BASICS ABOUT POINTS, VECTORS, AND GEOMETRY

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VECTORS AND POINTS

Two main ingredients in geometry

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(identified by coordinates $(480, 256)$)

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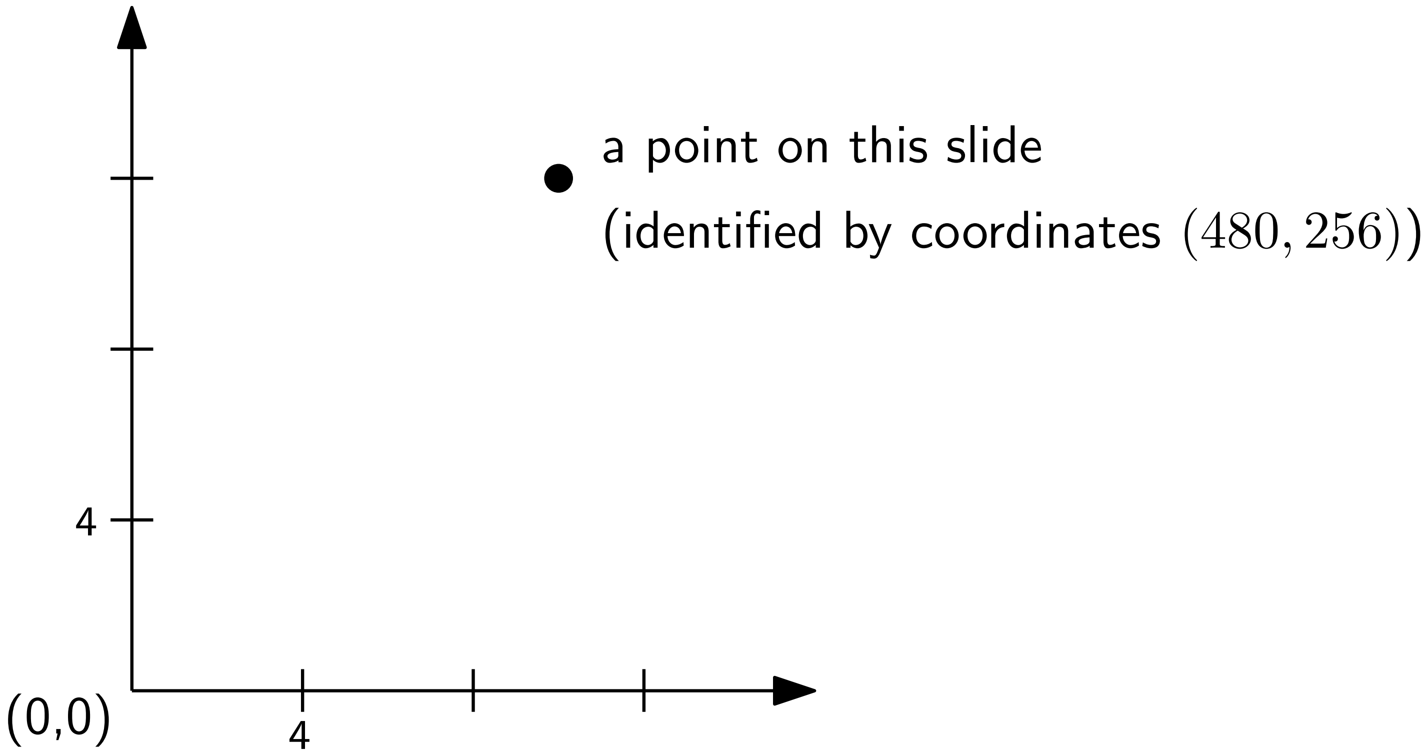

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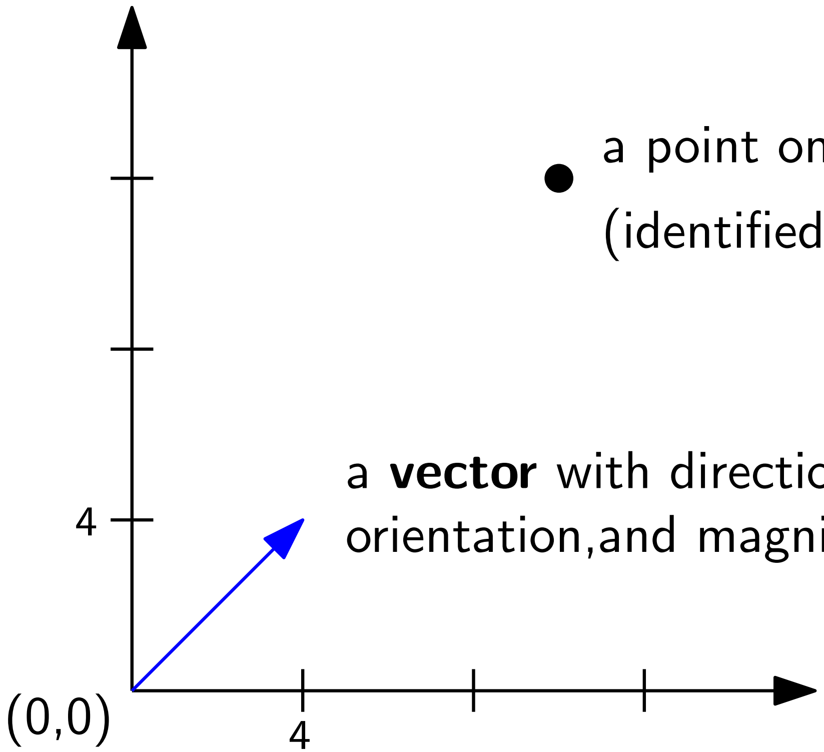

a point on this slide

(identified by coordinates $(480, 256)$)

a **vector** with direction of $45°$, positive orientation,and magnitude of $\approx 4 \cdot 1.41 = 5.64$

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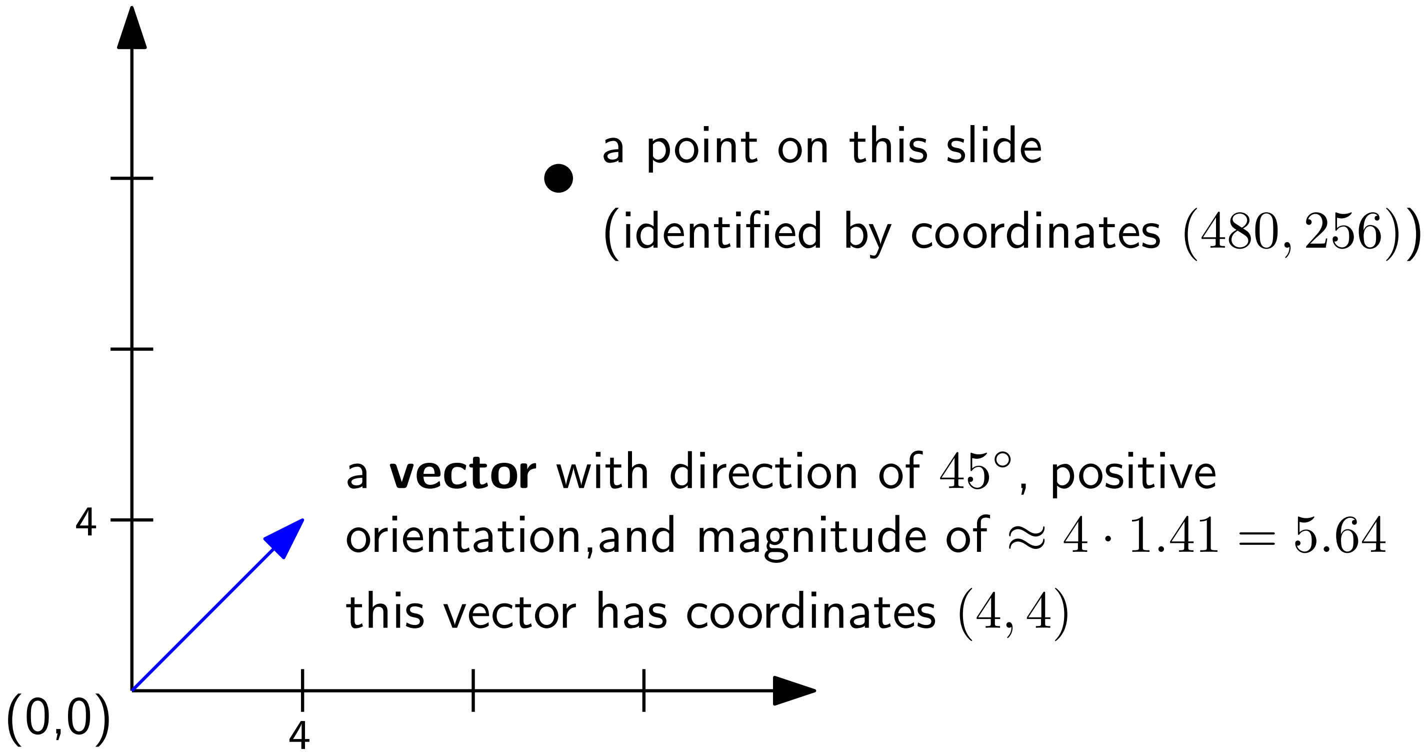

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Points: they represent **locations** in some space

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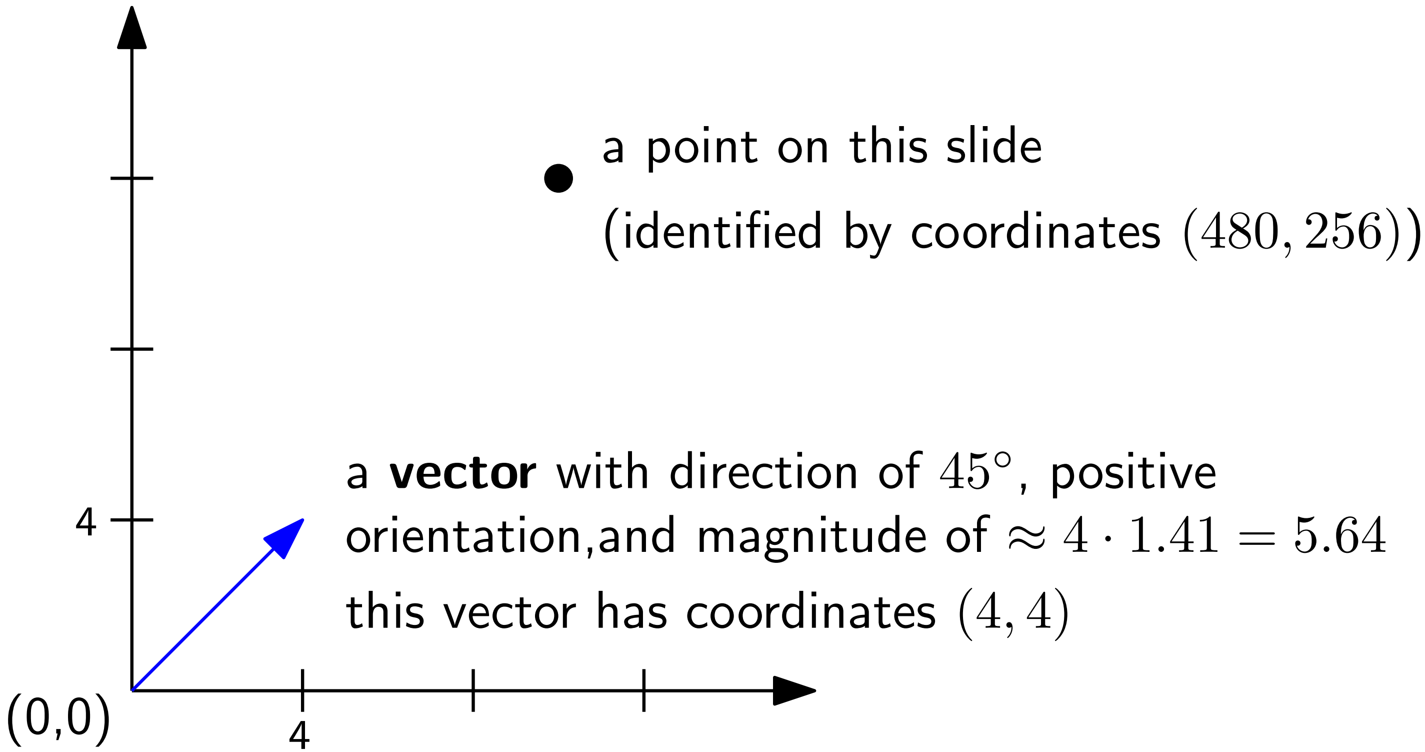
- vectors are always relative to a coordinate system



Vectors: basic elements of vector spaces

A *vector space* over $\mathbb{R}$ is a set V with two operations: vector addition and scalar multiplication

Vector addition: $V \times V \to V \quad (\vec{u}, \vec{v}) \to \vec{u} + \vec{v}$

Scalar multiplication: $\mathbb{R} \times V \to V \quad (\lambda, \vec{v}) \to \lambda\vec{v}$

These operations can be anything as long as they satisfy a series of axioms (next slide)

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The set of vectors can be any set that satisfies the axioms (in this course, $\mathbb{R}^2$ or $\mathbb{R}^3$)

The set of scalars can be any field ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, ..) (in this course, $\mathbb{R}$)

More examples of vector spaces?

Axioms for a vector space

The operations can be anything as long as they satisfy the following axioms:

- Closure under addition: $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$
- Commutativity of addition: $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u}$
- Associativity of addition: $\forall \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- Existence of additive identity: $\exists \vec{0} \in V, \forall \vec{v} \in V, \vec{v} + \vec{0} = \vec{v}$
- Existence of additive inverses: $\forall \vec{v} \in V, \exists - \vec{v} \in V, \vec{v} + (-\vec{v}) = \vec{0}$

- Closure under scalar multiplication: $\forall a \in \mathbb{R}, \vec{v} \in V, a\vec{v} \in V$
- Compatibility of scalar multiplication with field multiplication: $\forall a, b \in \mathbb{R}, \vec{v} \in V, (ab)\vec{v} = a(b\vec{v})$
- Distributivity of scalar multiplication over vector addition: $\forall a \in \mathbb{R}, \vec{u}, \vec{v} \in V, a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- Distributivity of scalar multiplication over field addition: $\forall a, b \in \mathbb{R}, \vec{v} \in V, (a + b)\vec{v} = a\vec{v} + b\vec{v}$
- Compatibility of scalar multiplication with field addition: $\forall a, b \in \mathbb{R}, \forall$ vectors u and $v \in$ vector space, $(a + b) \cdot v = (a \cdot v) + (b \cdot v)$

Combining vectors

The addition and scalar multiplication can be combined to produce new vectors

These are called *linear combinations*

Given vectors $\vec{v}_1, \ldots \vec{v}_n$ and scalars $\lambda_1, \ldots, \lambda_n$, $\vec{v}$ is a linear combination of $\vec{v}_1, \ldots \vec{v}_n$:

$$\vec{v} = \lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n = \sum_{i=1}^{n} \lambda_i \vec{v}_i$$

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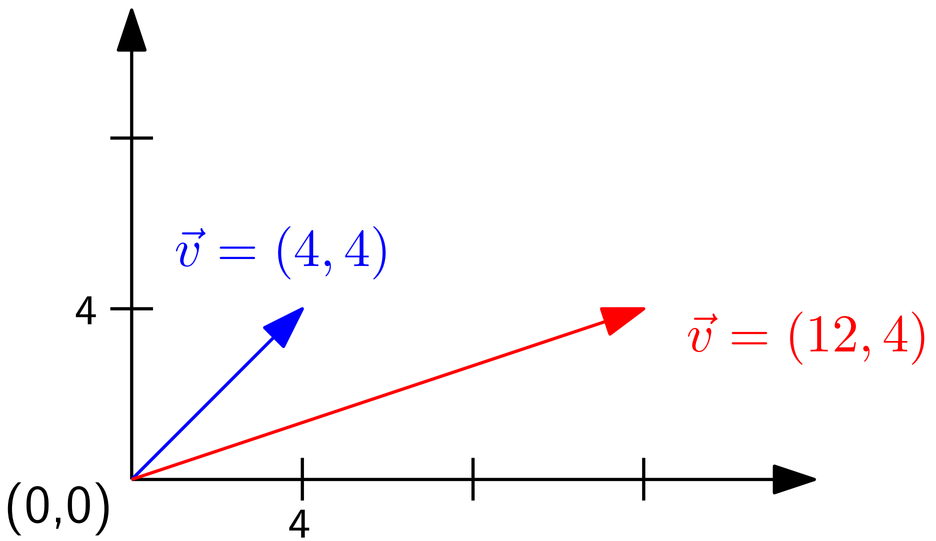

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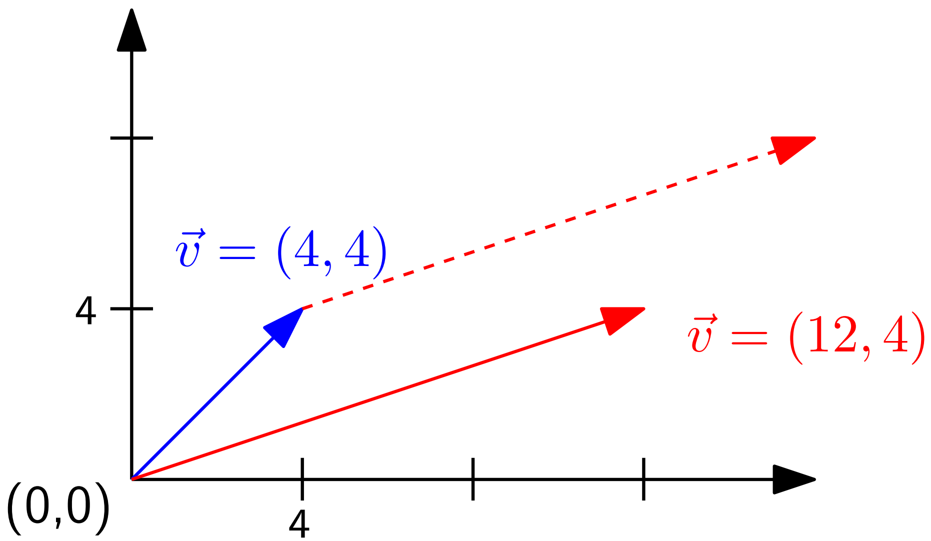

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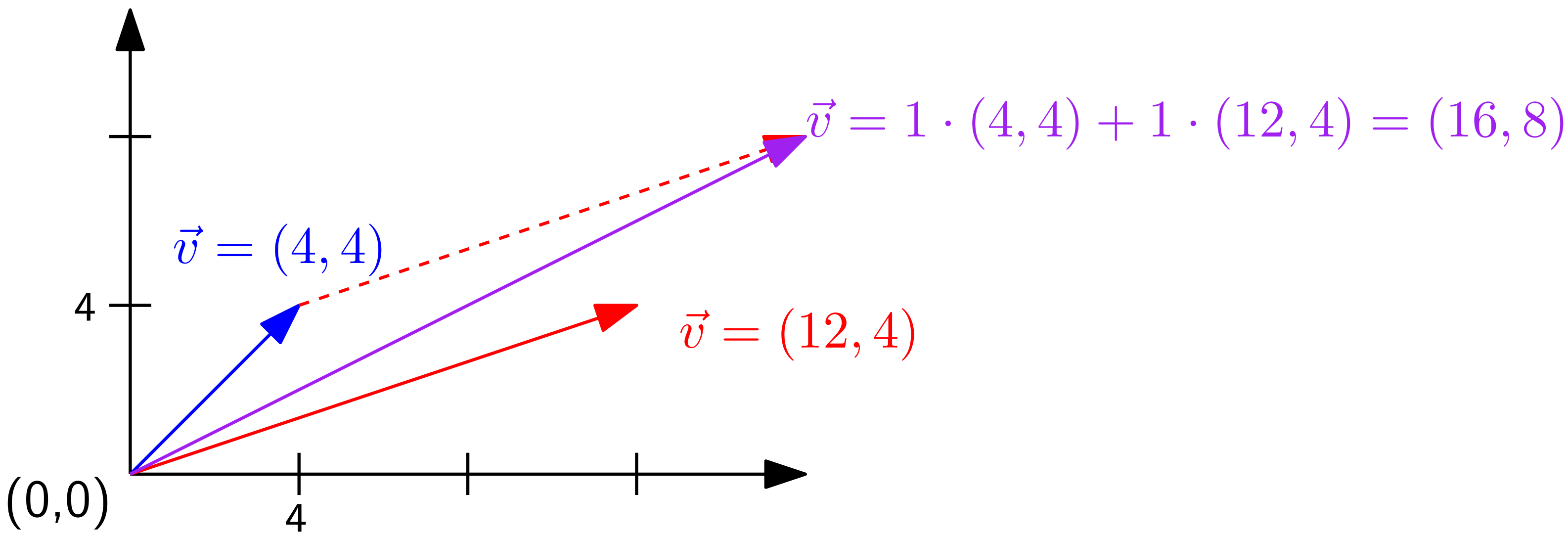

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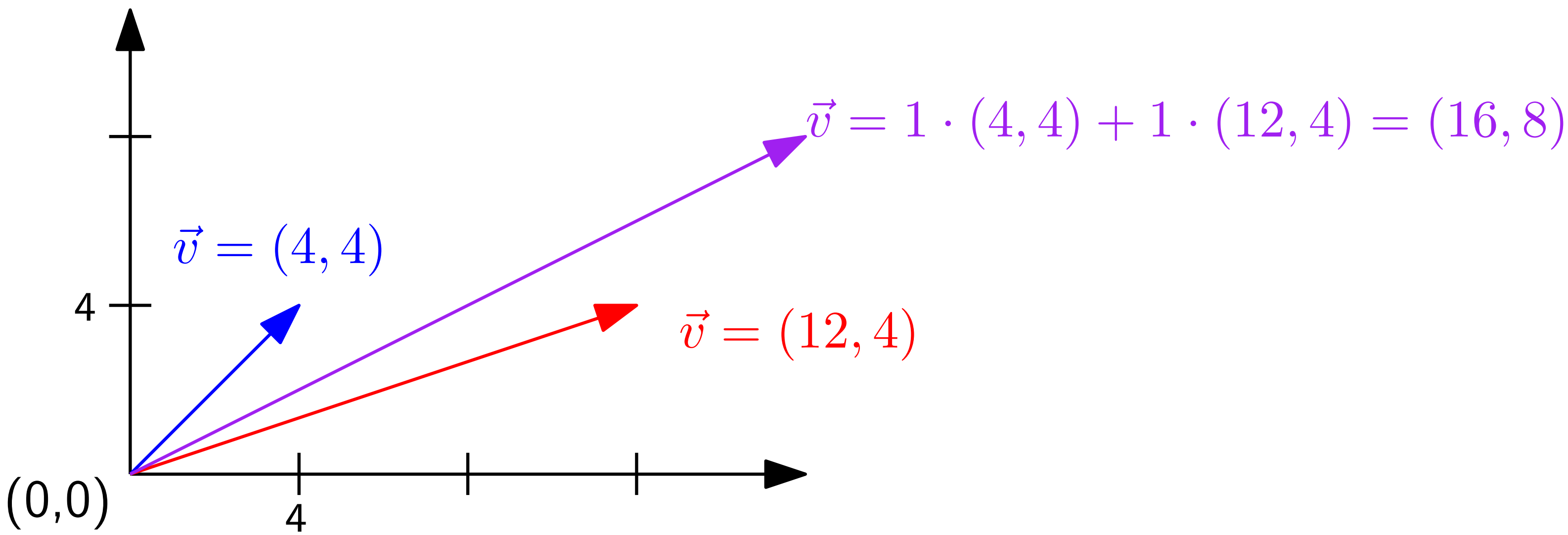

