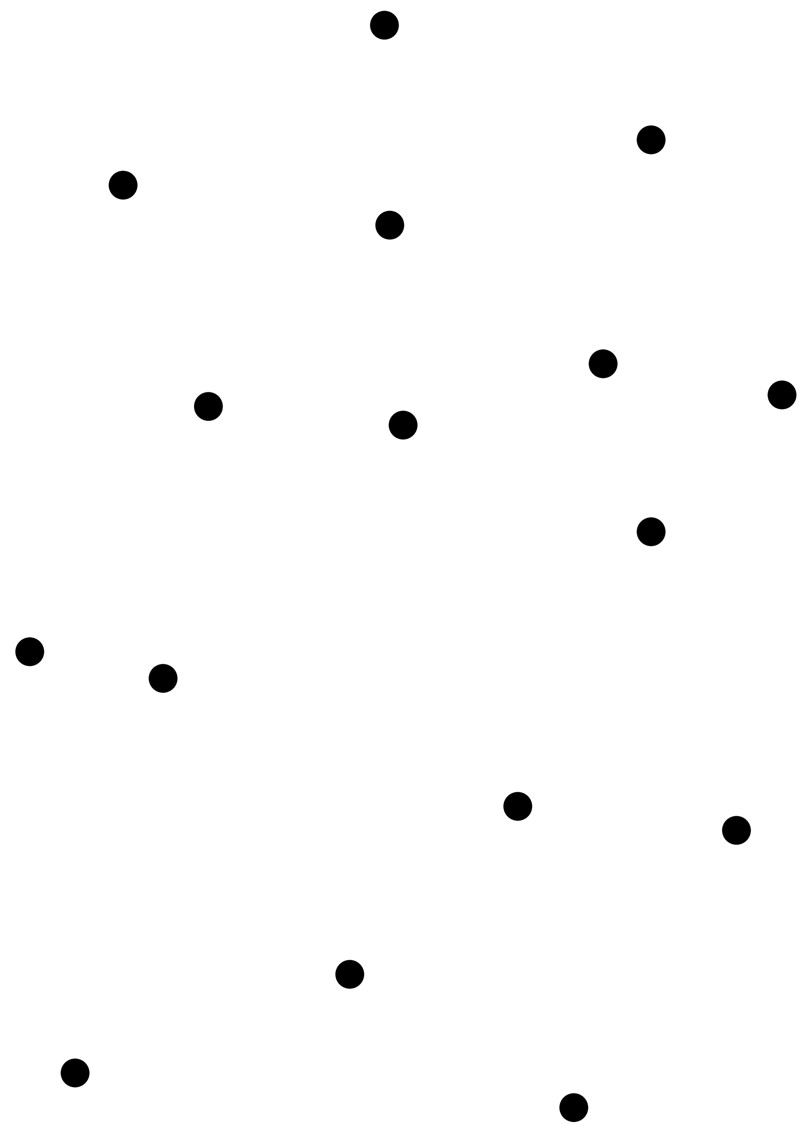
PROXIMITY

Vera Sacristán
Rodrigo Silveira

Universitat Politècnica de Catalunya

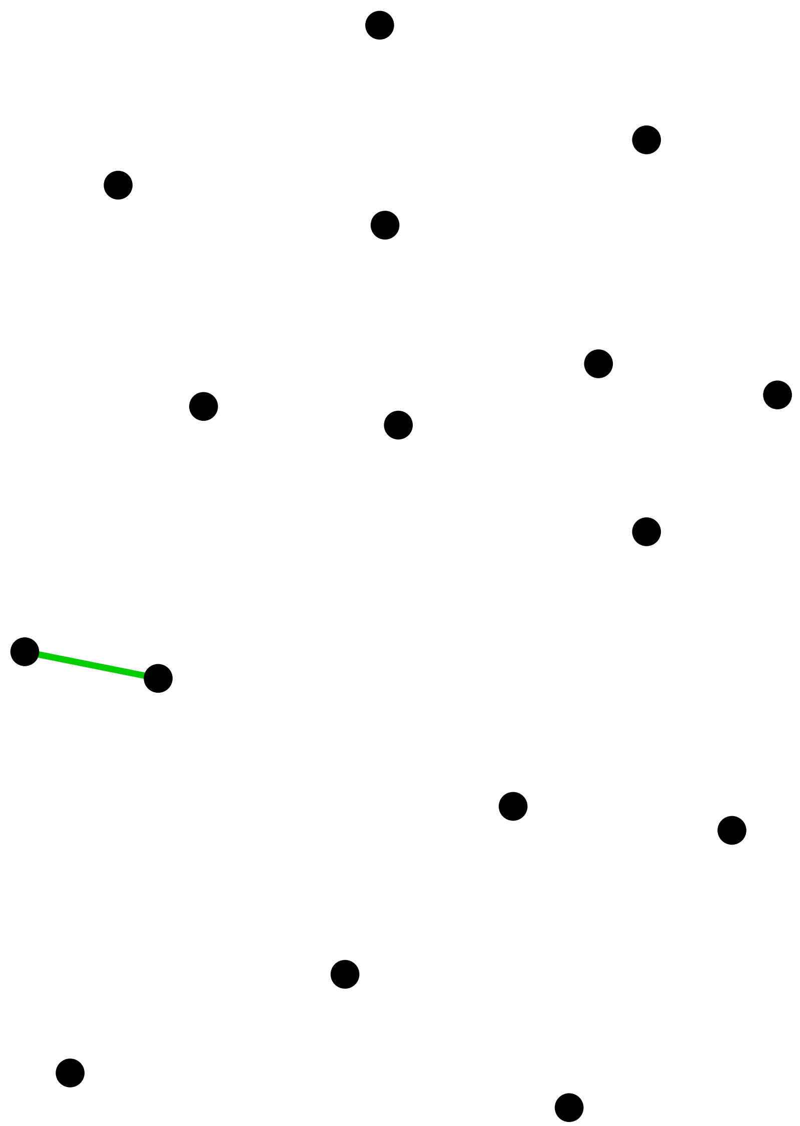
PROXIMITY

Let P be a set of n points in $\mathbb{R}^2$...



CLOSEST PAIR

Find the pair of points closest to each other



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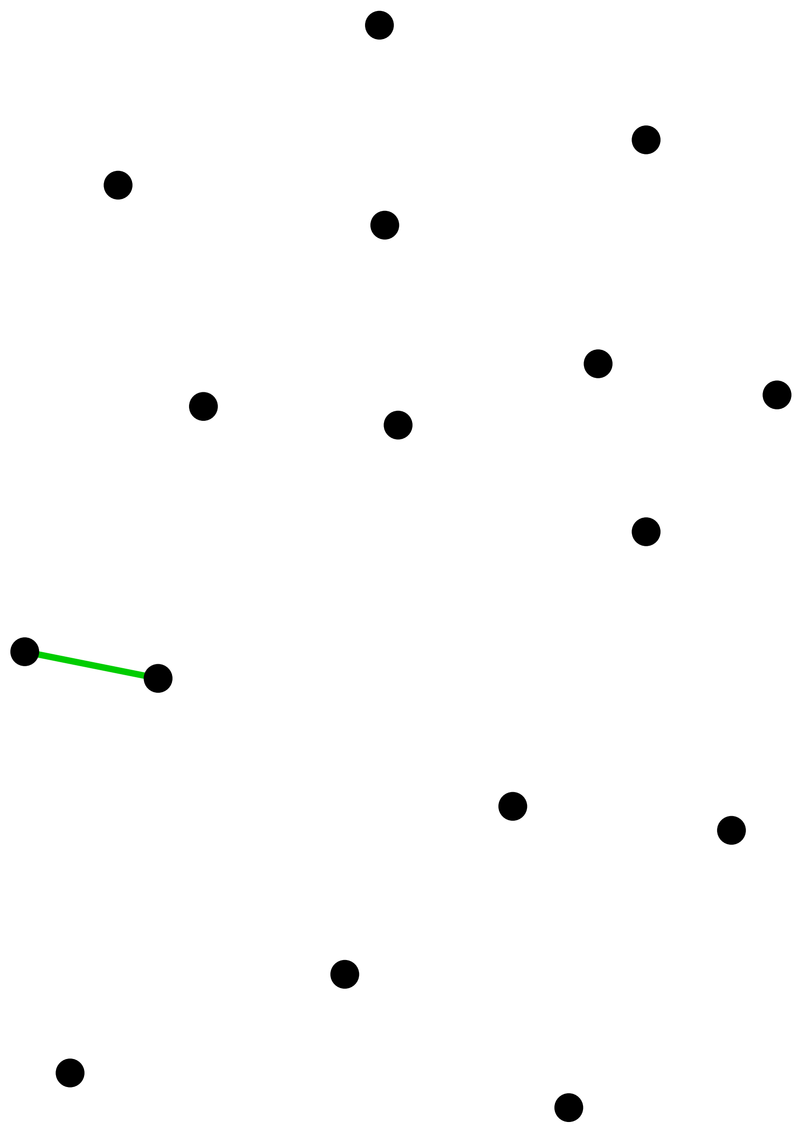
APPLICATION

Airplanes in danger of collision



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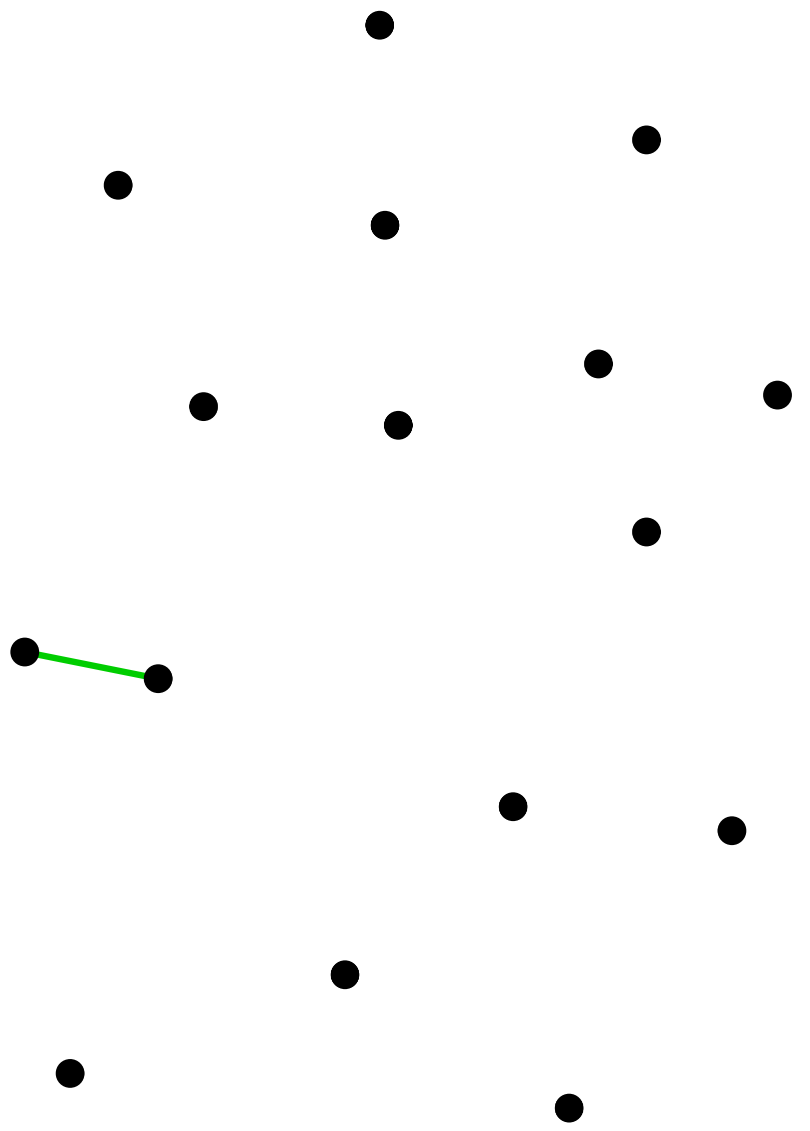
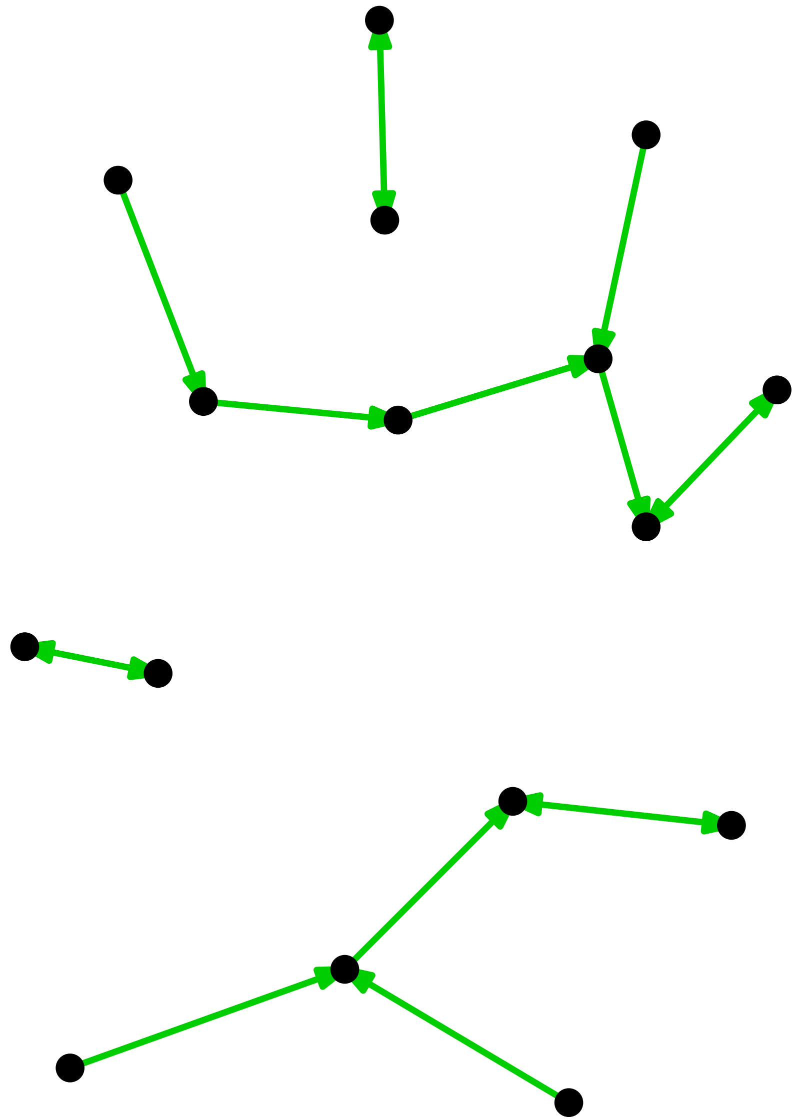


- Brute force solution: check every pair of points in $O(n^2)$ time
- This procedure does not exploit the metric properties
- In dimension 1, it can be solved in $O(n \log n)$ time by sorting the input first, and then using the fact that the solution must be a pair of consecutive points

PROXIMITY

All NEAREST NEIGHBORS

Build the directed graph where the existance of an (oriented) edge $\overrightarrow{p_i p_j}$ means that p_j is the point of P closest to p_i



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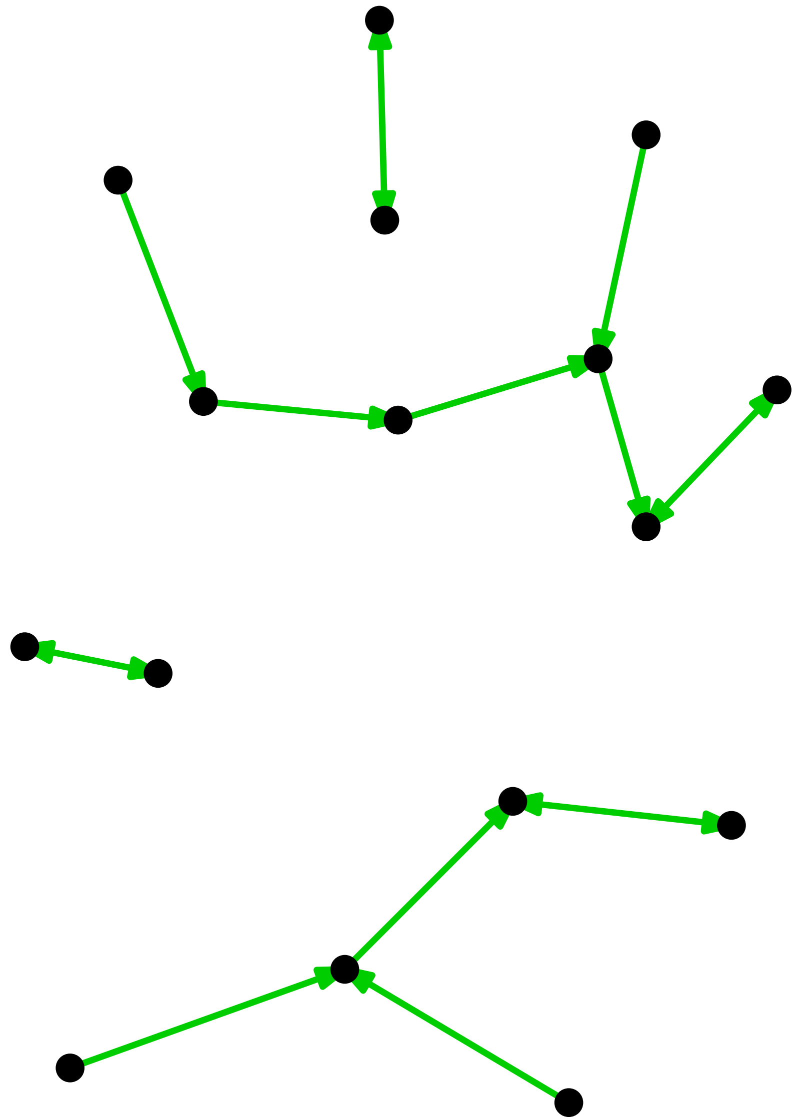
APPLICATION

In Ecology, to study the territoriality of species



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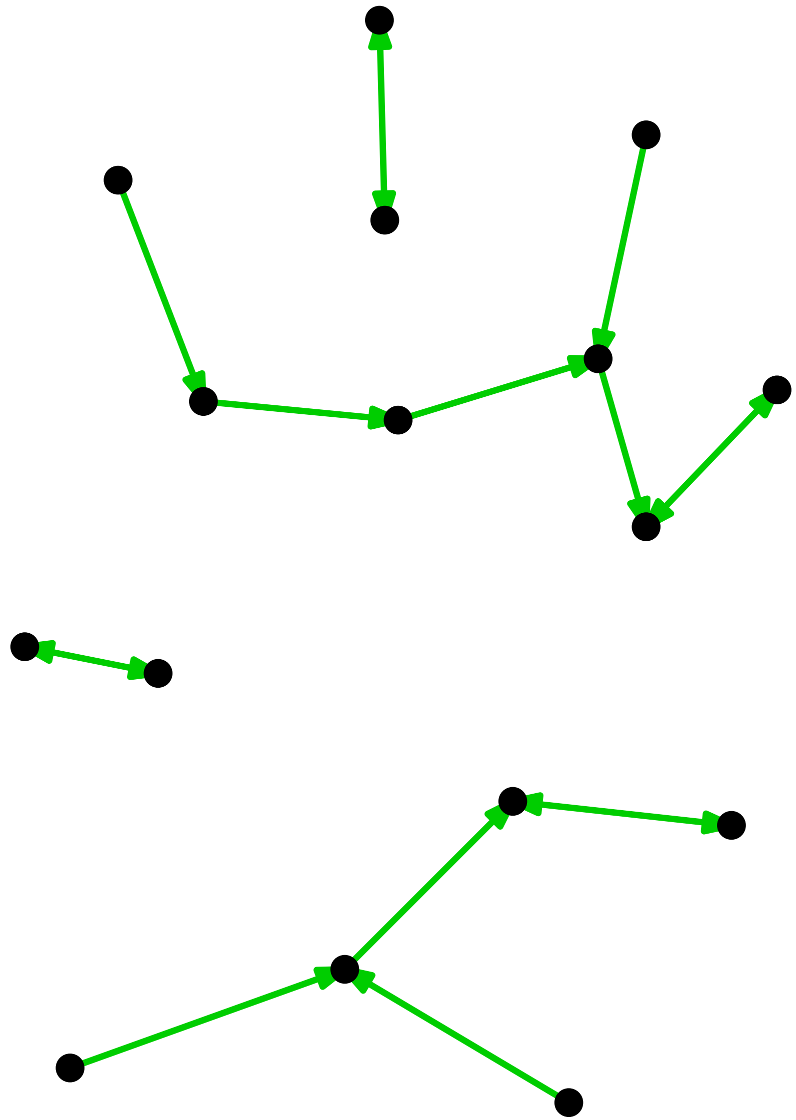


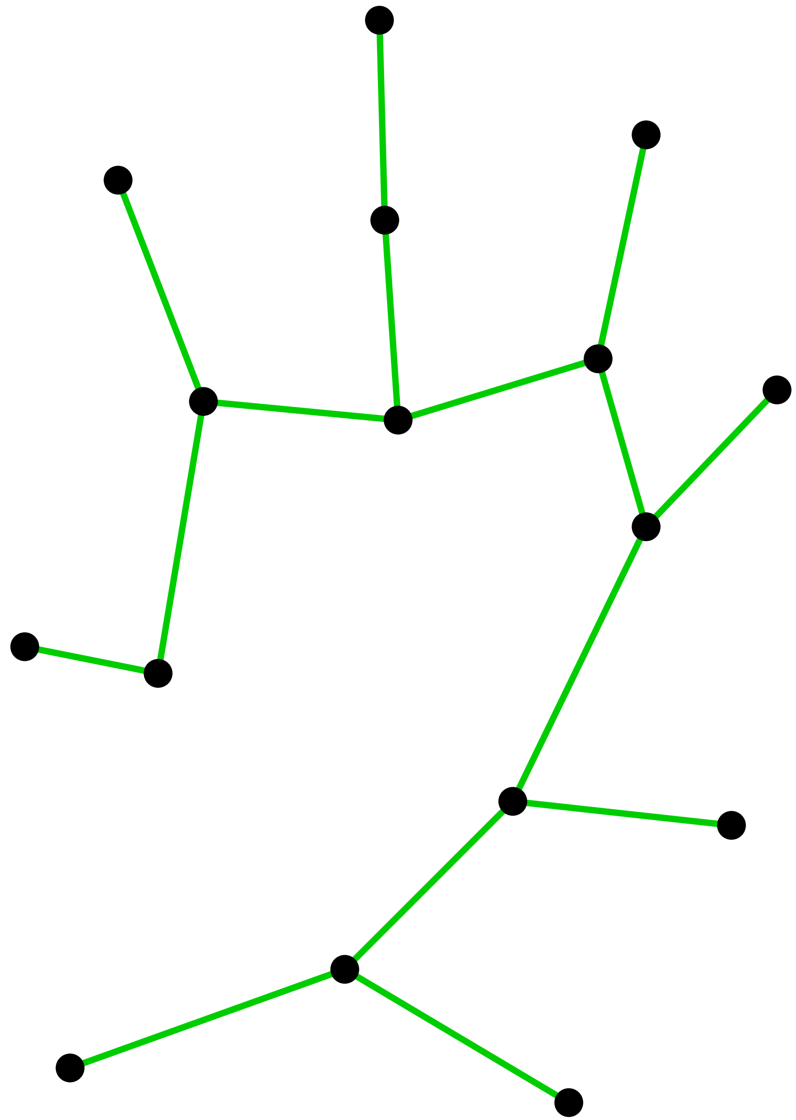
APPLICATION

In Ecology, to study the territoriality of species

- Brute force solution: check every pair of points in $O(n^2)$ time
- This process does not exploit the metric properties
- In dimension 2, at most 6 points of P have p_i as their closest point
- In dimension 1 it can be solved in $O(n \log n)$ time (same as before)

EUCLIDEAN MINIMUM SPANNING TREE

Build the tree connecting all points of P and minimizing the sum of the lenghts of its edges



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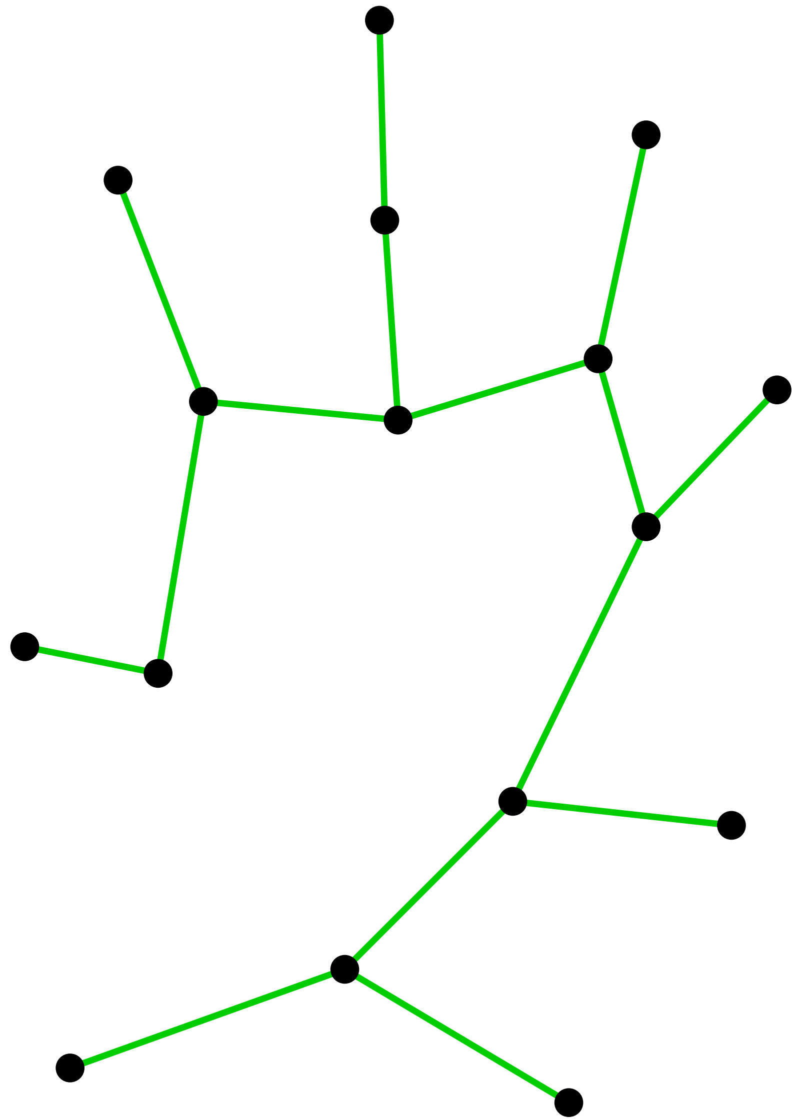
APPLICATION

Telecommunication networks



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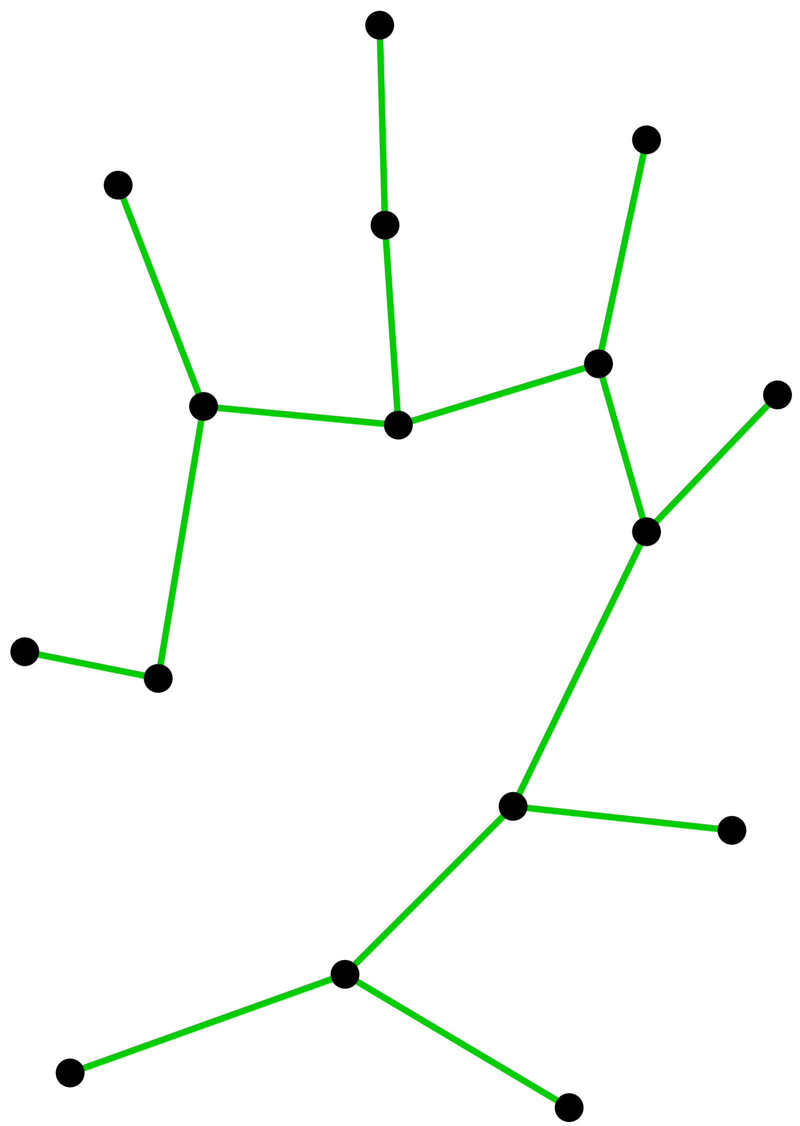
APPLICATION

Telecommunication networks

- Applying Prim's or Kruskal's algorithms to the complete graph of P, with Euclidean weights: $O(e \log e) = O(n^2 \log n)$ time
- Applying improved algorithms to the complete weighted graph: $O(e) = O(n^2)$ time
- These procedures do not exploit the metric properties
- In dimension 1 it can be done in $O(n \log n)$ time (same as before)

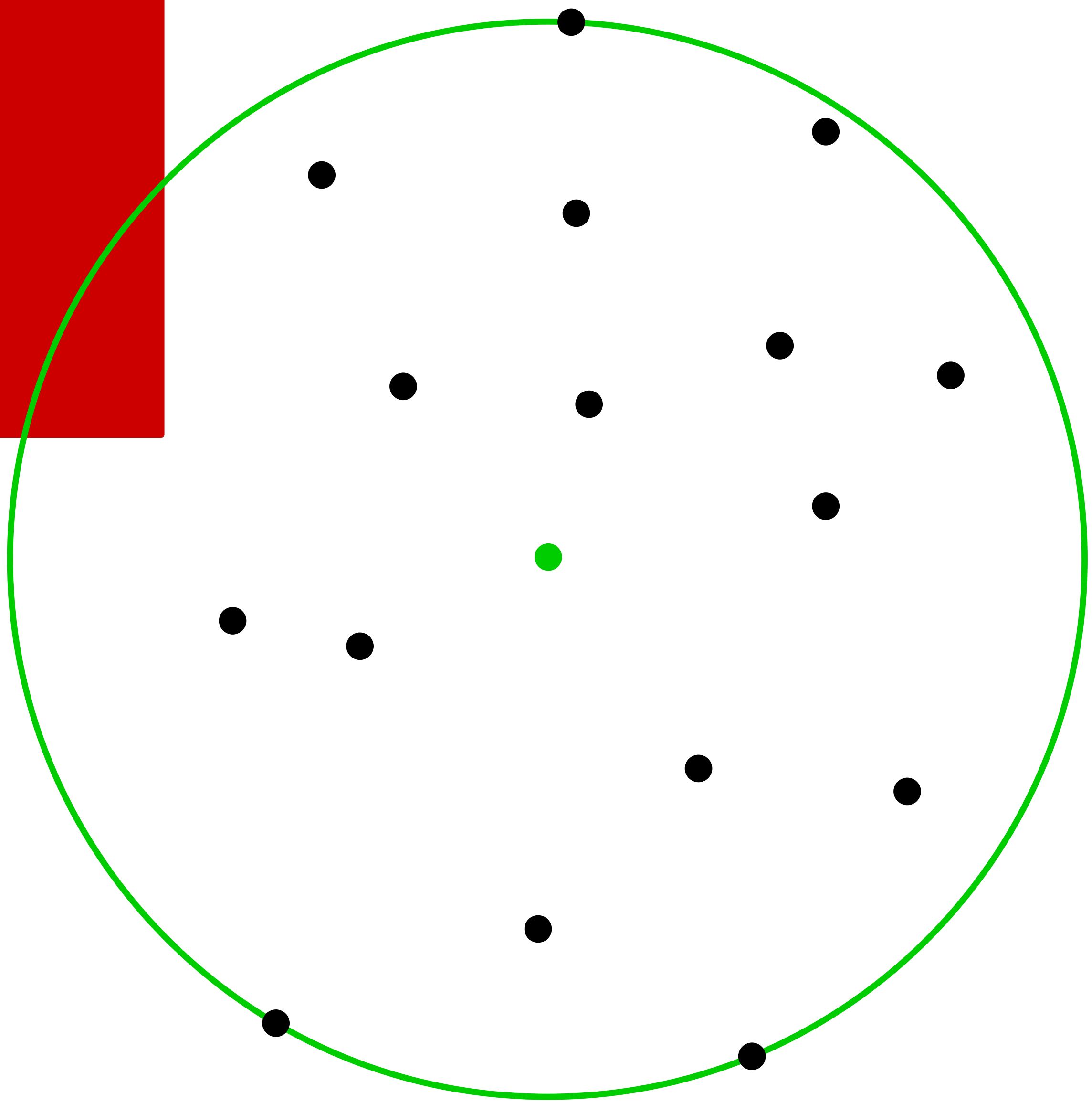


MIN-MAX FACILITY LOCATION: MINIMUM SPANNING CIRCLE

Find the point x on the plane achieving

$$\min_{x \in \mathbb{R}^2} \max_{p_i \in P} d(x, p_i)$$

Geometrically: find the center of the circle of minimum radius enclosing P



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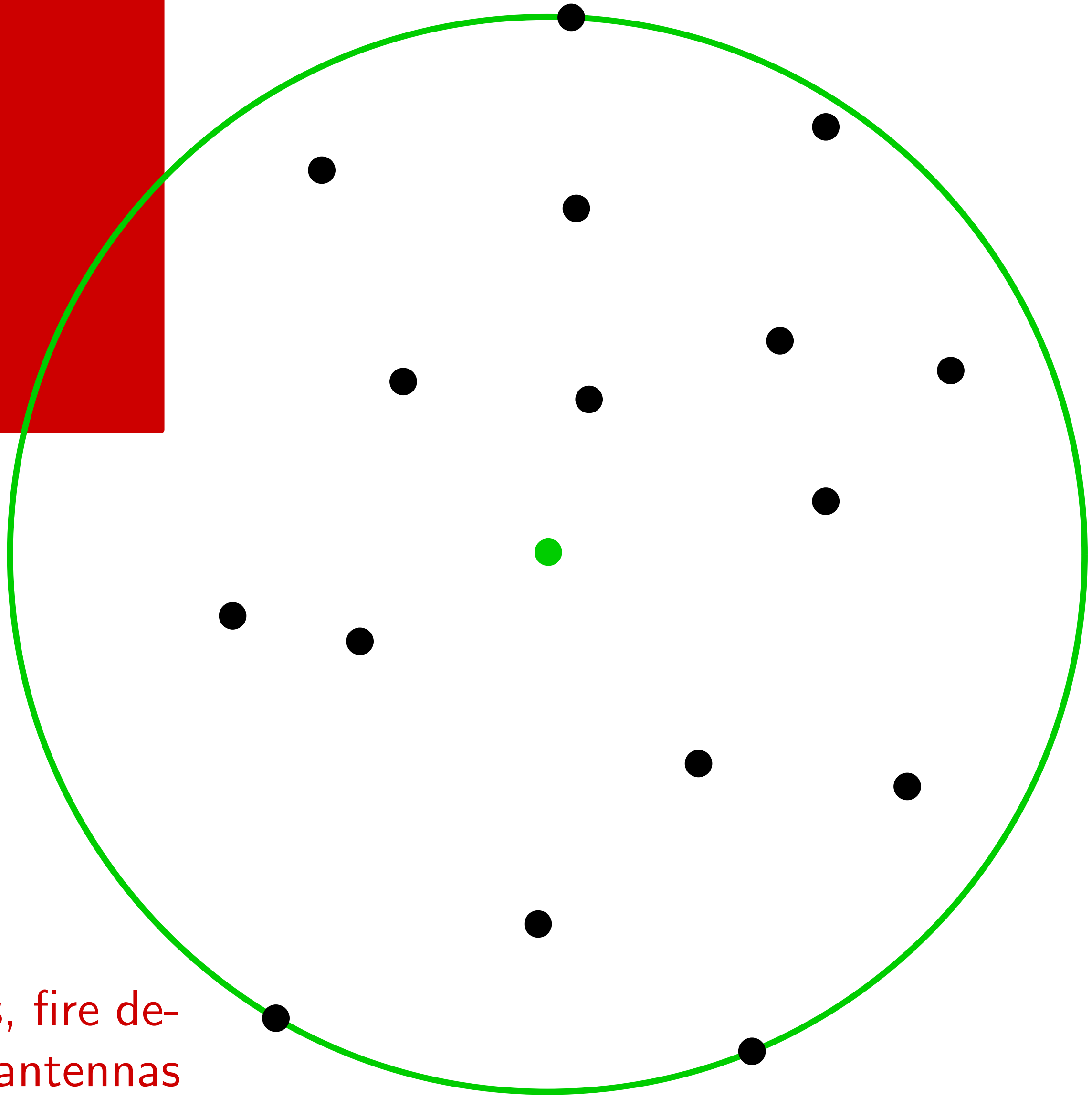

APPLICATION

Location of emergency services (hospitals, fire departments, ...), of radio and tv repeaters, antennas for mobile phones,...

