CURVES: BASIC REPRESENTATION

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HOW TO REPRESENT A CURVE?

Mathematical representations of curves (in 2D)

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Explicit equation

If $I \subseteq \mathbb{R}$ is any interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ is a continuous function, then the curve is

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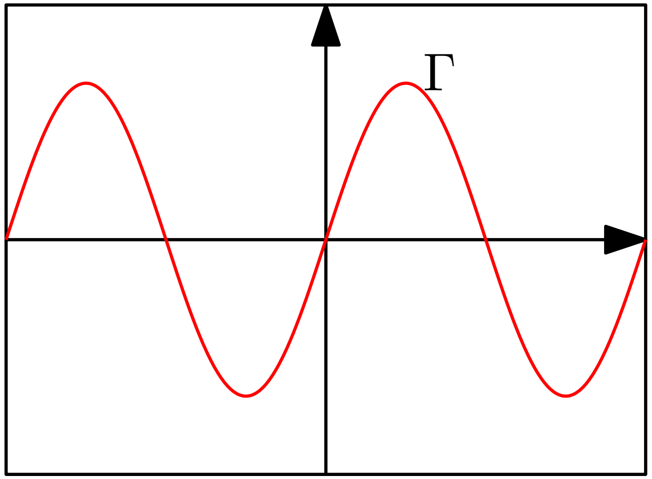

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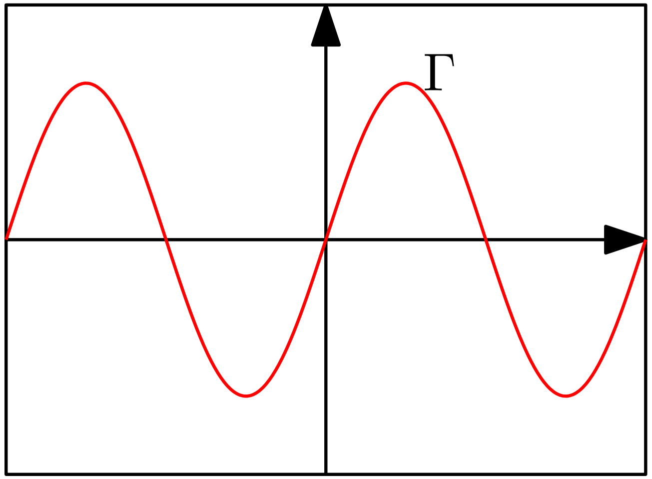

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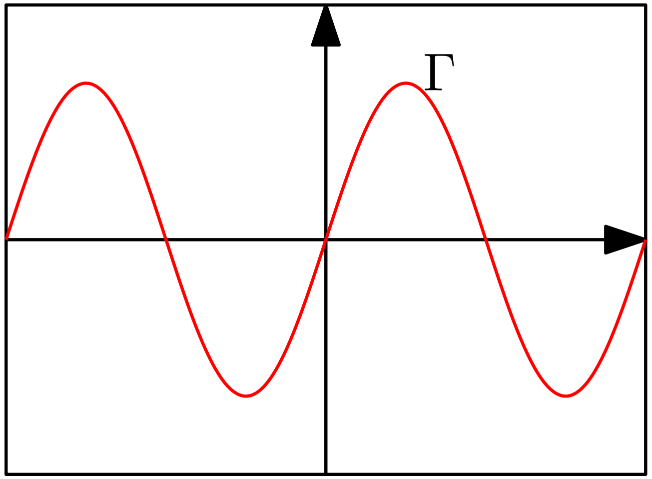

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Not so good:

- Doesn't extend easily to 3D
- Not all curves have one!

E.g., the unit circle, $x^2 + y^2 = 1$

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For some function $F : \mathbb{R}^2 \to \mathbb{R}$, the curve is $\Gamma = \{(x, y) : F(x, y) = 0\}$

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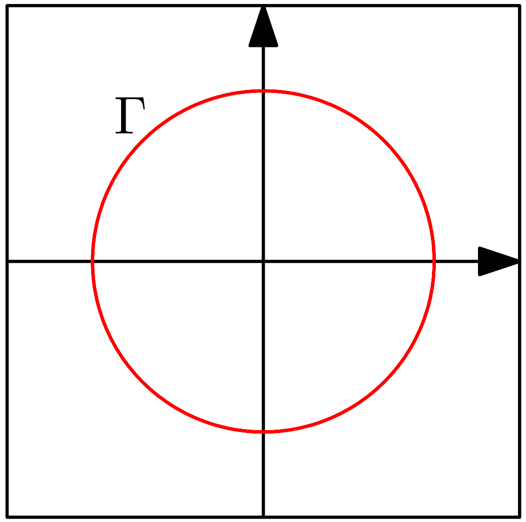

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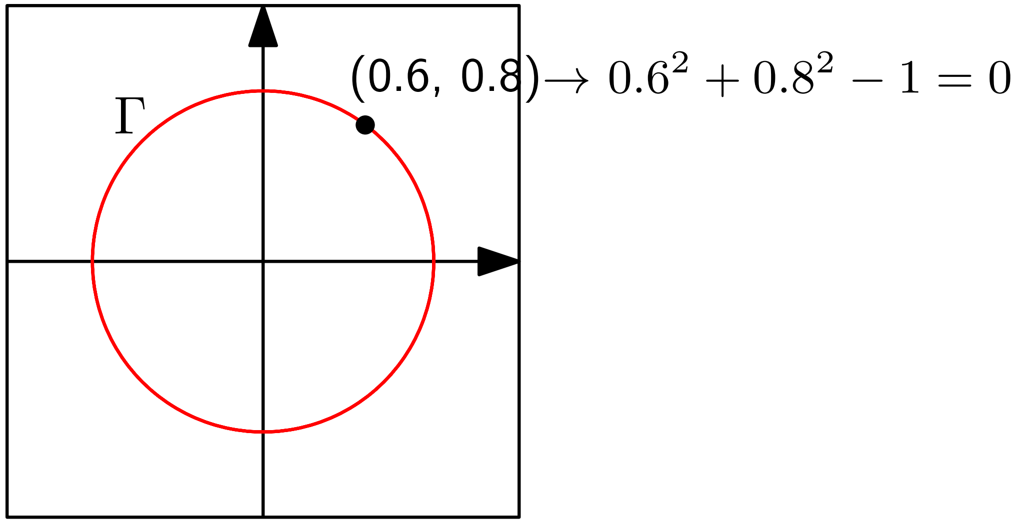

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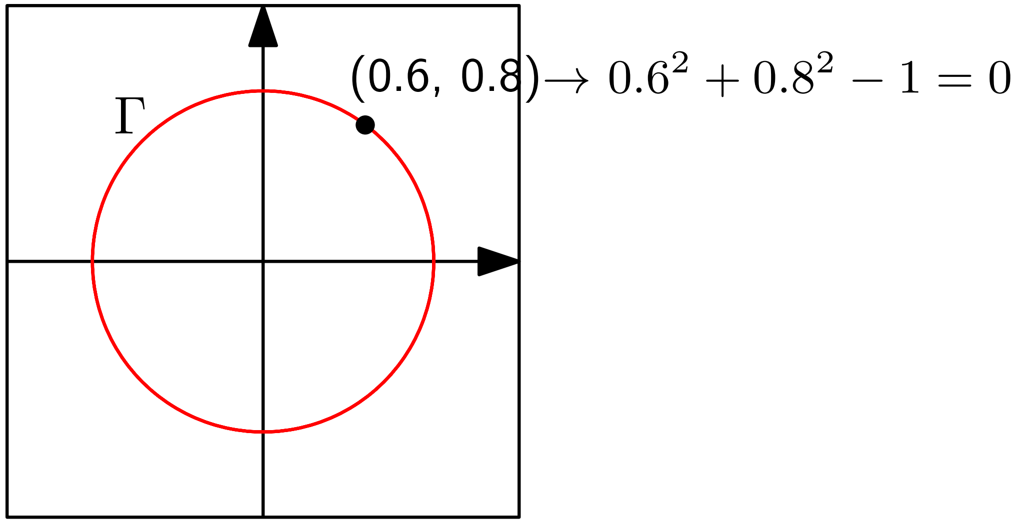

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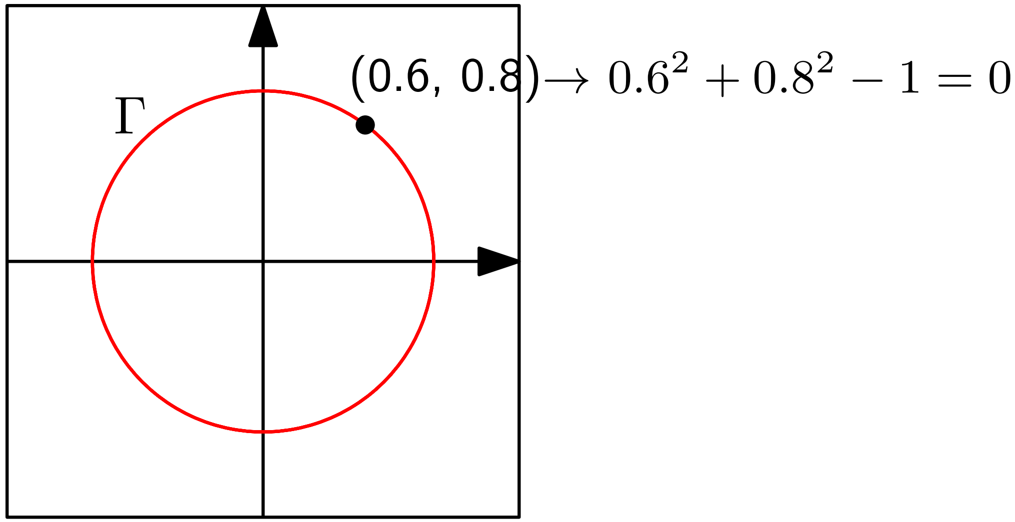

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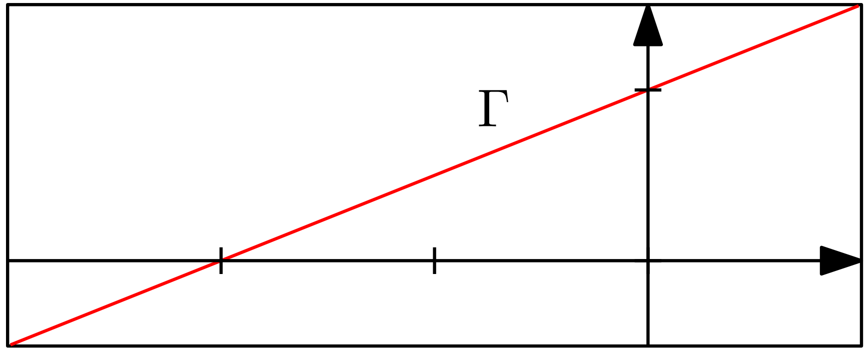

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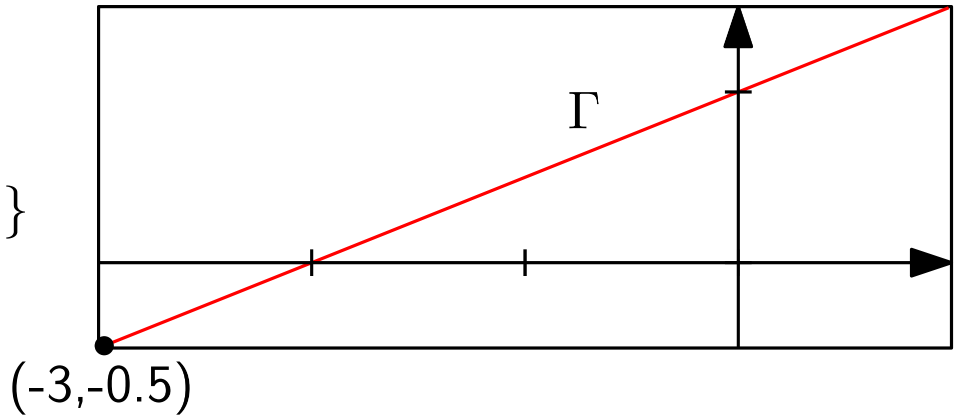

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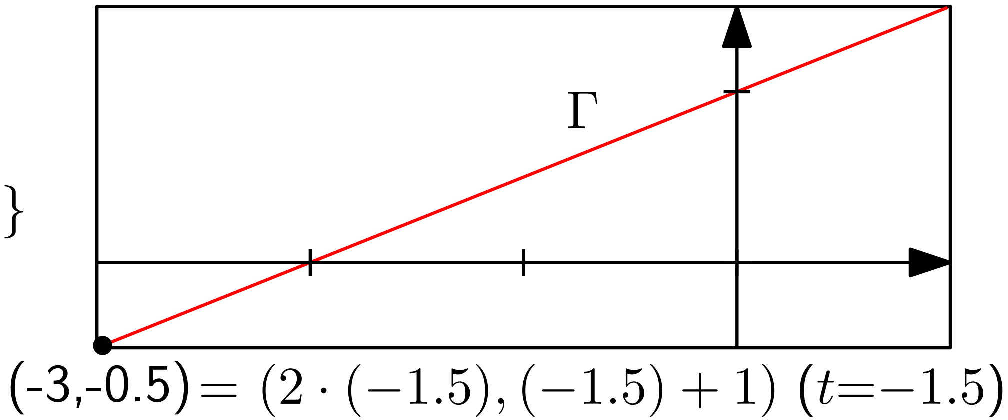

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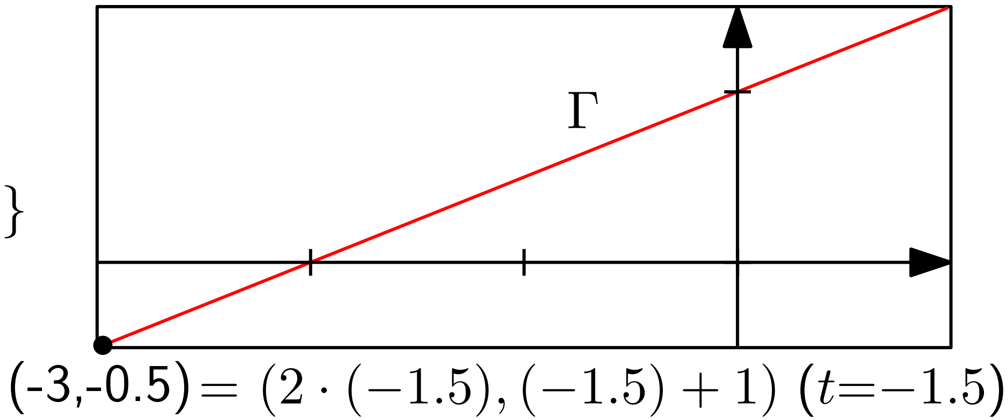

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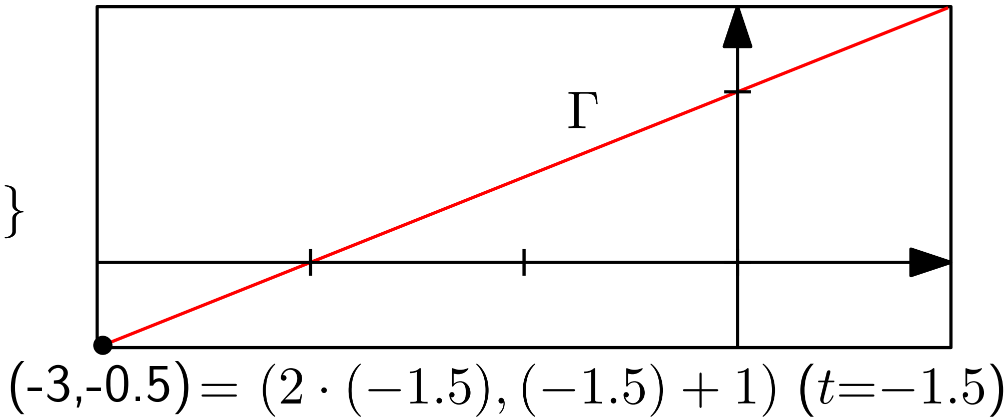

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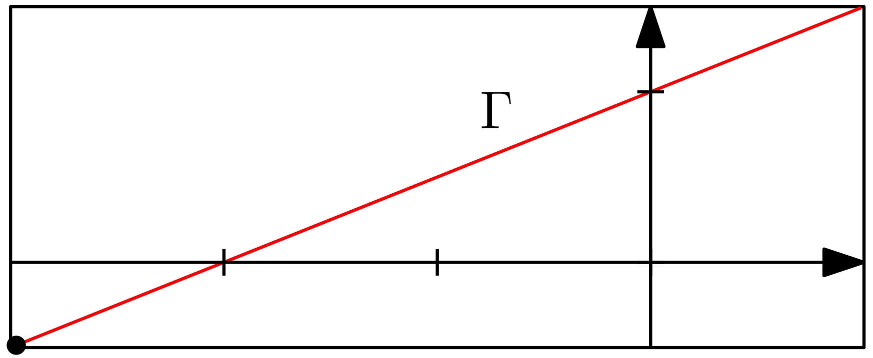

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In curve and surface design, we always use parametric equations!

EXAMPLES OF PARAMETRIC CURVES

1) Lines, halflines, and line segments

in 2D and 3D

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$$\gamma(\lambda) = P + \lambda\vec{v} = (x(\lambda), y(\lambda), z(\lambda)) \quad \lambda \in \mathbb{R}$$

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$y(\lambda) = y_0 + \lambda u_2$	$\gamma(\lambda) = (x_0 + \lambda u_1, y_0 + \lambda u_2, z_0 + \lambda u_3)$	$\lambda \in \mathbb{R}^+$	(halfline)
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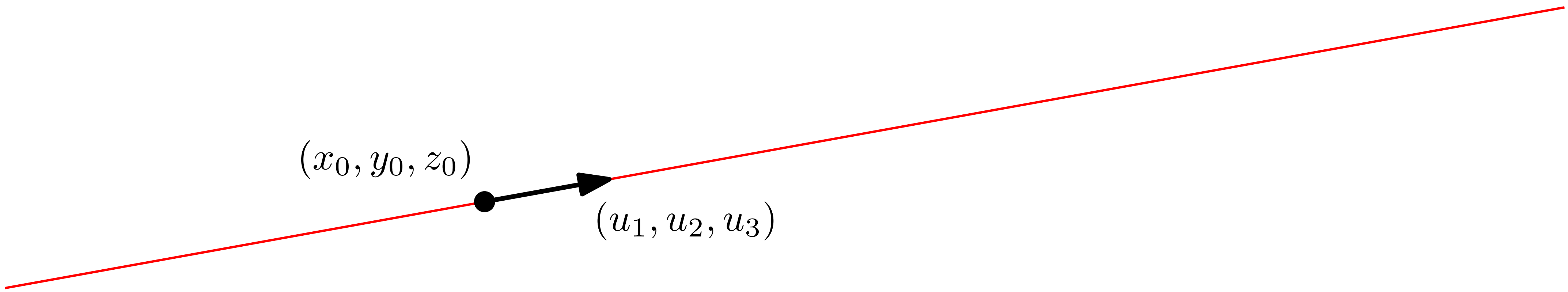

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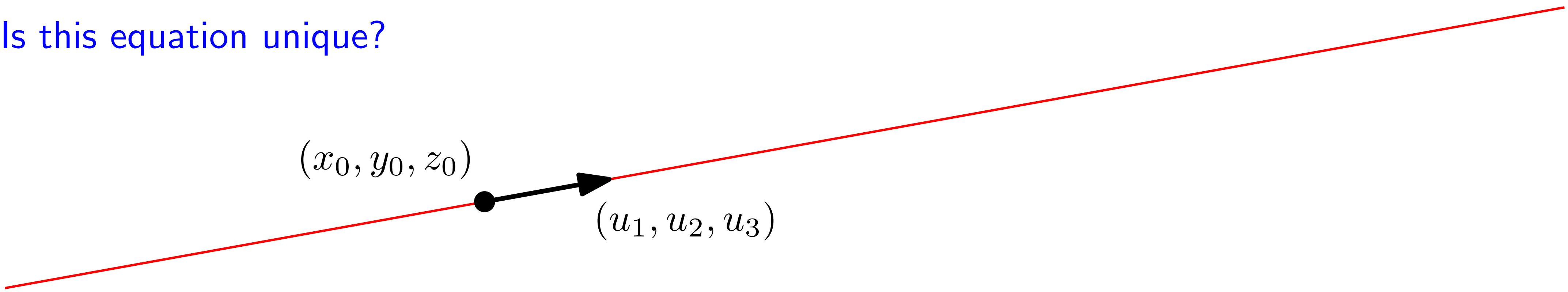

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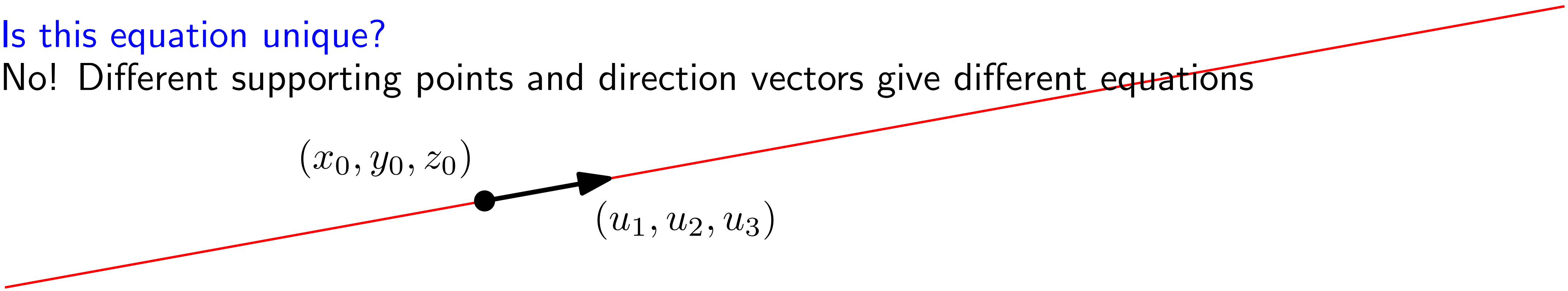

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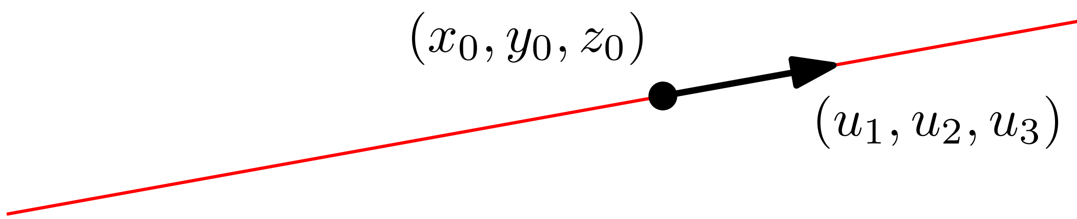

Great tool to play with parametric curves:
`desmos.com/calculator`

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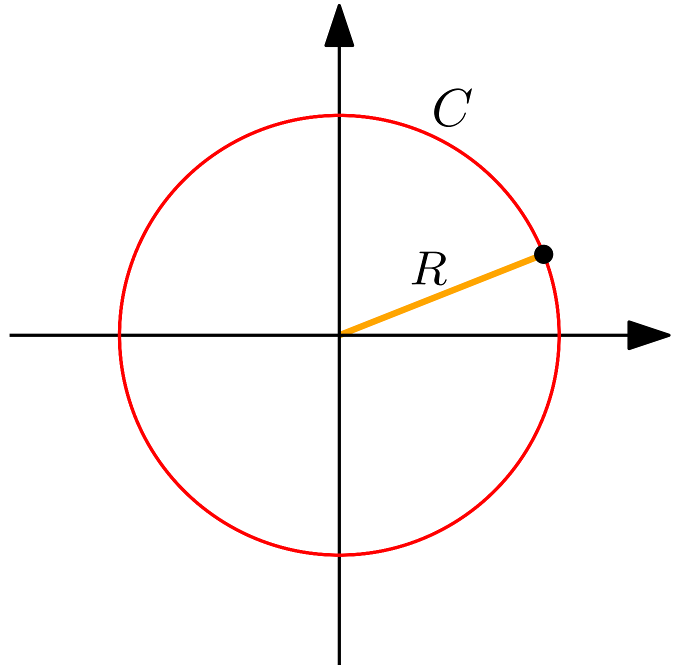

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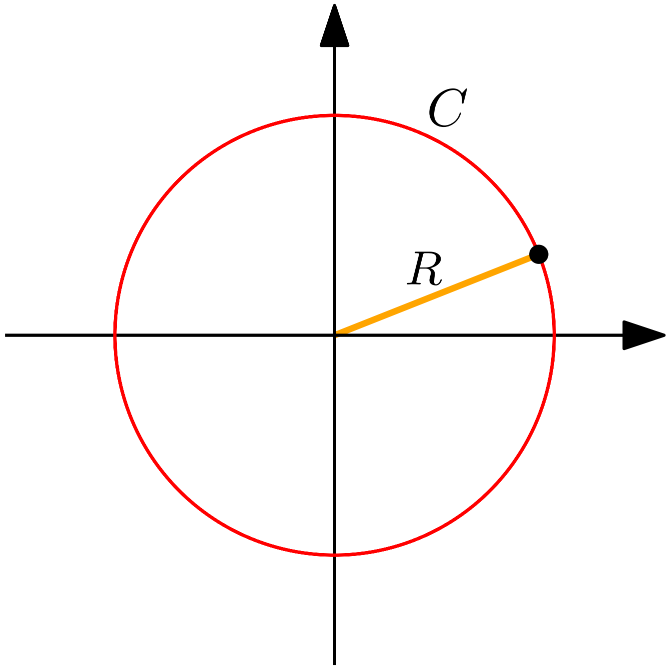

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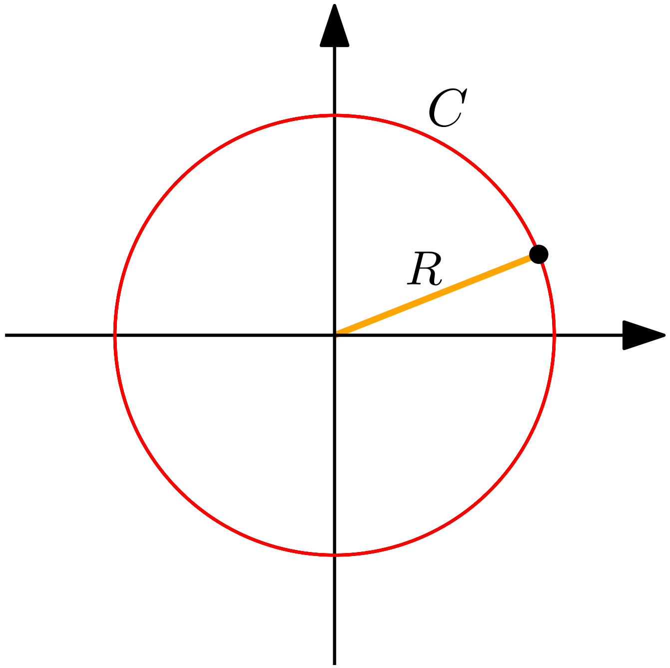

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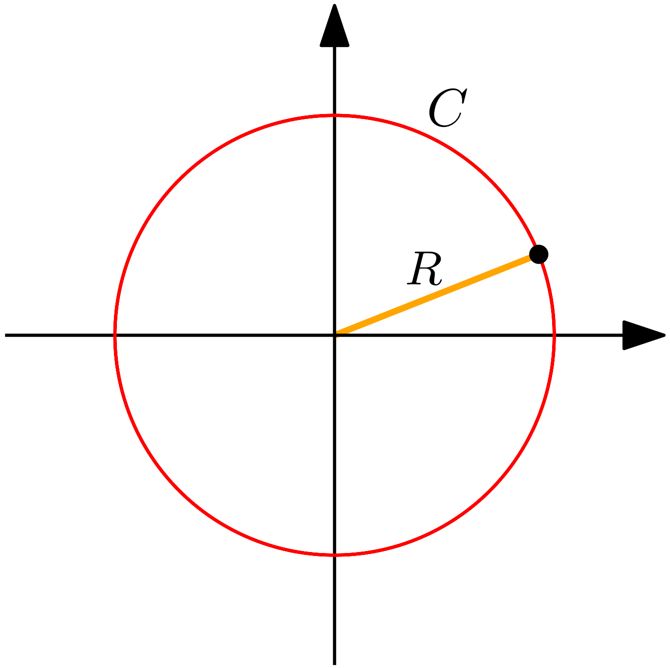

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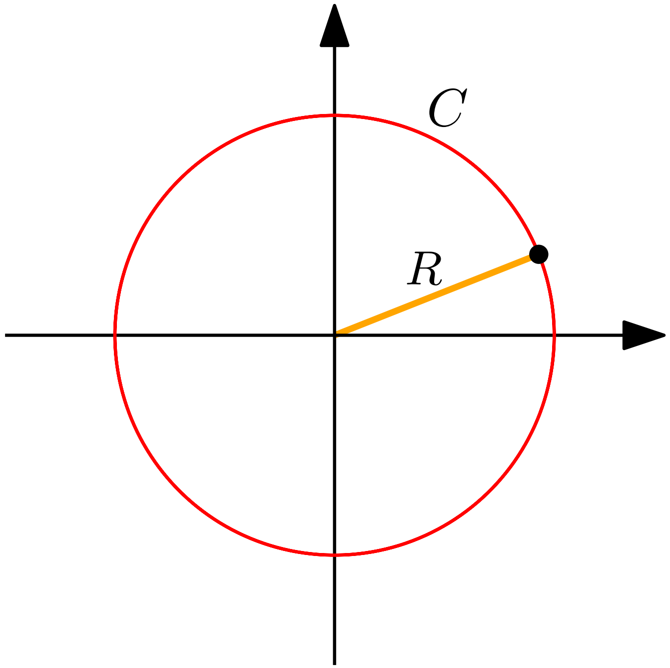

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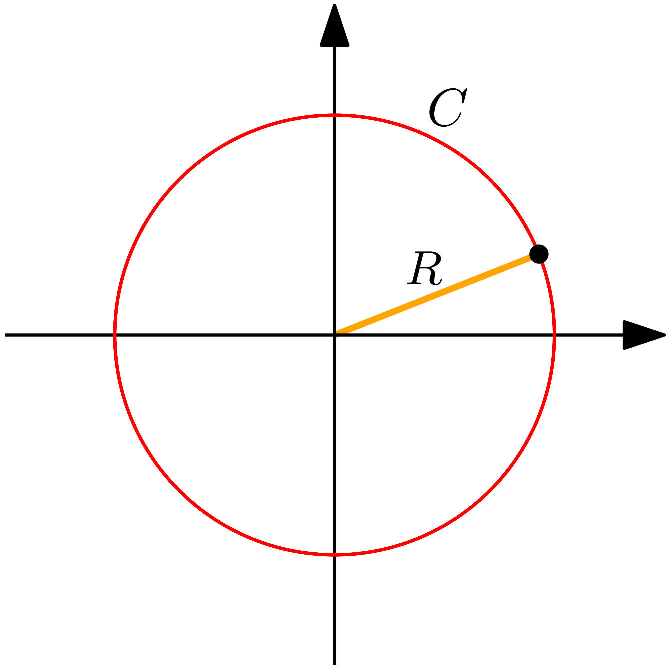

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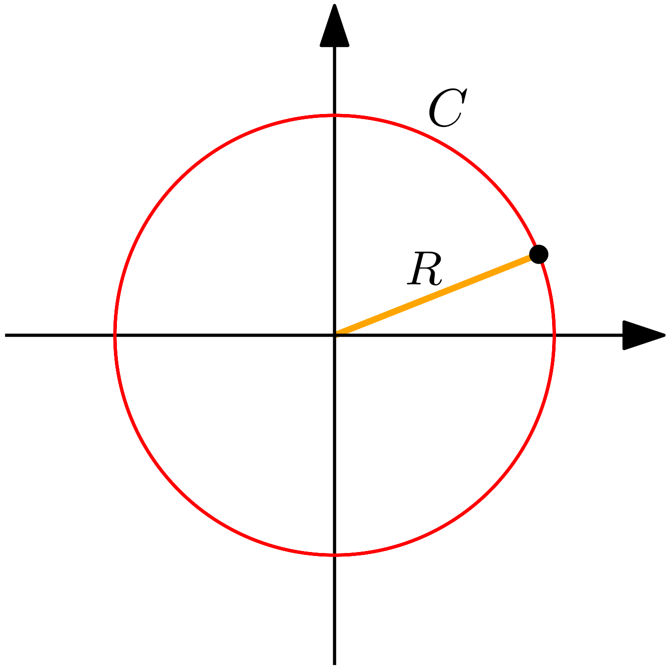

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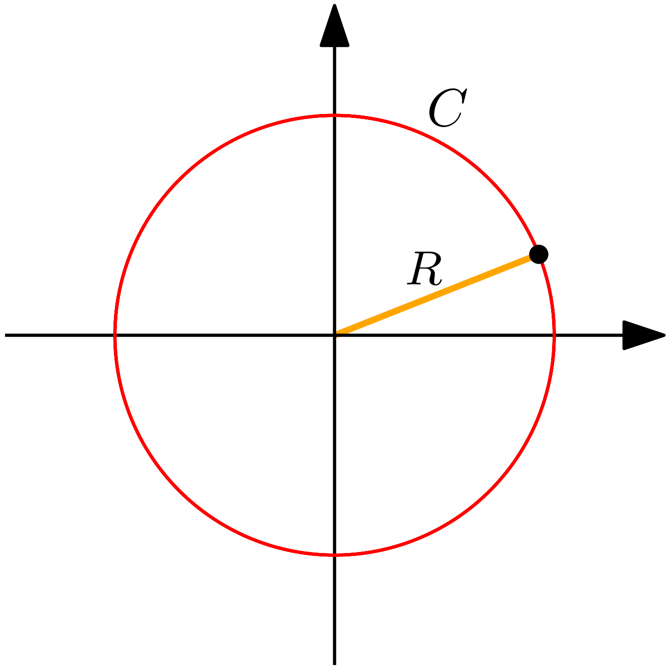

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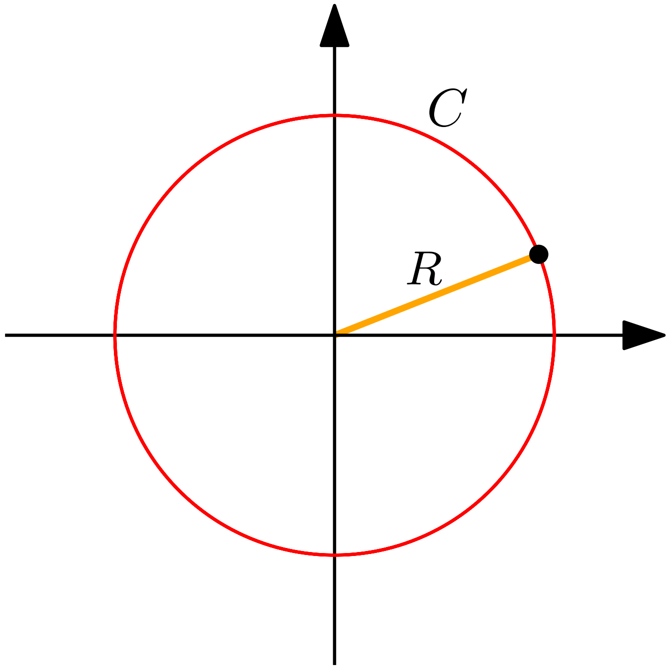

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Recall: to convert from Polar to Cartesian coordinates, we do $(r,\theta) \rightarrow (r\cos\theta, r\sin\theta)$

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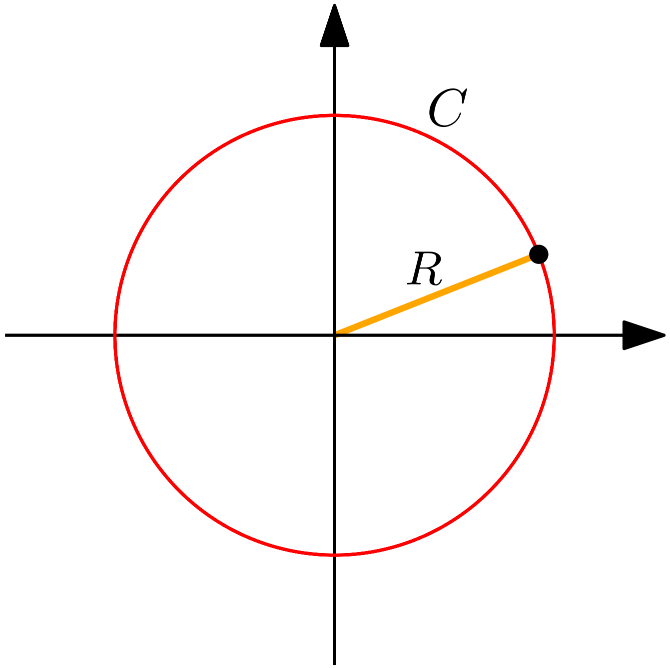

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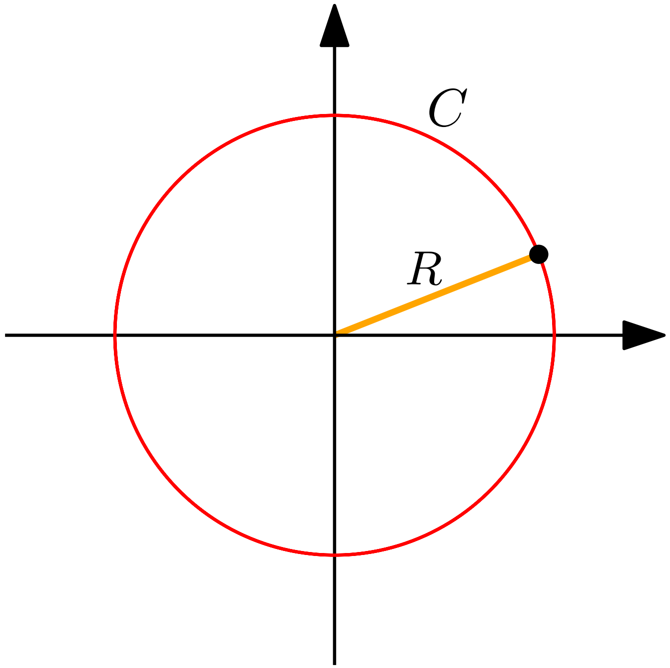