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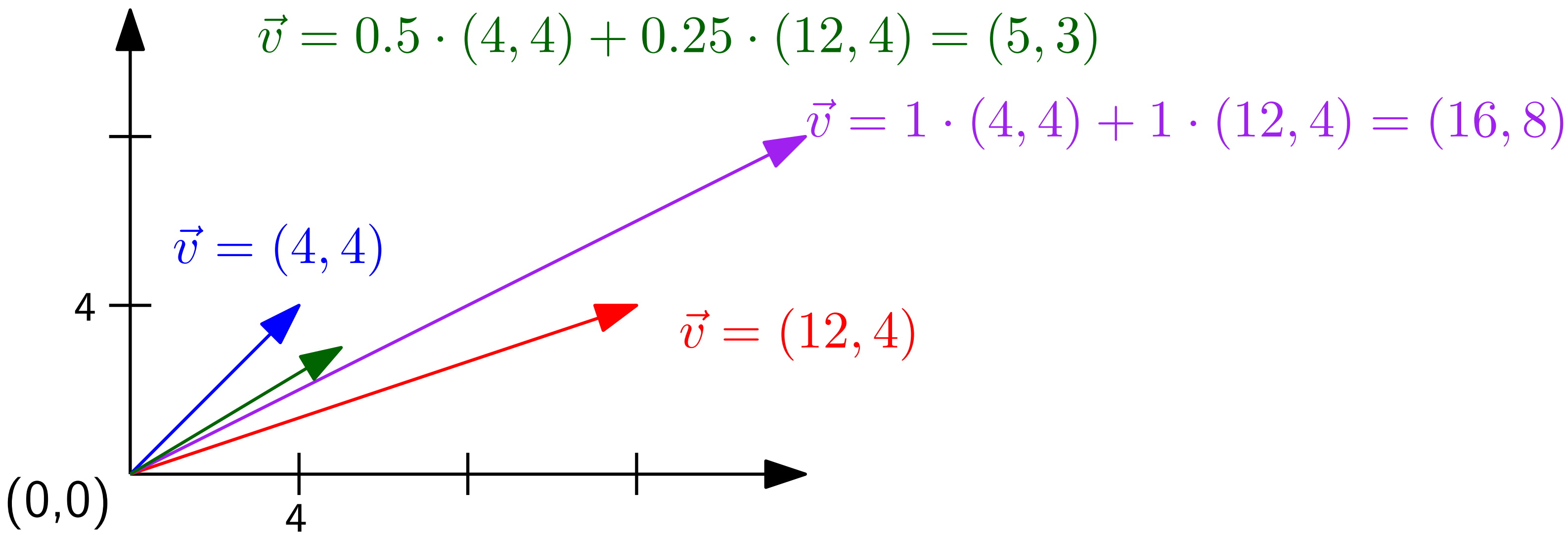

Basis of a vector space

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A *basis* of a vector space is an ordered sequence of vectors of V $\{\vec{e}_1, \ldots, \vec{e}_n\}$ such that:

1) $\vec{e}_1, \ldots, \vec{e}_n$ are linearly independent

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The coefficients ($x_1, \ldots, x_n$ above) of the unique combination are called the *coordinates* of the vector in that basis

Once a basis $\mathcal{B}$ is fixed, we have a bijection between vectors in V and coordinates in $\mathbb{R}^n$!

vector $\vec{x} \in V \leftrightarrow$ coordinates $(x_1, \ldots, x_n)$ in basis $\mathcal{B}$

Examples of bases and coordinates

$V = \mathbb{R}^3$ Canonical basis: $\mathcal{E} = \{e_1, e_2, e_3\} = \{(1,0,0),(0,1,0),(0,0,1)\}$

E.g., $(2,3,1) = 2 \cdot (1,0,0) + 3 \cdot (0,1,0) + 1 \cdot (0,0,1)$

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$V = \mathbb{P}_2$ (quadratic polynomials) Canonical basis $\mathcal{B} = \{e_1, e_2, e_3\} = \{1, x, x^2\}$

E.g., $5x^2 + 3x - 2 = (-2) \cdot e_1 + 3 \cdot e_2 + 5 \cdot e_3$

The coordinates of $5x^2 + 3x - 2$ in basis $\mathcal{B}$ are $(-2,3,5)$

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E.g., $5x^2 + 3x - 2 = 2 \cdot e_1 + (-2) \cdot e_2 + 5 \cdot e_3$

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Measuring vectors

Measuring vectors

An *inner product* is an operation that takes two vectors and returns a scalar value.
The *dot product* is the inner product usually used in $\mathbb{R}^n$:

For two vectors $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$, it is defined as the function

$$V \times V \to \mathbb{R} \quad (\vec{u}, \vec{v}) \to \vec{u}.\vec{v} = \langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i$$

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- Cauchy–Schwarz inequality $(\vec{u}.\vec{v})^2 \leq (\vec{u}.\vec{u})(\vec{v}.\vec{v})$

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Each inner product has an associated *norm*, a function from V to $\mathbb{R}$:

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For example, what's the associated norm for $\mathbb{R}^2$?

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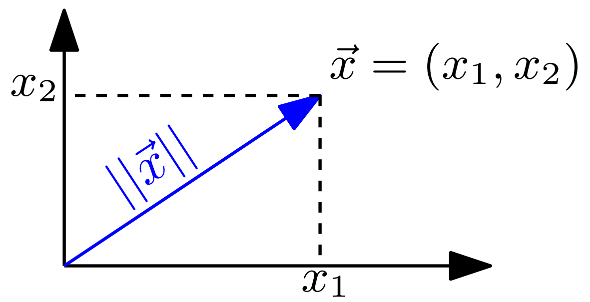

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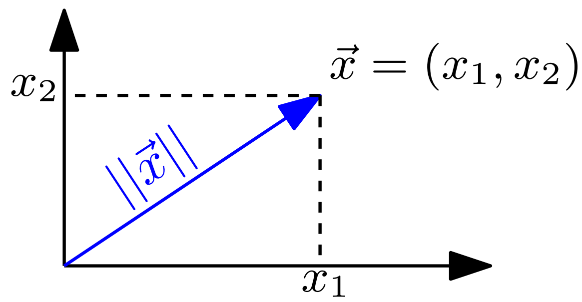

Norms have several nice properties

i) $||\vec{v}|| = 0 \iff \vec{v} = \vec{0}$ and $||\vec{v}|| \geq 0 \quad \forall v \in V$
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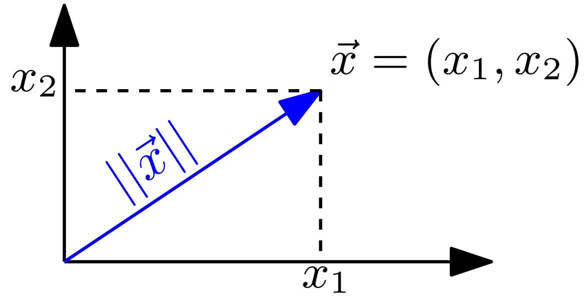

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Note that many other norms exist!

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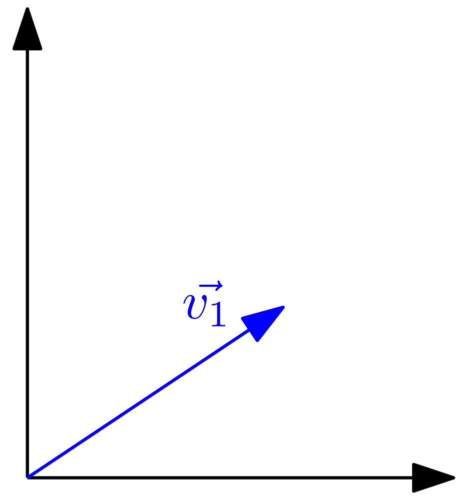

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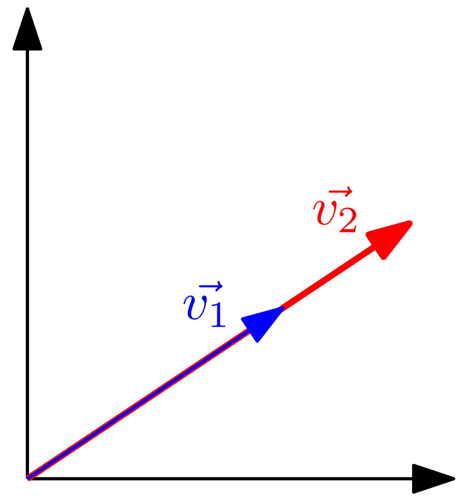

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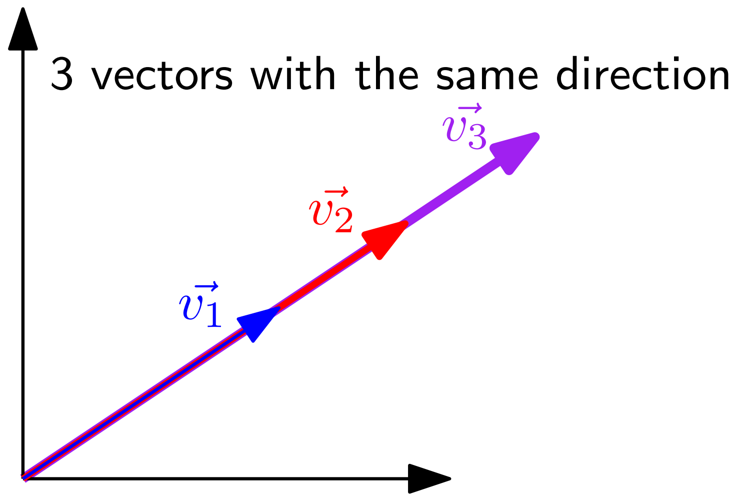

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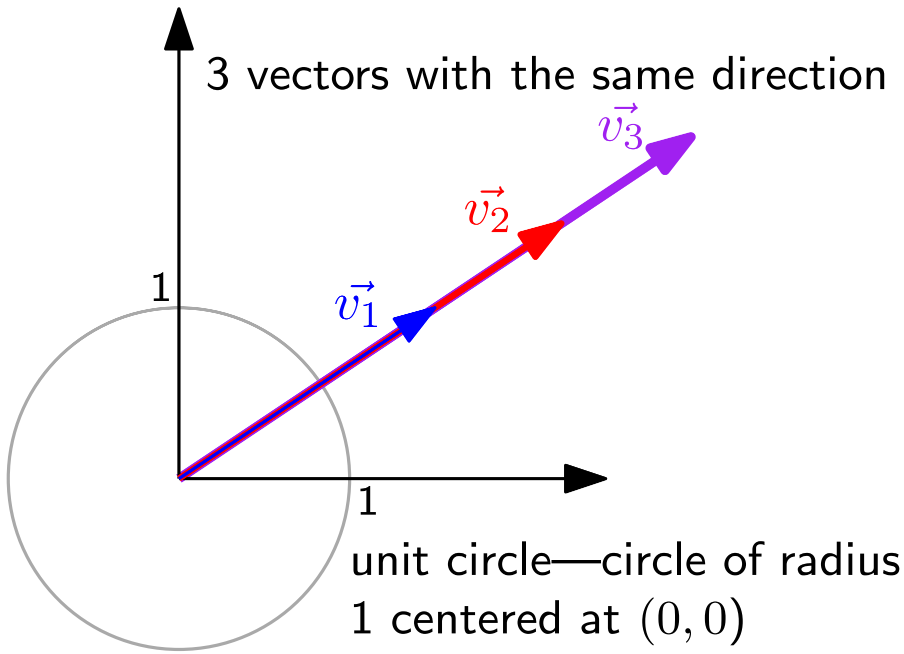

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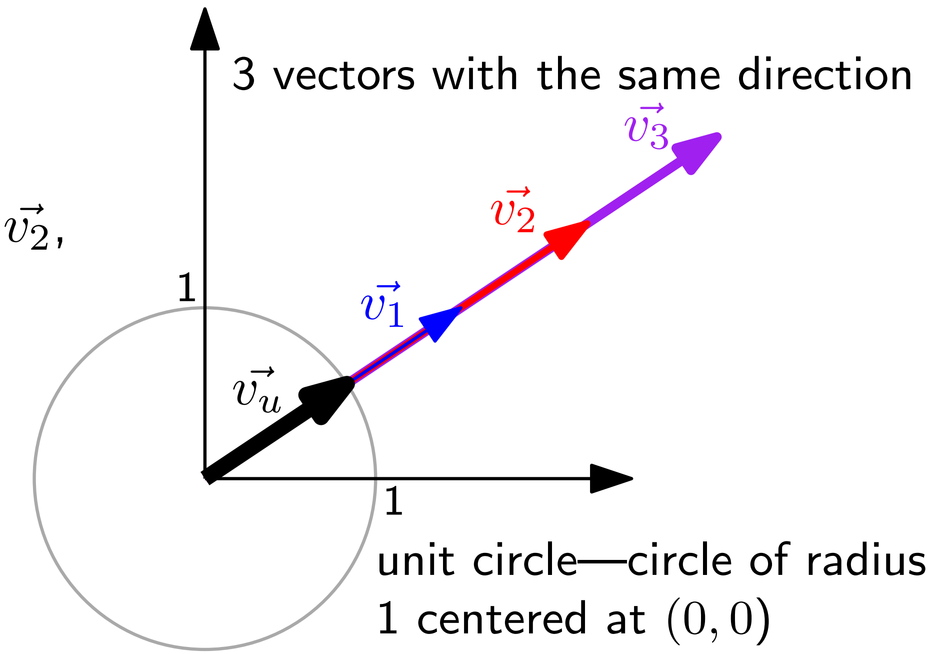
$\vec{v_u}$: unit vector with the same direction as $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$. That is: $\vec{v_u} = \frac{\vec{v_1}}{||\vec{v_1}||} = \frac{\vec{v_2}}{||\vec{v_2}||} = \frac{\vec{v_3}}{||\vec{v_3}||}$



Making a vector unit-length by dividing it by its norm is called *normalizing* the vector

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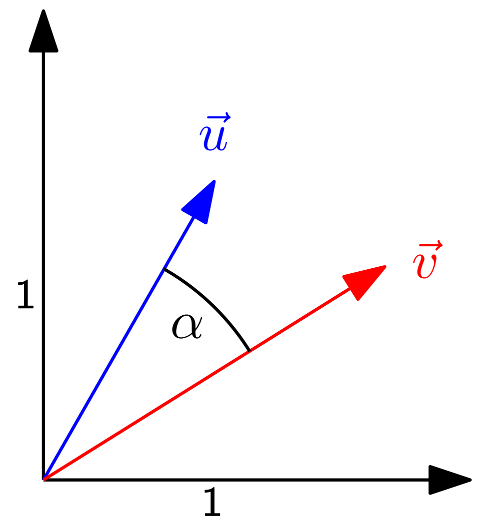

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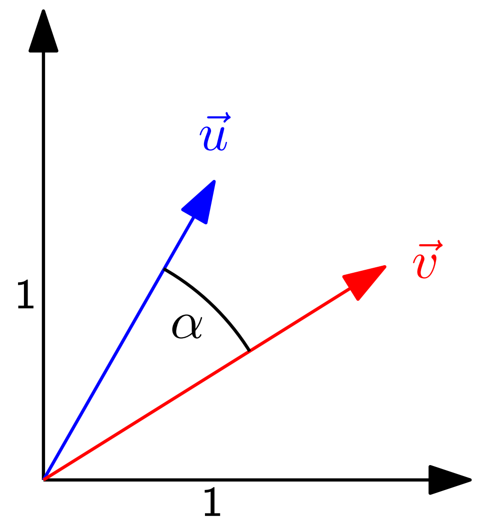
What is $\cos\alpha$, geometrically?



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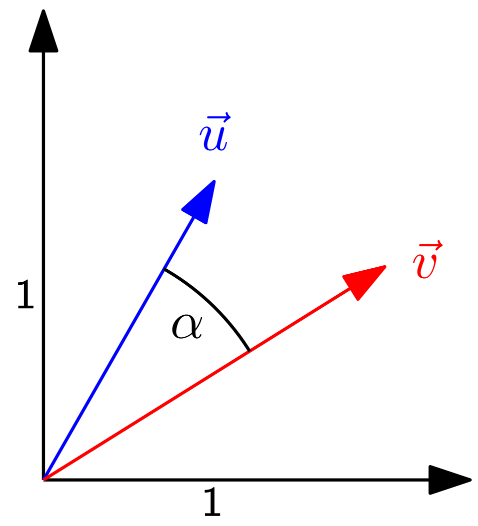

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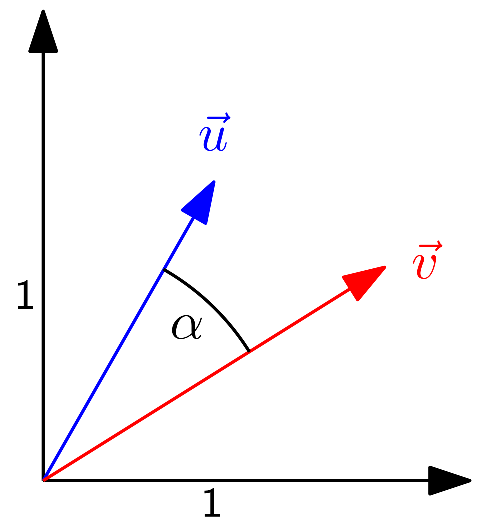

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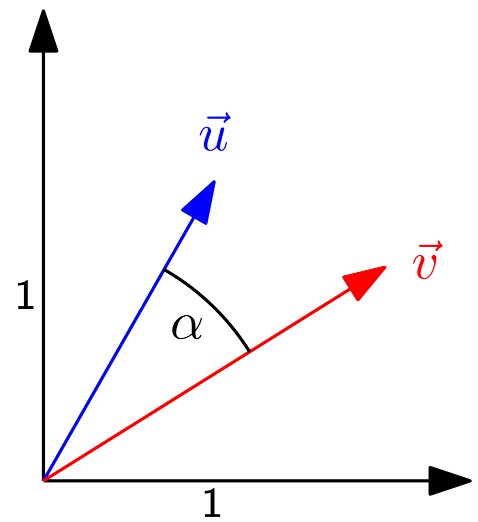

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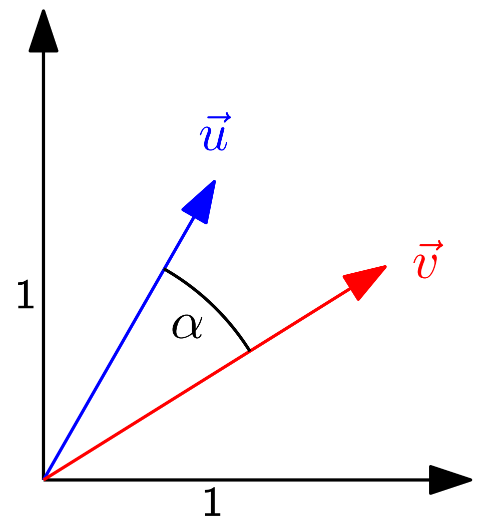

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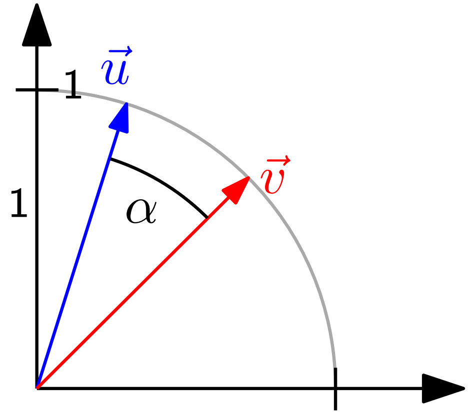

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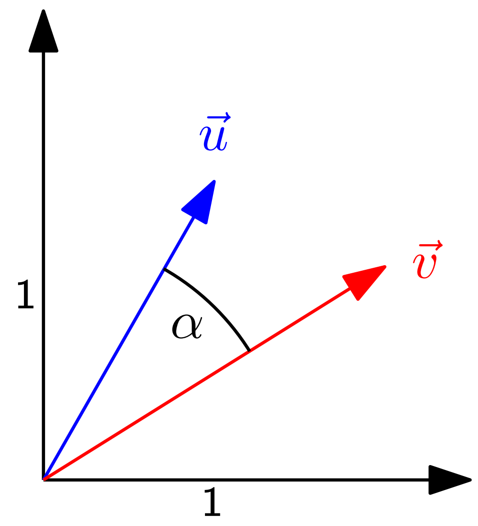

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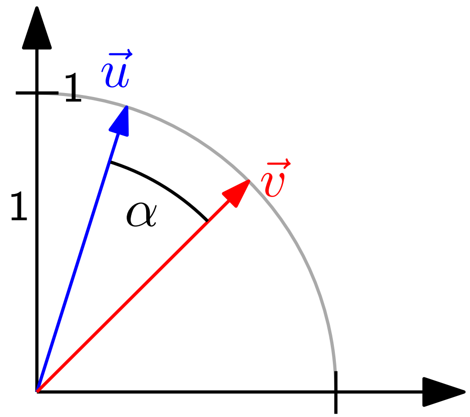

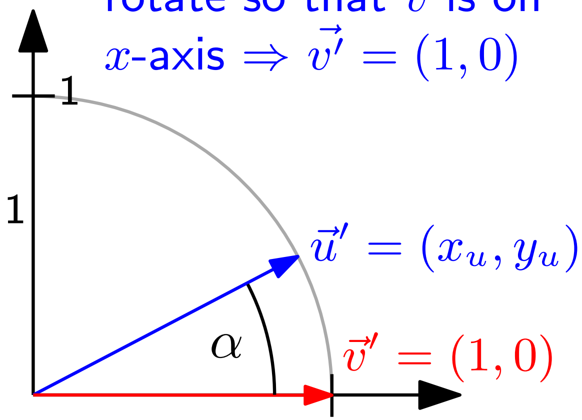

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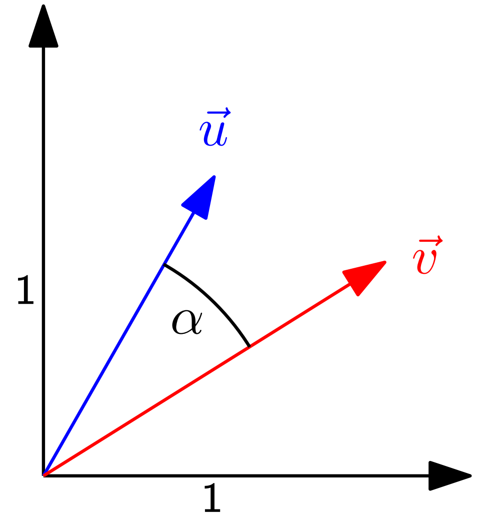

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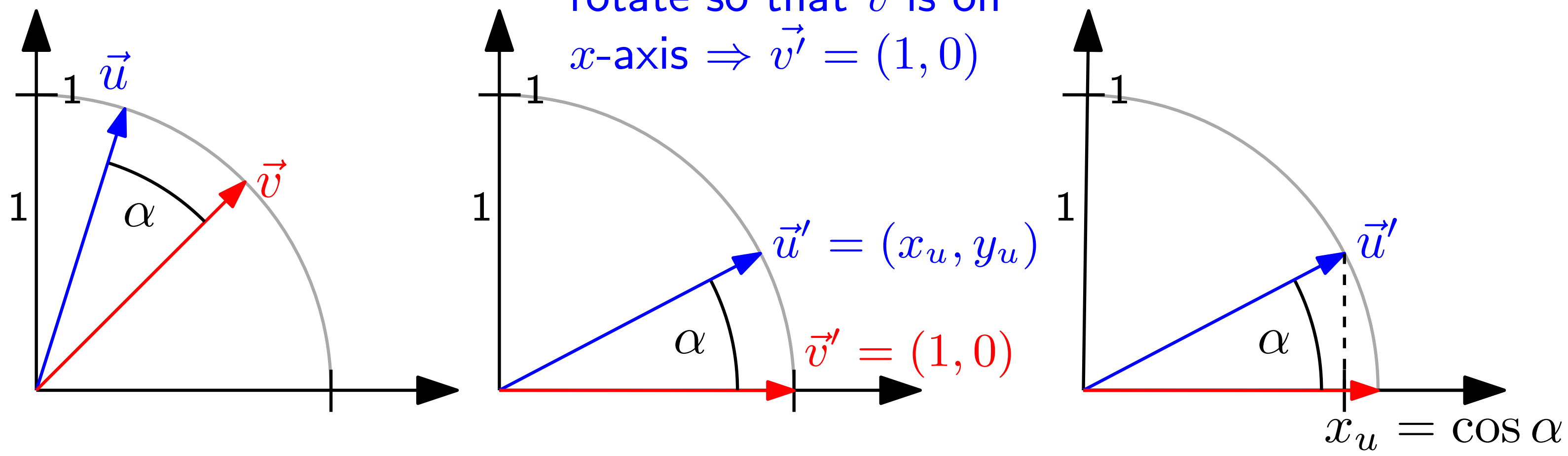

