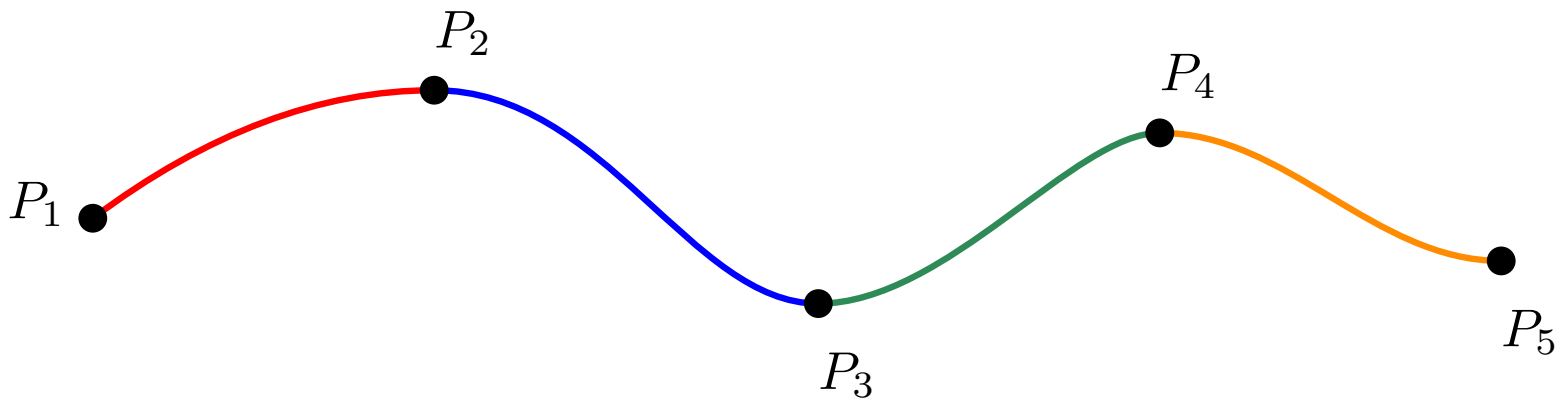

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  this is referred to as being *C^2-continuous*

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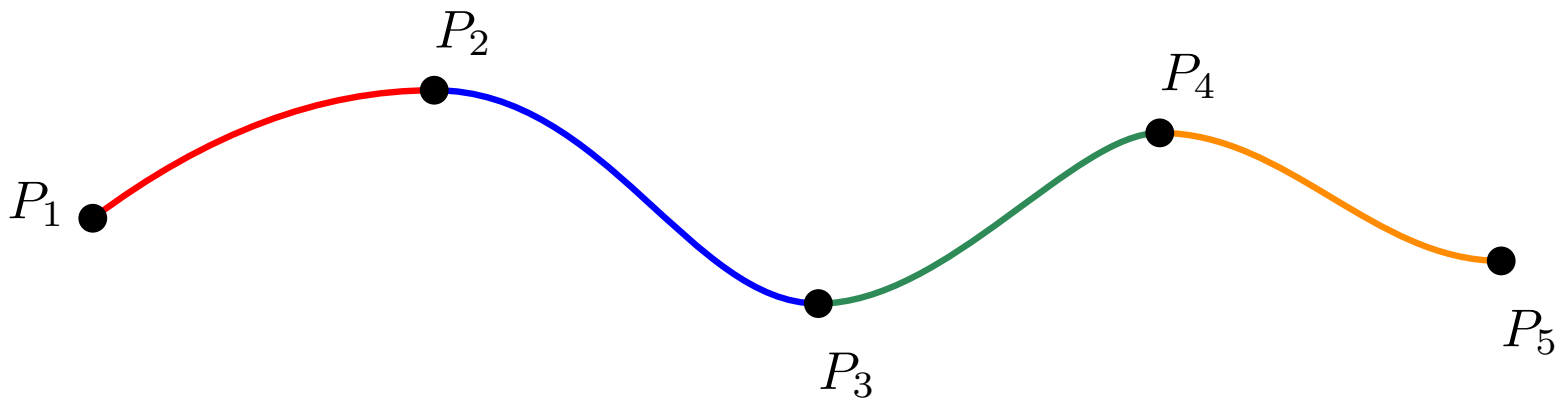

Computing cubic splines

Consider three consecutive input points P_i, P_{i+1}, P_{i+2} for $i = 1, \ldots, n-2$.