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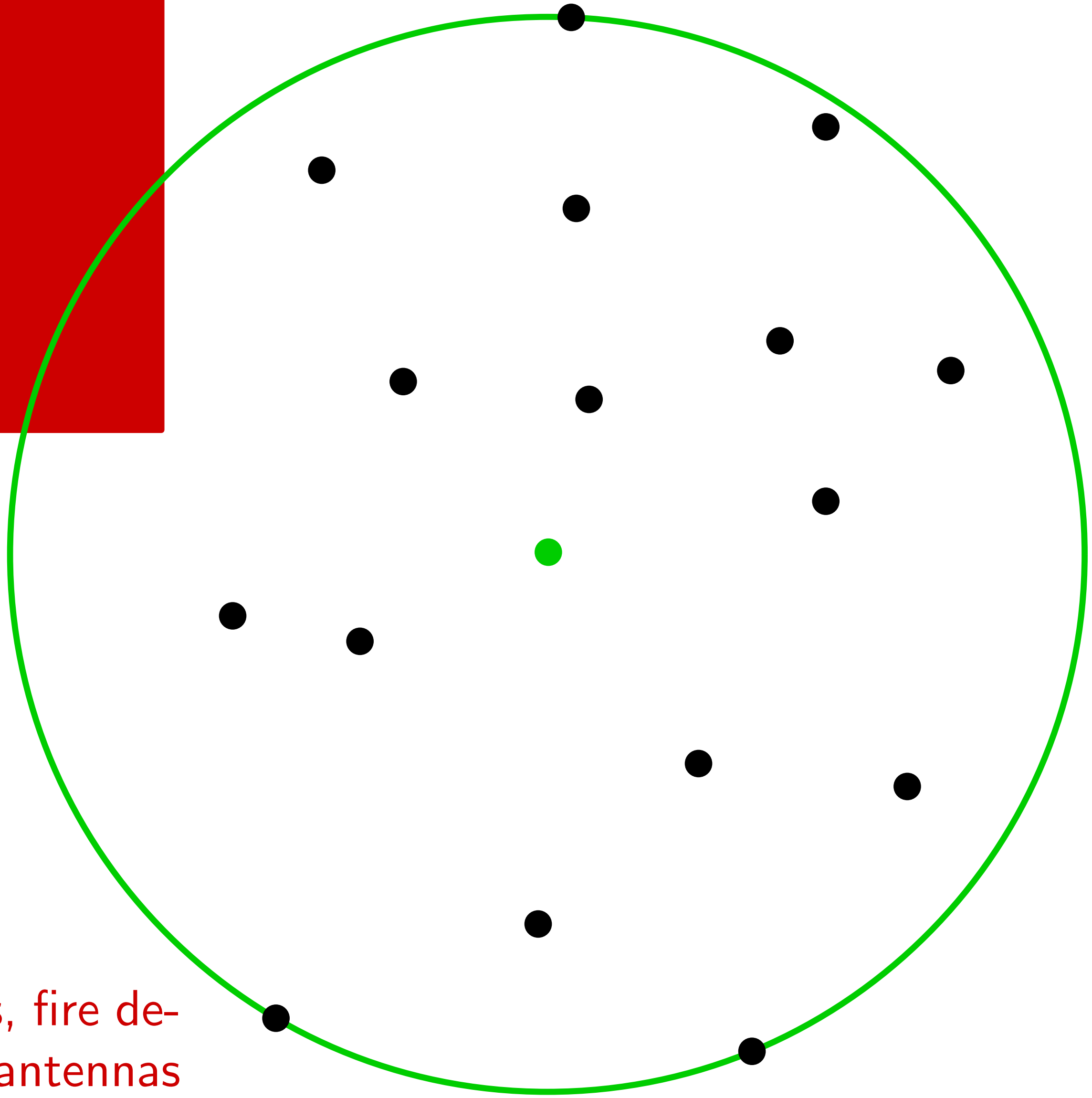


- Brute force solution: consider every set of three points, in $O(n^4)$ time
- In dimension 1 it can be solved in $O(n)$ time: the solution is the midpoint of $\min(P)$ and $\max(P)$

APPLICATION

Location of emergency services (hospitals, fire departments, ...), of radio and tv repeaters, antennas for mobile phones,...

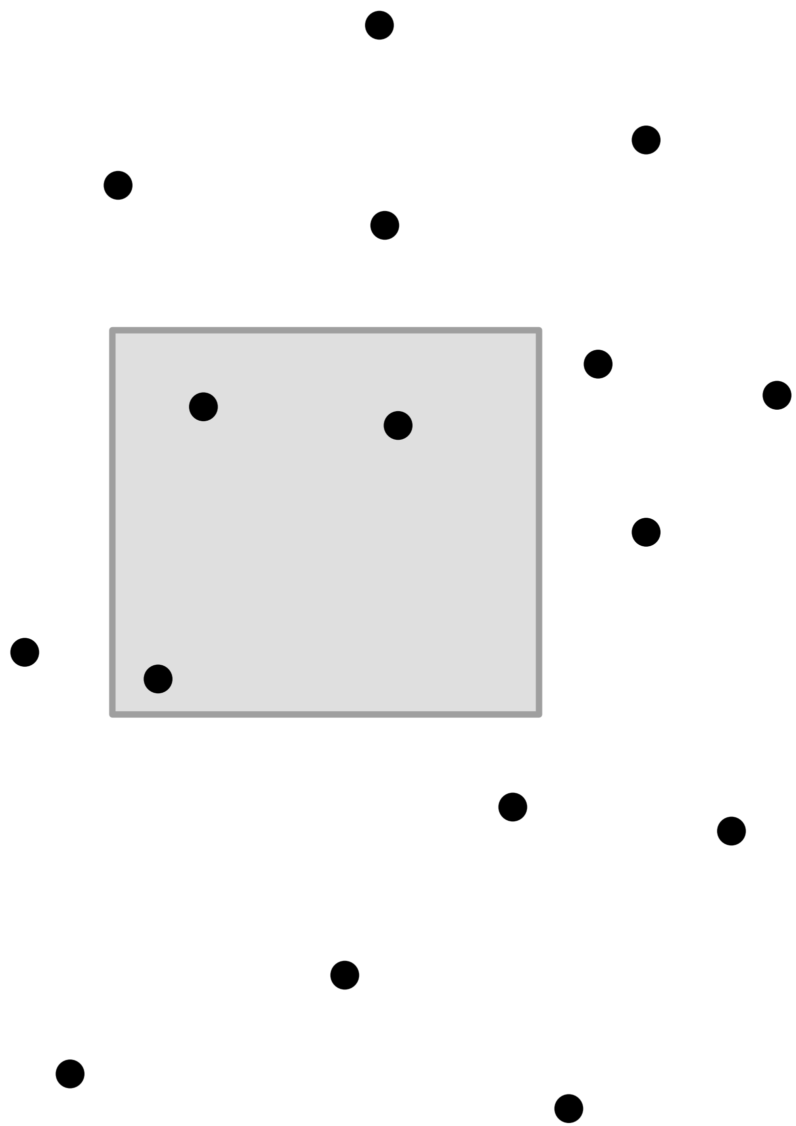
PROXIMITY

MAX-MIN FACILITY LOCATION: MAXIMUM EMPTY CIRCLE

Find the point x in the region A achieving

$$\max_{x \in A} \min_{p_i \in P} d(x, p_i)$$

Geometrically: find in A the center of circle of maximum radius not enclosing any point of P



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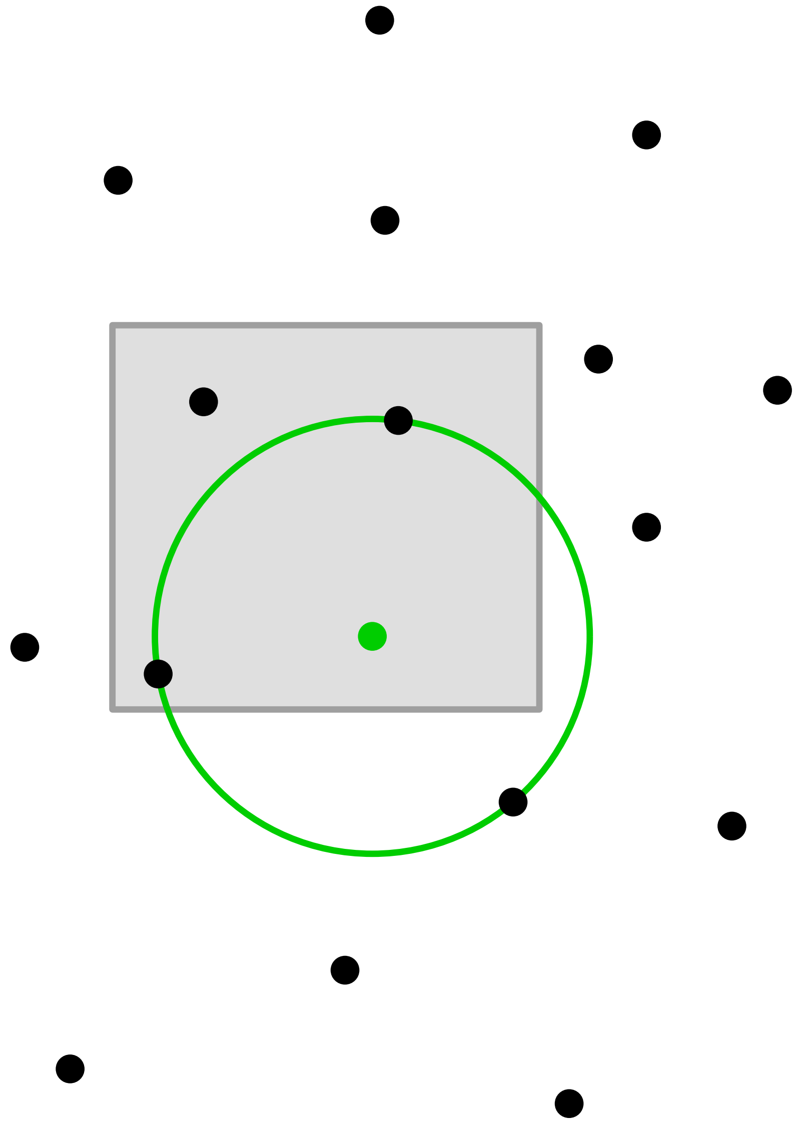

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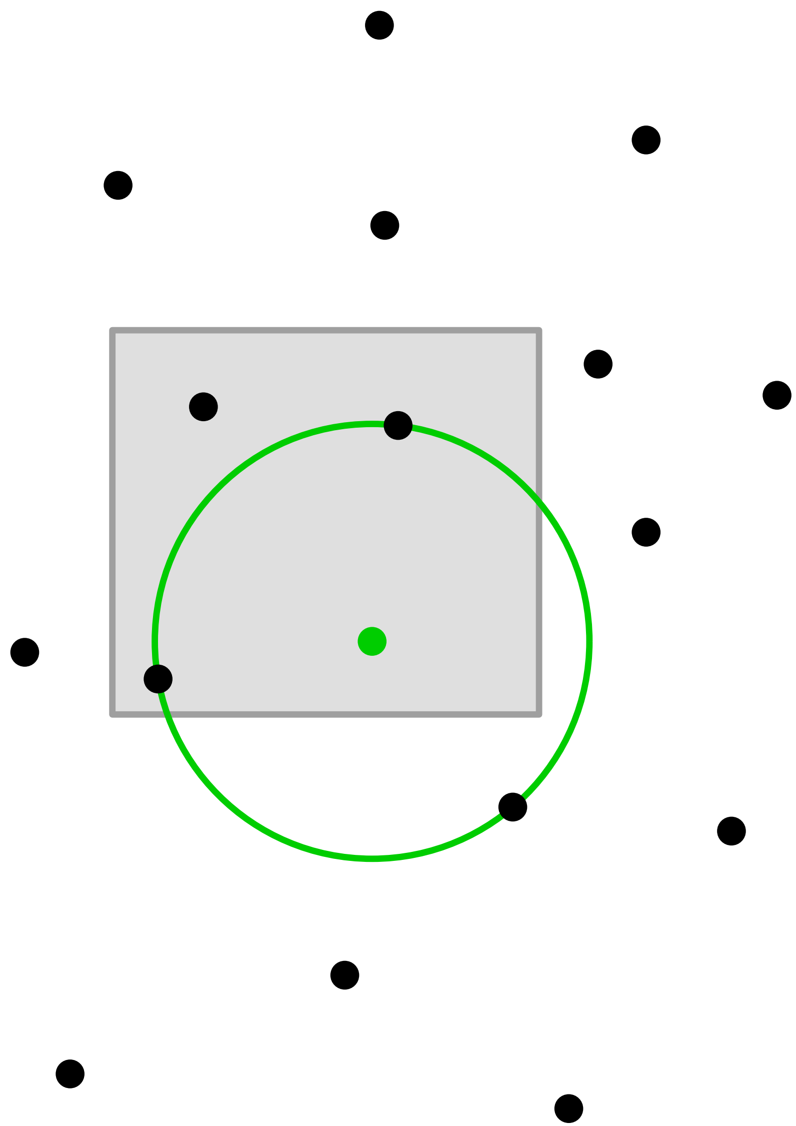

APPLICATION

Location of undesired services (polluting industry, landfills, ...) or new shopping malls in competence with preexistent centers

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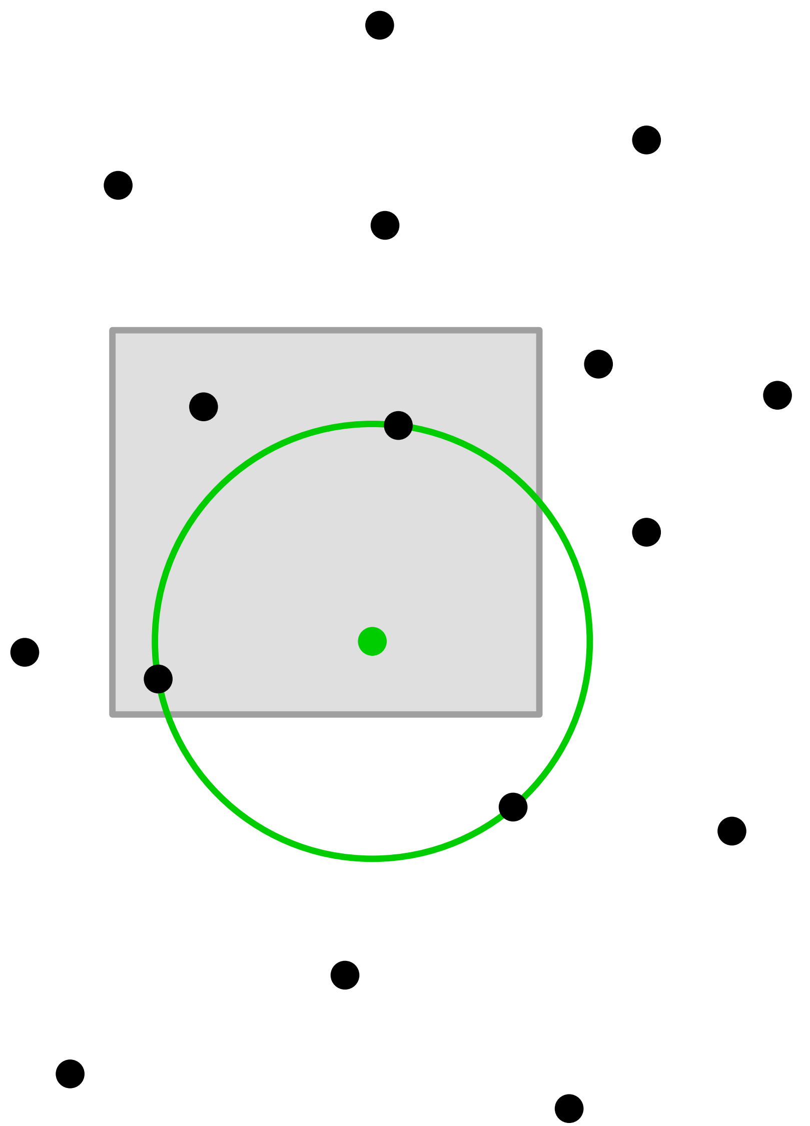

- Brute force solution: $O(n^4)$ time (as before)
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APPLICATION

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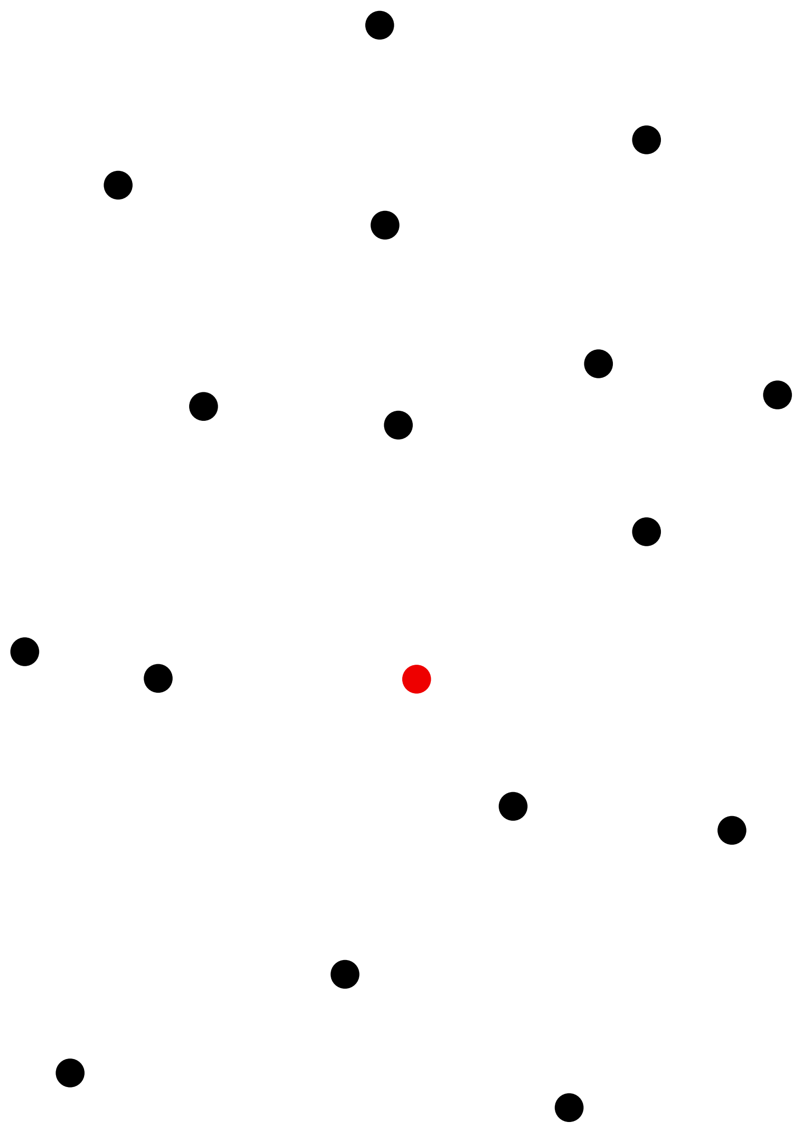
PROXIMITY

NEAREST NEIGHBOR QUERY

Given a new point q, quickly find its closest site p_i



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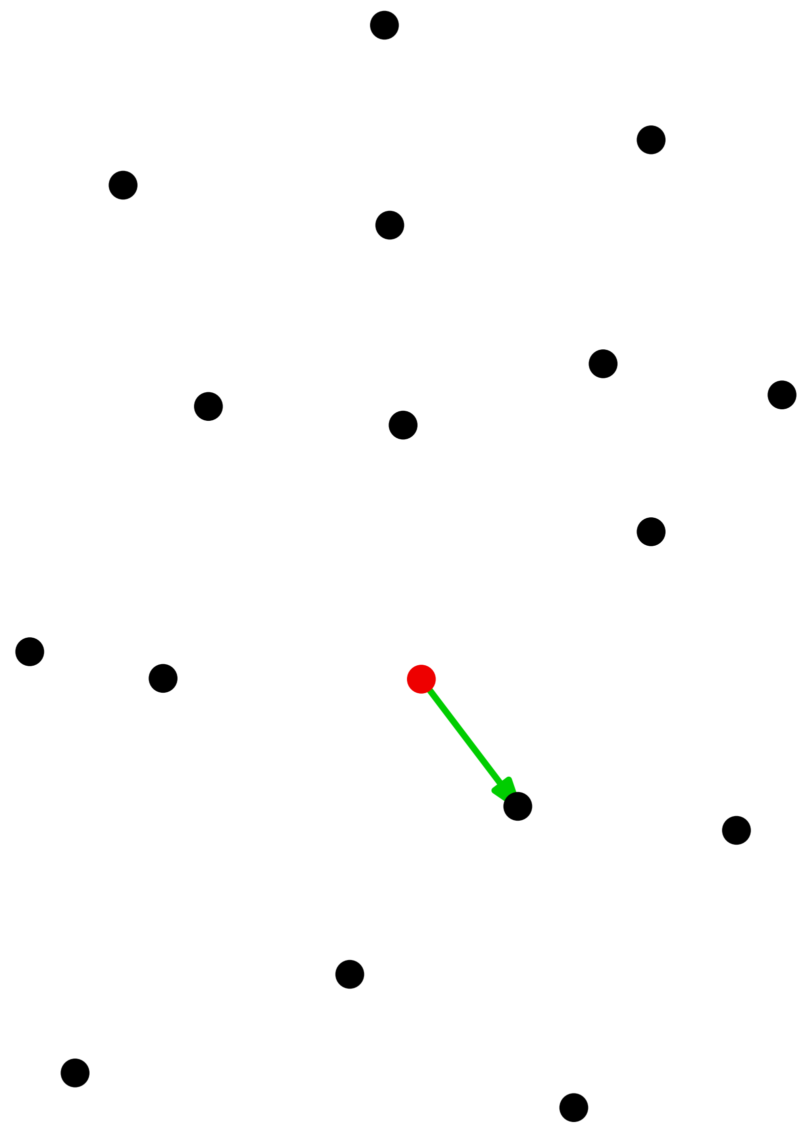

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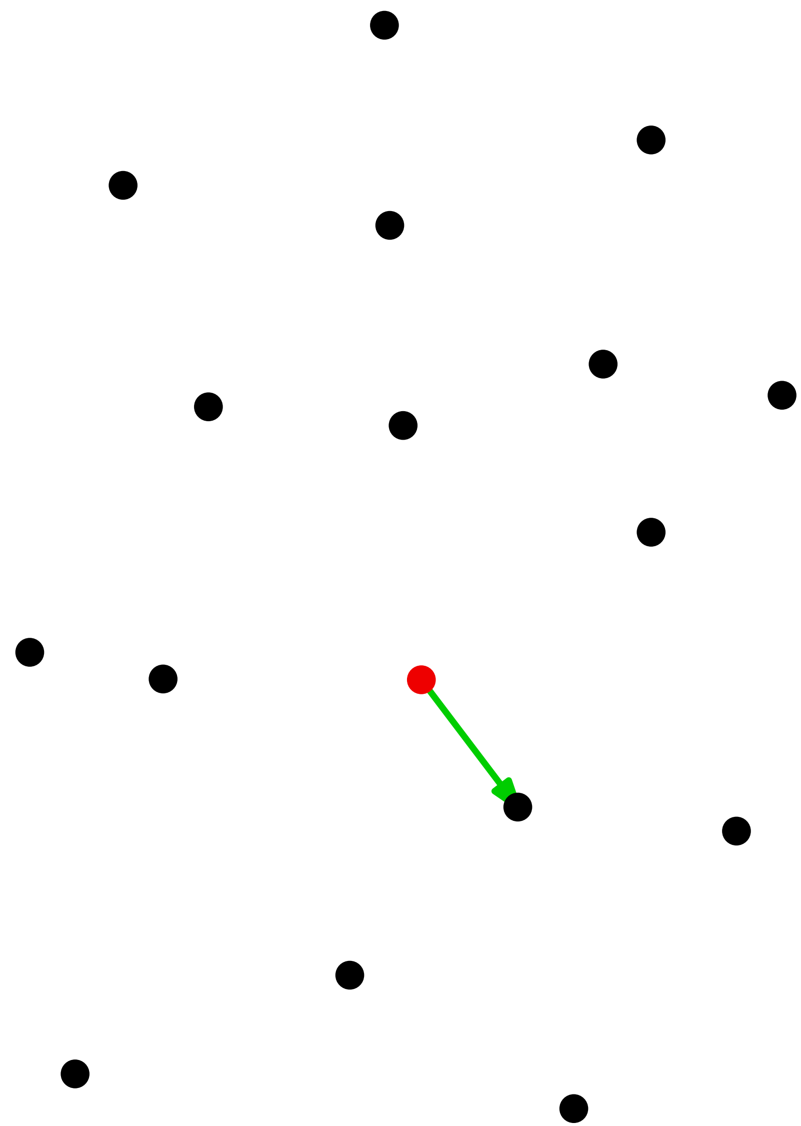

APPLICATION

The post office problem, pattern (voice) recognition and, in general, matching problems

PROXIMITY

NEAREST NEIGHBOR QUERY

Given a new point q, quickly find its closest site p_i

- Brute force: compare q with every site p_i, in $O(n)$ time
- Preprocessing P in order to speed up the queries is worth the effort when the query is to be repeatedly called
- In dimension 1, first sort the points of P in $O(n \log n)$ time, and then each query will be aswered in $O(\log n)$ time by binary search

APPLICATION

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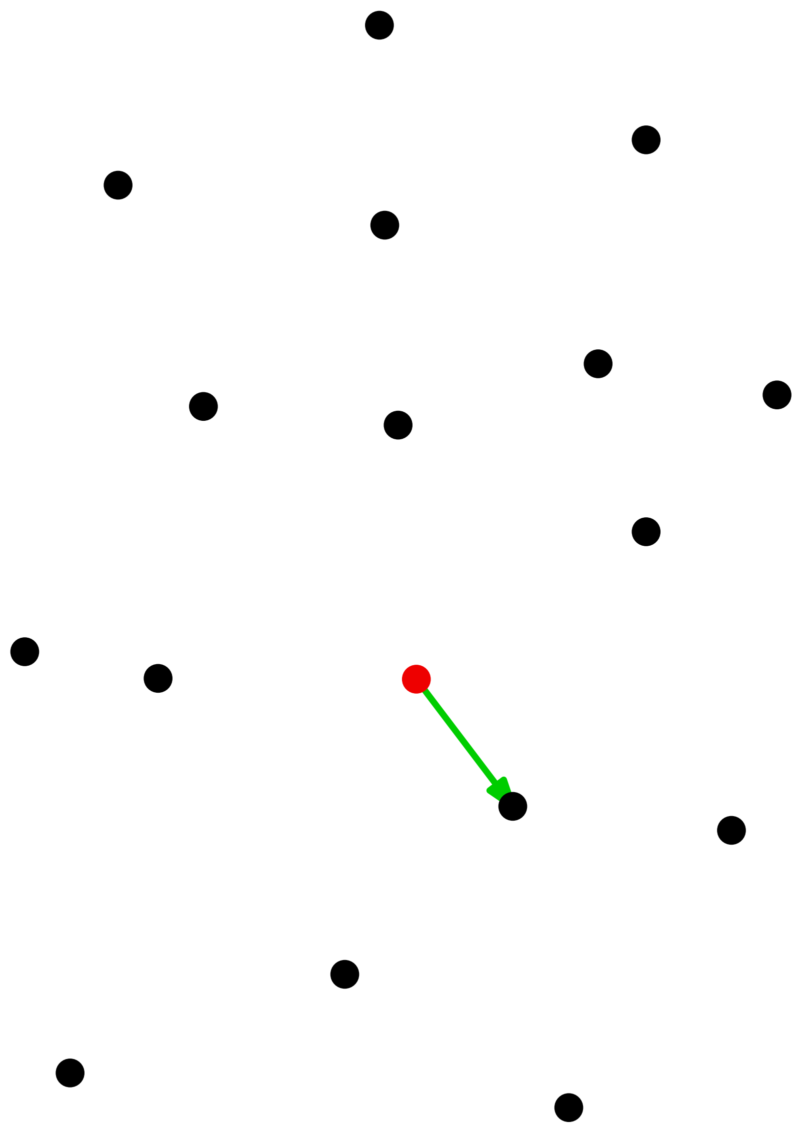
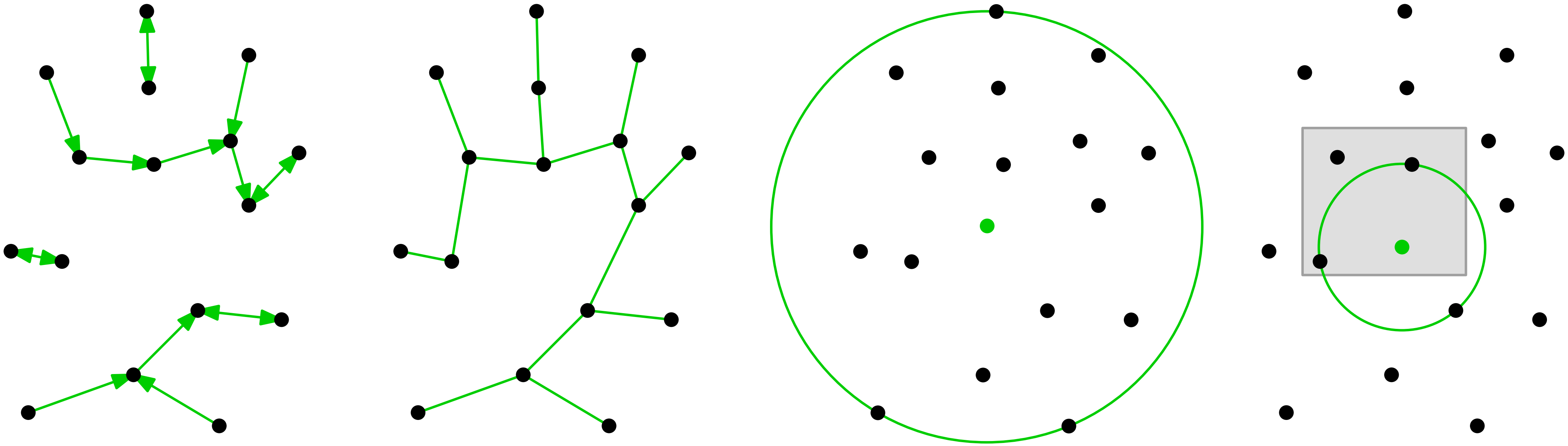


PROXIMITY

MANY DIFFERENT PROBLEMS



ALL OF THEM INVOLVE DISTANCES AND PROXIMITY

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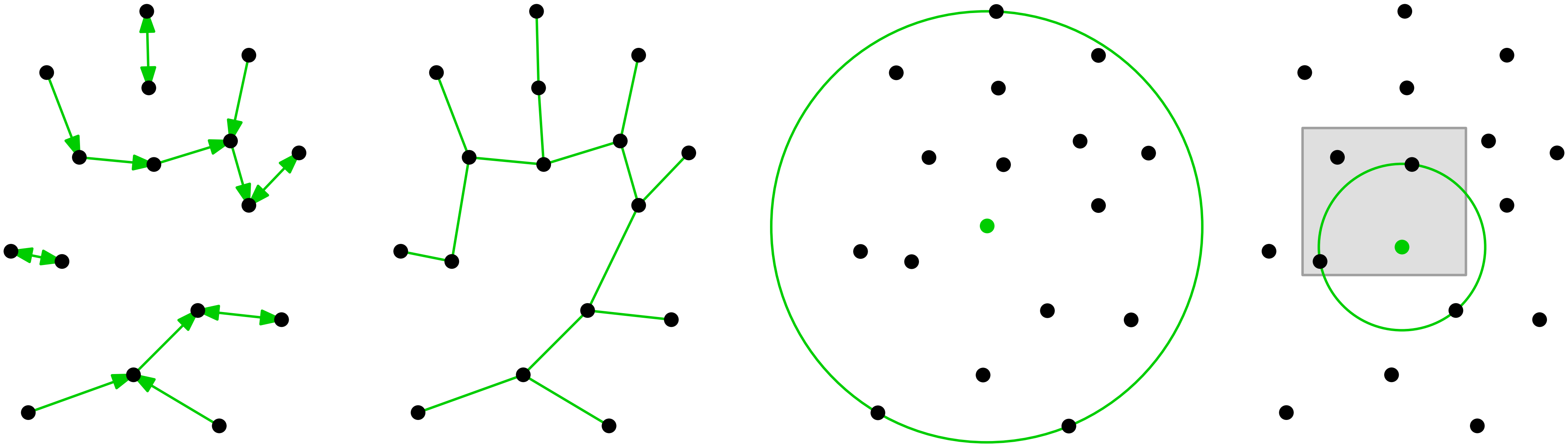

ALL OF THEM INVOLVE DISTANCES AND PROXIMITY

ONE COMMON SOLUTION:

PROXIMITY

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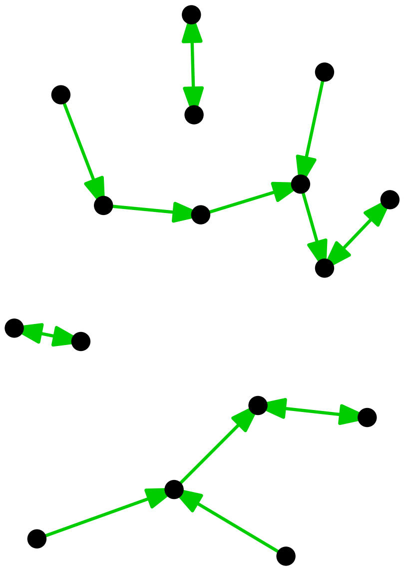
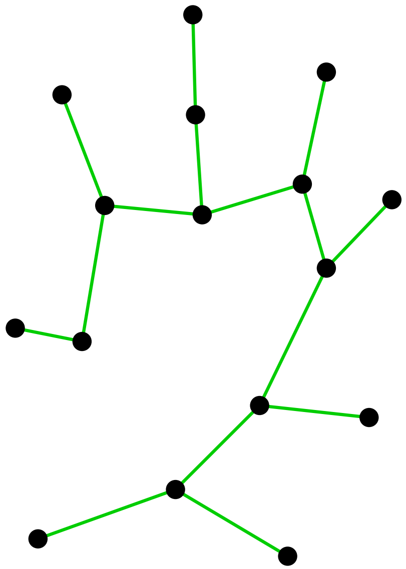
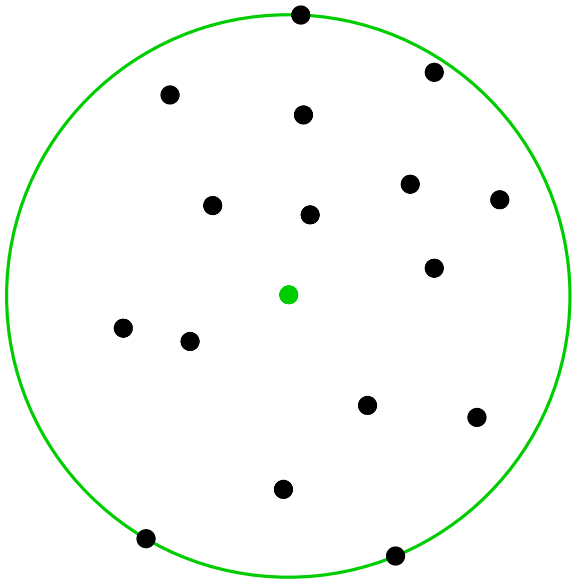
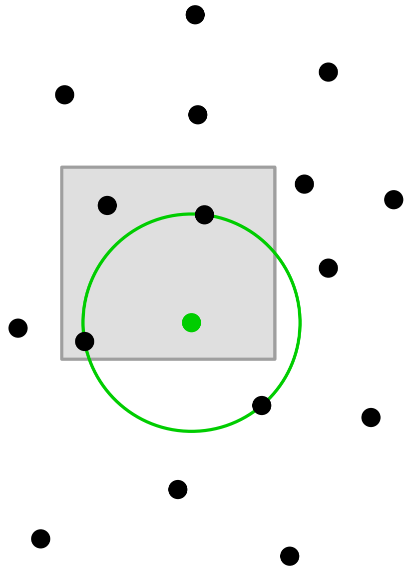