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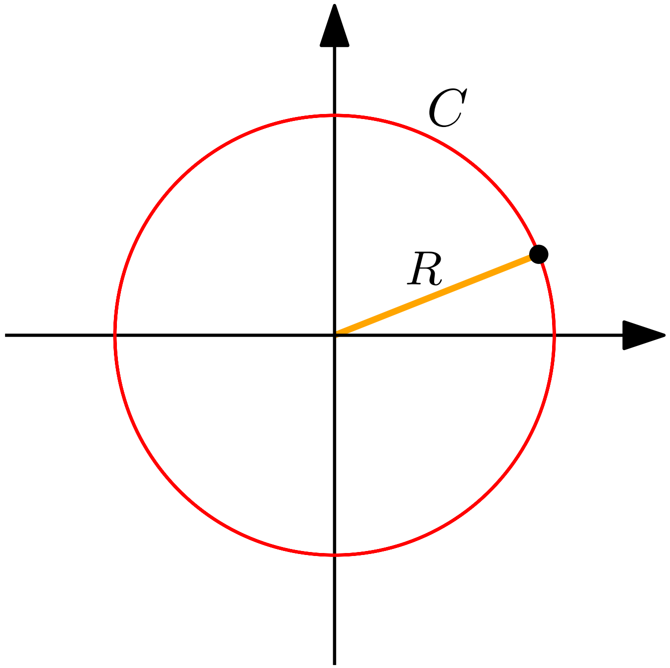

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In fact, $\theta \in [0, 2\pi)$ is enough

2) Circles (2D) (cont'd)

$$\gamma(\theta) = (R\cos\theta, R\sin\theta) \quad \theta \in [0, 2\pi)$$

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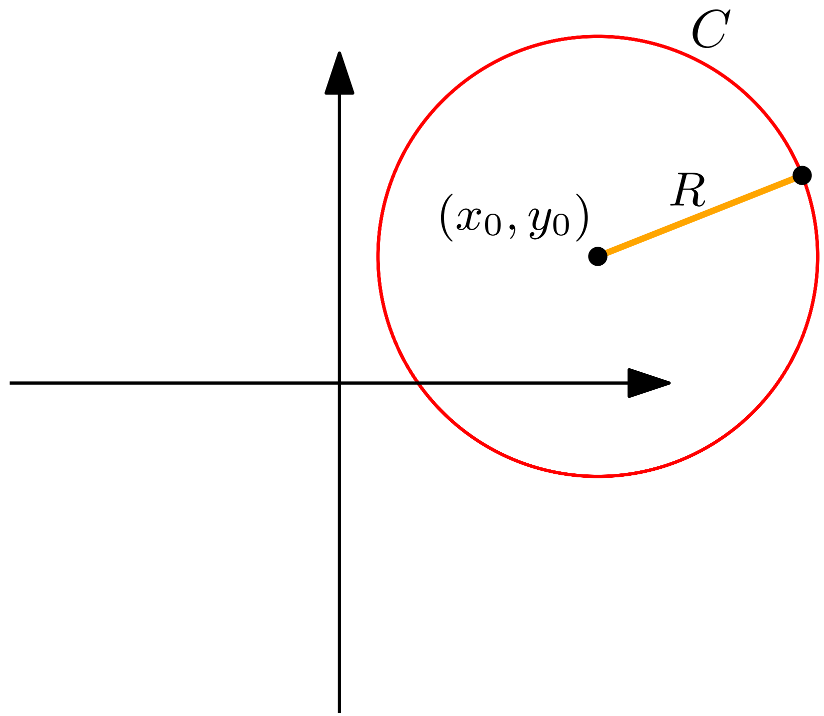

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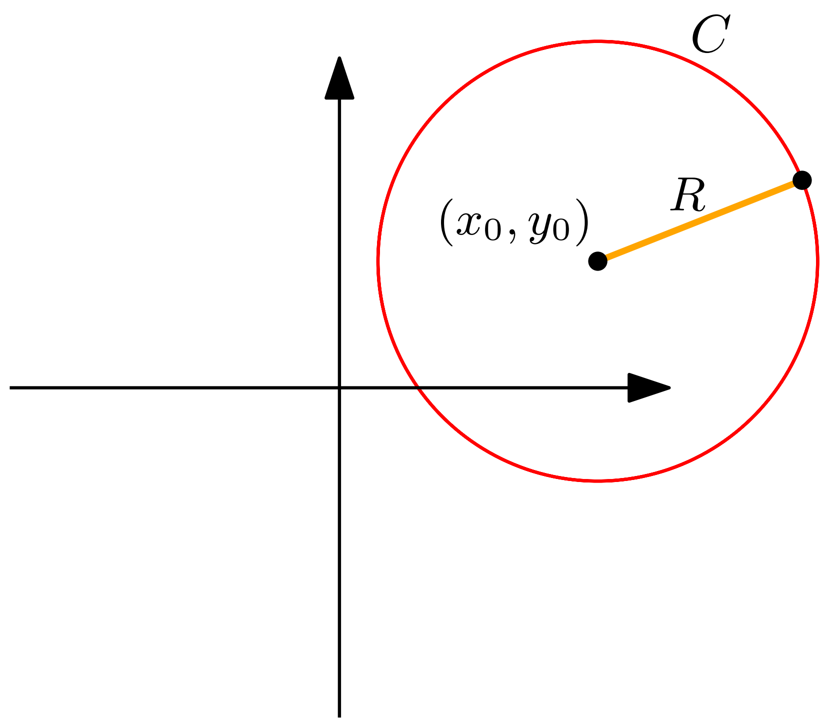

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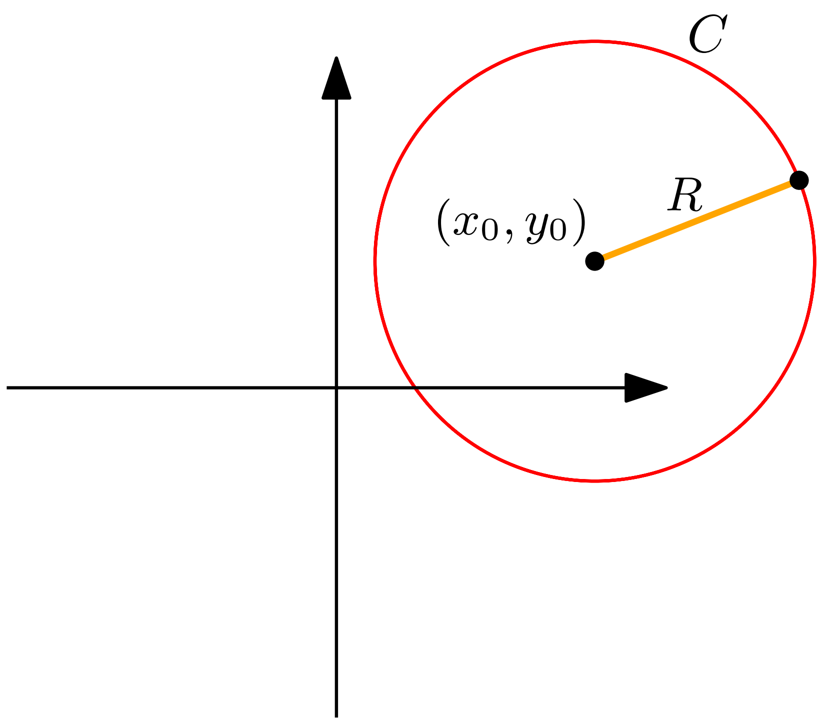

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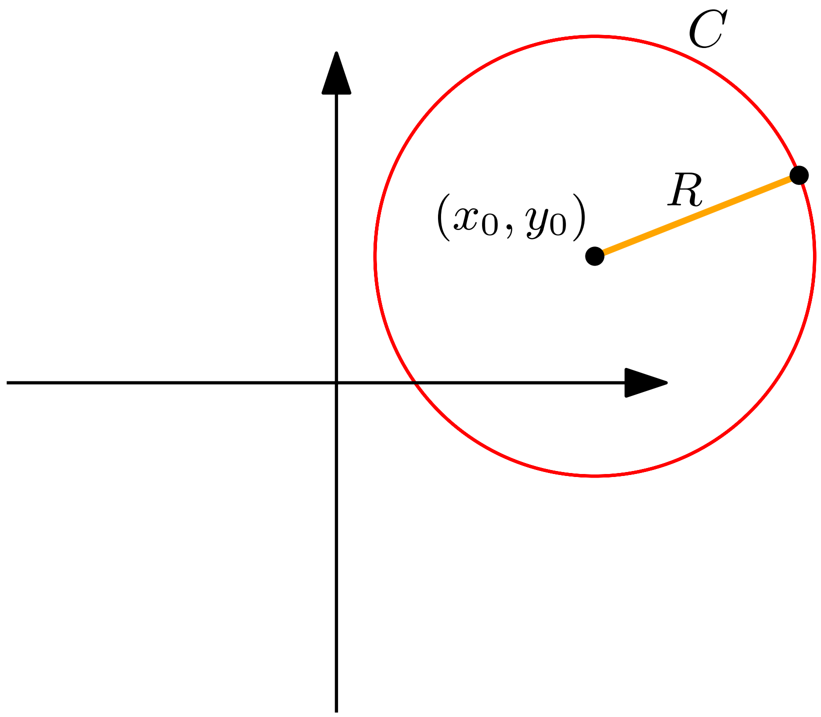

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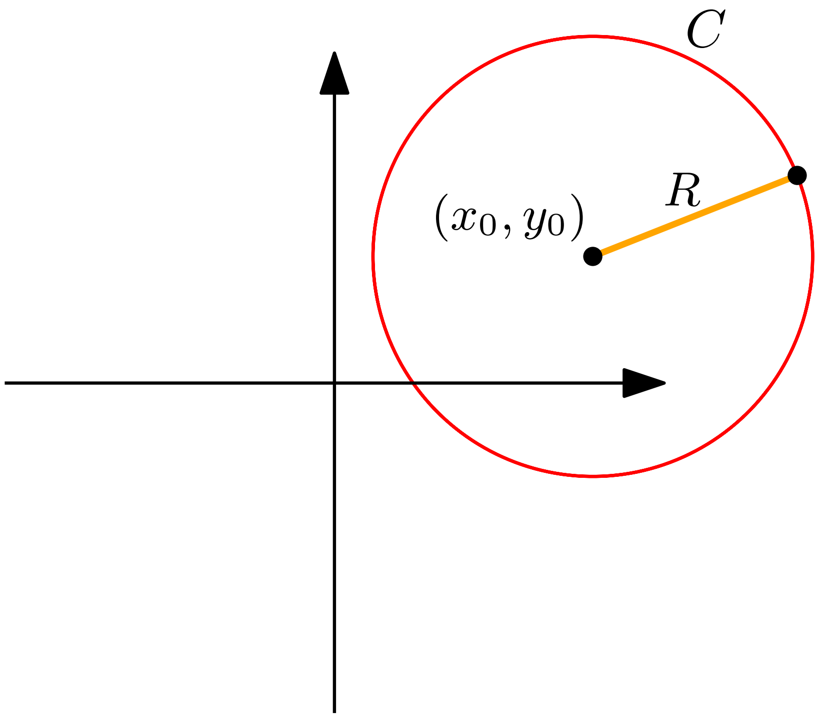

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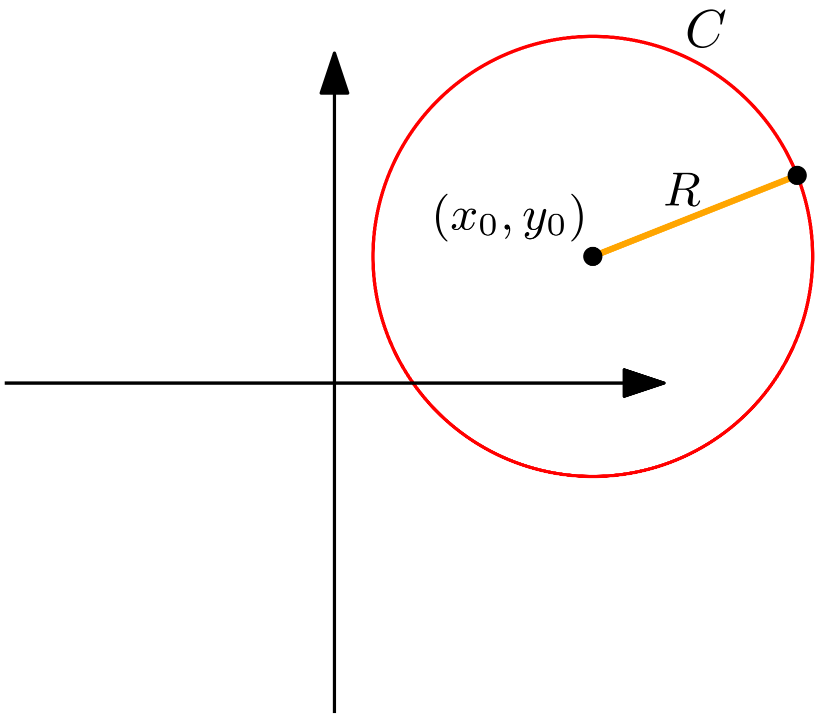

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- Parametrizations of the same curve can be very different!

$(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi/2]$ $\quad$ $(\cos 2\alpha, \sin 2\alpha)$ for $\alpha \in [0, \pi/4]$ $\quad$ $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ for $t \in [0, 1]$

3) Ellipses (2D)

Consider an ellipse centered at $(0, 0)$ with semi-axes a and b (at the x and y axis, resp.)

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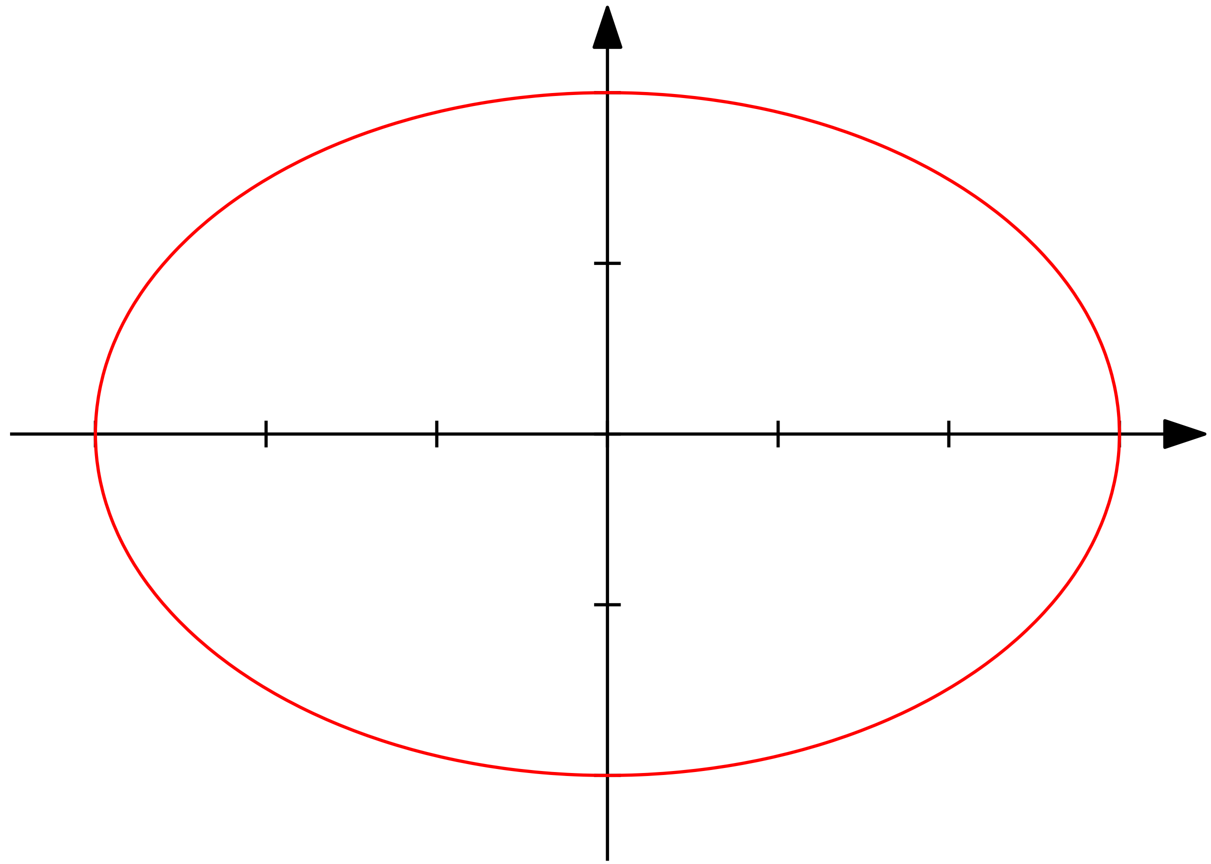

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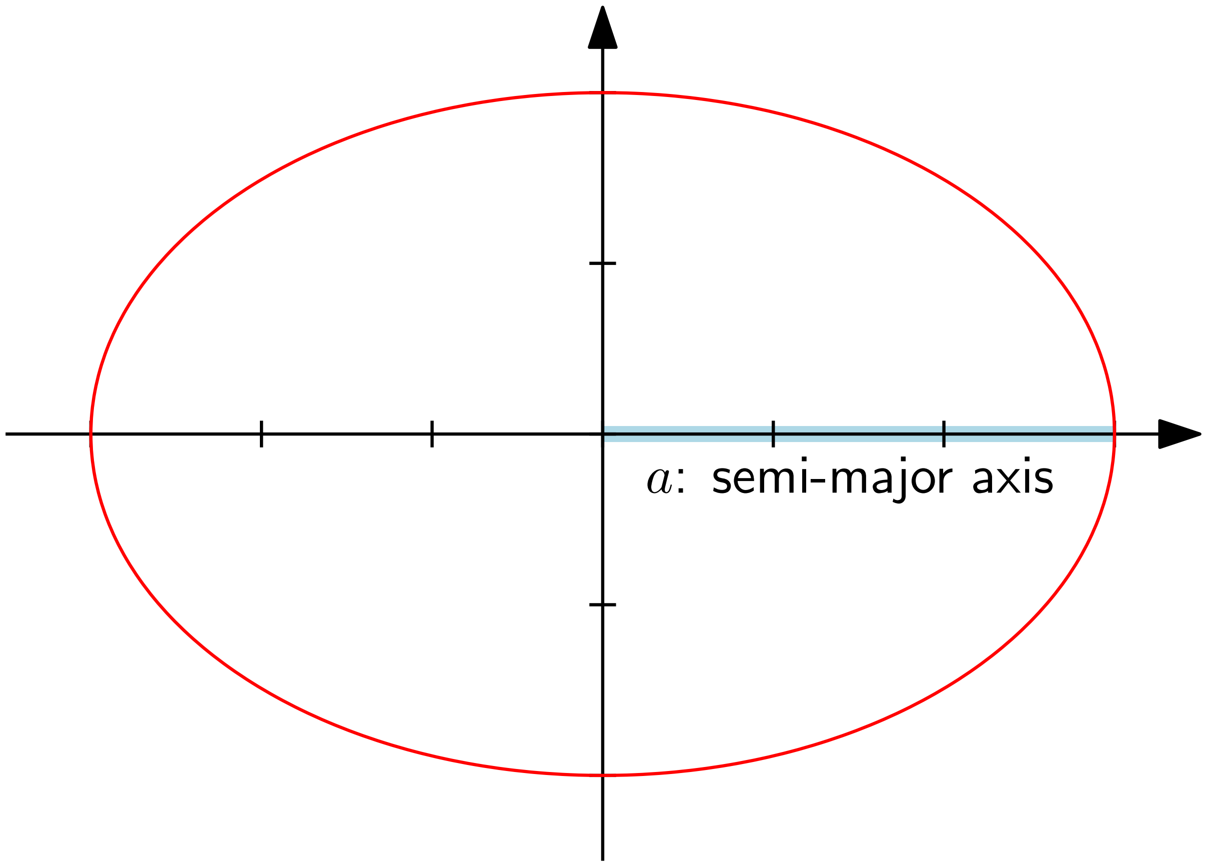

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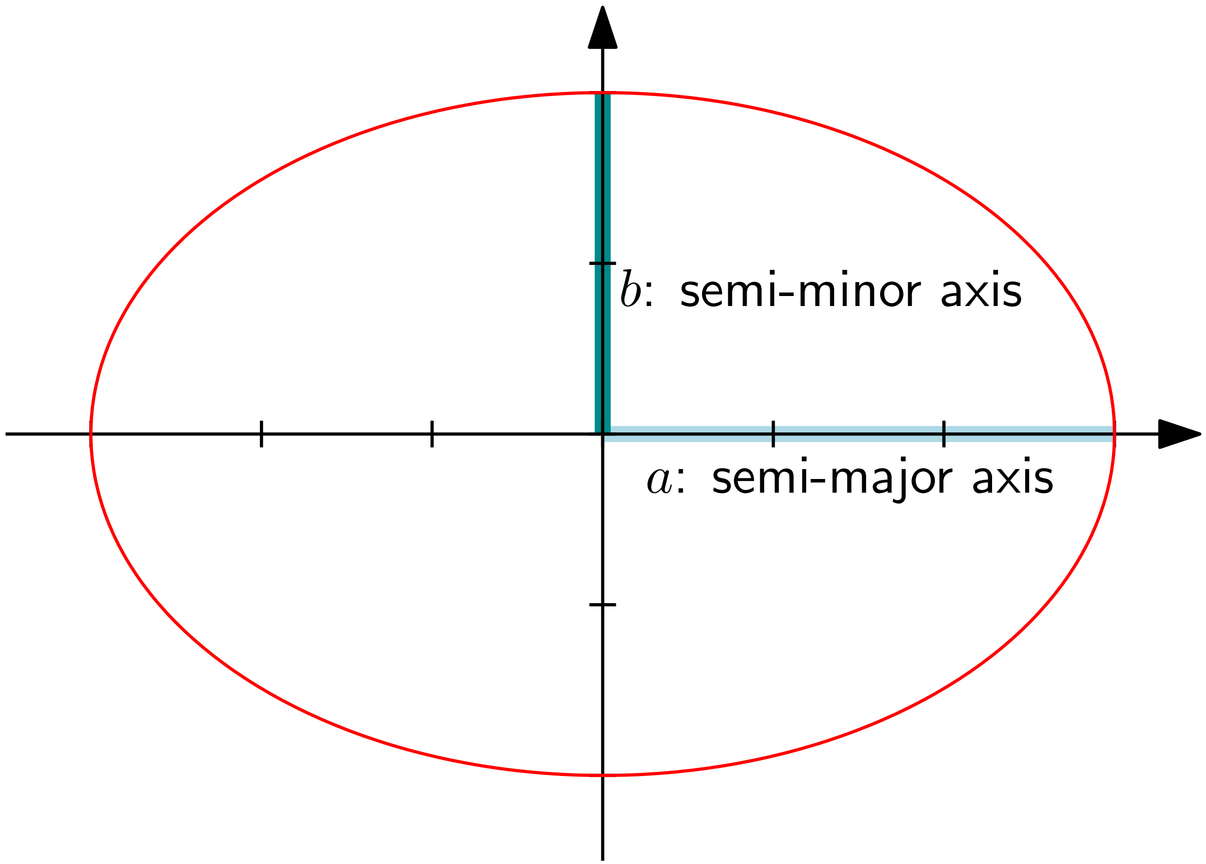

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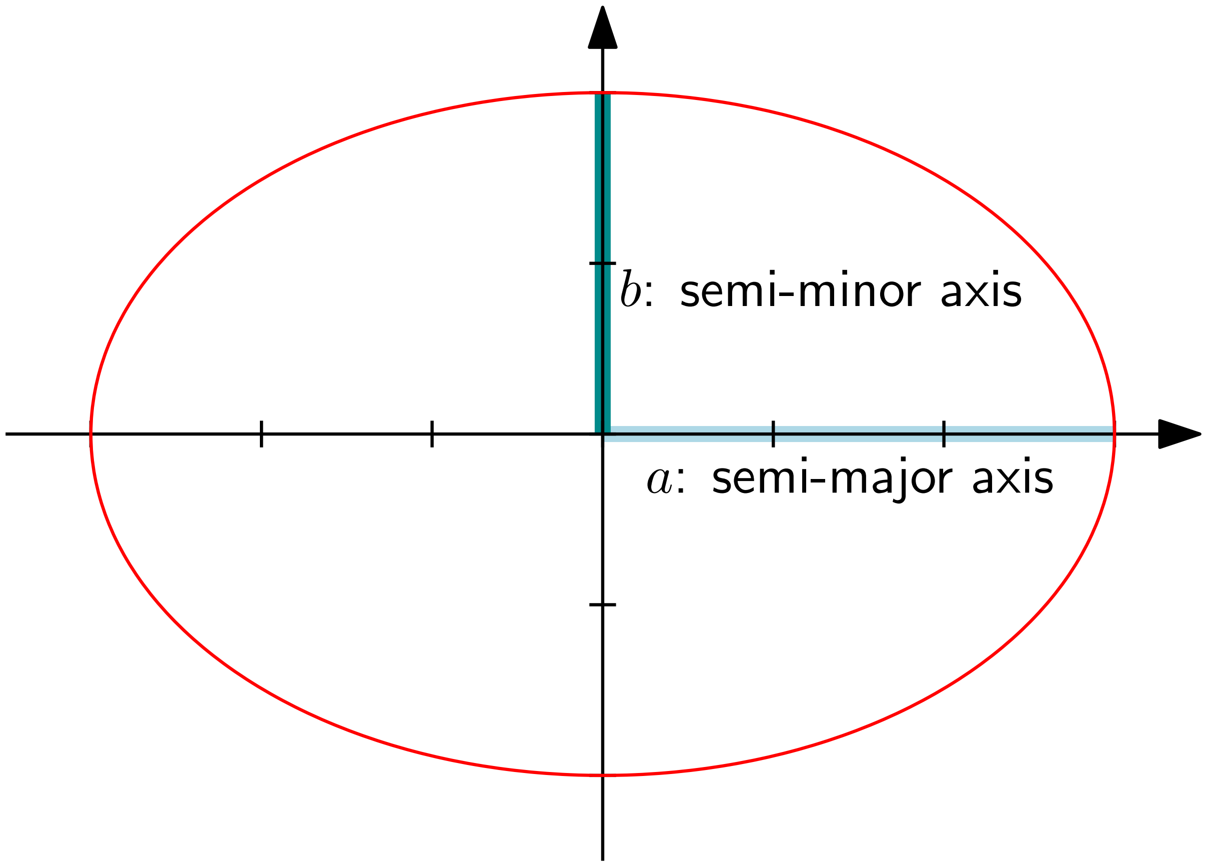

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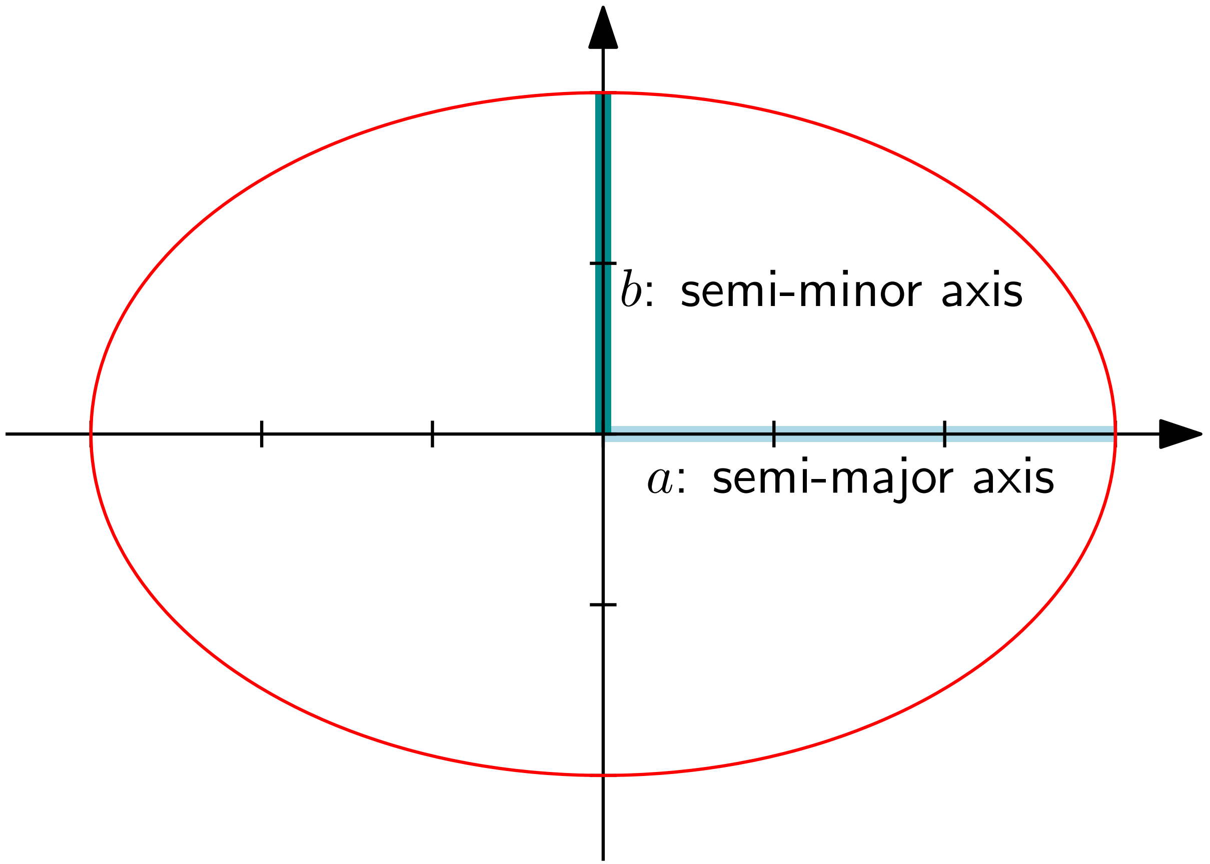

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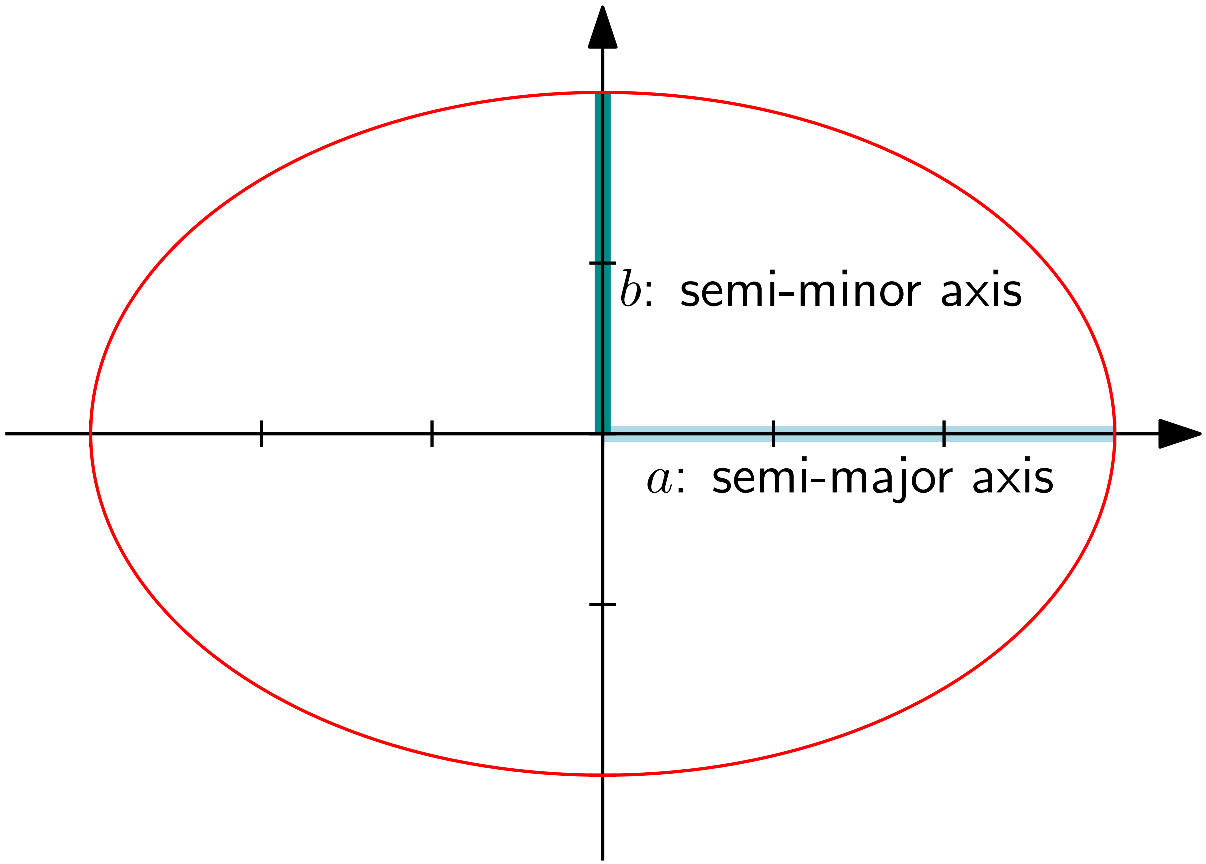
$\gamma(\theta) = (a\cos\theta, b\sin\theta) \qquad \theta \in [0, 2\pi)$

(a circle when $a = b$)

Its implicit equation is $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$



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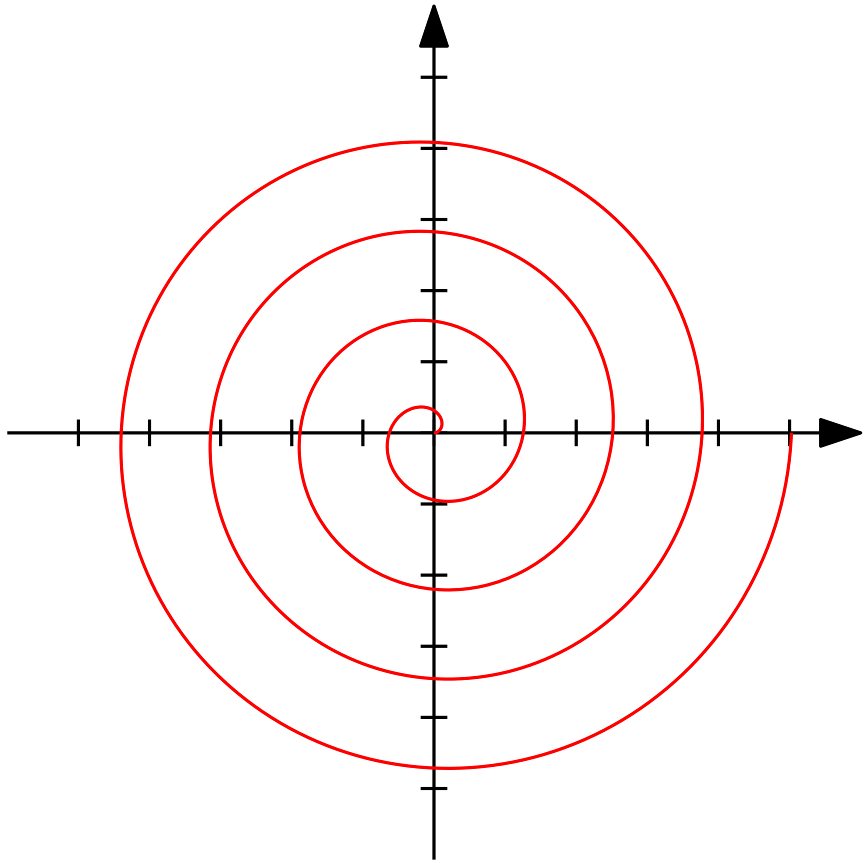

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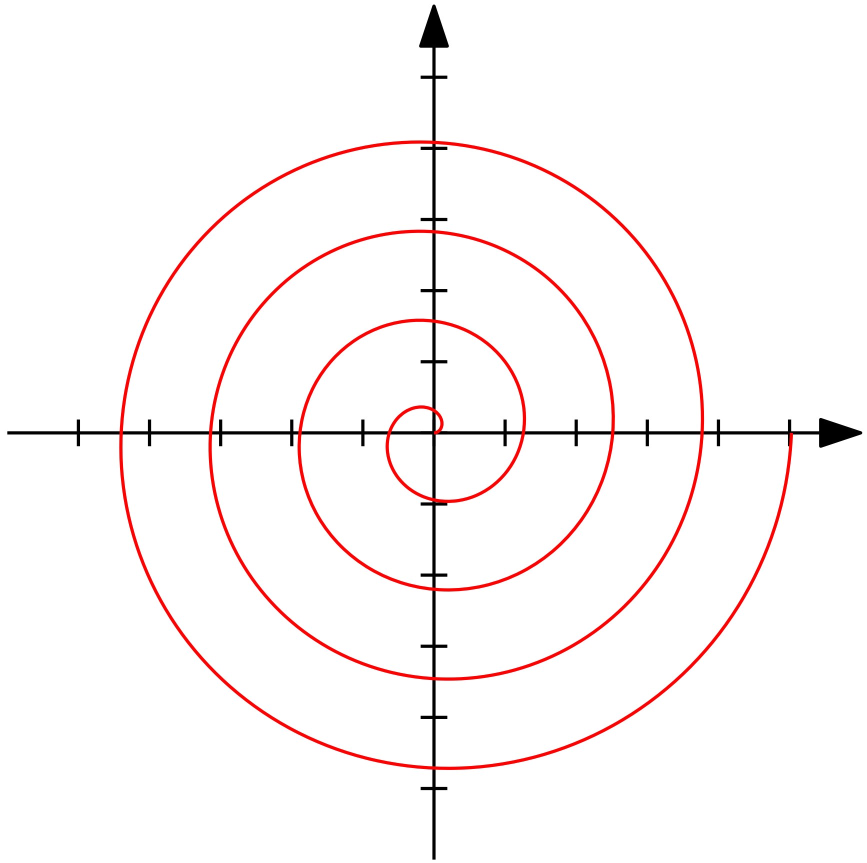
How can we find a parametrization?



