

INTERPOLATING CURVES

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INTERPOLATING A SET OF POINTS

Interpolation problem

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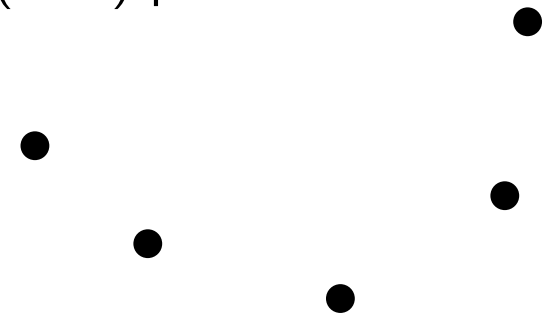
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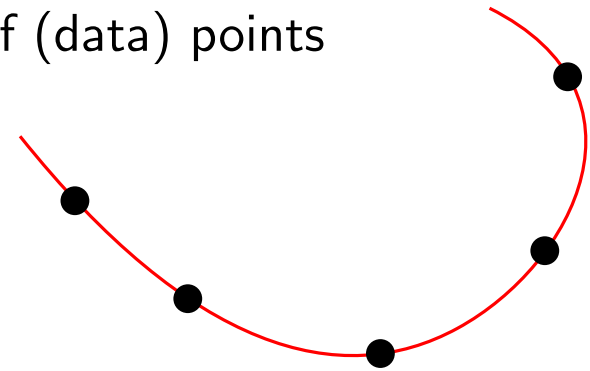


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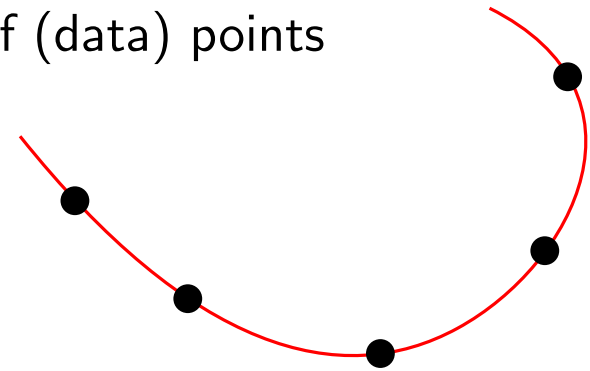


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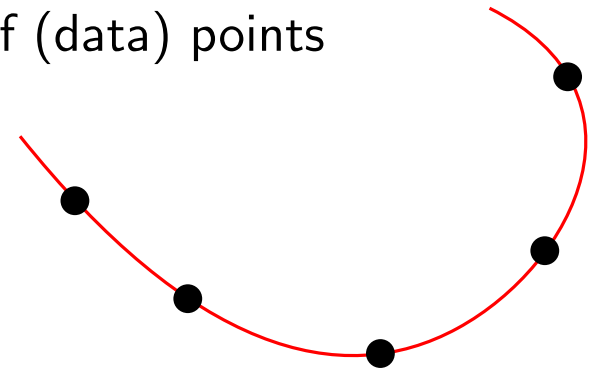
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In other (more precise) words:

Given P_0, \dots, P_n in \mathbb{R}^d our goal is to find a curve $\gamma(t)$ such that $\gamma(t_i) = P_i$ for some values of t_0, \dots, t_n .