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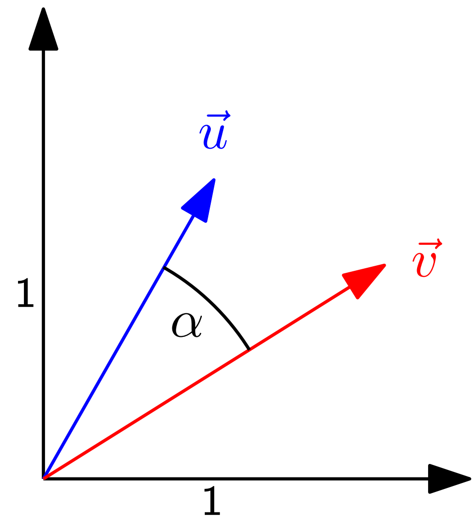

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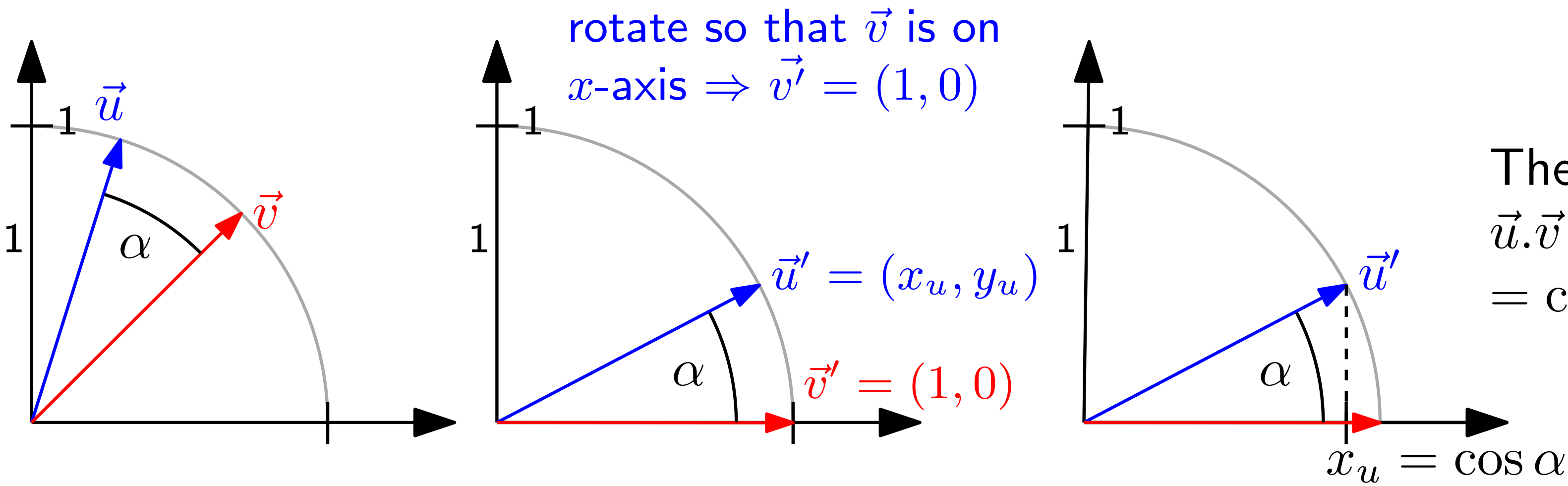

Therefore, we have:
$\vec{u}.\vec{v} = \langle (x_u, y_u), (1, 0) \rangle = x_u$
$= \cos\alpha$

Points

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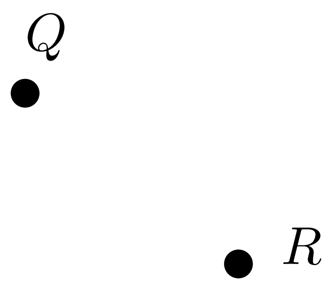

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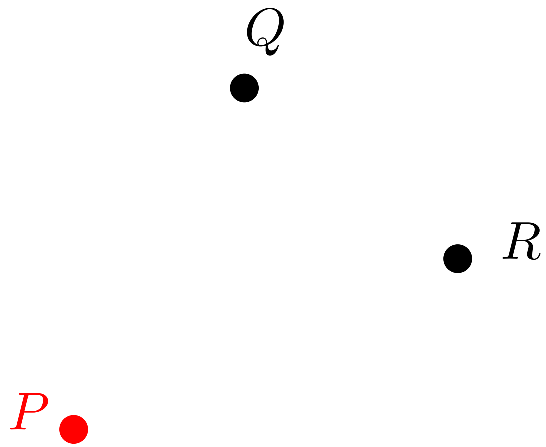

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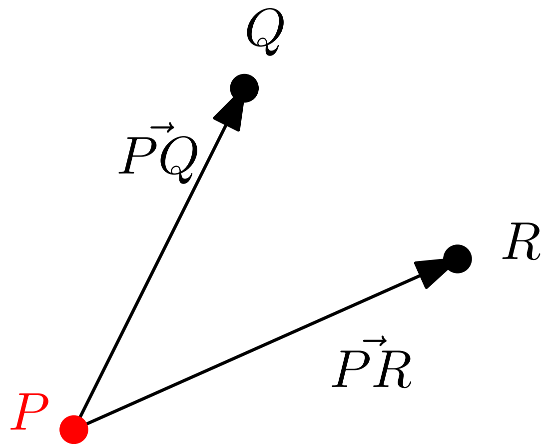

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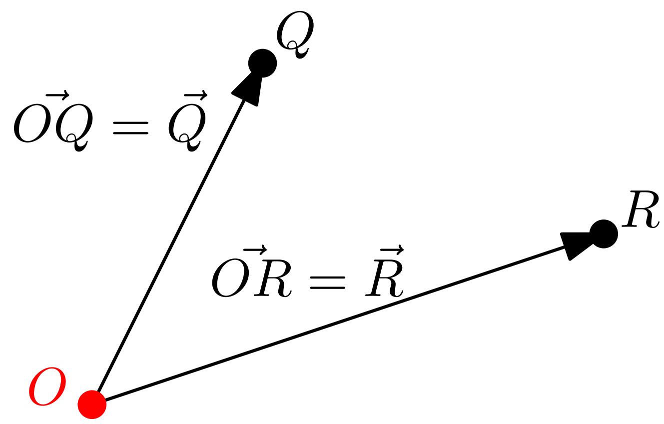

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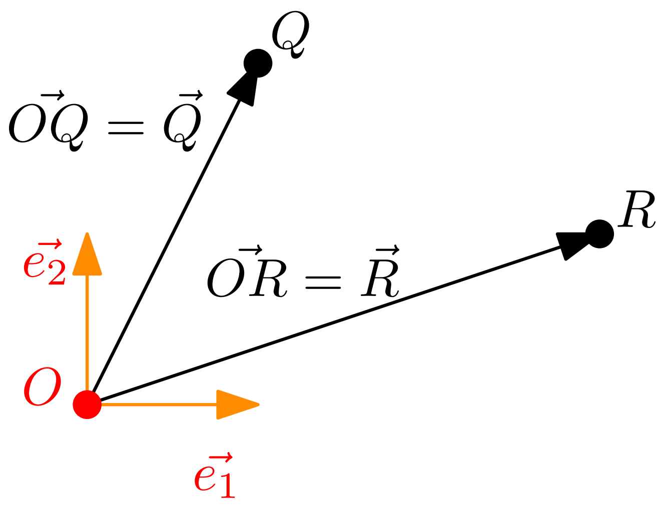

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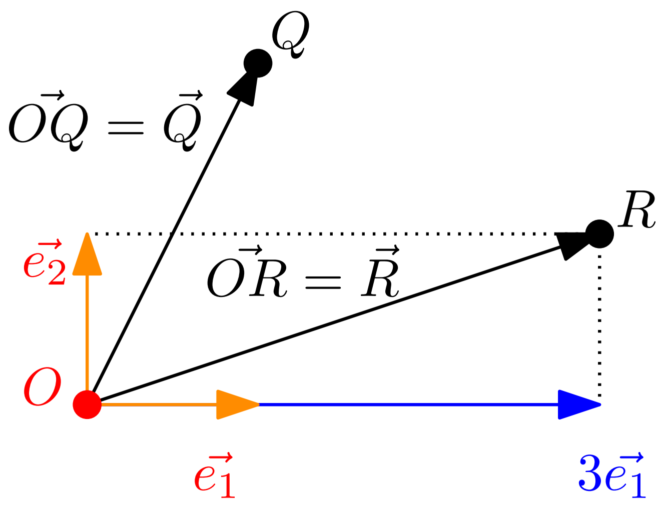

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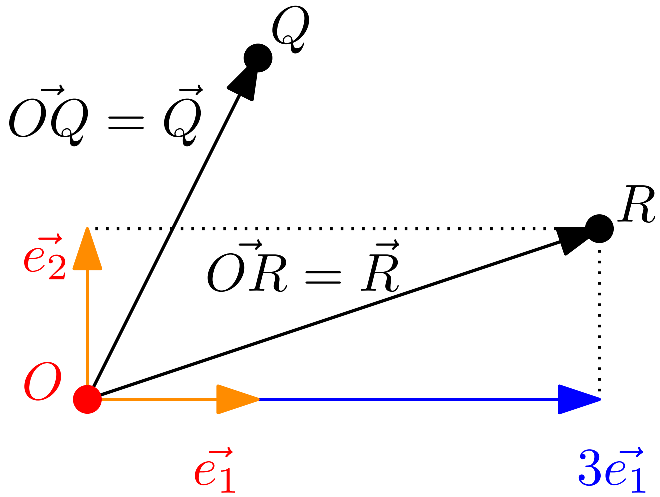

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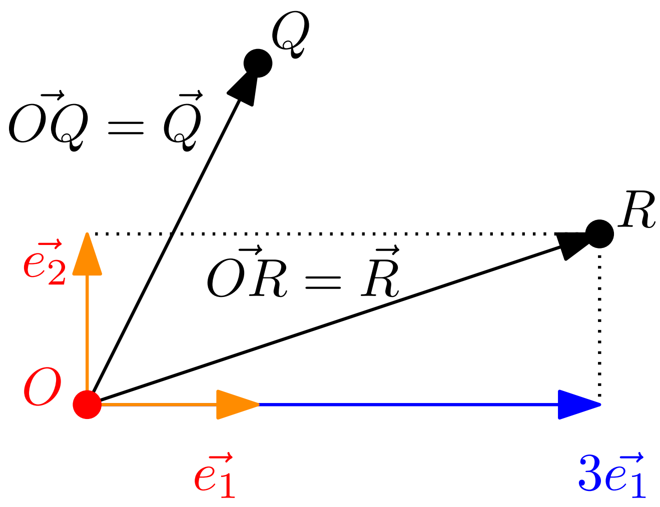

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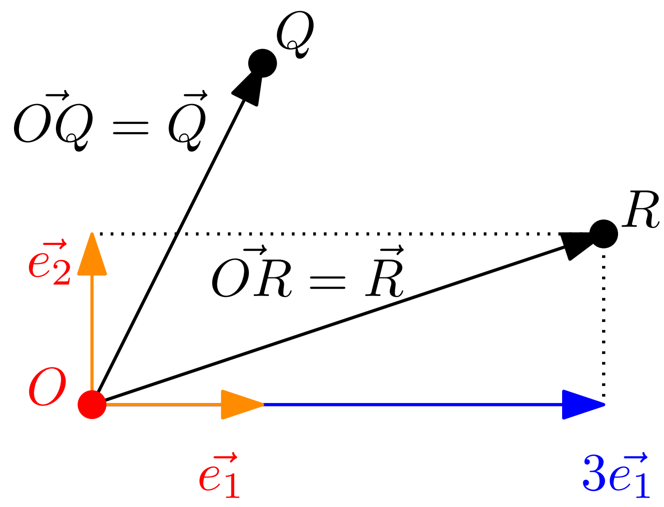
We have the following situation:

A	V	$\mathbb{R}^n$
(points)	(vectors)	(coordinates)

Coordinate systems

A *coordinate system* in A is composed of:

i) A point $O \in A$, called *origin*
ii) A basis $\{\vec{e_1}, \ldots, \vec{e_n}\}$ of V



$R = 3\vec{e_1} + \vec{e_2}$

The coordinates of R
(wrt O, $\vec{e_1}$, and $\vec{e_2}$) are $(3, 1)$

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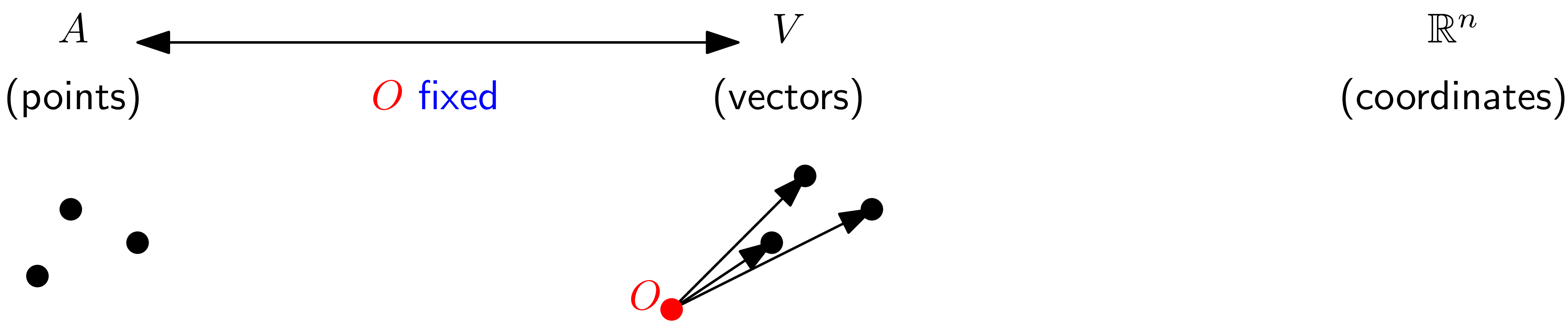

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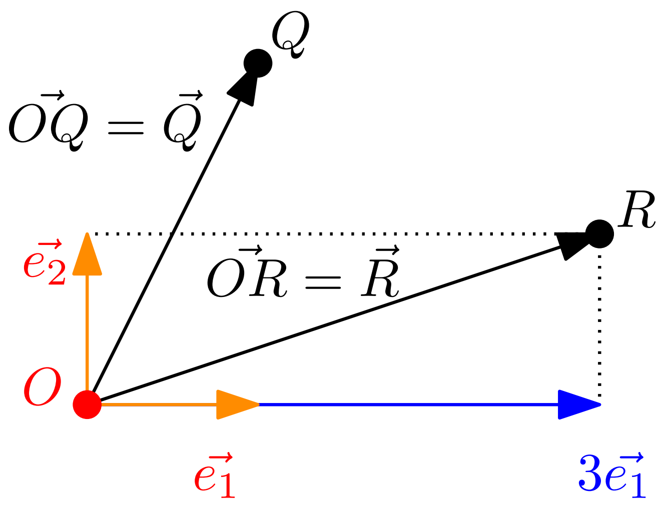

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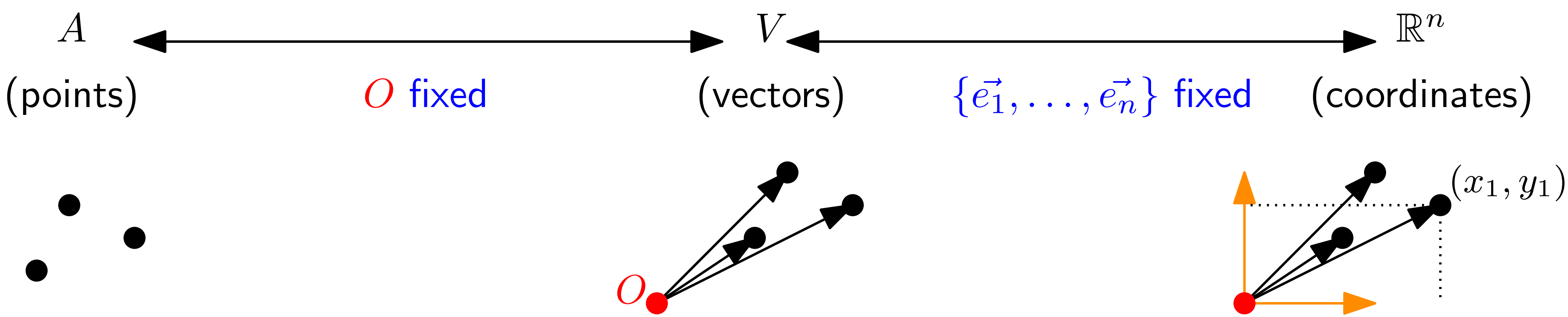

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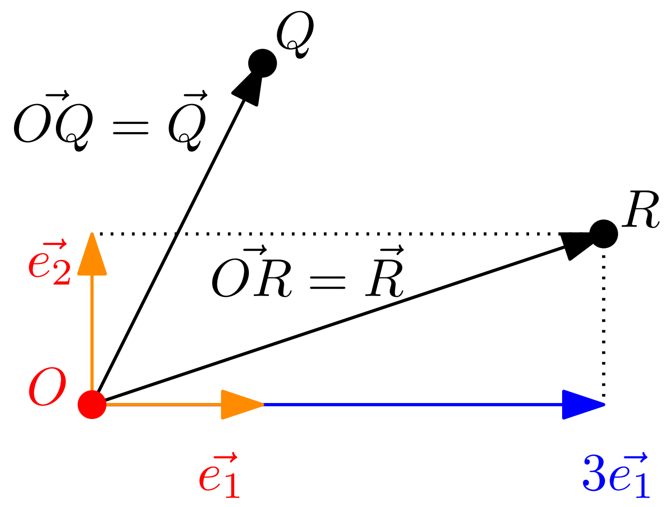
With a coordinate system, we can write both vectors and points as tuples of real numbers (but the meaning remains different!)



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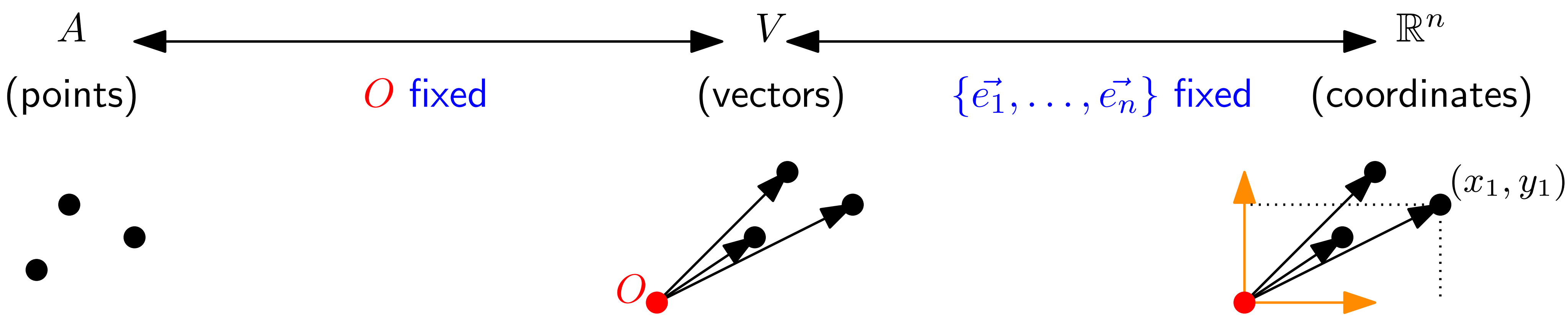

Examples of coordinate systems

Our “usual” system is the *Cartesian coordinate system*

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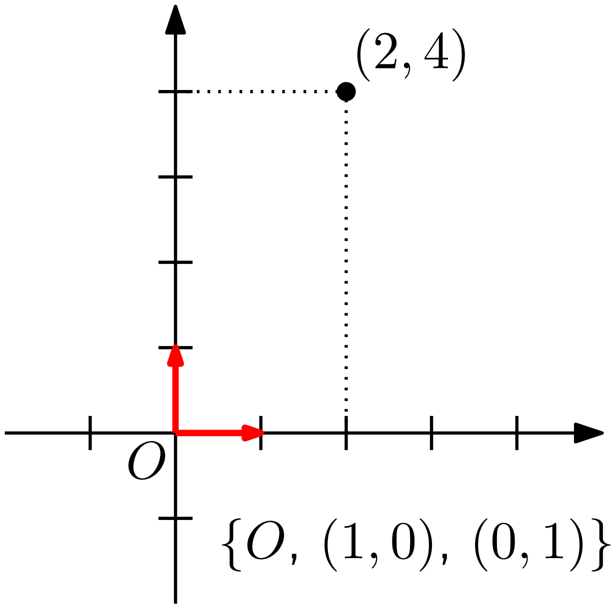

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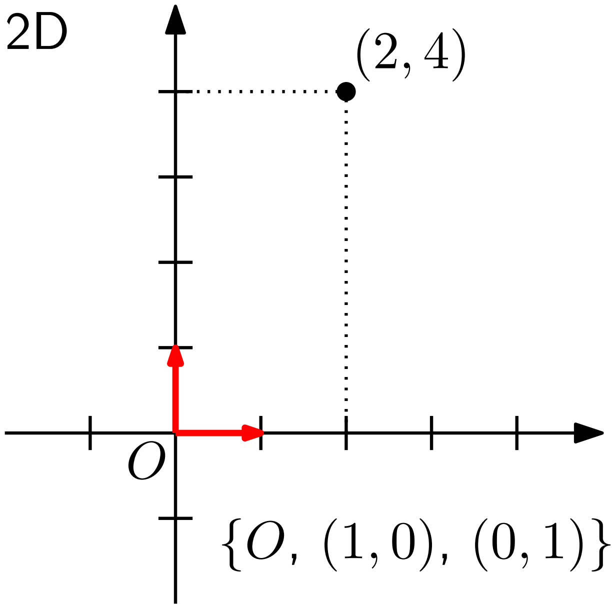

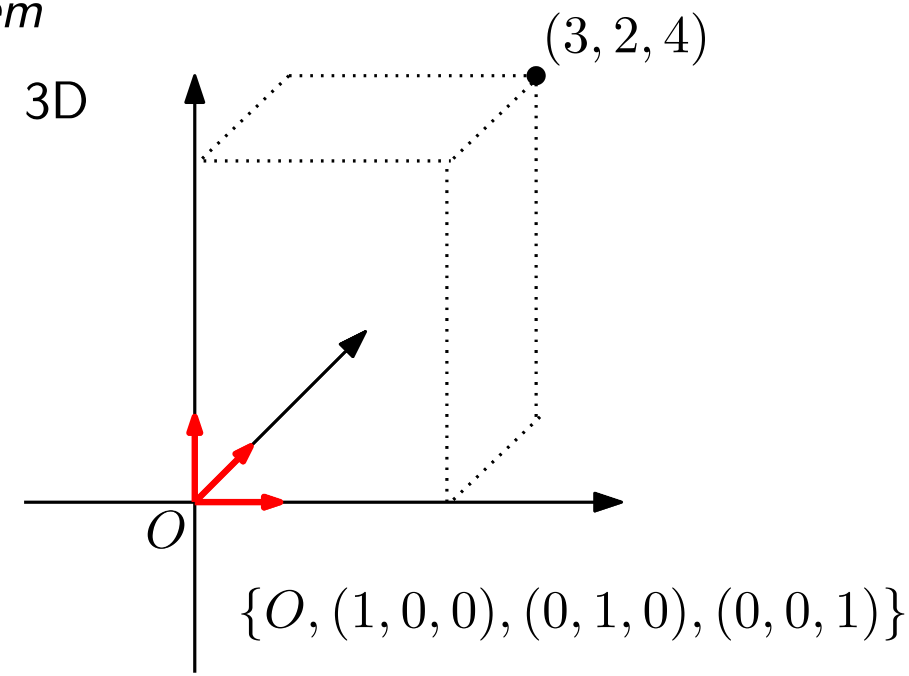

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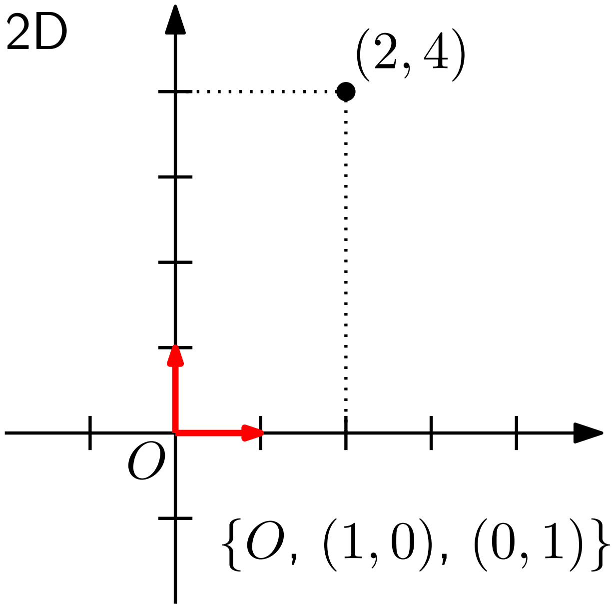