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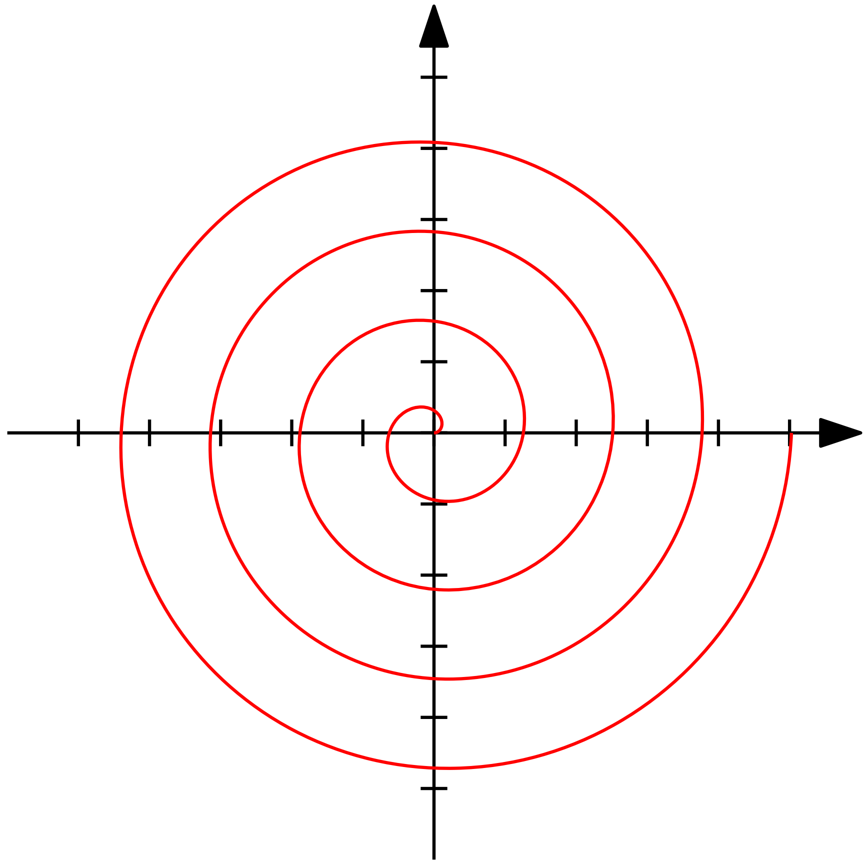

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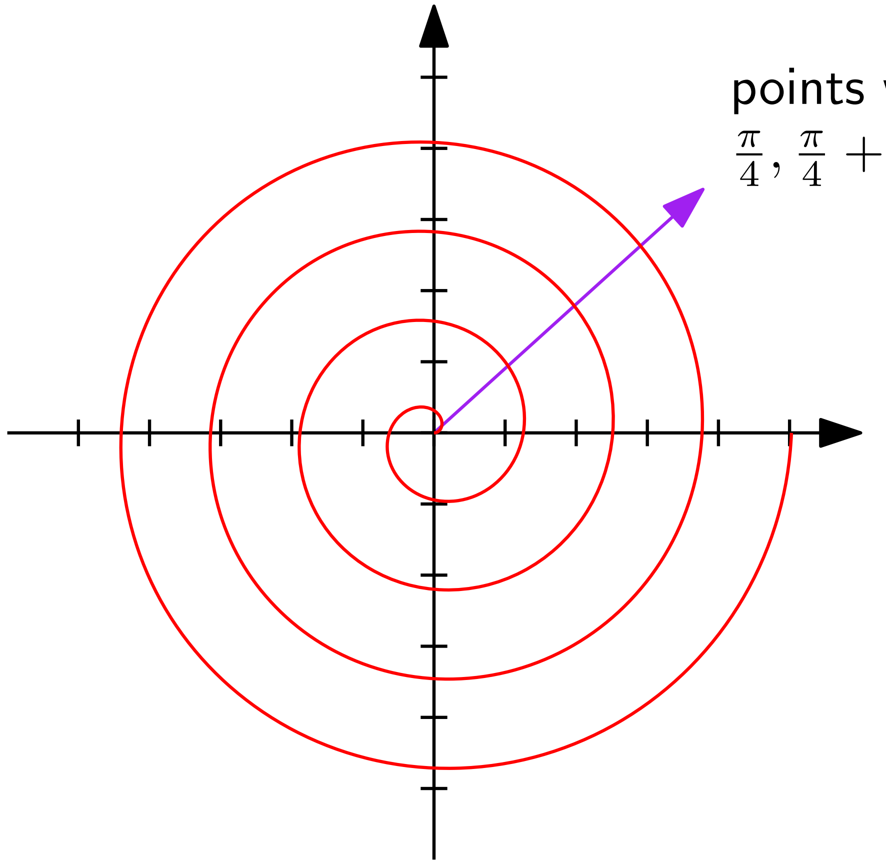

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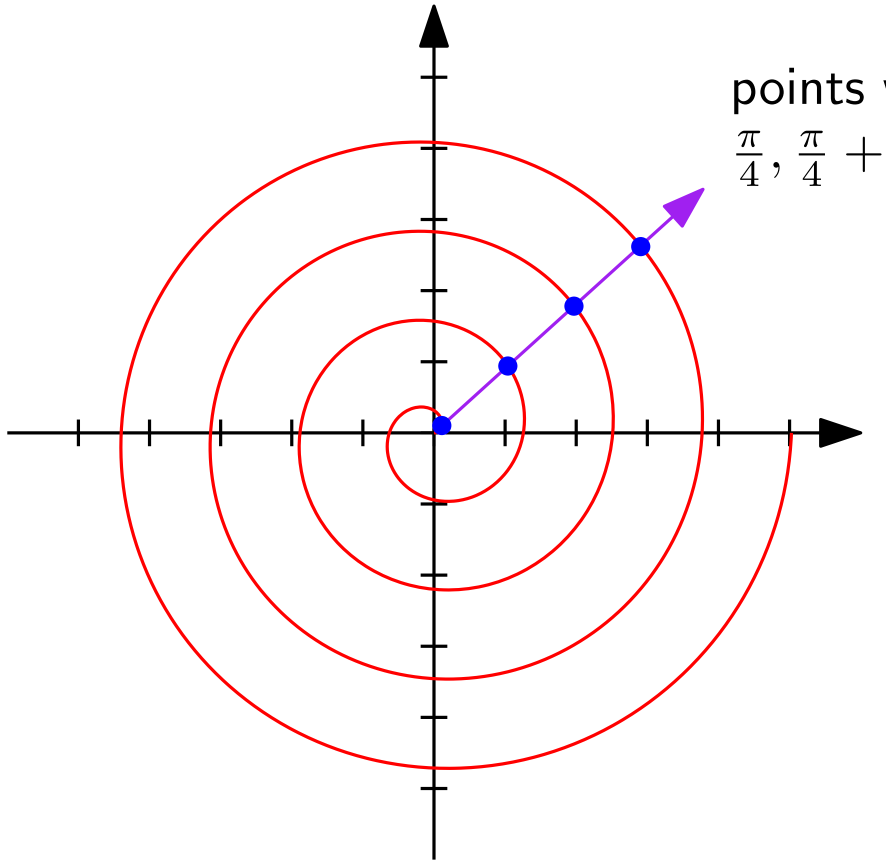

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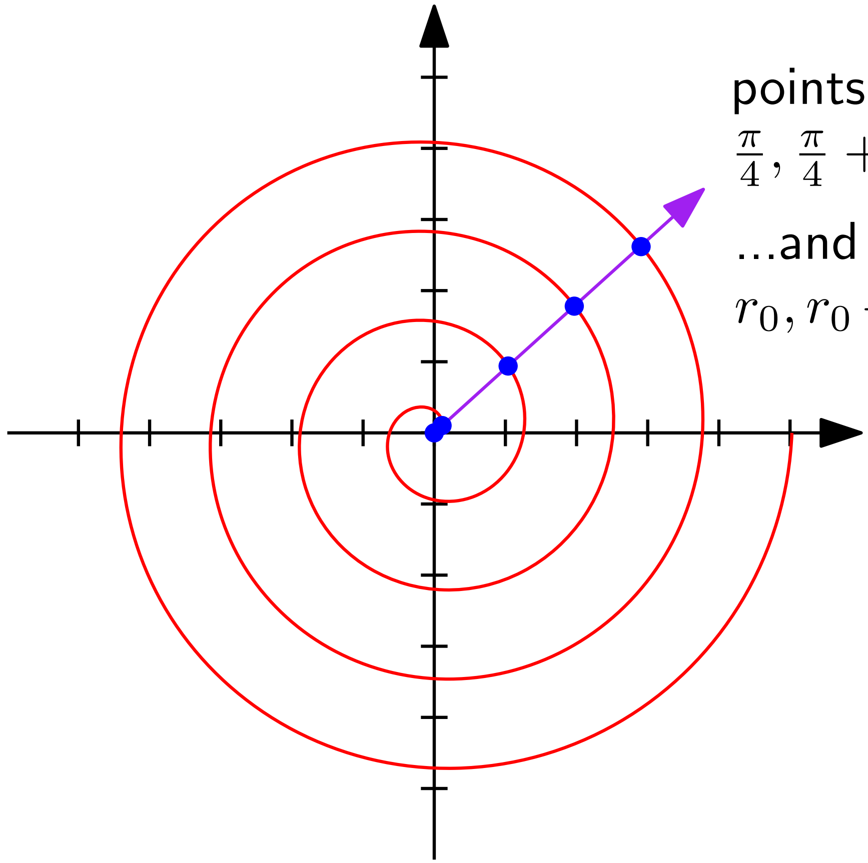

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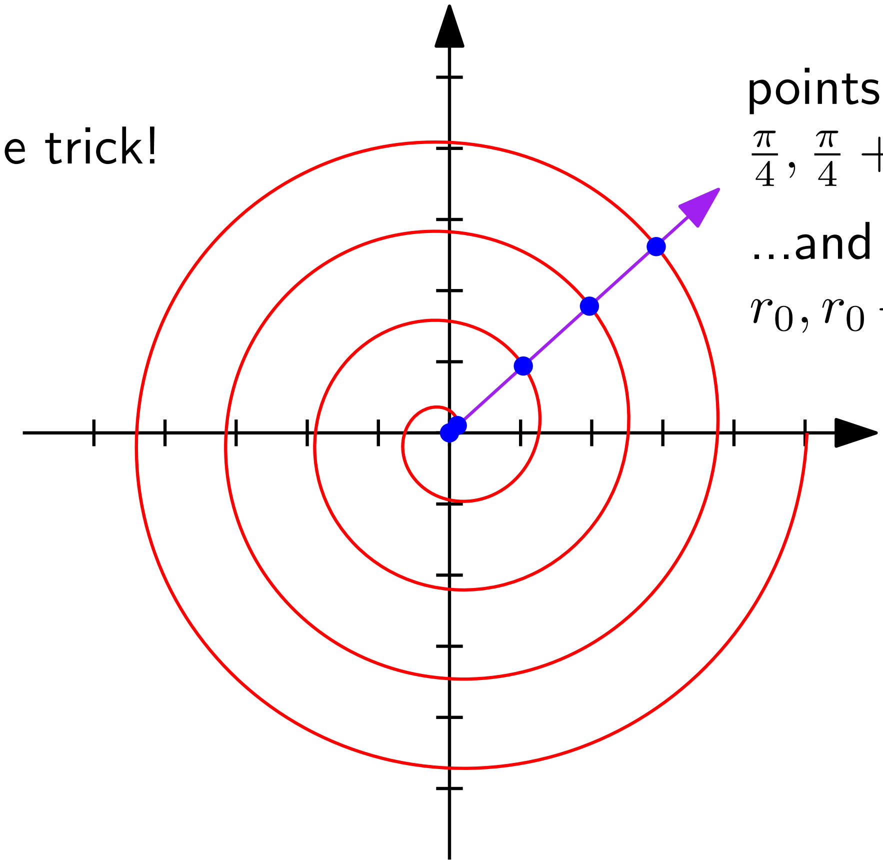

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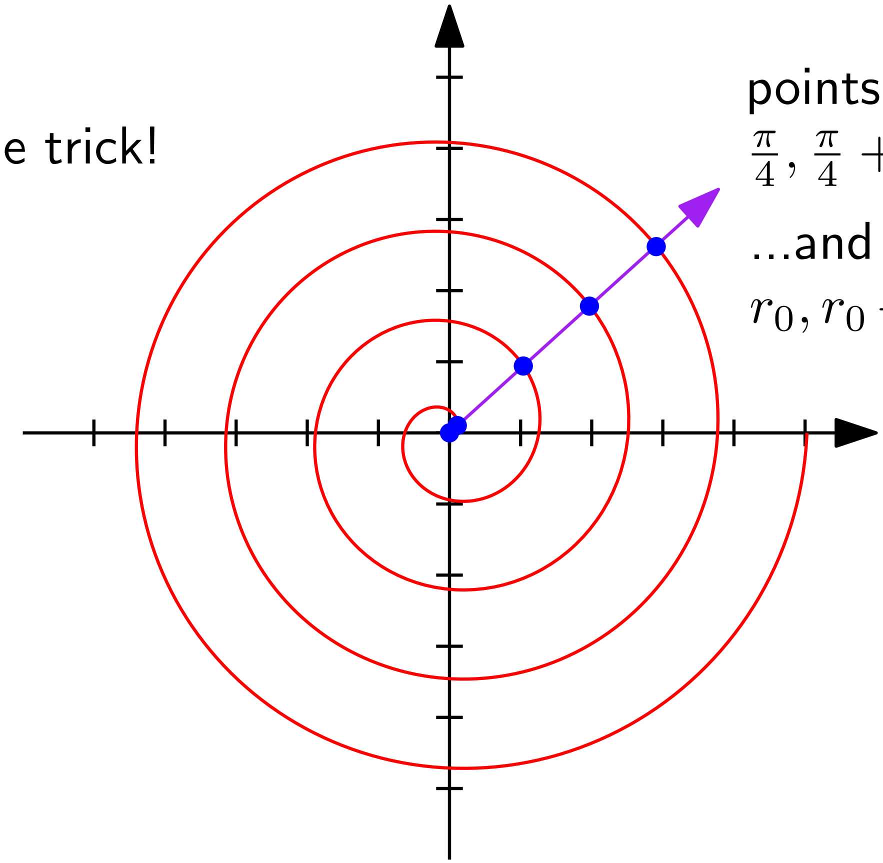
(thus $d = 0.2 \cdot 2\pi \approx 1.25$)



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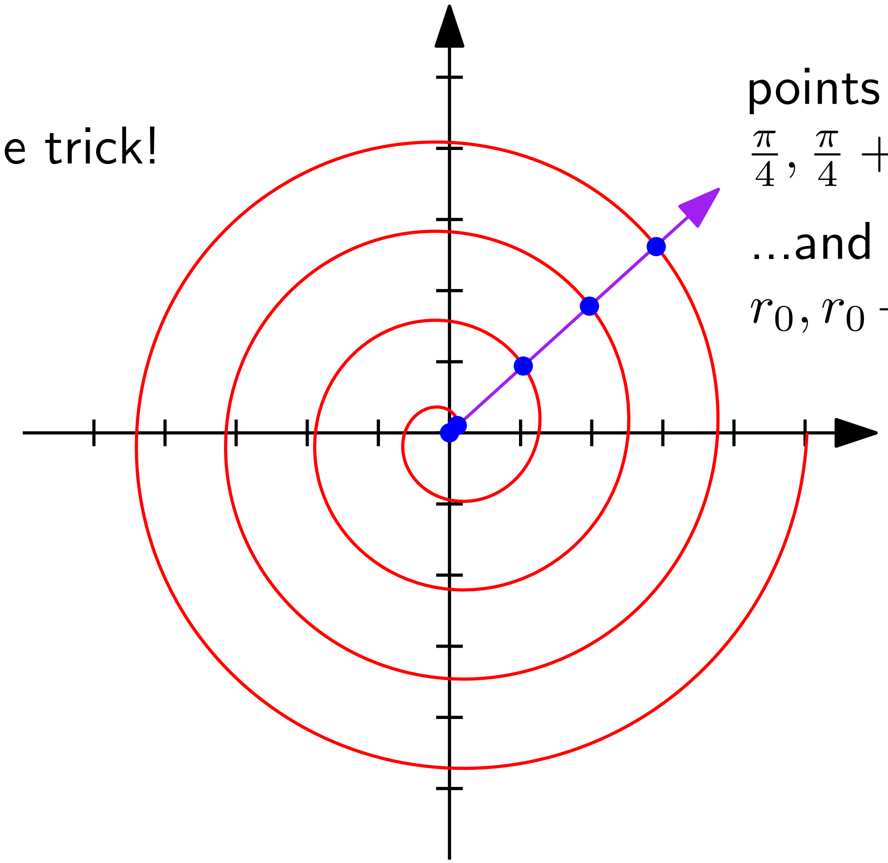

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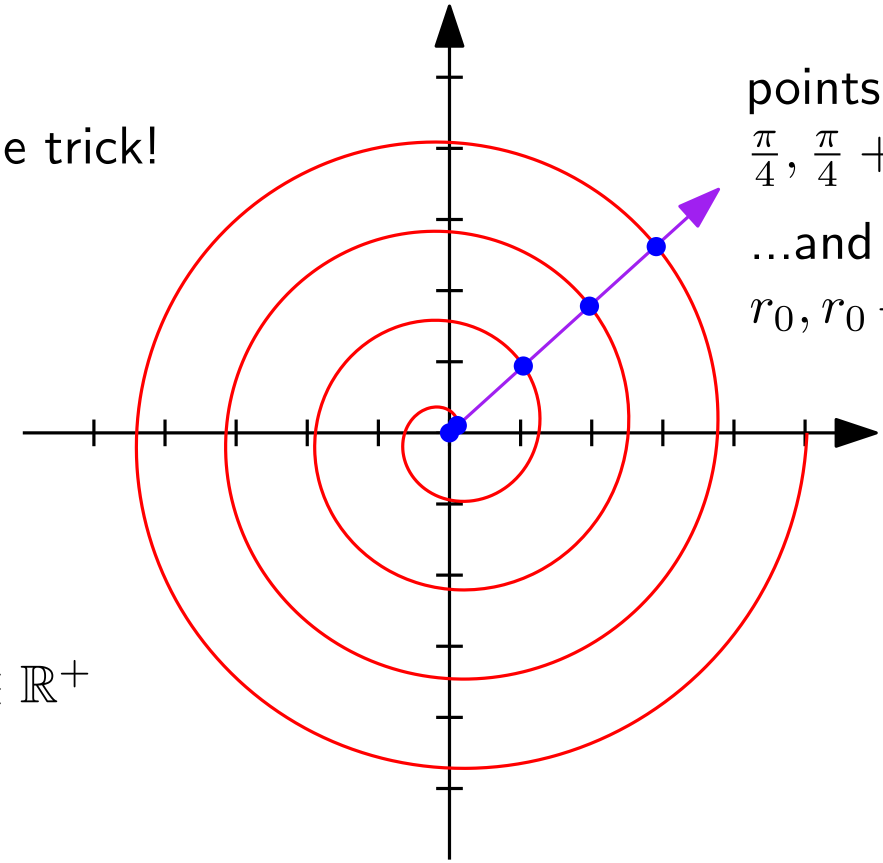

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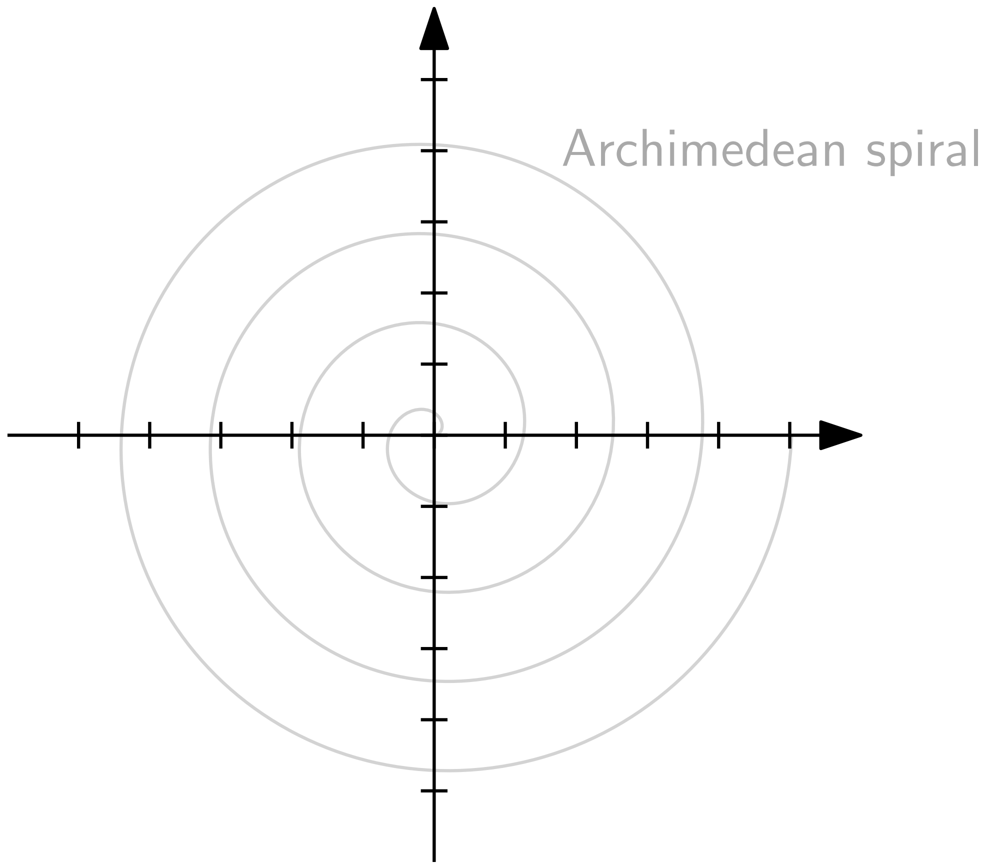
Distances between points intersected by the same ray grow exponentially

Appears a lot in nature, and it is related to Fibonacci numbers



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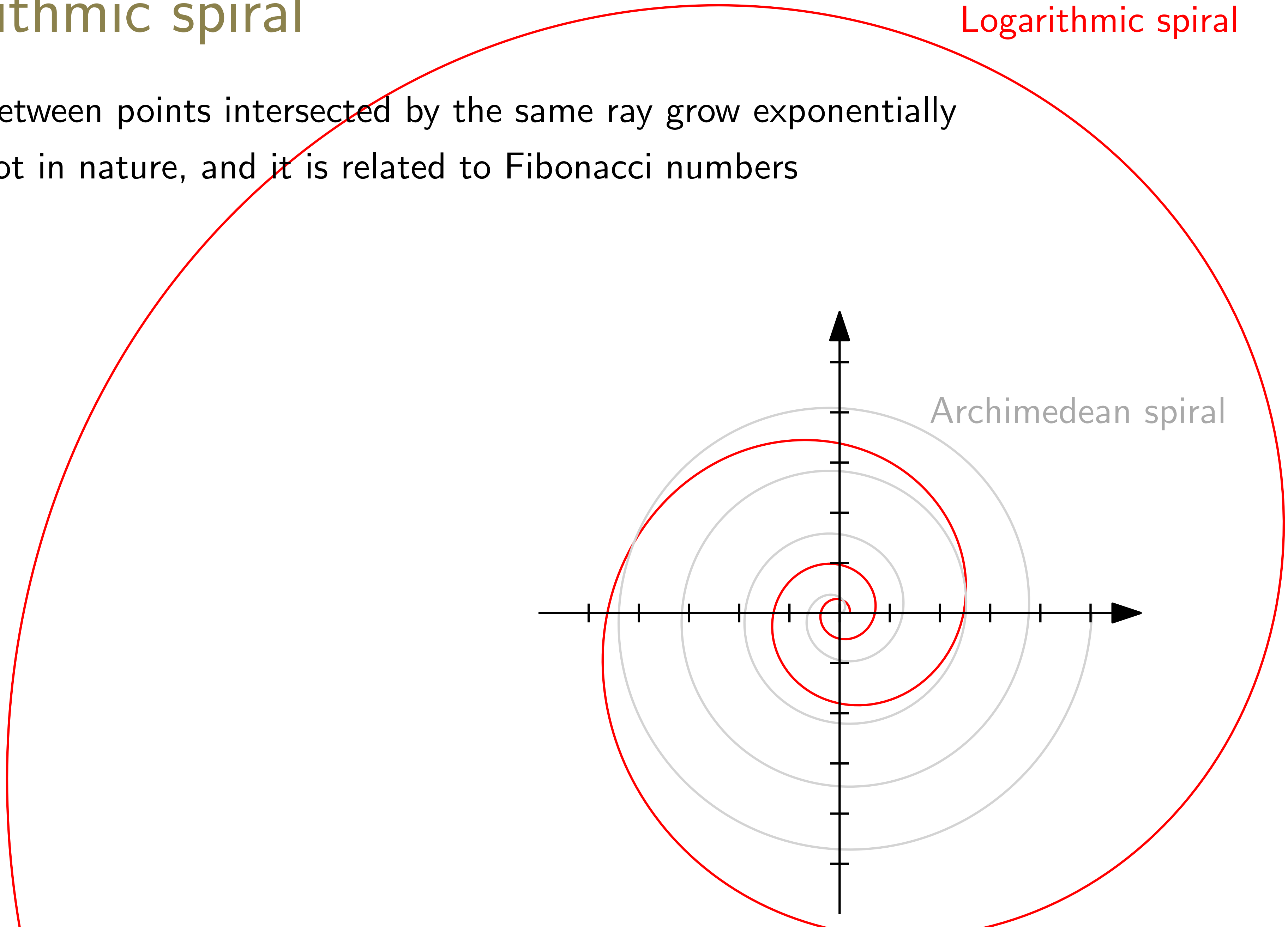

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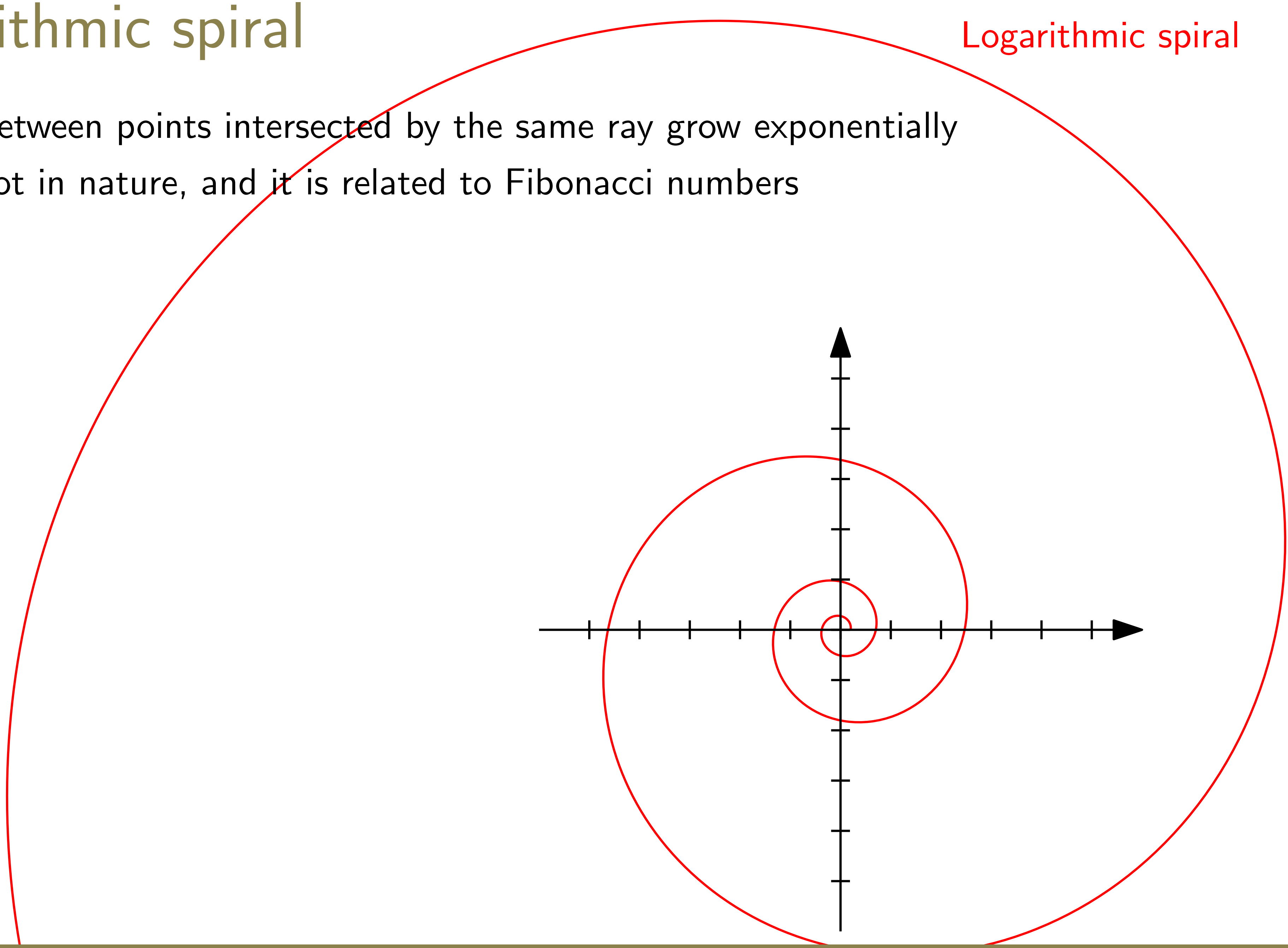

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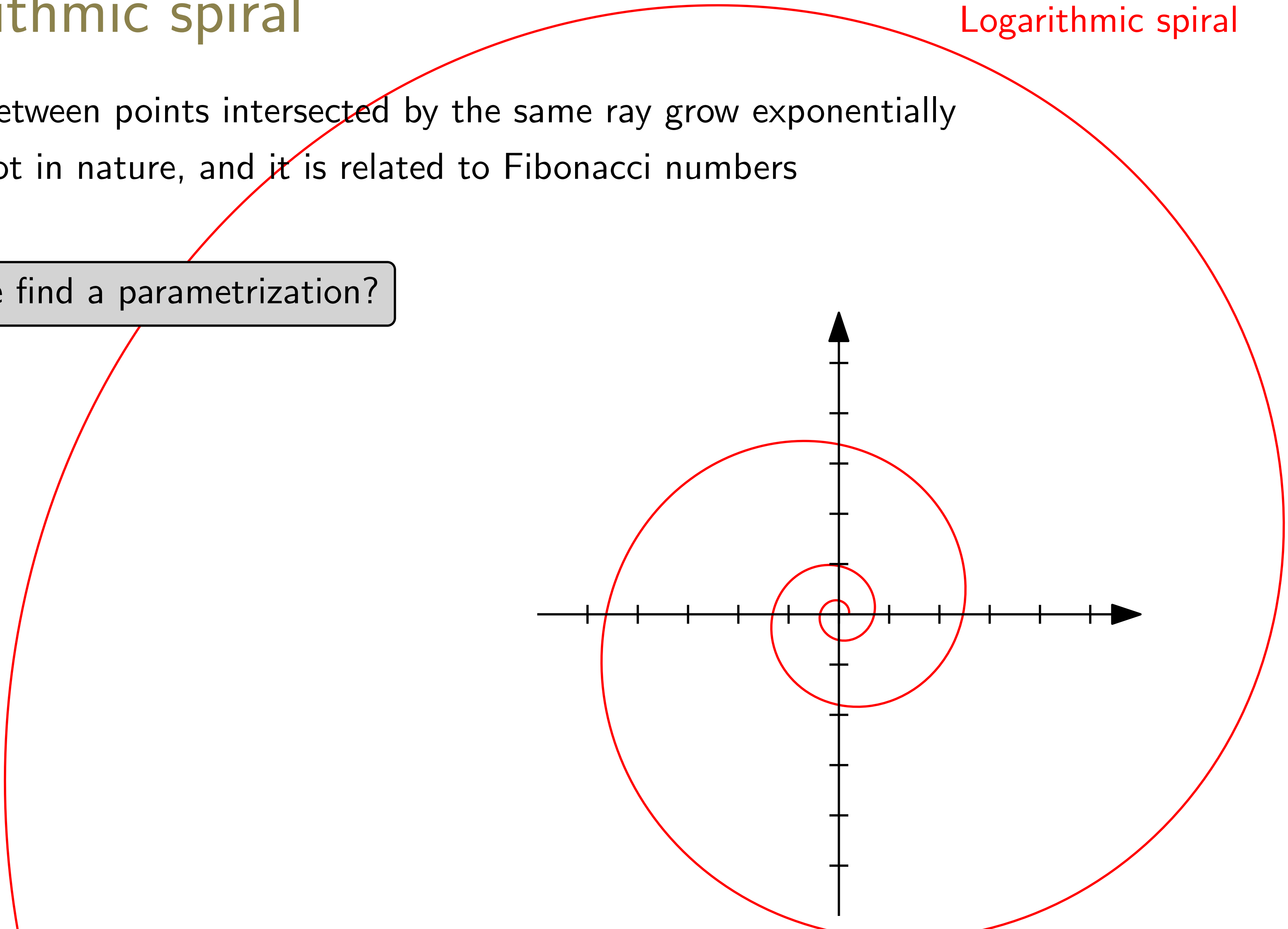

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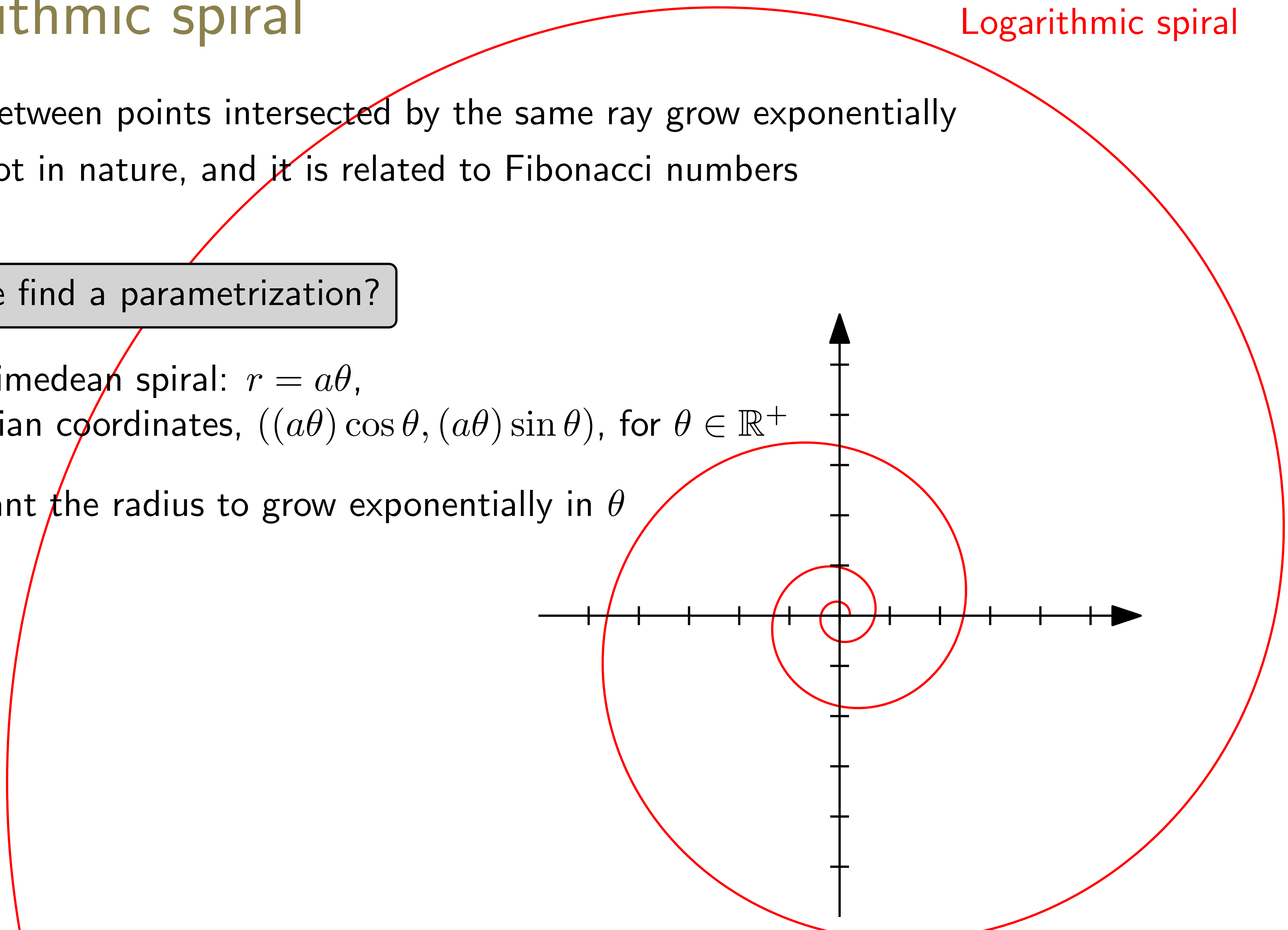

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Recall Archimedean spiral: $r = a\theta$,
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Now, we want the radius to grow exponentially in θ

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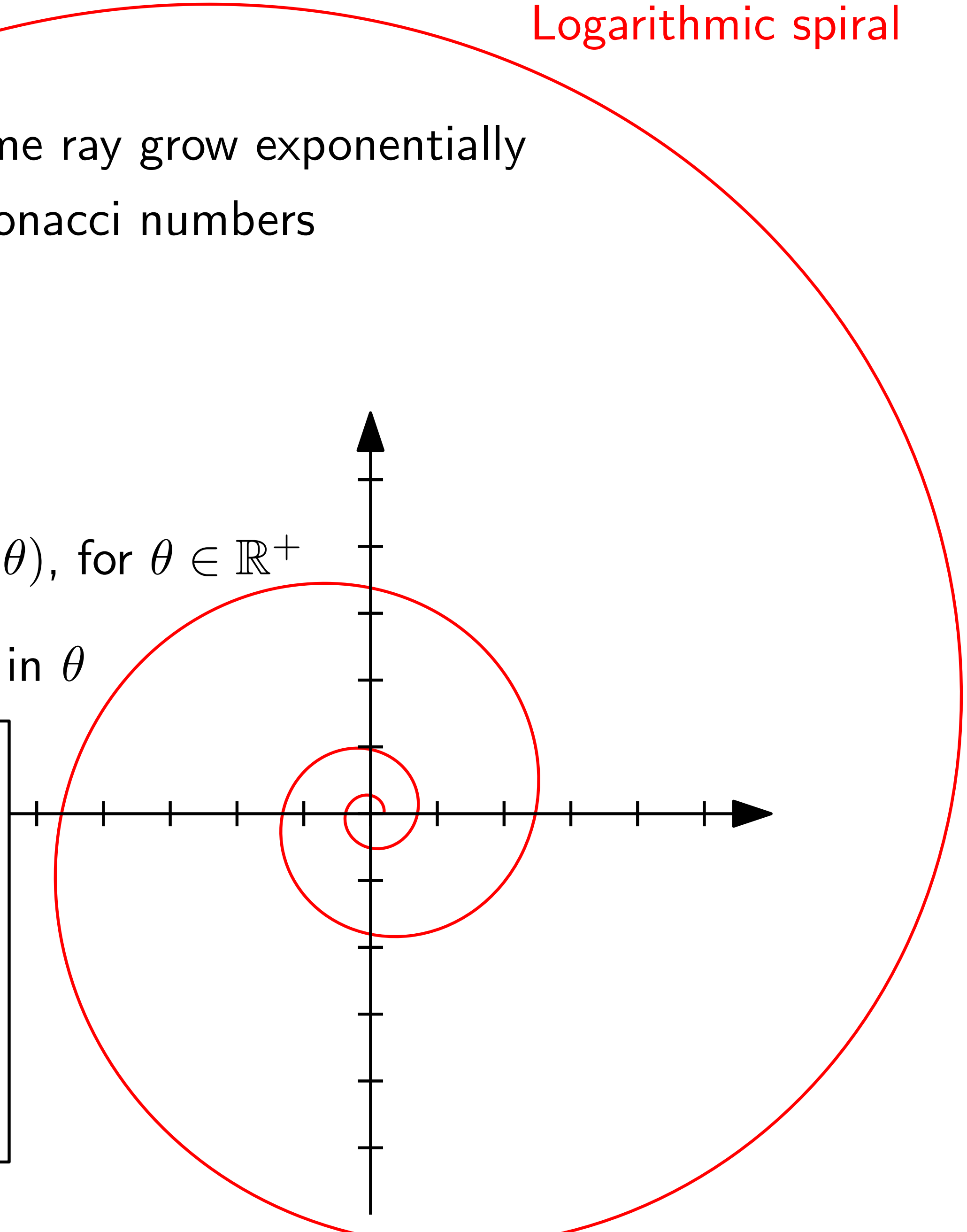
In general, in Polar coordinates:
$r = ae^{b\theta}$, so a parametrization is $(ae^{b\theta}, \theta)$