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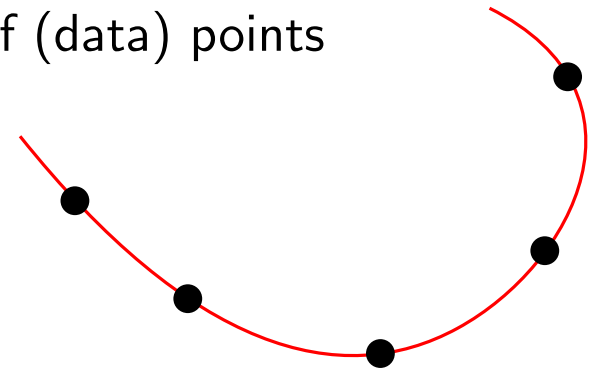
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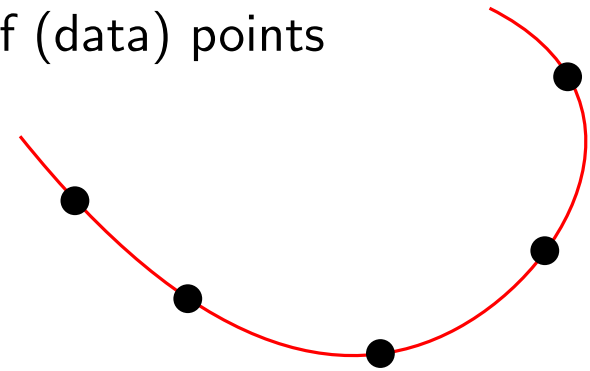
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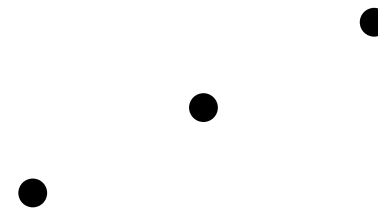
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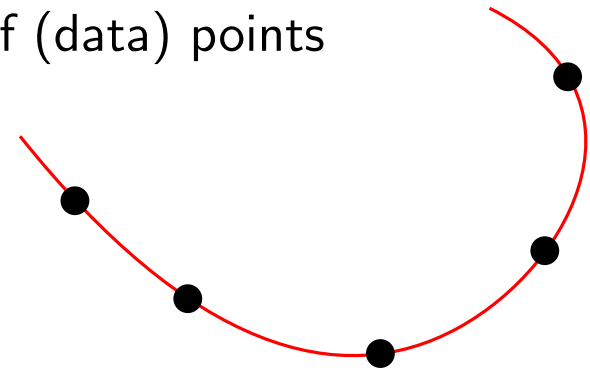
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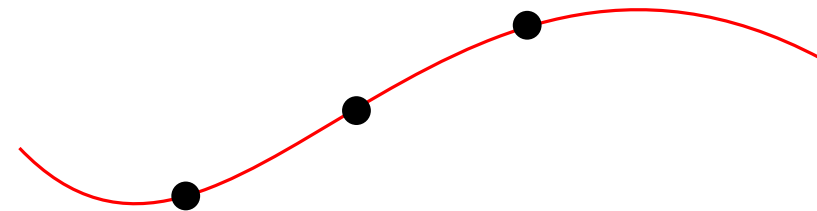
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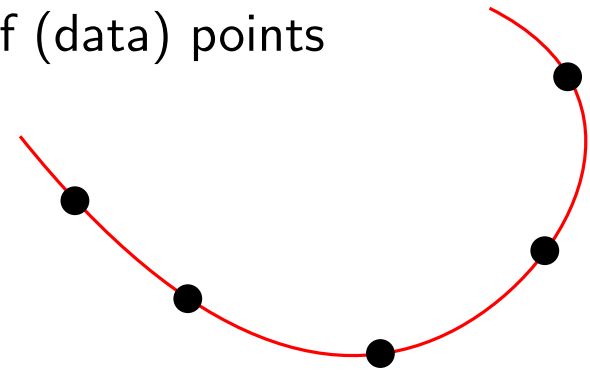
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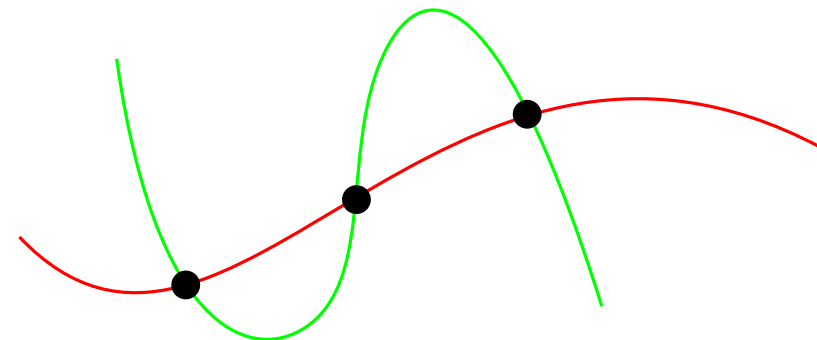
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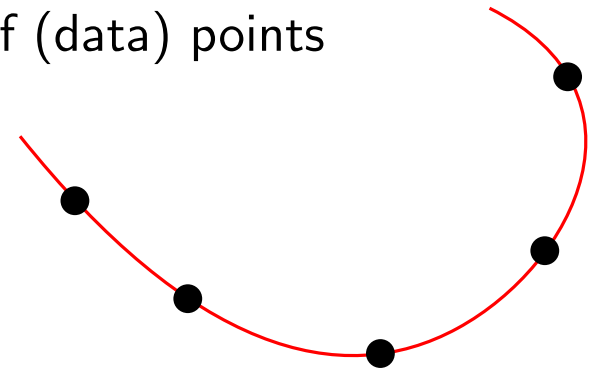
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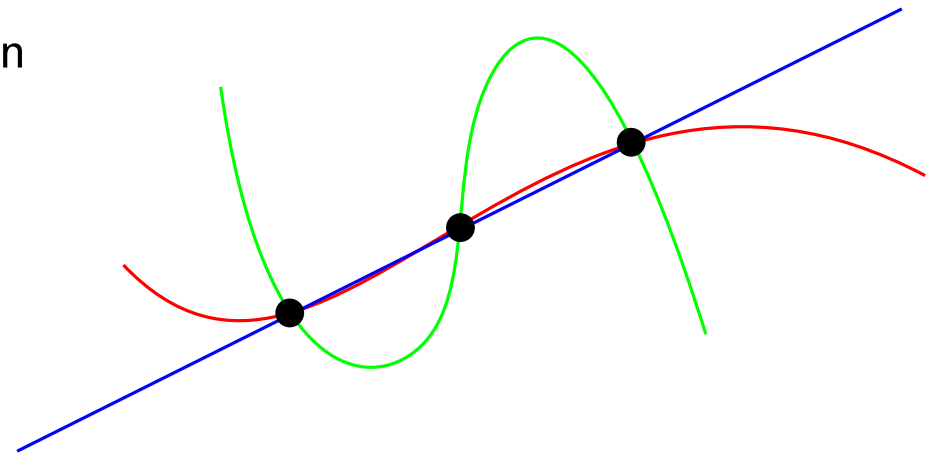
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LINEAR INTERPOLATION