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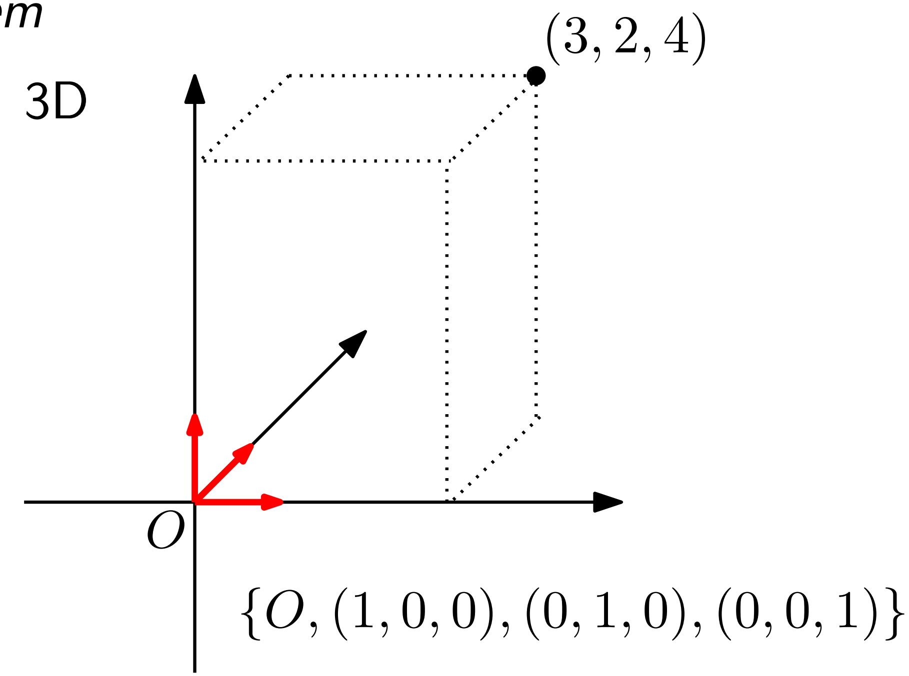
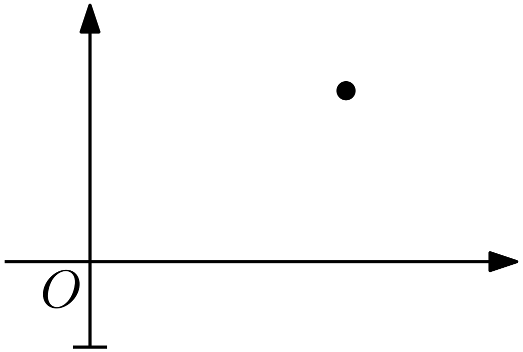


Polar coordinates (2D)



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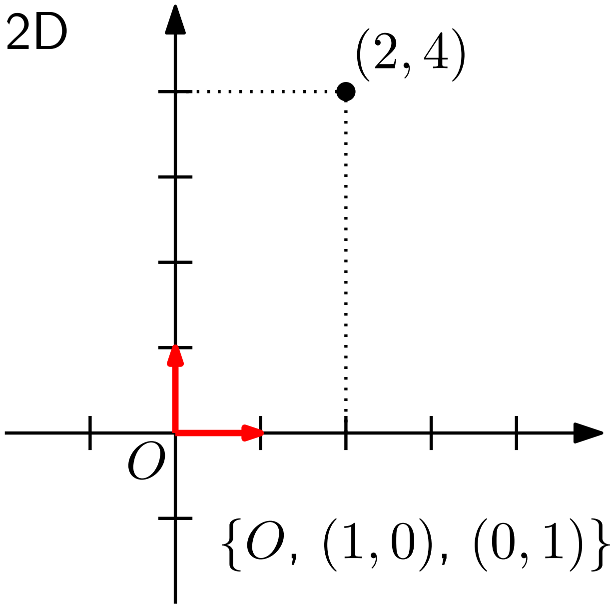

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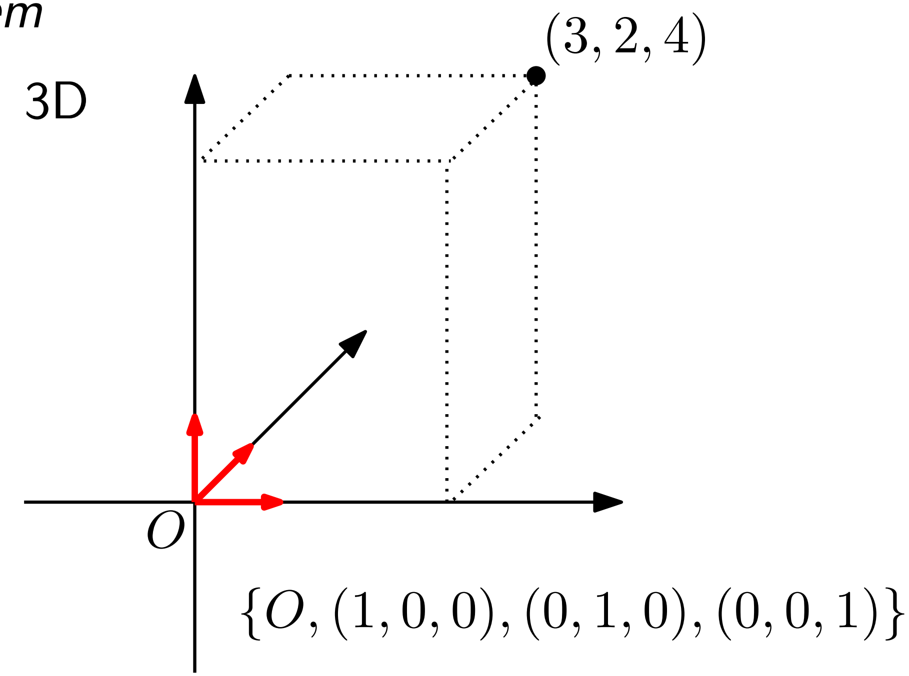

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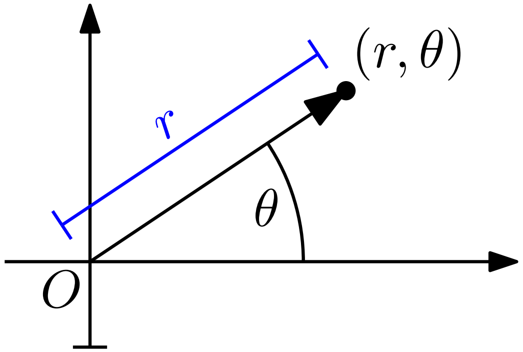

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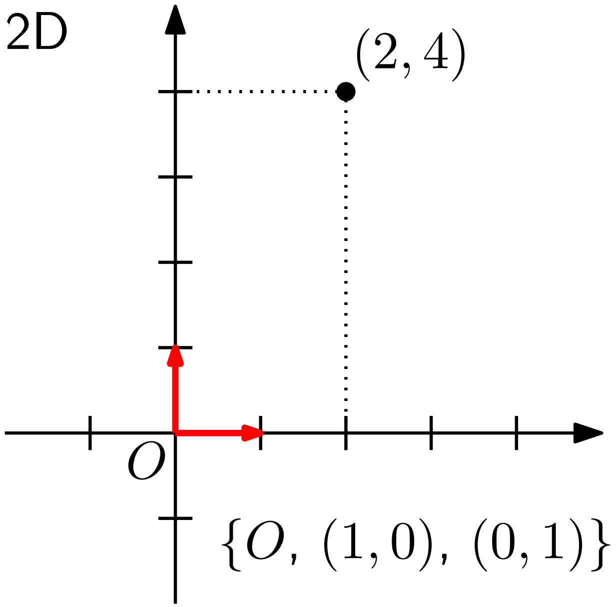

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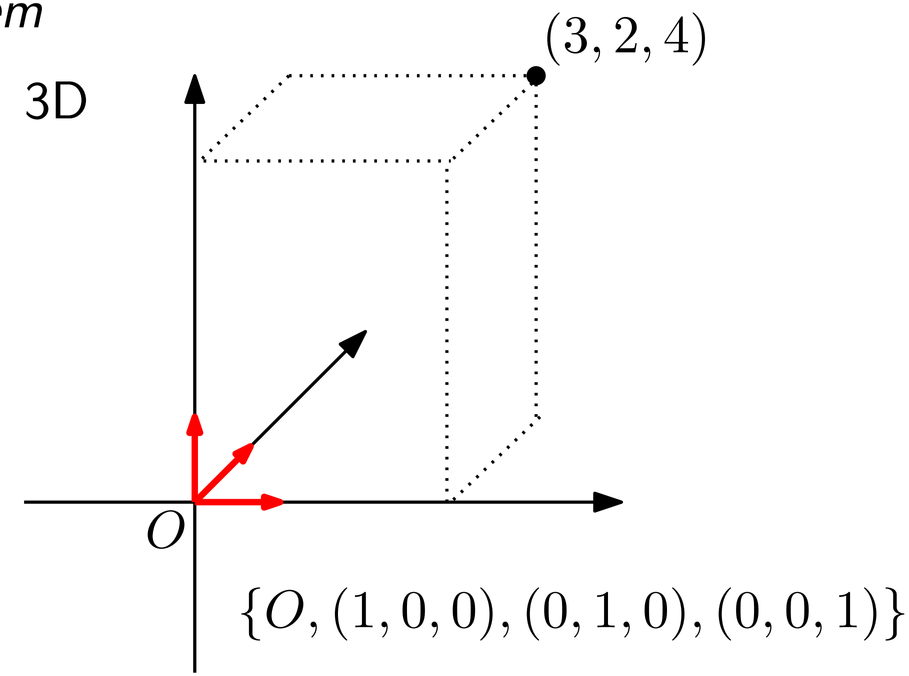

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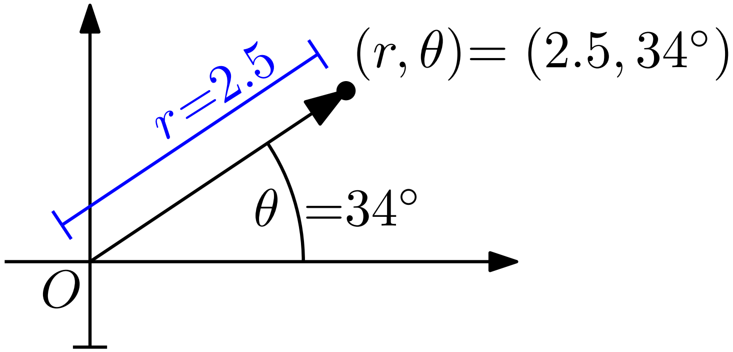

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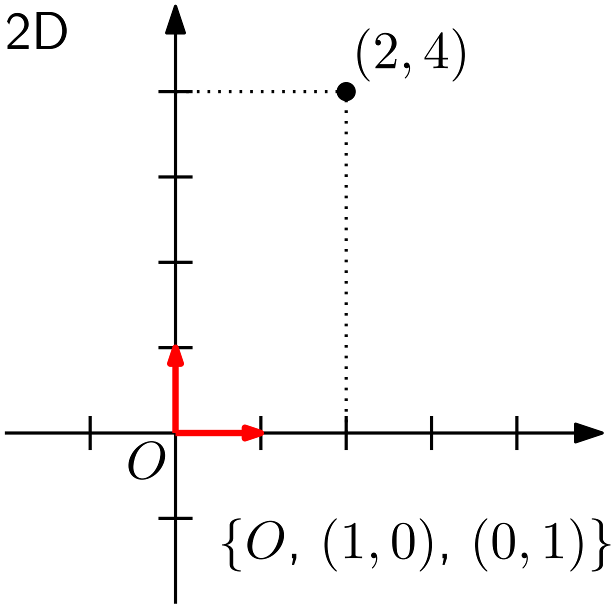

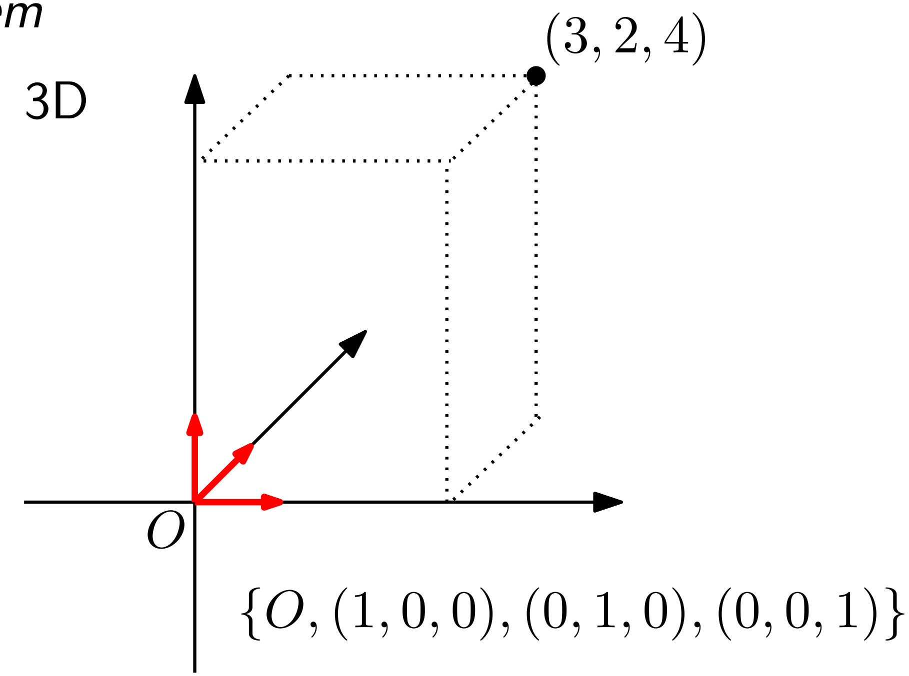

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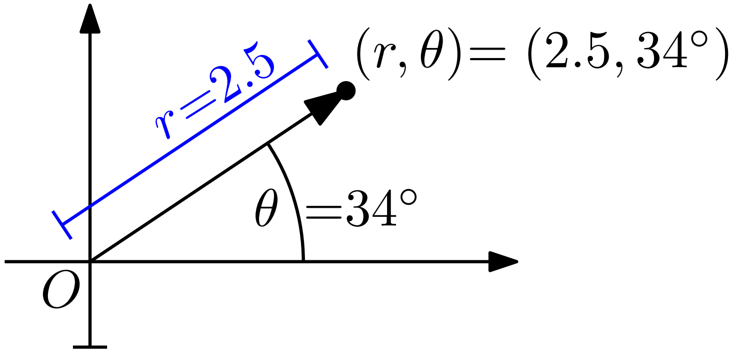

Conversion between Cartesian and polar coordinates

$x = r\cos\theta$ ⟷ $r = \sqrt{x^2 + y^2}$

$y = r\sin\theta$

$\theta = \text{atan}\,\frac{y}{x}$ (if $x > 0$)

$\theta = \text{atan2}(y, x)$ (in general)

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Two ways to generalize polar coordinates to 3D: *spherical* and *cylindrical coordinates*

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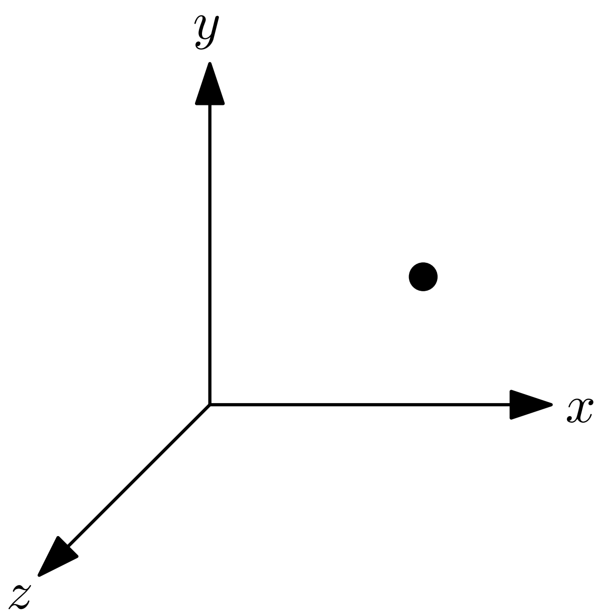

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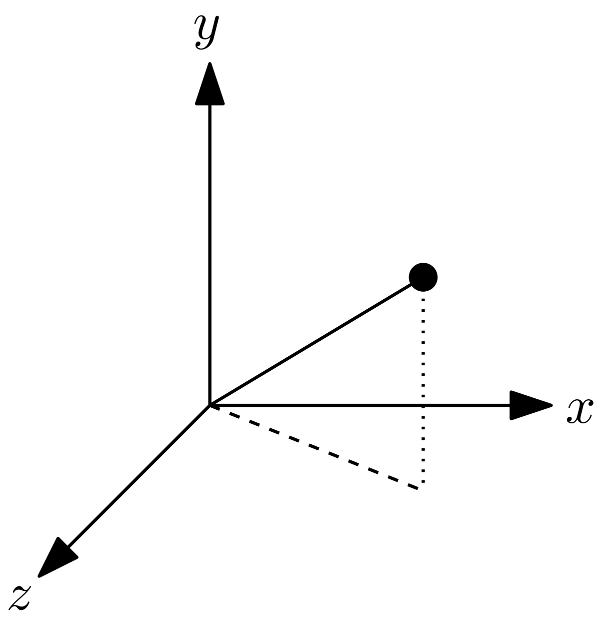

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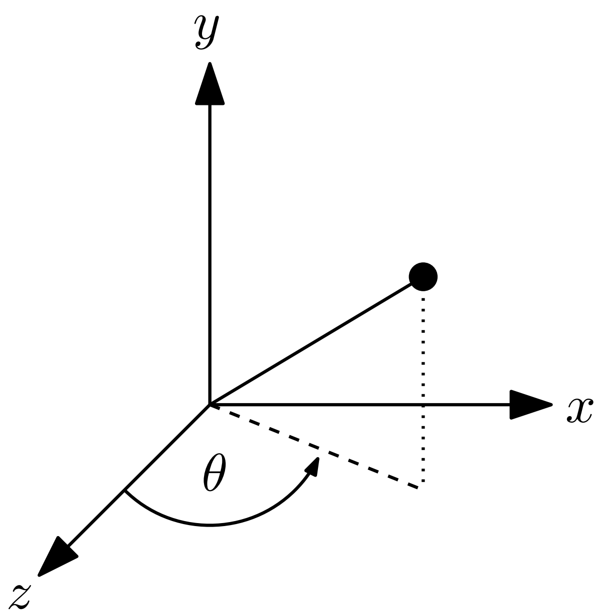

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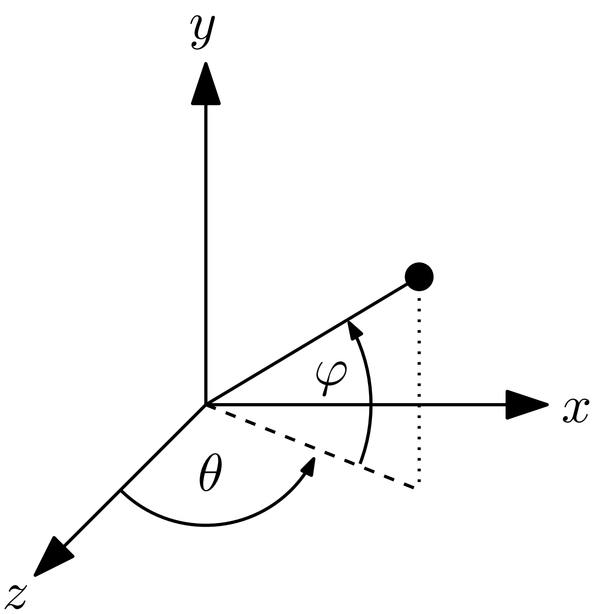

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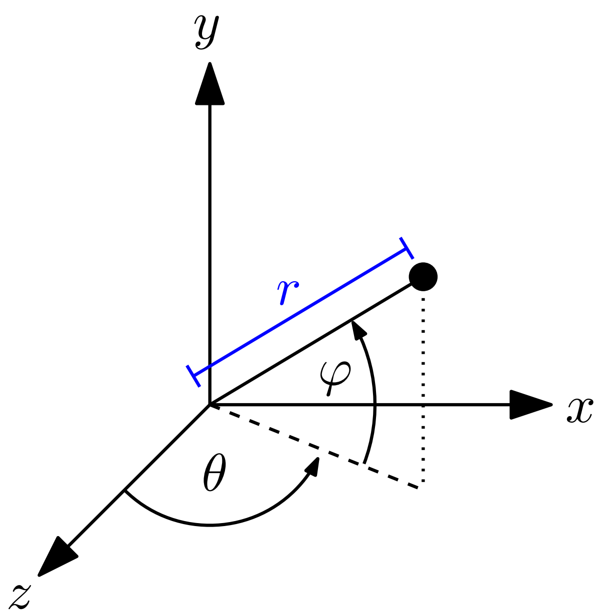

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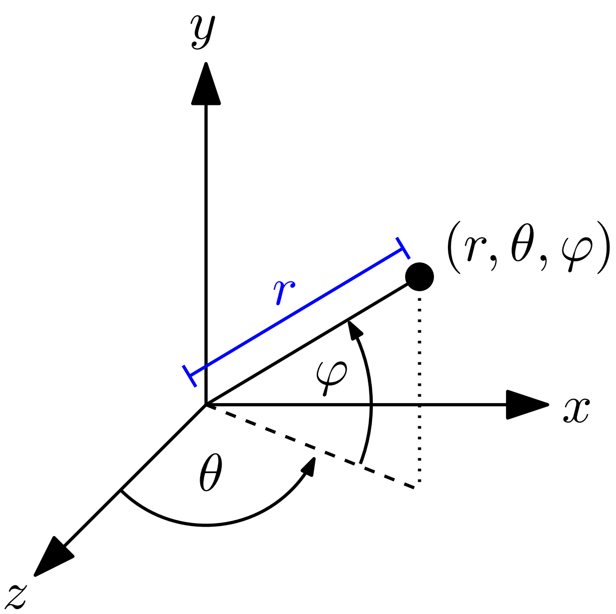

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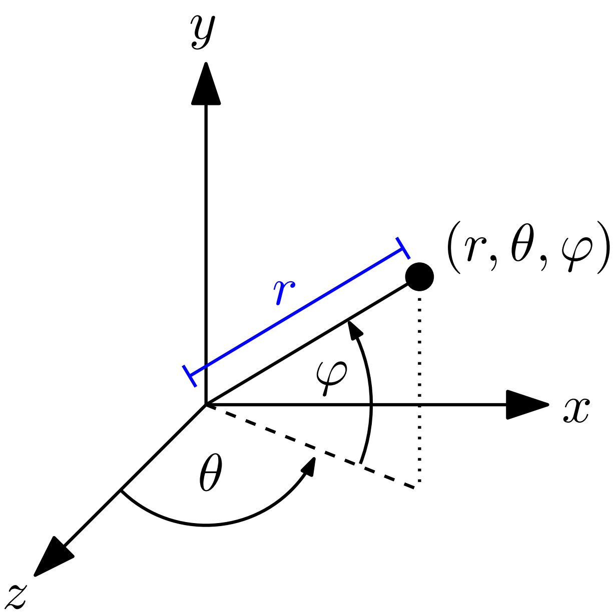