Equivalently, in Cartesian coordinates:
$\gamma(\theta) = ((ae^{b\theta})\cos\theta, (ae^{b\theta})\sin\theta$, for $\theta \in \mathbb{R}$

for two parameters $a, b \in \mathbb{R}$ (non-zero)



5) Logarithmic spiral (cont'd)

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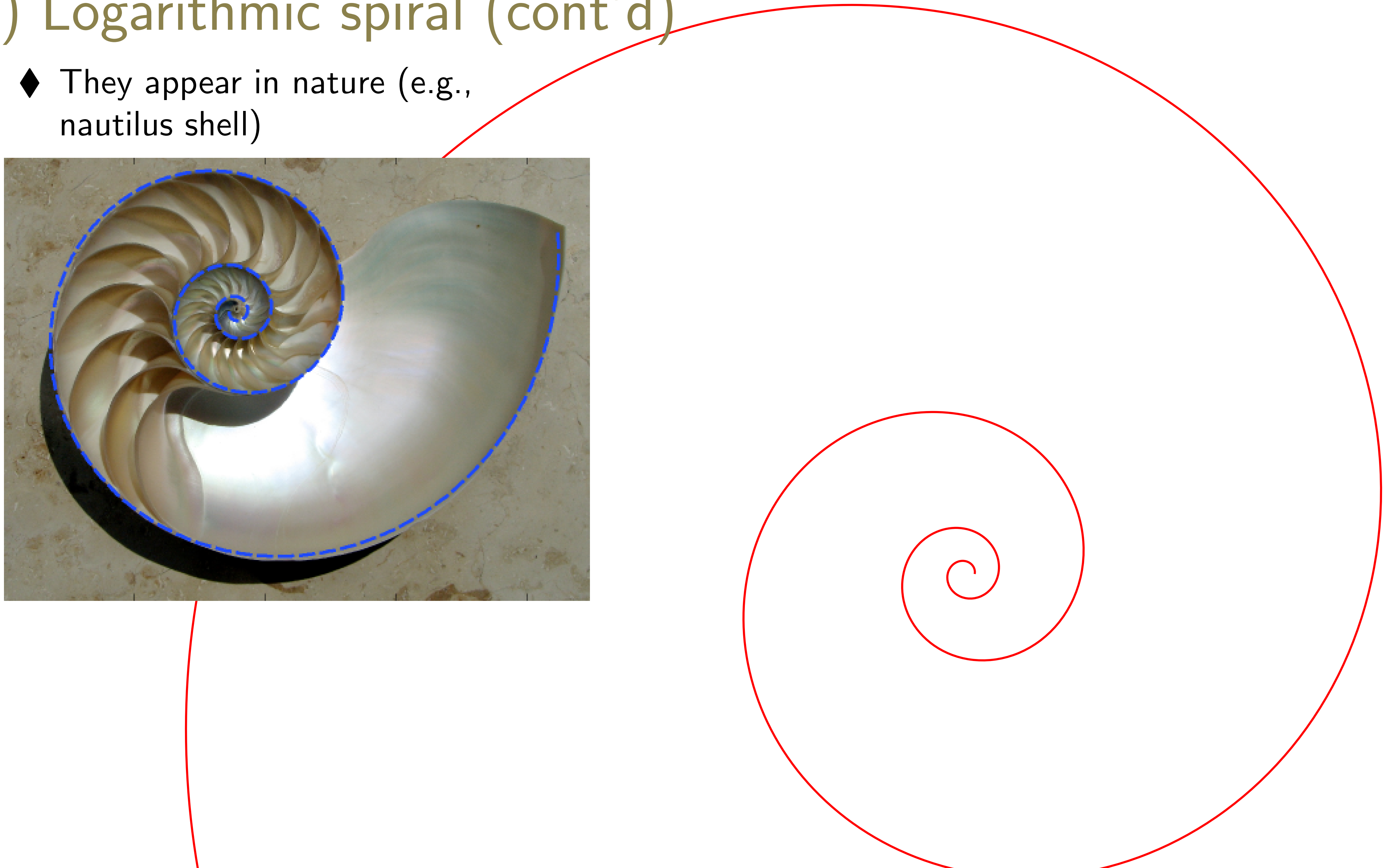

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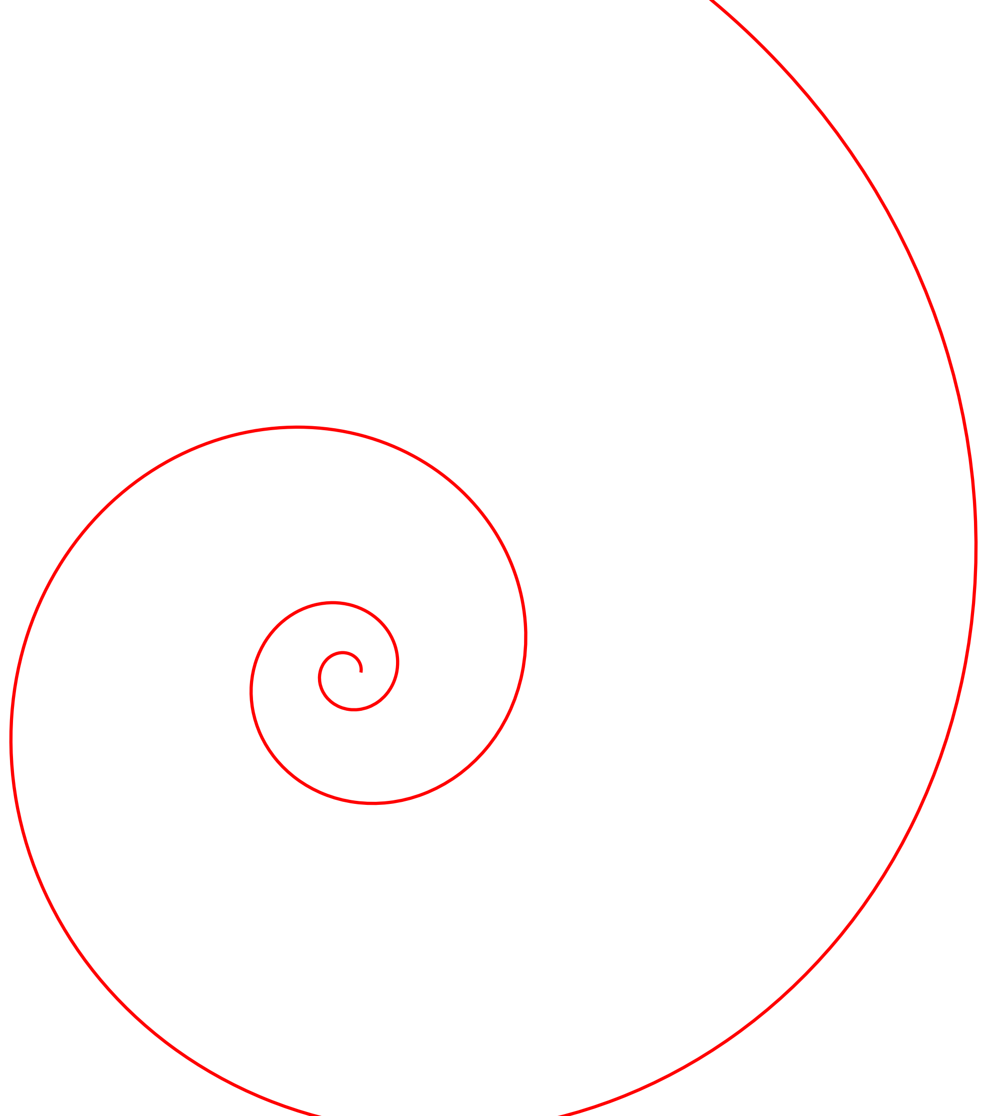

EXAMPLES OF PARAMETRIC CURVES

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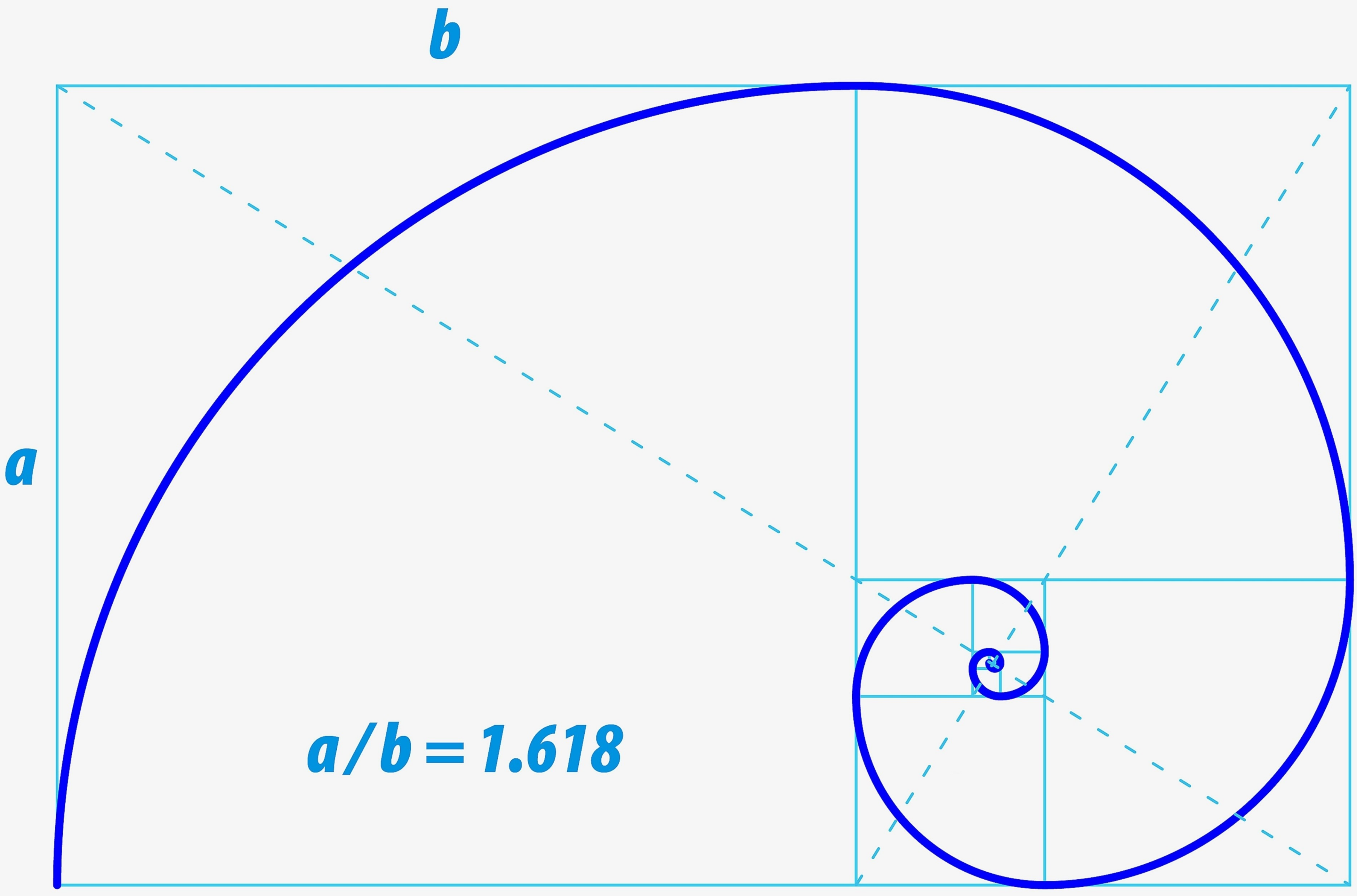
b
a
a/b=1.618

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By mistake, an Archimedean spiral was carved instead!

6) Helices (3D)

Helix: a curve that goes around a central tube or cone shape in the form of a spiral (From Cambridge Dictionary)

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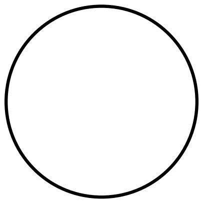
Circular (or elliptical) helix

Base: circle



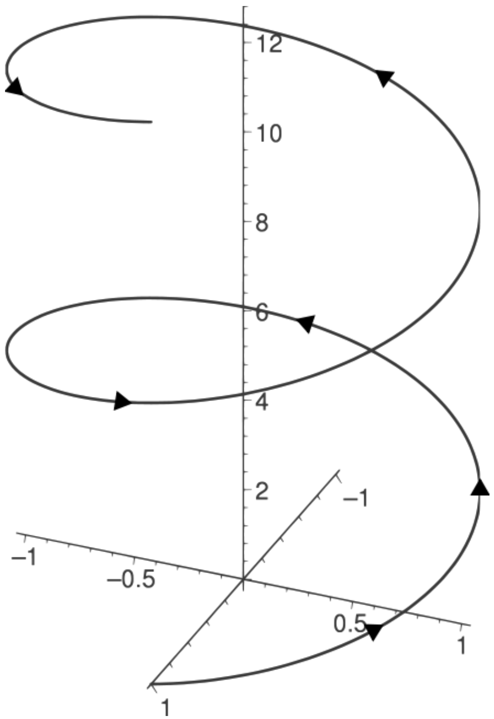


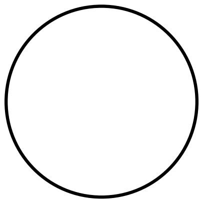
Image source: Wikipedia

6) Helices (3D)

Helix: a curve that goes around a central tube or cone shape in the form of a spiral (From Cambridge Dictionary)

Circular (or elliptical) helix

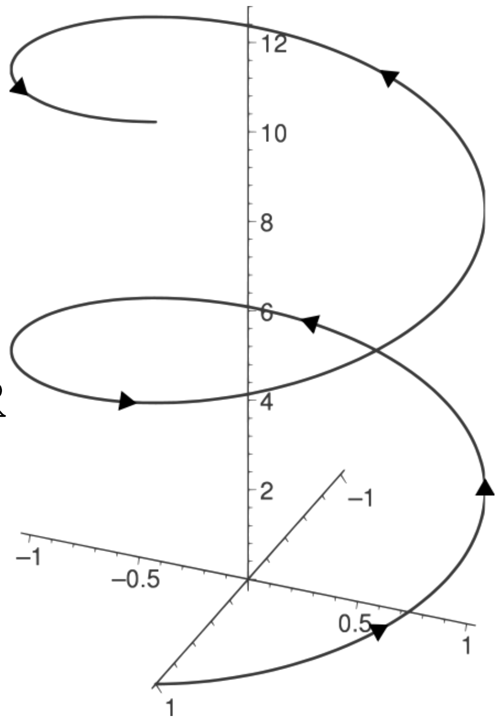
Base: circle



Parametric formula:

$(a\cos t, a\sin t, bt)$, $t \in \mathbb{R}$

a: radius
b: pitch
($b > 0$ right-handed,
$b < 0$ left-handed)



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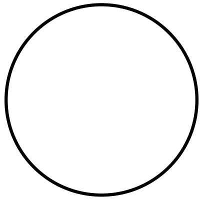

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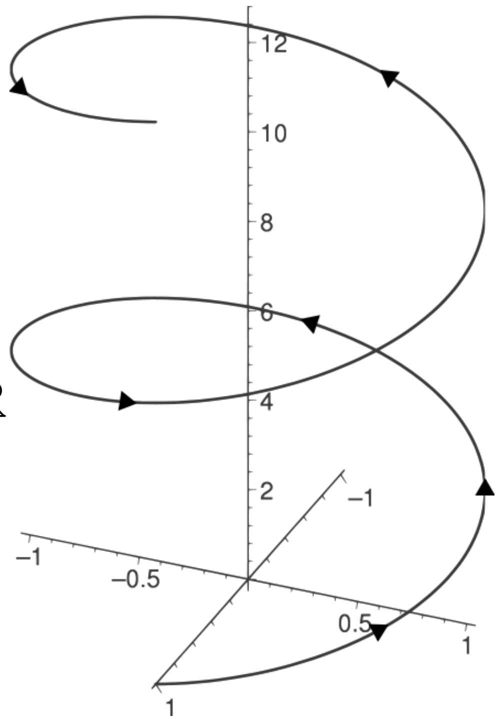
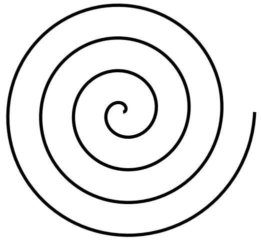


Image source: Wikipedia

Spiral helix

Base: Archimedean spiral



Formula:

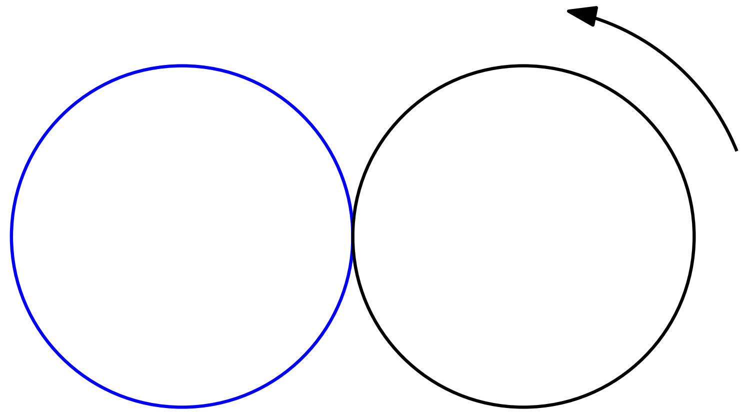
Lab assignment 2

Image source: Wikipedia

7) Cardioid

7) Cardioid

Curve traced by a point on the black circle as it rolls around the blue circle



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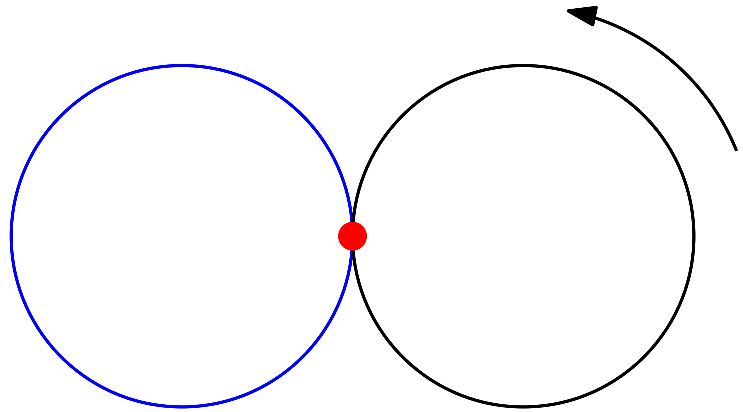

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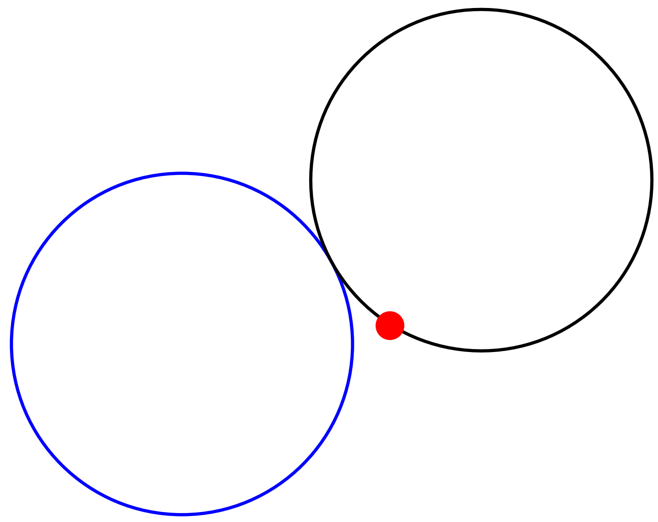

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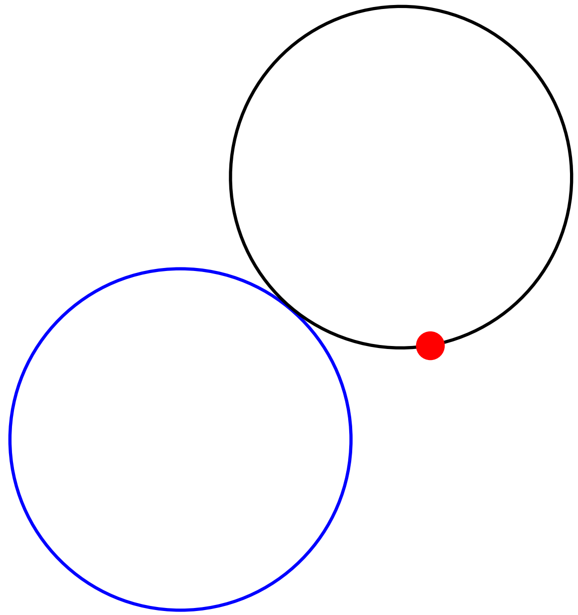

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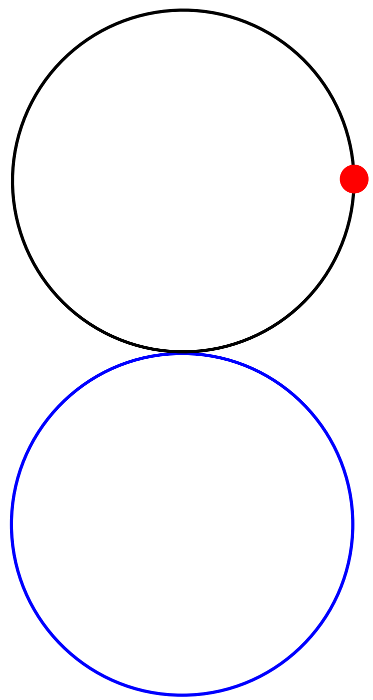

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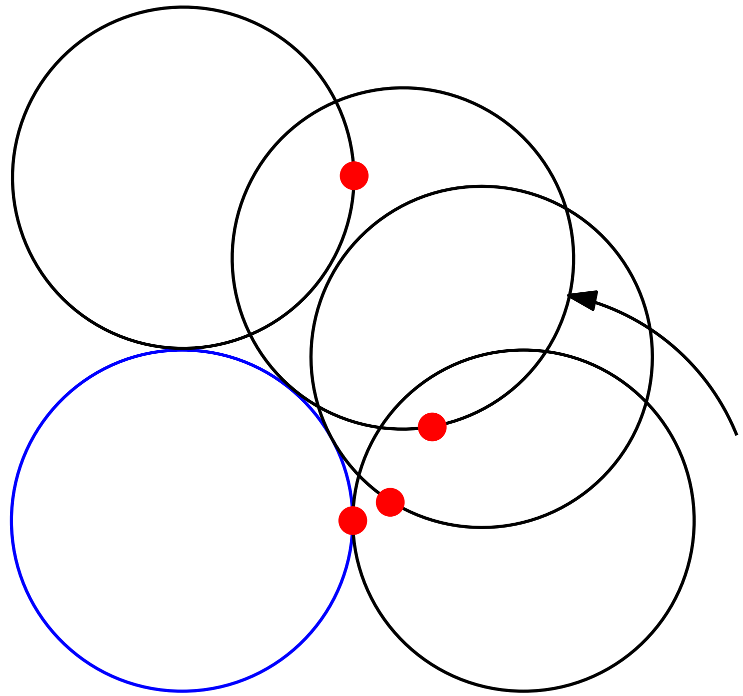

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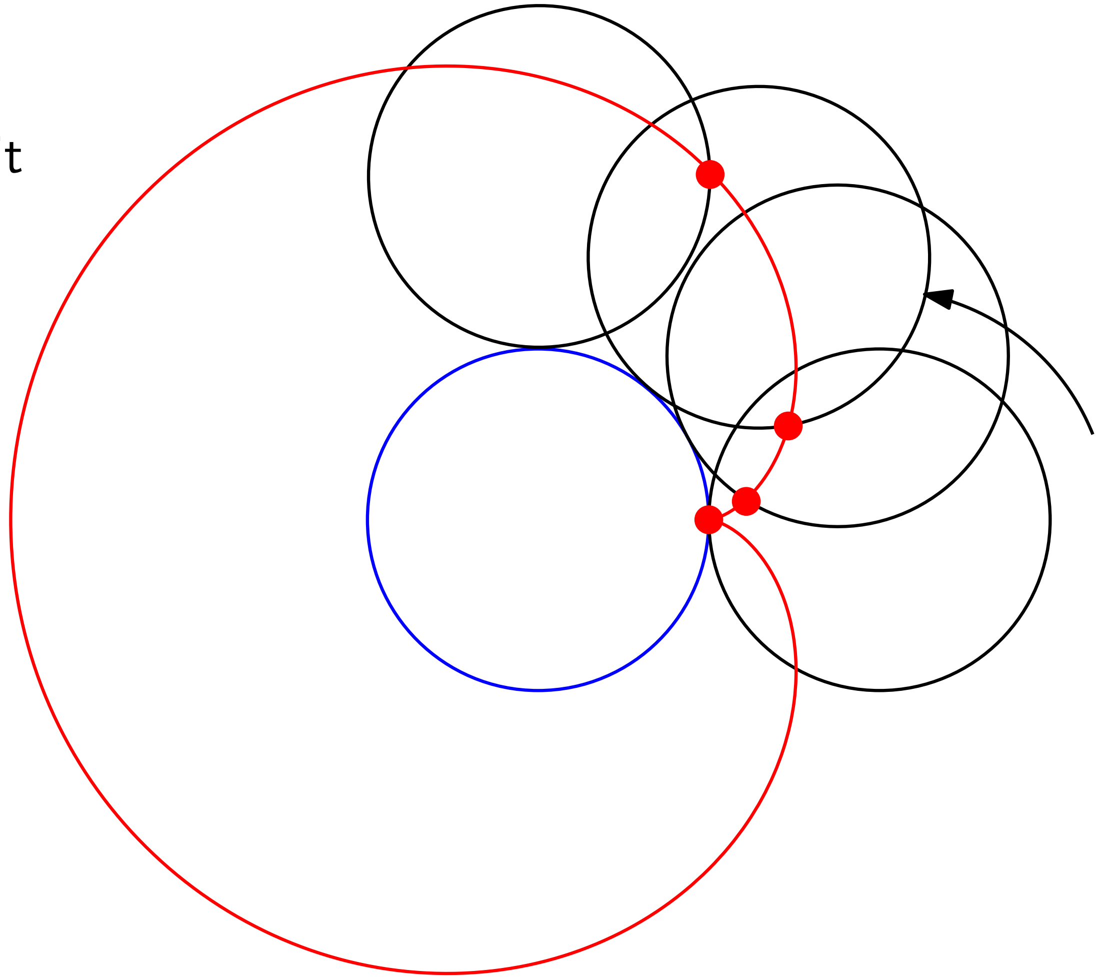

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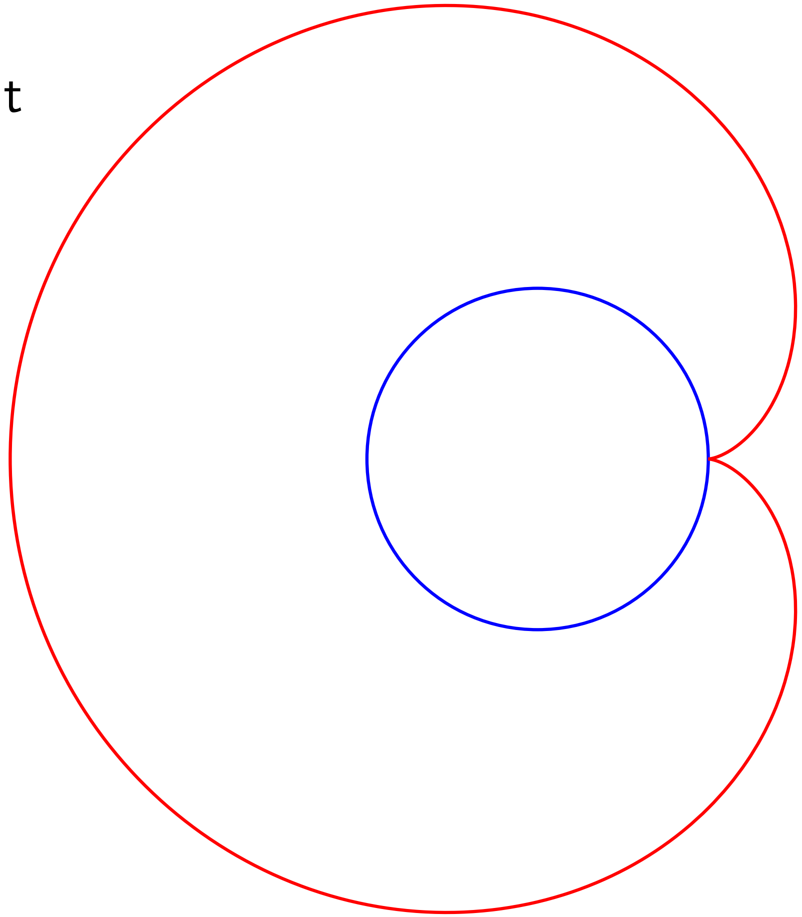
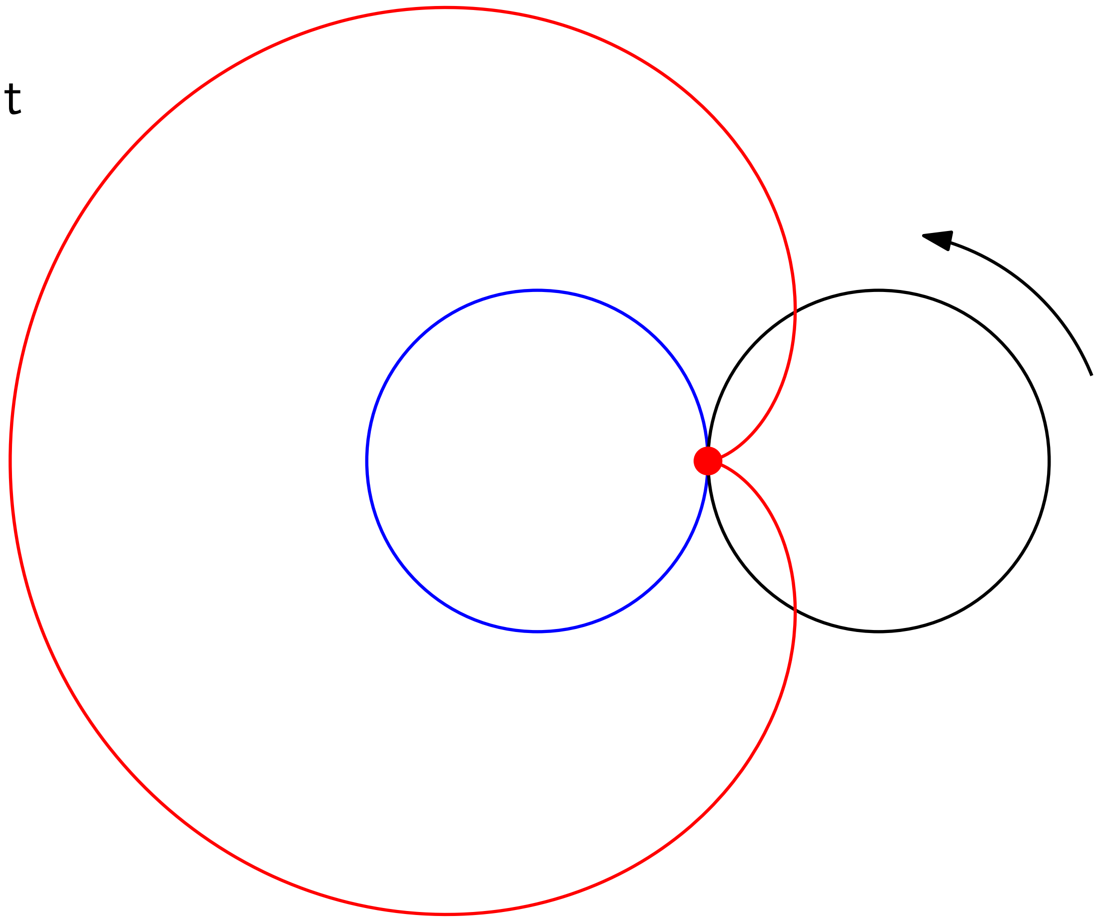


7) Cardioid

Curve traced by a point on the black circle as it rolls around the blue circle

How can we parametrize it?



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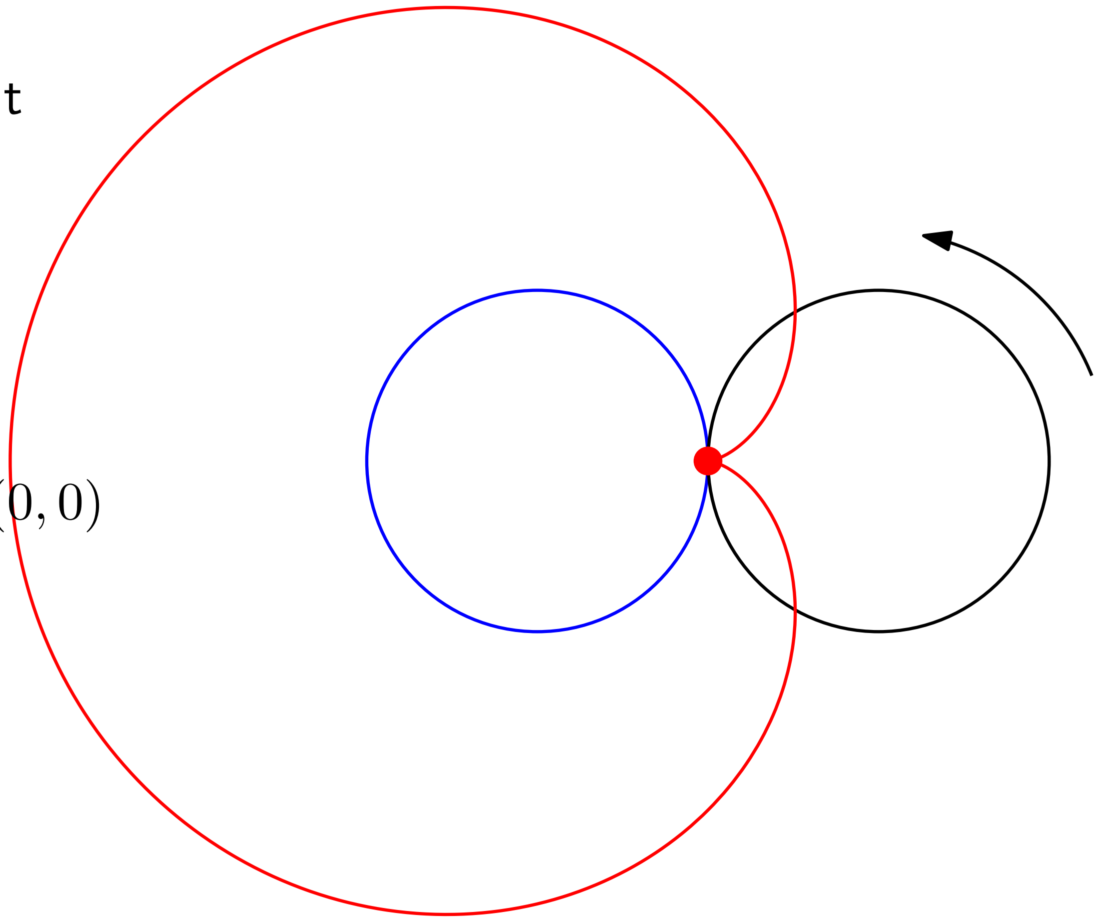

How can we parametrize it?