Piecewise linear interpolation

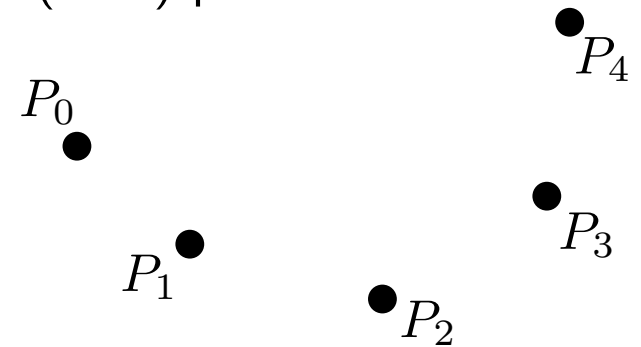
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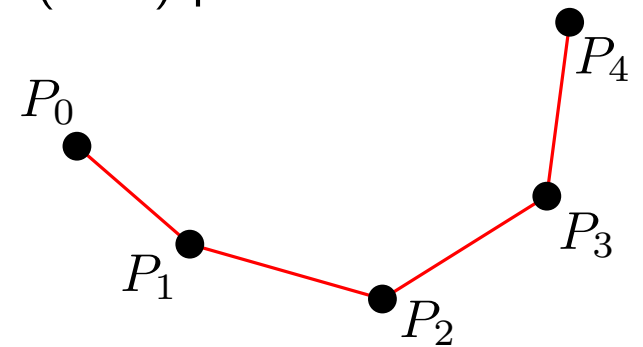


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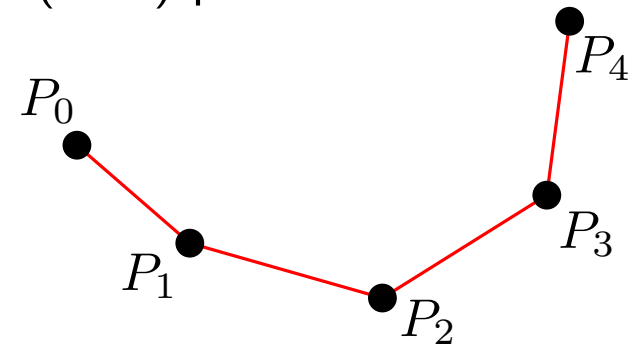
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How can we parametrize this curve?

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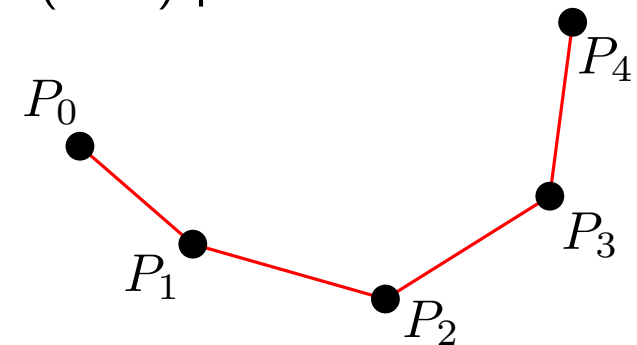
Parametrization

Given a set P of $n + 1$ points P_0, P_1, \dots, P_n in \mathbb{R}^d , and an increasing sequence of $n + 1$ real values $t_0 < t_1 < \dots < t_n$, the following curve interpolates the points in P :

$$\gamma : [t_0, t_n] \rightarrow \mathbb{R}^d$$

$$\gamma(t) = \frac{t_i - t}{t_i - t_{i-1}} P_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} P_i \quad \text{if } t \in [t_{i-1}, t_i] \text{ for all } i = 1, \dots, n$$