ALL OF THEM INVOLVE DISTANCES AND PROXIMITY

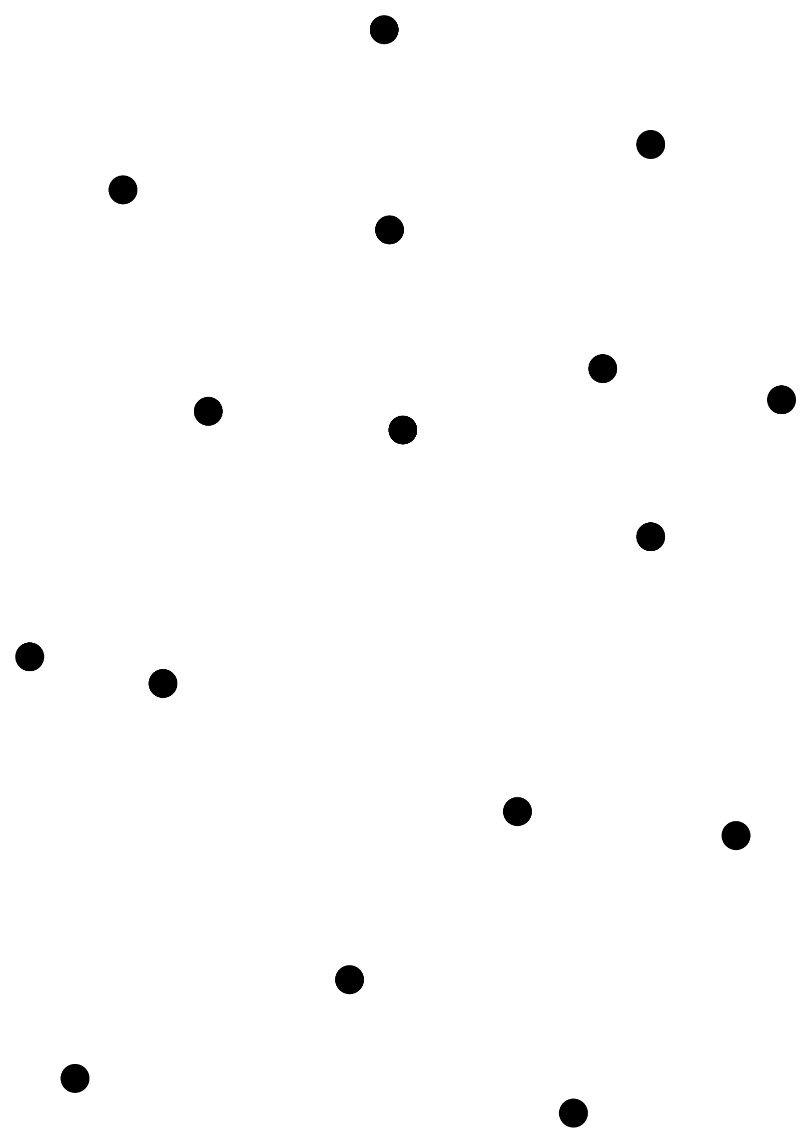
ONE COMMON SOLUTION:

Voronoi diagram

PROXIMITY

VORONOI DIAGRAM

- Structure capturing the proximity information
- Decomposes the plane into regions, each one is associated by proximity to one of the points of P
- Given two points, the boundary separating the portion of the plane closer to one point than to the other is the perpendicular bisector of the segment connecting the two points



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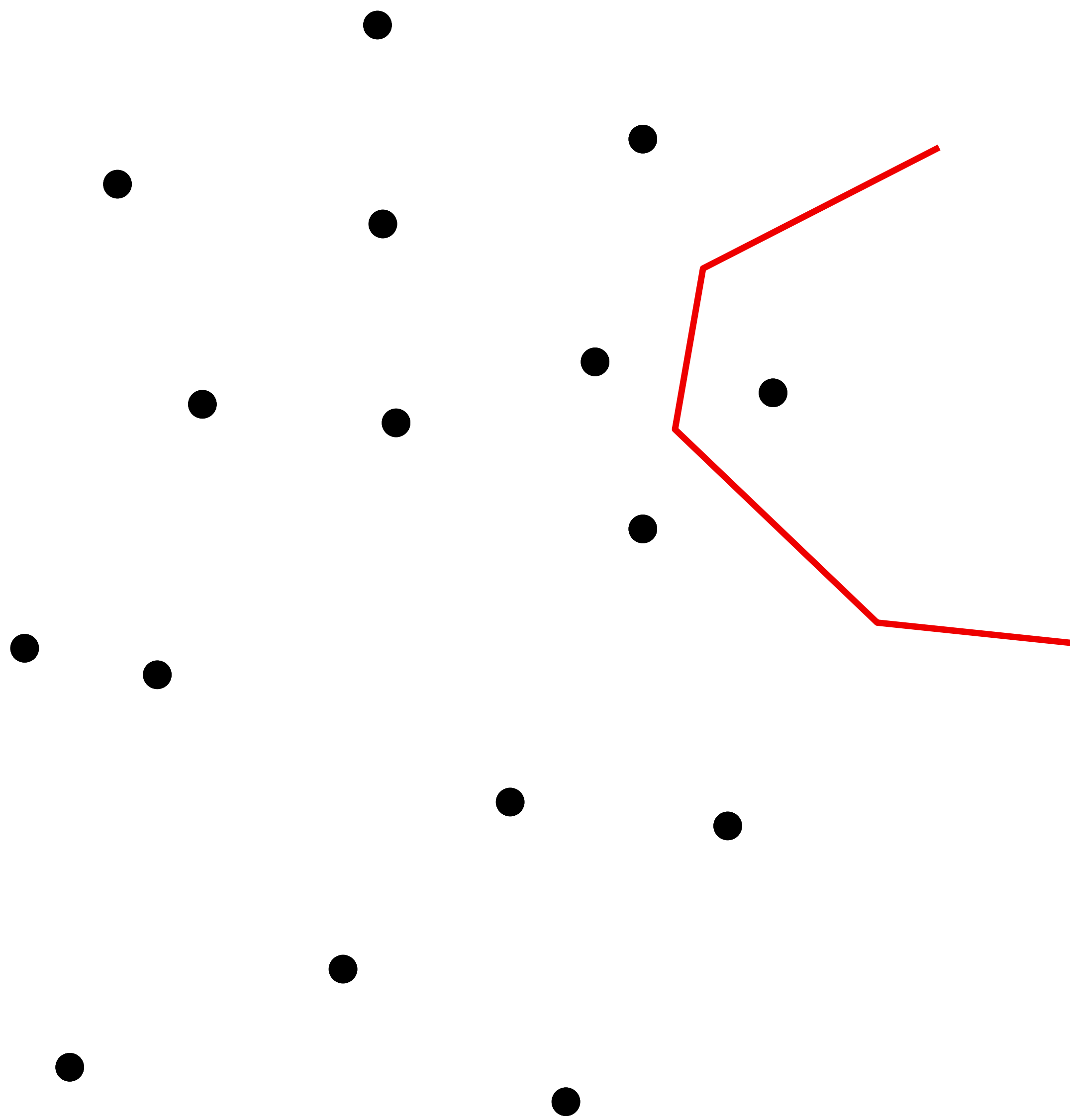

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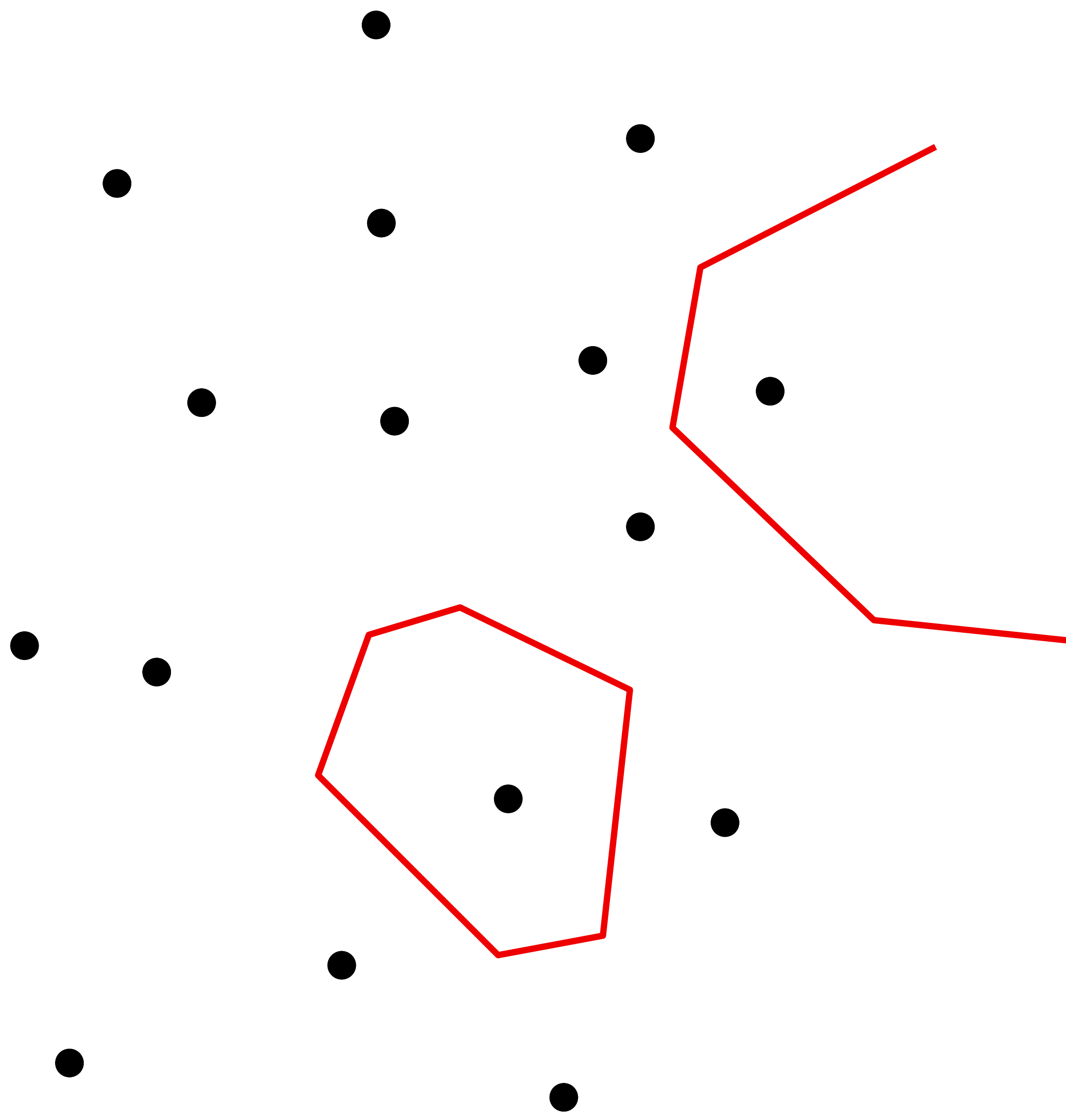

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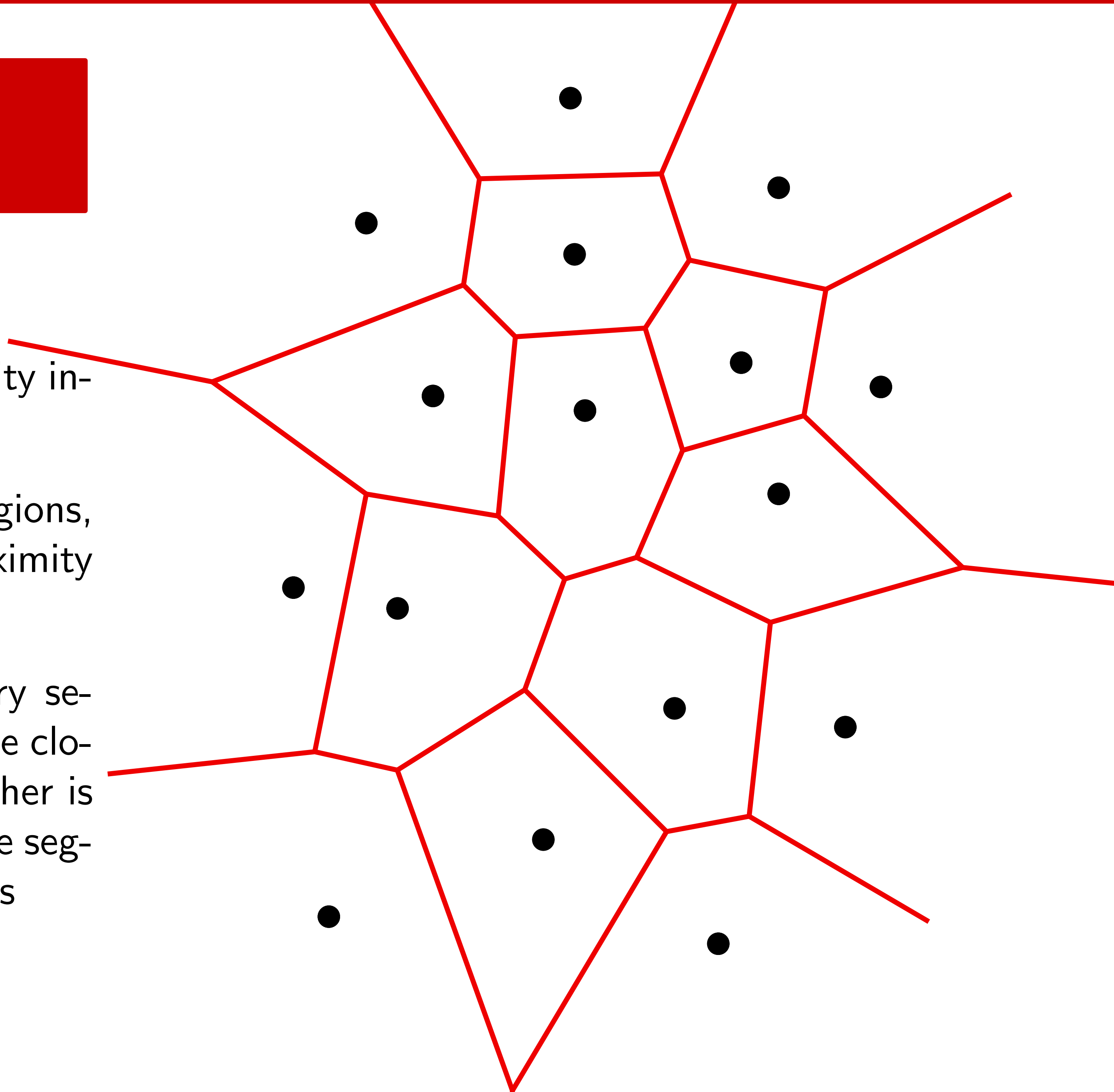
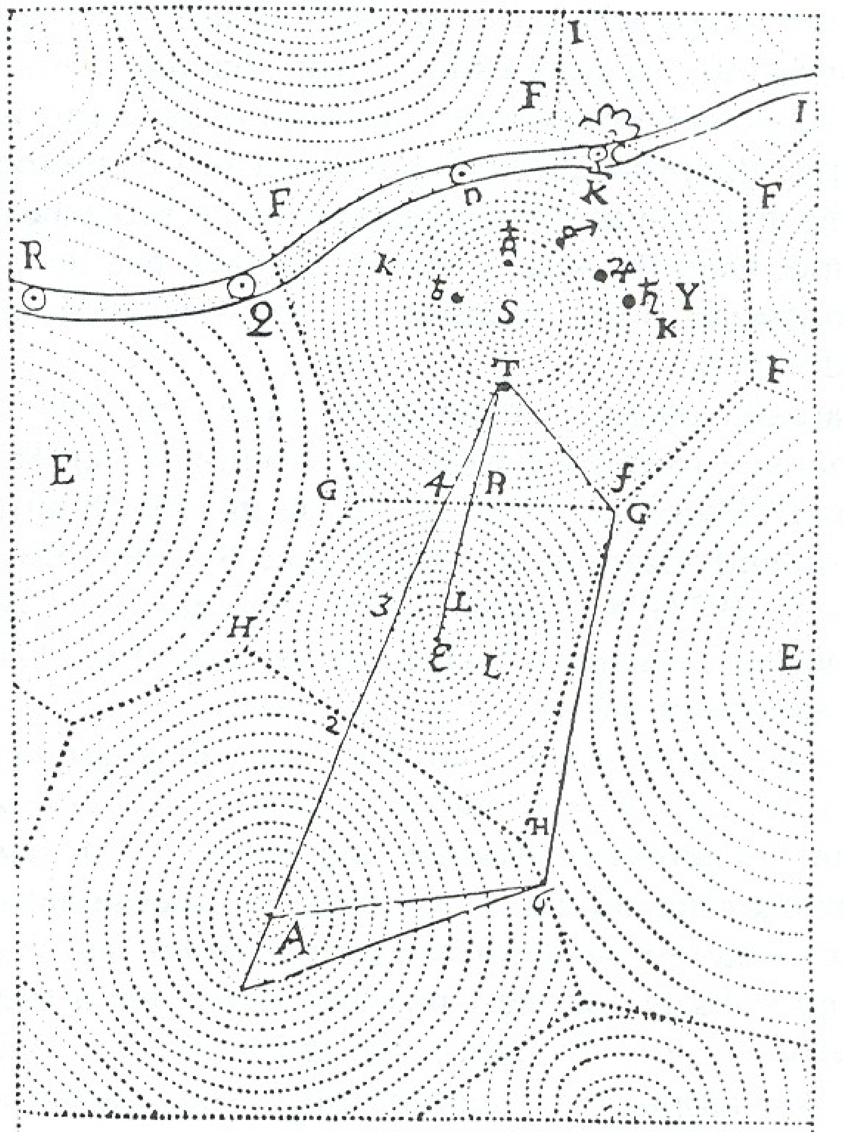


History of the rediscoveries of the Voronoi diagram

- **Descartes:** distribution of matter in the solar system and surroundings (1644).



Copy from the original book of René Descartes *Le Monde, ou Traité de la Lumière* (1644), reproduced in the edition of M. S. Mahoney, 1979.

Quoted in A. Okabe, B. Boots, K. Sugihara, S. N. Chiu, *Spatial Tessellations*, 2nd ed., J. Wiley & Sons, 2000.

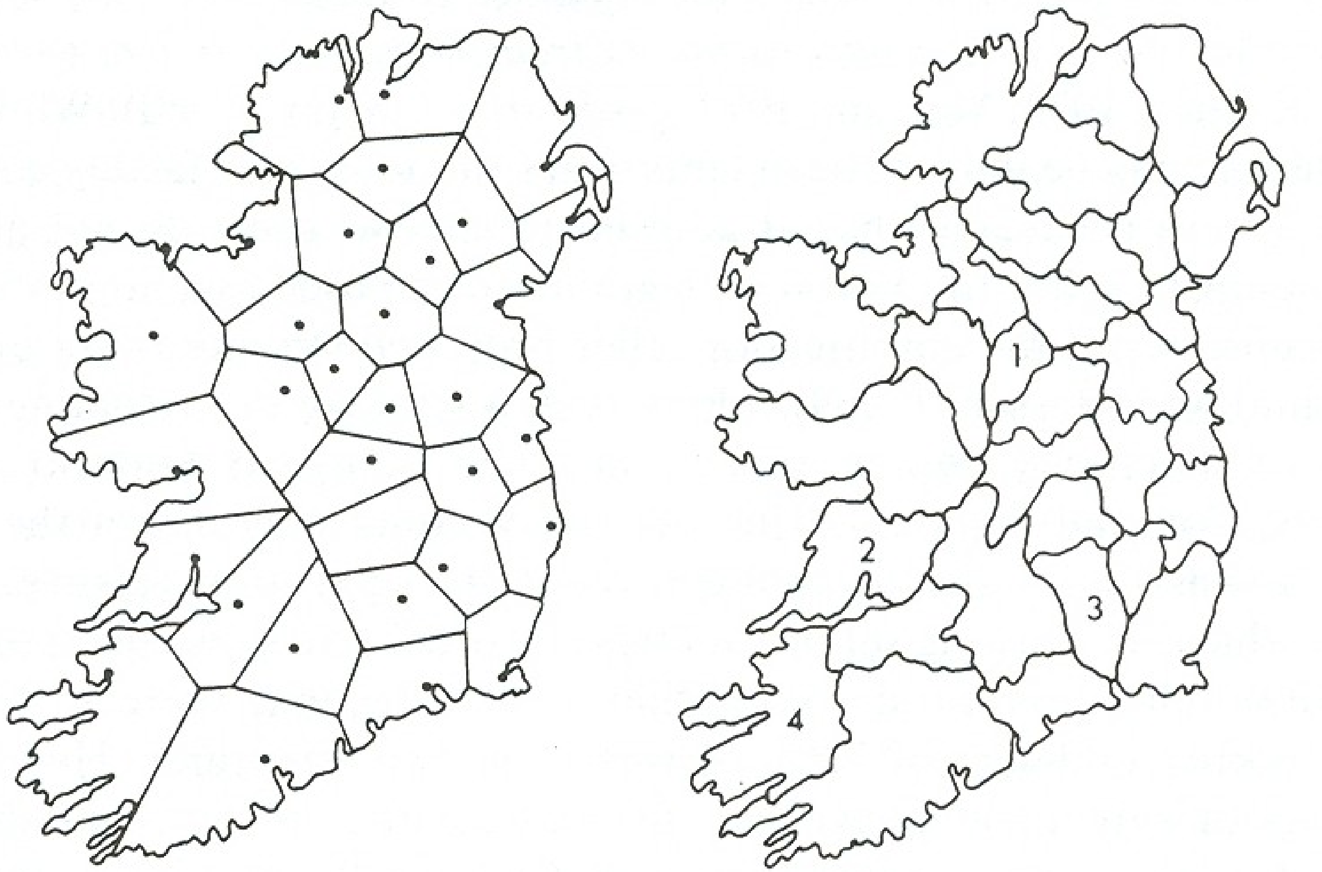
History of the rediscoveries of the Voronoi diagram

- **Descartes:** distribution of matter in the solar system and surroundings (1644).
- **Dirichlet** in dim 2 and 3, and **Voronoi** in dim d. Study of positive definite quadratic forms. *Dirichlet tessellation* (1850), *Voronoi Diagram* (1908).
- **Crystallography:** regular distribution of the points *Wirkung Sbereich, action domain, activity area, influence area* (end of the XIX century).
- **Meteorology:** rainfall average estimation based on local data, *Thiessen's polygons* (1911).
- **Geology:** estimation of the presence of gold in a sediment, based on trial pits, *influence area polygons* (1909).
- **Physics and Chemistry:** study of the chemical properties of some elements, *Wigner-Seitz's regions*(1933); alloy structure modelling, *atom domain* (1958).
- **Ecology:** survival of organisms in competition for nourishment or light, like trees in a forest, *area potentially available* (1965), *plants polygons* (1966)
- **Medicine:** region of muscle tissue supplied by a capillary, *capillarity domains* (1985)

History of the rediscoveries of the Voronoi diagram (continued)

- **Economics:** market areas of centers in competition, under various conditions of market prices and transportation costs (since the mid XIX century)
- **Archaeology:** study of the diffusion of the use of tools in order to analyze the influence of the civilization centers (sixties-seventies)
- **Anthropology:** modelling territorial human organization under several aspects (sixties-seventies)

More recent applications of the Voronoi diagram



Ireland counties. On the left, theoretical partition created with the Voronoi diagram of the counties capitals. On the right, the real counties.

K. R. Cox, J. A. Agnew, *The spatial correspondence of theoretical partitions*, Discussion paper, SyracuseUniversity, Dept. of Geography, 1976. Quoted in A. Okabe, B. Boots, K. Sugihara, S. N. Chiu, *Spatial Tessellations*, 2nd ed., J. Wiley & Sons, 2000.

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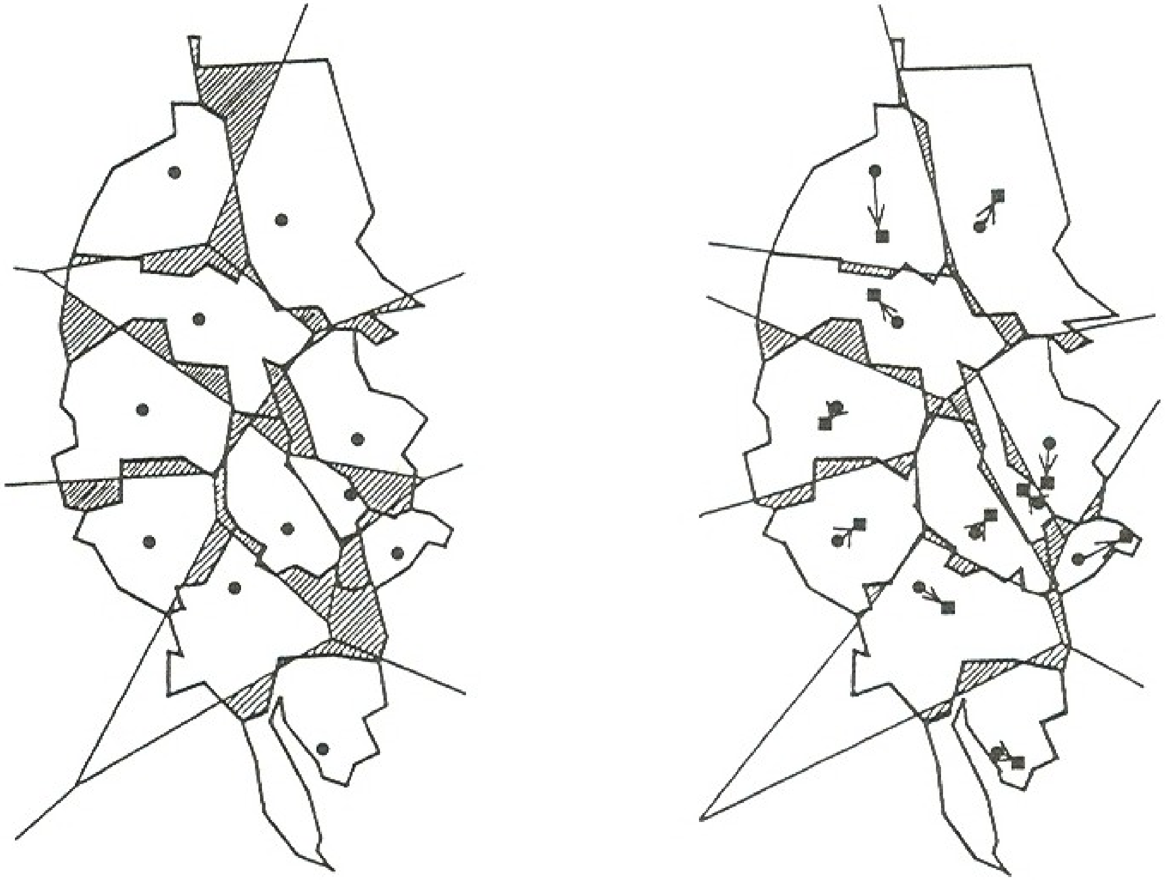

The school districts at Tsukuba. Left, real districts and the location of the schools (circles). Right, relocation of the schools (squares) as to minimizing the area of the students that do not attend their closest school.

A. Suzuki, M. Iri, *Aproximation on a tessellation of the plane by a Voronoi diagram*,journal of the Operation Research Society of Japan, 29, 1986. Quoted in A. Okabe, B. Boots, K. Sugihara, S. N. Chiu, *Spatial Tessellations*, 2nd ed., J. Wiley & Sons, 2000.

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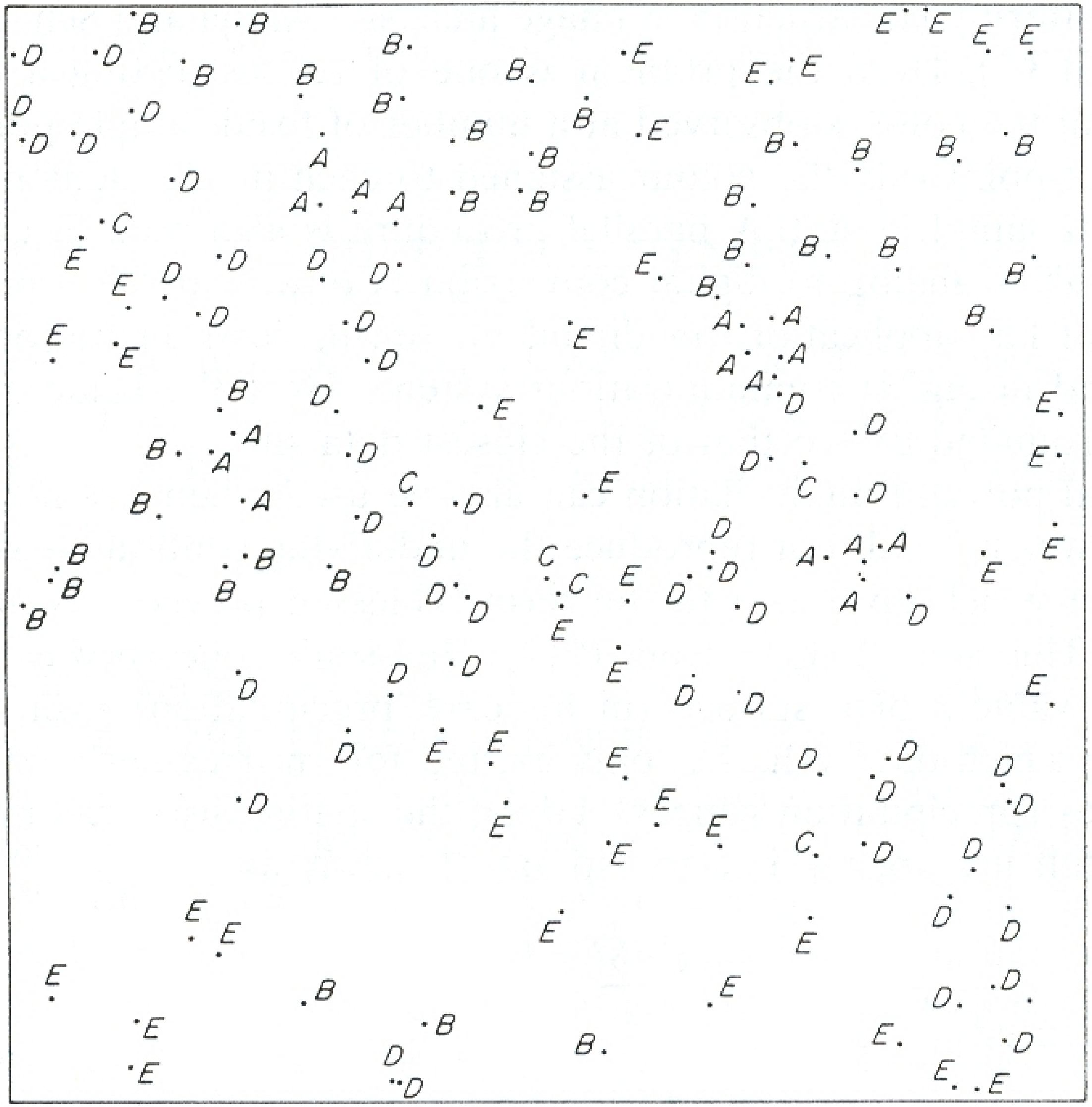

Interpolation of discrete sample values

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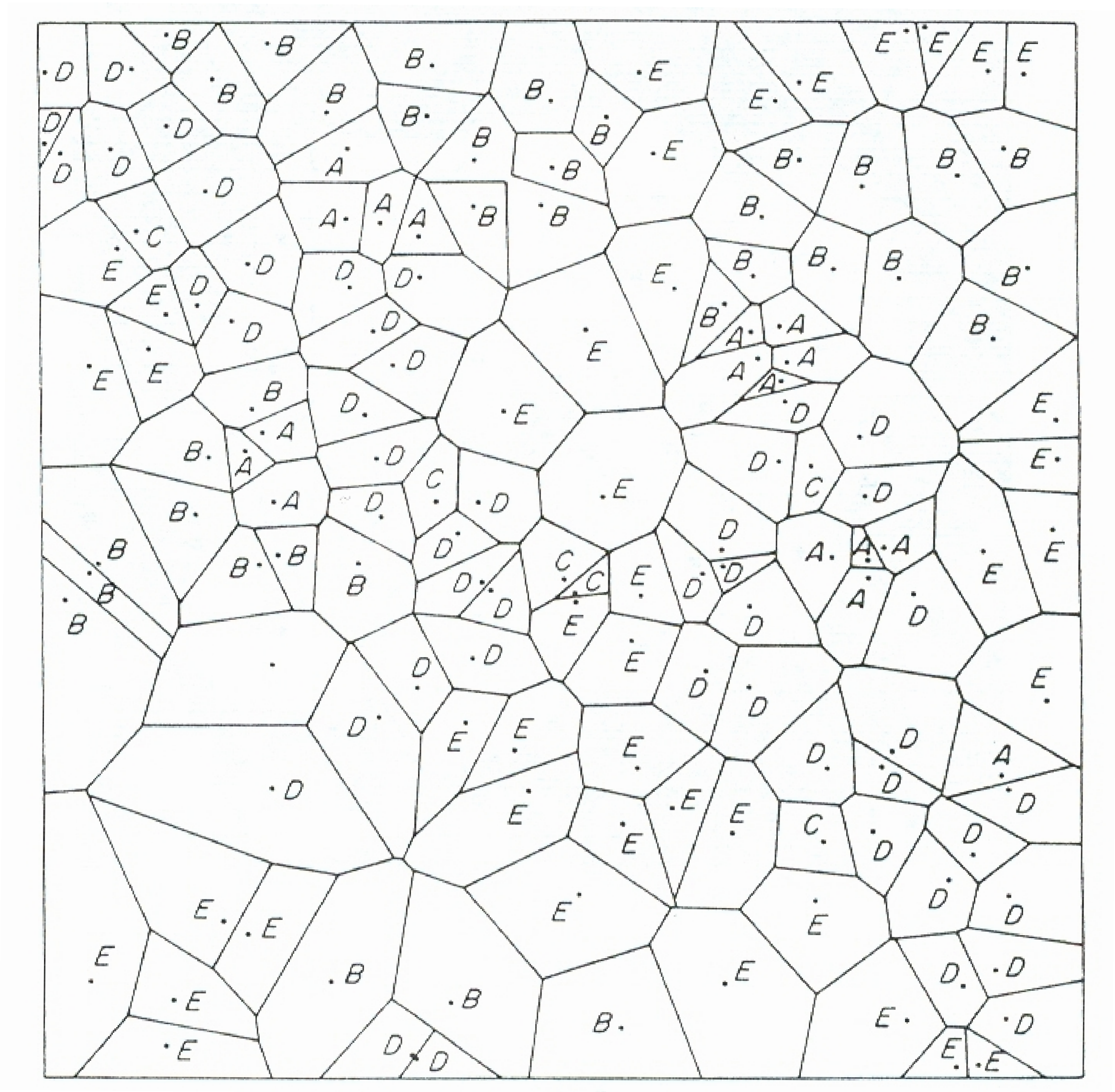

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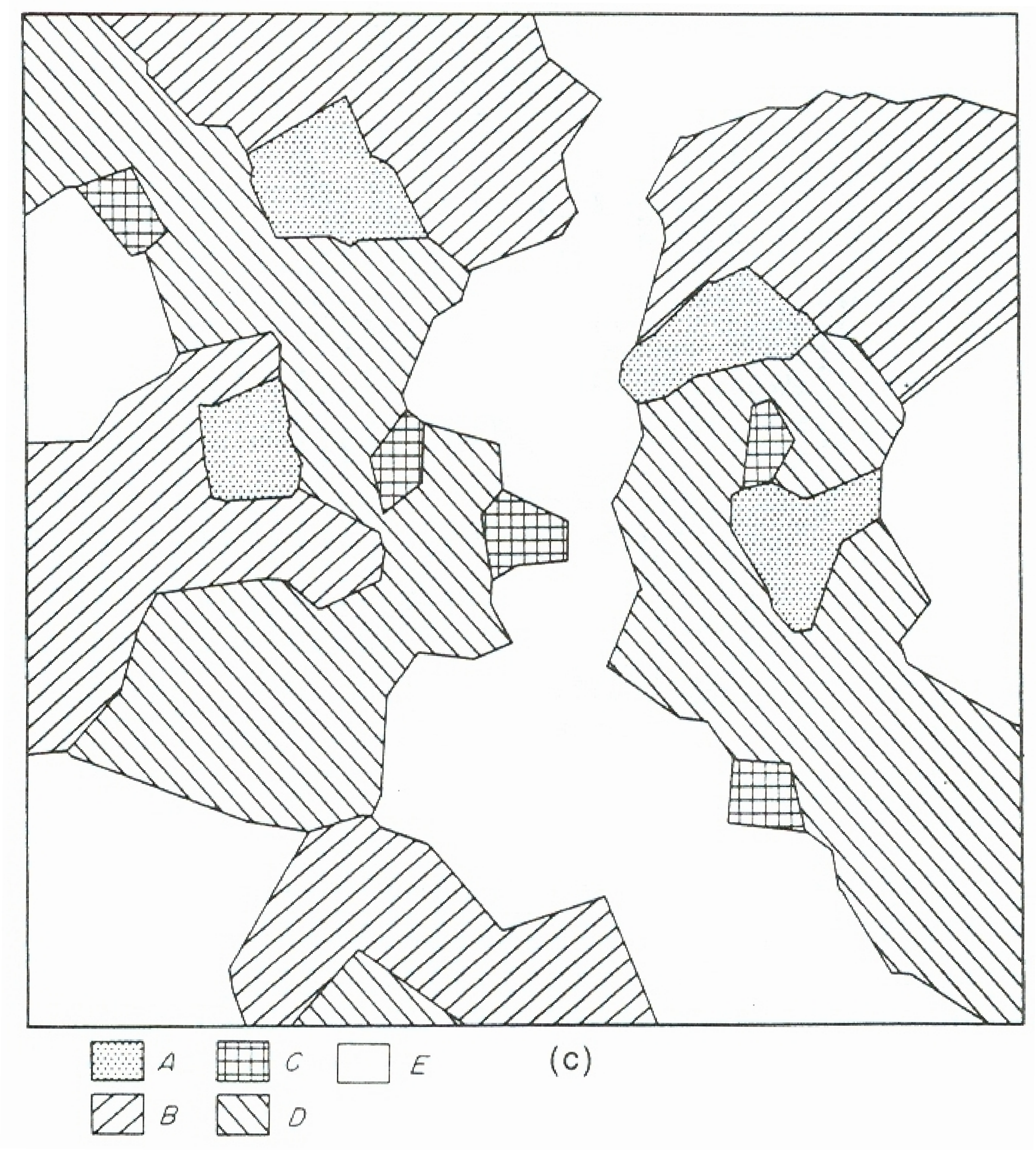

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Constructing the Voronoi diagram

Sixties

- Application concepts clear
- Lack of tools to build the diagrams

Seventies

- Start of the development of algorithm to compute the diagrams, thanks to computer science
- Discovery of new propierties associated to the diagrams

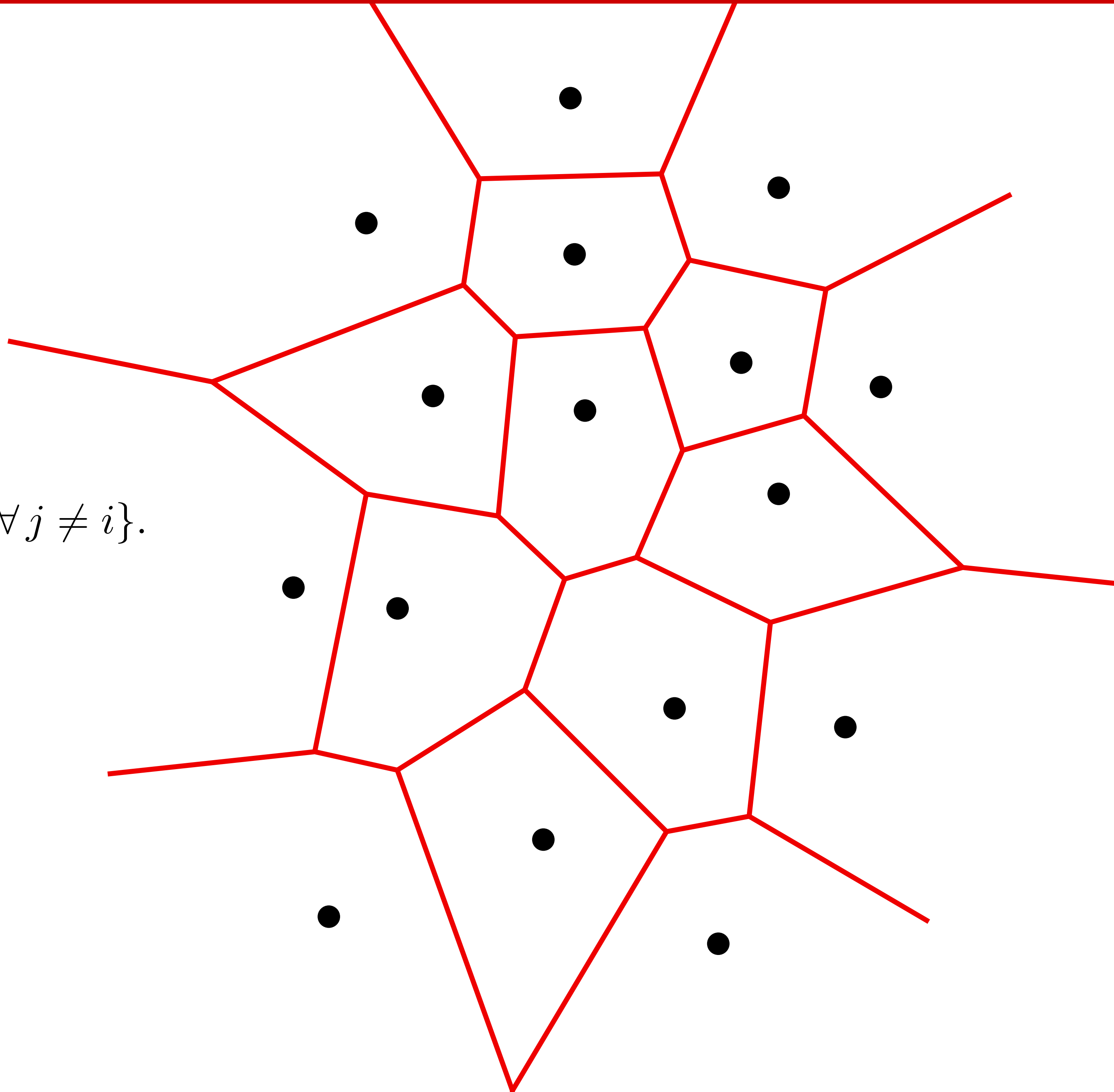
PROXIMITY

DEFINITION

Let $P = \{p_1, \ldots, p_n\}$ be a finite set of points in the plane.