Assume, without loss of generality:
circles of radius 1, and blue circle centered at $(0, 0)$

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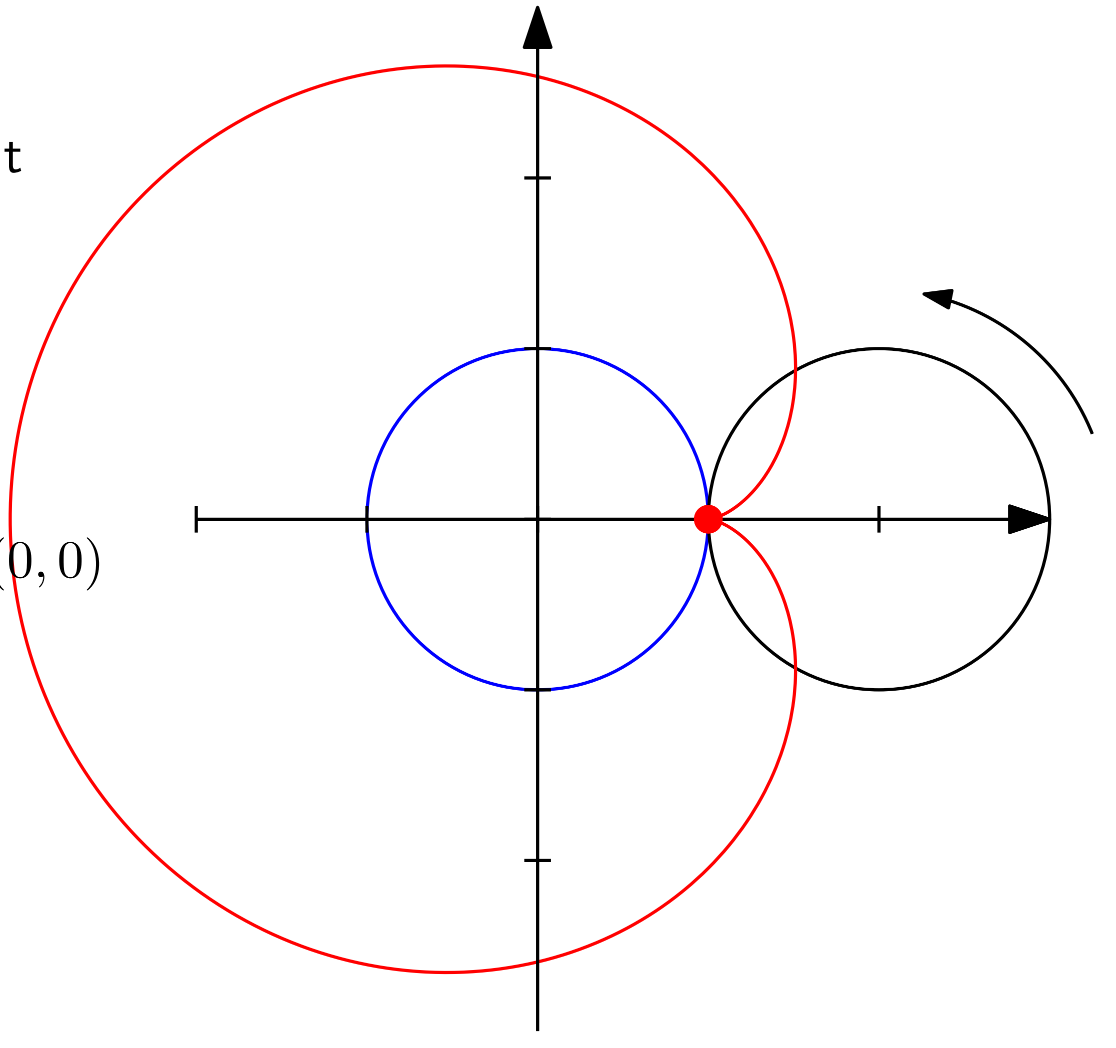

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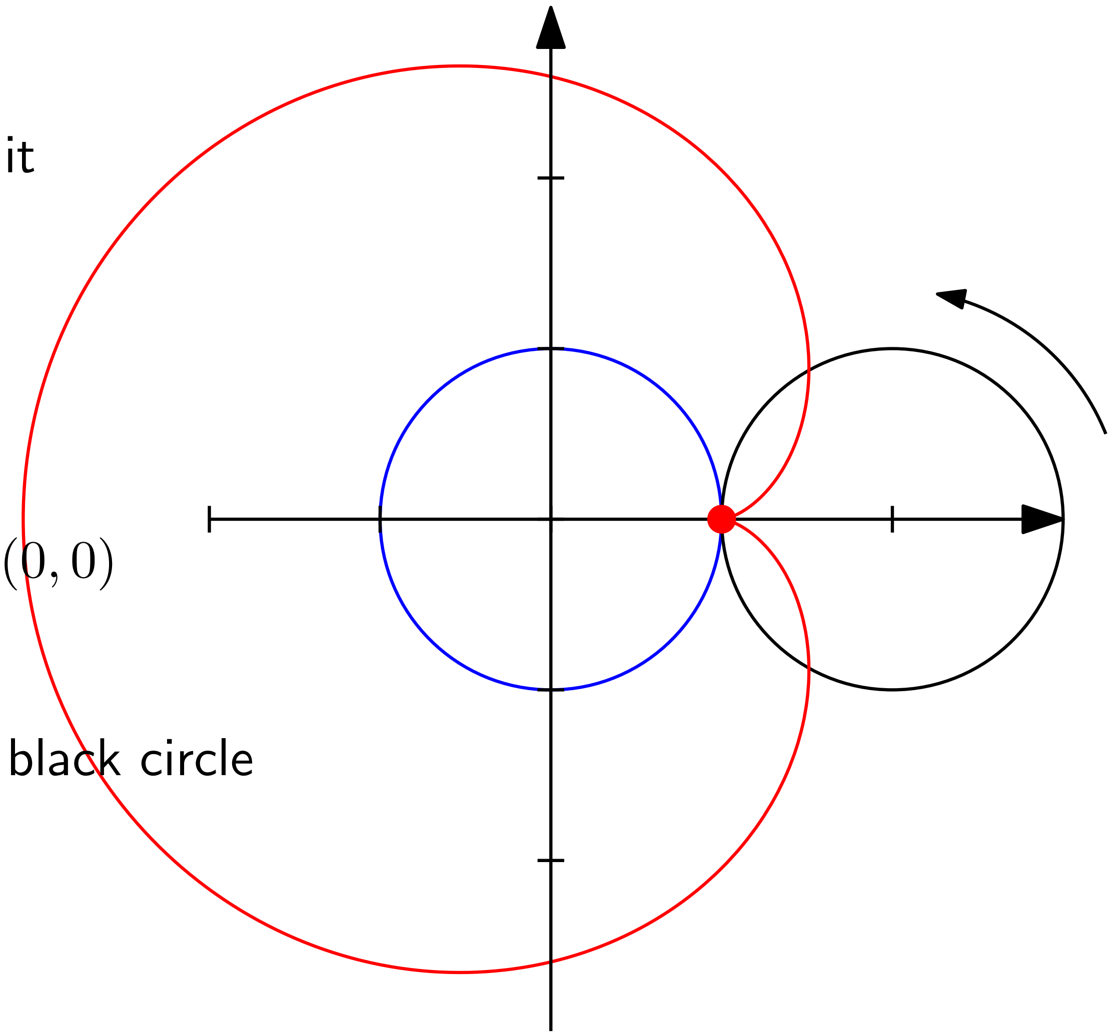
Possible approach:

1) Find parametric equation $C(t)$ of center of black circle



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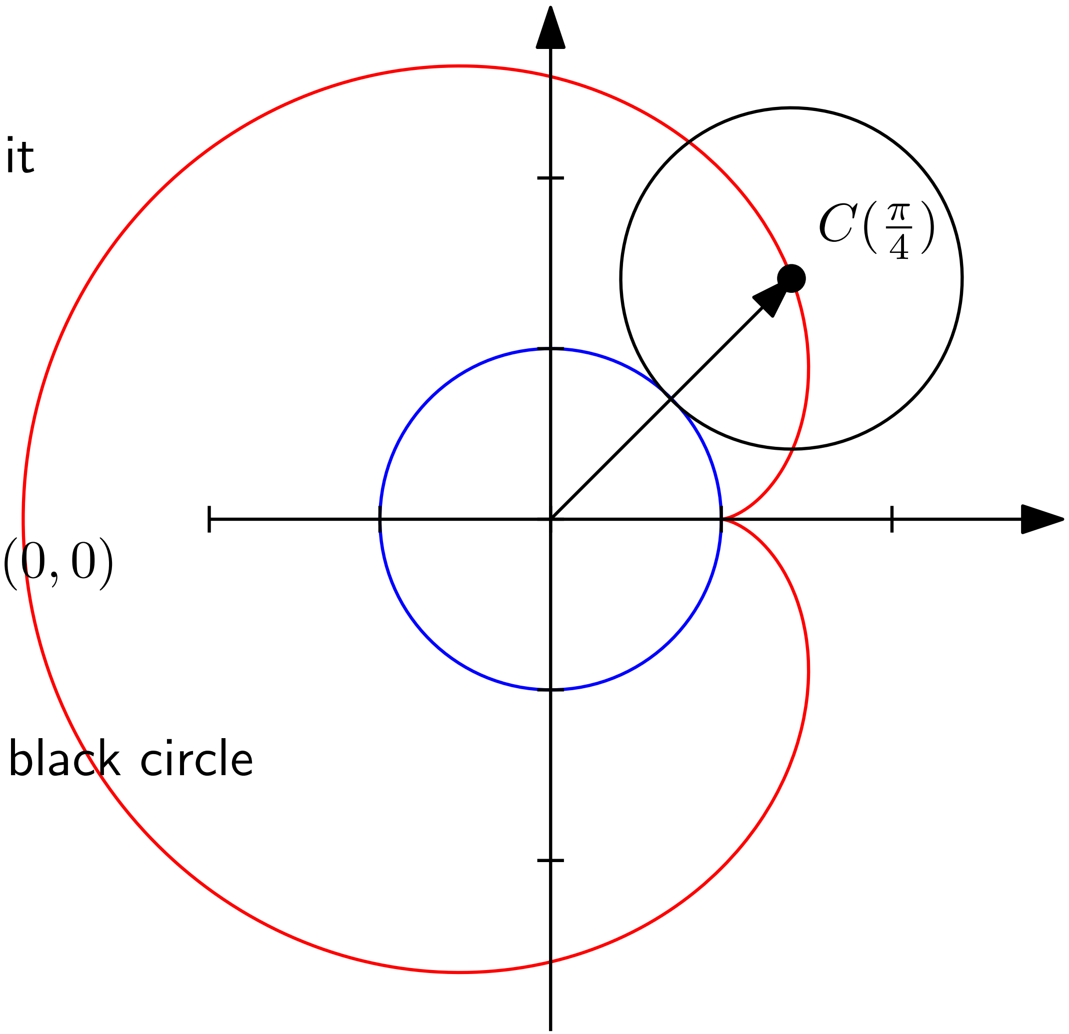

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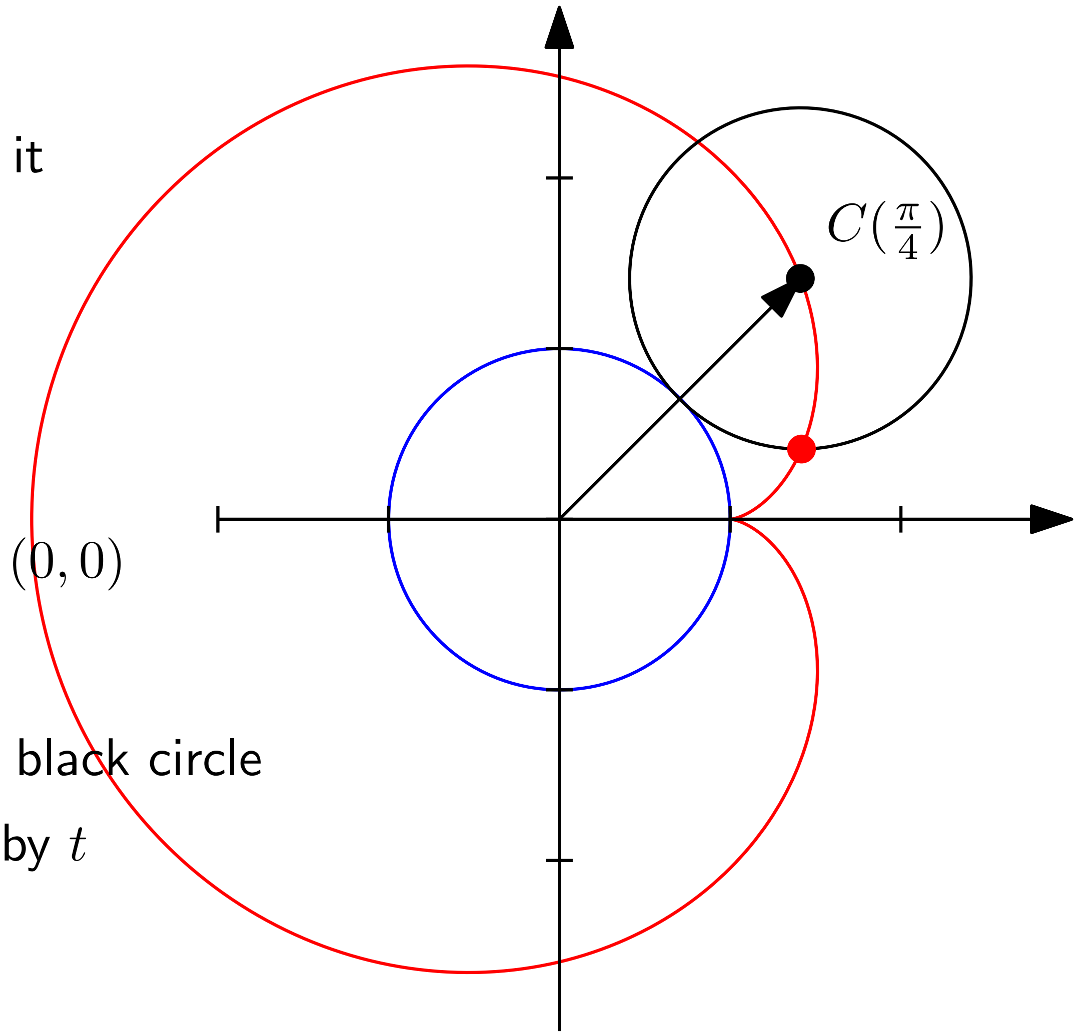

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Possible approach:

1) Find parametric equation $C(t)$ of center of black circle

2) Observe that red point rotates w.r.t. $C(t)$ by t

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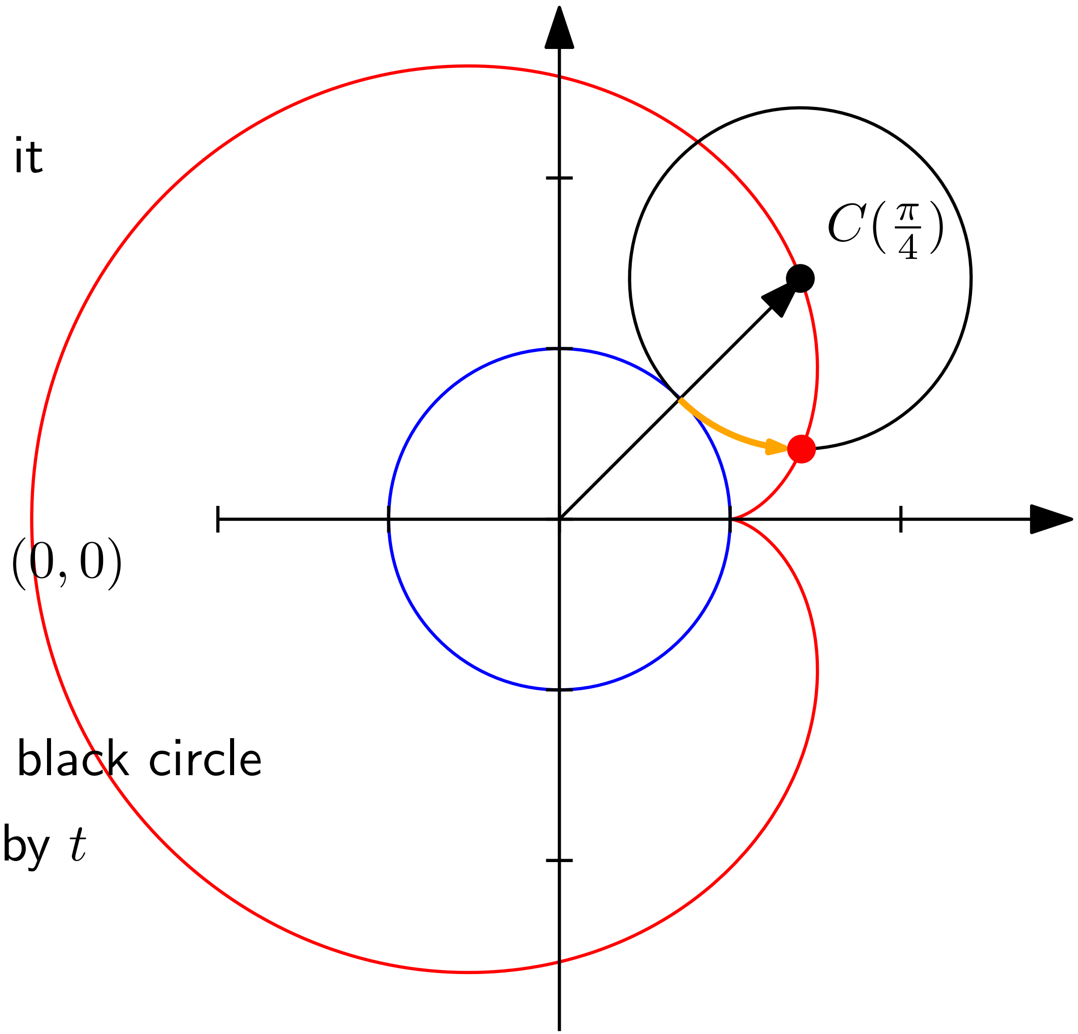

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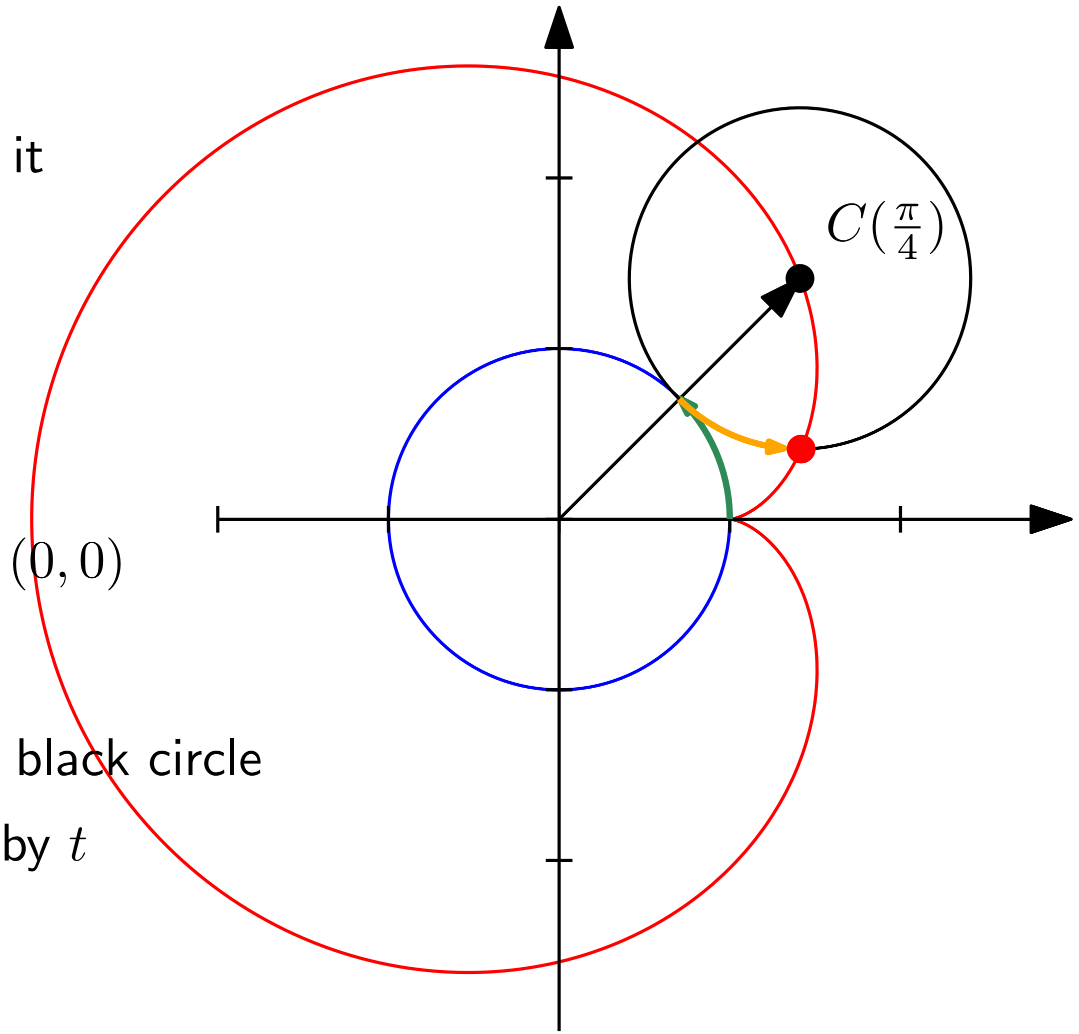

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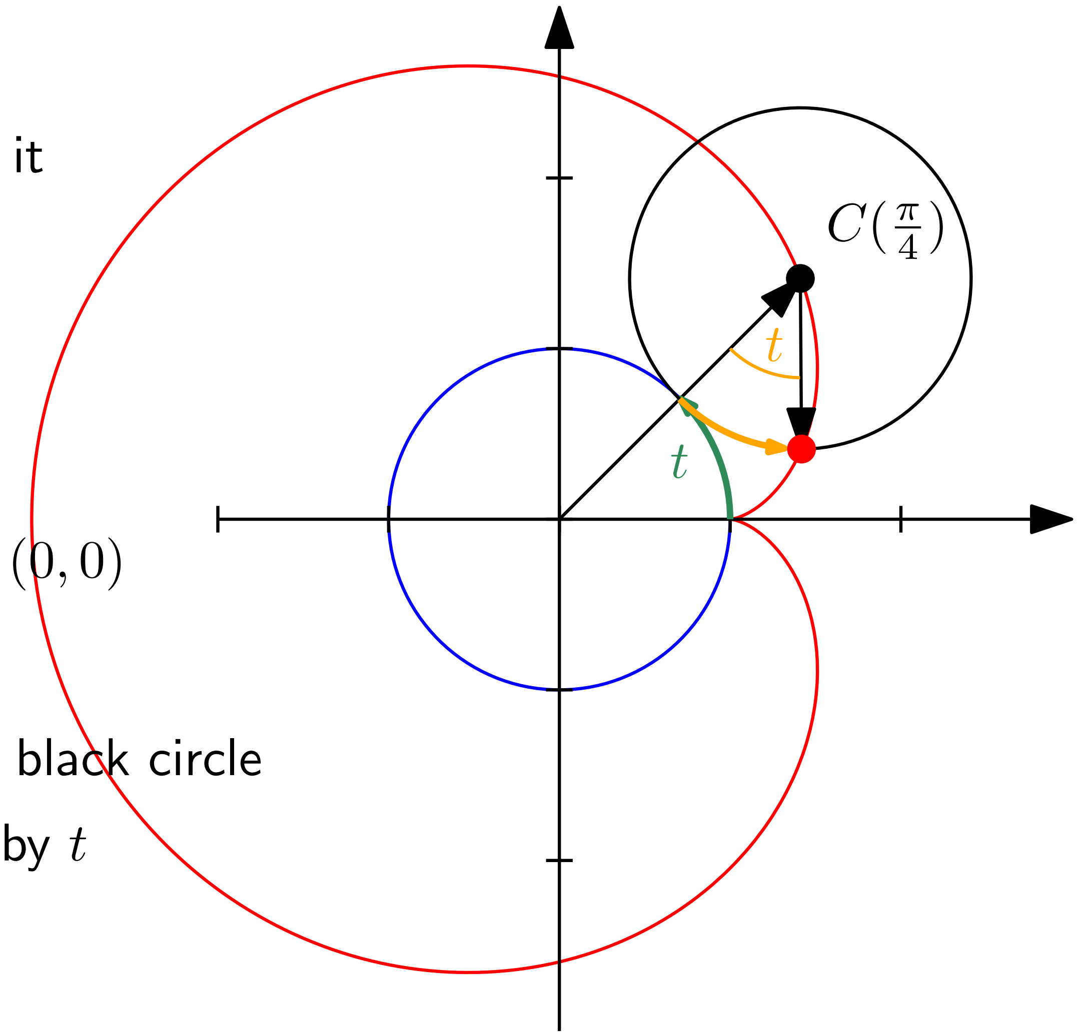

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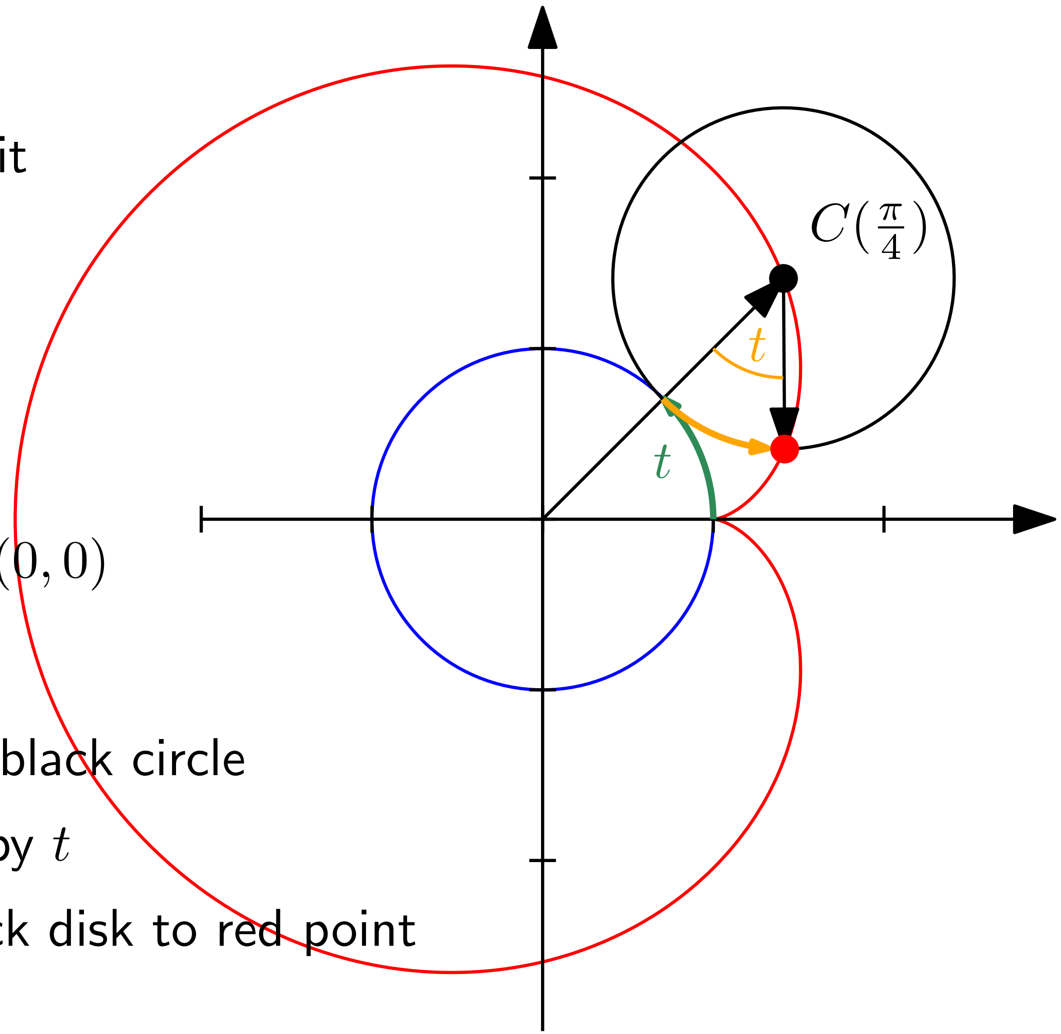
3) Find equation for vector from center of black disk to red point



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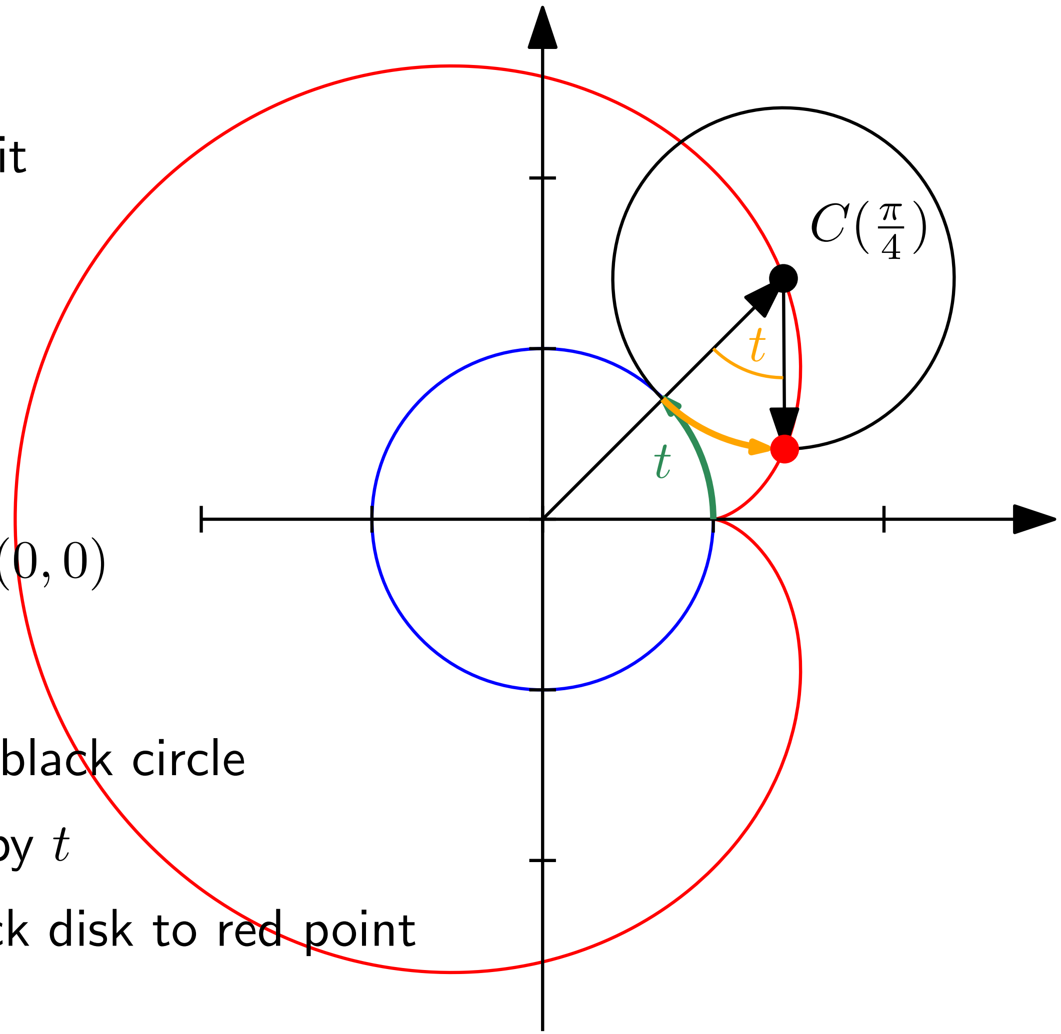

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3) Find equation for vector from center of black disk to red point

4) Compute equation of red point $\gamma(t)$

7) Cardioid: solution

Approach:

1) Find parametric equation $C(t)$ of center of black circle

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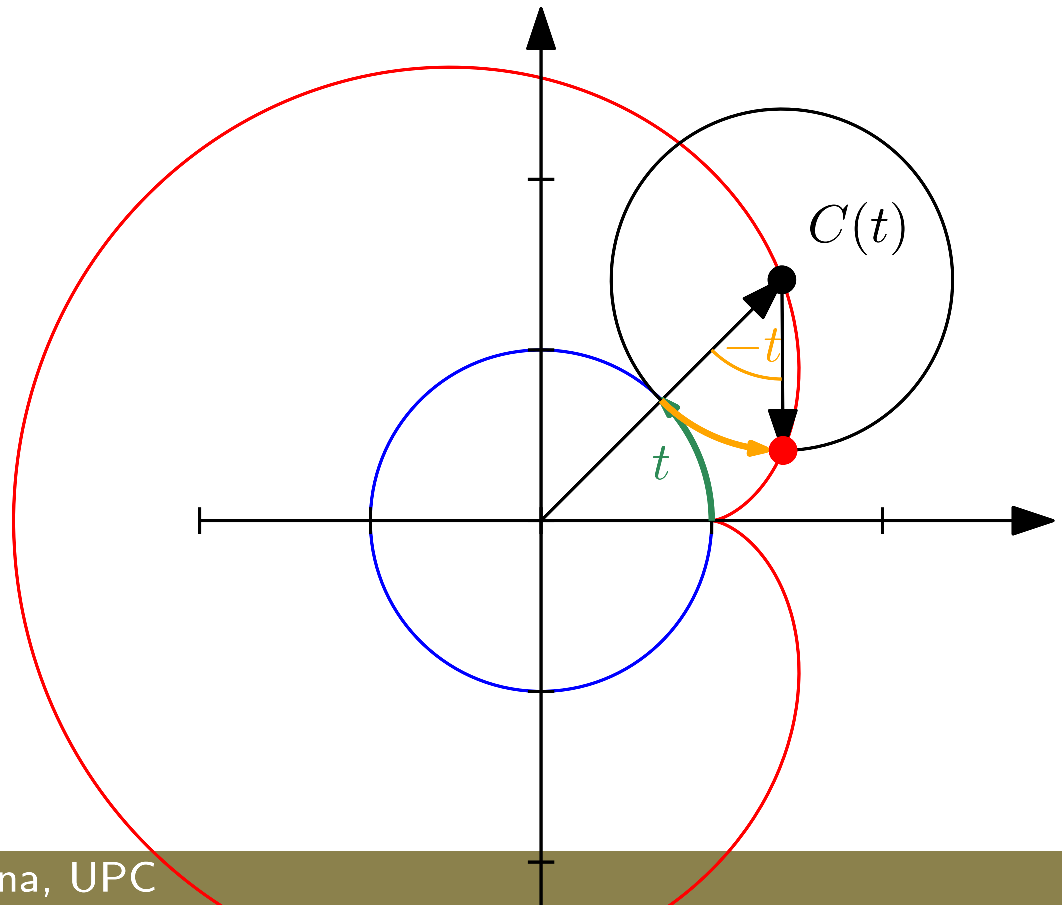

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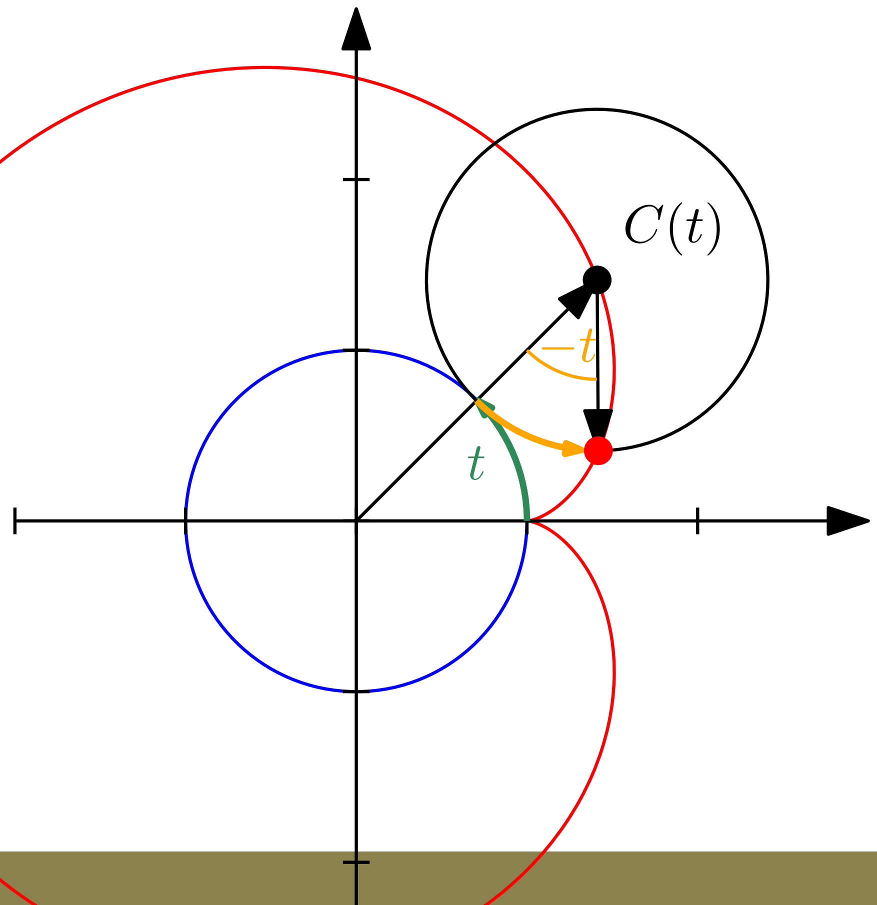

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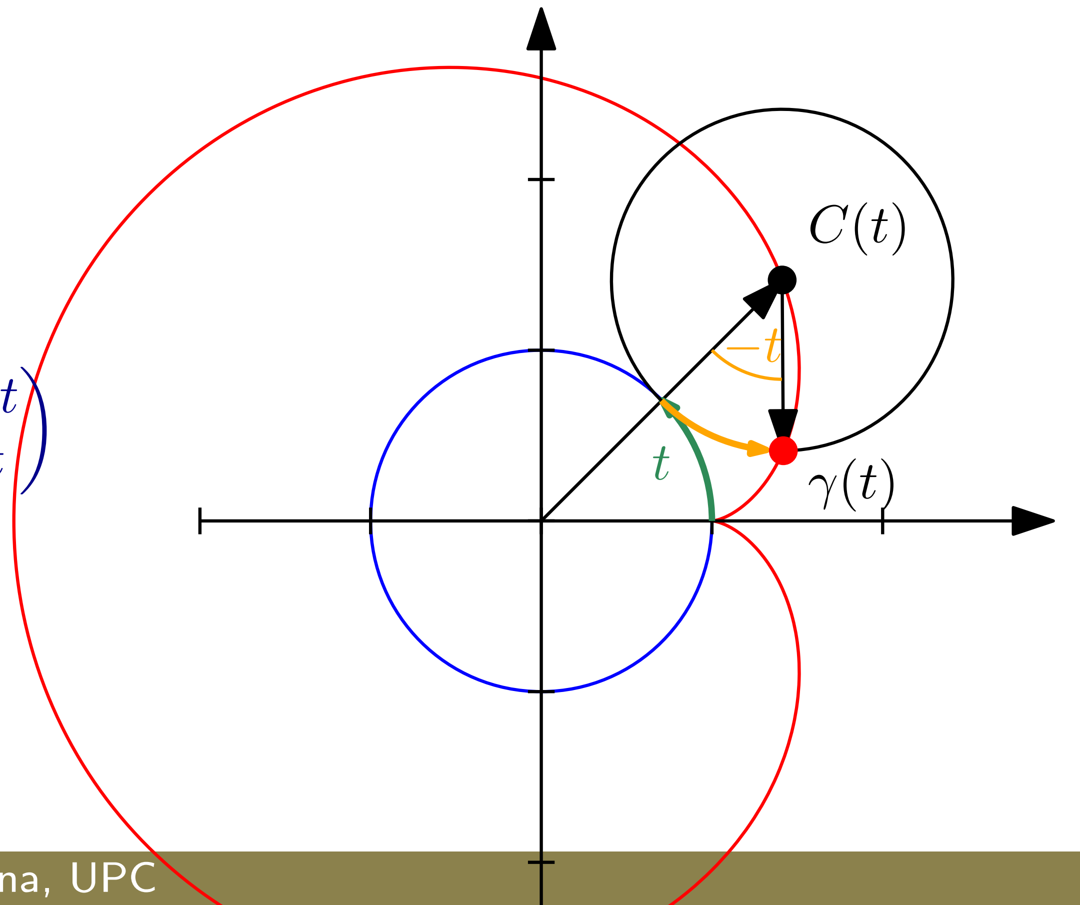
4) Compute equation of red point $\gamma(t)$