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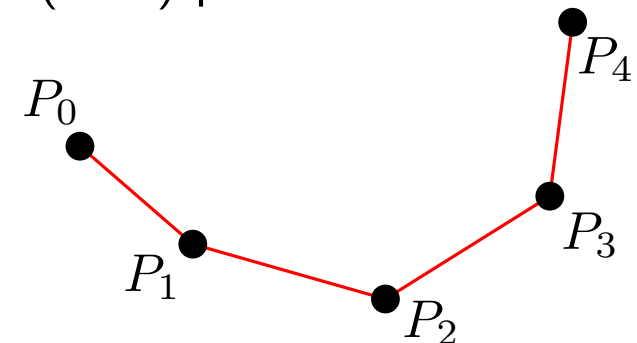
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Observations

- γ_{t-1} is continuous: trivially in (t_{i-1}, t_i) for all i , and also at each t_i for all i because $\gamma(t_i)$ is well defined (i.e., consecutive line segments coincide at data points)
- γ_{t-1} is not differentiable at the points t_i (unless three consecutive points are aligned)

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Try it in Desmos!

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Variation diminishing property

An important property for any interpolating curve

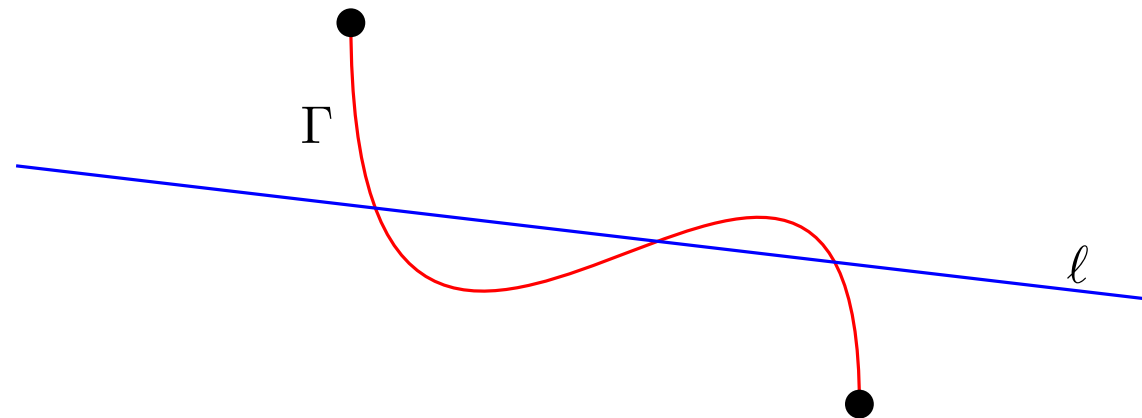
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Suppose that the points you want to interpolate are samples from an unknown curve Γ

Given a curve Γ in \mathbb{R}^2 or \mathbb{R}^3 , and **any** line ℓ (in \mathbb{R}^2) or plane π (in \mathbb{R}^3), let us denote $cross(\Gamma, \ell)$ or $cross(\Gamma, \pi)$ the number of crossings (i.e., intersection points) of Γ and ℓ or π .



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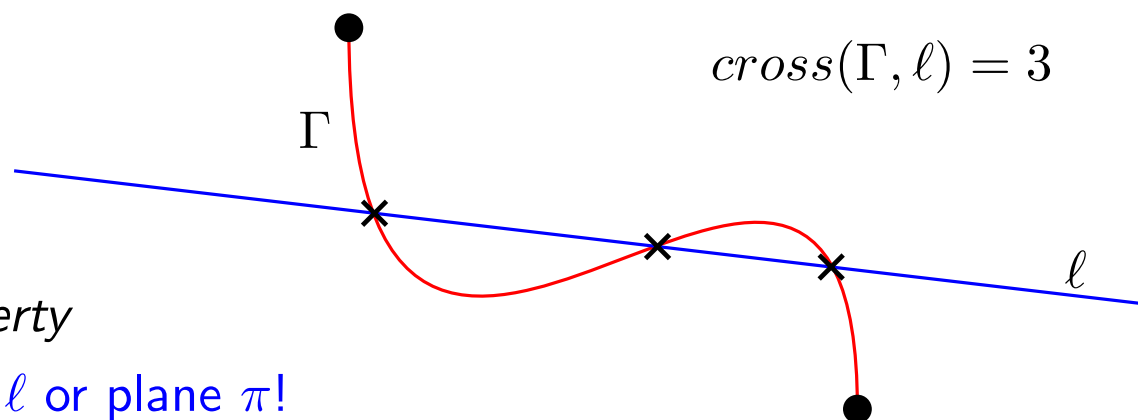
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$$cross(p, \ell) \leq cross(\Gamma, \ell) \text{ (in } \mathbb{R}^2)$$