Let γ_i be the Hermite curve connecting P_i to P_{i+1}, and γ_{i+1} the one connecting P_{i+1} to P_{i+2}

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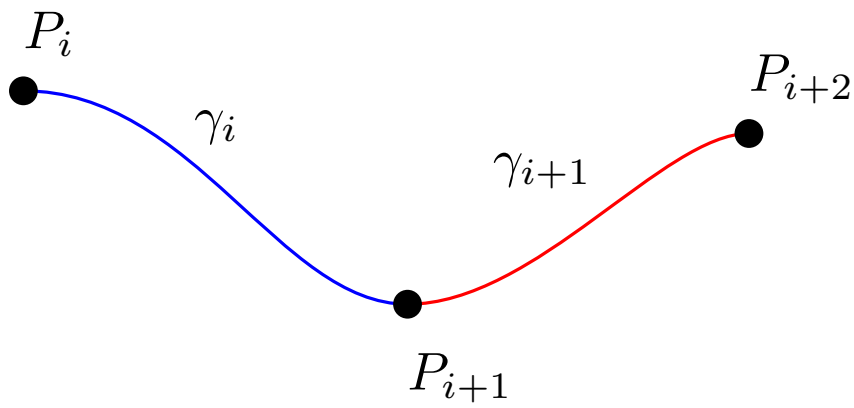

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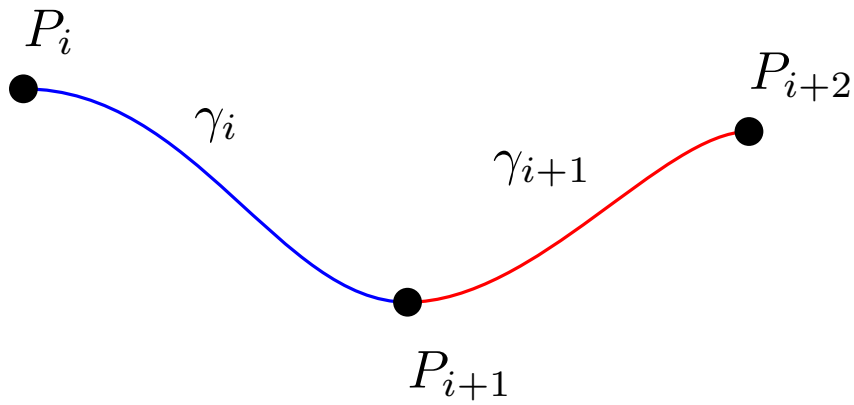
Strategy: we will specify a system of equations with the conditions that our curves should satisfy, and solve the system to find out how they must look like



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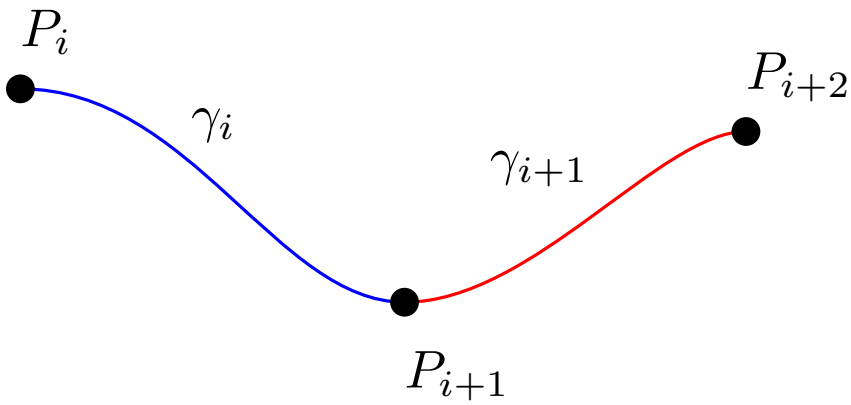