The *Voronoi diagram* of P is the decomposition of the plane $V(P) = \{V(p_1), \ldots, V(p_n)\}$ such that

$V(p_i) = \{x \mid d(x, p_i) \leq d(x, p_j) \; \forall \, j \neq i\}$.



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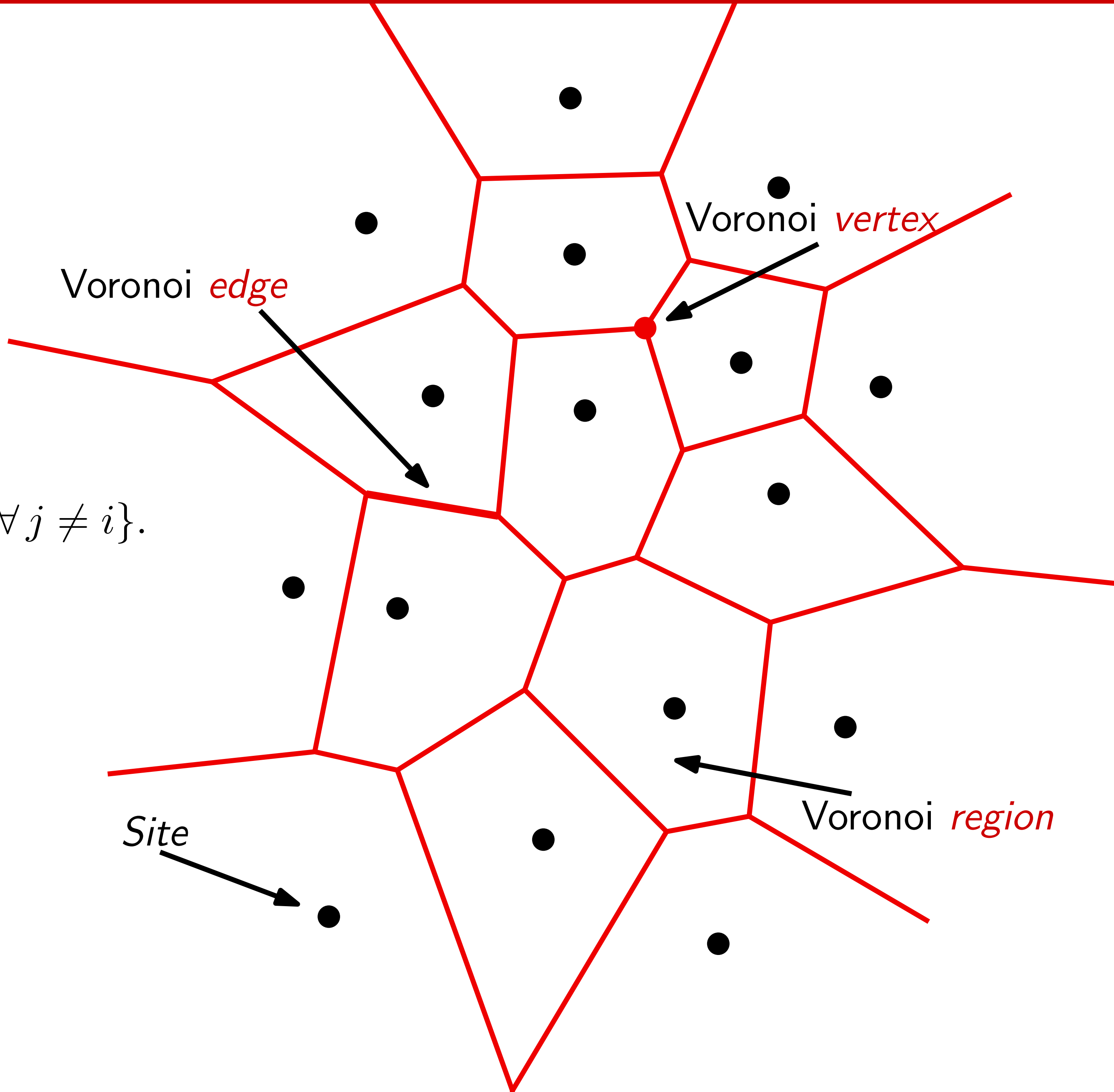

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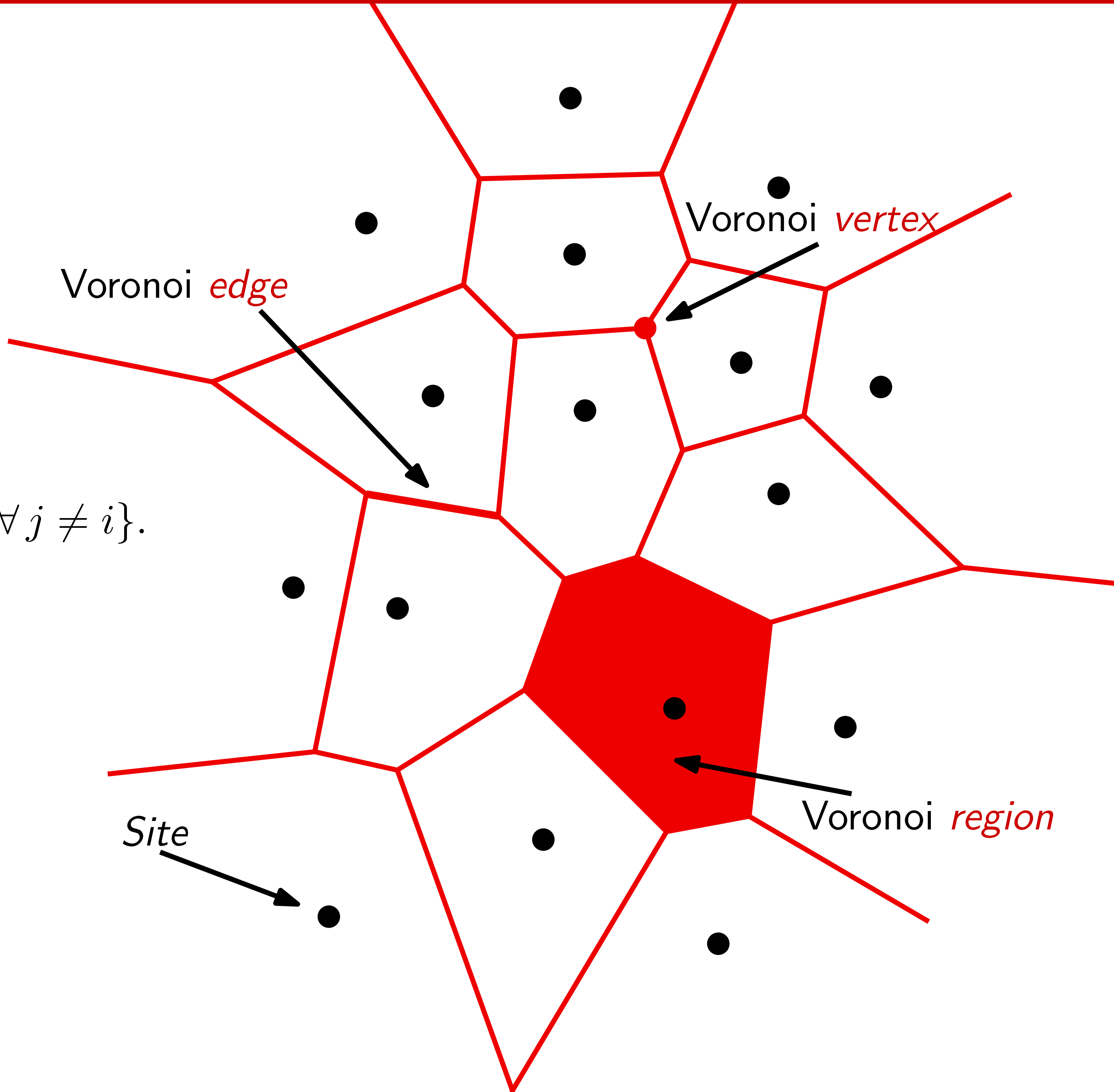
CHARACTERIZATION

If b_{ij} is the perpendicular bisector of the segment $p_i p_j$, and H_{ij} is the halfplane defined by b_{ij} enclosing p_i, then

$$V(p_i) = \bigcap_{j \neq i} H_{ij}.$$



PROXIMITY

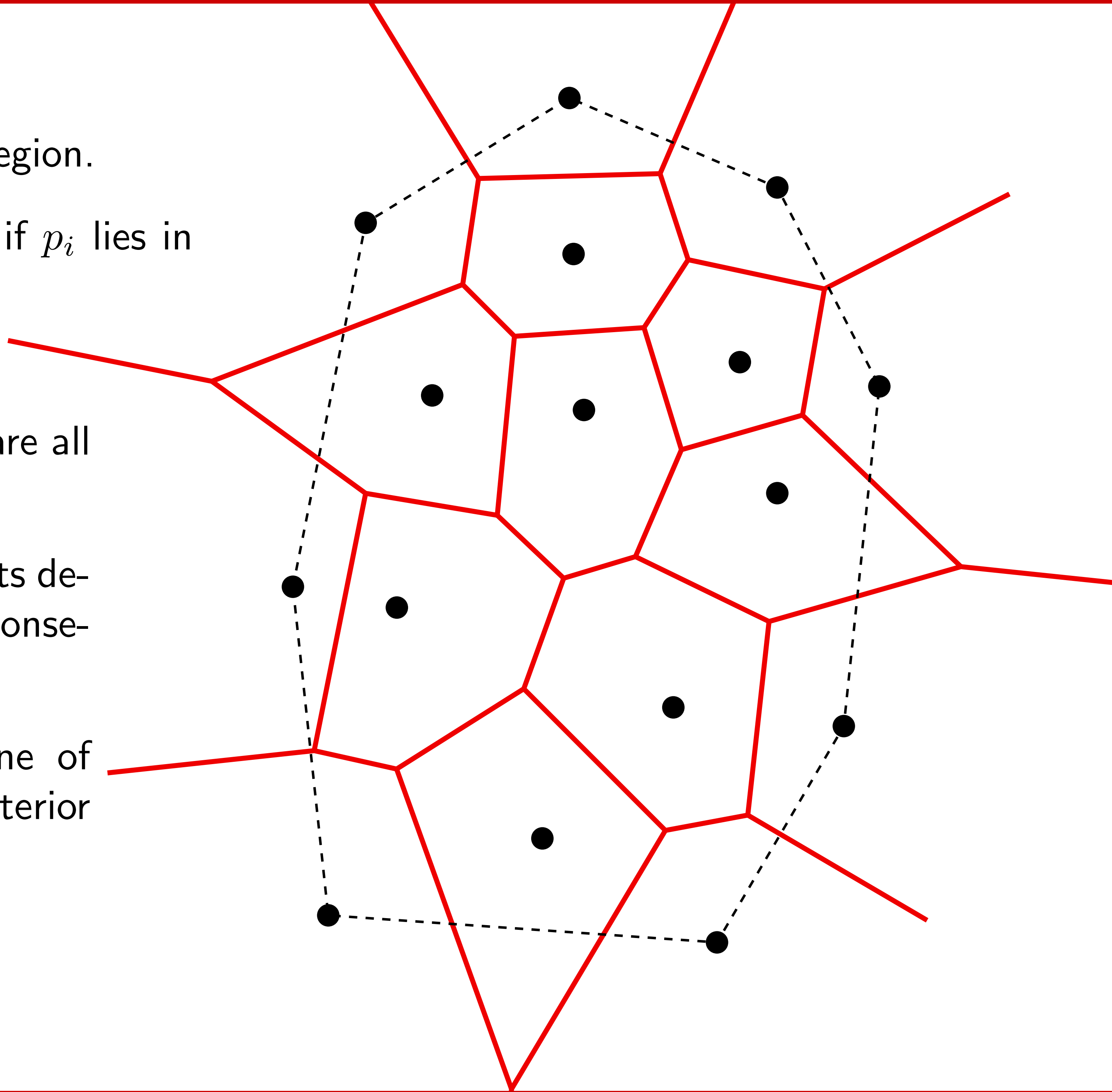
PROPERTIES

1. $V(p_i)$ is a convex polygonal region.

2. $V(p_i)$ is bounded if and only if p_i lies in the interior of $ch(P)$.

3. Voronoi edges can be:

    - lines, if the points of P are all aligned;
    - half-lines, if the two points determining the edge are consecutive vertices of $ch(P)$;
    - segments, if at least one of the points lies in the interior of $ch(P)$.



PROXIMITY

PROPERTIES (continued)

4. The planarity of the Voronoi diagram allows applying Euler's formula to the graph adding a vertex at infinity:

   $$v + n = e + 1.$$

   Since each vertex is incident to at least 3 edges, and each edge has exactly 2 endpoints,

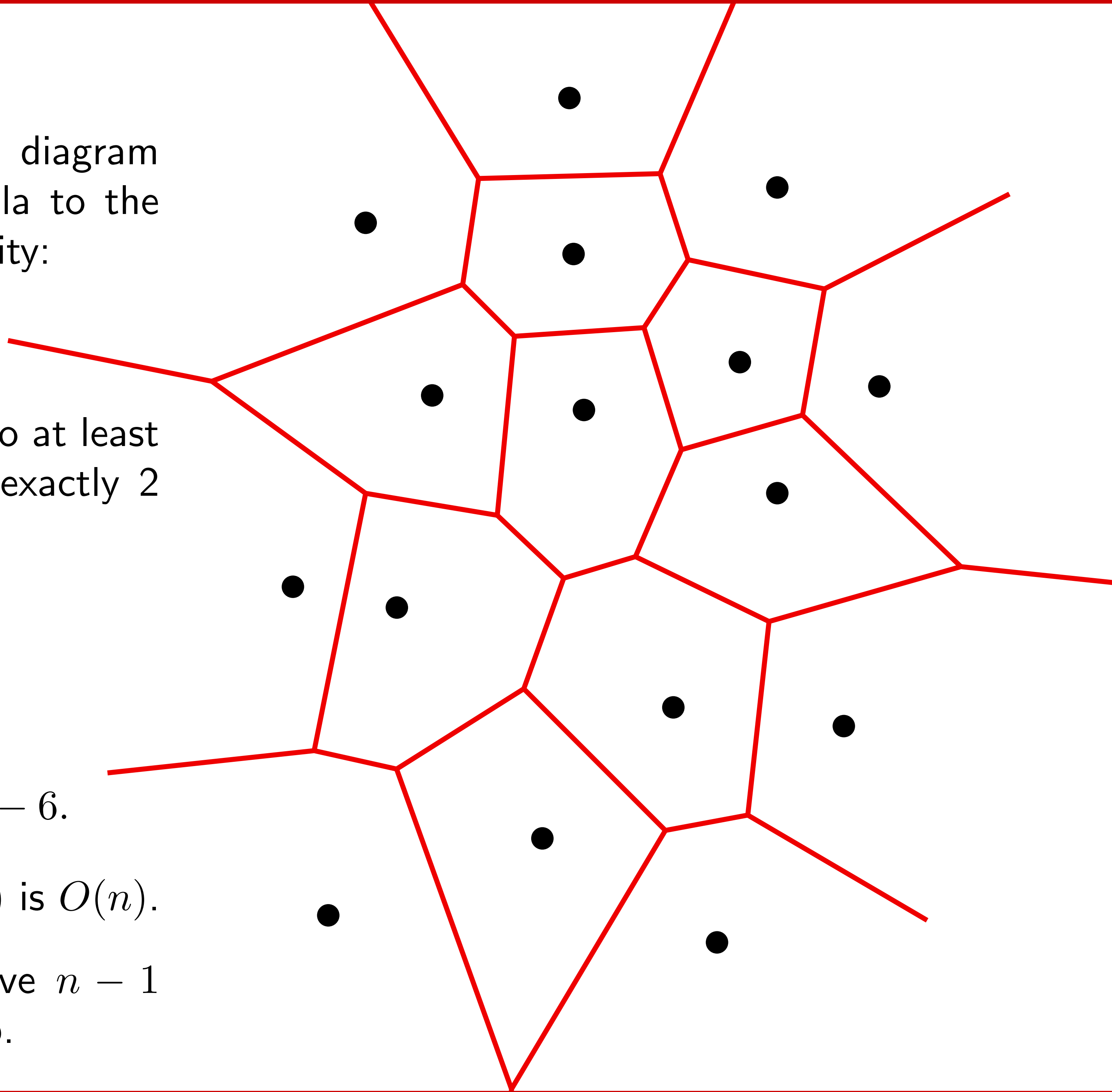
   $$2e \geq 3(v + 1),$$

   from where we get:

   $$v \leq 2n - 5, \qquad e \leq 3n - 6.$$

   Thus, the complexity of $V(P)$ is $O(n)$.

5. Although each $V(p_i)$ can have $n - 1$ edges, on average they have 6.



PROXIMITY

THE DUAL GRAPH

Proximity between points: two points of P are neighbors if they share an edge in the Voronoi diagram of P.

A proximity graph is obtained by connecting each point to its Voronoi neighbors. This graph is called *Delaunay graph*.

The Delaunay graph is the rectiliniear dual of the Voronoi diagram!

In general, it is a triangulation of P, although sometimes it is not:

- when the points are aligned,
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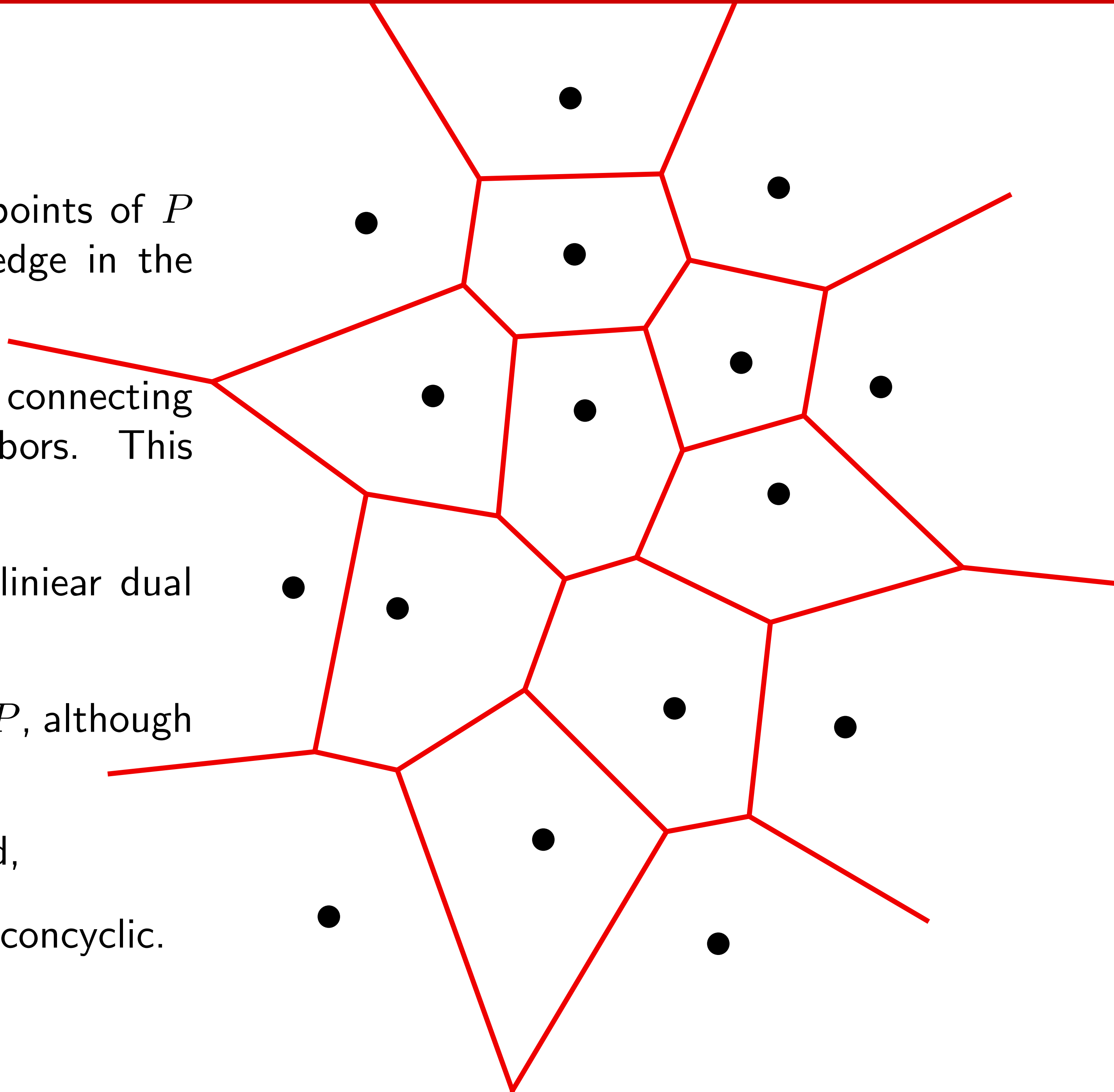

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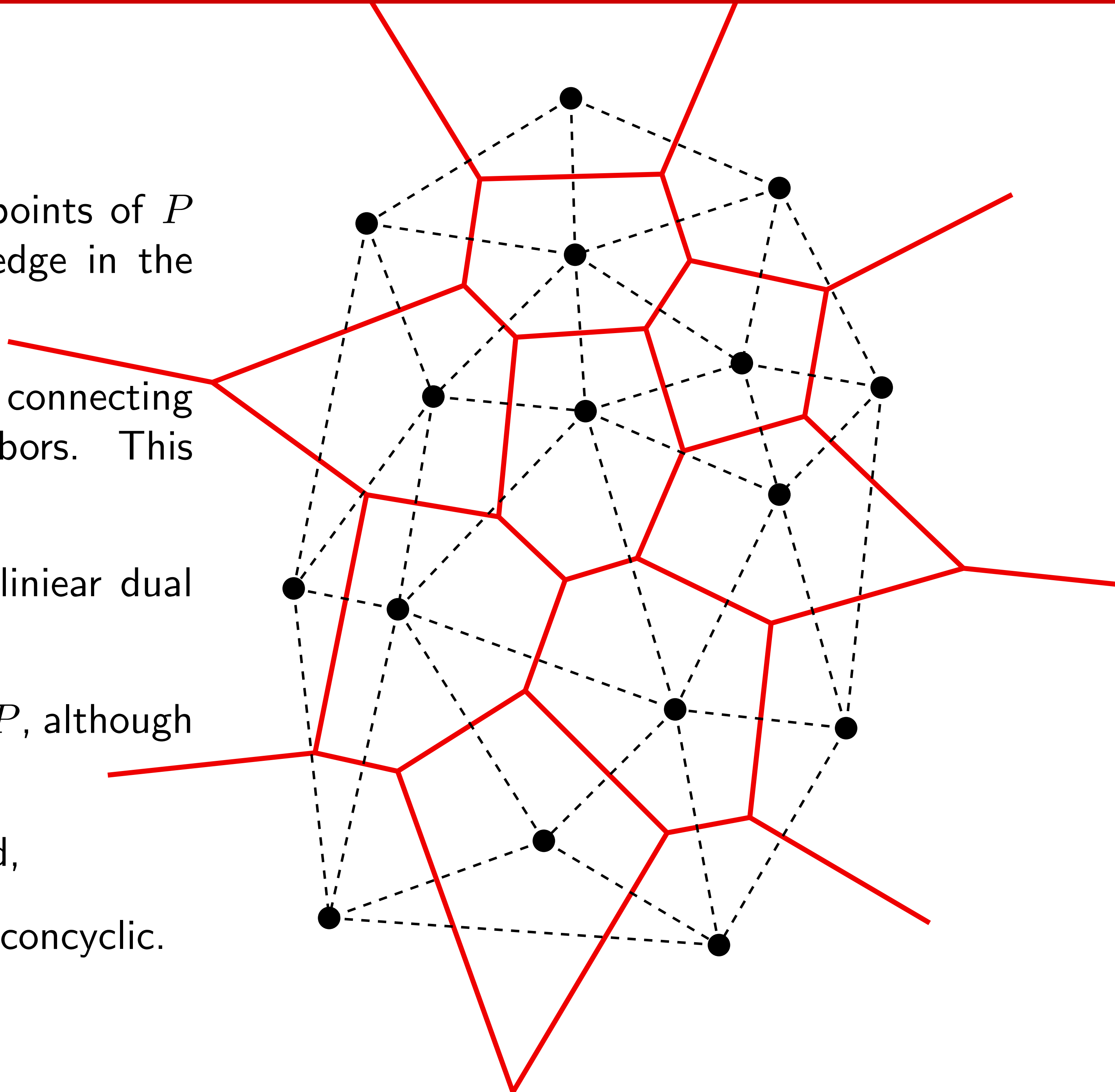

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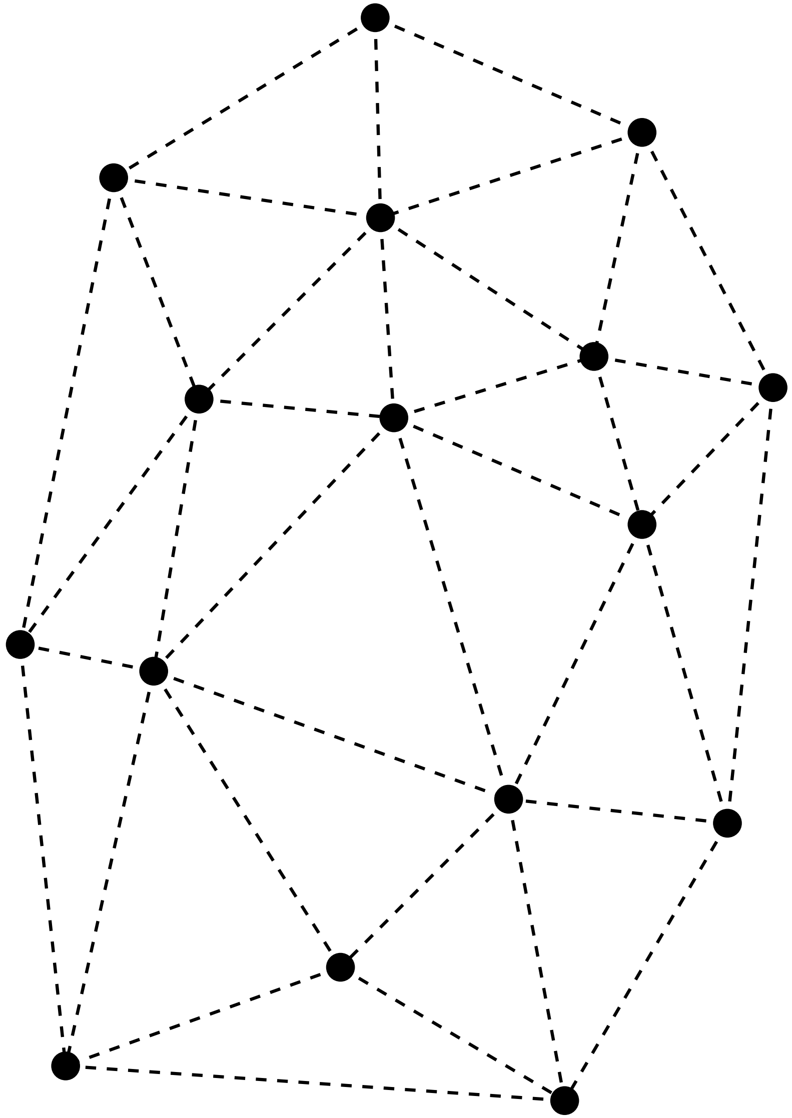

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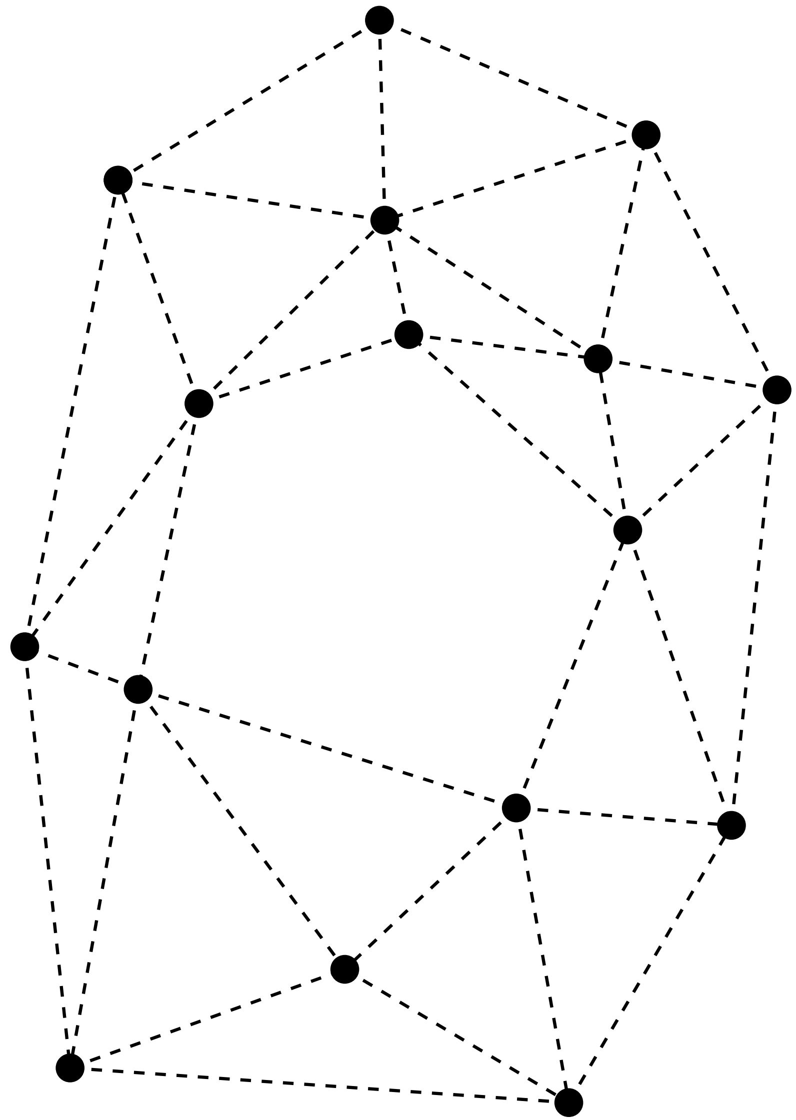

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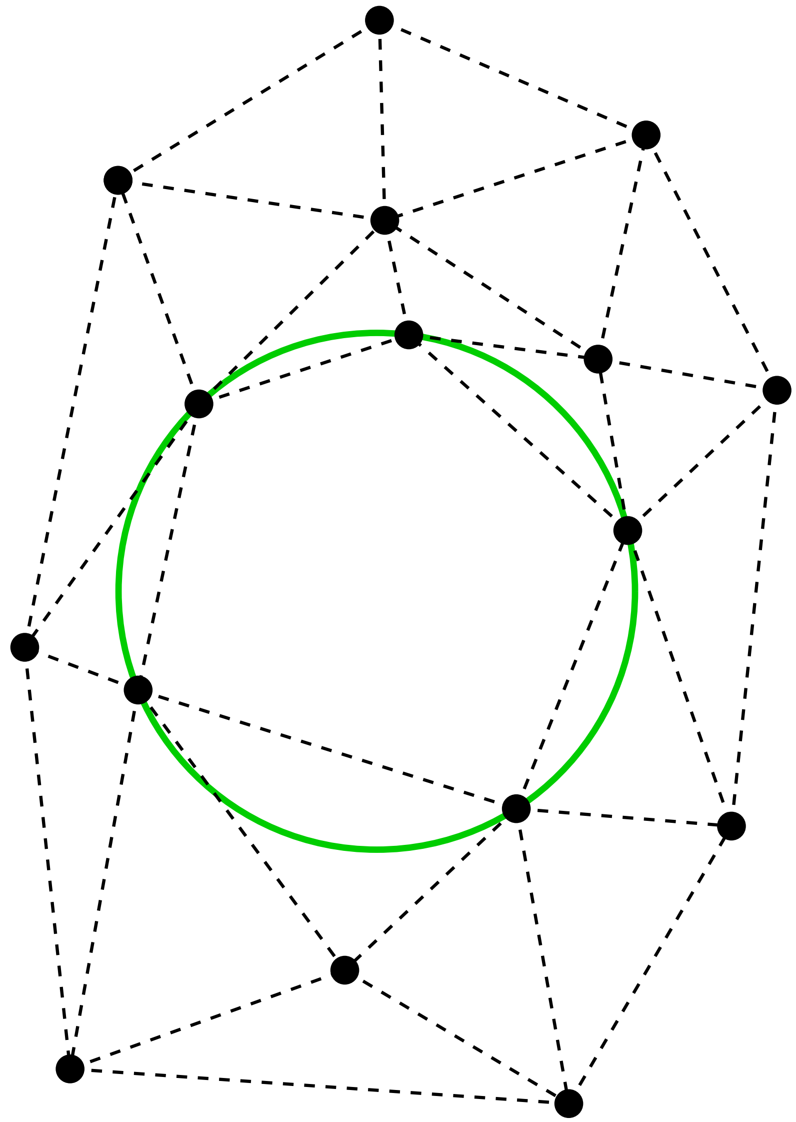