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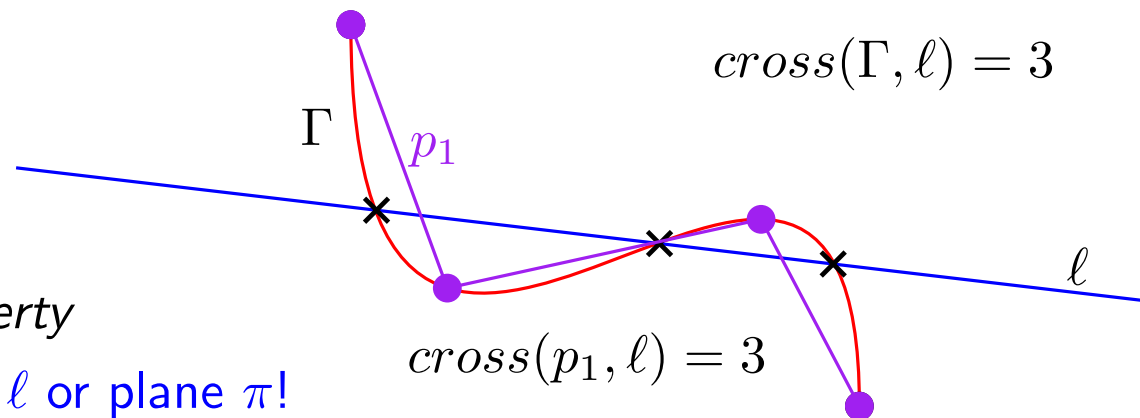
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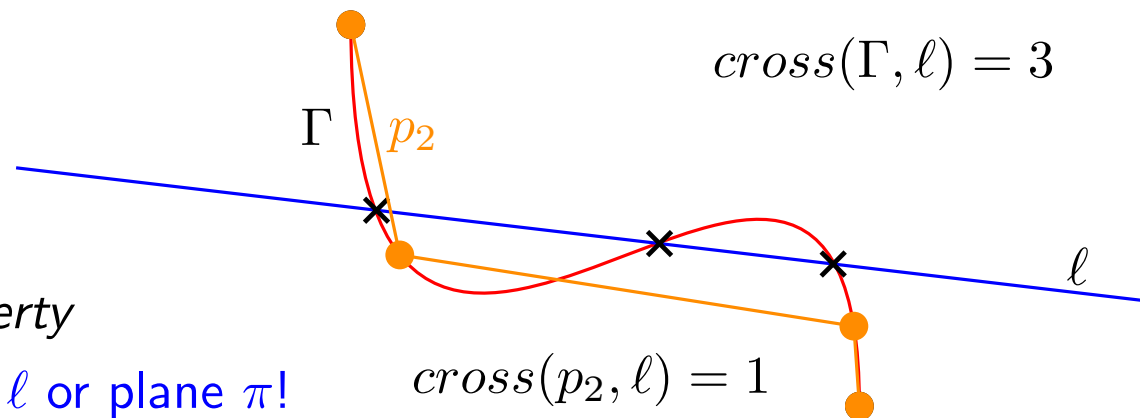
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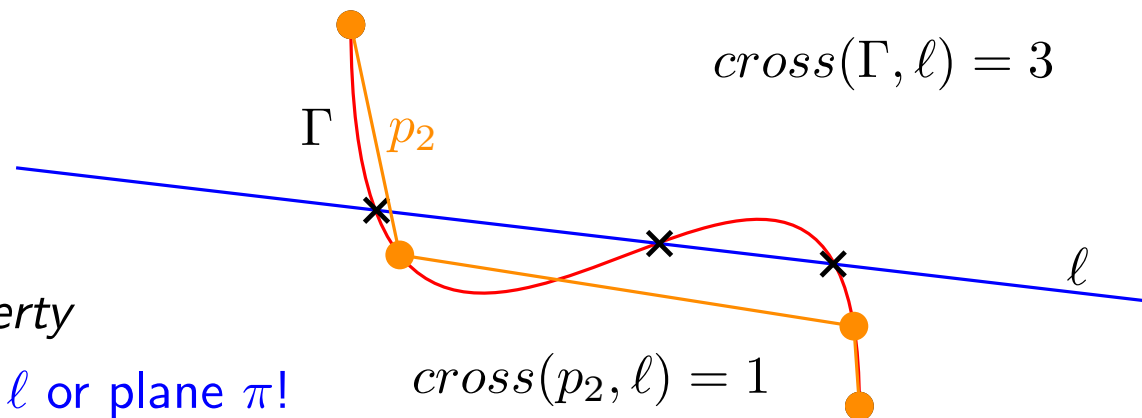
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Implies that the interpolating curve (p) does not wiggle much more than the original one (Γ)