Curves γ_i and γ_{i+1} should satisfy the following:

$$\gamma_i(0) = P_i$$
$$\left.\begin{aligned}\gamma_i(1) &= P_{i+1}\\ \gamma_{i+1}(0) &= P_{i+1}\end{aligned}\right\} =$$
$$\gamma_{i+1}(1) = P_{i+2}$$

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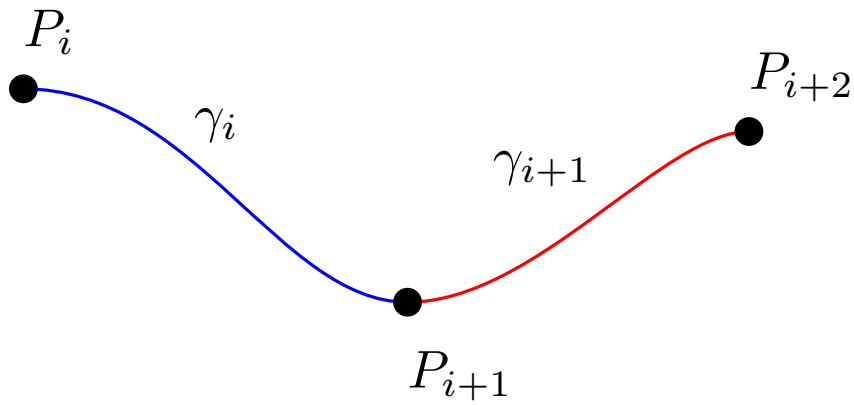

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$$\gamma_{i+1}(1) = P_{i+2} \qquad \gamma_{i+1}'(1) = \overrightarrow{v_{i+2}}$$

$$\gamma_i''(1) = \gamma_{i+1}''(0)$$

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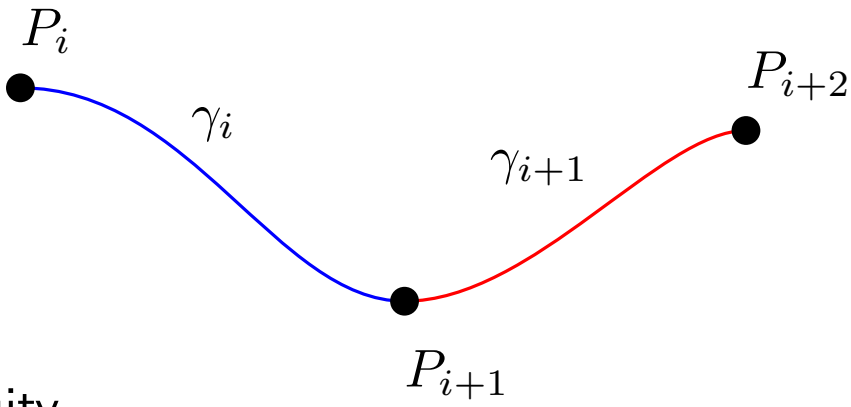

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$$\left.\begin{array}{l}\gamma_i(0) = P_i \\ \gamma_i(1) = P_{i+1} \\ \gamma_{i+1}(0) = P_{i+1} \\ \gamma_{i+1}(1) = P_{i+2}\end{array}\right\} C^0\text{-continuity}$$

$$\left.\begin{array}{l}\gamma_i'(0) = \overrightarrow{v_i} \\ \gamma_i'(1) = \overrightarrow{v_{i+1}} \\ \gamma_{i+1}'(0) = \overrightarrow{v_{i+1}} \\ \gamma_{i+1}'(1) = \overrightarrow{v_{i+2}}\end{array}\right\} C^1\text{-continuity}$$

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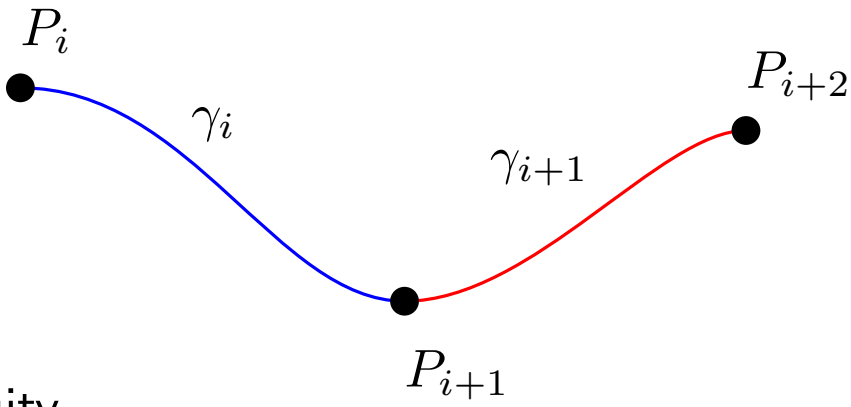