3) We want the vector from $C(t)$ to $\gamma(t)$

It is -half the vector from the origin to $C(t)$,
rotated by t, i.e., $-\frac{C(t)}{2}$ rotated by t

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} \sin^2 t - \cos^2 t \\ -2\sin t \cos t \end{pmatrix}$$



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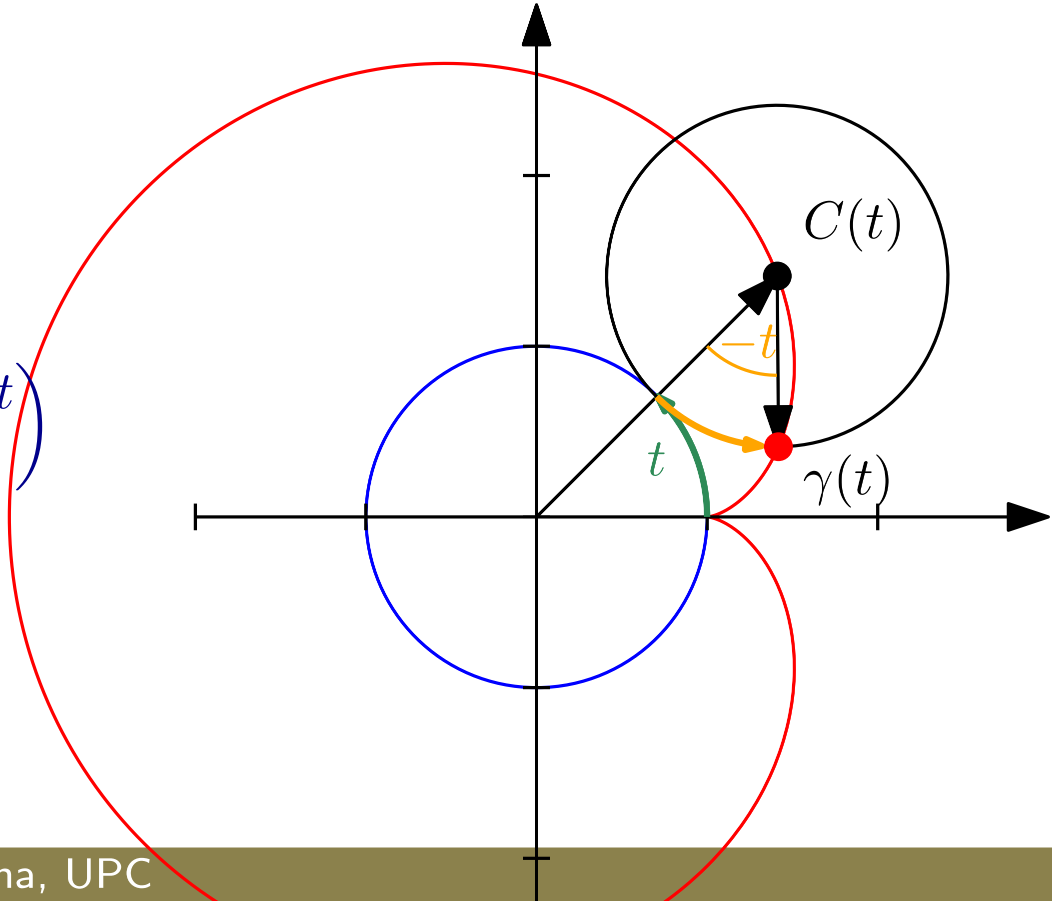

4) $\gamma(t)$ is $C(t)$ plus the vector computed in 3)

$$\gamma(t) = \begin{pmatrix} 2\cos t + \sin^2 t - \cos^2 t \\ 2\sin t - 2\sin t \cos t \end{pmatrix}$$

$$t \in [0, 2\pi)$$

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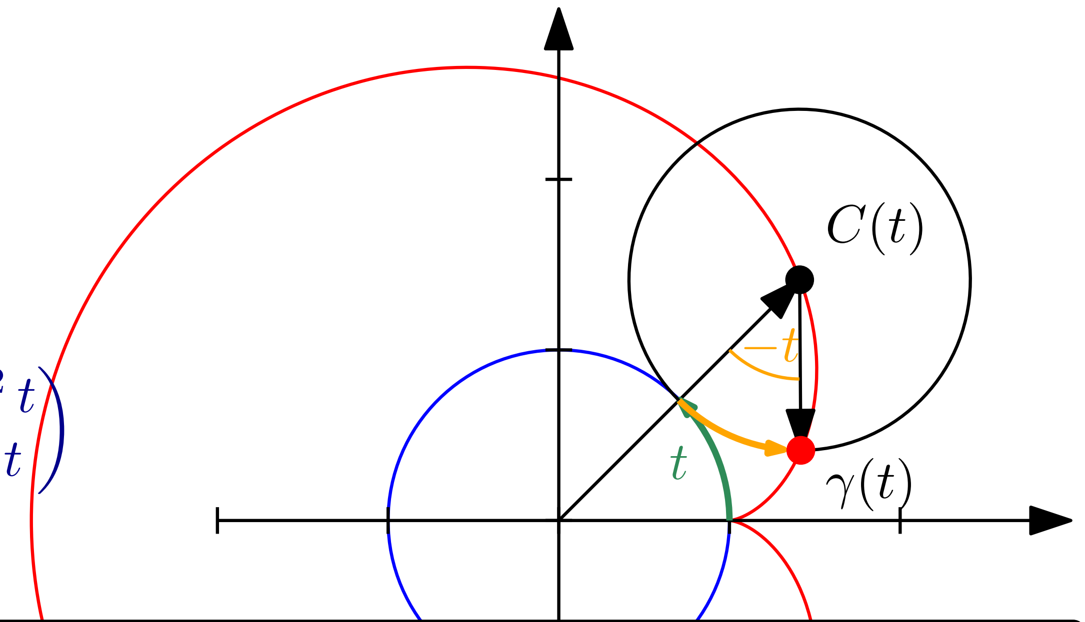

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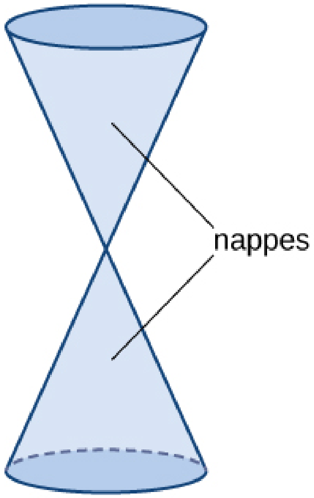
$$\gamma(t) = \begin{pmatrix} 2\cos t + \sin^2 t - \cos^2 t \\ 2\sin t - 2\sin t \cos t \end{pmatrix} \quad t \in [0, 2\pi)$$

Translating by $(-1, 0)$ and using that $\sin^2 t = 1 - \cos^2 t$, we obtain the more usual equation $\gamma(t) = \begin{pmatrix} 2(1-\cos t)\cos t \\ 2(1-\cos t)\sin t \end{pmatrix}$

Conics: ellipses, hyperbolas and parabolas

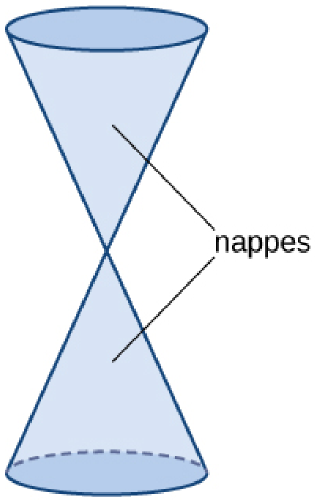
Curves that result from intersecting a cone with a plane



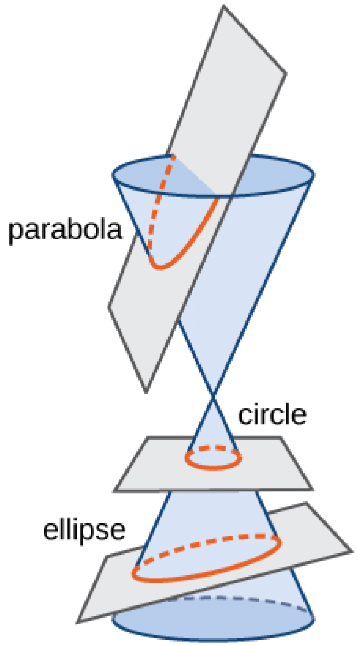
A cone

Conics: ellipses, hyperbolas and parabolas

Curves that result from intersecting a cone with a plane



A cone



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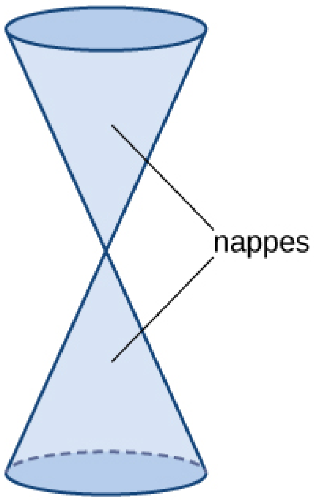
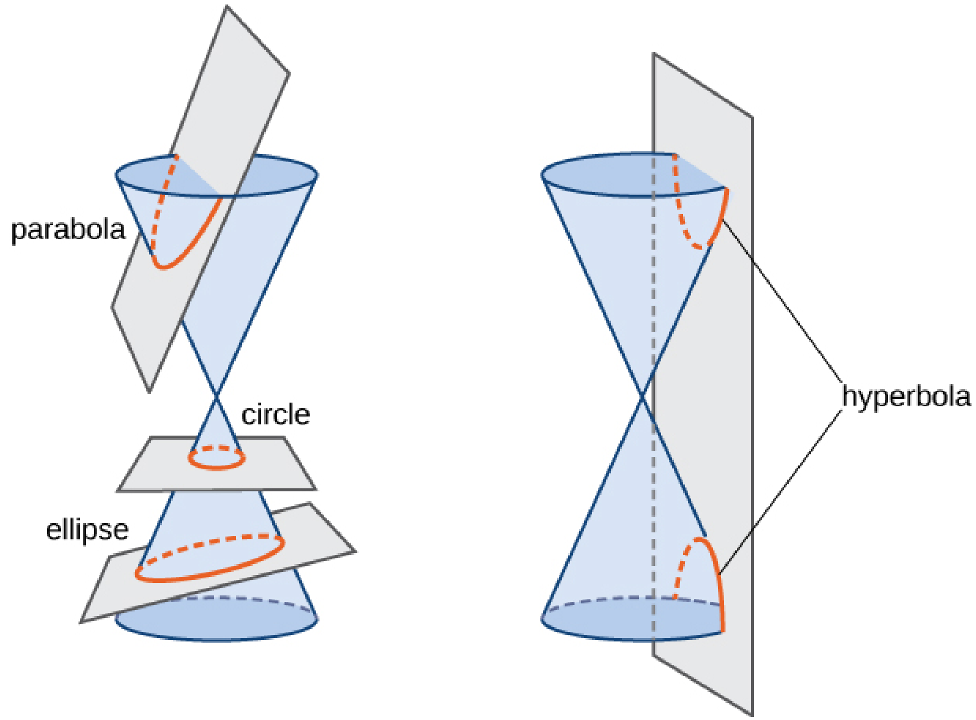




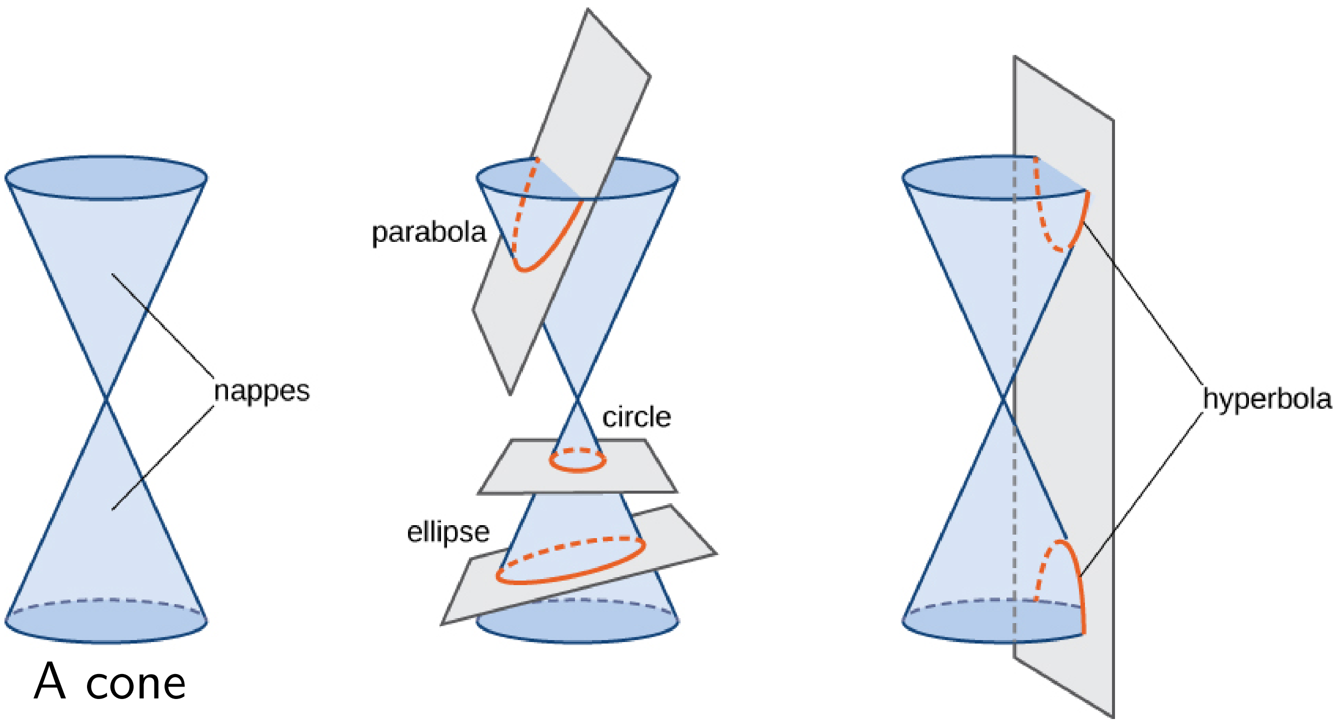
Conics: ellipses, hyperbolas and parabolas

Curves that result from intersecting a cone with a plane

- They appear all around us (planet trajectories are ellipses, antennas use parabolic shapes, etc.)

- They are easy to deal with (plane curves given by low-degree polynomial)

- They have several useful properties (e.g., reflection, focal, and excentricity properties)



Parabolas

Set of points in the plane that are at equal distance from a fixed line (directrix) and a fixed point F (focus)

Image source: Wikipedia

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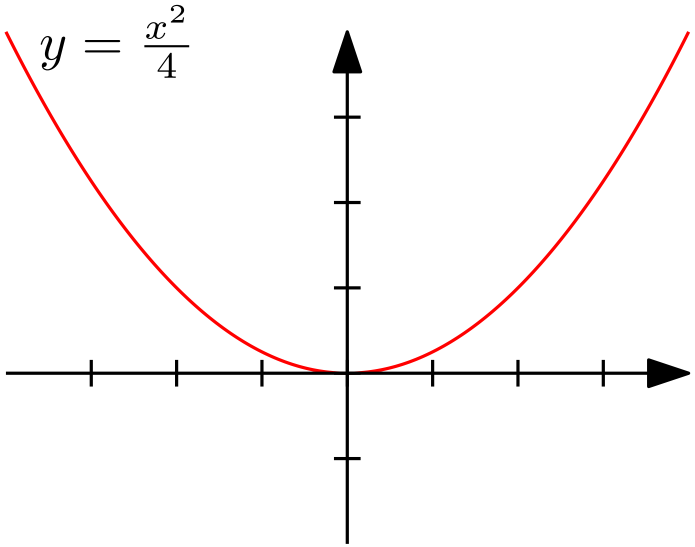

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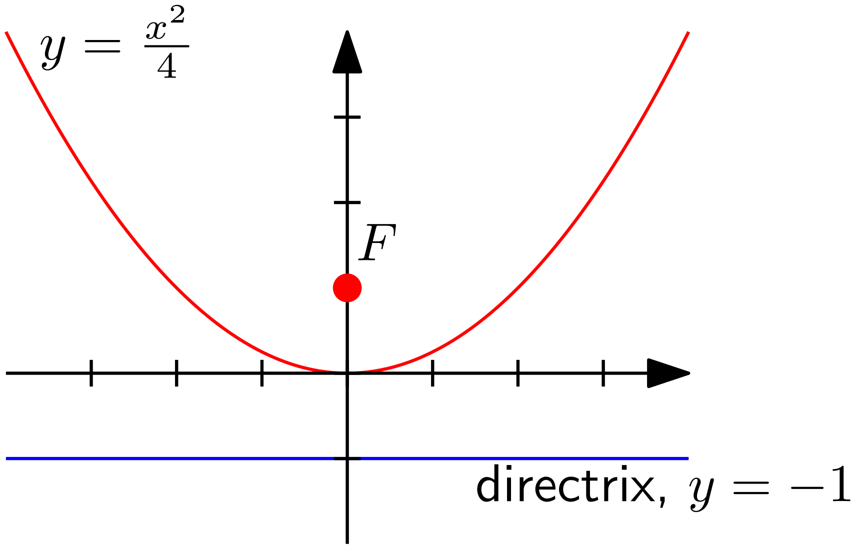

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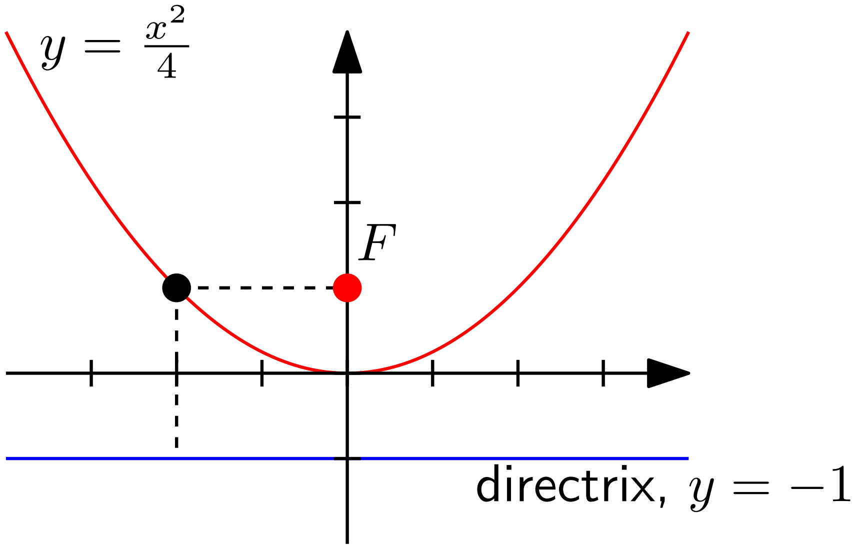

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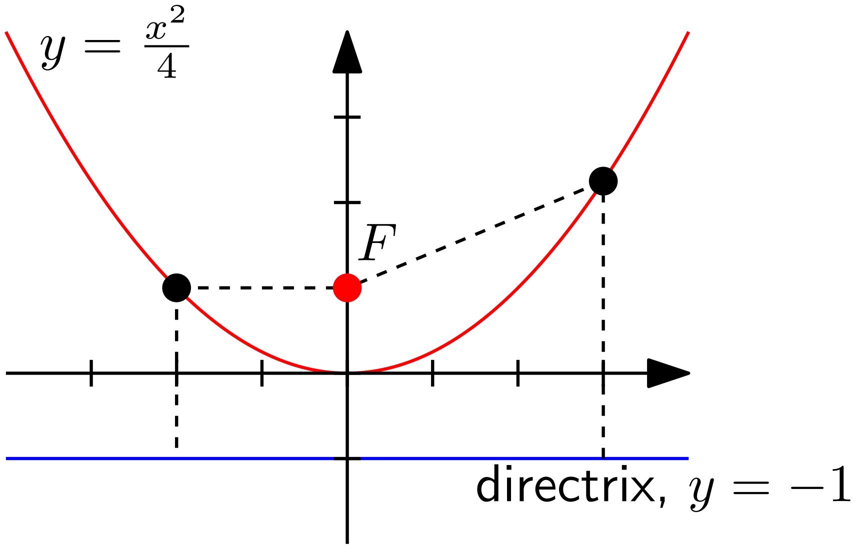

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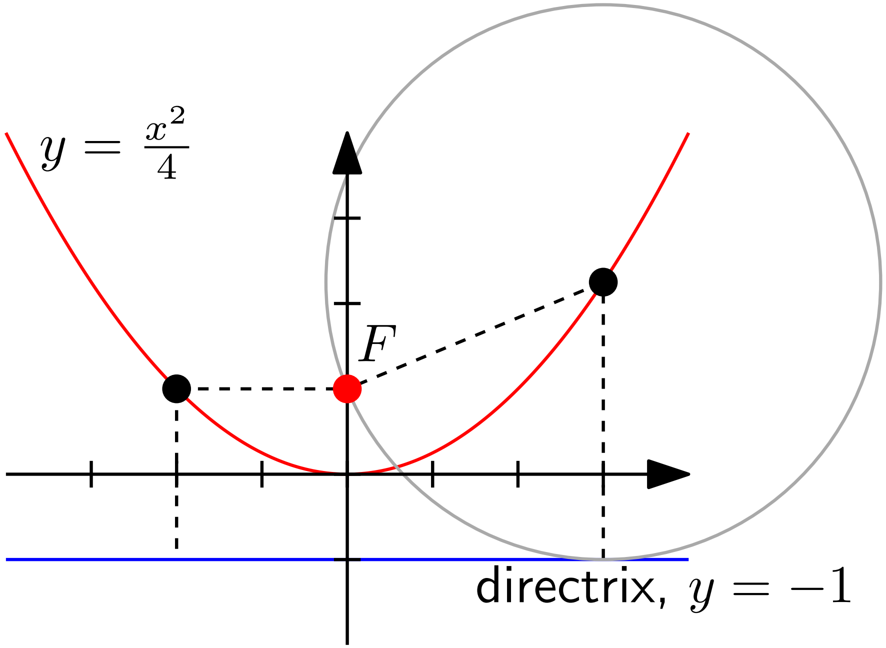

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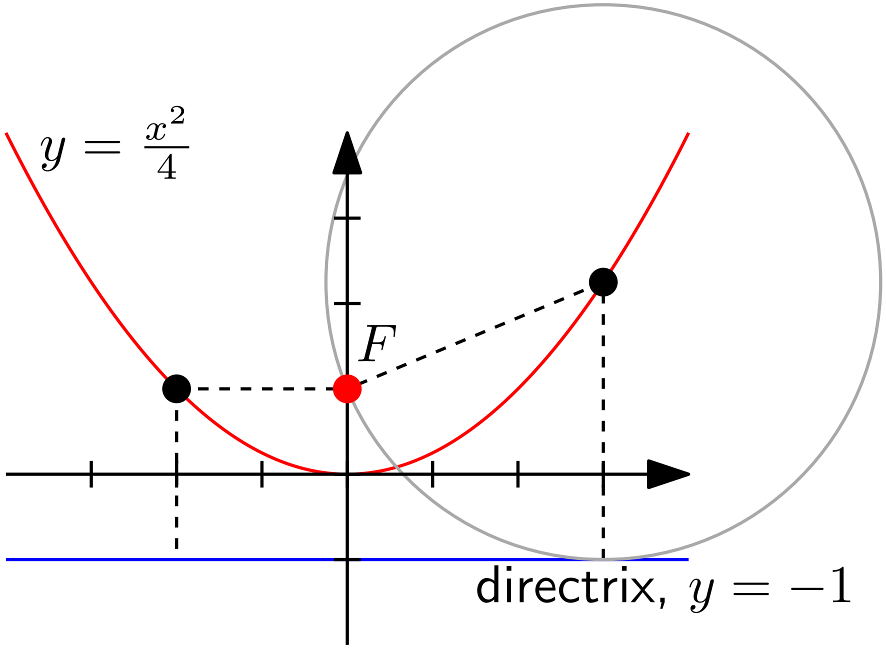


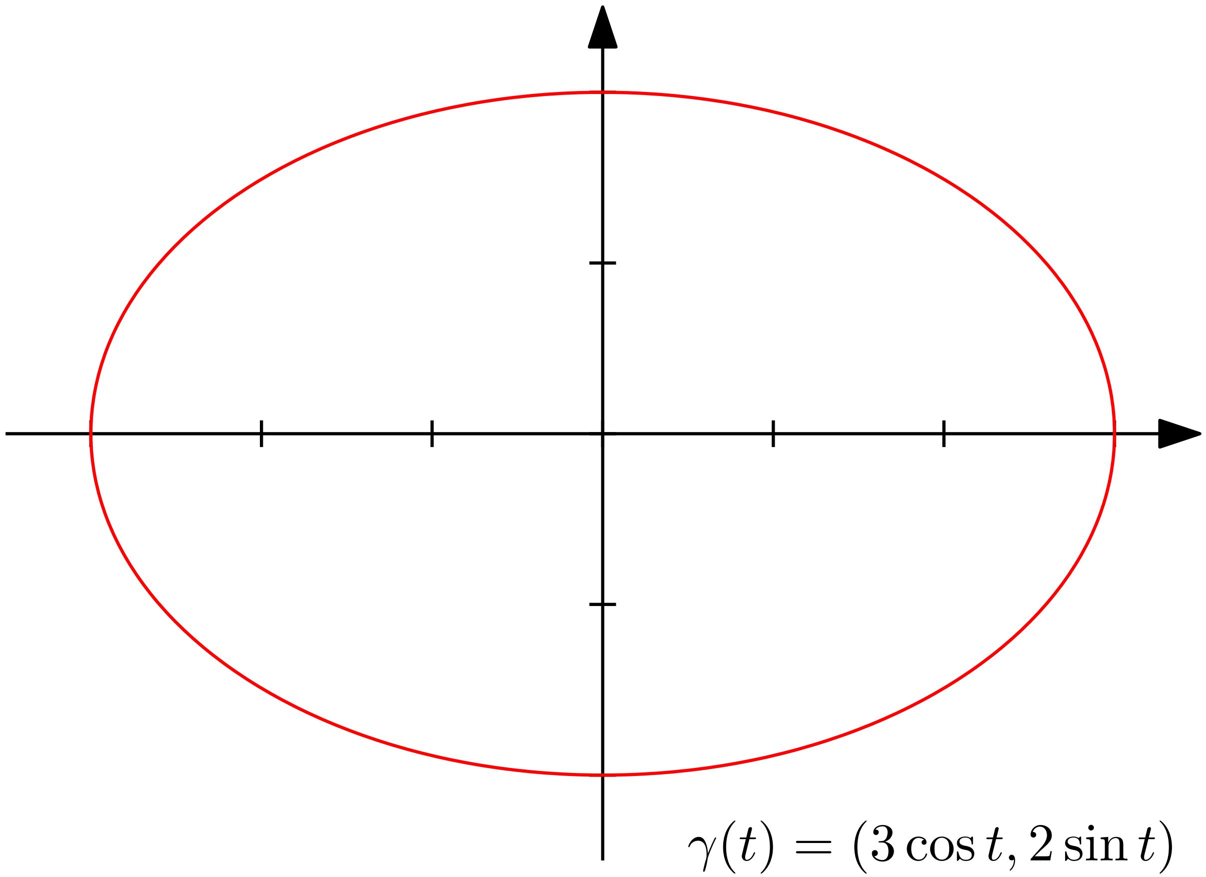
When the focus is $(0, p)$ and the directrix $y = -p$:

Explicit equation: $y = \frac{x^2}{4p}$

Parametric equation: $(t, \frac{t^2}{4p})$, for $t \in \mathbb{R}$

Ellipses

Set of points in the plane the **sum** of whose distances from two fixed *focus points* F_1 and F_2 is constant.



$$\gamma(t) = (3\cos t, 2\sin t)$$
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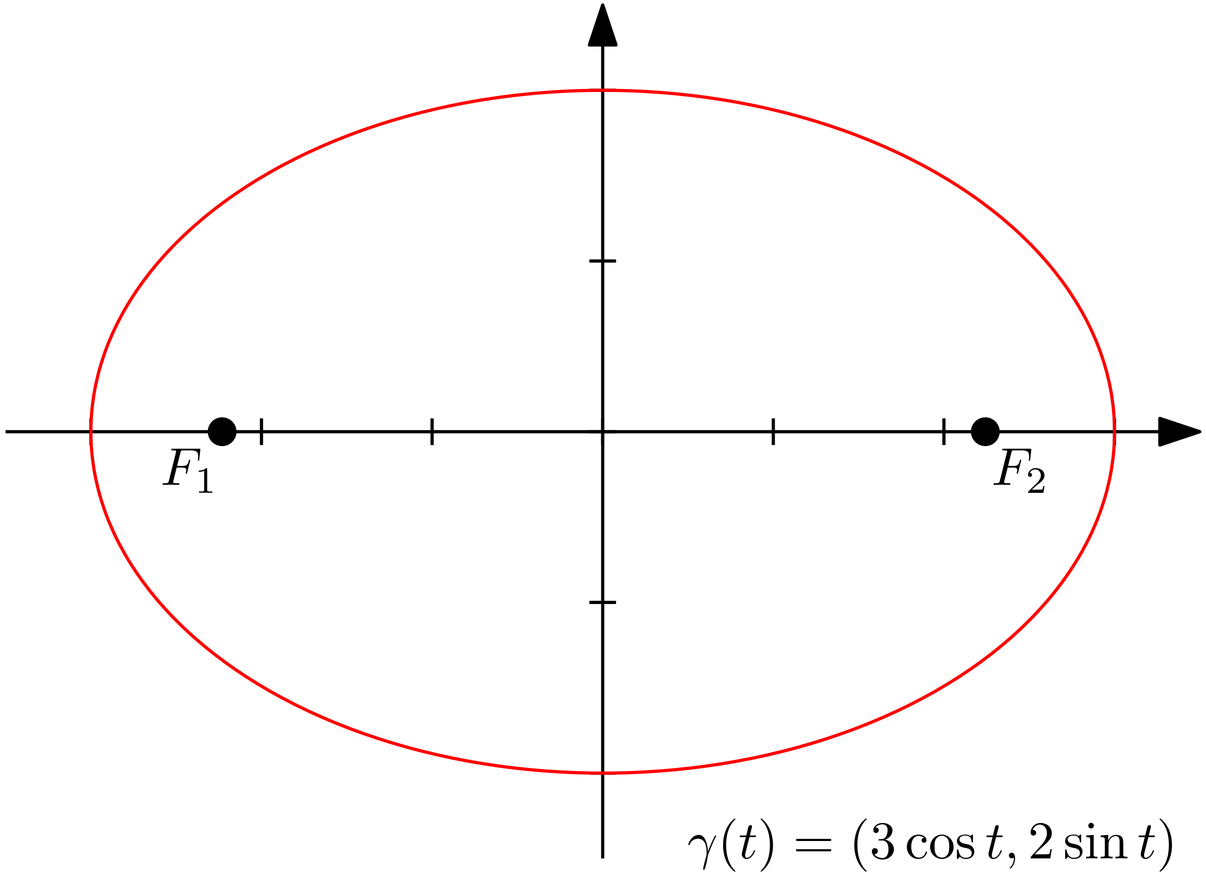

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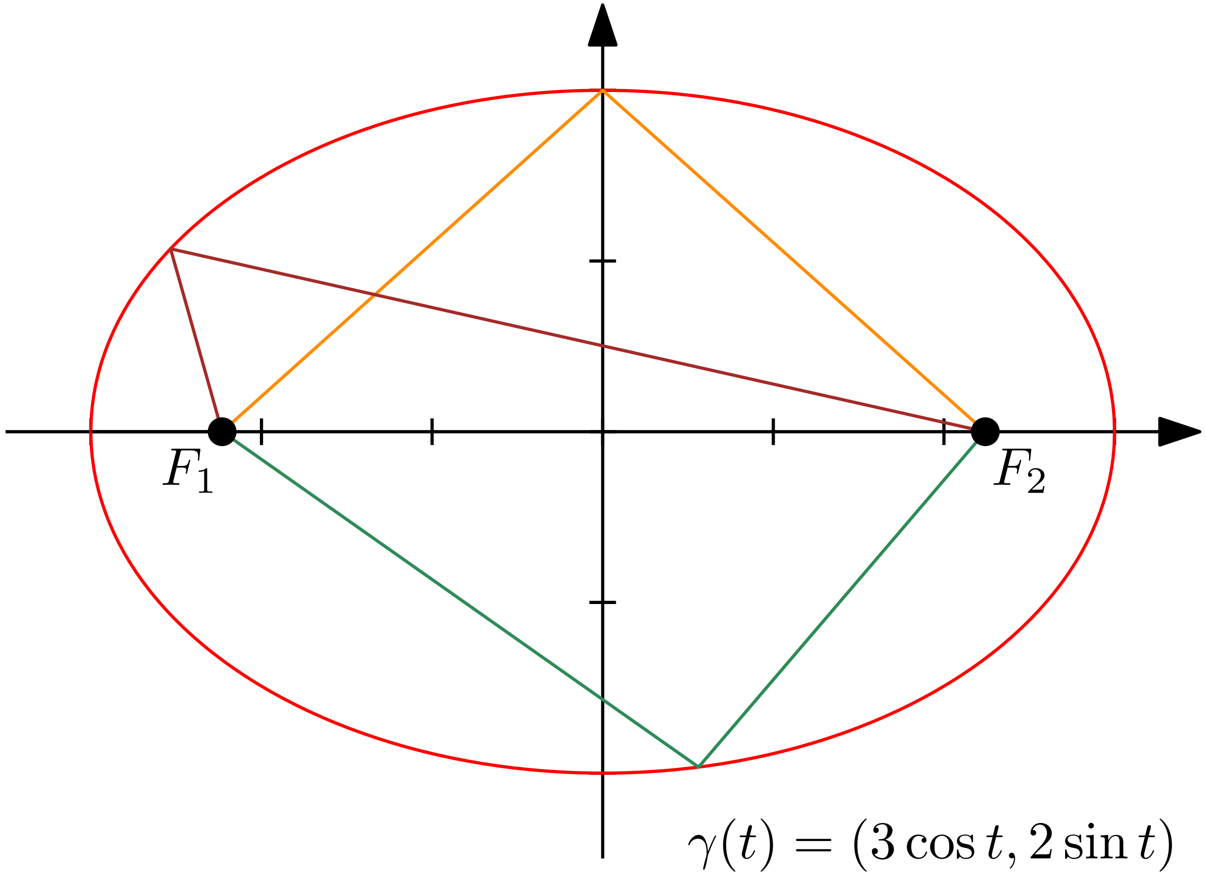
The three (orange, brown, green) pairs of segments have the same total length (6)



$$\gamma(t) = (3\cos t, 2\sin t)$$
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Ellipses

Set of points in the plane the **sum** of whose distances from two fixed *focus points* F_1 and F_2 is constant.