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Affine invariance of piecewise linear interpolation

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Affine invariance of piecewise linear interpolation

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Thus it is the same to (i) first linearly interpolate, then apply affine transformation, as (ii) first apply affine transformation, then linearly interpolate

POLYNOMIAL INTERPOLATION

Using higher degree polynomials

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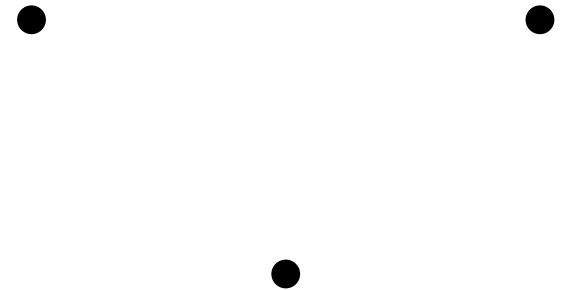
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Idea: find a higher degree polynomial that interpolates the points in a smoother way

POLYNOMIAL INTERPOLATION

Using higher degree polynomials

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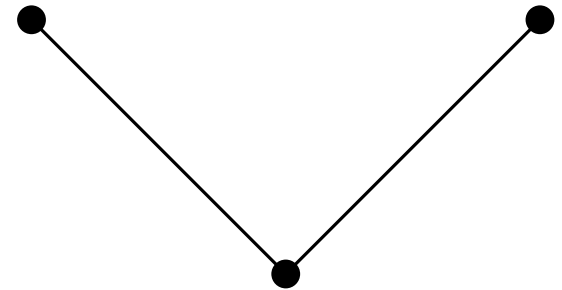


POLYNOMIAL INTERPOLATION

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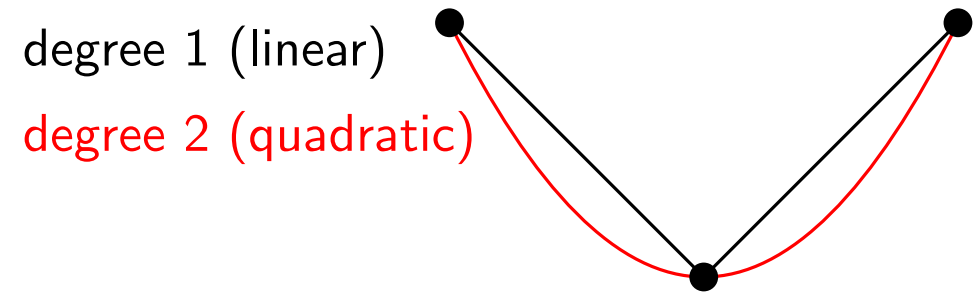
degree 1 (linear)



POLYNOMIAL INTERPOLATION

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POLYNOMIAL INTERPOLATION

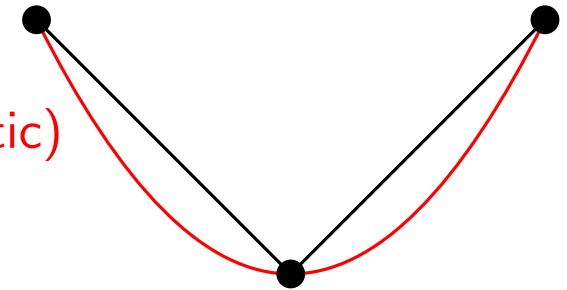
Using higher degree polynomials

Idea: find a higher degree polynomial that interpolates the points in a smoother way

Let P_0, P_1, \dots, P_n with $P_i = (x_i, y_i)$, be $n + 1$ points in \mathbb{R}^2

degree 1 (linear)

degree 2 (quadratic)



POLYNOMIAL INTERPOLATION

Using higher degree polynomials

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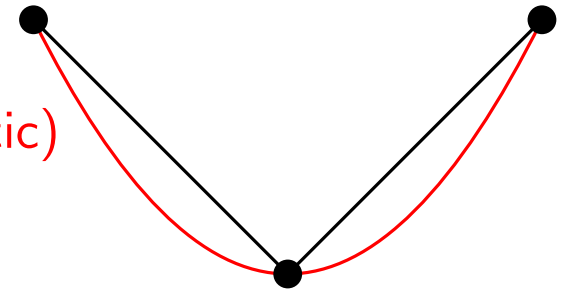
Let P_0, P_1, \dots, P_n with $P_i = (x_i, y_i)$, be $n + 1$ points in \mathbb{R}^2

Interpolation theorem

There exists a unique polynomial of degree at most n that passes through P_0, P_1, \dots, P_n