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known

C^0-continuity

$$\gamma_i'(0) = \overrightarrow{v_i}$$
$$\gamma_i'(1) = \overrightarrow{v_{i+1}} = \gamma_{i+1}'(0)$$
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C^1-continuity

$$\gamma_i''(1) = \gamma_{i+1}''(0)$$

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C^1-continuity

unknown!

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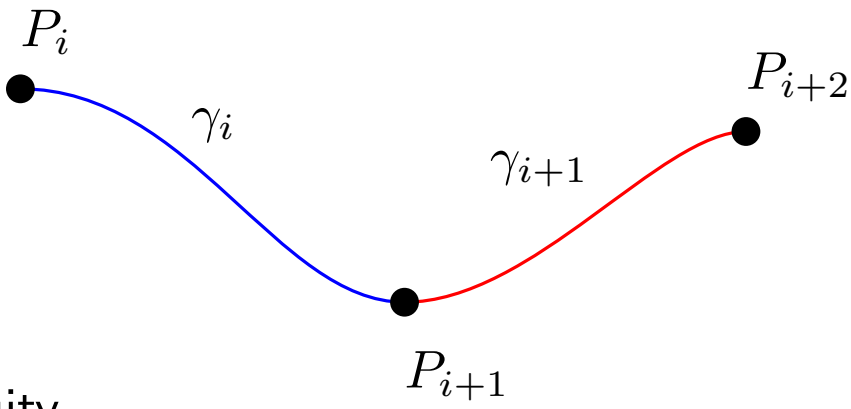

$\rightarrow$ We don't know the tangent and second derivative vectors at the interior points! That is all that we need for the Hermite curves to be fully defined

Computing cubic splines

We begin from the equation of the Hermite curve γ_i and its first two derivatives

$$\gamma_i(t) = (2t^3 - 3t^2 + 1)P_i + (-2t^3 + 3t^2)P_{i+1} + (t^3 - 2t^2 + t)\overrightarrow{v_i} + (t^3 - t^2)\overrightarrow{v_{i+1}}$$

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$$\gamma_i''(t) = (12t - 6)P_i + (-12t + 6)P_{i+1} + (6t - 4)\overrightarrow{v_i} + (6t - 2)\overrightarrow{v_{i+1}}$$

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$$\iff \overrightarrow{v_i} + 4\overrightarrow{v_{i+1}} + \overrightarrow{v_{i+2}} = 3(P_{i+2} - P_i)$$

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$$\iff \overrightarrow{v_i} + 4\overrightarrow{v_{i+1}} + \overrightarrow{v_{i+2}} = 3(P_{i+2} - P_i) \qquad \text{for } i = 1, \ldots, n-2$$

unknown known

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We have $n-2$ equations like this. That gives a system of $n-2$ equations where the variables are the $n-2$ unknown tangent vectors $\overrightarrow{v_2}, \ldots, \overrightarrow{v_{n-1}}$ (recall that $\overrightarrow{v_1}, \overrightarrow{v_n}$ are given)

Computing cubic splines

Each equation has shape $\overrightarrow{v_i} + 4\overrightarrow{v_{i+1}} + \overrightarrow{v_{i+2}} = 3(P_{i+2} - P_i)$ for $i = 1, \ldots, n-2$

We can write the equation system in matrix form as follows:

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Note that this includes the two input tangent vectors, thus it has n equations and n variables

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Each equation has shape $\overrightarrow{v_i} + 4\overrightarrow{v_{i+1}} + \overrightarrow{v_{i+2}} = 3(P_{i+2} - P_i)$ for $i = 1, \ldots, n-2$

We can write the equation system in matrix form as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{v_1} \\ \overrightarrow{v_2} \\ \overrightarrow{v_3} \\ \vdots \\ \overrightarrow{v_{n-1}} \\ \overrightarrow{v_n} \end{pmatrix} = \begin{pmatrix} \underline{\text{user value}} \\ 3(P_3 - P_1) \\ 3(P_4 - P_2) \\ \vdots \\ 3(P_n - P_{n-2}) \\ \underline{\text{user value}} \end{pmatrix}$$

Note that this includes the two input tangent vectors, thus it has n equations and n variables

This system has rank n, so it has a unique solution!