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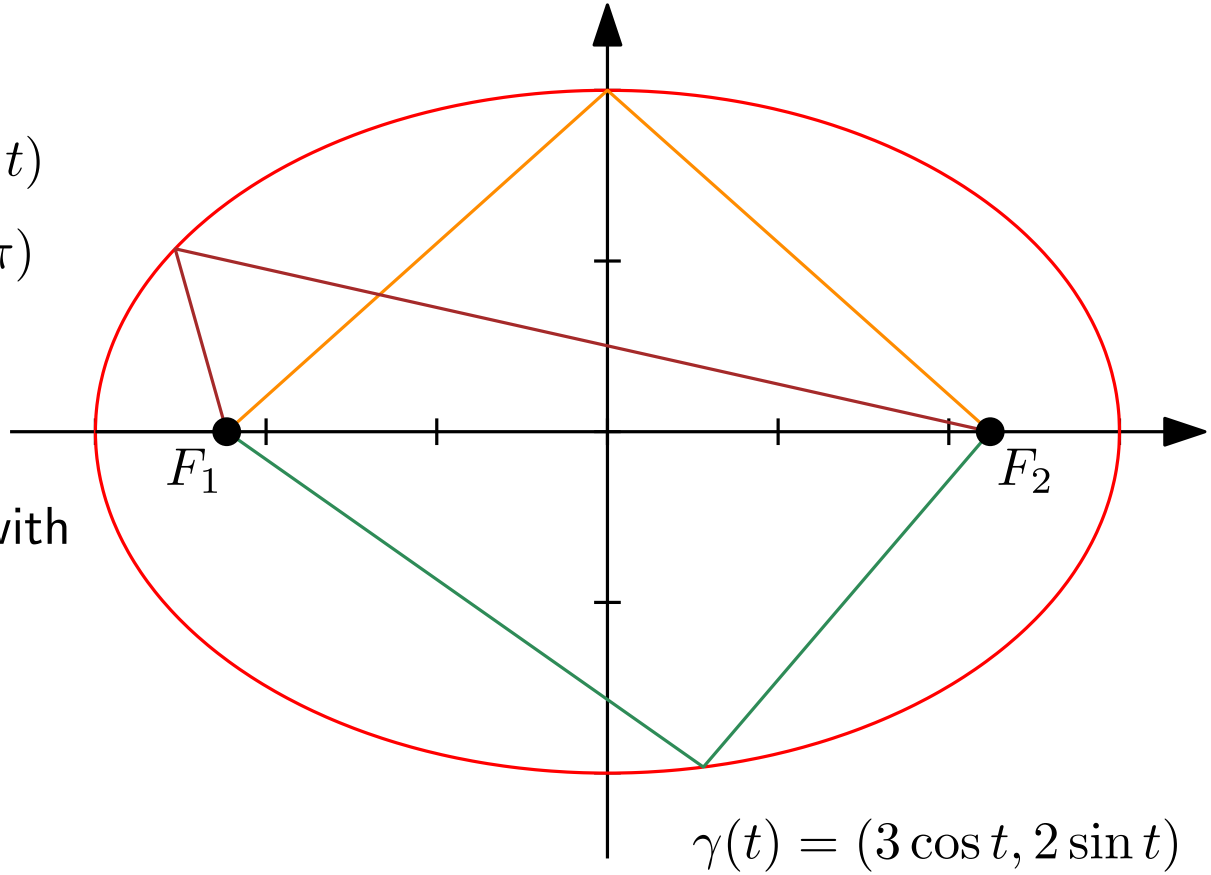

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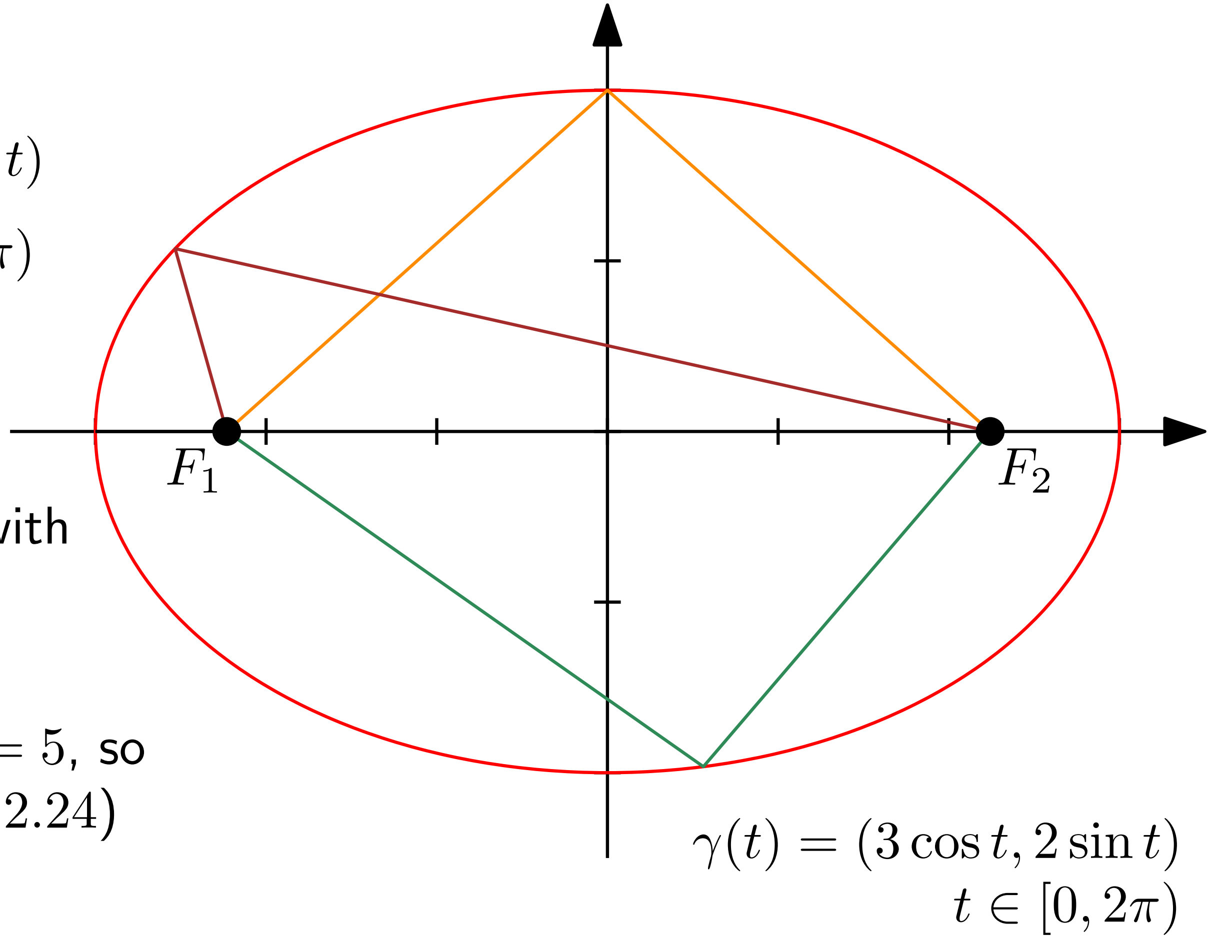

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In the example, $a = 3, b = 2$, $c^2 = 9 - 4 = 5$, so the foci are $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$ ($\sqrt{5} \approx 2.24$)

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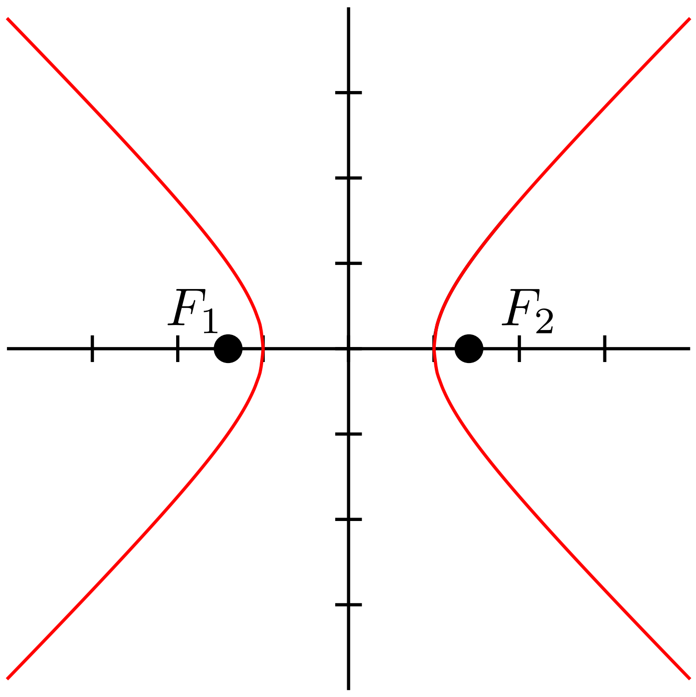

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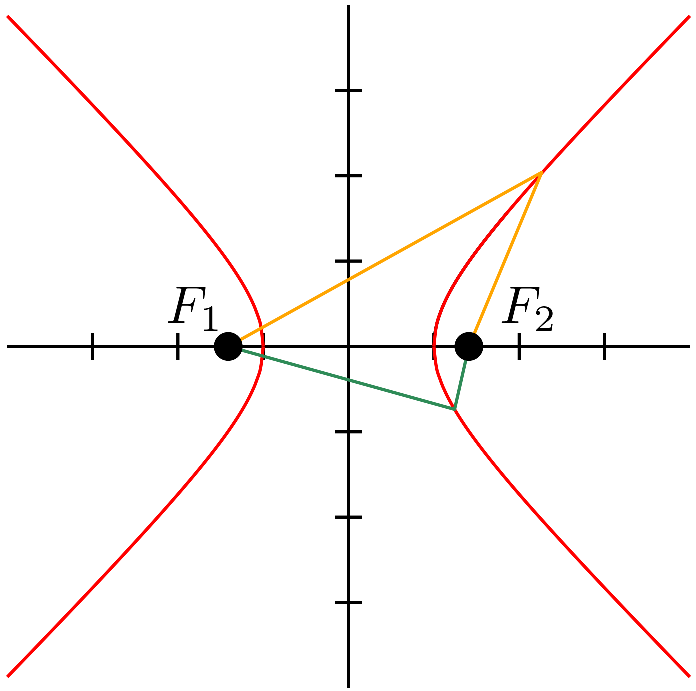

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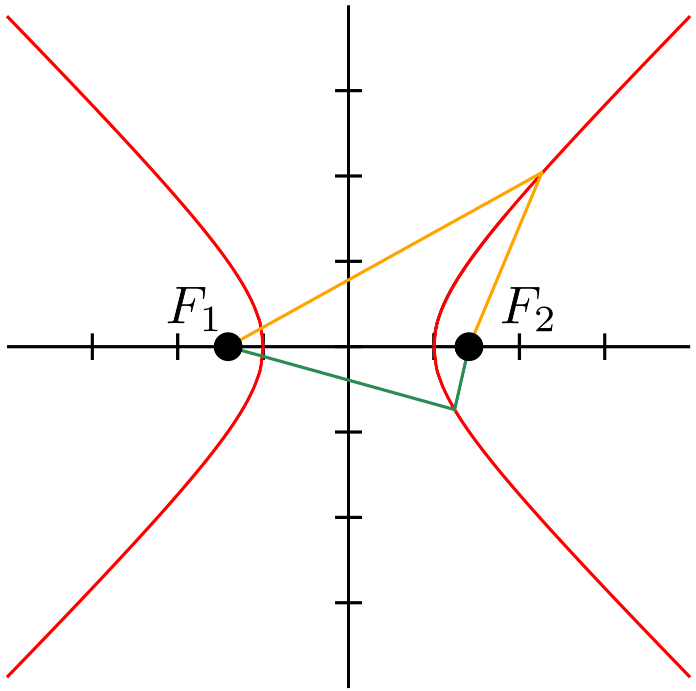

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Equivalently:

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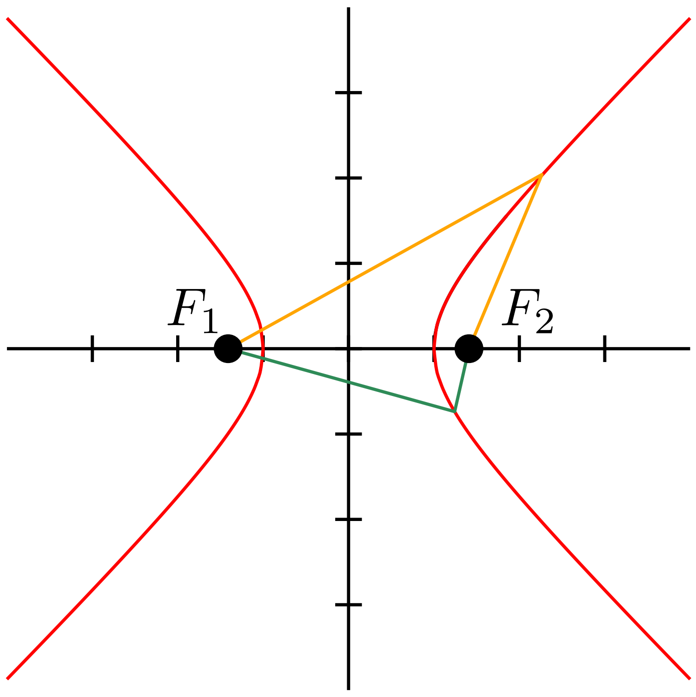
$(\pm a \cosh t, b \sinh t)$ for $t \in \mathbb{R}$



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General formula

Conics can be defined in a unified way:

A conic is the locus of all the points P that satisfy the following:

- The distance from P to a fixed point F (focus) is *proportional* to the distance from P to a fixed line D (directrix)

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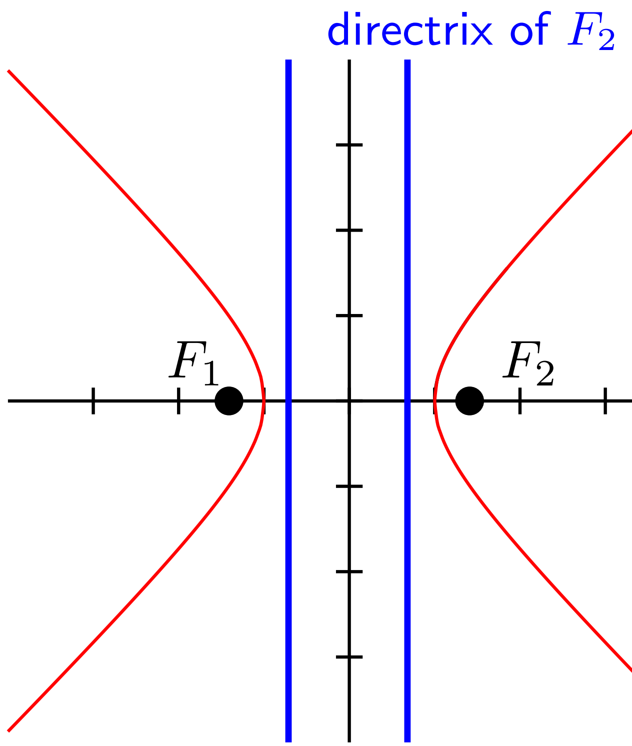

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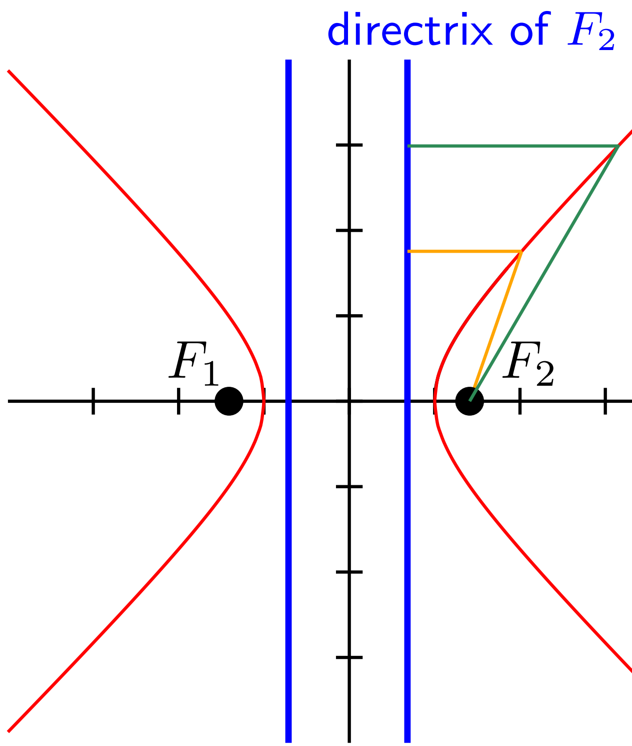

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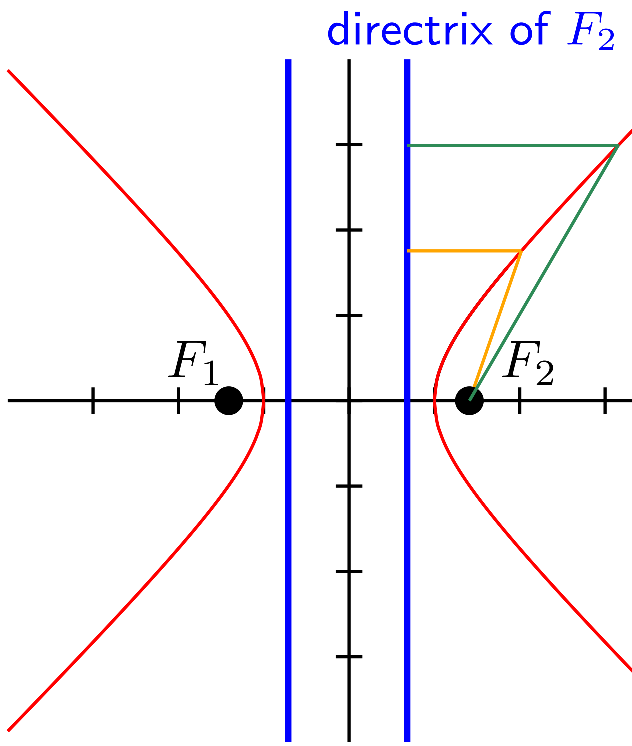

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For the particular case where D is the y-axis and $P = (k, 0)$, this gives the general formula:

$\frac{\sqrt{(x-k)^2+y^2}}{|x|} = \lambda$ or equiv. $(1 - \lambda^2)x^2 - 2kx + y^2 + k^2 = 0$

See [Salomon, pg 364] for a classification of conics

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Properties that depend only on the *shape* of the curve, and not on the coordinate system

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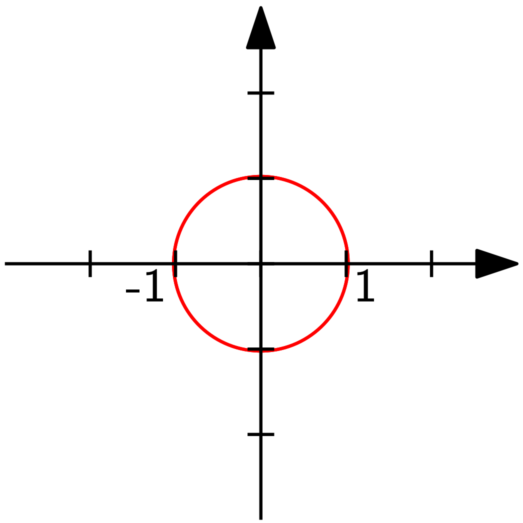
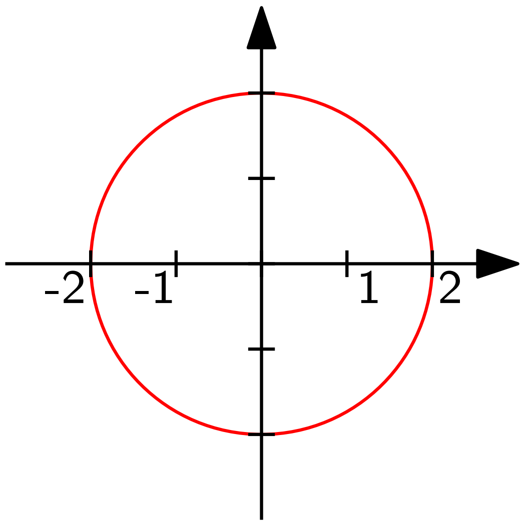
- The fact that rectangles have four equal angles, is intrinsec
- Area and length of a curve, however, are not (they are *extrinsic* properties)



Perimeter: $2\pi r = 2\pi$



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Tangent vector

Intrinsec property when normalized (i.e., as unit tangent vector)

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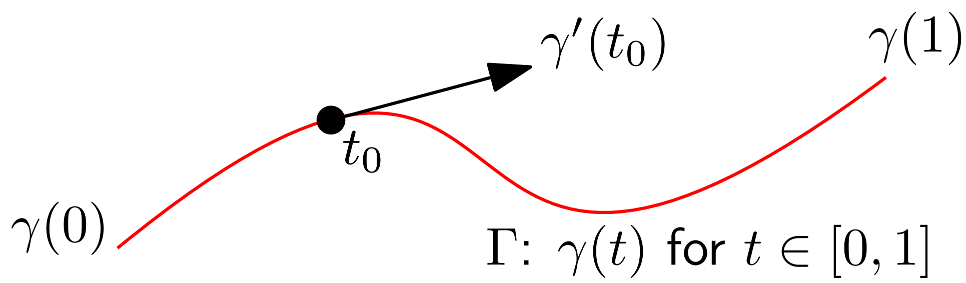
Let Γ be a curve parametrized by $\gamma : [a, b] \to \mathbb{R}^d$, such that:

- γ is differentiable in t
- $\gamma'(t) \neq 0$

$\gamma'(t)$ is the *tangent vector* of Γ at point $\gamma(t)$



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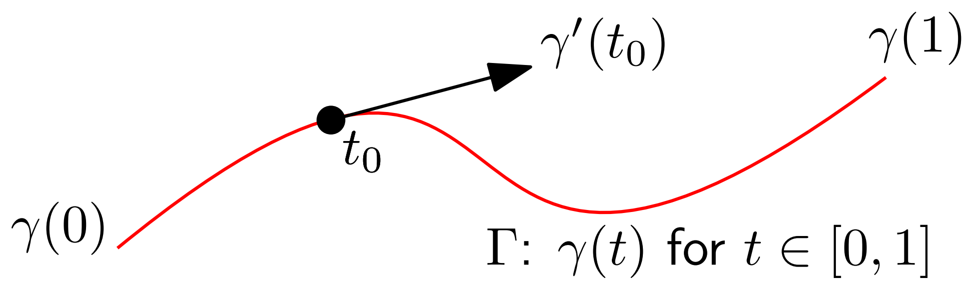
In other words, if Γ is parametrized as $(x(t), y(t), z(t))$, for $t \in [a, b]$, then the tangent vector of Γ at $t_0 \in [a, b]$ is $(x'(t_0), y'(t_0), z'(t_0))$, as long as it exists and does not vanish

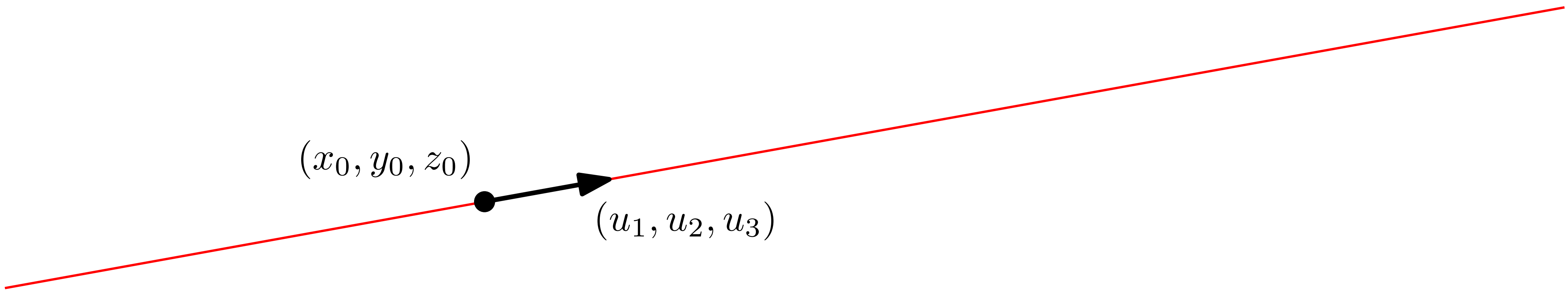


Tangent vector: examples

1) Line

In 3D, with supporting point (x_0, y_0, z_0) and direction vector (u_1, u_2, u_3)

$$\gamma(t) = (x_0 + tu_1, y_0 + tu_2, z_0 + tu_3) \qquad \text{for } t \in \mathbb{R}$$



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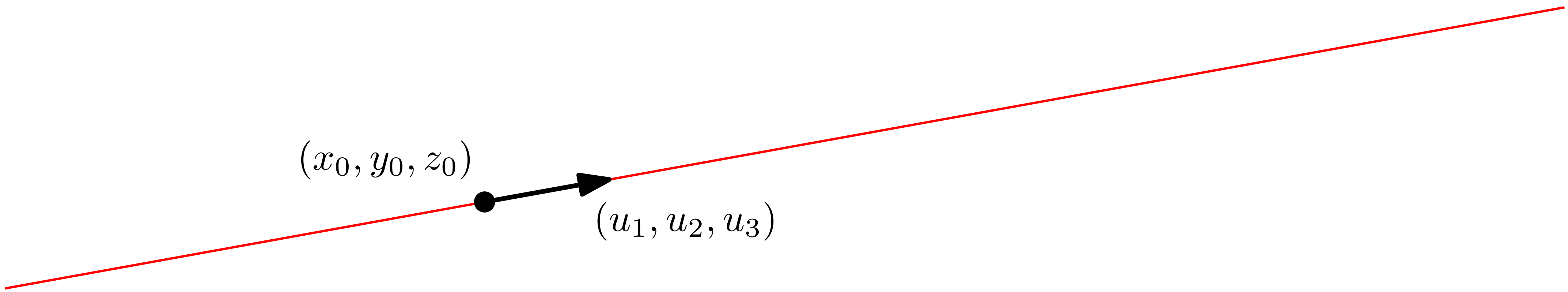
$$\gamma'(t) = (u_1, u_2, u_3)$$ Constant! And equal to direction vector (as expected)



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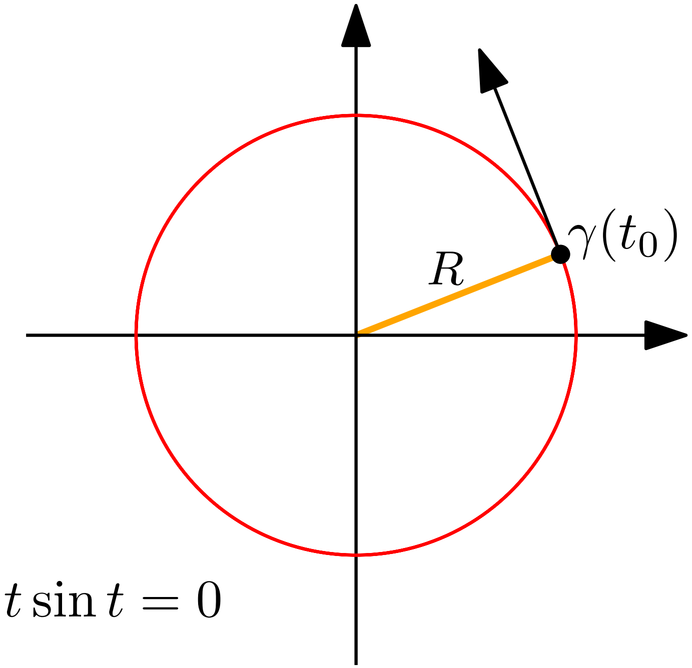

2) Circle

$\gamma(t) = (R\cos t, R\sin t)$ for $t \in [0, 2\pi)$

Tangent vector

$$\gamma'(t) = (-R\sin t, R\cos t)$$

Observe that $\gamma(t) \perp \gamma'(t)$: $\quad \gamma(t)\gamma'(t) = -R^2 \cos t \sin t + R^2 \cos t \sin t = 0$

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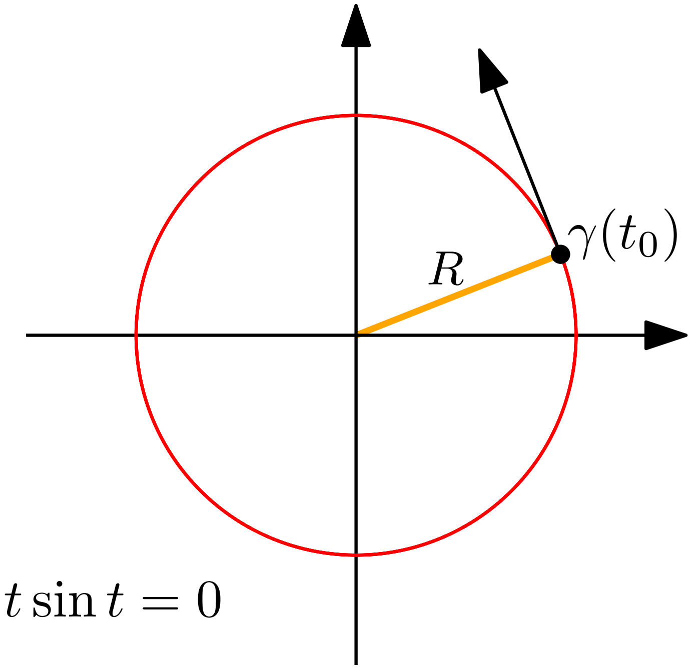

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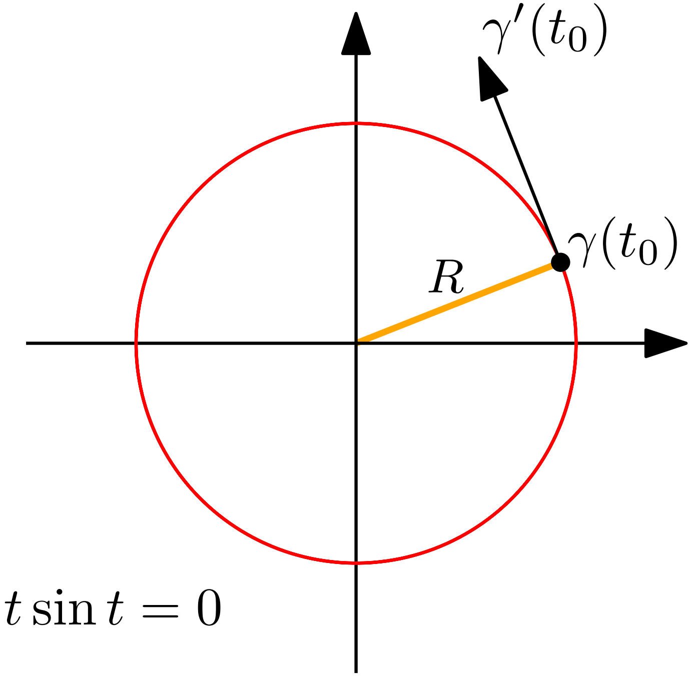

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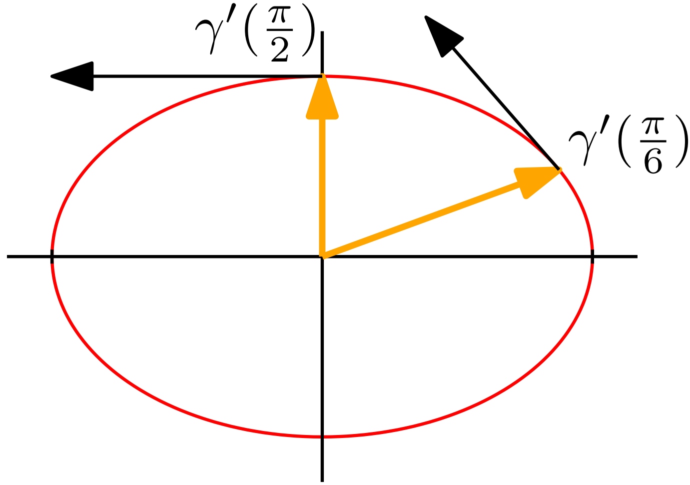

Observe that $\gamma(t)$ and $\gamma'(t)$ are not always perpendicular

Tangent vector: examples

4) Archimedean spiral

$\gamma(t) = (at\cos t, bt\sin t)$ for $t \in [0, +\infty)$

Tangent vector

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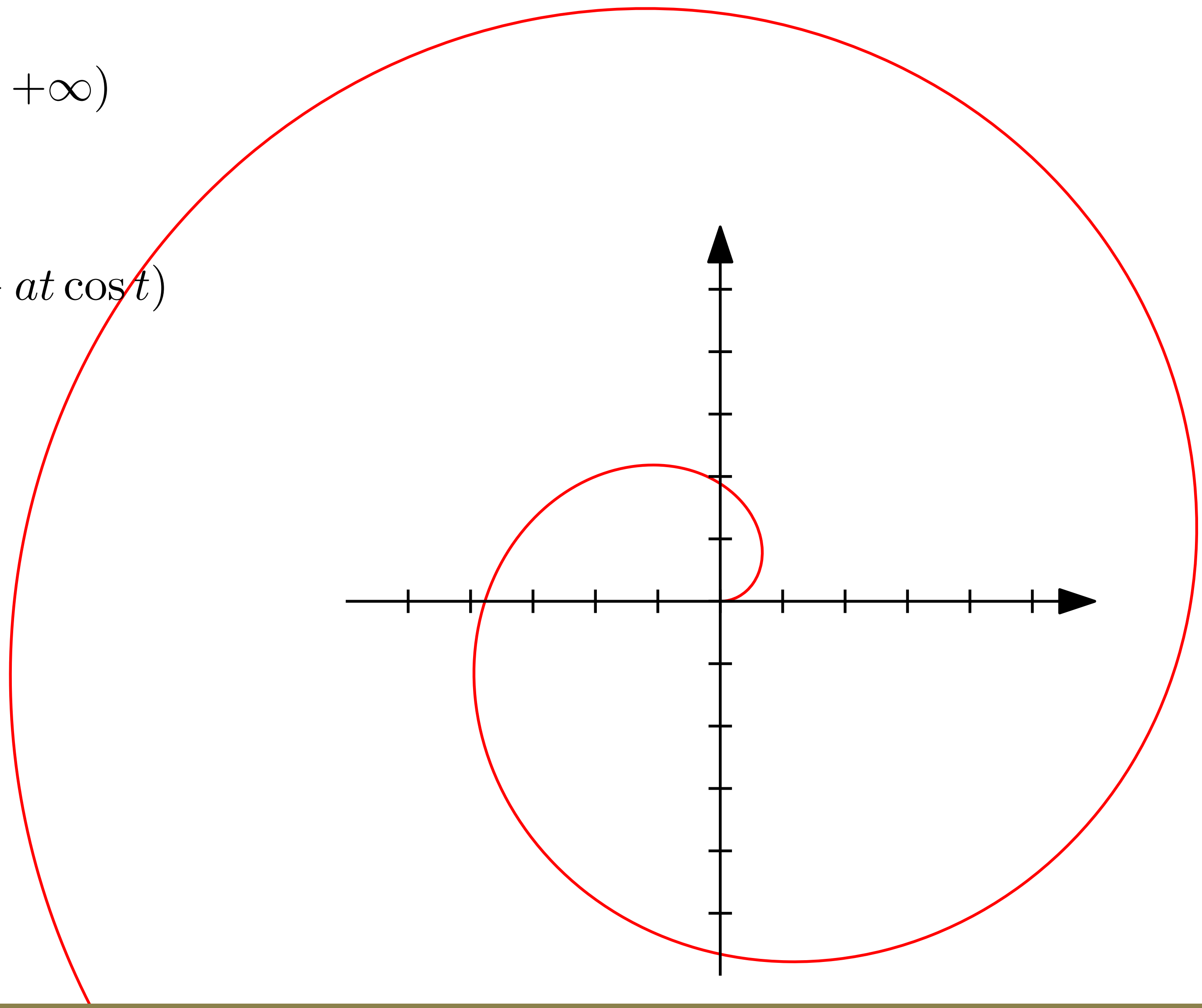

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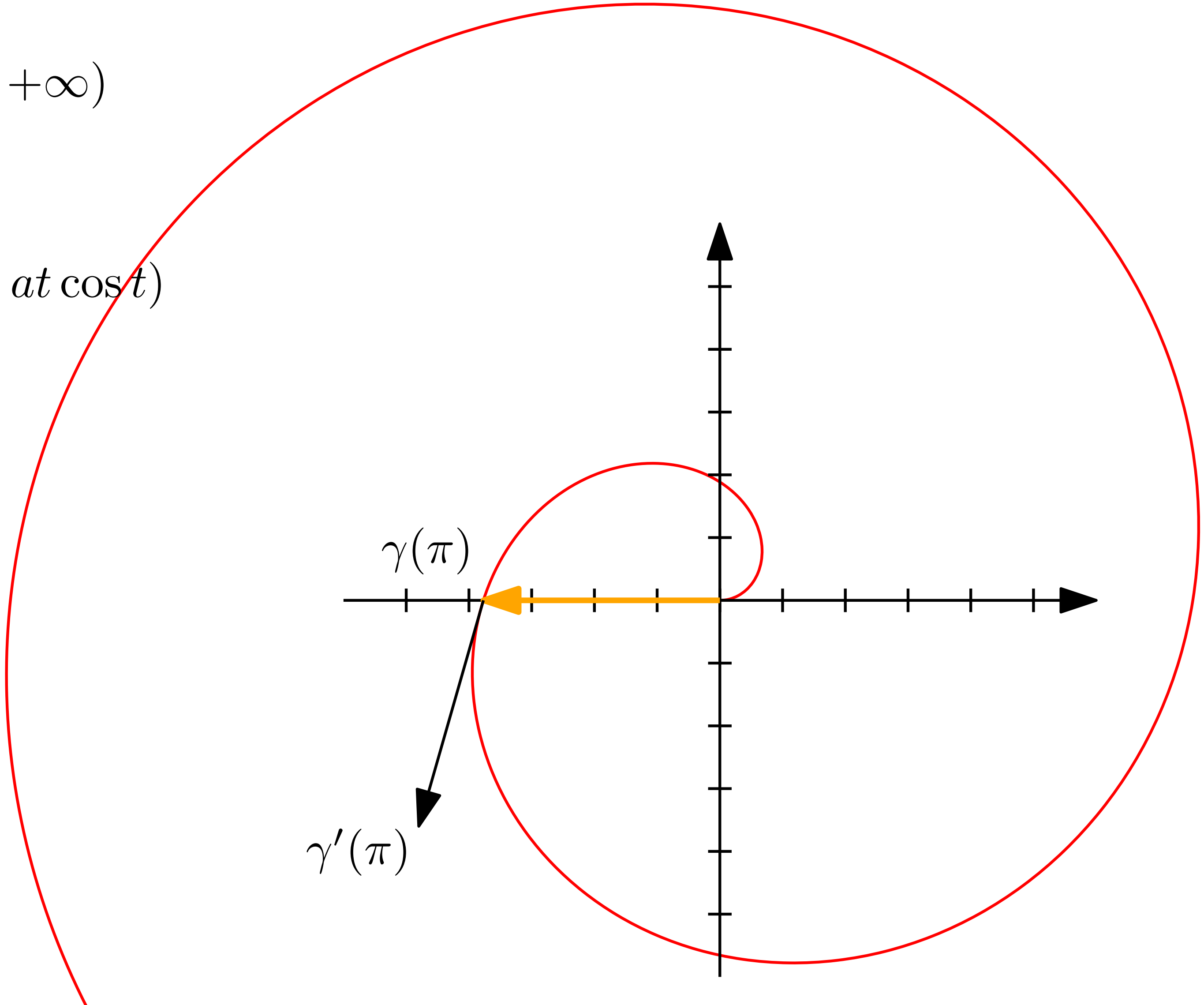