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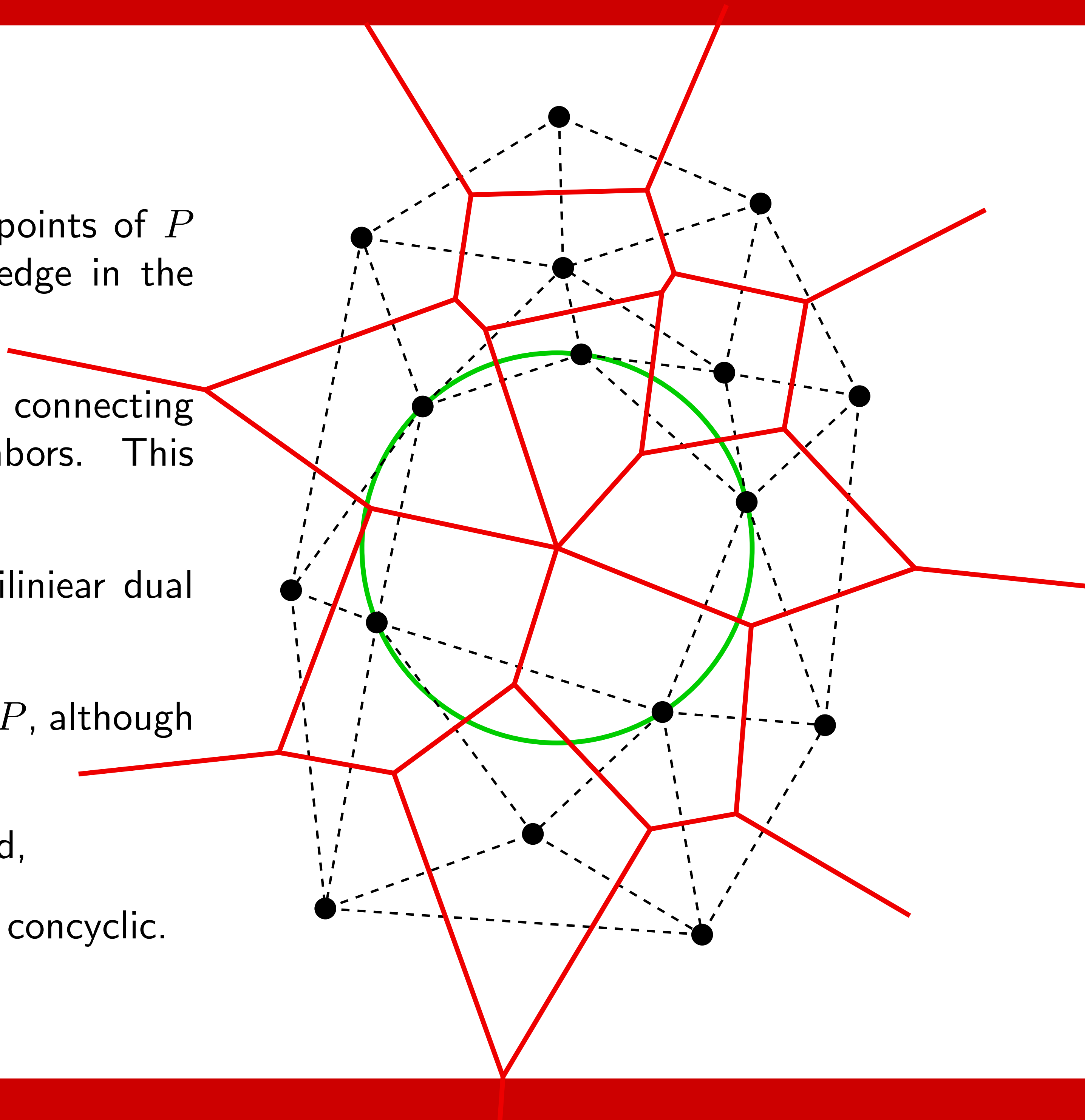
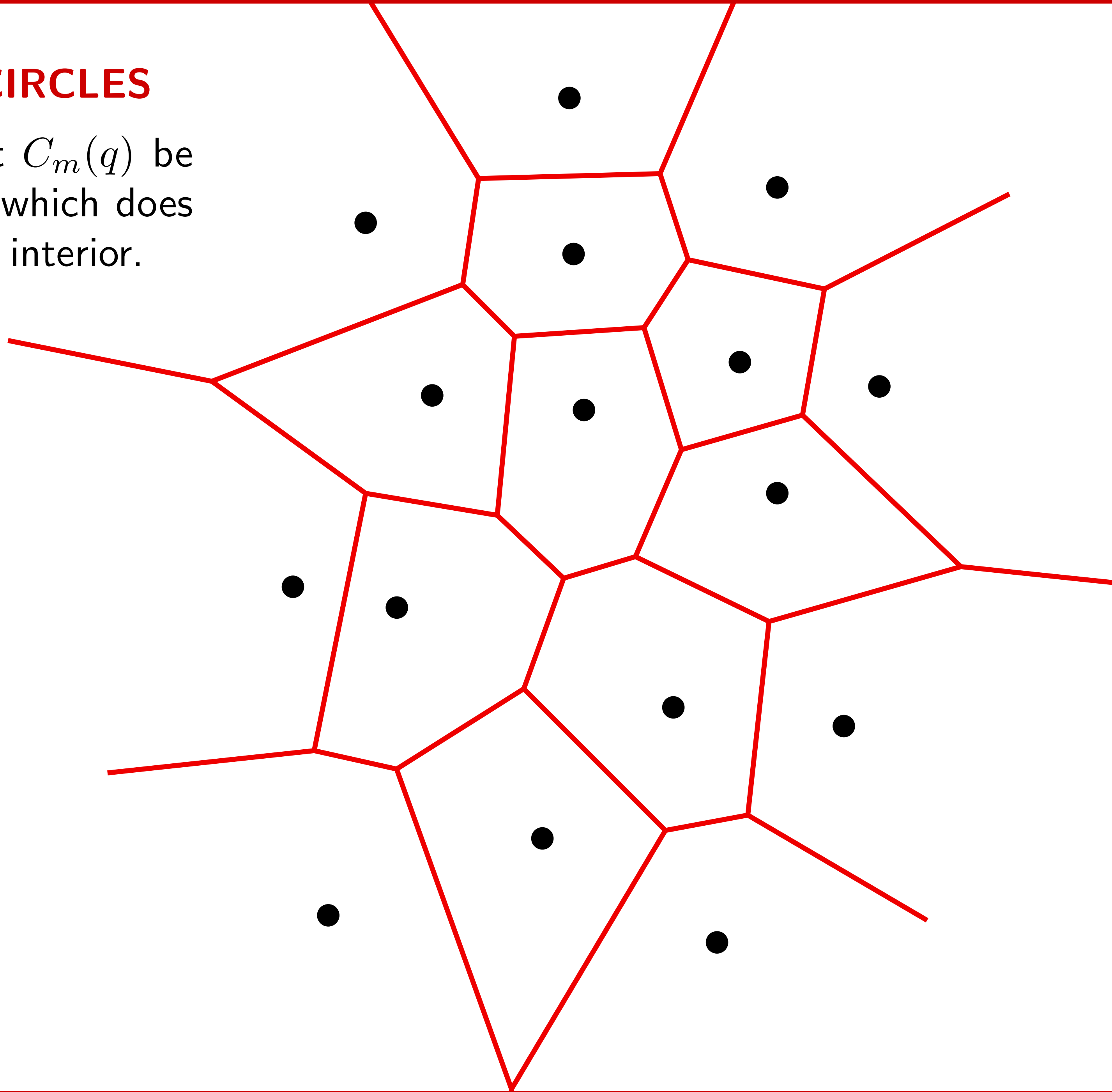


PROXIMITY AND EMPTY CIRCLES

For any point q in the plane, let $C_m(q)$ be the largest circle, centered at q, which does not contain any point of P in its interior.



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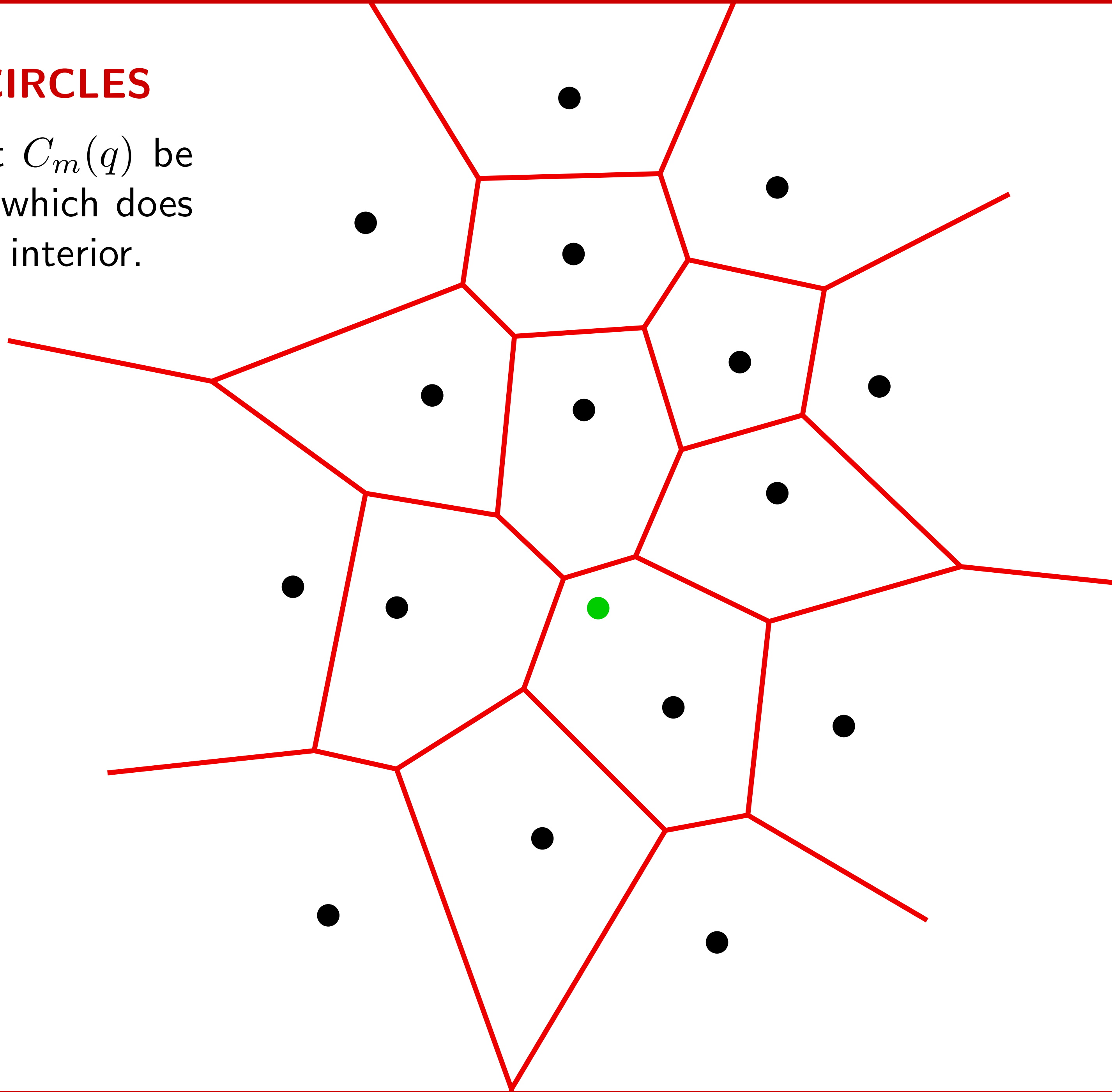

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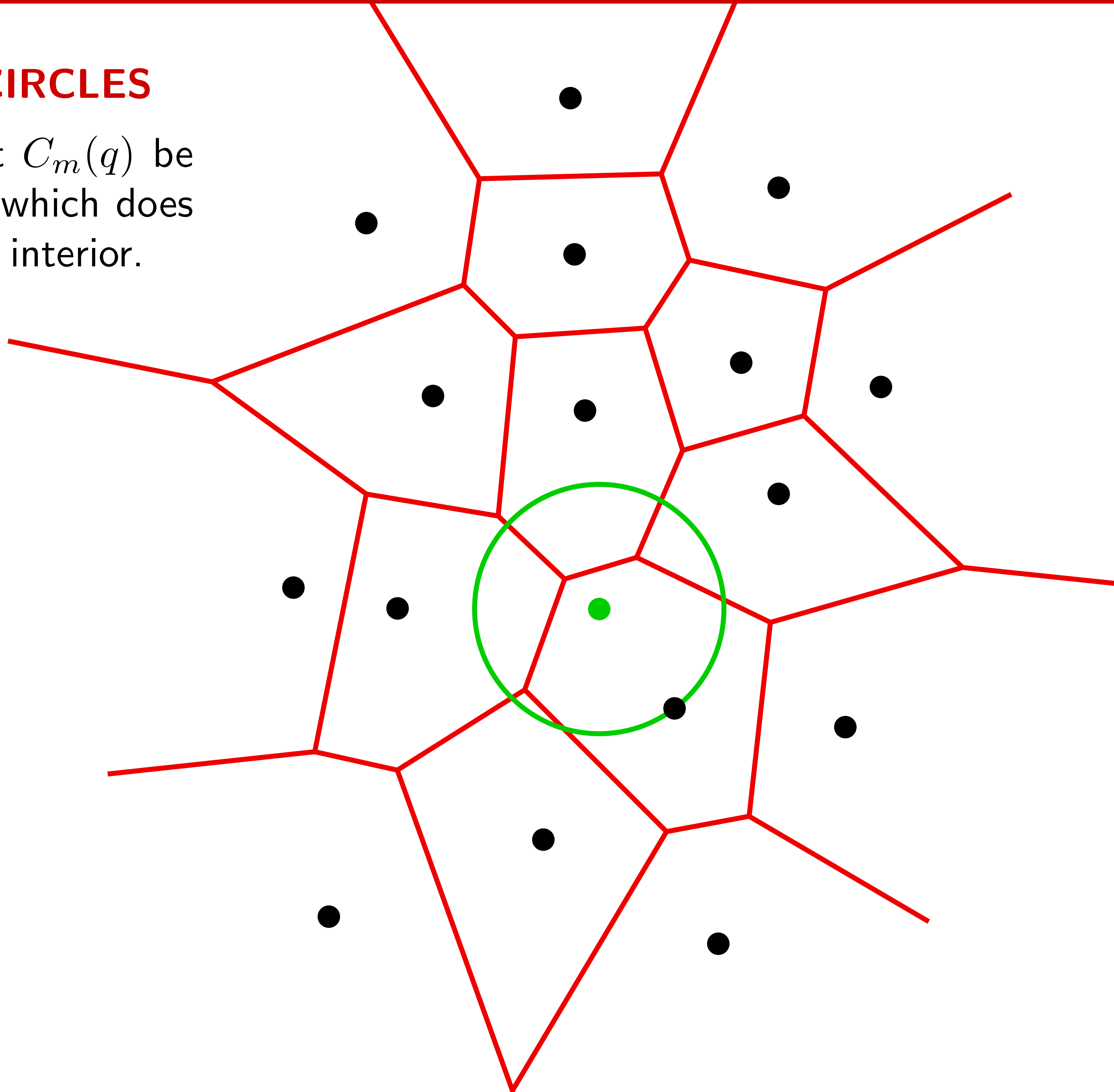

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For any point q in the plane, let $C_m(q)$ be the largest circle, centered at q, which does not contain any point of P in its interior.

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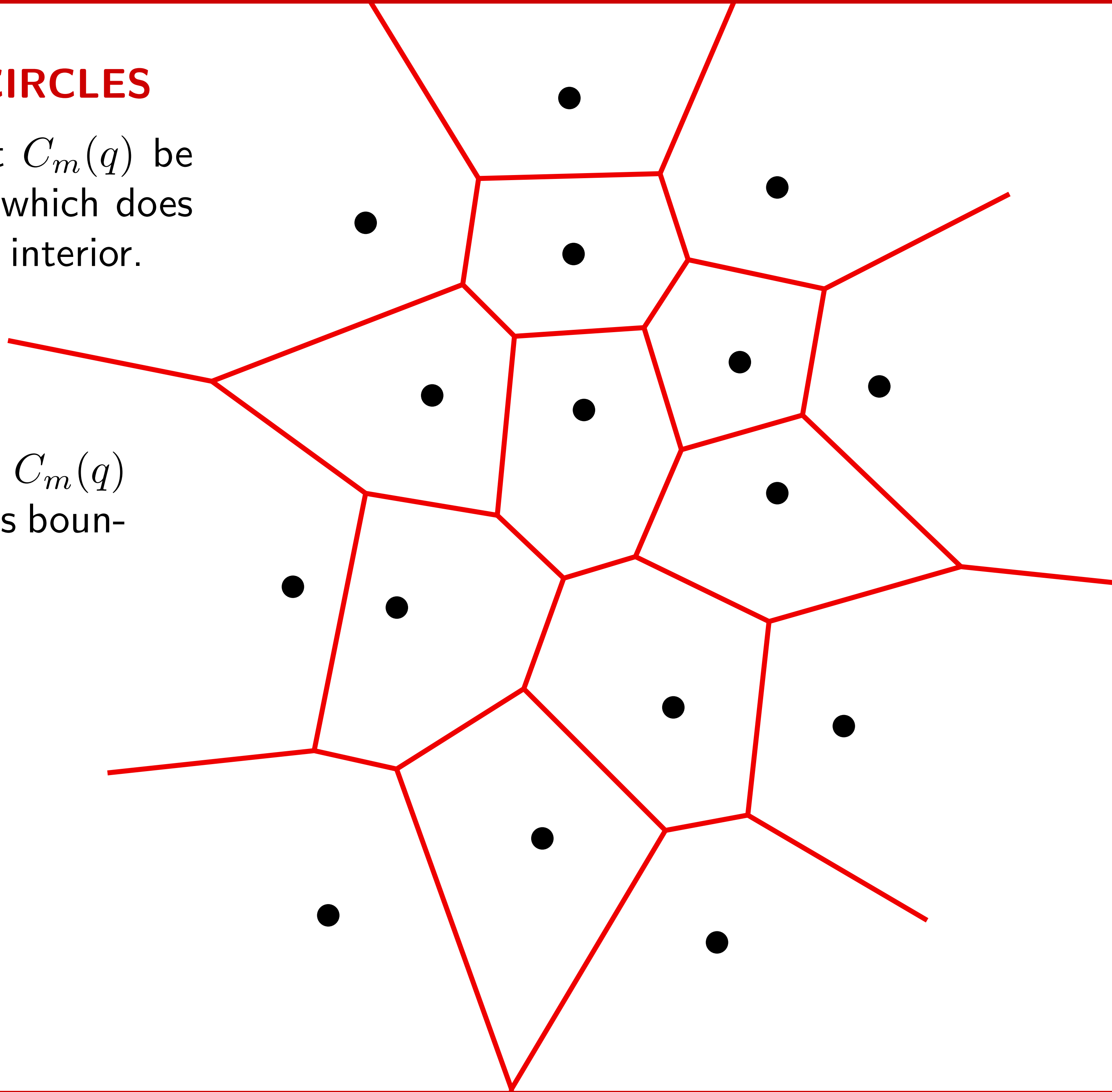
A point q is a Voronoi vertex iff $C_m(q)$ has three or more points of P in its boundary.



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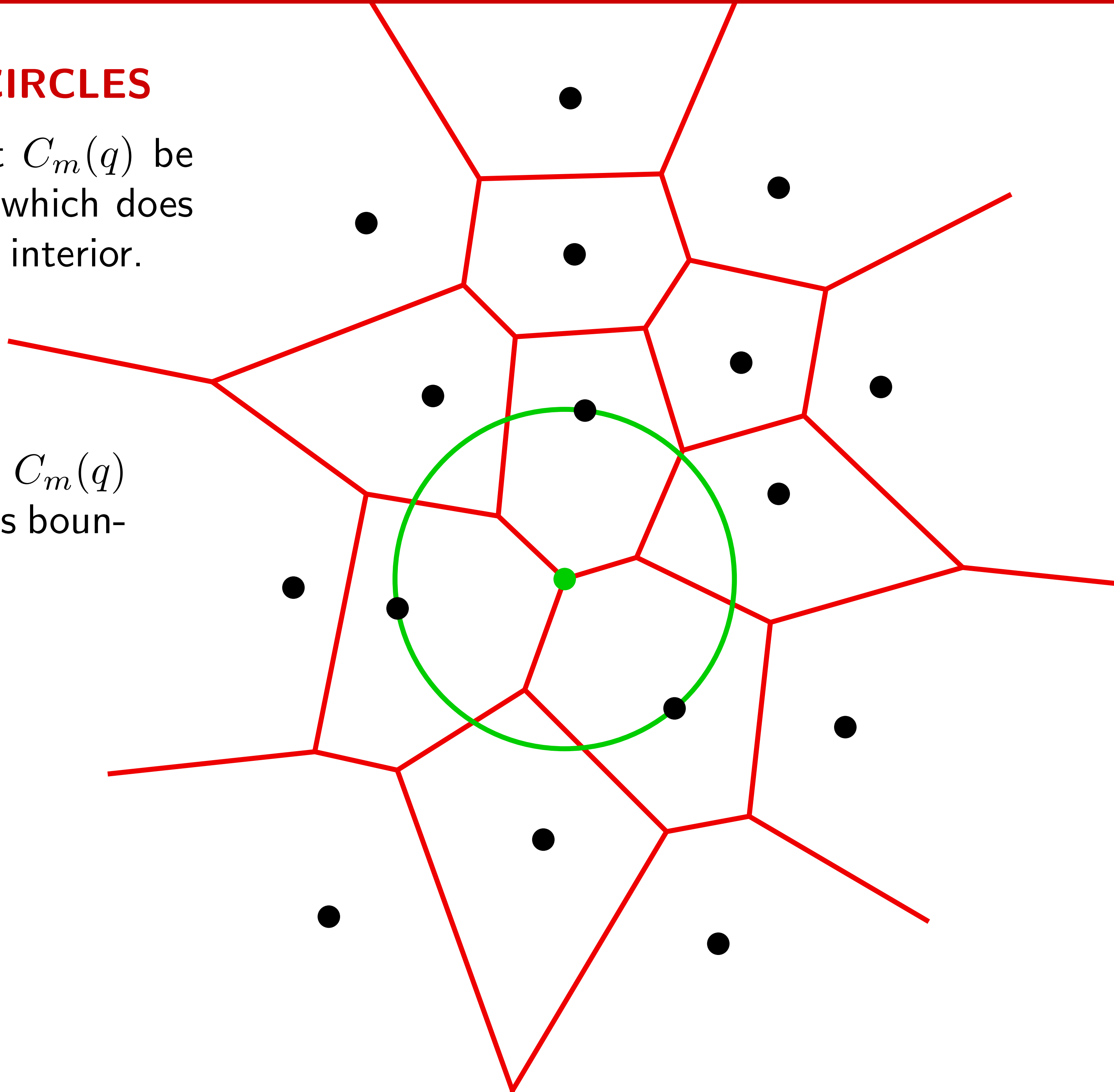

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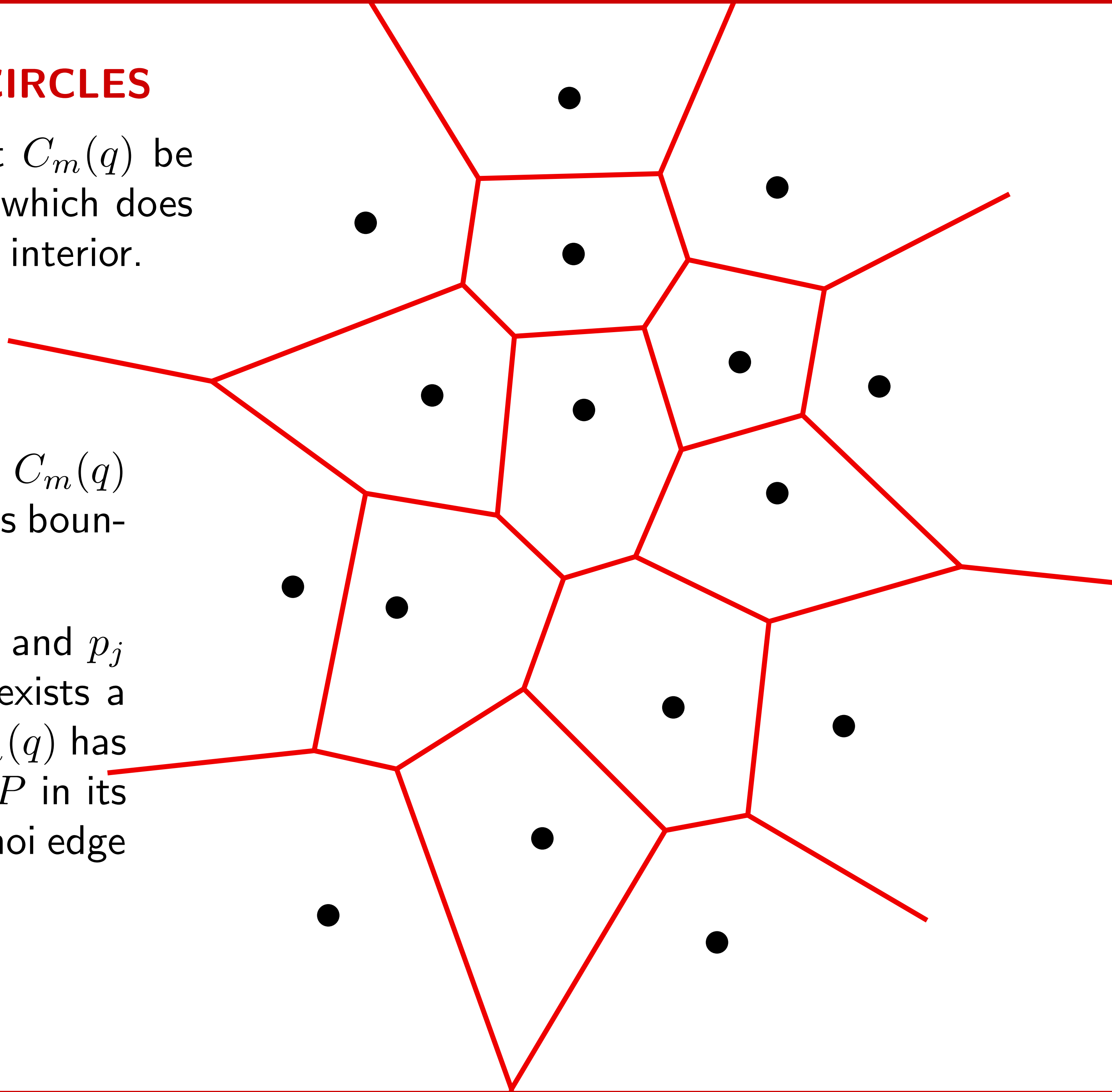

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The bisector b_{ij} of two points p_i and p_j defines a Voronoi edge iff there exists a point q of the plane such that $C_m(q)$ has p_i and p_j and no other point of P in its boundary. In such case, the Voronoi edge is the set of all these points q.

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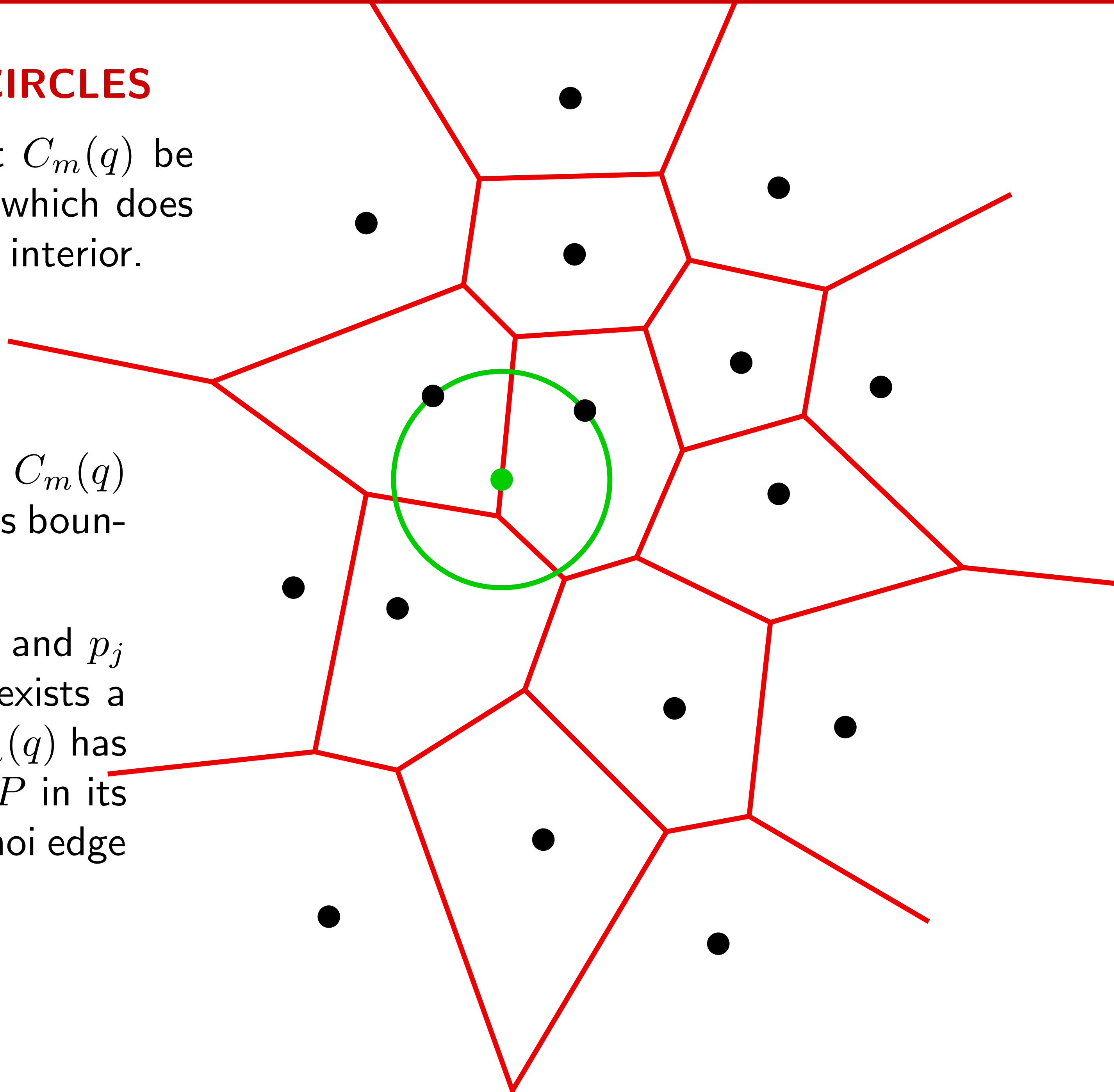

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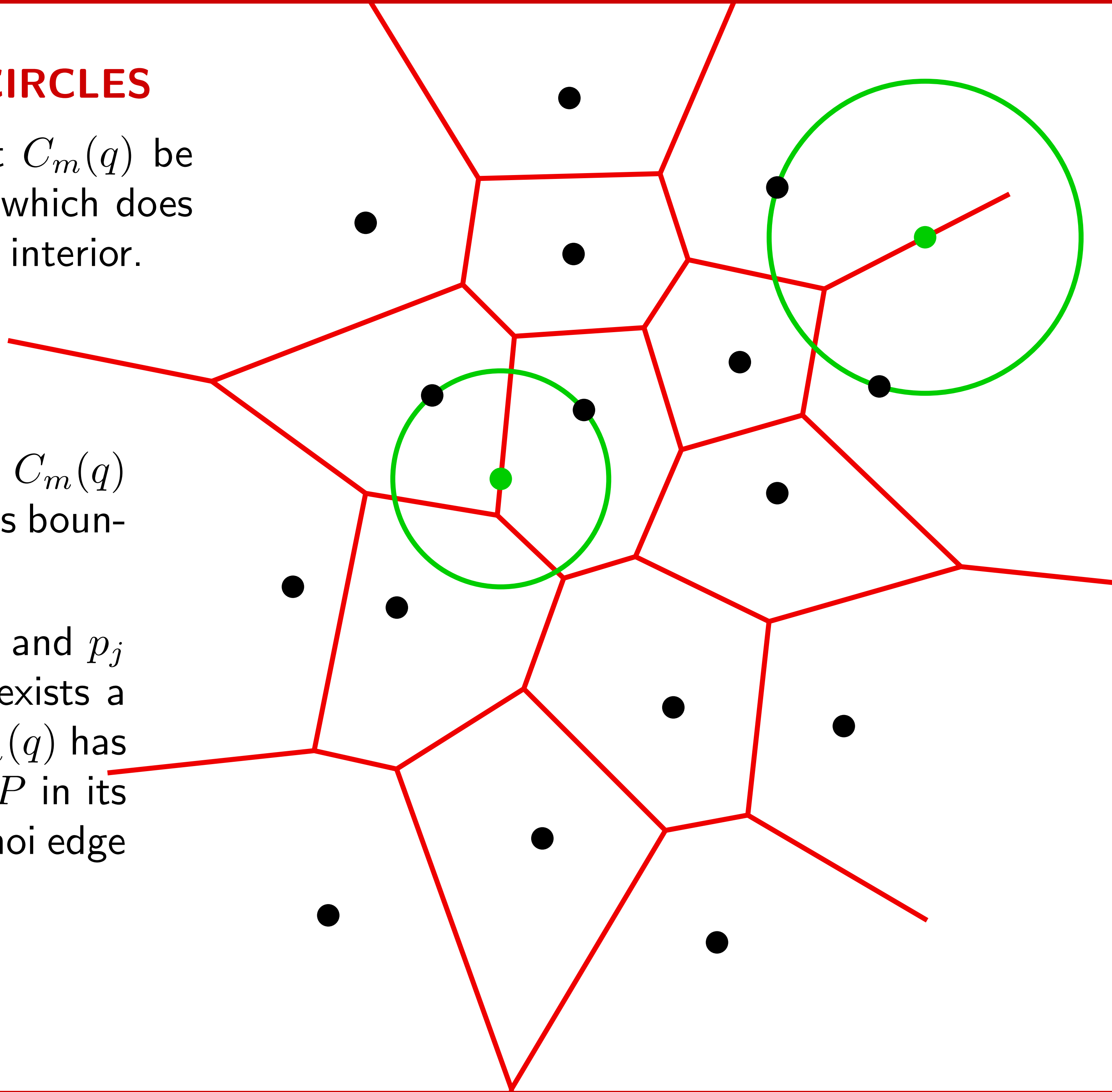

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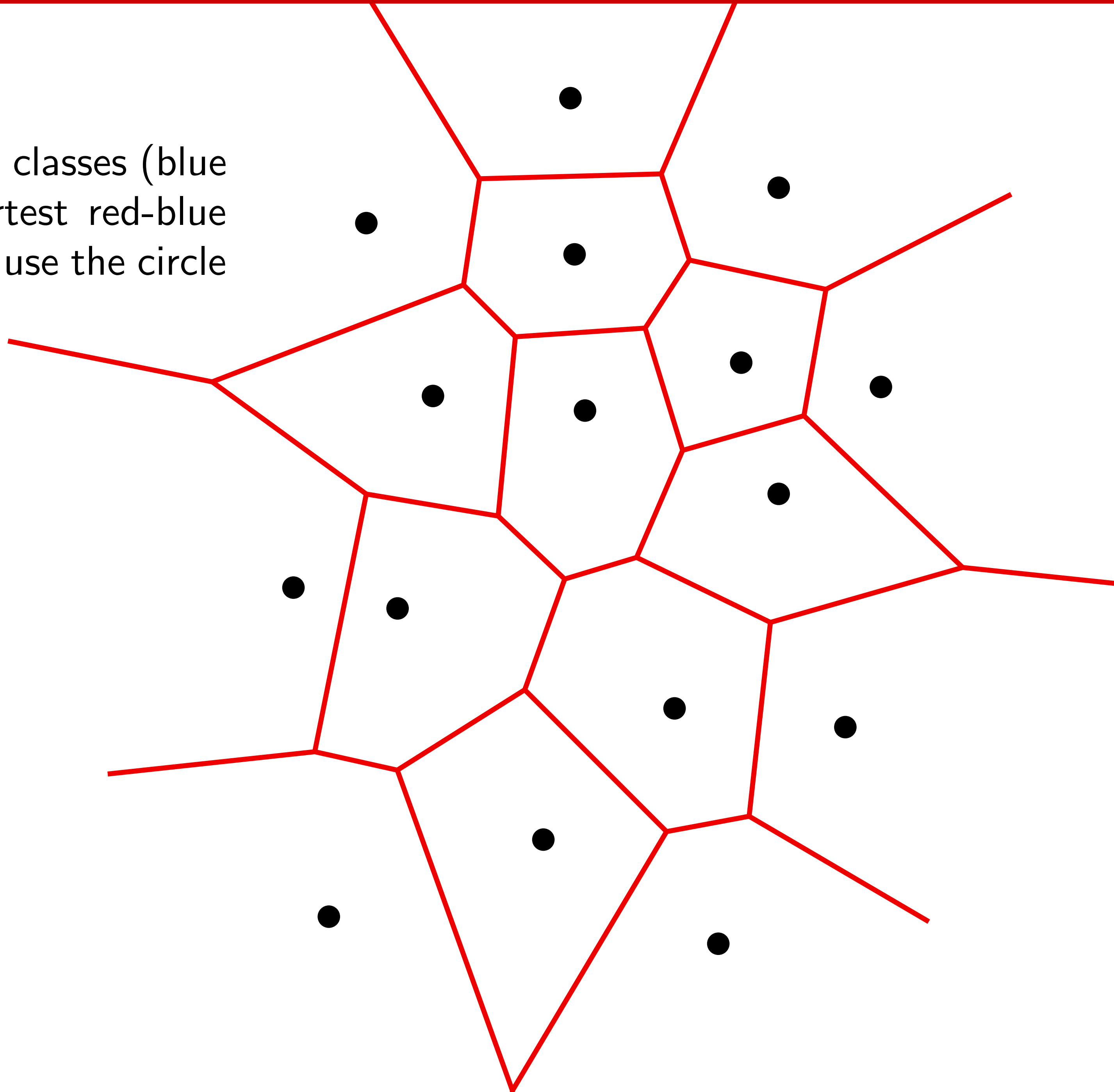
Corollary (blue and red)

Given any partition of P into two classes (blue points and red points), the shortest red-blue segment is a Delaunay edge, because the circle having it as diameter is empty.