Summary: to produce the cubic splines one has to:
(i) Solve the system to obtain the $n-2$ unknown tangent vectors
(ii) Use them to build the $n-1$ Hermite curves connecting each pair of consecutive points P_i, P_{i+1}, for $i = 1, \ldots, n-1$

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See [Salomon], sections 5.1.2, 5.1.3, 5.1.5 for more details

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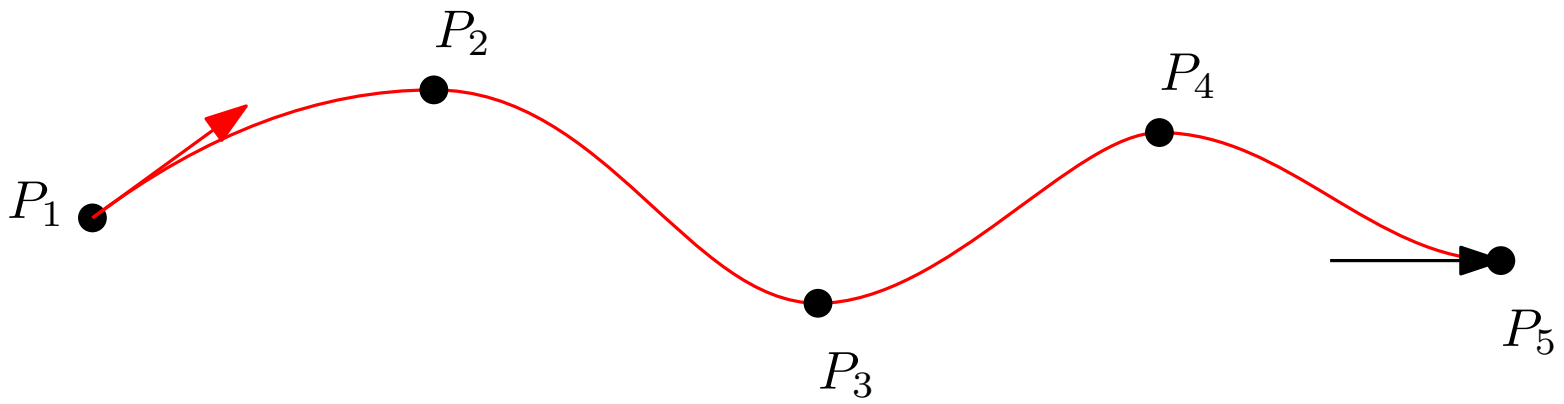