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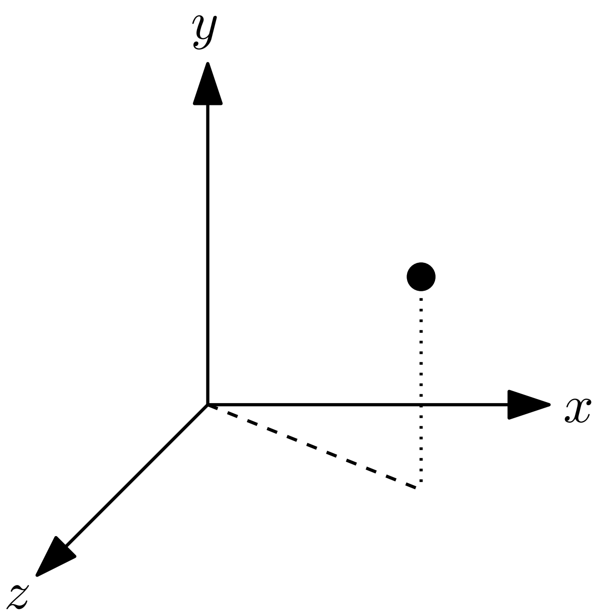

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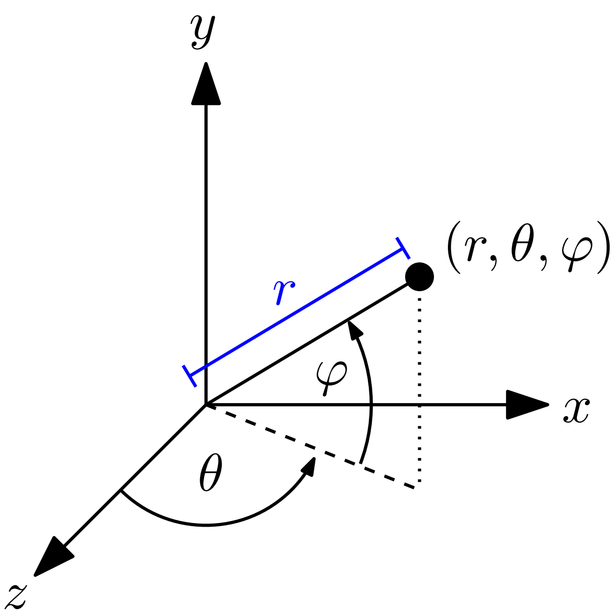

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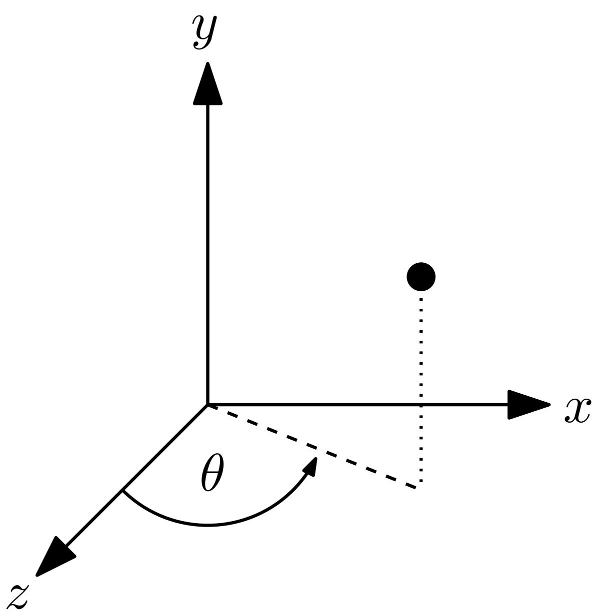

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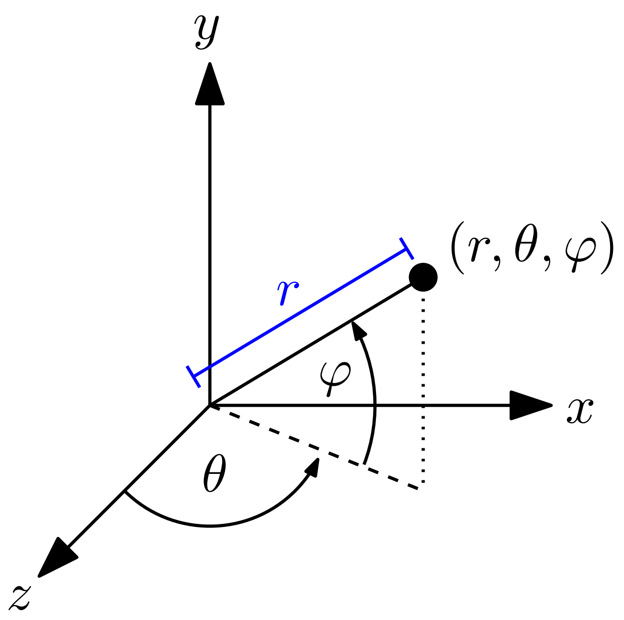

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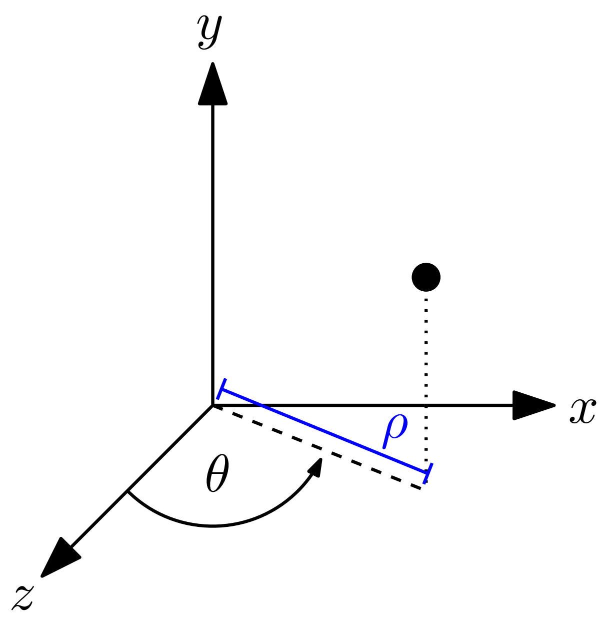

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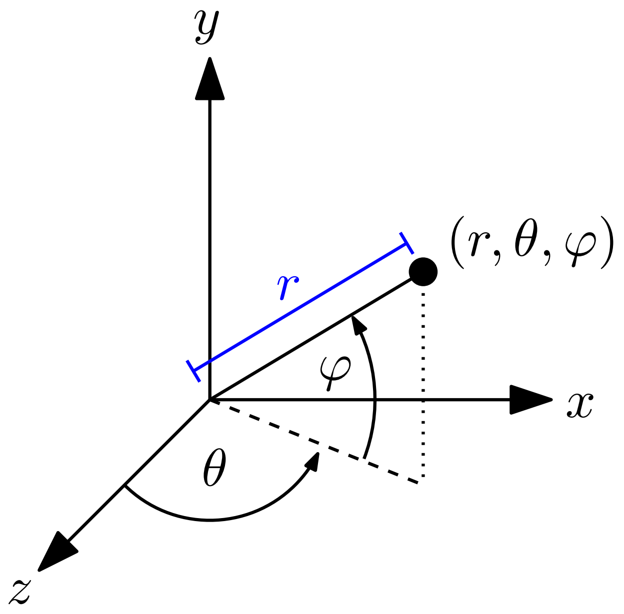

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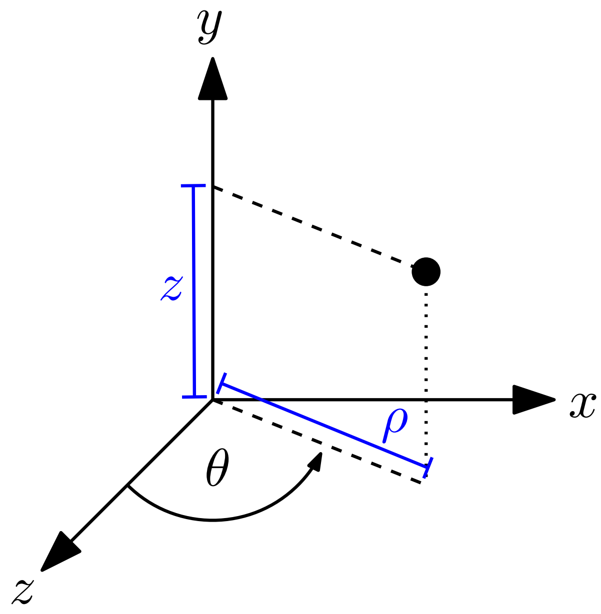

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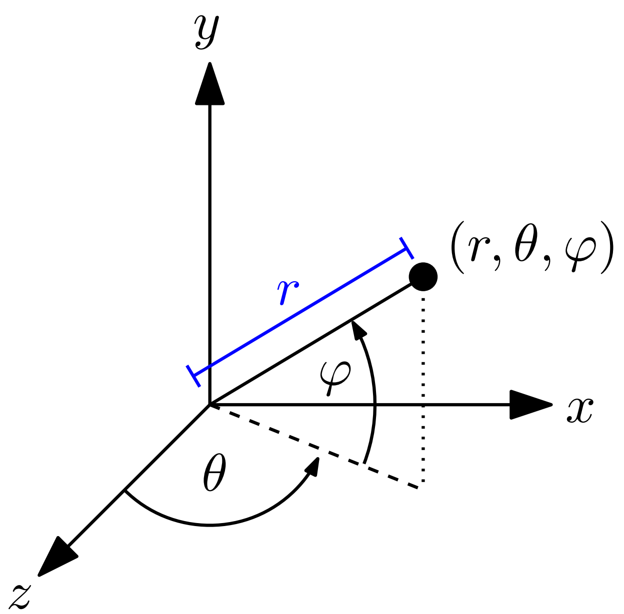

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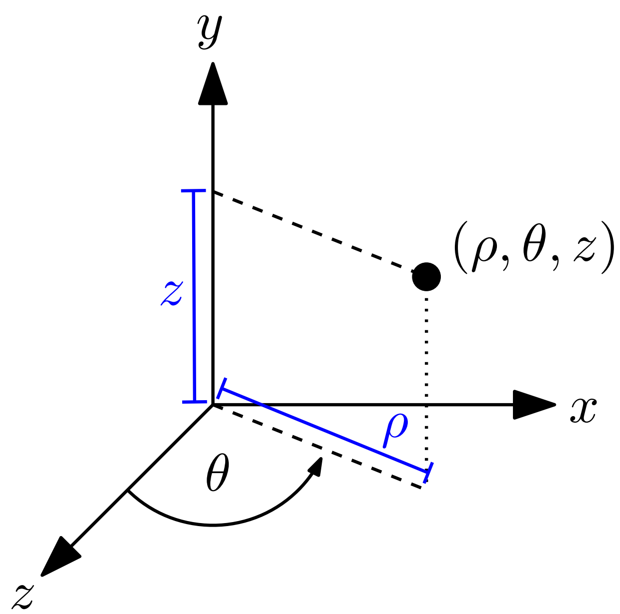

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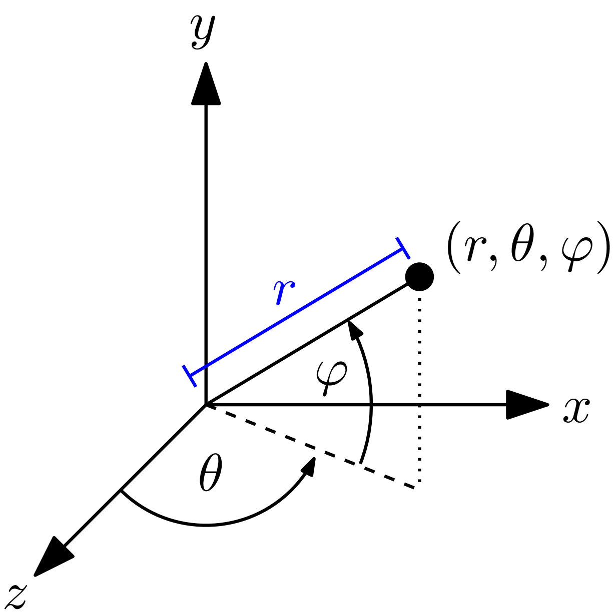

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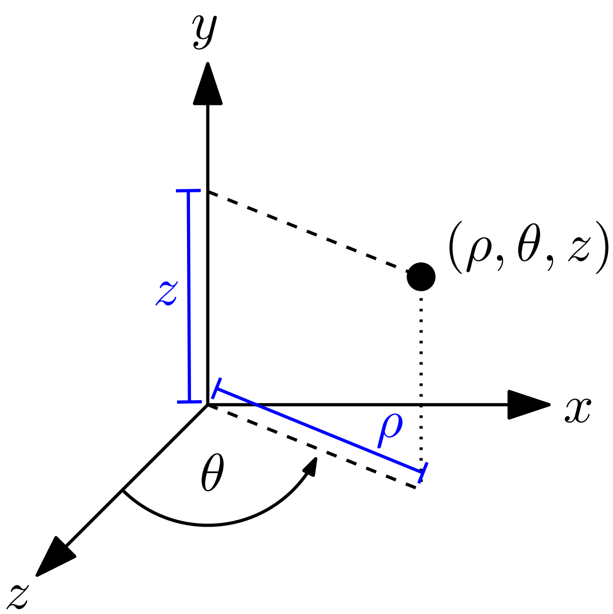

Formulas to convert to and from Cartesian coordinates are well-known

Distance between points

The *distance* between two points P and Q is a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.
For two points $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$, it can be defined as:

$$(P, Q) \to d(P, Q) = ||\vec{PQ}|| = ||Q - P|| = \sqrt{\sum_{i=1}^{n} (q_i - p_i)^2}$$

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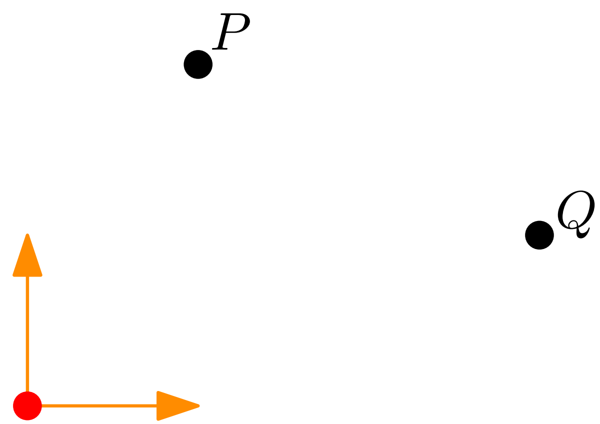

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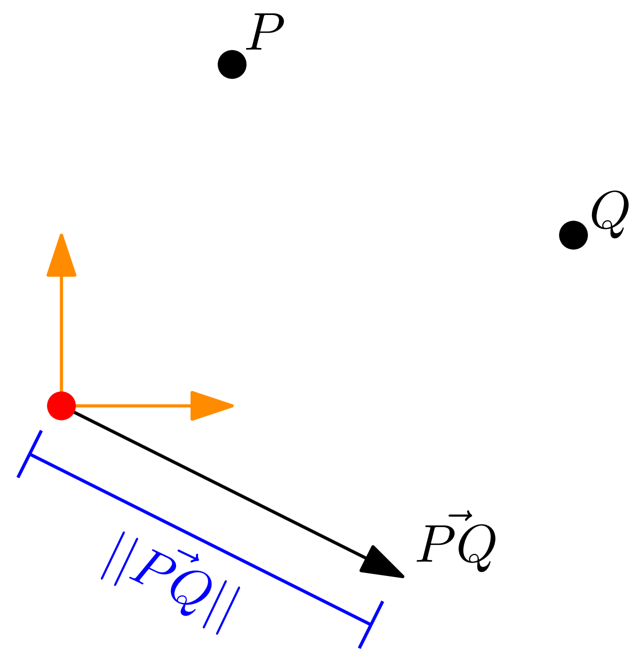

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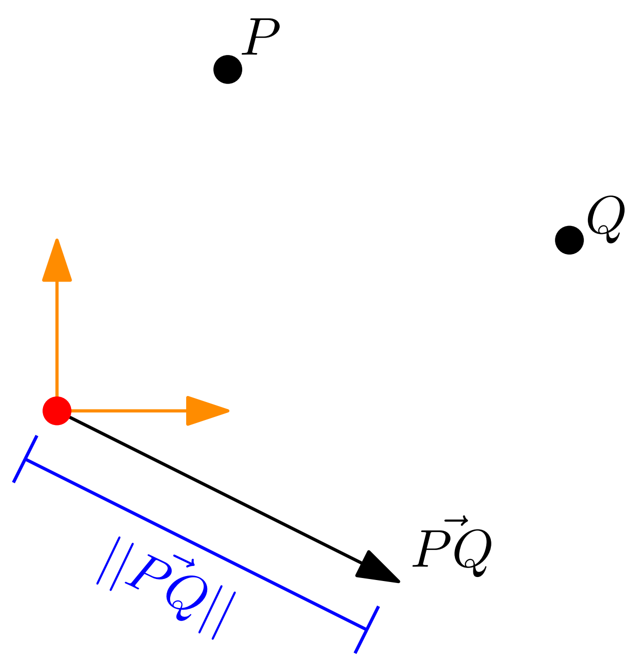

This distance inherits several nice properties

For all $P, Q, R \in \mathbb{R}^n$, we have:

i) Positivity $d(P, Q) \geq 0$, and $d(P, Q) = 0 \iff P = Q$
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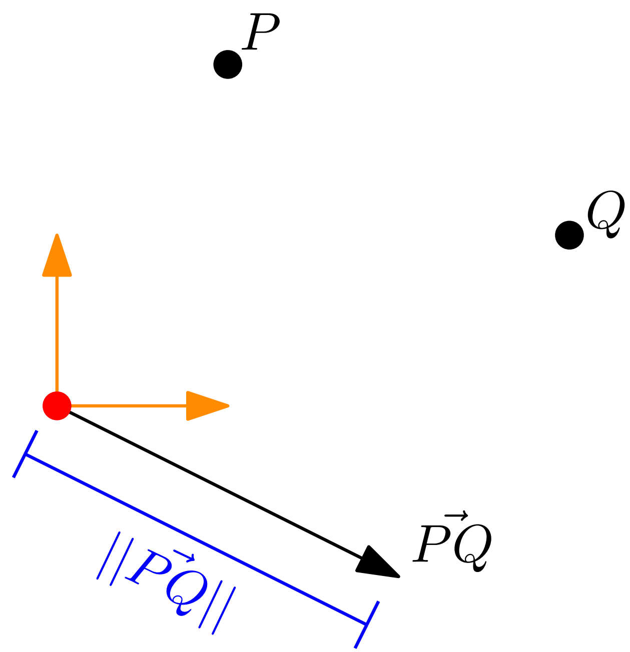

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Any set of points with a distance function that satisfies these properties forms a *metric space*. As you can guess, there are many different metric spaces!

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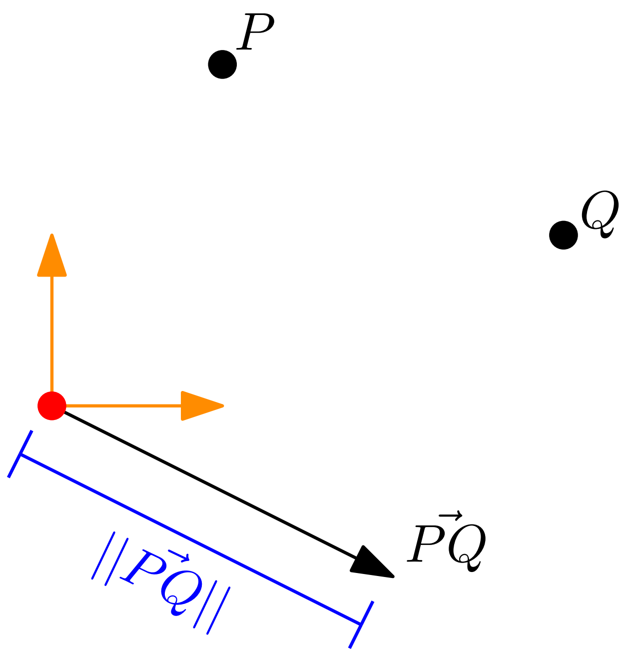

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Do you know any other distance functions?

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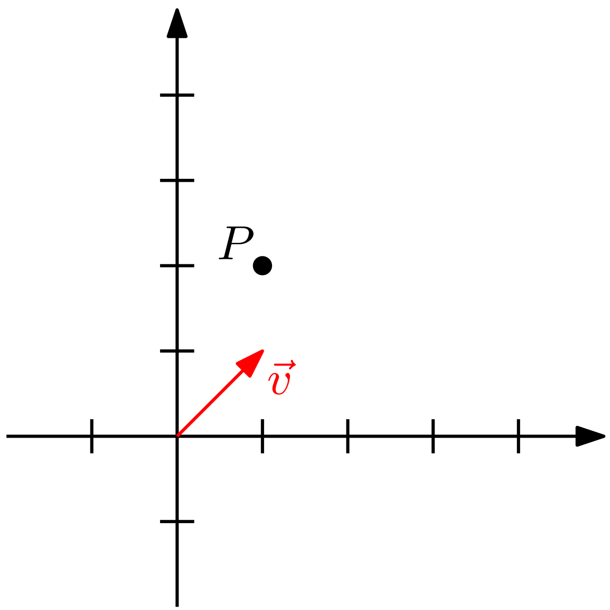

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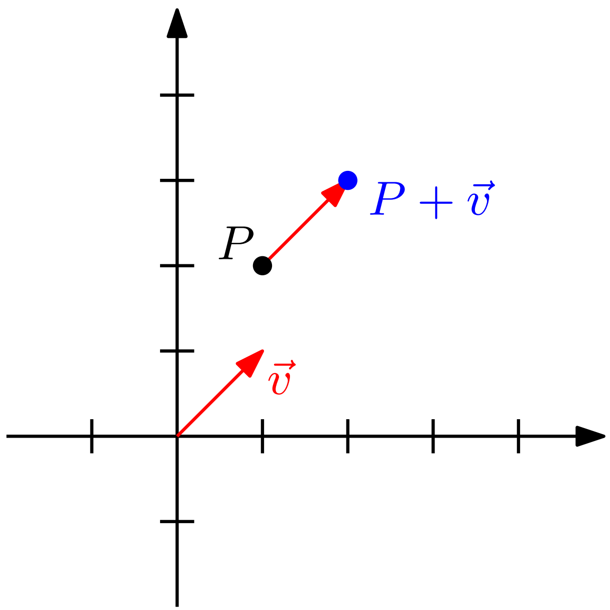

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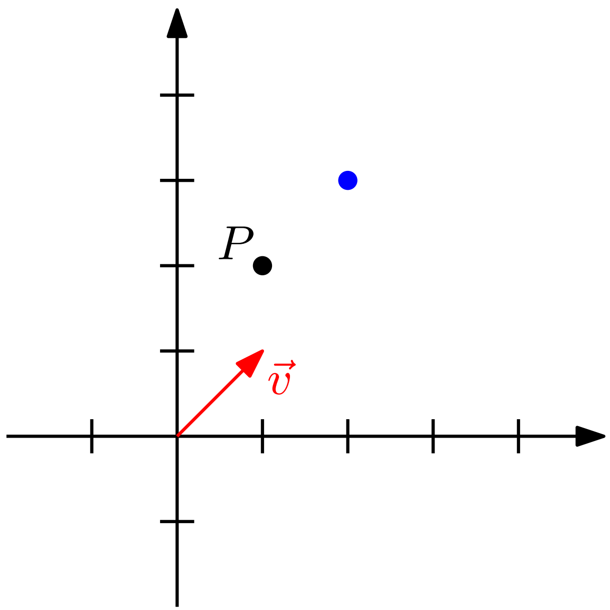

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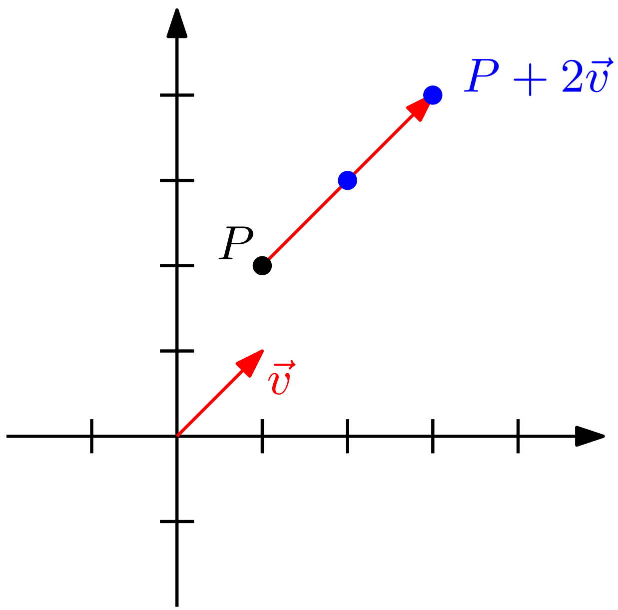