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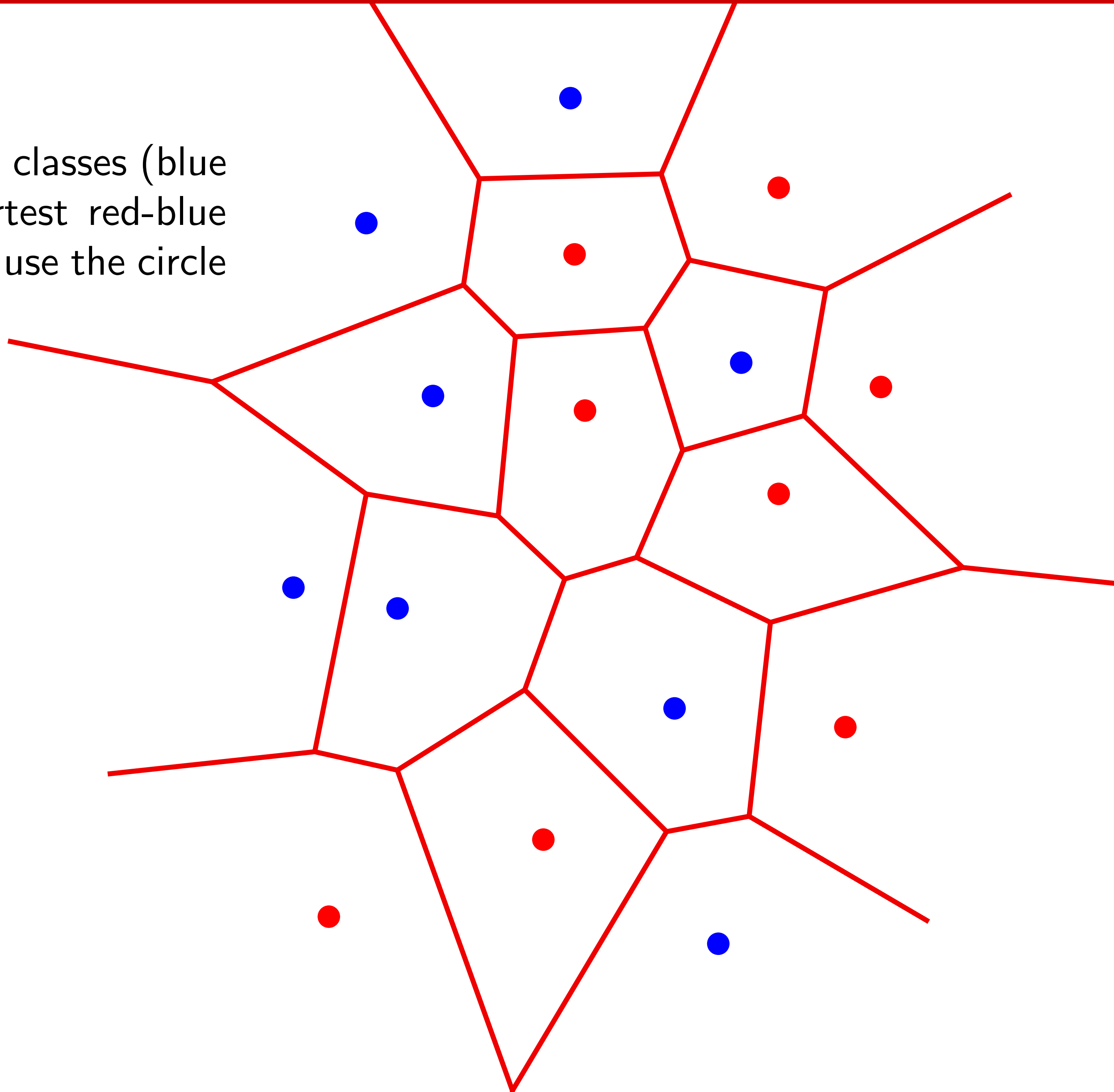

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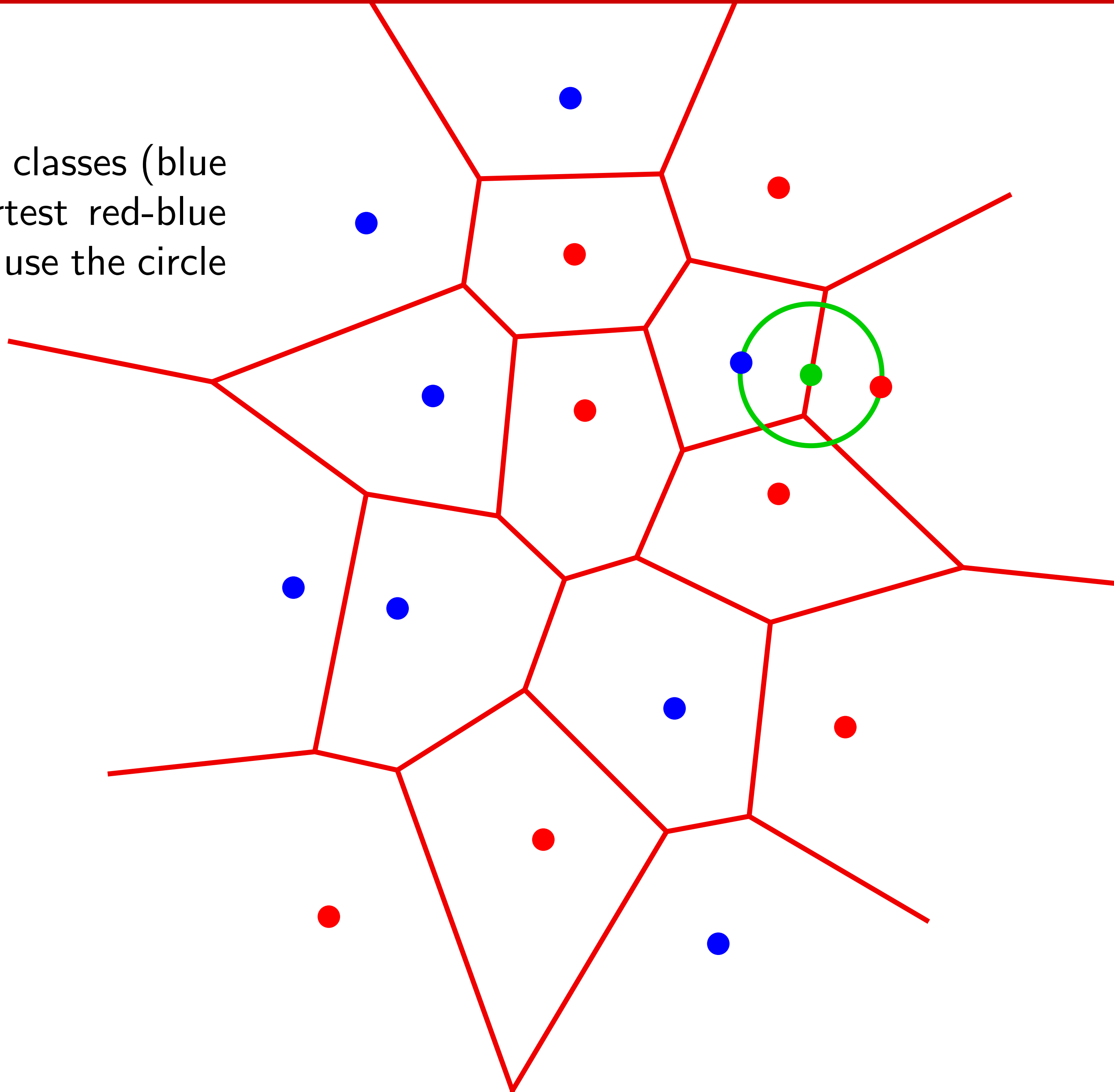

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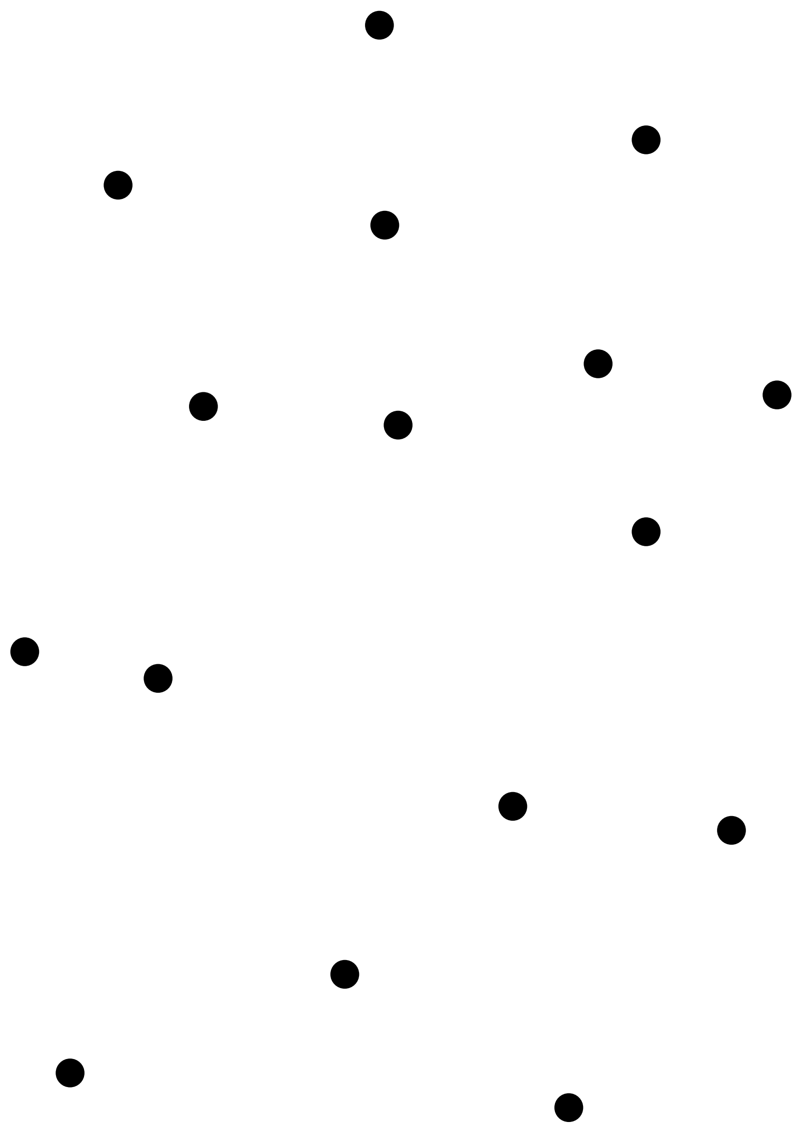

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Closest neighbors digraph

For each point $p_i \in P$, the closest point of P is a Voronoi neighbor. Therefore, the closest neighbors digraph is a subgraph of the Delaunay graph.

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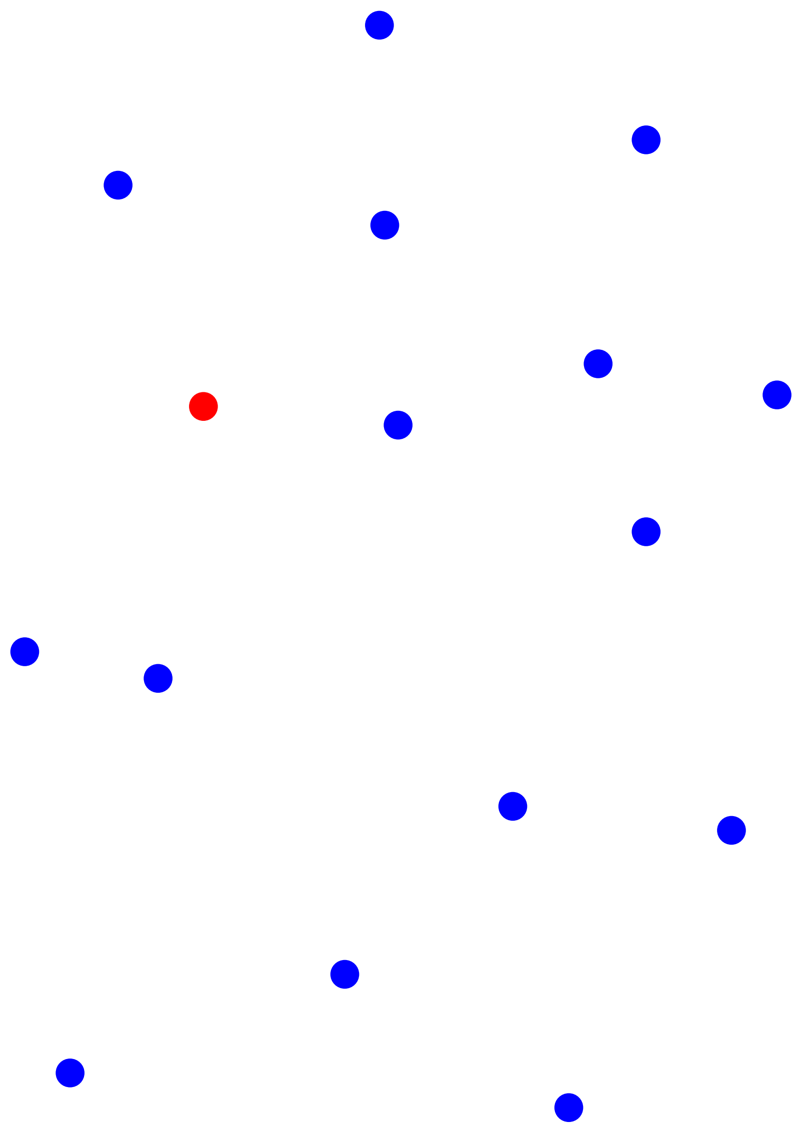

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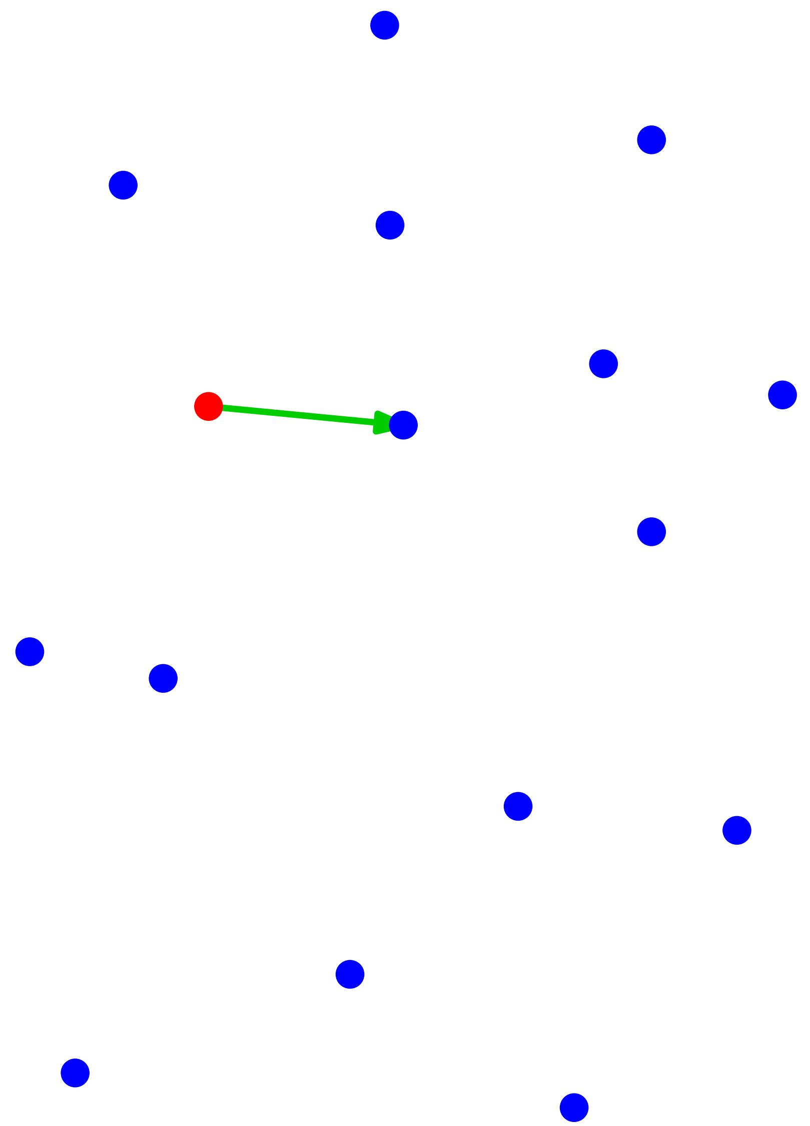

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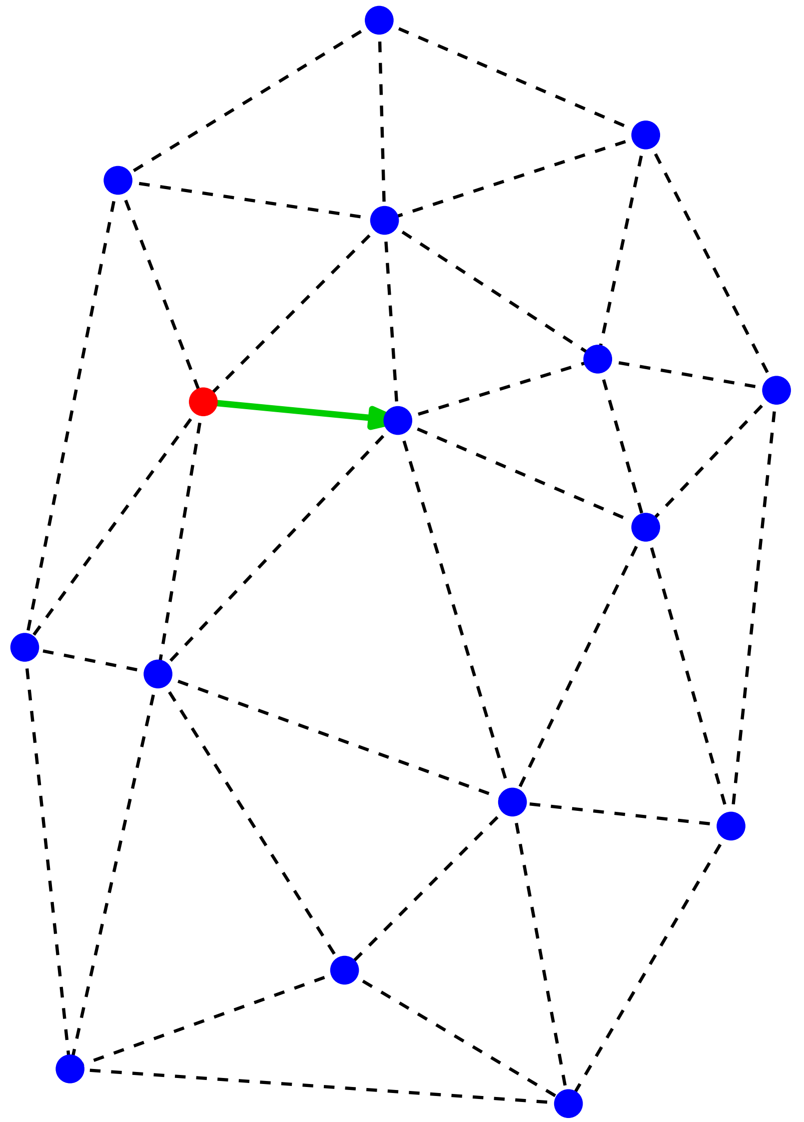

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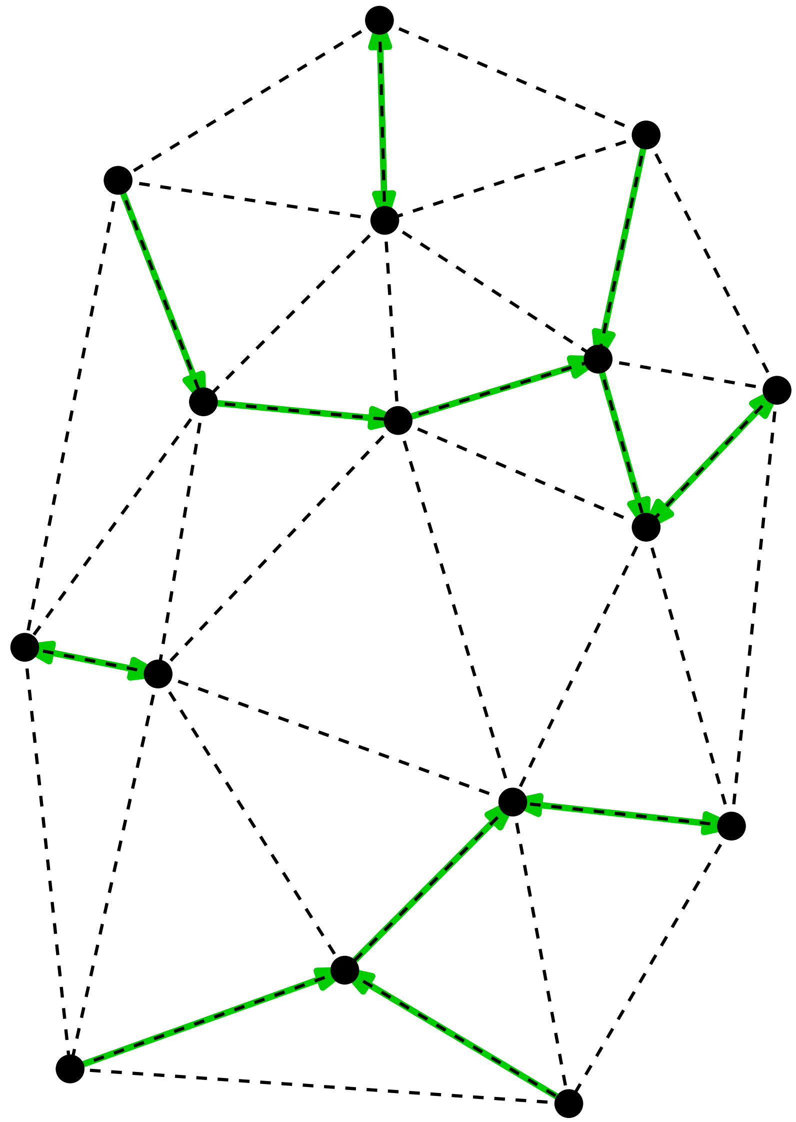

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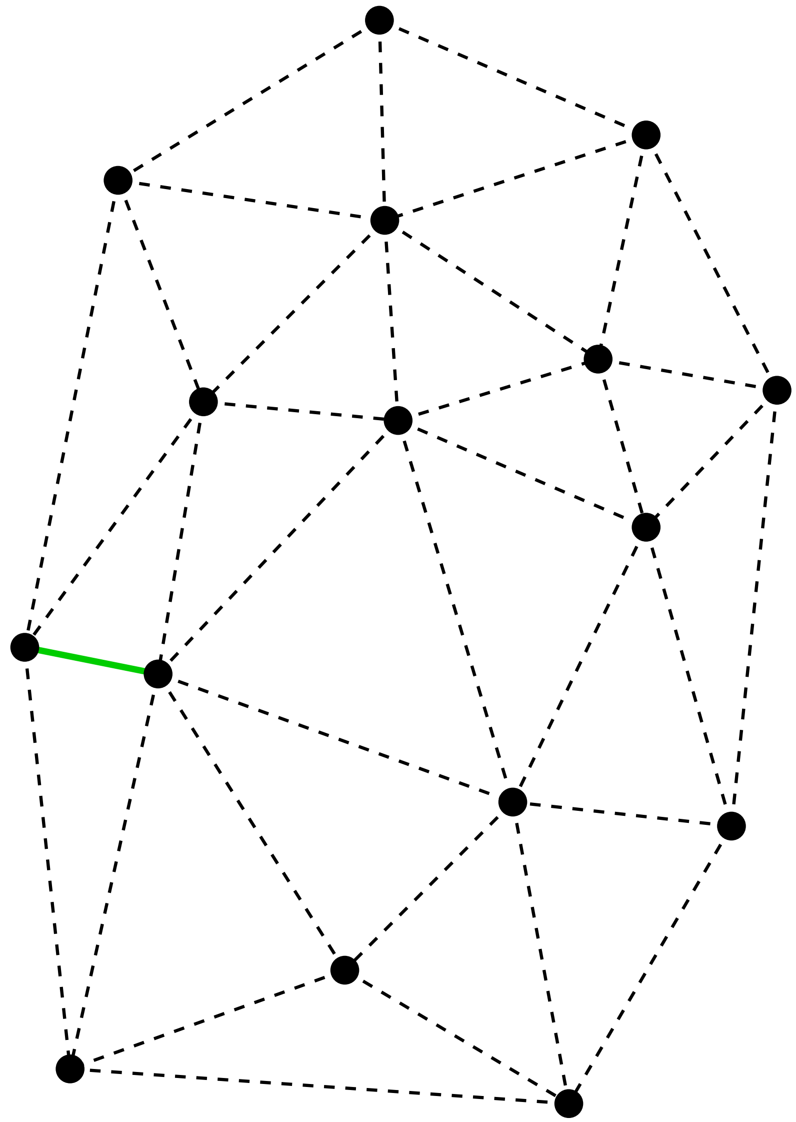
Application 2
The closest pair

In particular, the shortest edge p_ip_j belongs to the Delaunay graph.



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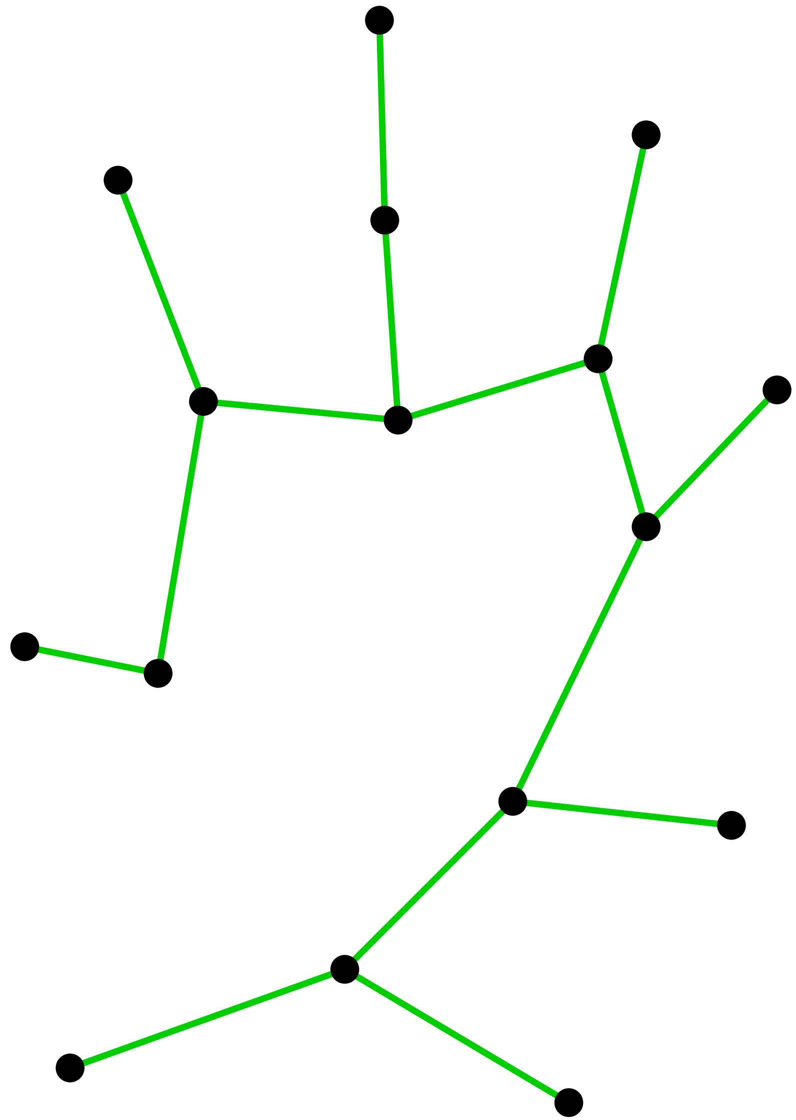
Application 3
Euclidean minimum spanning tree

The euclidean minimum spanning tree is a subgraph of the Delaunay graph.



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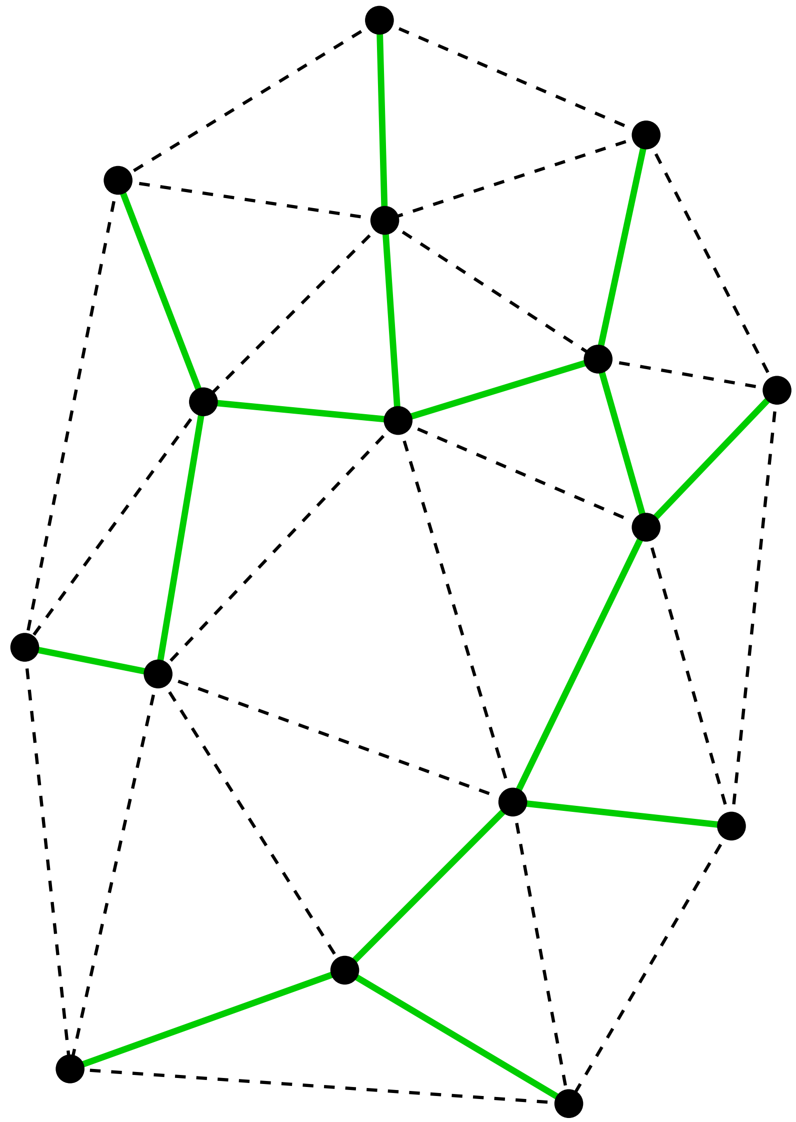

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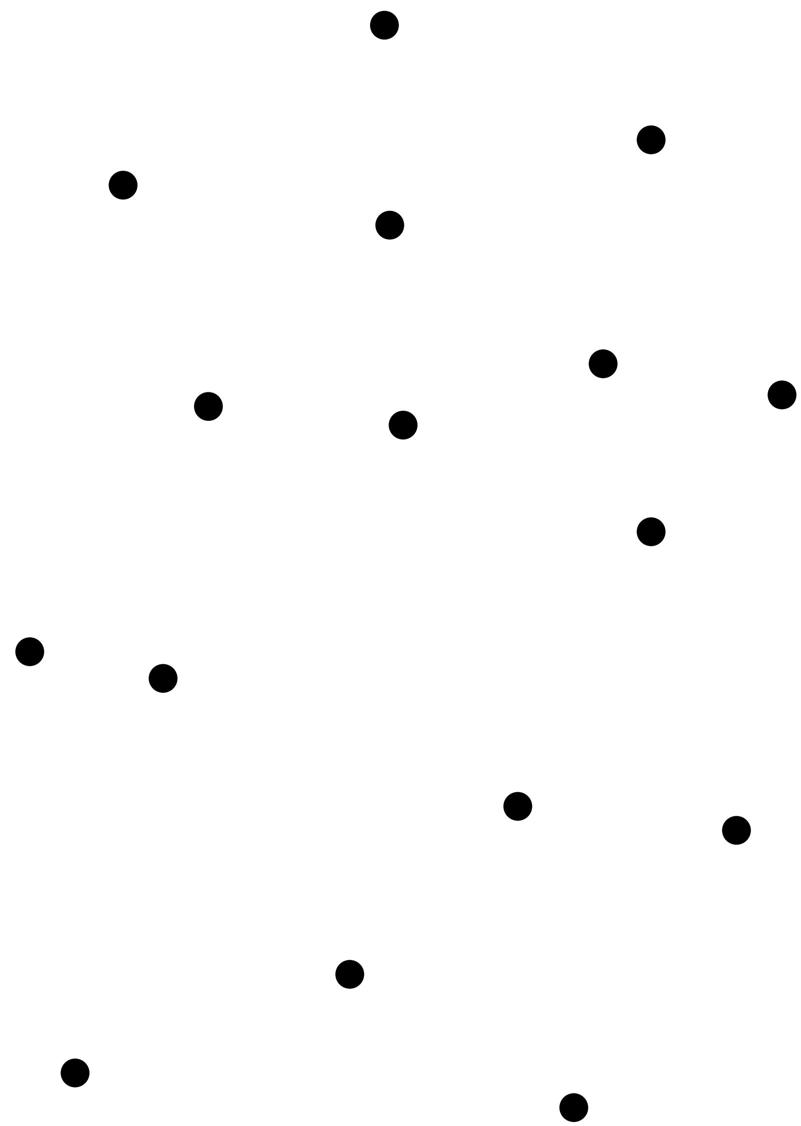
Prim's algorithm to compute the minimum spanning tree in a weighted graph builds the tree T by adding, at each iteration, the edge of minimum weight incident to T which does not produce a cycle when added to T.