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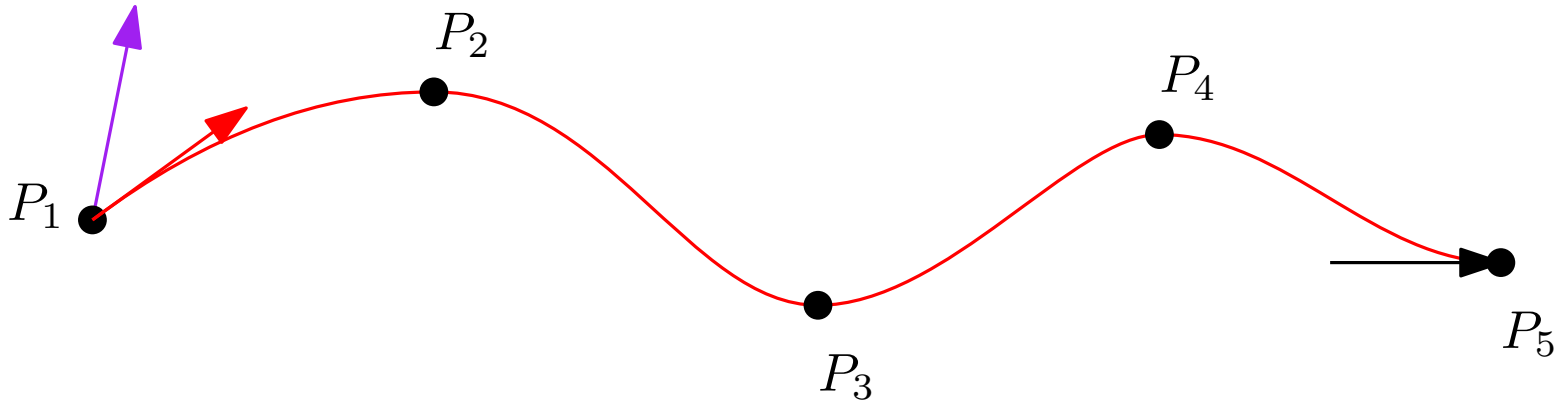

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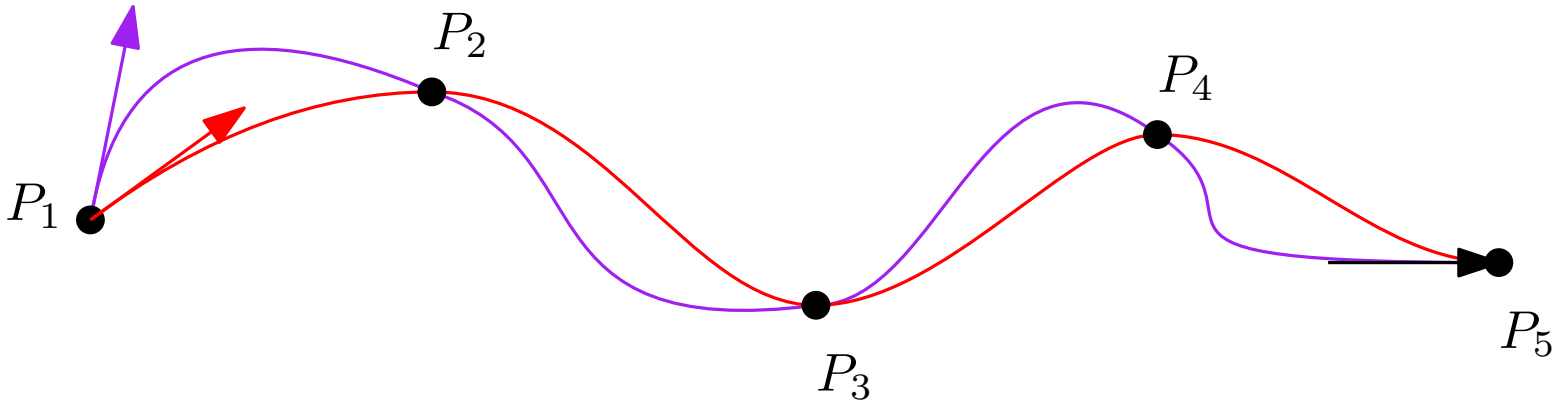

(the example is only approximate)

Cardinal splines: a way to overcome these limitations

A different way to apply Hermite interpolation to construct a spline

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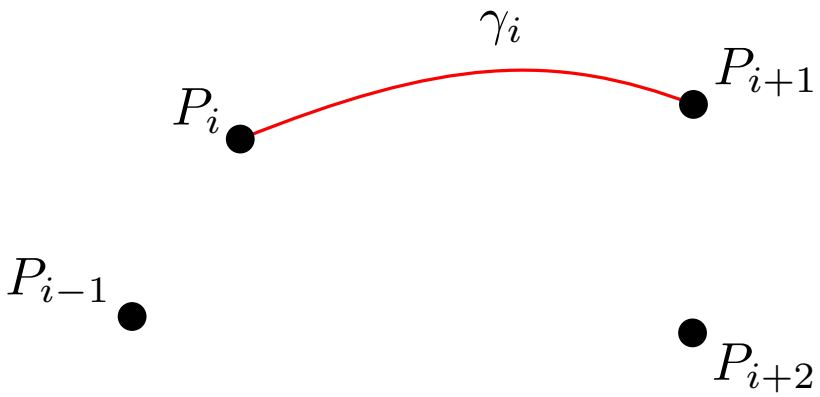