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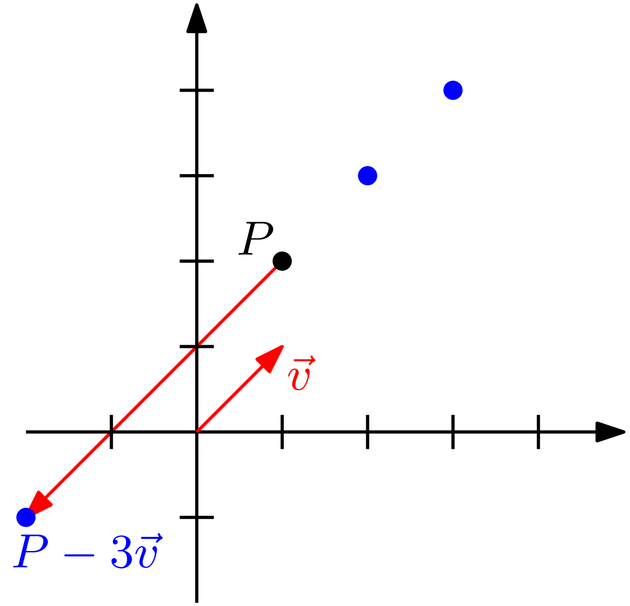

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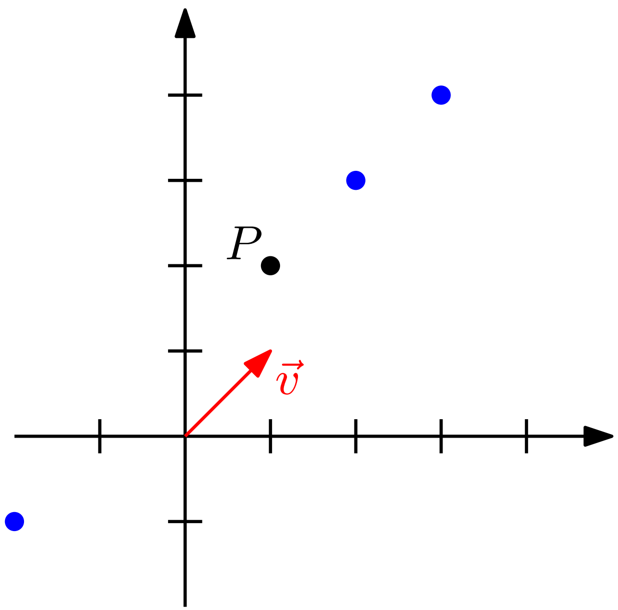

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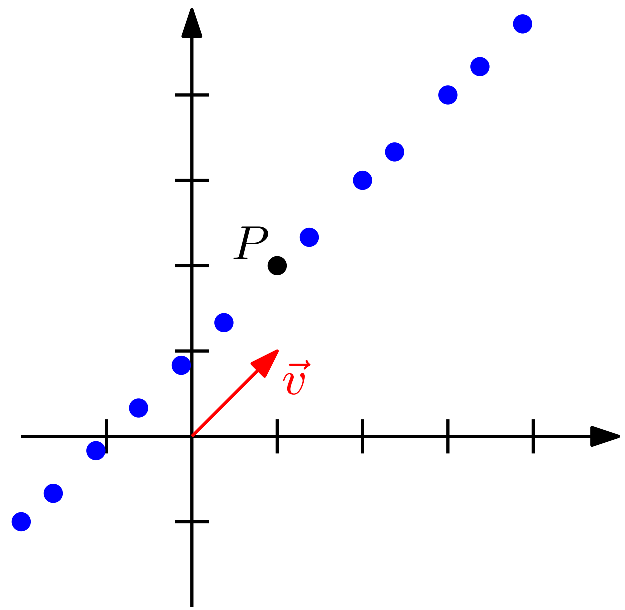

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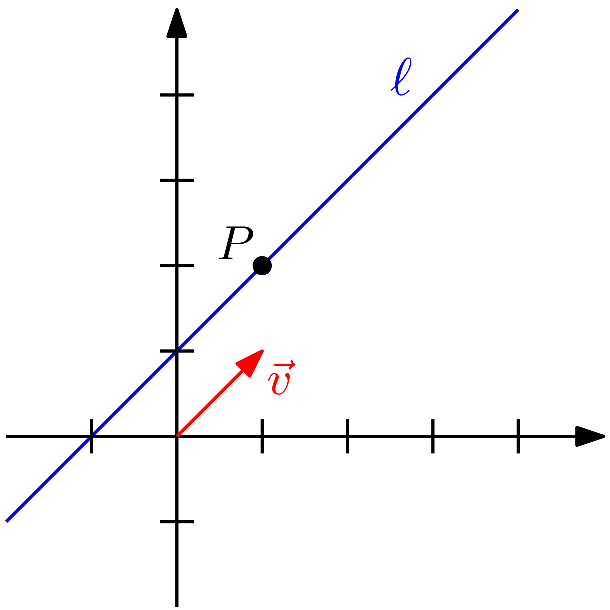

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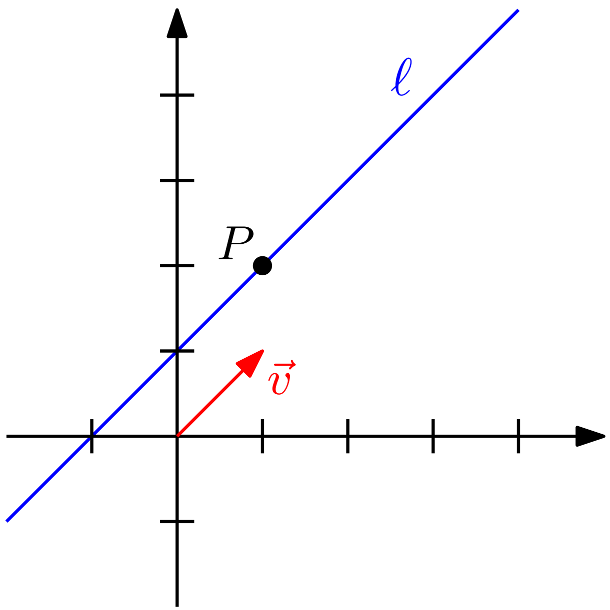

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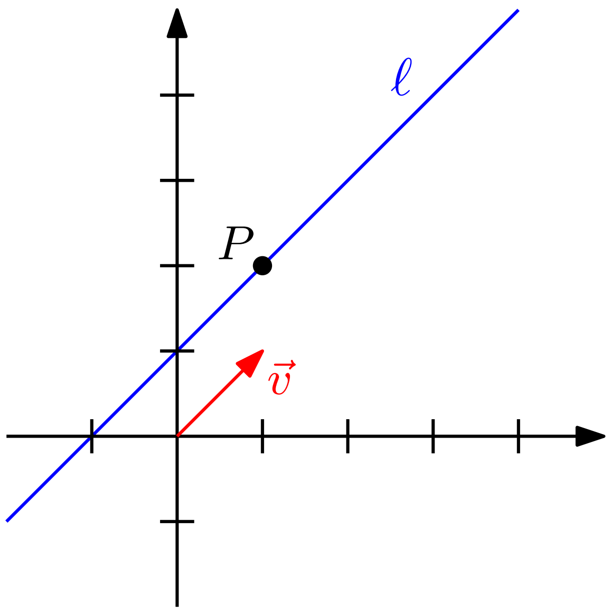

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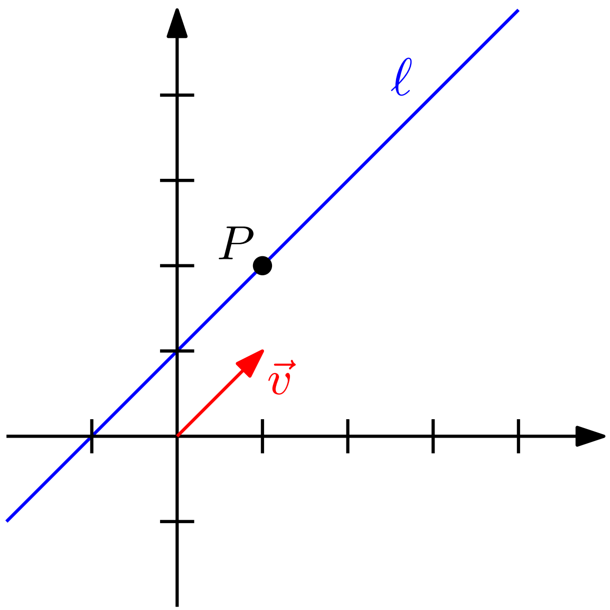

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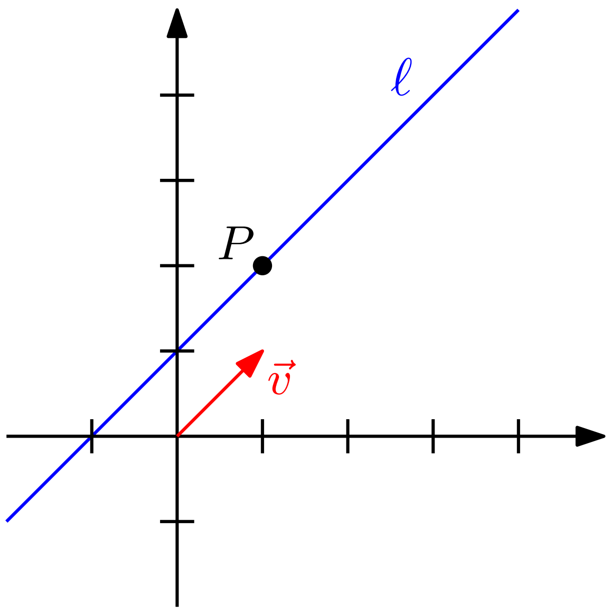

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$(1 - \lambda)P + \lambda Q \quad \lambda \in [0, 1]$ (since $\overrightarrow{PQ} = \vec{Q} - \vec{P}$)

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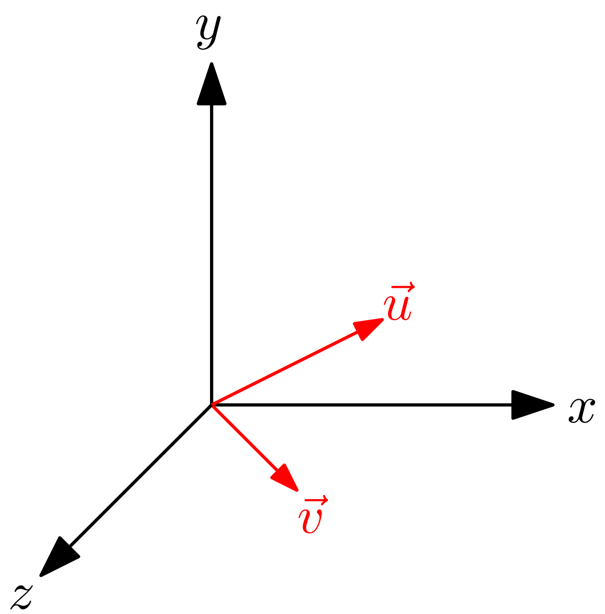

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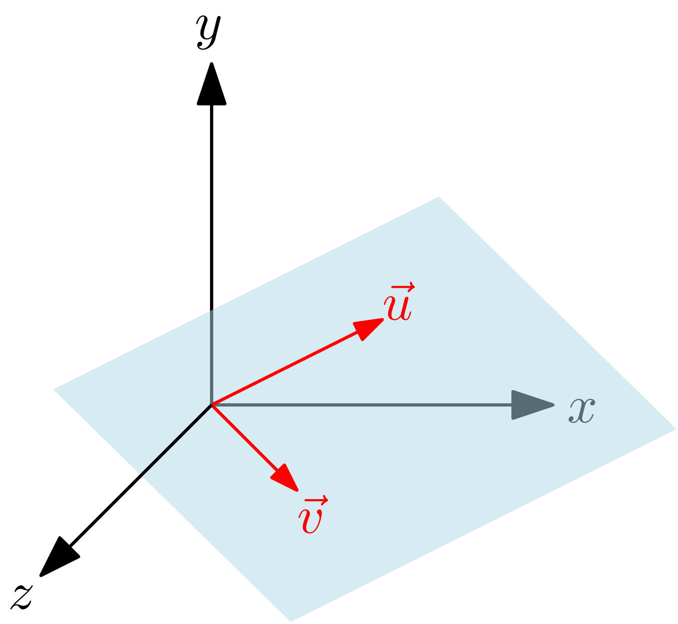

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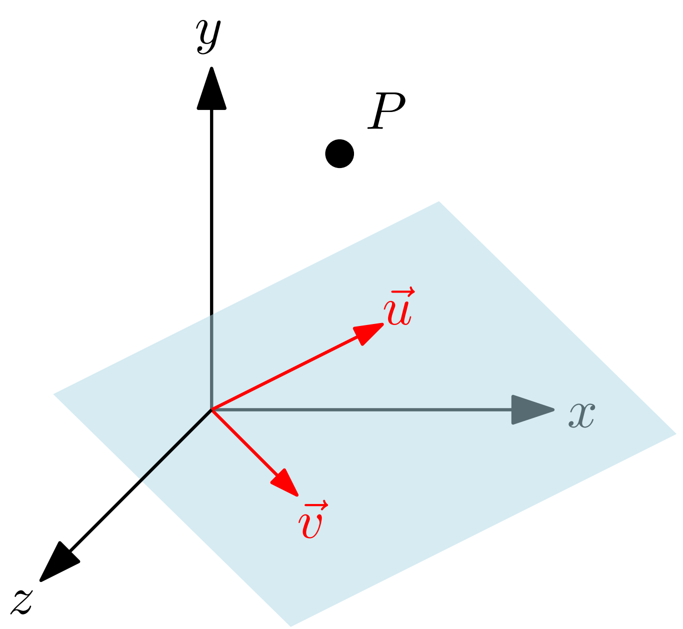

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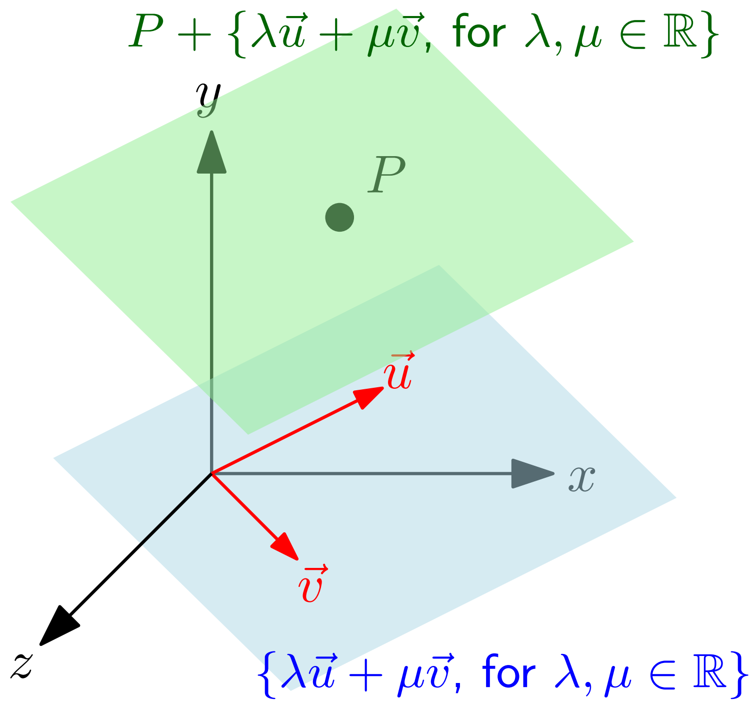

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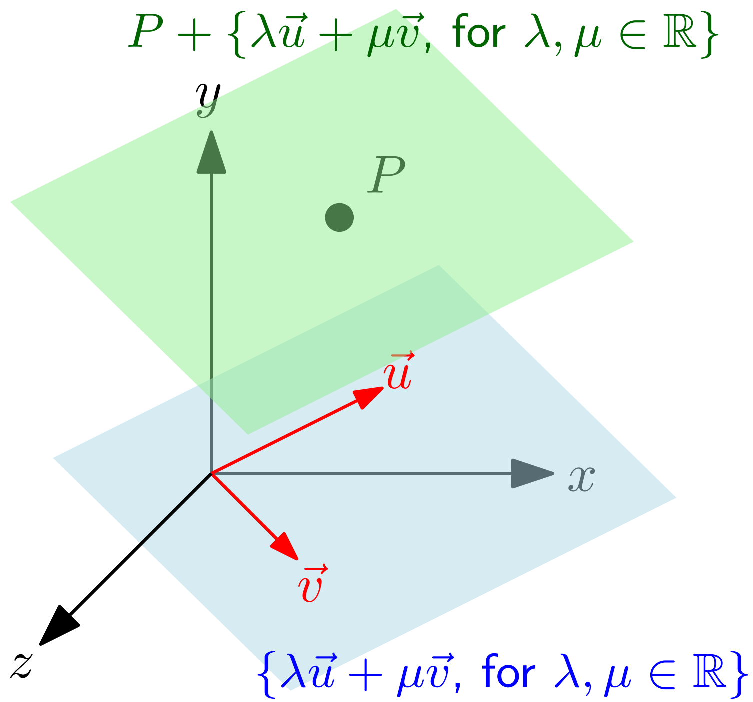

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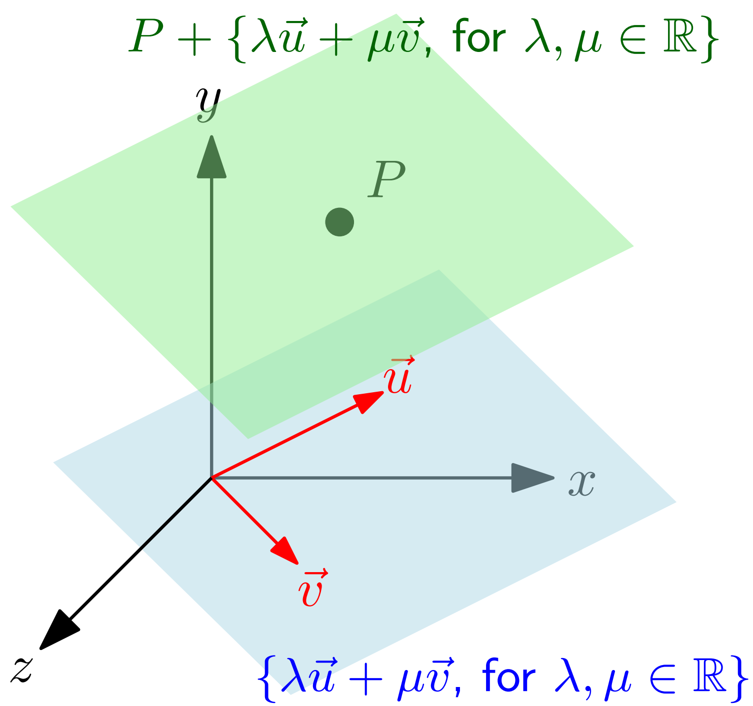

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Given three points P, Q, R, if they are not collinear, they define a unique plane:

$$\pi = P + \lambda\overrightarrow{PQ} + \mu\overrightarrow{PR}, \text{ for } \lambda, \mu \in \mathbb{R}$$

Or, equivalently:

$$\pi = (1 - \lambda - \mu)P + \lambda Q + \mu R, \text{ for } \lambda, \mu \in \mathbb{R}$$

Combining vectors and points

Consider n vectors $\vec{v}_1, \ldots \vec{v}_n$ and n real scalars values $\lambda_1, \ldots, \lambda_n$. Then

$$\vec{v} = \lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n = \sum_{i=1}^{n} \lambda_i \vec{v}_i \quad \text{is a}$$

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What does it mean geometrically?

Linear and affine transformations

Let U and V be vector spaces over $\mathbb{R}$. A *linear transformation* is a function from U to V s.t.:
For all $x, y \in U$ and $\lambda \in \mathbb{R}$,

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Affine transformations

An *affine transformation* is a function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that, once a reference system is fixed, we have

$$f(X) = AX + \vec{W}$$

for A a matrix in $\mathbb{R}^{m \times n}$, $\vec{W} \in \mathbb{R}^m$ a fixed vector, and $X \in \mathbb{R}^n$ a vector or point

Some useful linear transformations

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Scaling (by factors $\lambda_1, \dots, \lambda_n, \neq 0$, with respect to the origin)

$$f_{\lambda_1,\dots,\lambda_n}(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n)$$

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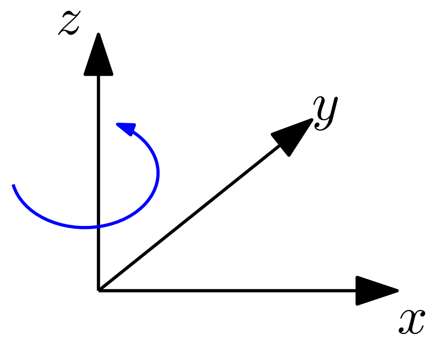
What is R_α^{-1}? What does it mean geometrically?