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POLYNOMIAL INTERPOLATION

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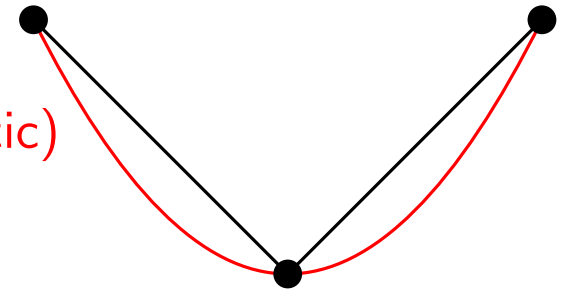
Proof

1) Uniqueness (if it exists \Rightarrow it is unique)

- Suppose that there are two polynomials $p(x)$ and $q(x)$ that interpolate the points. Consider then $r(x) = p(x) - q(x)$

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POLYNOMIAL INTERPOLATION

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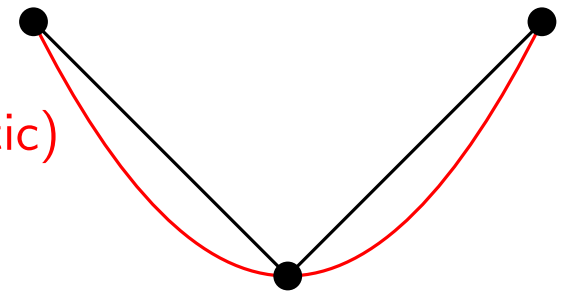
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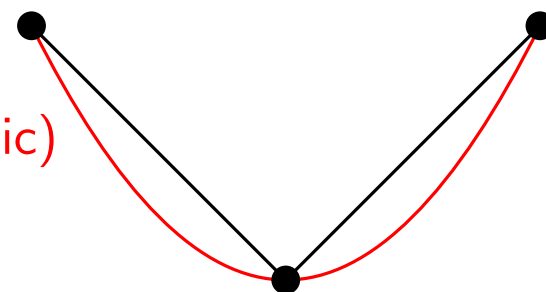
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- But a degree- n polynomial different from zero can have at most n roots! Then $r(x)$ must be the zero polynomial, i.e., $r(x) = 0$, implying that $p(x) = q(x)$!

degree 1 (linear)

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POLYNOMIAL INTERPOLATION

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Proof (cont'd)

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- We define the following auxiliary polynomials (known as *Lagrange weights*)

$$L_i^n(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

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POLYNOMIAL INTERPOLATION

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(other ways exist, but they are reformulations of the same—unique!—polynomial)

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That is: the only degree that *always* allows to interpolate $n + 1$ points is degree n

The exception is when two or more data points lie on a low-degree polynomial.

Example: n points on a line can be interpolated with a polynomial of degree just 1



POLYNOMIAL INTERPOLATION

Lagrange polynomial

Lemma: n is the minimum degree that guarantees the existence of an interpolating polynomial for *any* set of $n + 1$ distinct points.

Why? **Proof sketch:** (by induction on n)

- Base case: $n = 1$. Then we have only two points P_0, P_1 , and we know that two points are required to determine a line (i.e., a polynomial of degree 1)

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- Induction step: assume that there exists a set of n points P_1, \dots, P_n whose interpolating polynomial has degree exactly $n - 1$ (i.e., with lower degree it is not possible)

Let P_0 be a point that does not lie on the polynomial curve that interpolates P_1, \dots, P_n . Since that polynomial of degree $n - 1$ is unique, and it does not go through P_0 , then the polynomial through P_0, P_1, \dots, P_n must be different, and thus must have higher degree.

POLYNOMIAL INTERPOLATION

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$$p(x) = \sum_{i=0}^n L_i^n(x) y_i$$

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Clever idea: let's choose a really simple set of points to interpolate (we can choose any, because the $L_i^n(x)$ weights are independent of the y -values)

Consider $n + 1$ distinct points on the function $f(x) = 1$, that is: $P_i = (x_i, 1)$ for $i = 0, \dots, n$. Then the unique polynomial of degree at most n that interpolates them is easy: $p(x) = 1$

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$$1 = p(x) = \sum_{i=0}^n L_i^n(x) y_i = \sum_{i=0}^n L_i^n(x) \cdot 1 = \sum_{i=0}^n L_i^n(x) \rightarrow \sum_{i=0}^n L_i^n(x) = 1$$

Therefore, the Lagrange polynomial is affine invariant

POLYNOMIAL INTERPOLATION

Parametric version of Lagrange polynomial

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POLYNOMIAL INTERPOLATION

Parametric version of Lagrange polynomial

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This only works to interpolate values from a function, but doesn't work for any set of points (e.g., as soon as two points have the same x coordinate, it doesn't work)

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Let P_0, P_1, \dots, P_n be $n + 1$ points in R^d , and consider $0 = t_0 < t_1 < \dots < t_n = 1$.