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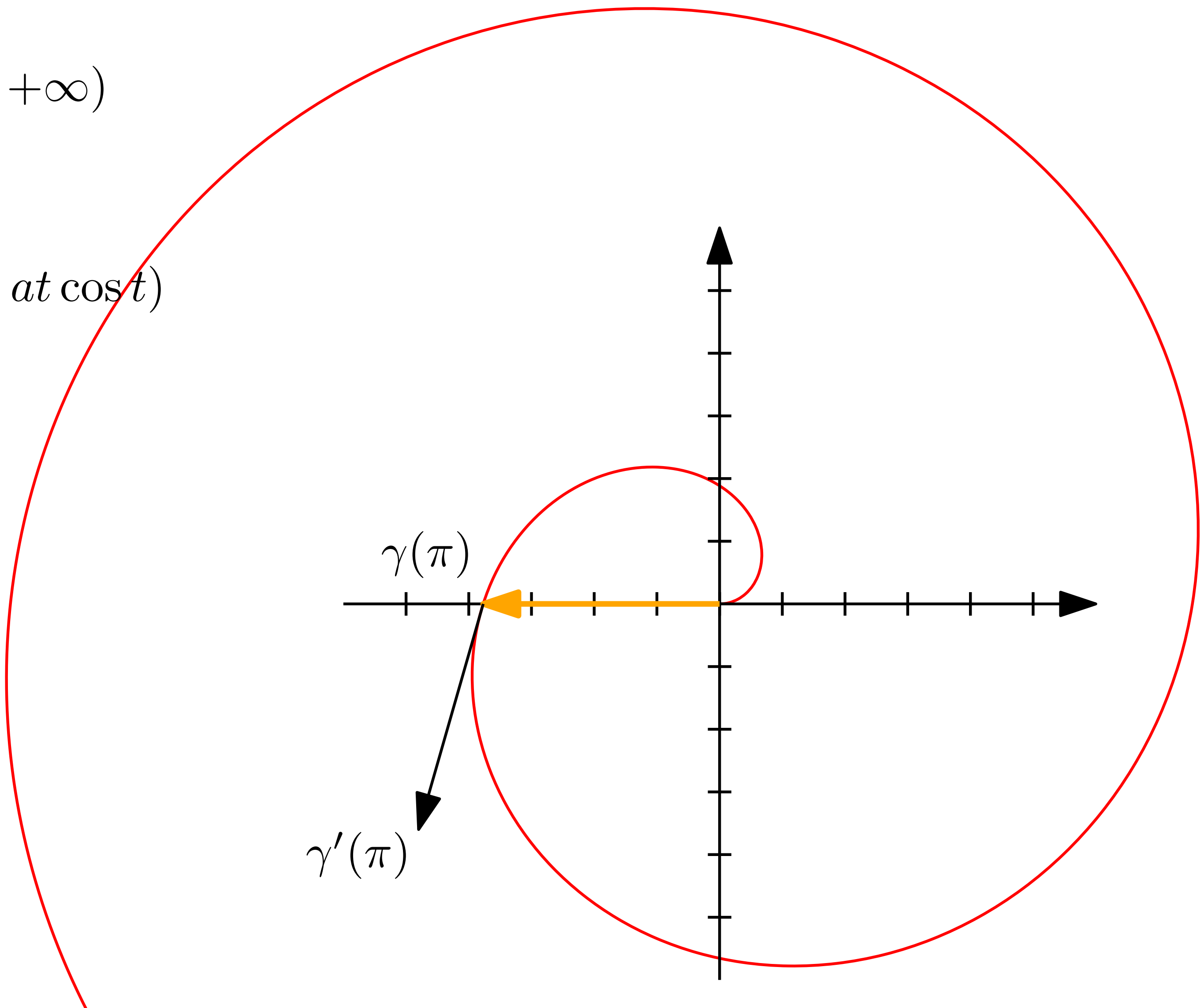

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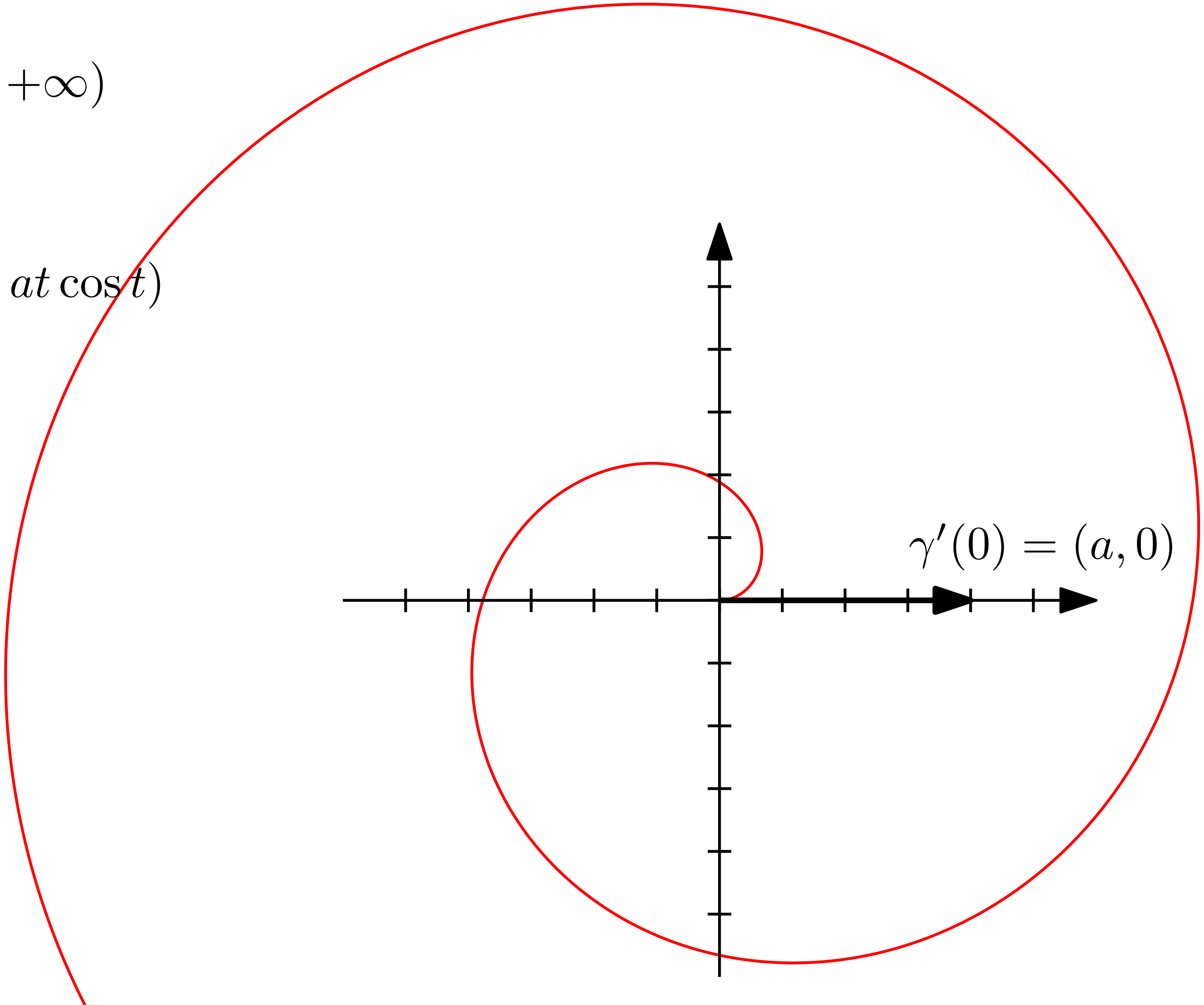

Tangent vector: examples

5) An example where $\gamma'(t) = (0, 0)$

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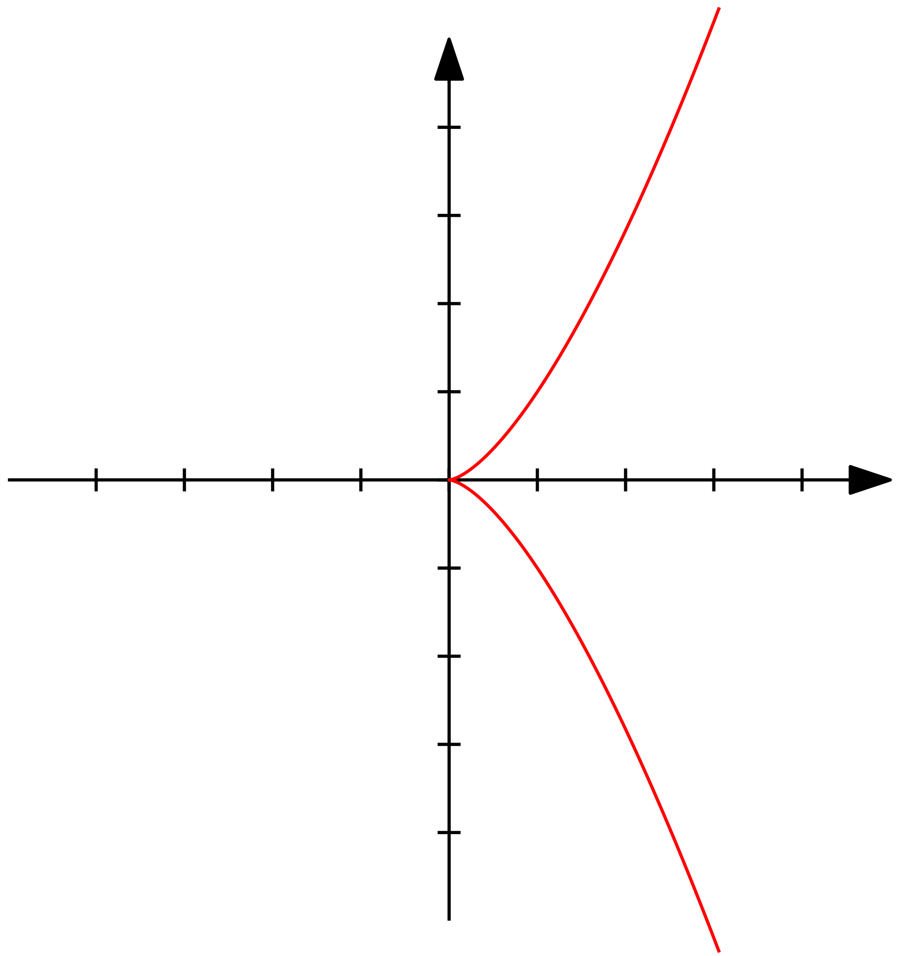
$\gamma'(t) = (2t, 3t^2)$ for $t \in \mathbb{R}$



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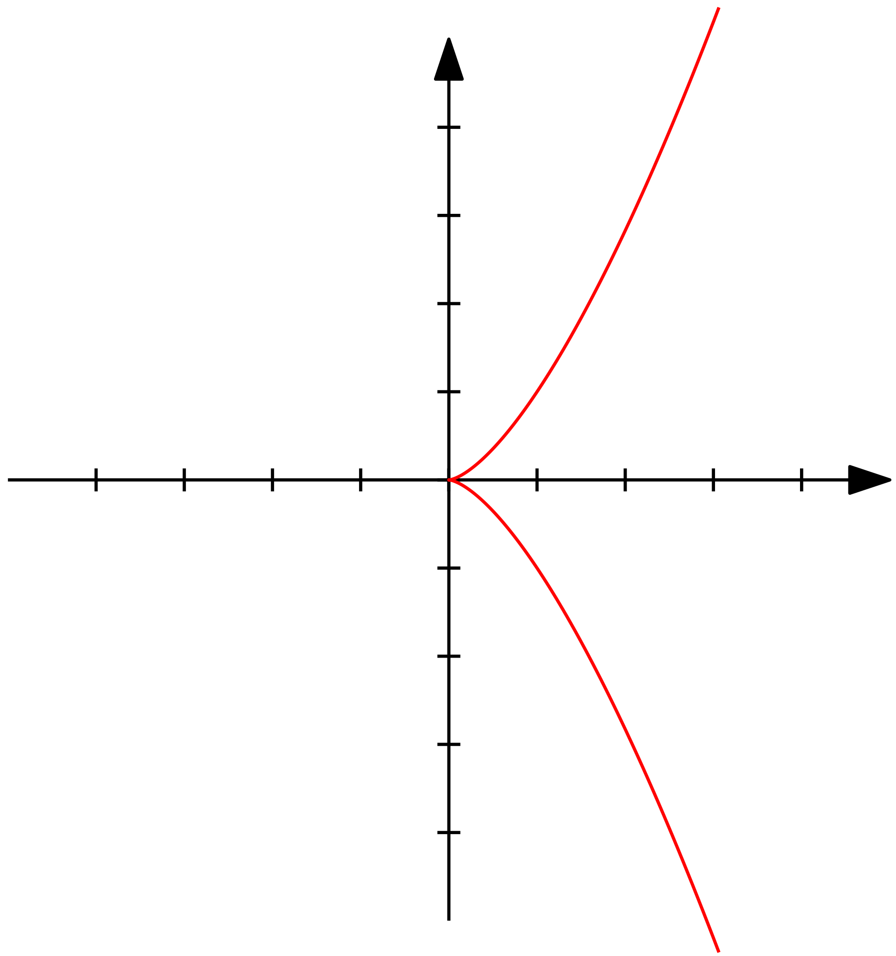

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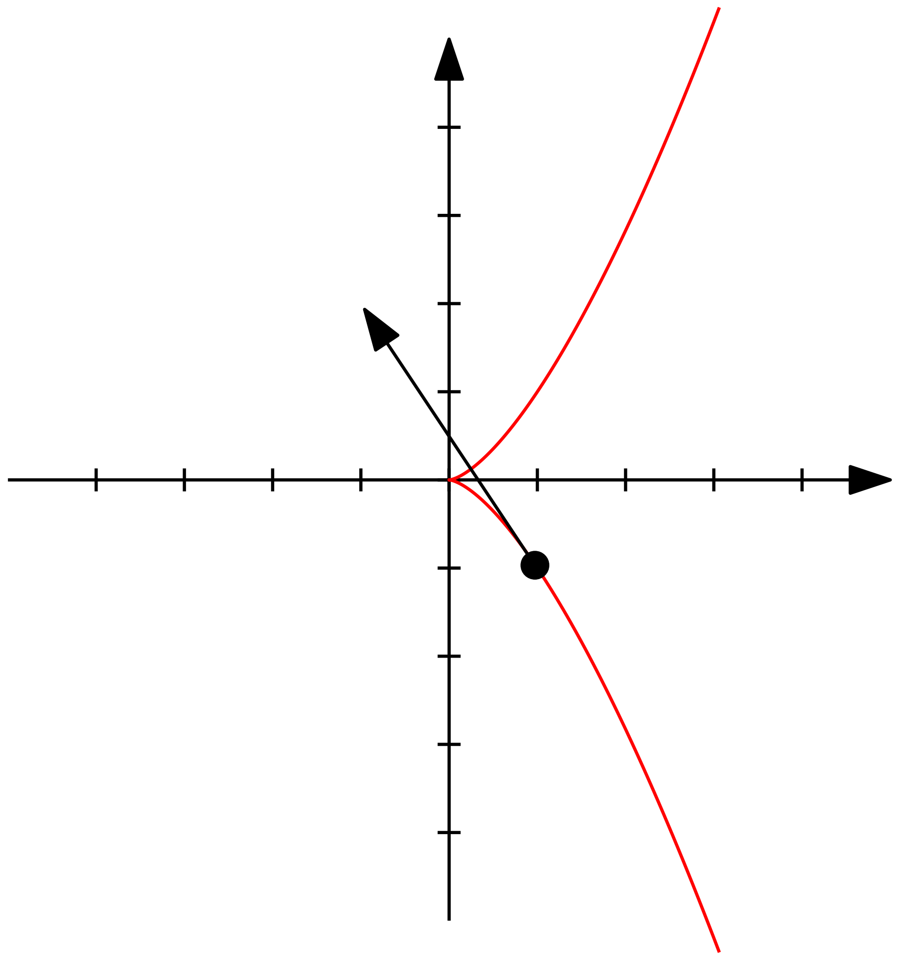

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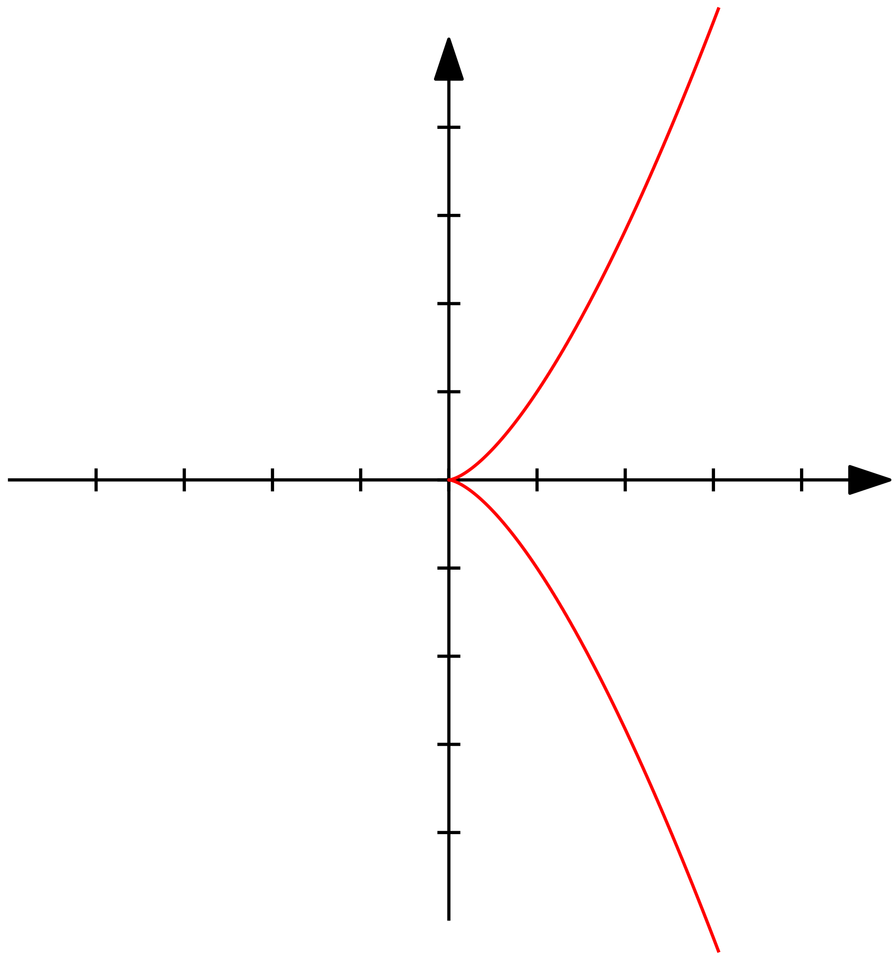

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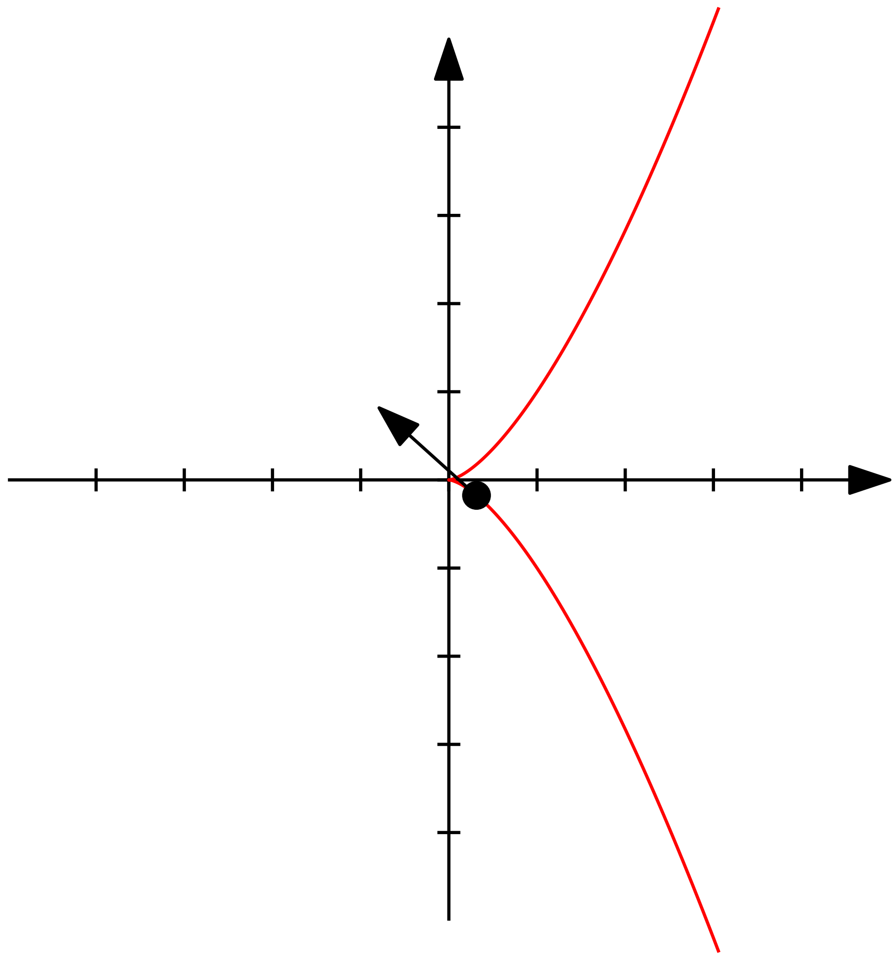

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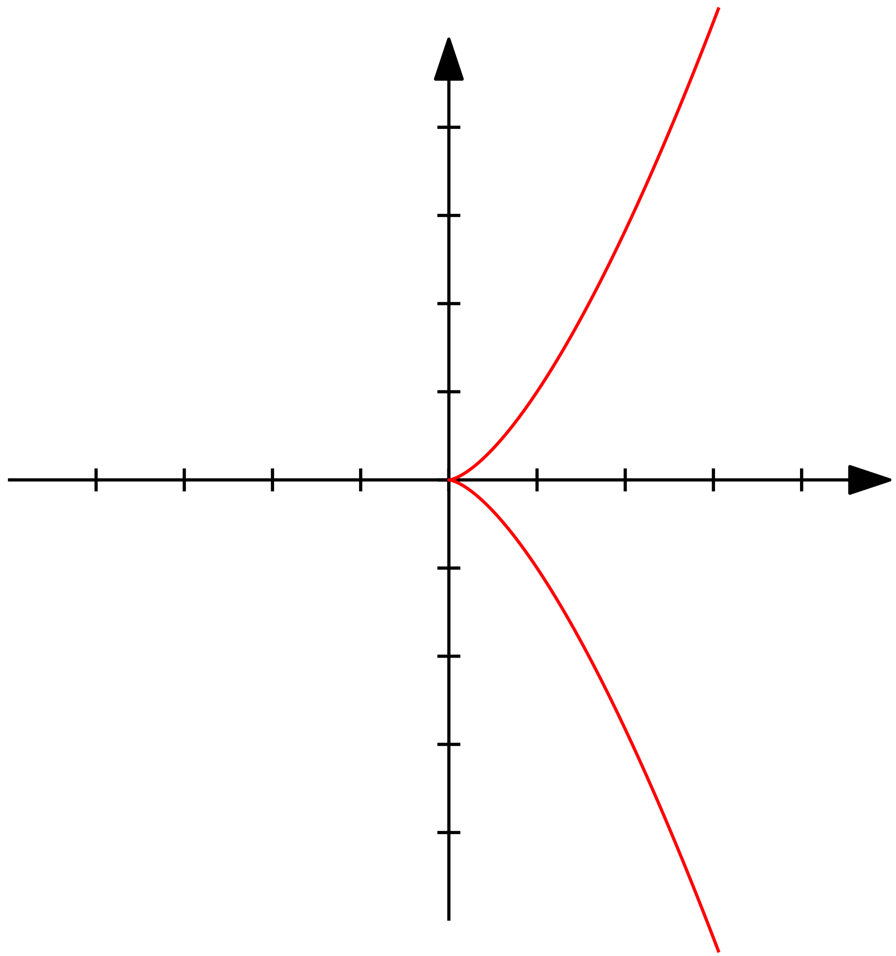

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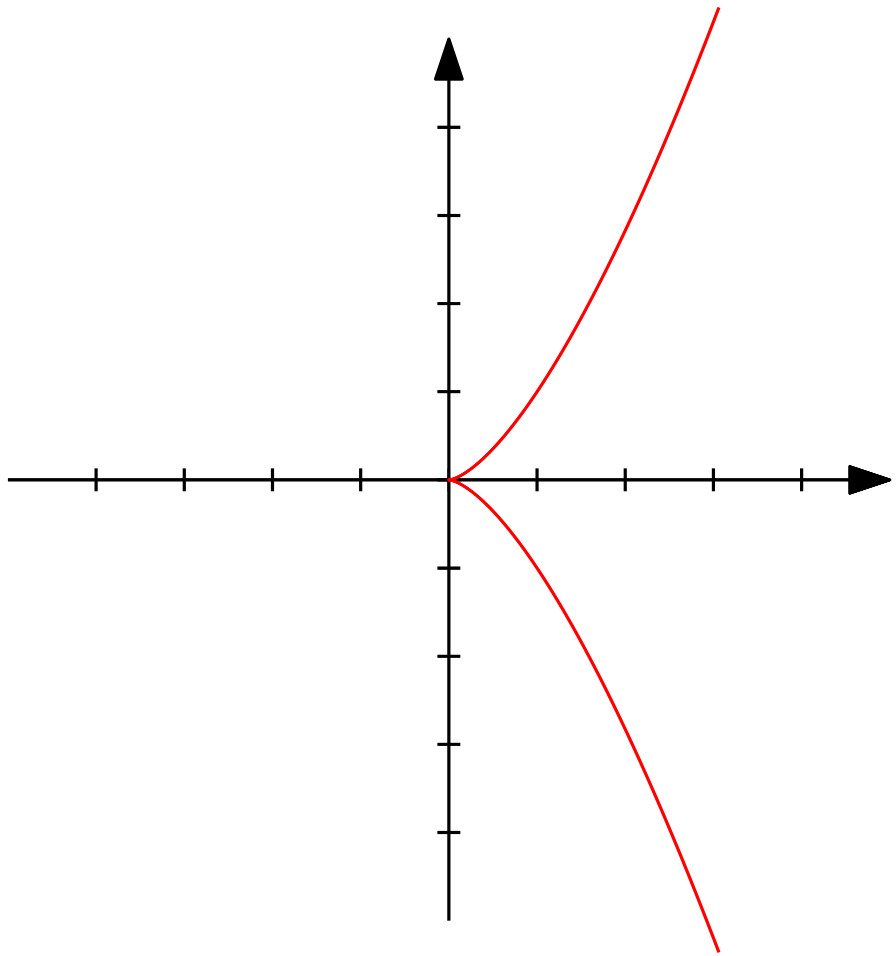

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$\gamma'(0) = (0, 0)$

tangent vector vanishes at $t = 0$

Tangent vector: effect of parametrizations

Consider different parametrizations of the same curve, for example:

$(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi/2]$	$(\cos 2\alpha, \sin 2\alpha)$ for $\alpha \in [0, \pi/4]$	$(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$ for $t \in [0, 1]$

Same curve, with $\theta = 2\alpha$ and $t = \tan\alpha = \tan\frac{\theta}{2}$

$\rightarrow$ different parametrizations can be seen as different **speeds**

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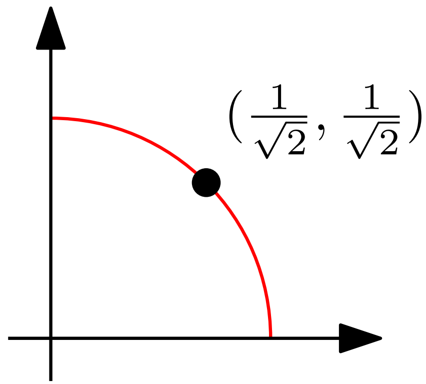
(it belongs to the curve, for $\theta = \frac{\pi}{4}$, $\delta = \frac{\pi}{8}$ and $t = \sqrt{2} - 1$)



Tangent vector: effect of parametrizations

Consider different parametrizations of the same curve, for example:

$(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi/2]$	$(\cos 2\alpha, \sin 2\alpha)$ for $\alpha \in [0, \pi/4]$	$(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$ for $t \in [0, 1]$

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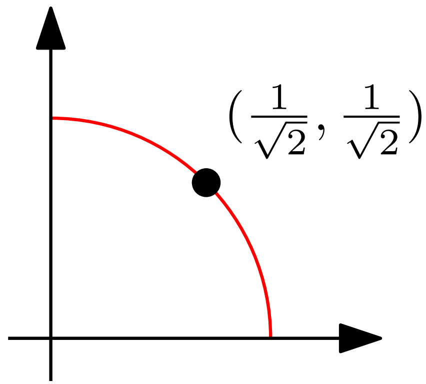

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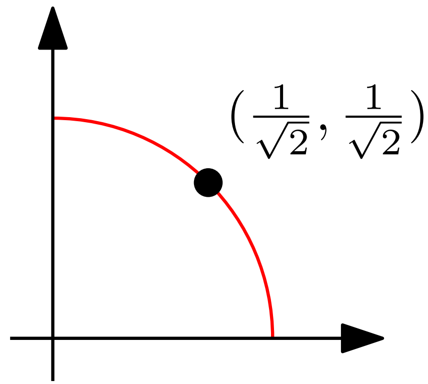

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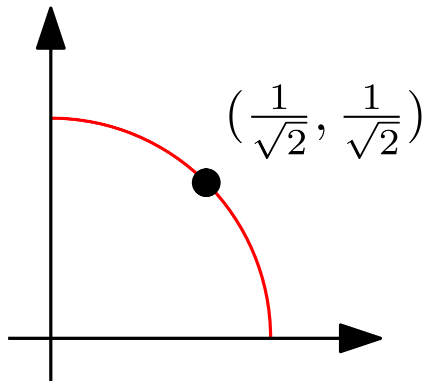

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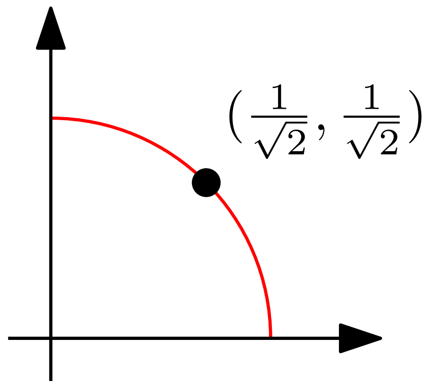

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The three vectors are proportional!

This is always the case, as long as the parametrizations are "nice" (i.e., they are always differentiable, and the tangent never vanishes)

Unit tangent vector

Often only the direction of the tangent vector matters, so we can use the unit tangent vector:

$$T(t) = \frac{\gamma'(t)}{||\gamma'(t)||}$$

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Example: $\gamma(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [0, \pi/2]$

$$\gamma'(\theta) = (-\sin\theta, \cos\theta)$$

$$||\gamma'(\theta)|| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

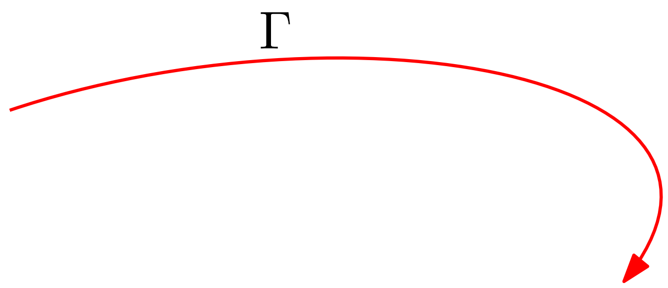
$\rightarrow \gamma(\theta)$ is a unit-speed parametrization

Normal vector and curvature

Another important vector associated with a curve Γ is the *principal normal vector* $N(t)$

If $\gamma : I \to \mathbb{R}^d$ is a unit-speed parametrization of a curve Γ, then:

$$N(t) = \gamma''(t)$$



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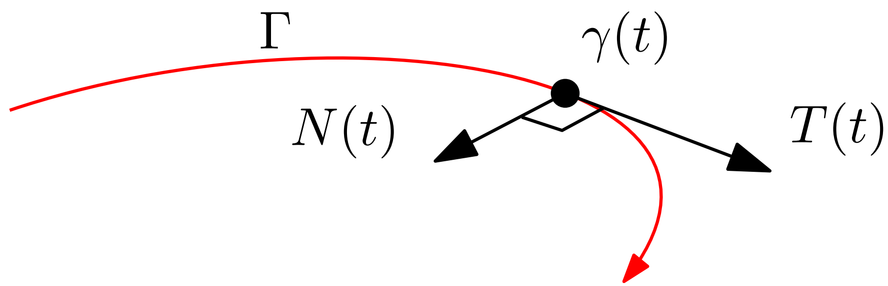

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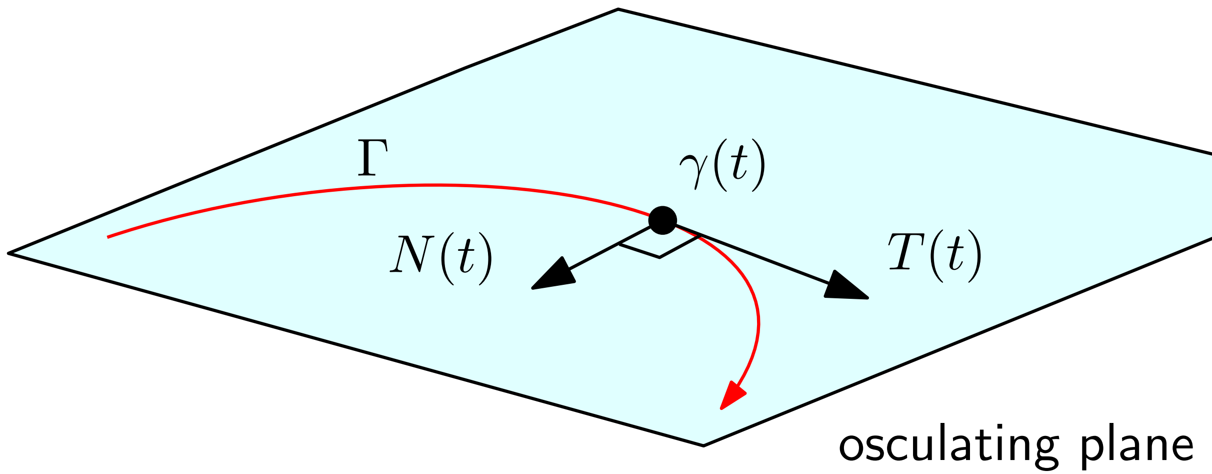

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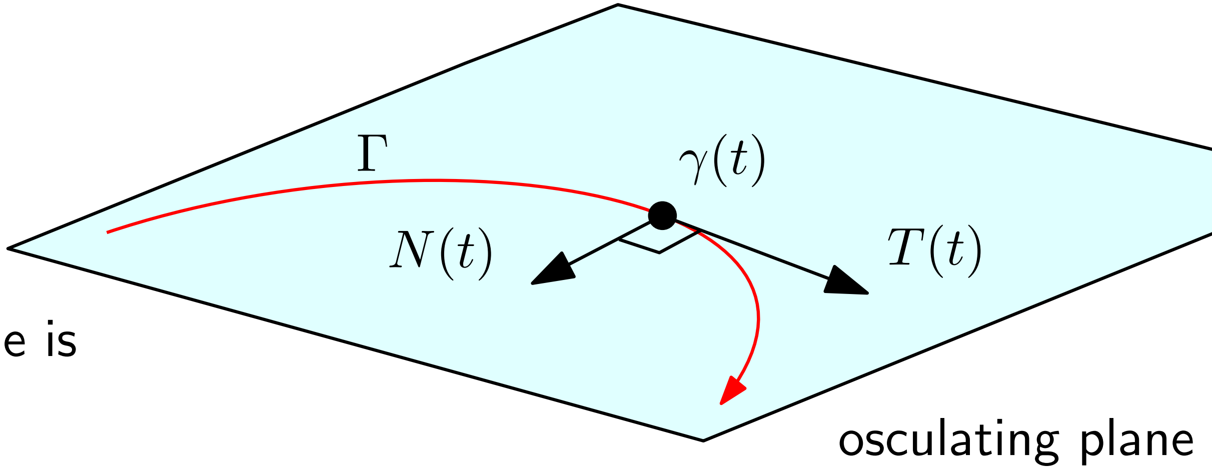

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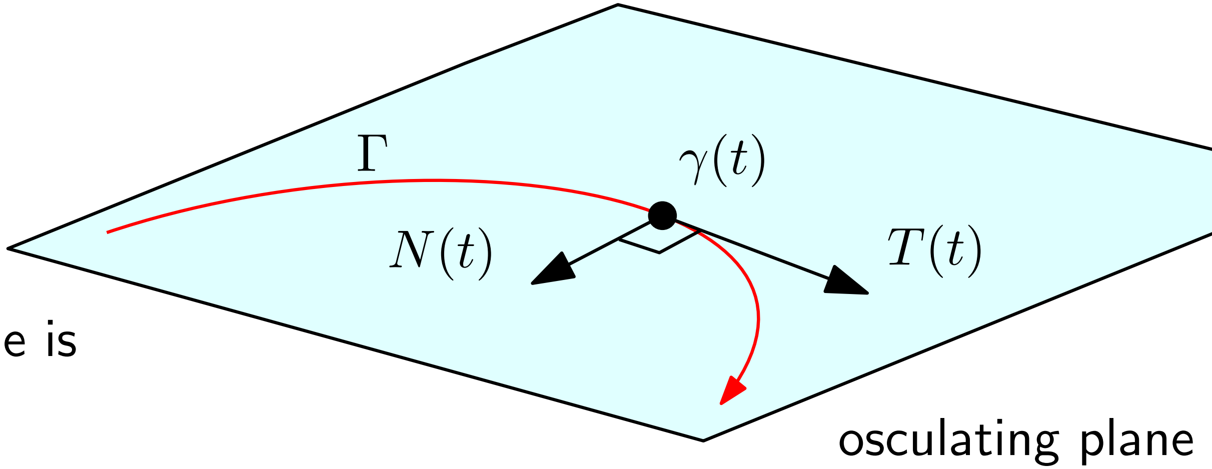

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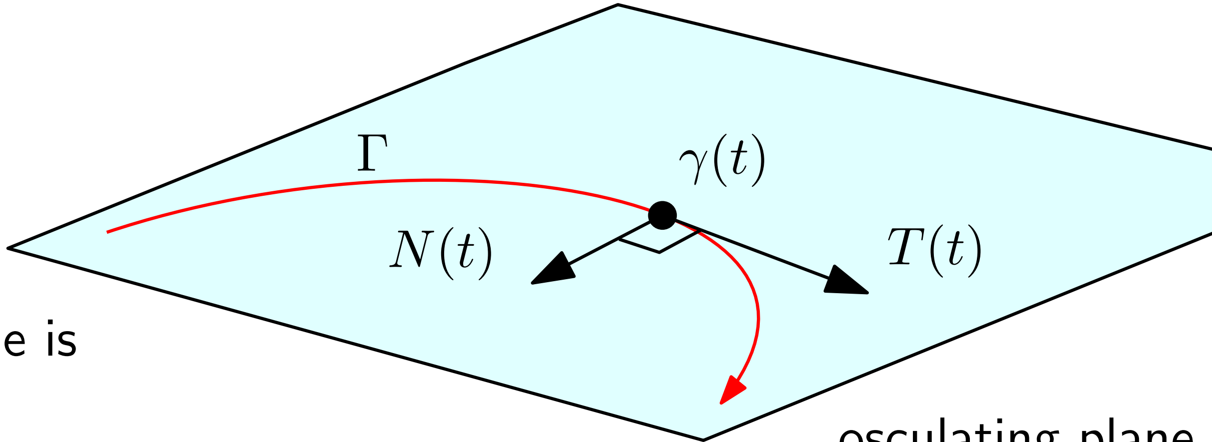

Exercise: compute the curvature of $\gamma(t) = (\cos t, \sin t)$