CCW rotation around z-axis



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Around x-axis

$$R_\alpha^x \vec{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

Around y-axis

$$R_\alpha^y \vec{v} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Around z-axis

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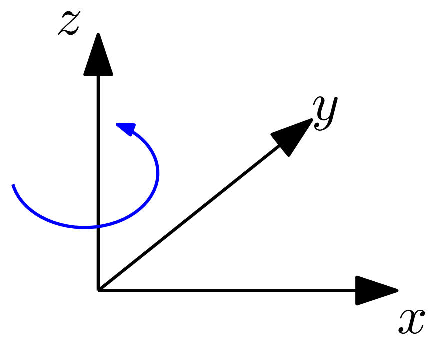

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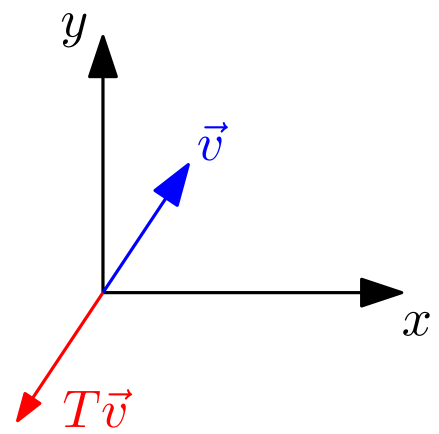
Around z-axis

$$R^z_\alpha \vec{v} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General rotation: $(R^x_\alpha R^y_\beta R^z_\gamma)\vec{v}$

More useful linear transformations

2D central inversion (or reflection w.r.t. origin)



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Can be seen as a rotation by π, or as uniform scaling by -1

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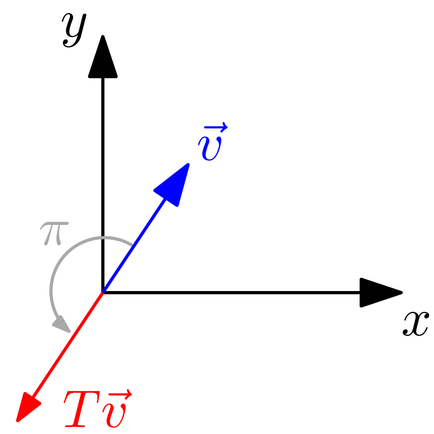

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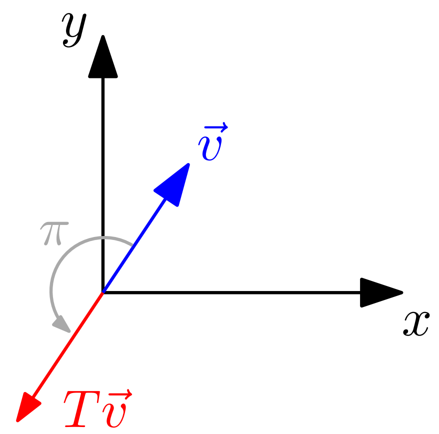

2D reflection across a line

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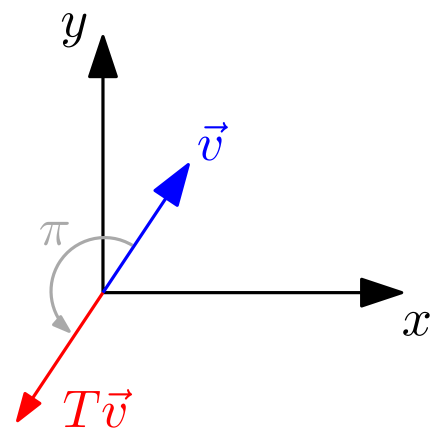

2D reflection across a line

Easy if the line ℓ is the horizontal axis: just negate the y-coordinate:

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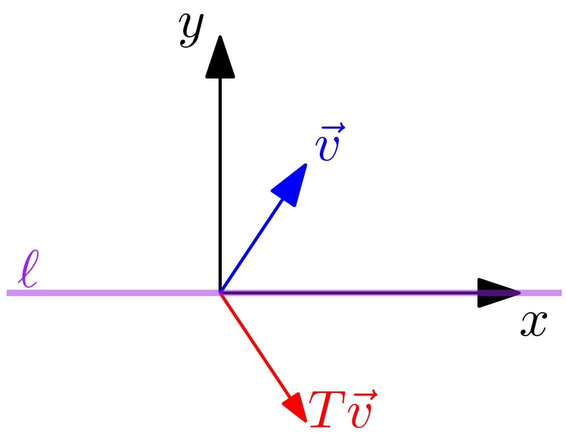

More useful linear transformations

2D central inversion (or reflection w.r.t. origin)

Can be seen as a rotation by π, or as uniform scaling by -1

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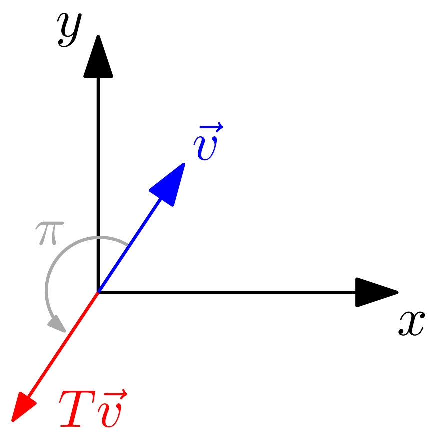

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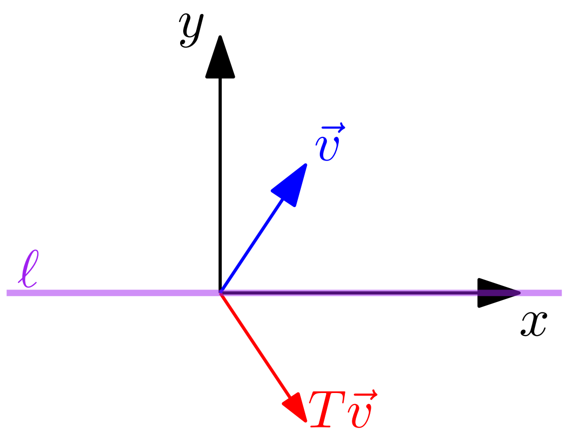

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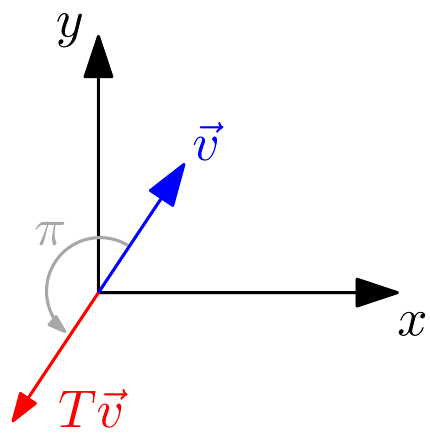

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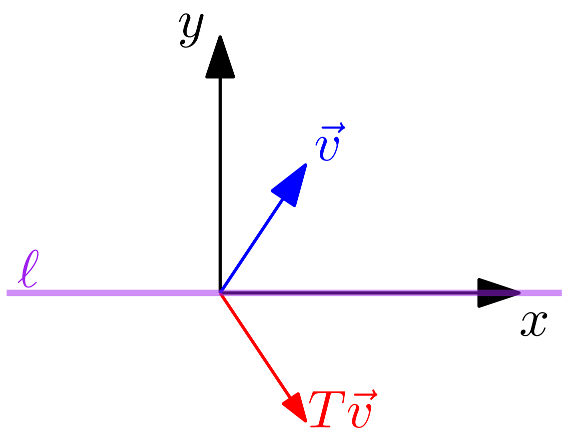

What if ℓ is some other line (through the origin)?

1) Rotate everything so that ℓ becomes the x-axis.
2) Reflect as above.
3) Rotate back with the opposite of the angle used in 1)

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Parallel projection–3D→2D

Projection onto plane $z = 0$ w.r.t. a direction $\vec{u} = (u_1, u_2, u_3)$

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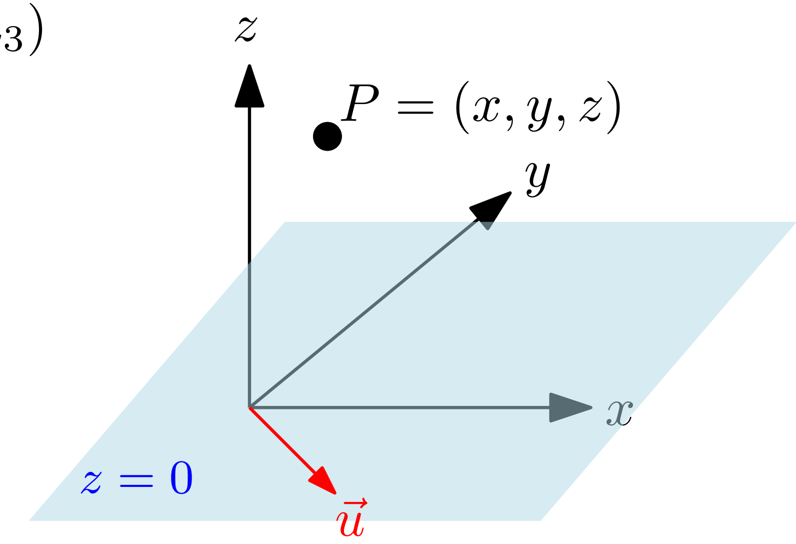

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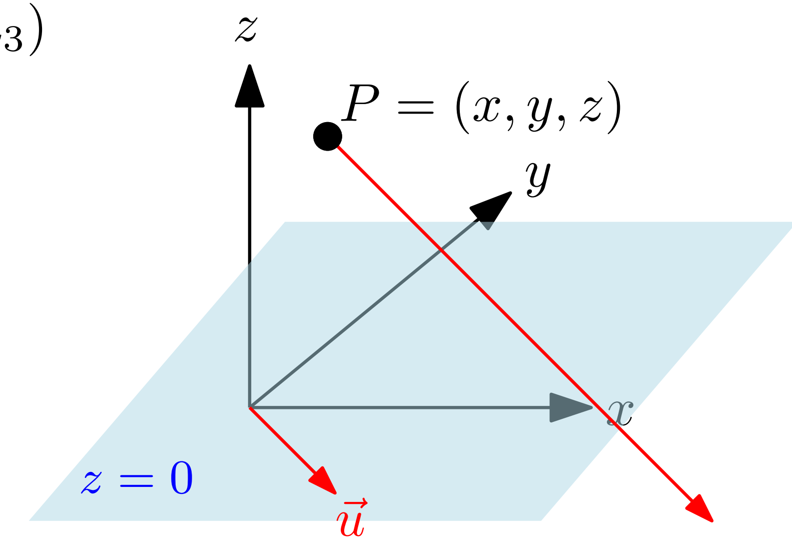

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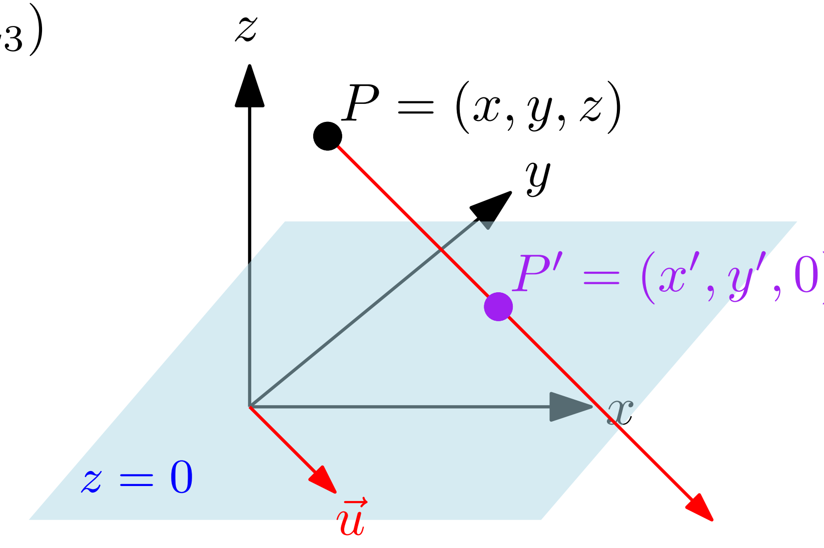

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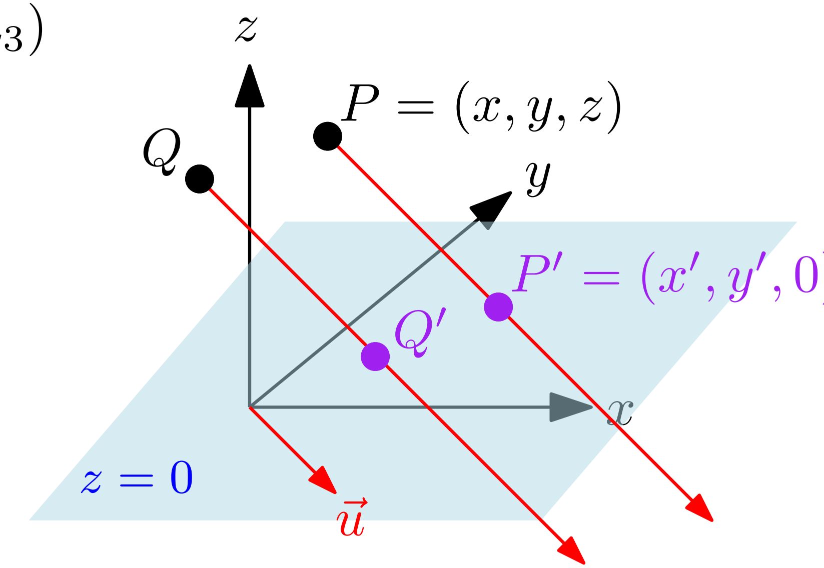

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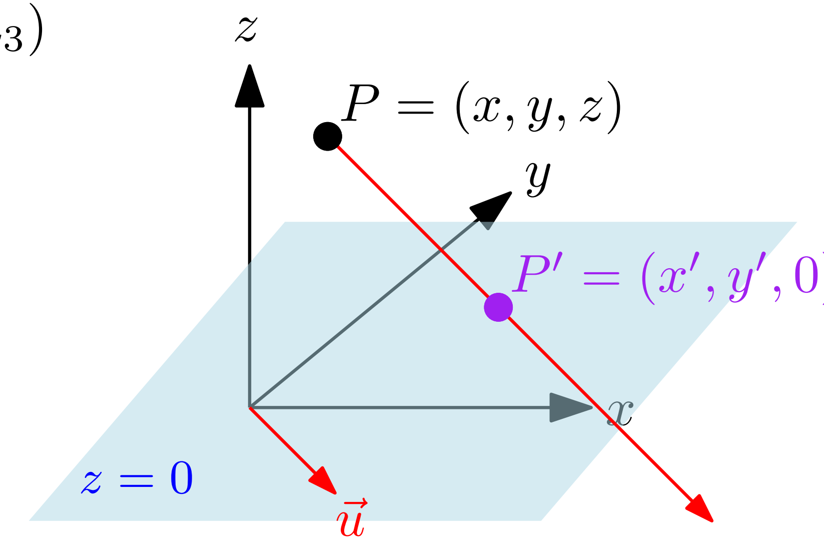

Function $(x, y, z) \rightarrow (x', y', z')$ such that:

- $x' = x + \lambda u_1$
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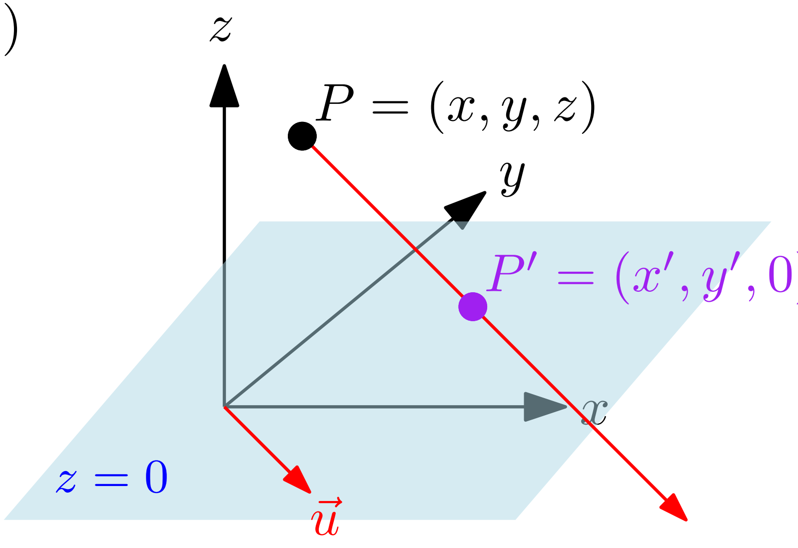

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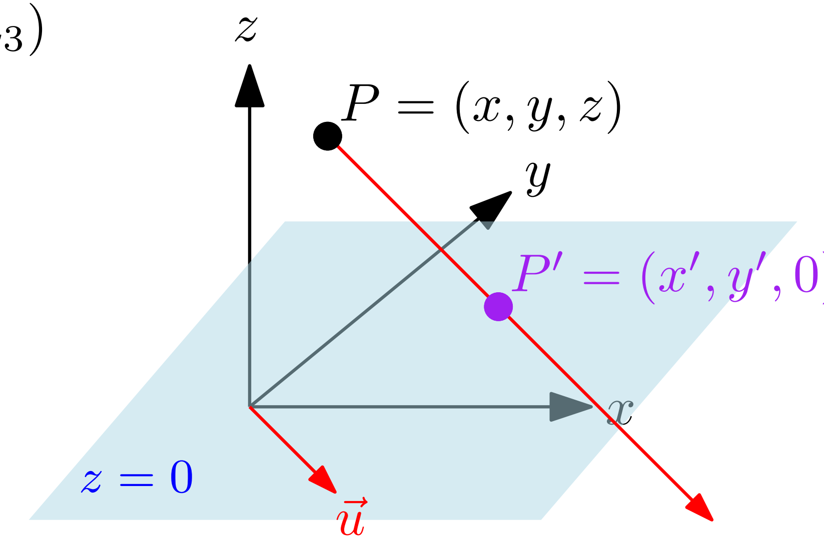

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How can you project onto an arbitrary given plane?

The most useful affine transformation

Translation

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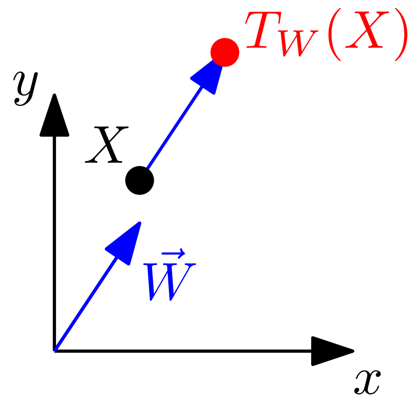
Translation of a point X by a fixed vector $\vec{W}$



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For example, in $\mathbb{R}^2$ with $\vec{W} = (w_x, w_y)$, $T_W(x, y) = (x + w_x, y + w_y)$.

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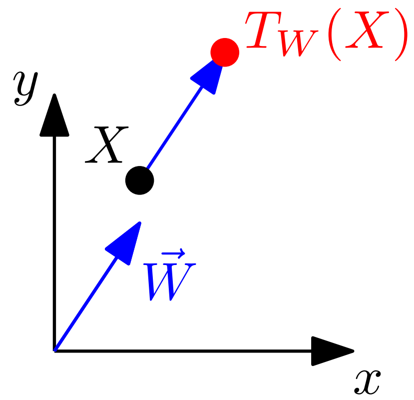

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Note that this transformation is not linear
(i.e., it cannot be written as AX for a fixed matrix $A \in \mathbb{R}^{2\times 2}$)

Any affine transformation can be seen as a linear transformation followed by a translation

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$\vec{v} = \lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n = \sum_{i=1}^{n} \lambda_i \vec{v}_i$ is an affine combination of $\vec{v}_1, \ldots \vec{v}_n$, if $\lambda_1, \ldots, \lambda_n$, satisfy that $\sum_{i=1}^{n} \lambda_i = 1$

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Then we have:

f is affine invariant if and only if

$$f(\lambda_1 \vec{v}_1 + \cdots + \lambda_n \vec{v}_n) = \lambda_1 f(\vec{v}_1) + \cdots + \lambda_n f(\vec{v}_n)$$

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Therefore, f is affine invariant!

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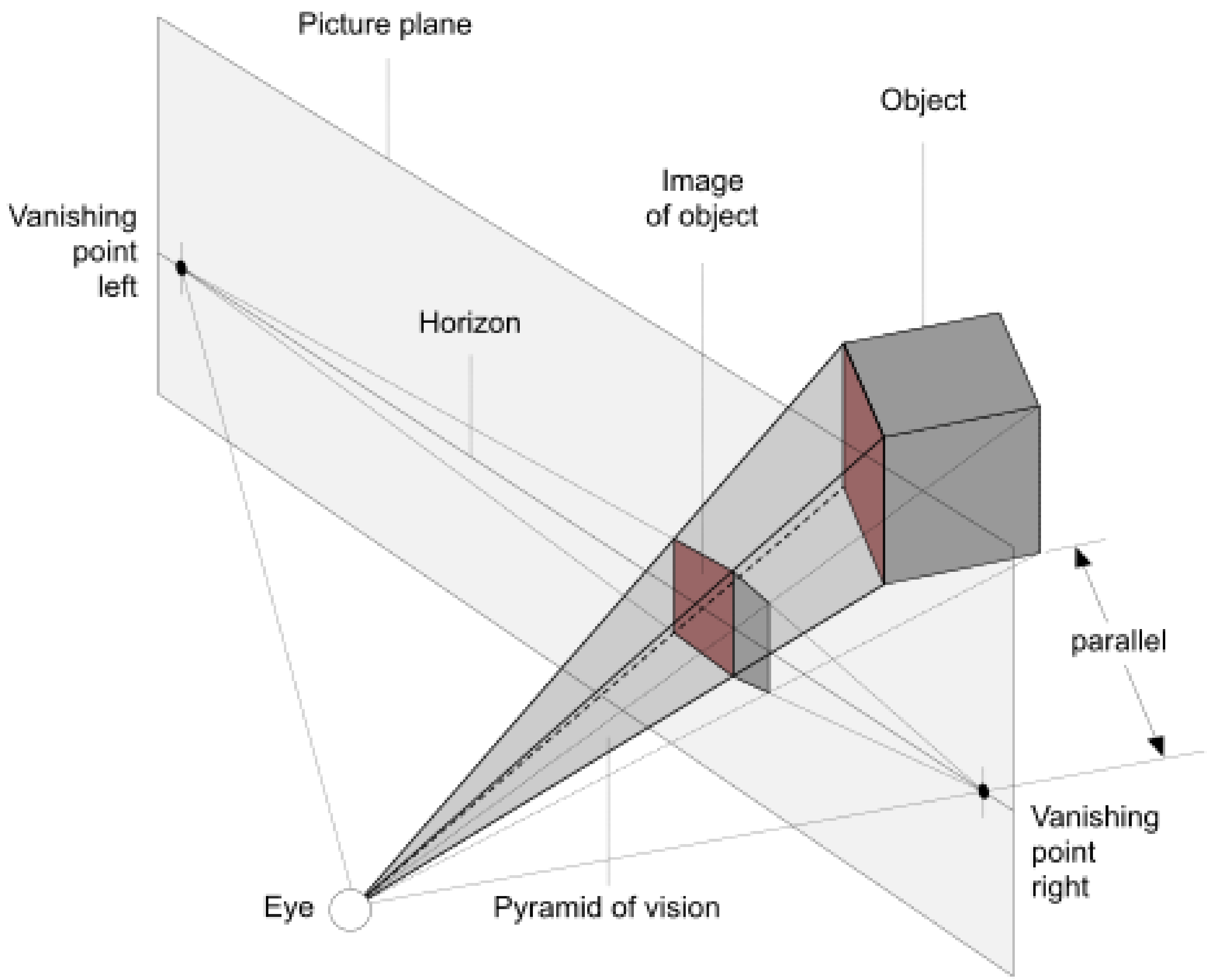

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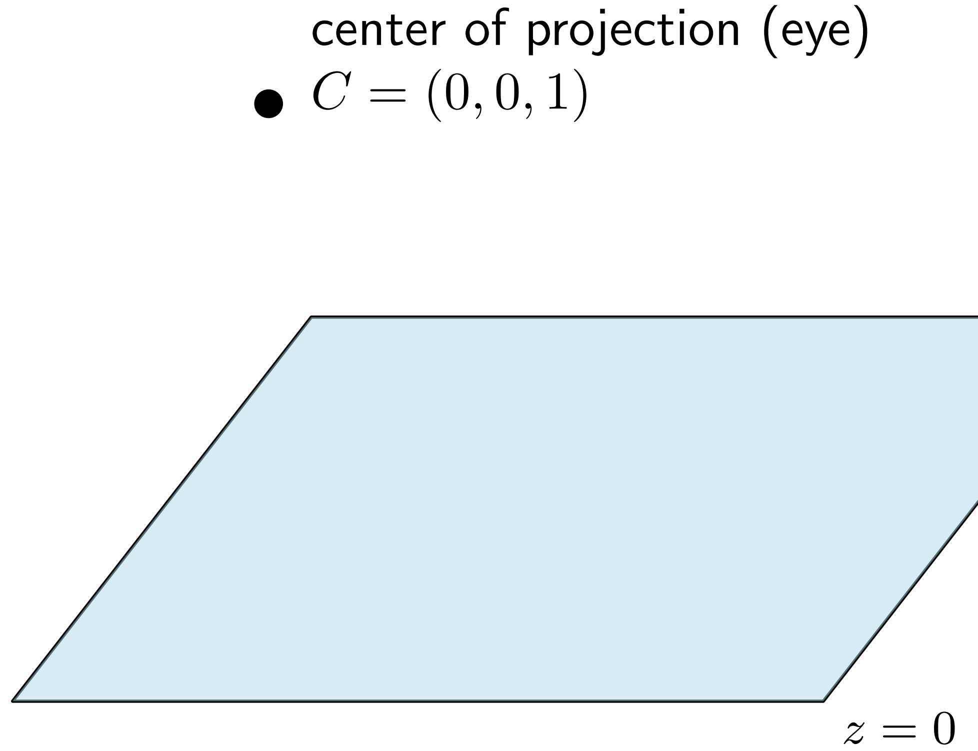