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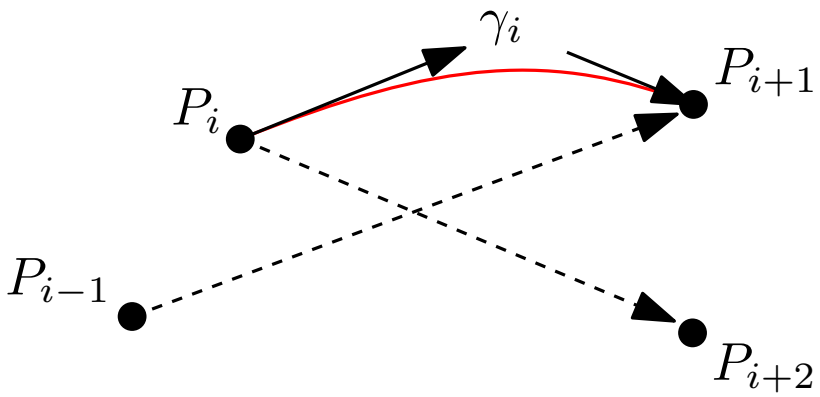

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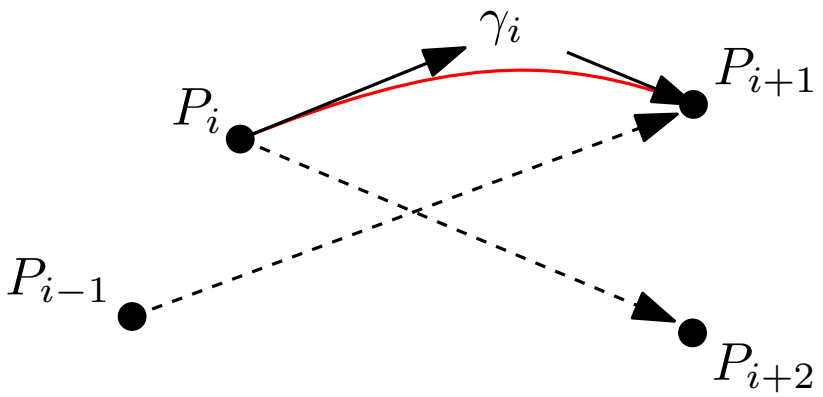

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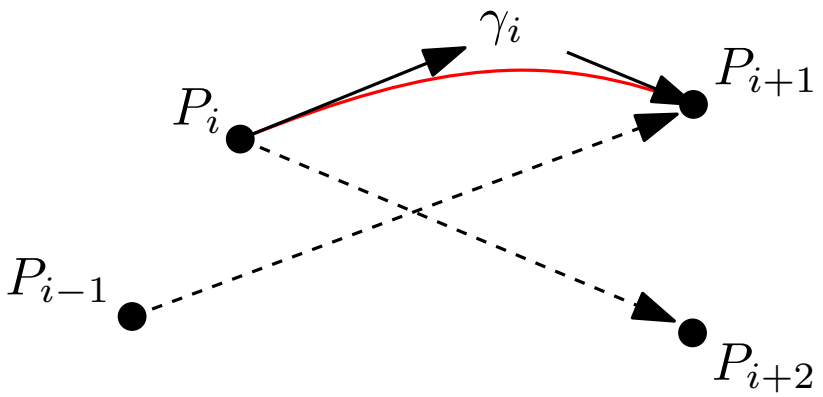

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Note that each Hermite curve depends only on four points, avoiding the $n \times n$ linear system

THIS CONCLUDES INTERPOLATION

This concludes our study of *interpolation* methods. Next, we will see that one doesn't need to interpolate to produce nice curves in an interactive way. The remaining of the course will focus on *approximation* methods.

Most of this and two previous presentations follow [Salomon], Section 3.2 (Lagrange interpolation) and Section 5 (Hermite interpolation)