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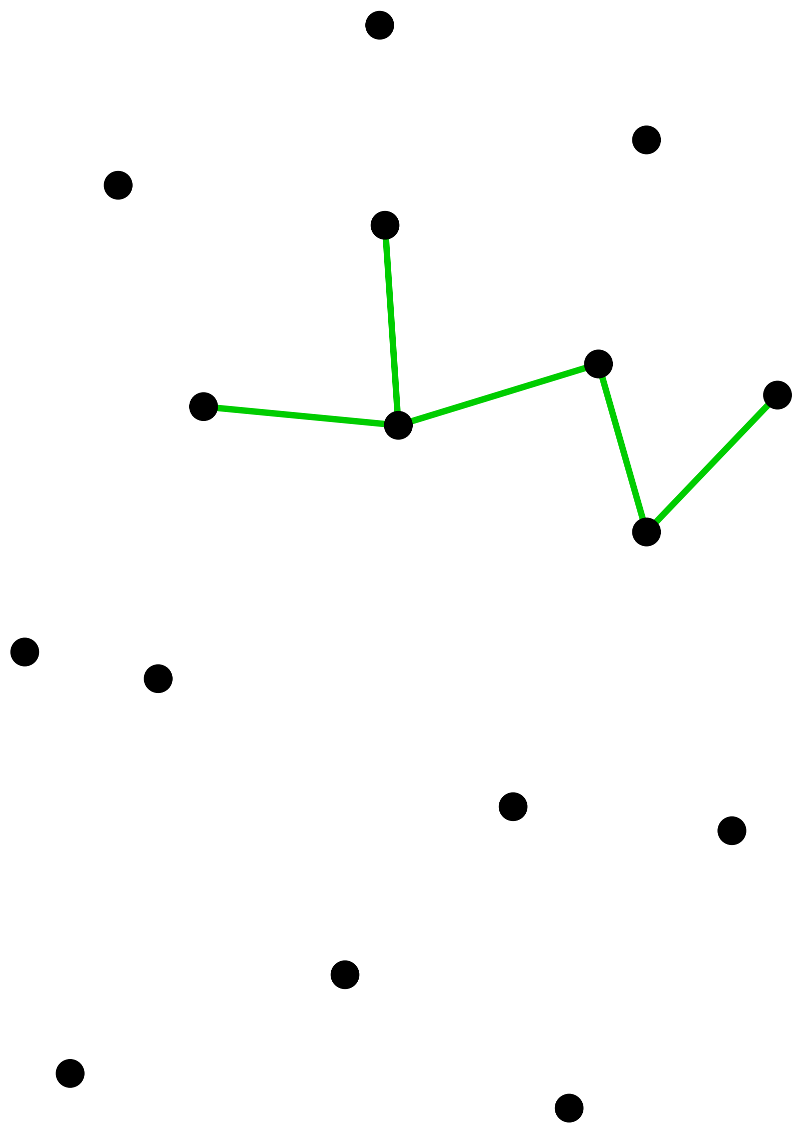

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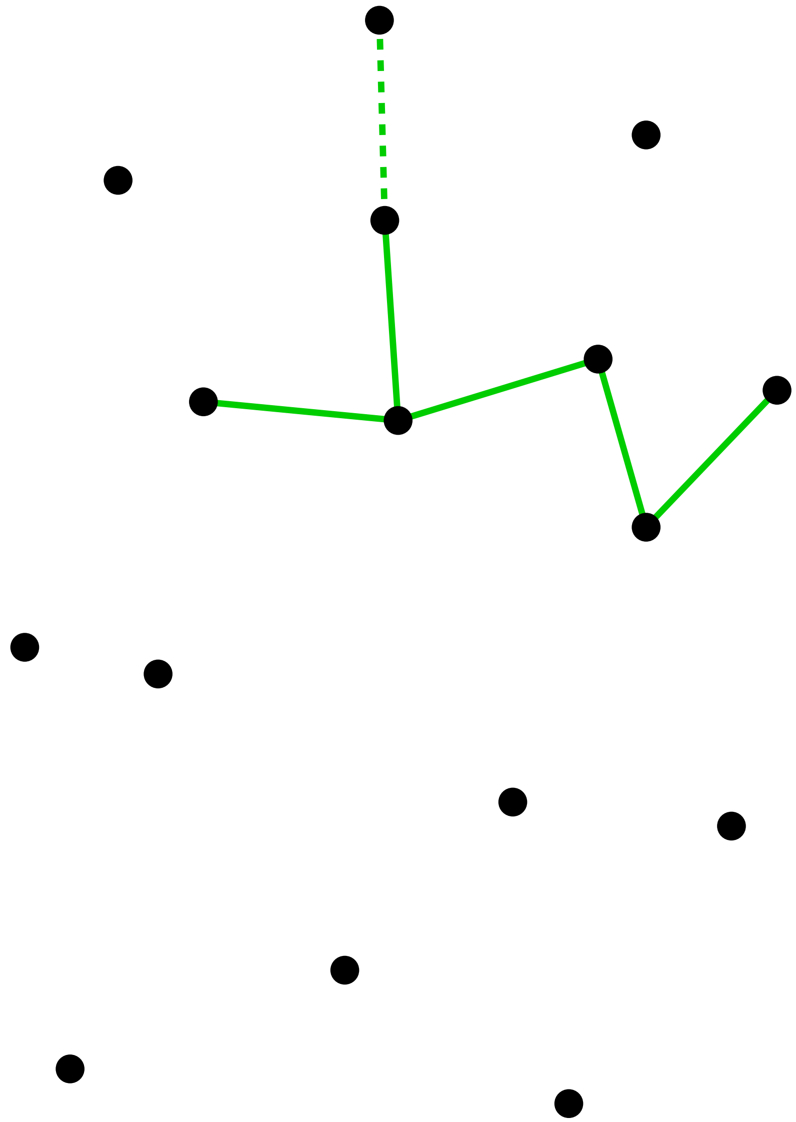

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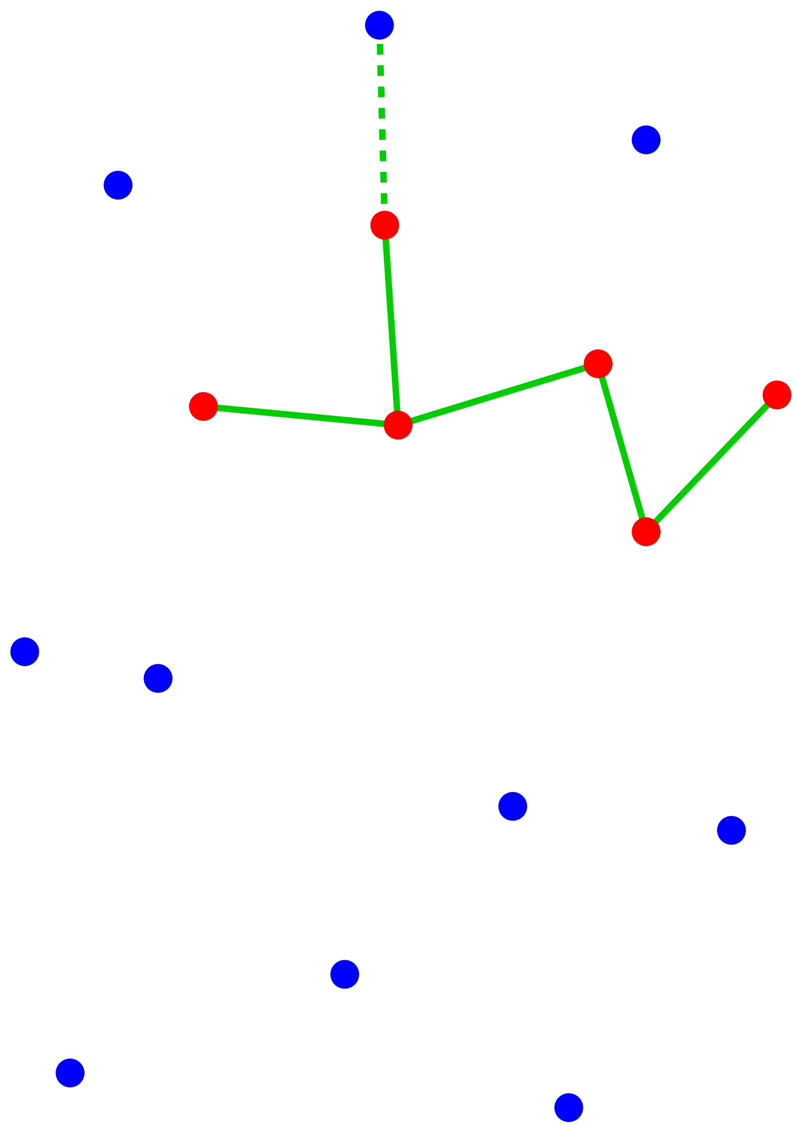
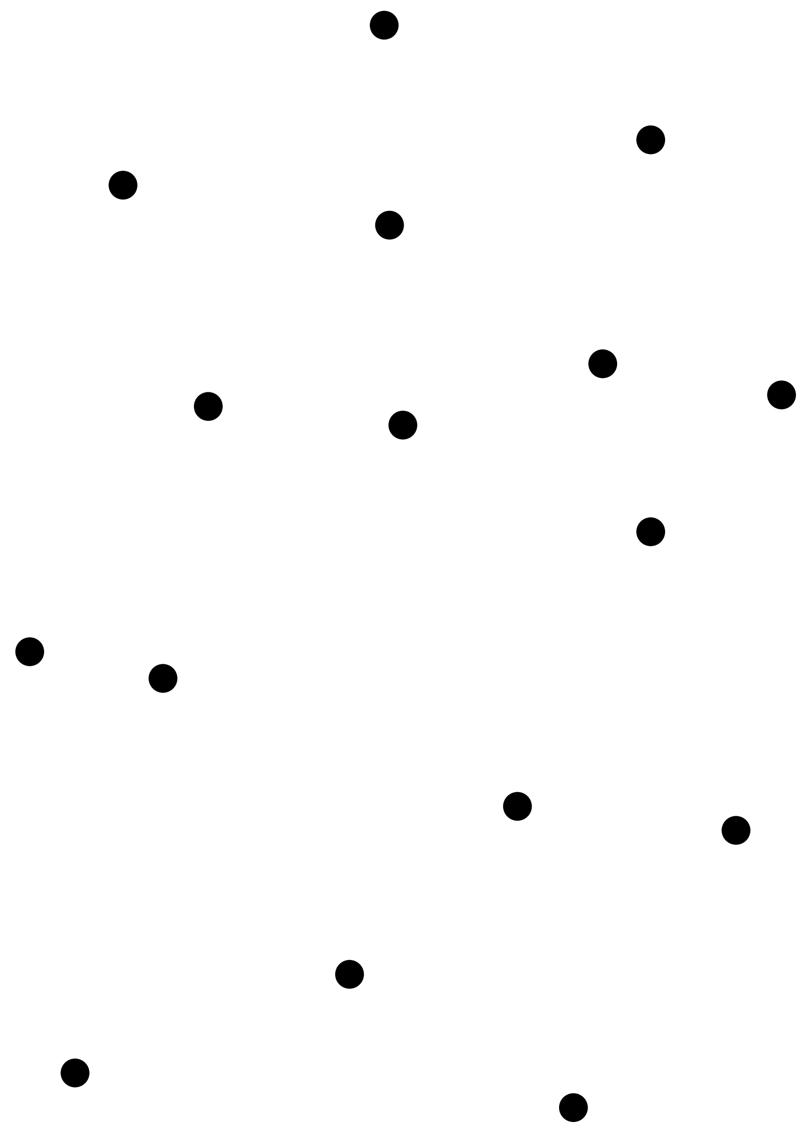


PROXIMITY

Application 4
Finding the closest site

Given any point q in the plane, finding its closest site $p_i \in P$ means detecting the Voronoi region $V(p_i)$ containing q.



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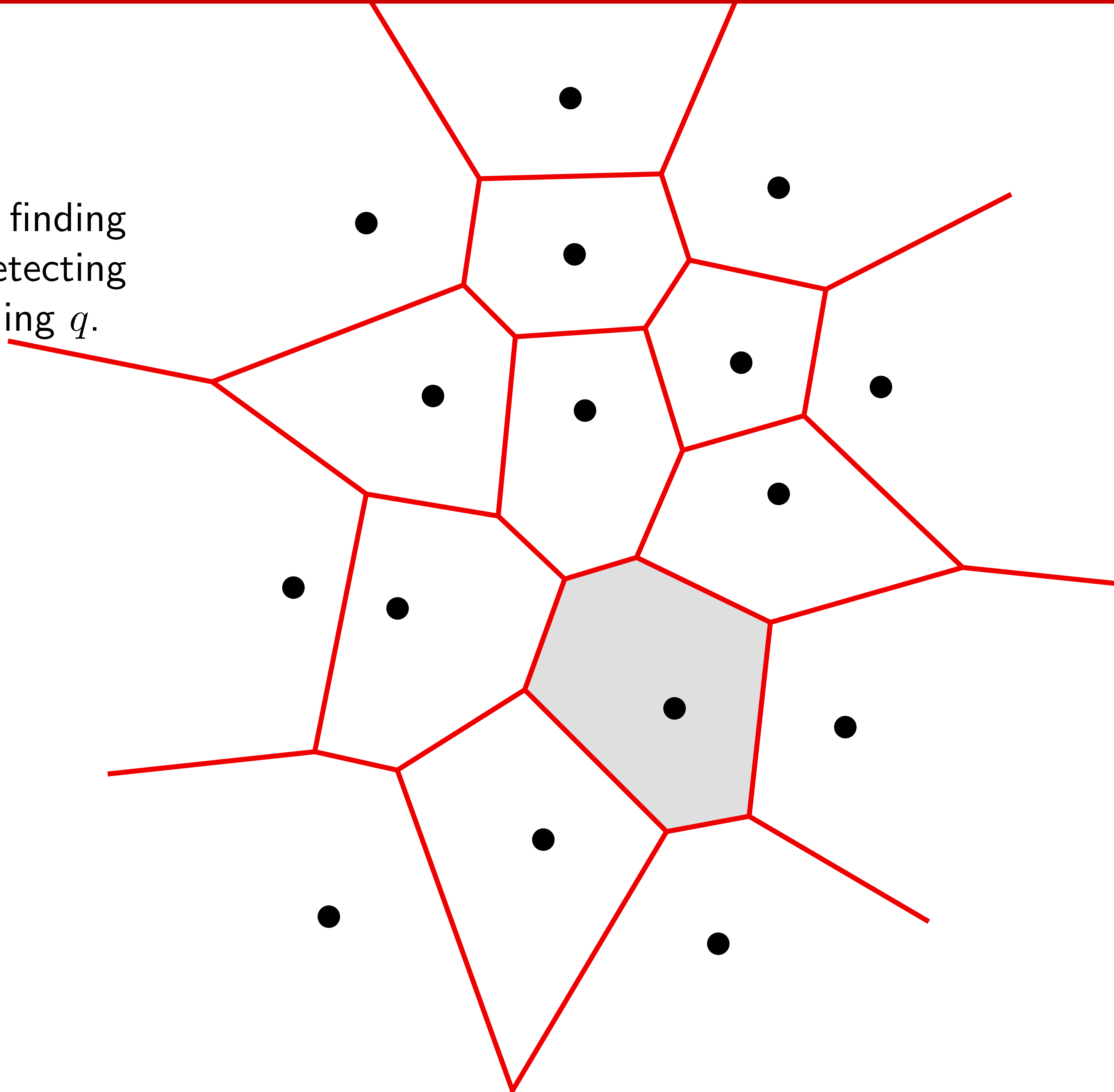

Application
Max-min facility location

The center of the largest empty circle, restricted to a given region A, can be located at:

- a Voronoi vertex,
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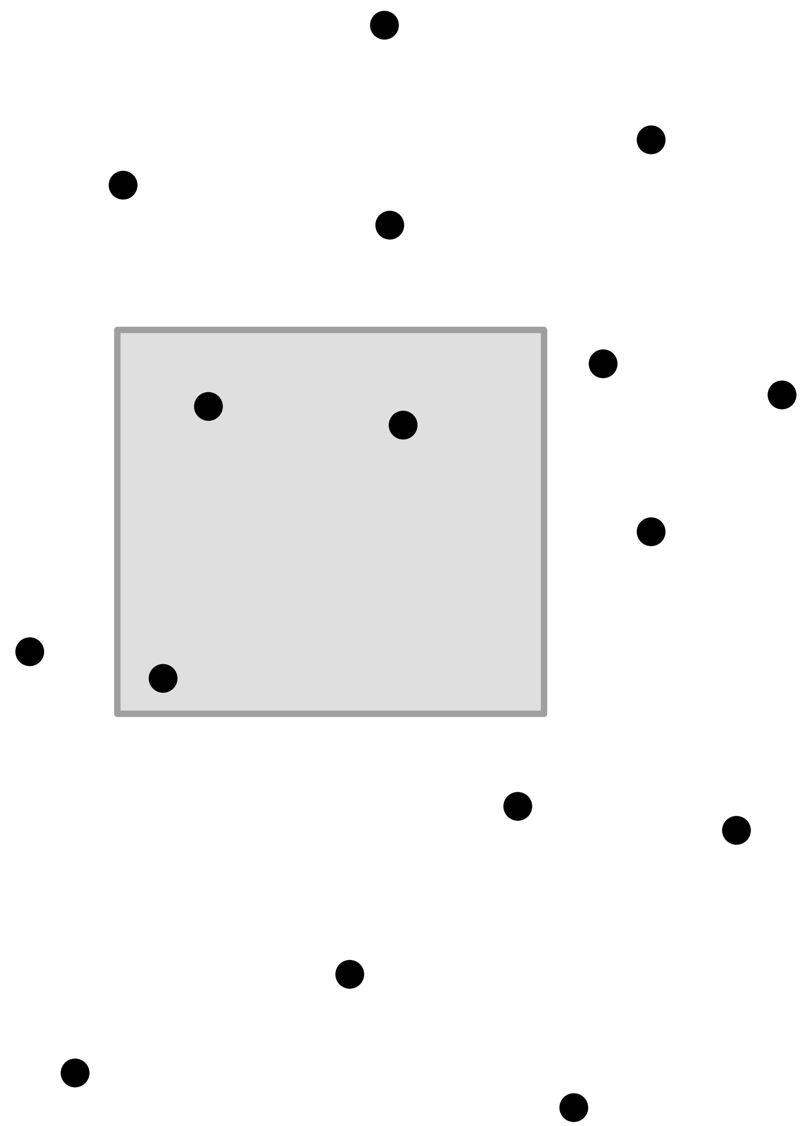
- a vertex of A.



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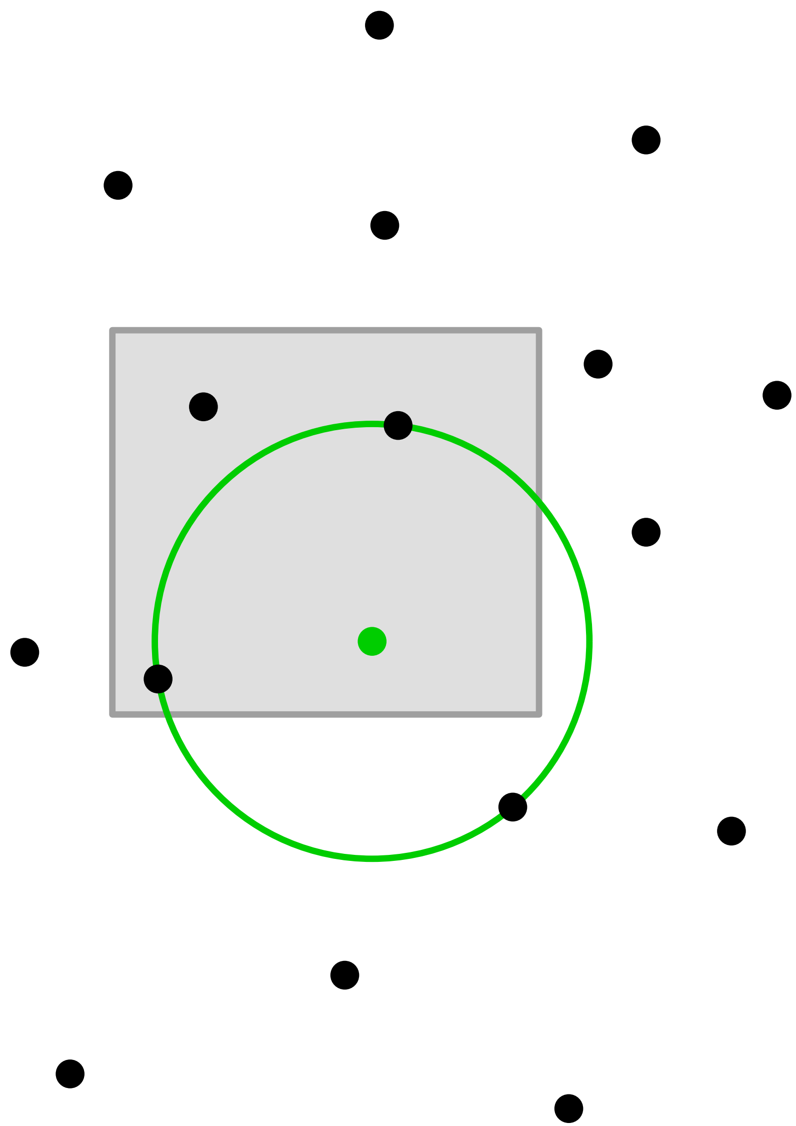

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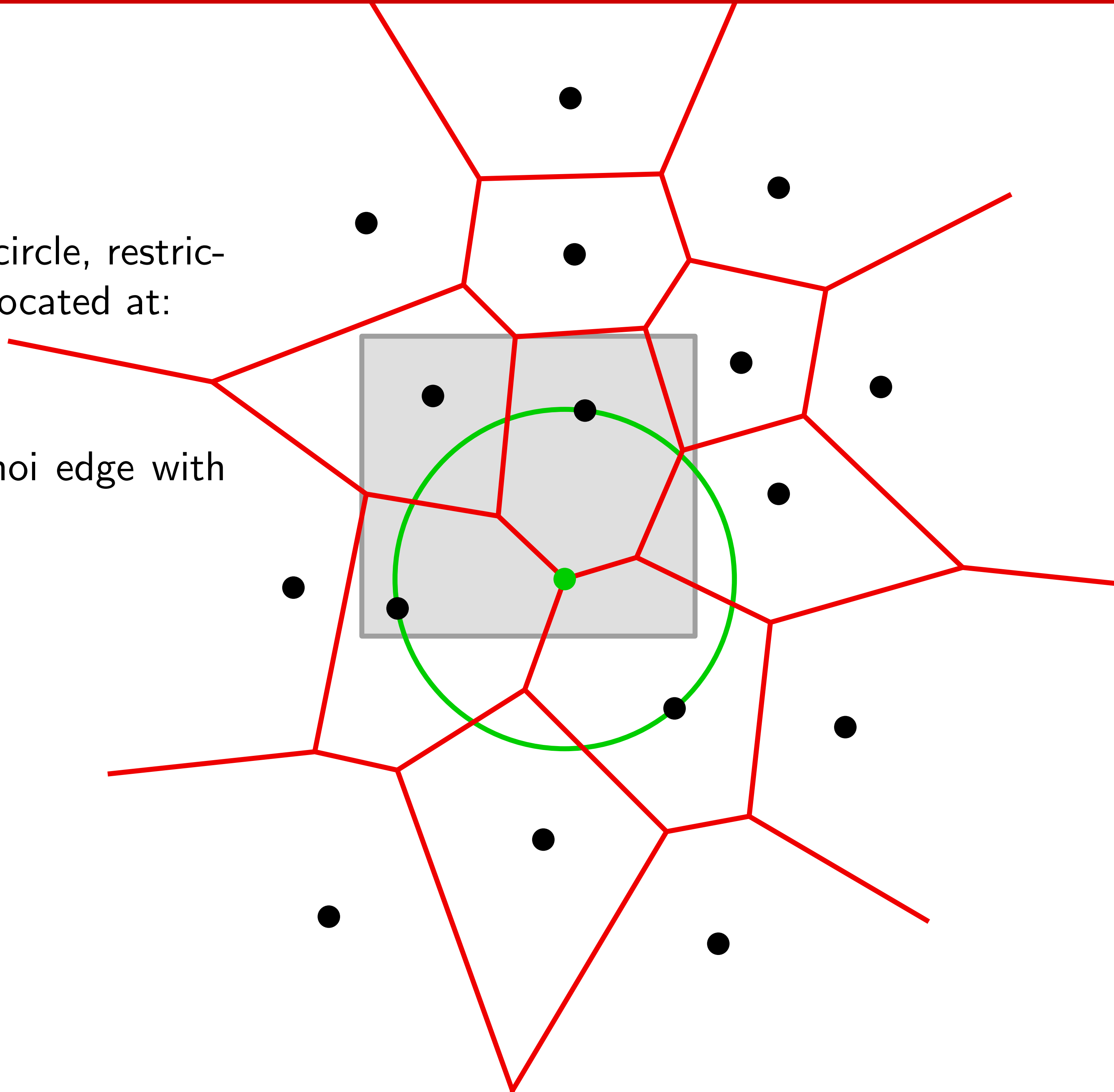

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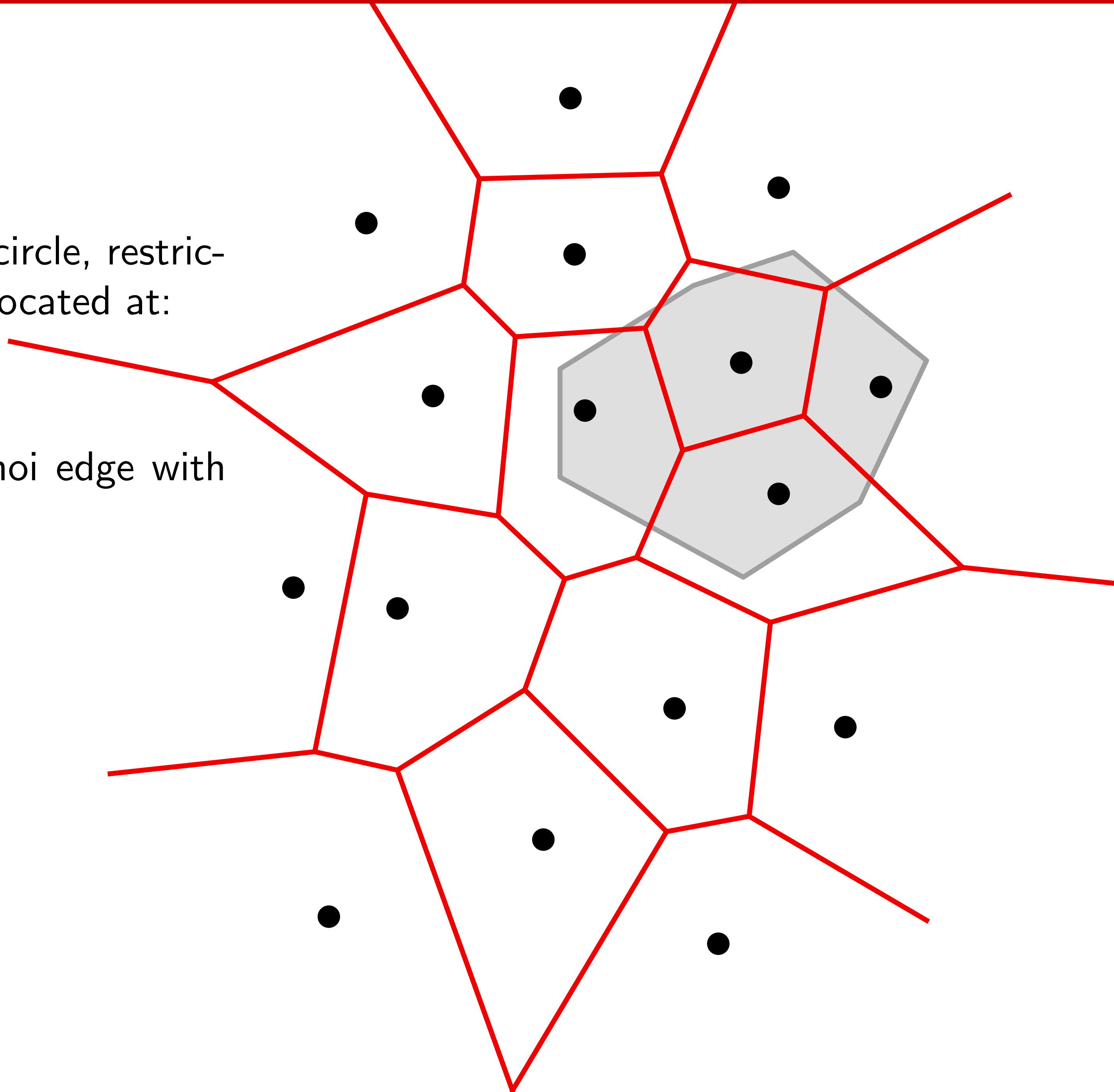

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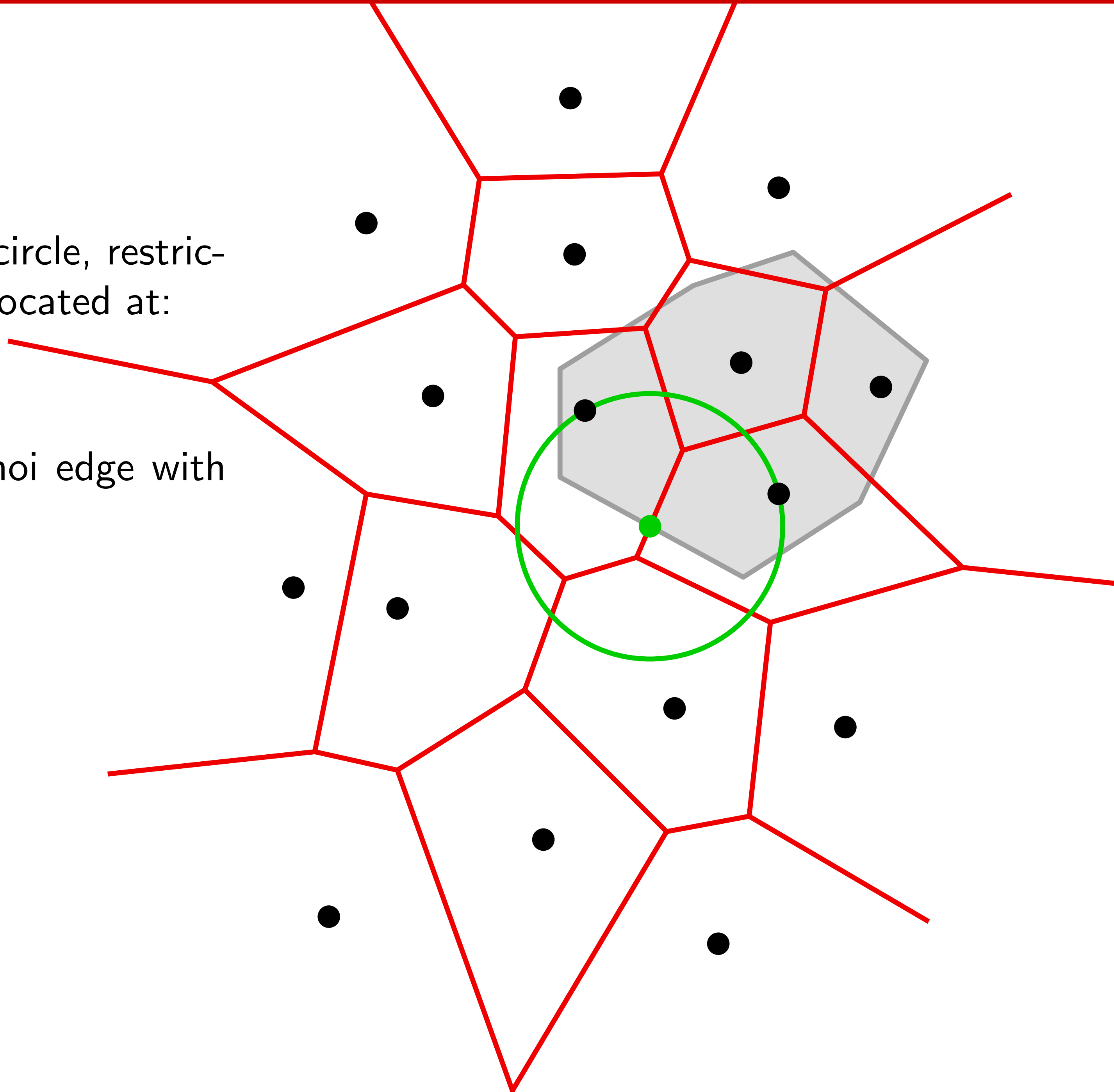

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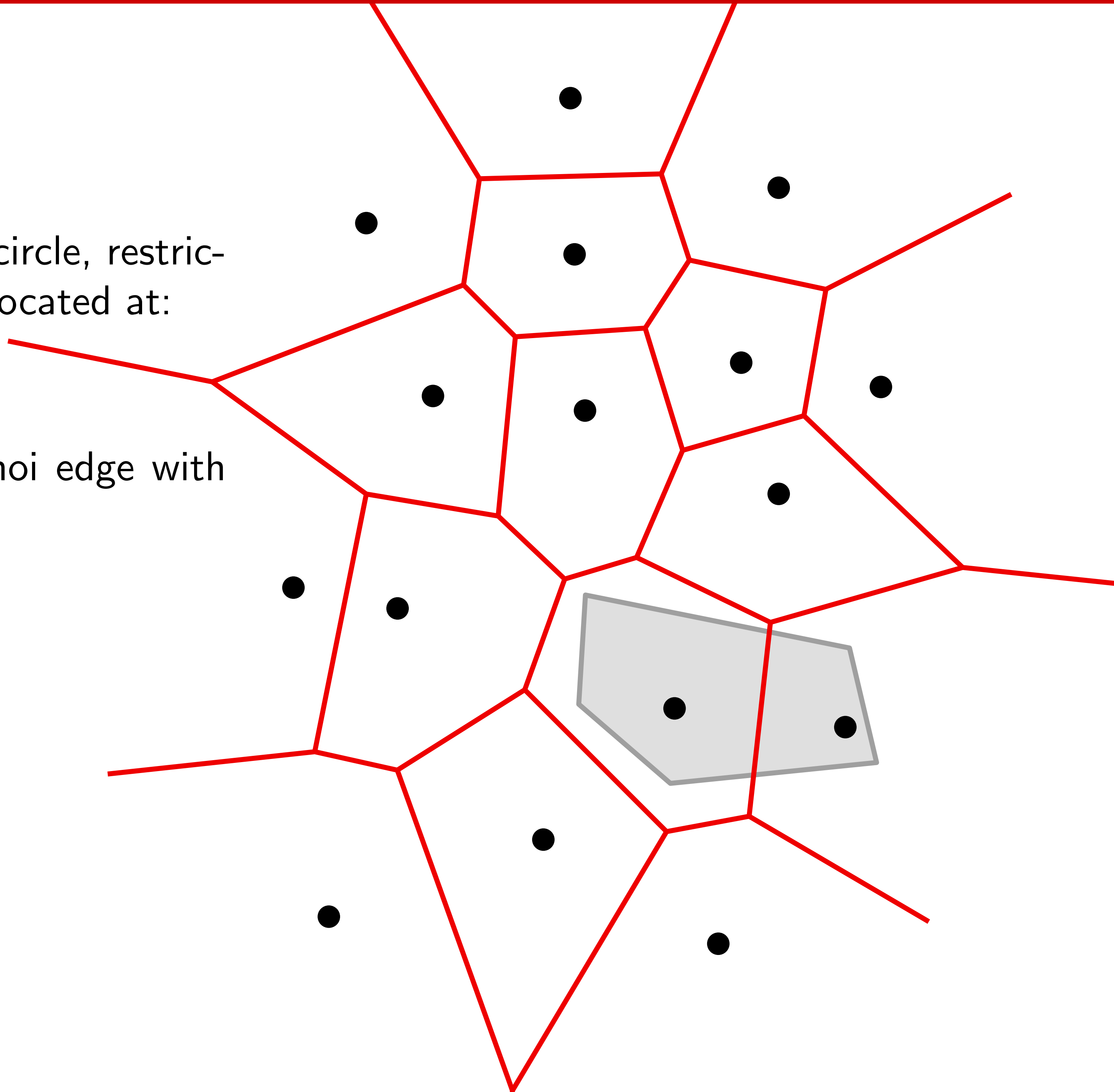