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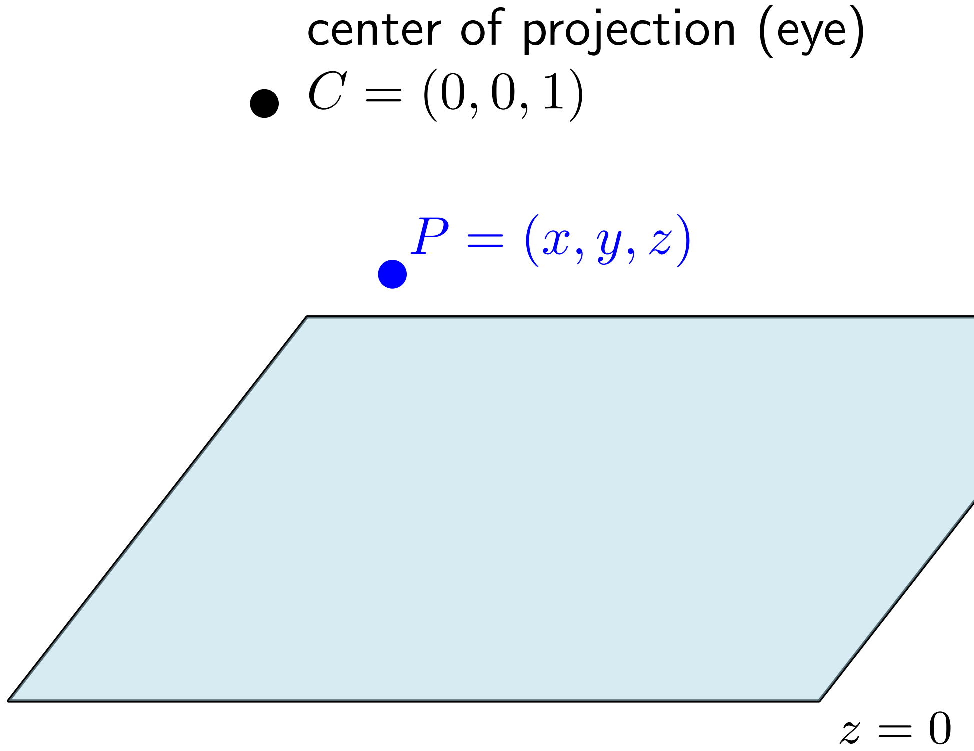

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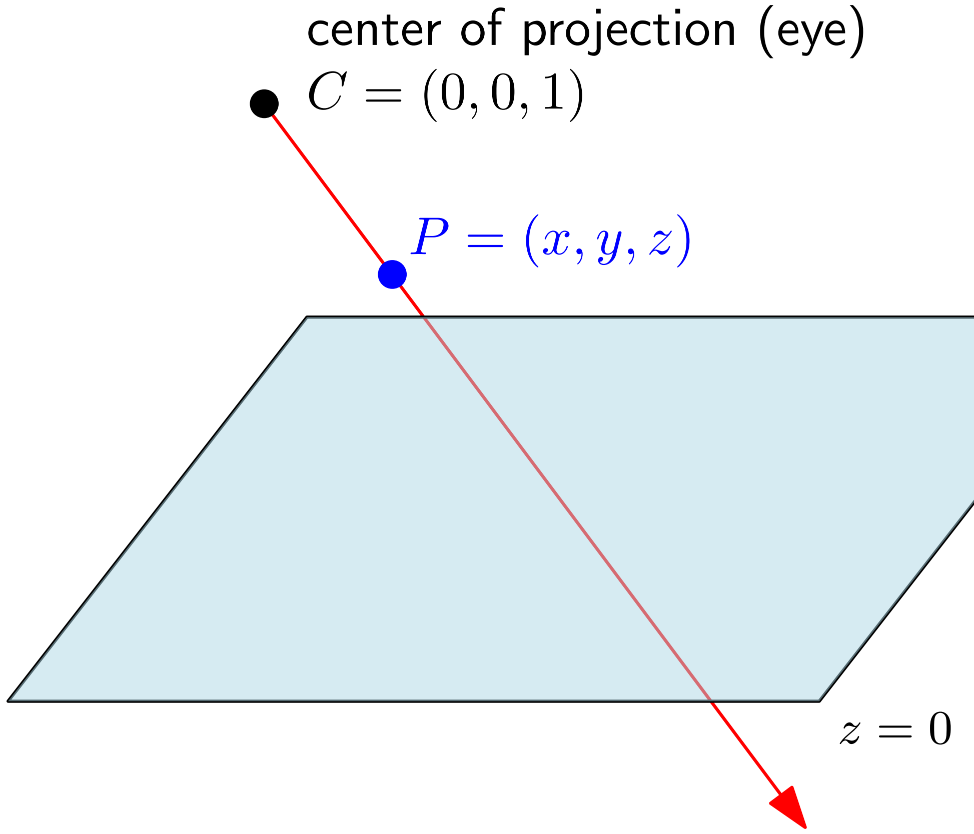

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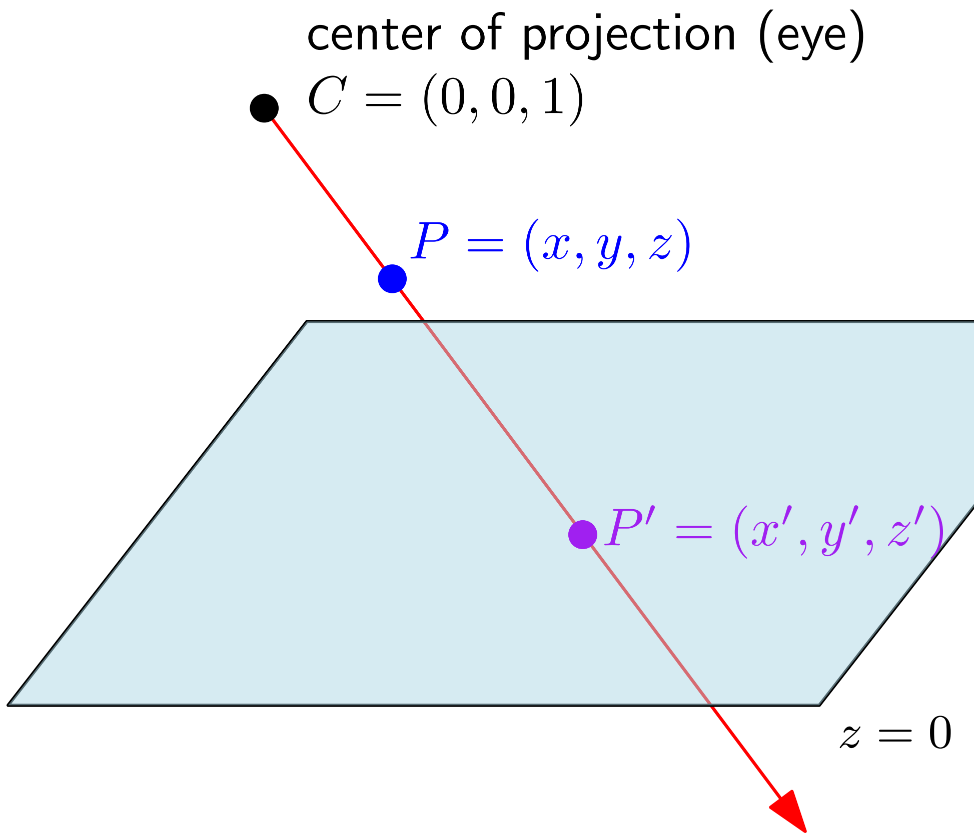

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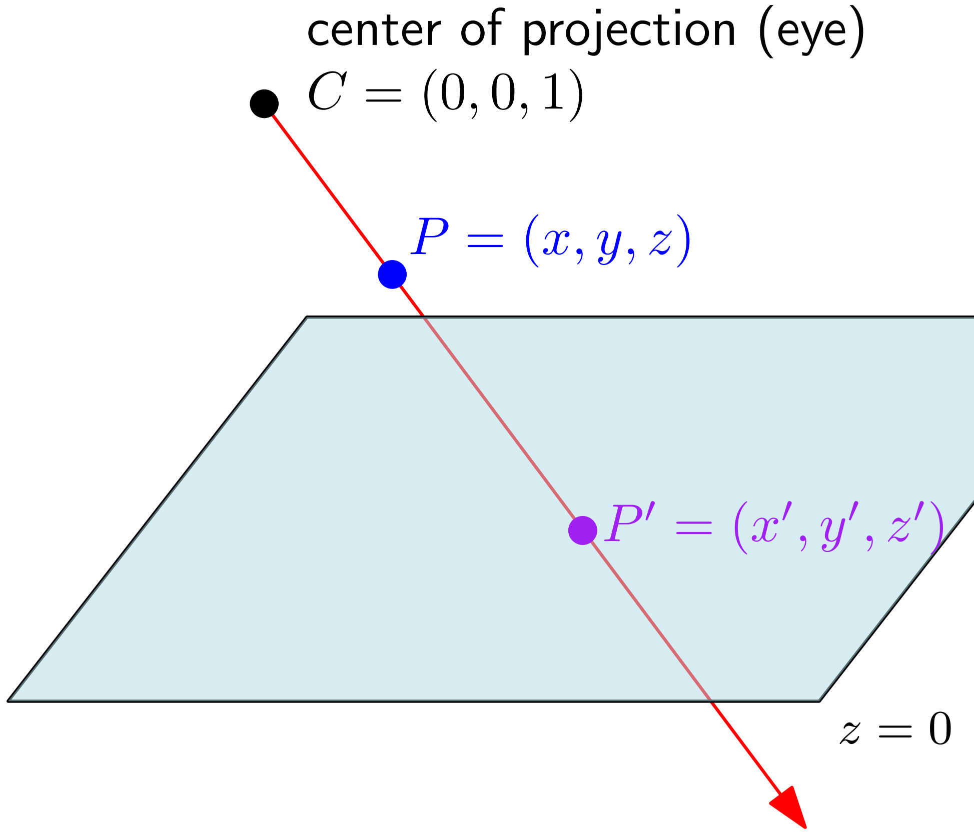

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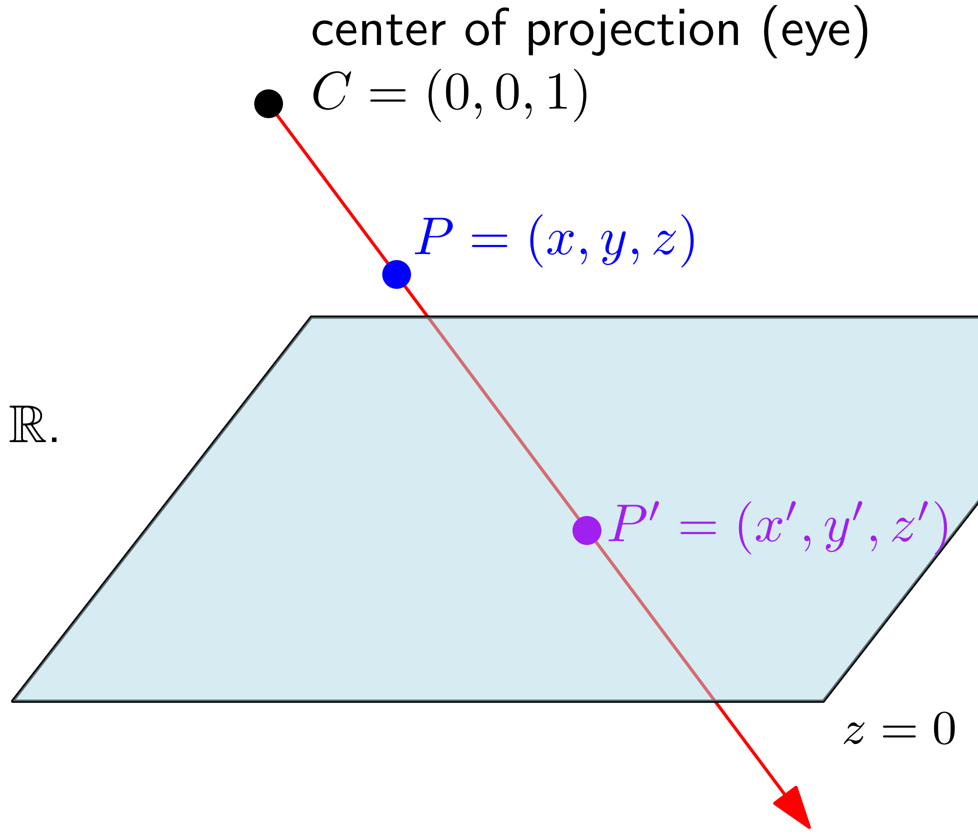

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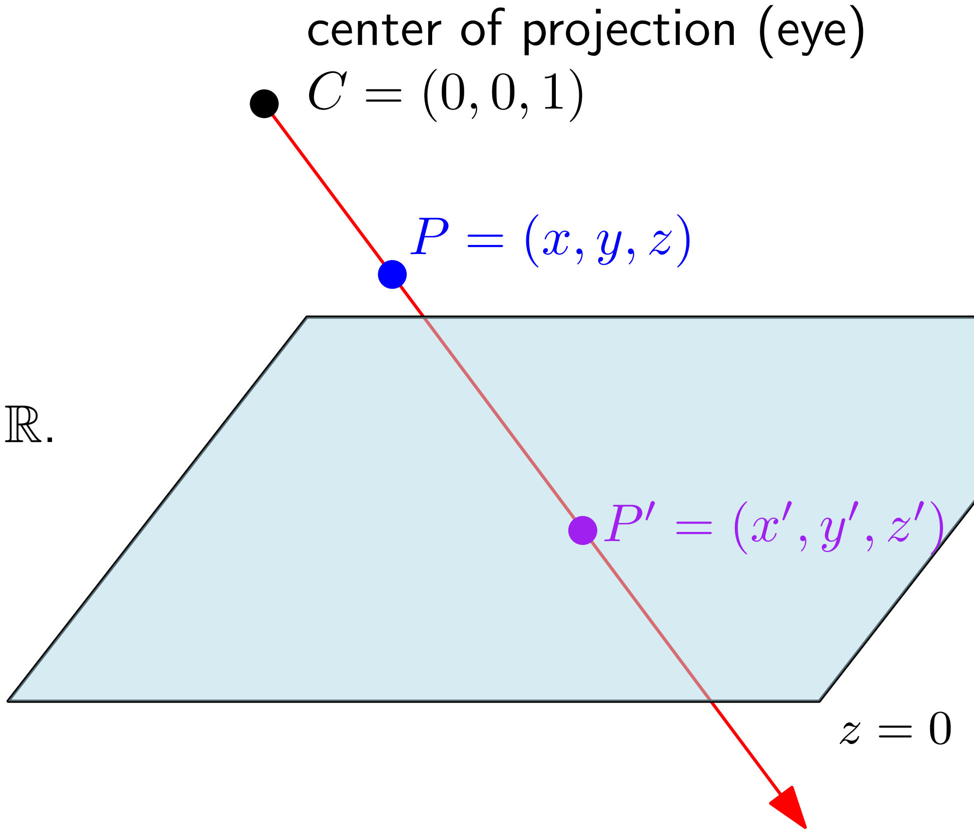

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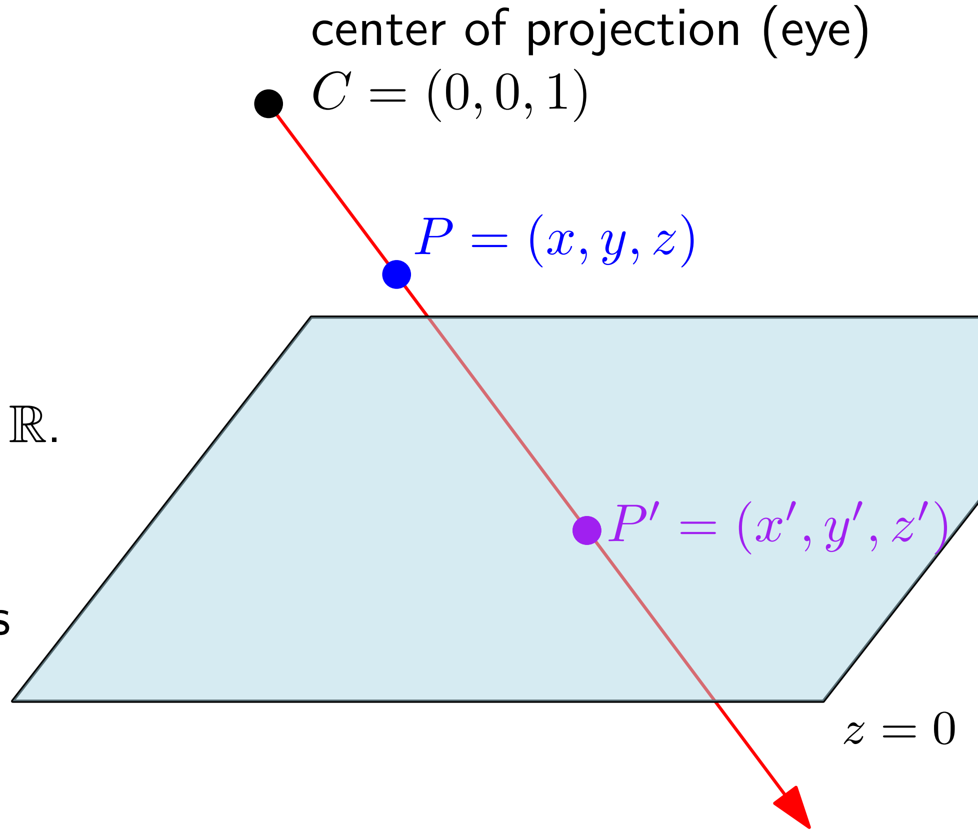

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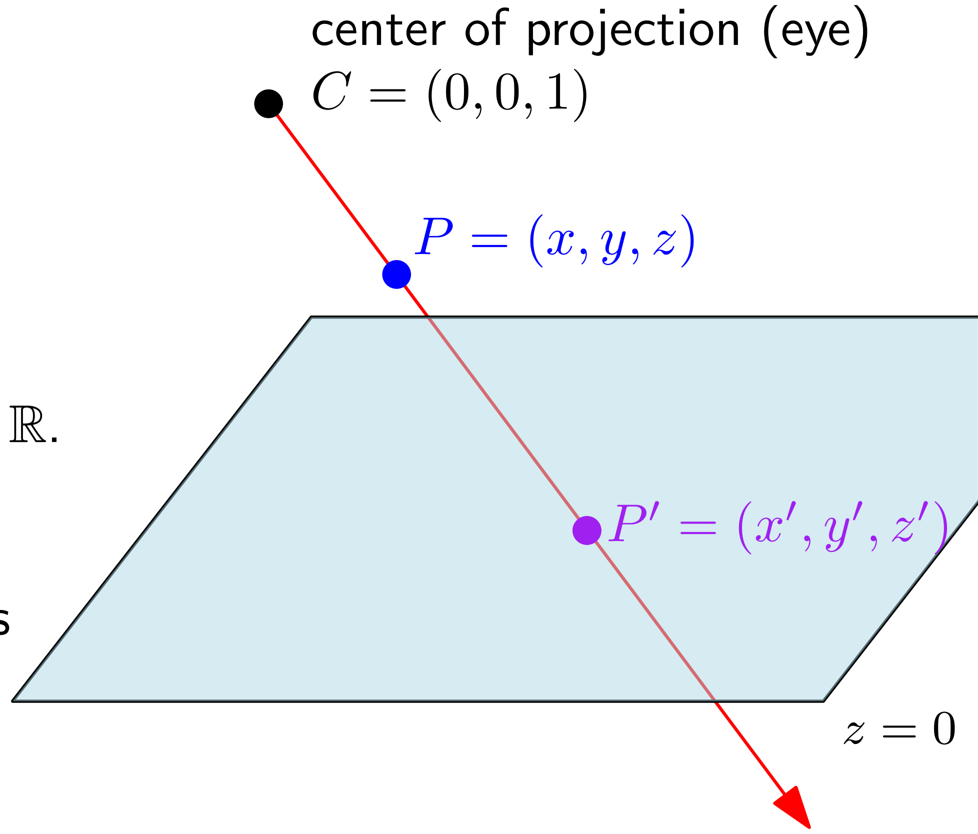

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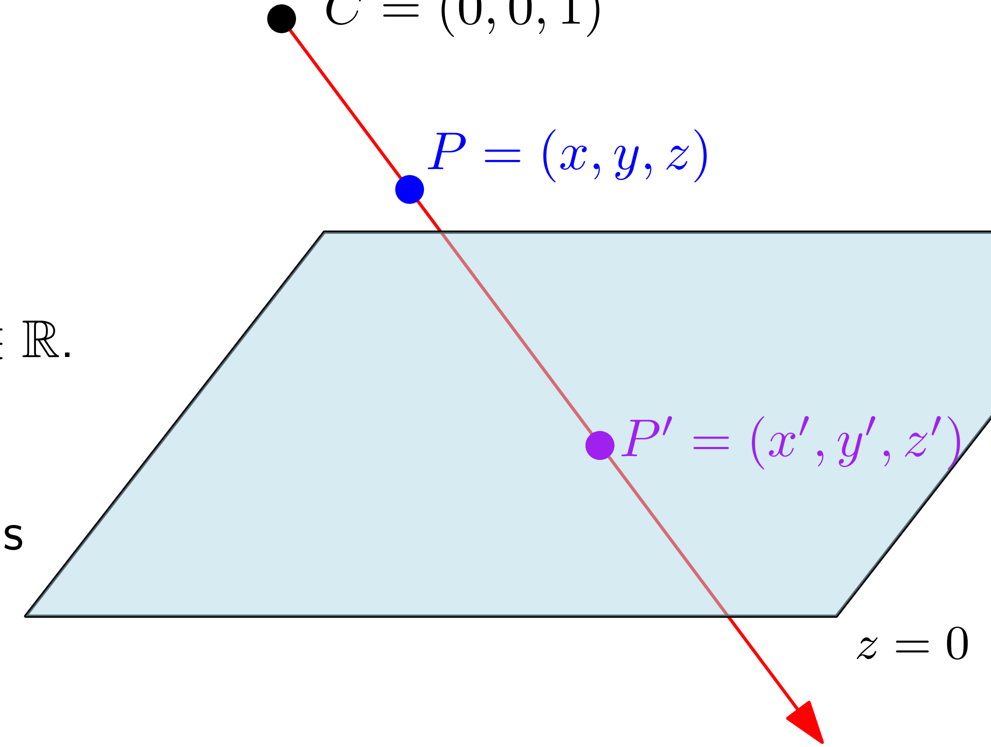

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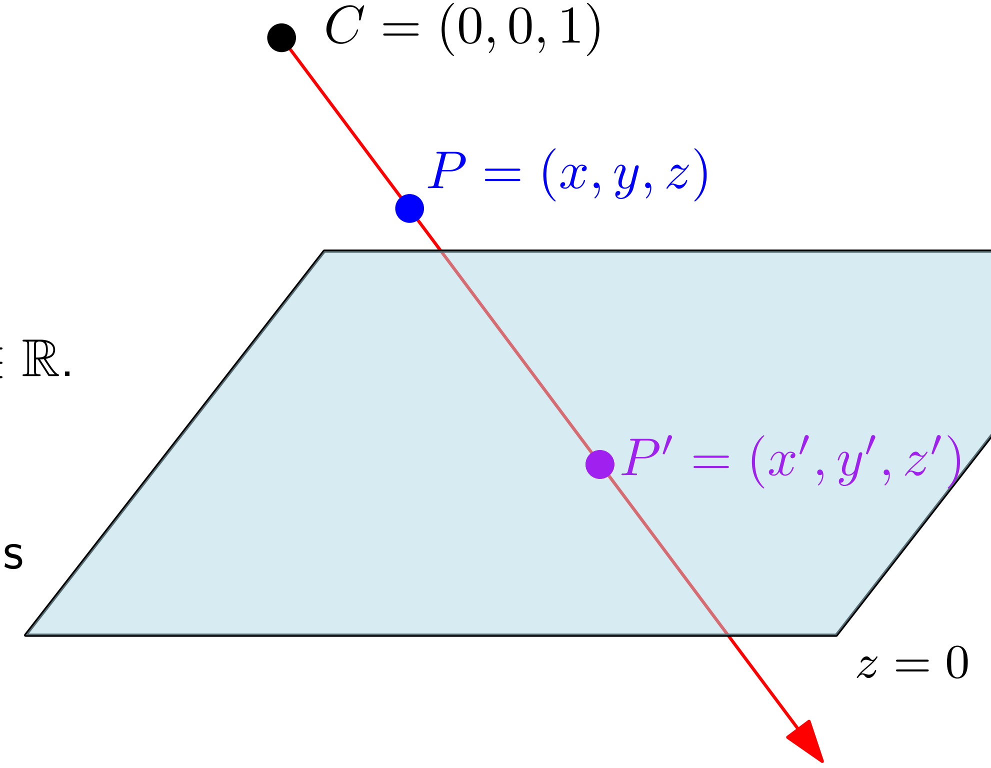

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You will play with this in Lab 1 (from 2D to 1D)