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- $L_i^n(t)$ is a polynomial of degree n in $t \rightarrow$ so is $\gamma(t)$
- $L_i^n(t_j) = 1$ if $j = i$, and $L_i^n(t_j) = 0$ if $j \neq i$
- $\gamma(t_i) = P_i$
- For each value of t , $\gamma(t)$ is an affine combination of P_0, \dots, P_n , with weights $L_i^n(t)$ that (you can prove) add up to 1 \rightarrow this interpolation is also affine invariant

Parametric version of Lagrange polynomial

Remarks

- As before, the parametric Lagrange interpolation is not unique: we are free to choose the parameter values t_1, \dots, t_{n-1} .
 - Uniform version: $t_i - t_{i-1} = 1$ or $= \frac{1}{n}$
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For example, for only two points P_0, P_1

We have $t_0 = 0, t_1 = 1,$

$$L_0^1(t) = \frac{t-t_1}{t_0-t_1} = \frac{t-1}{-1} = -t + 1 = (t, 1)(-1, 1)^t$$

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$$p(t) = L_0^1(t)P_0 + L_1^1(t)P_1 =$$

$$= (t, 1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \end{pmatrix}$$

See bibliography for the matrix version for larger n

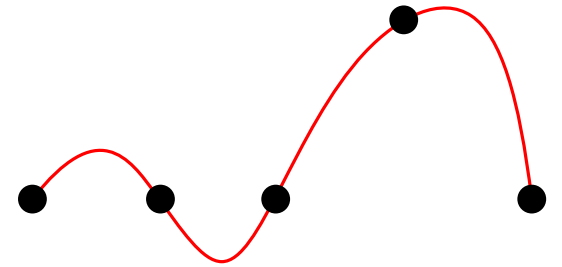
POLYNOMIAL INTERPOLATION

Issues with polynomial interpolation

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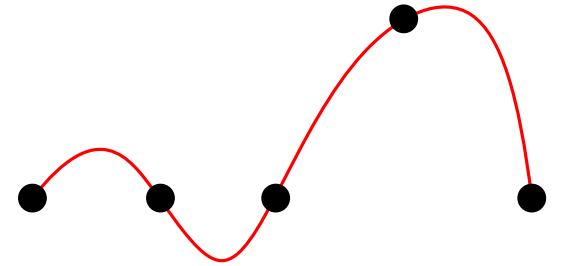
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POLYNOMIAL INTERPOLATION

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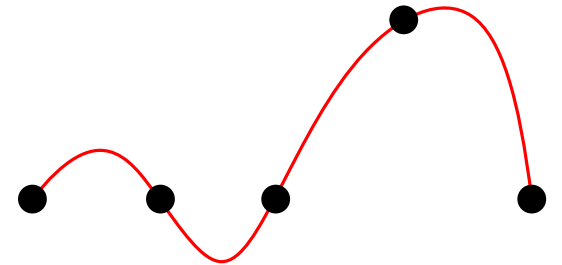
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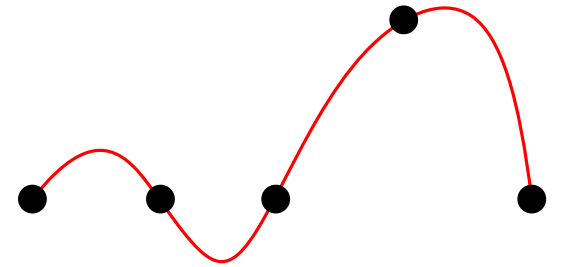
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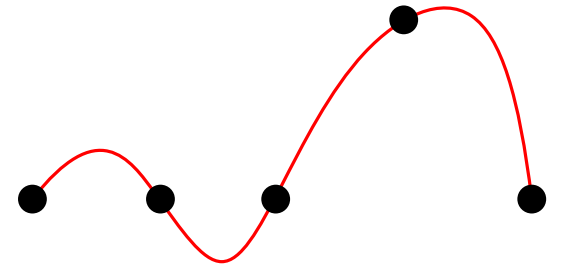


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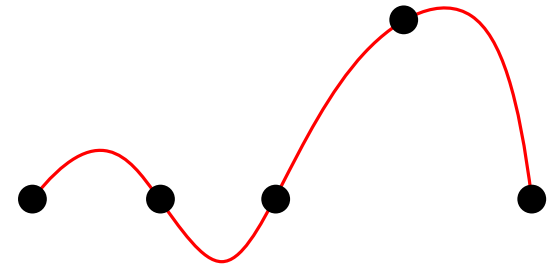


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- If one has computed $\gamma(t)$ for n points and needs to add one extra point, everything needs to be recomputed
- Lagrange's formula is not numerically stable: small variations in the input points can produce large variations in the final curve
- The method is not easy to make interactive: if the curve is not what one wants, (and you cannot modify the data points) all you can do is to add more points



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