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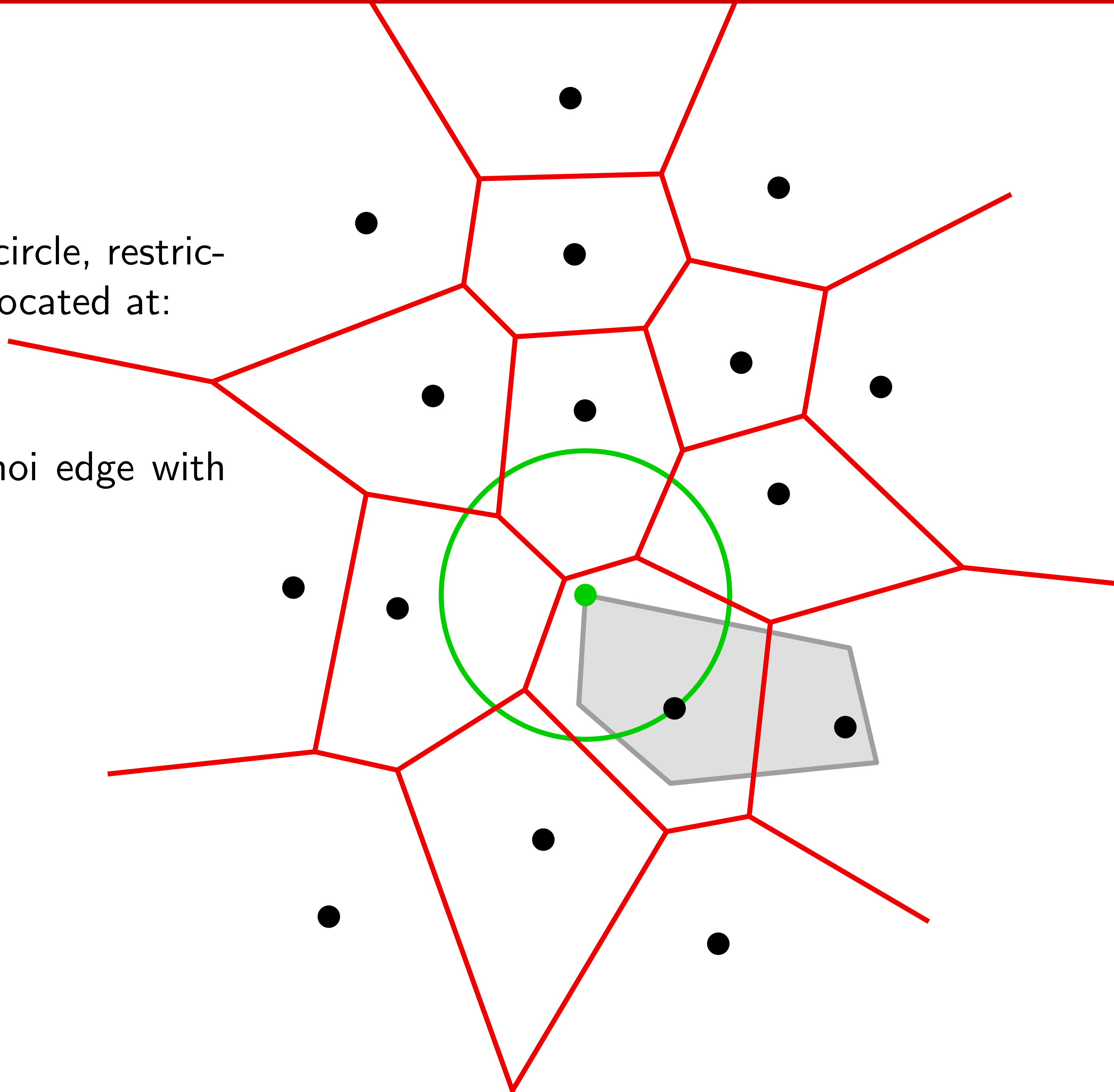
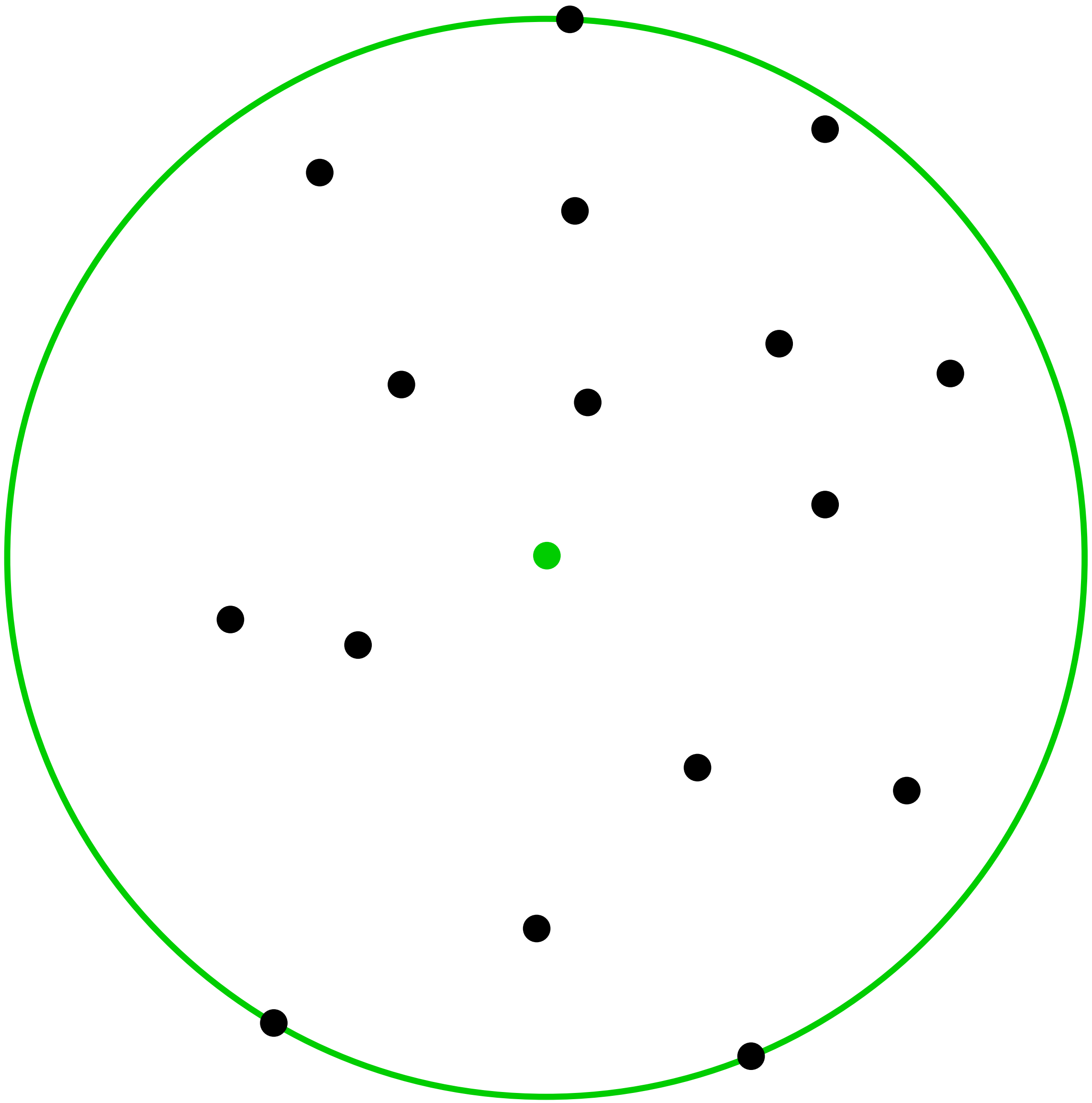


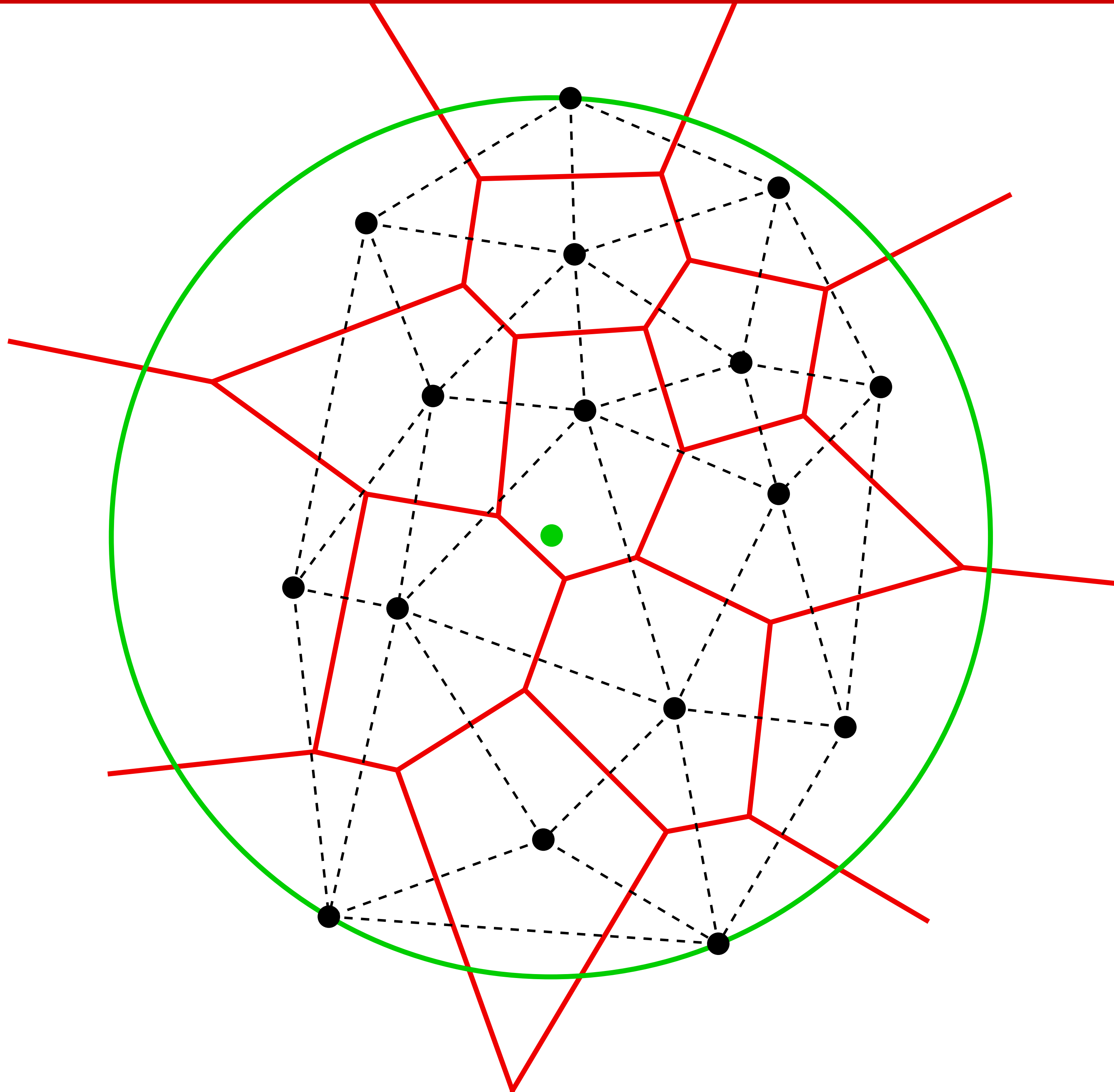
Application 6
Min-max facility location

The center of the minimum spanning circle



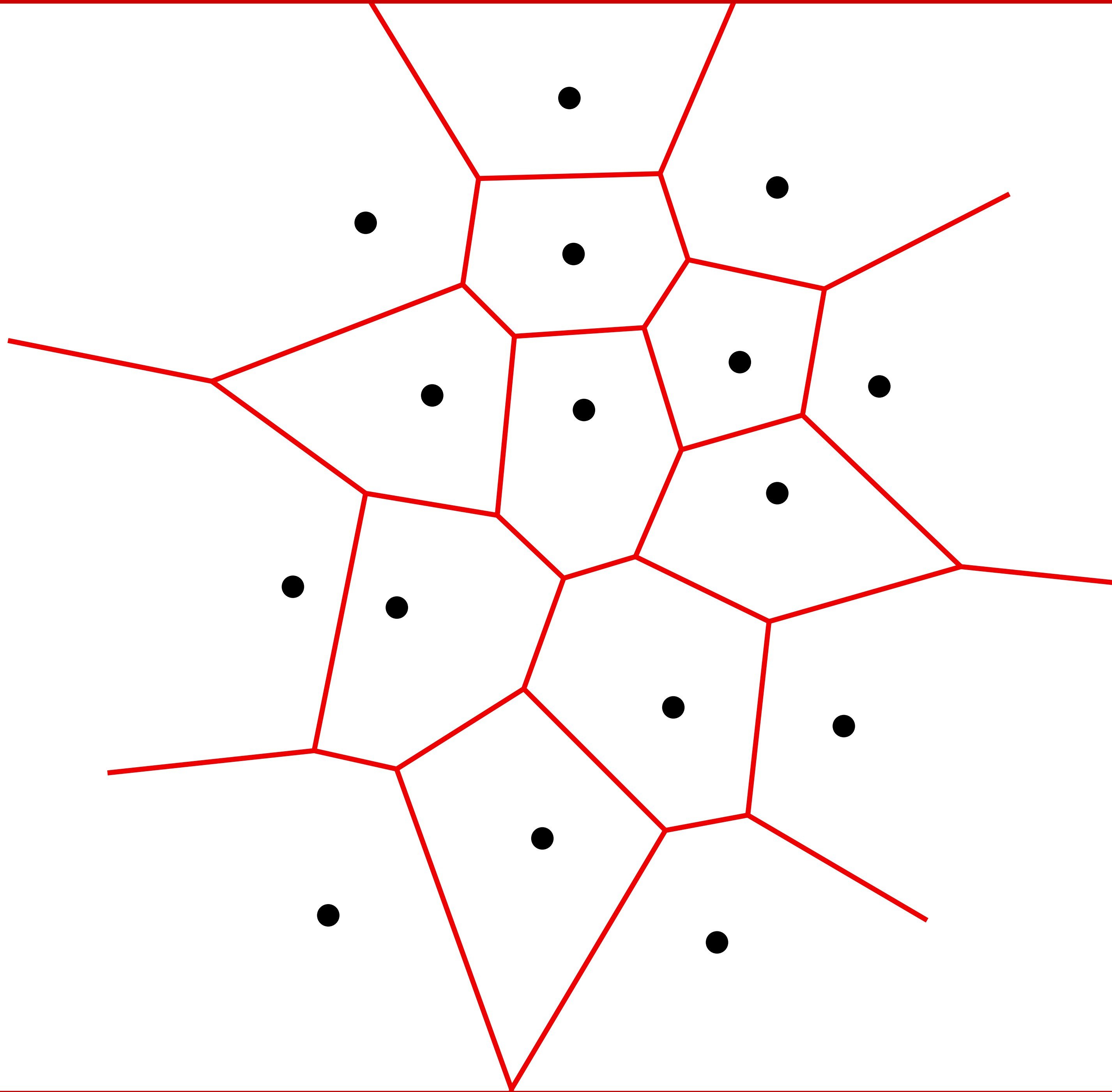
Application 6
Min-max facility location

The center of the minimum spanning circle does not seem to be related with the Voronoi diagram! How strange!



GENERALIZATIONS OF THE VORONOI DIAGRAM

The Voronoi diagram of a finite set P of points in the plane is a partition of the plane such that each region is the locus of the points that lie closer to one element of P than to the remaining ones.



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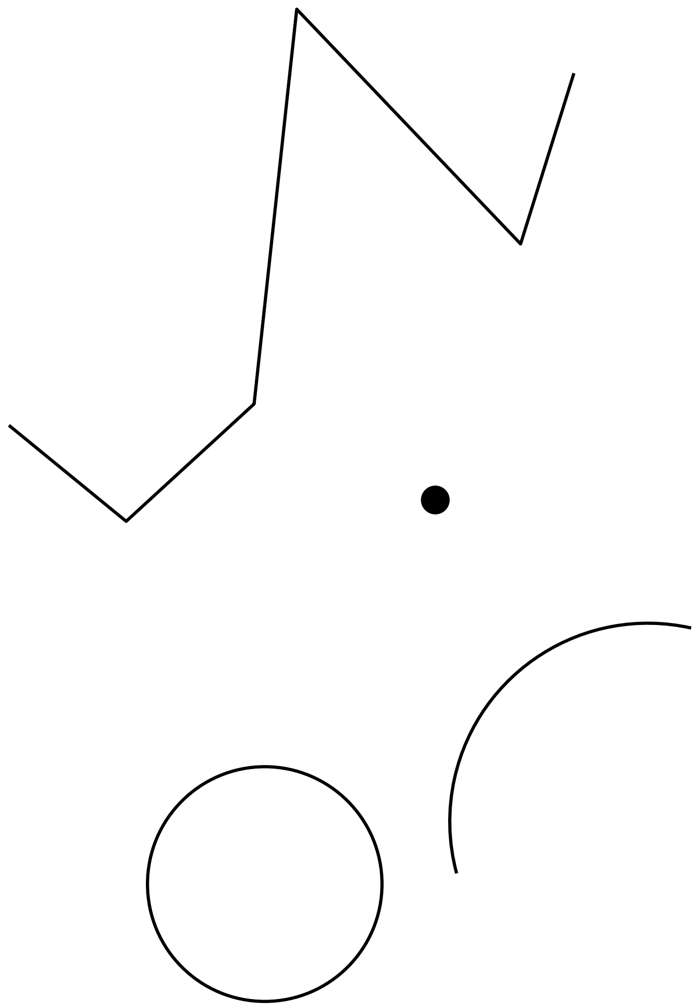
segments
disks
⋮
barriers



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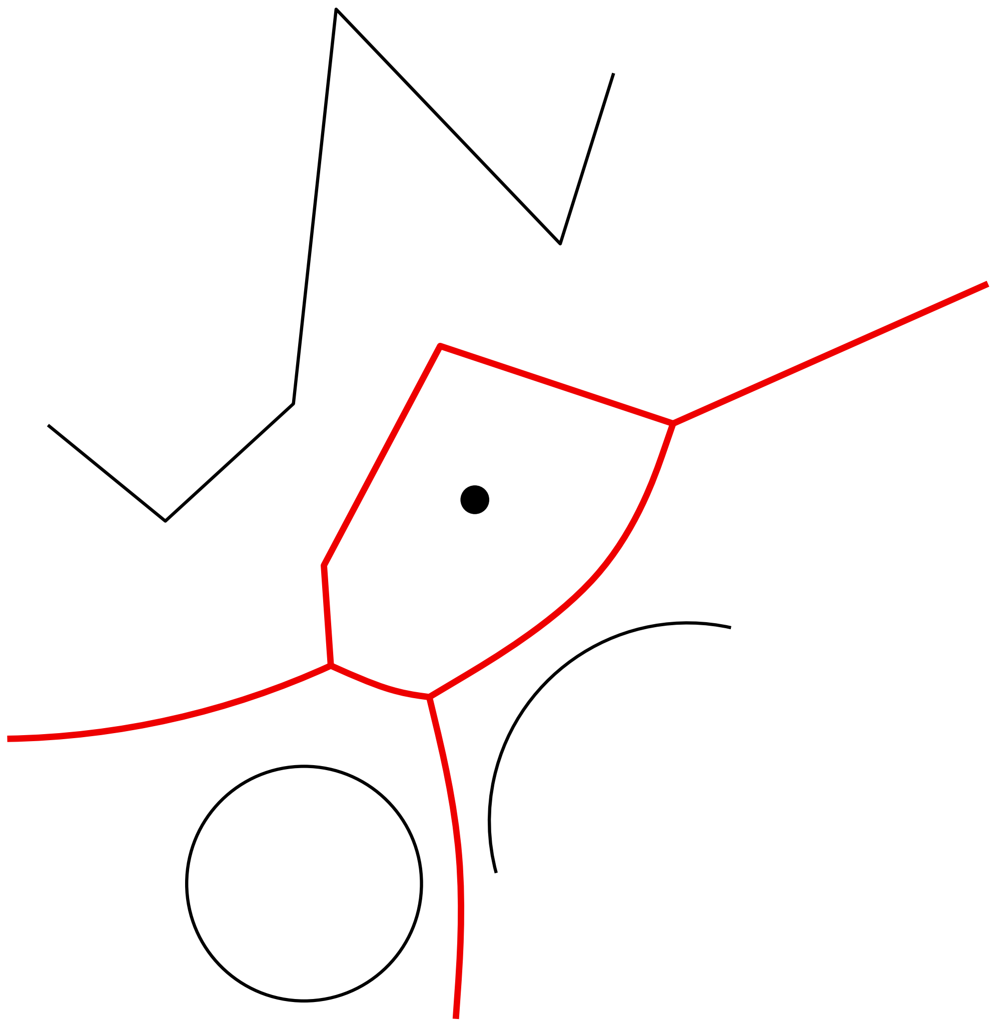

PROXIMITY

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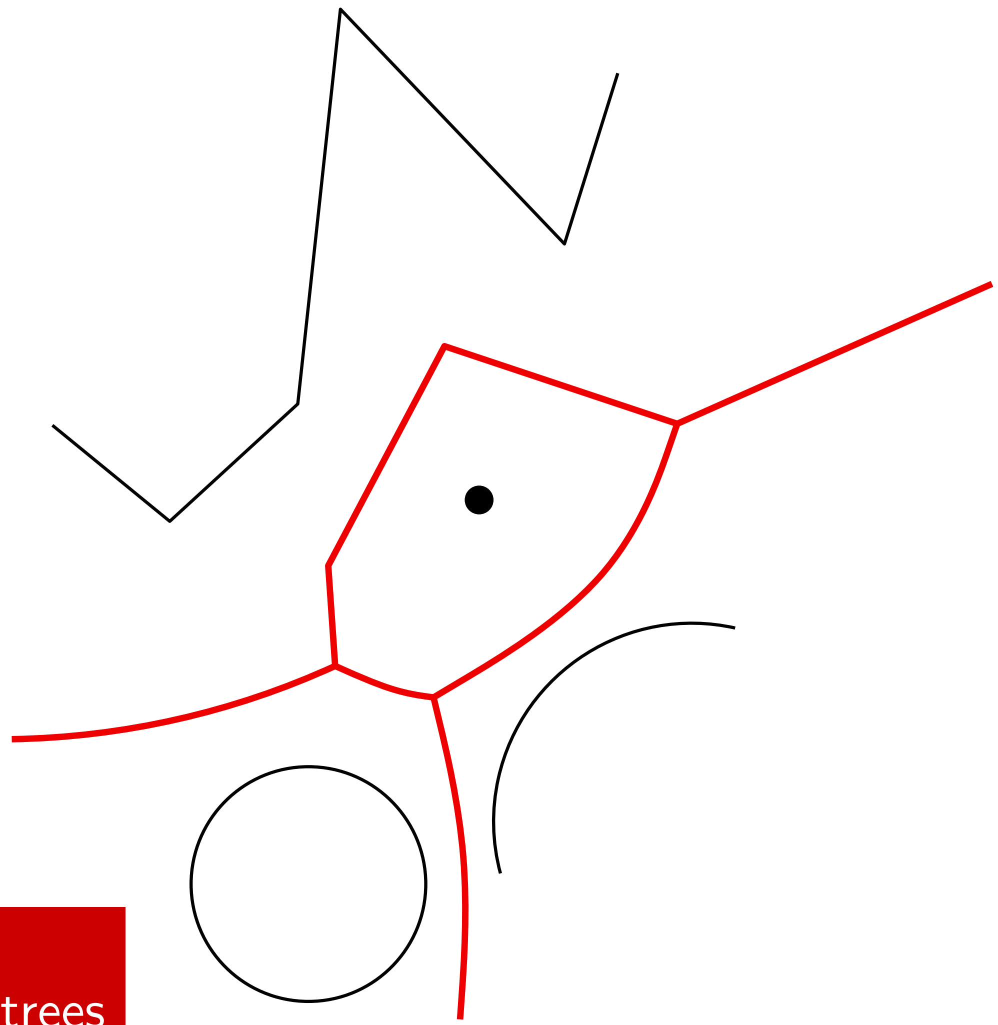

APPLICATIONS
- How do car emissions affect the trees along the highways.
- Rivers and mountains influence in determining school districts.

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sphere
cylinder
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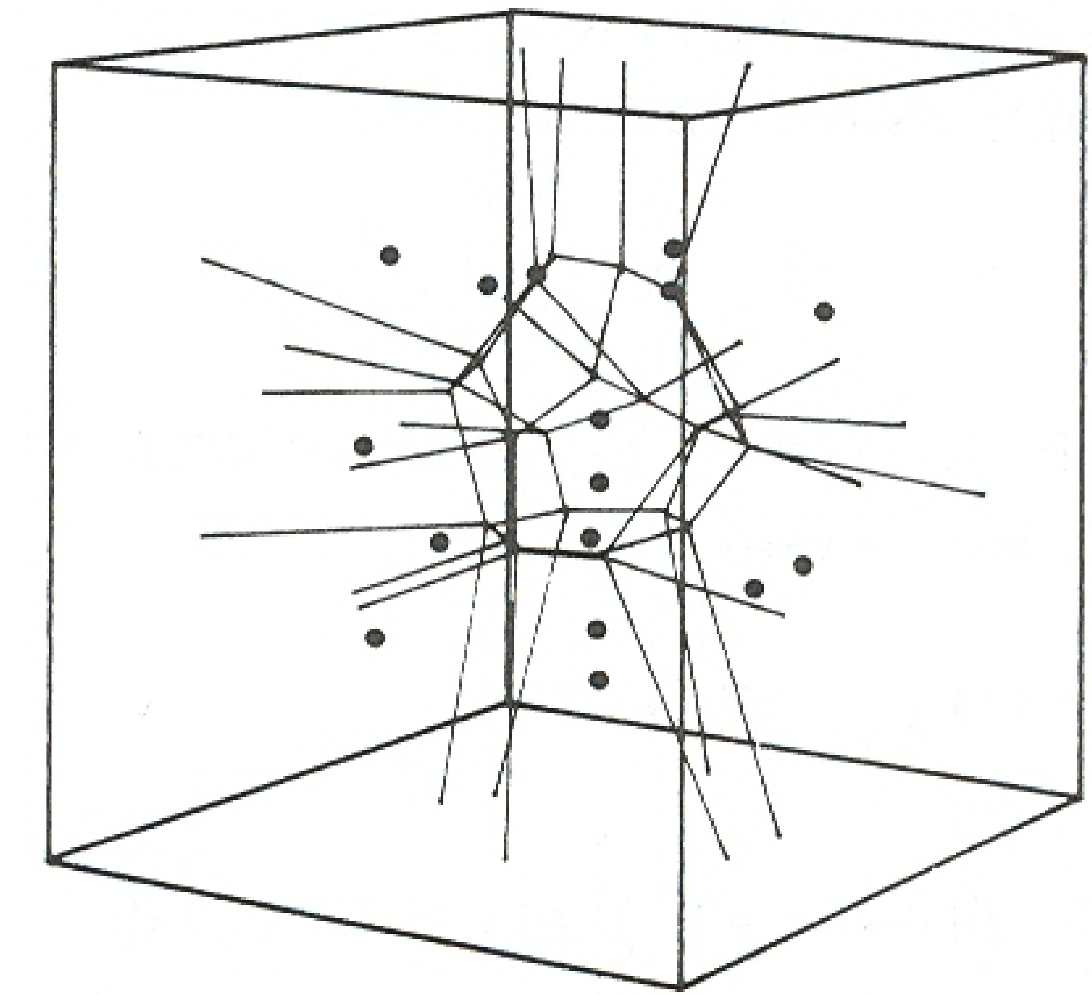

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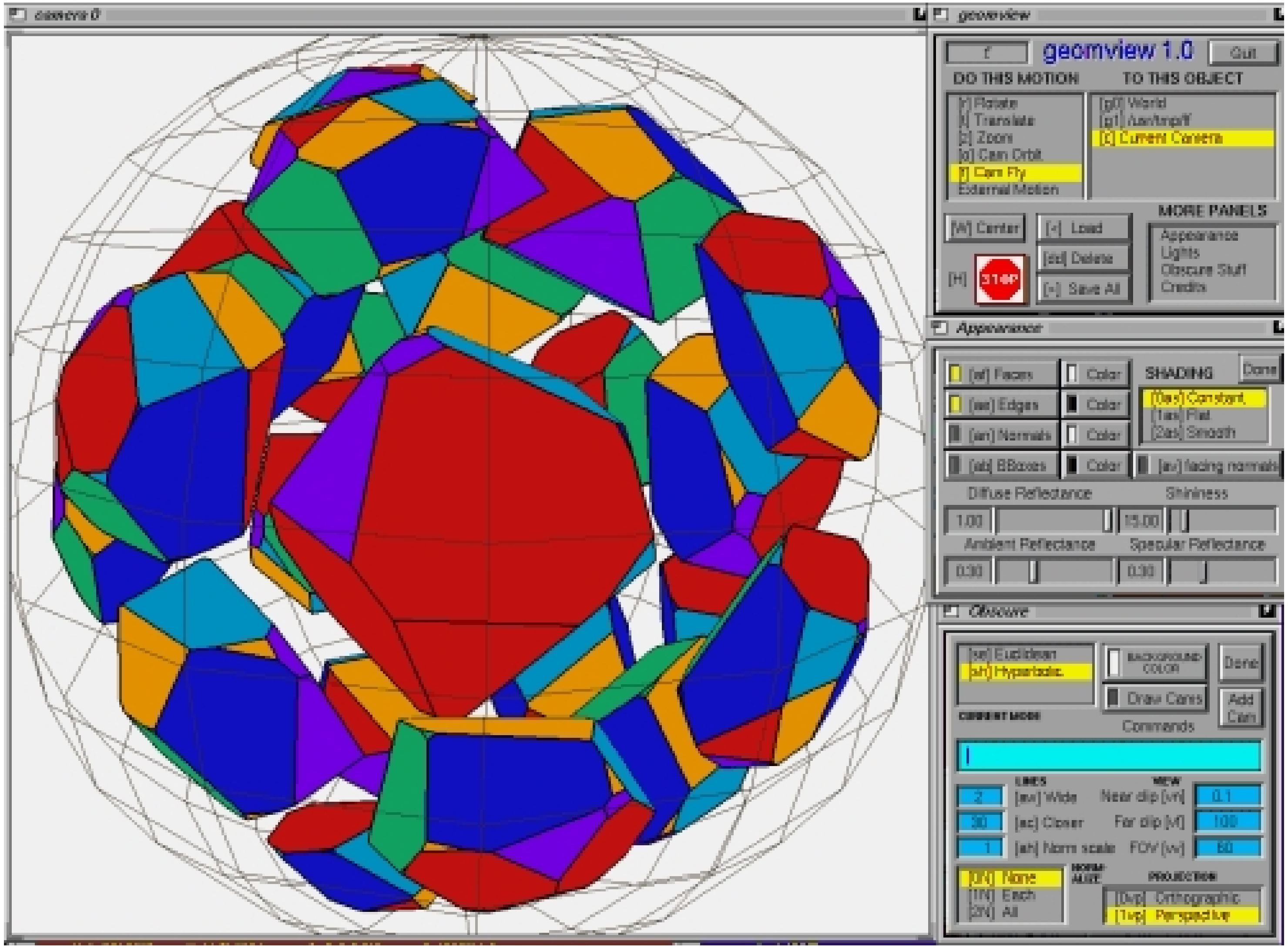

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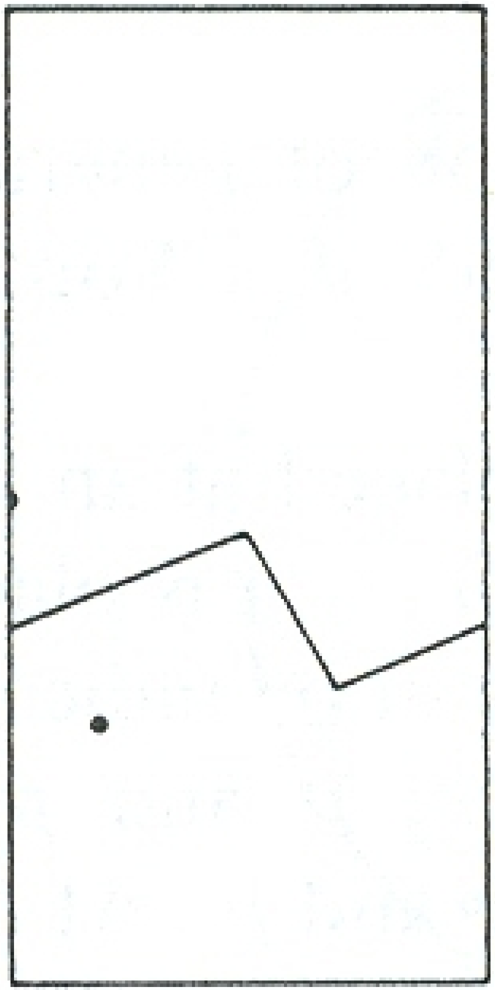

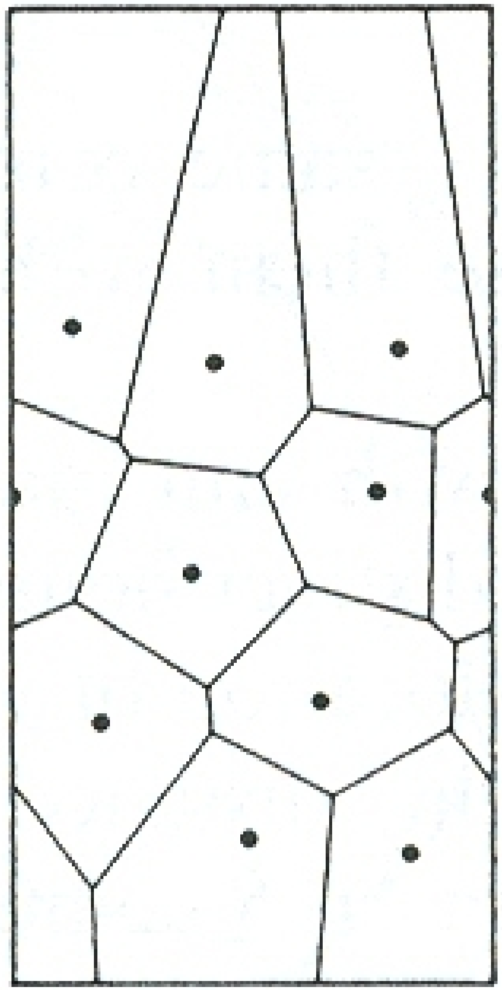

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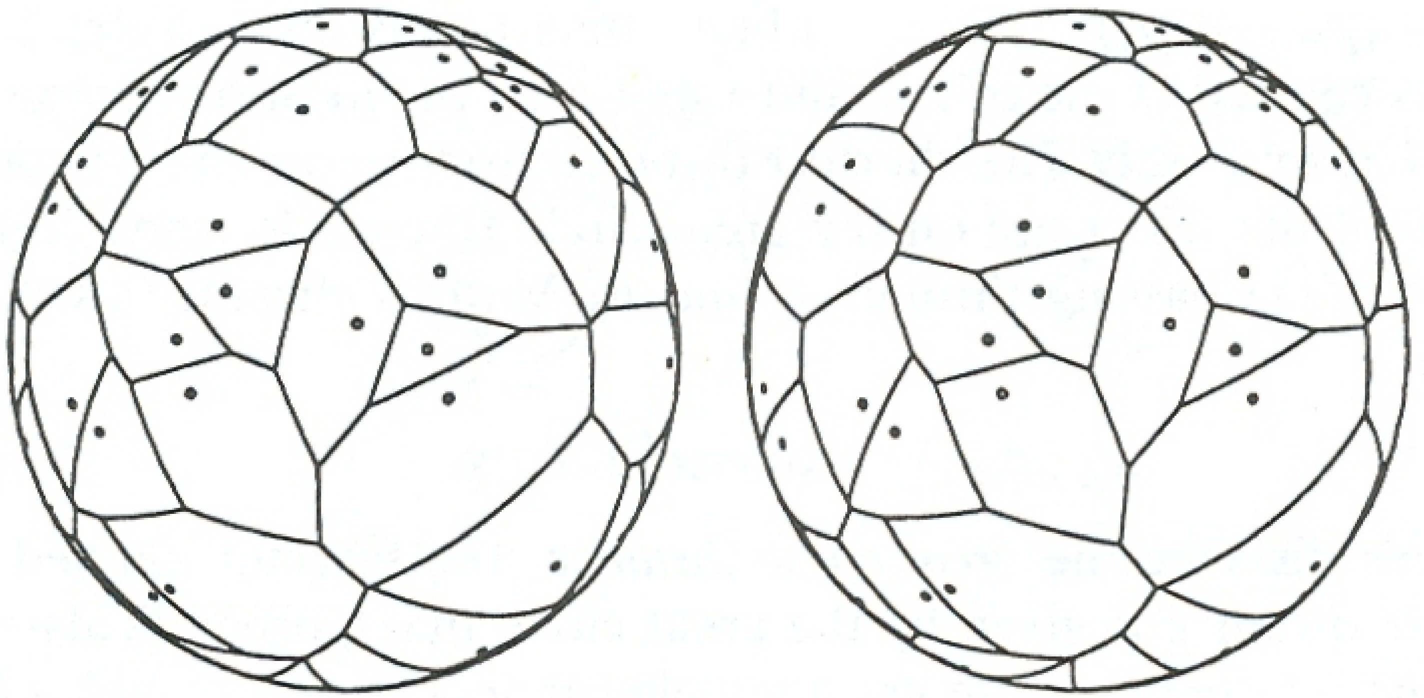

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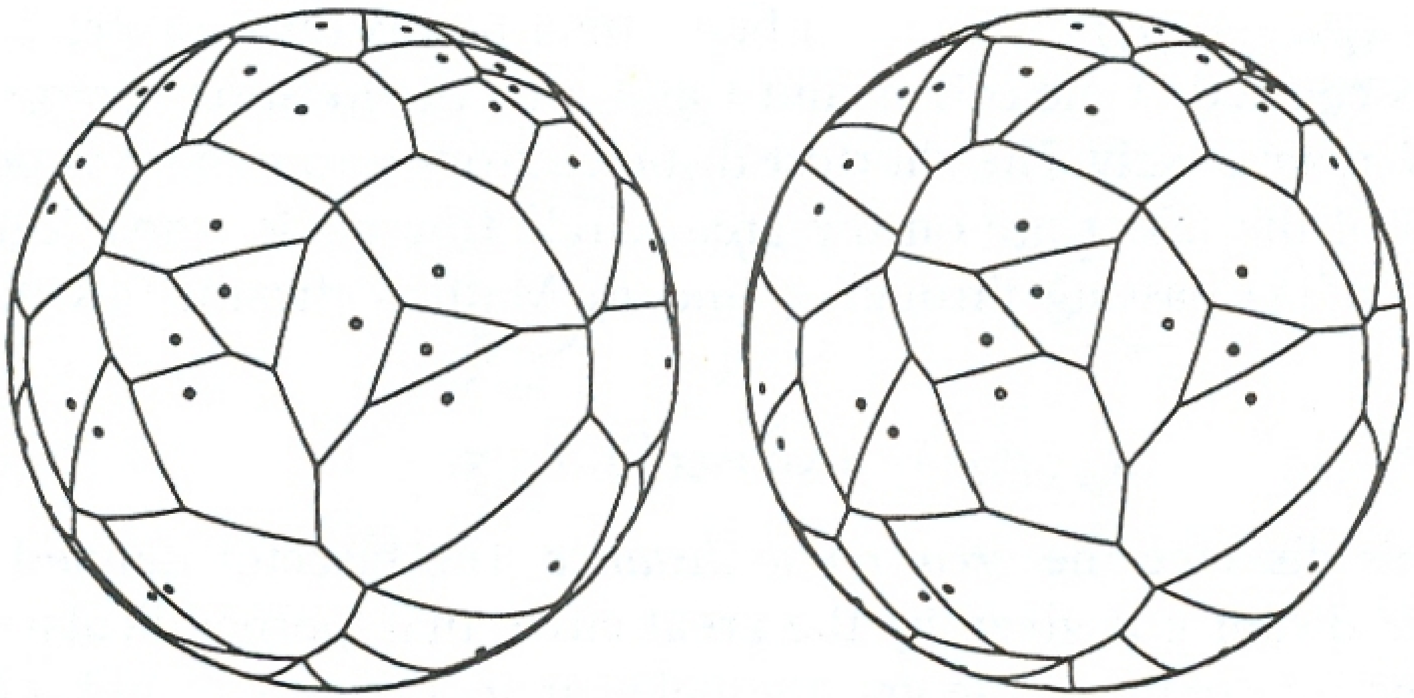
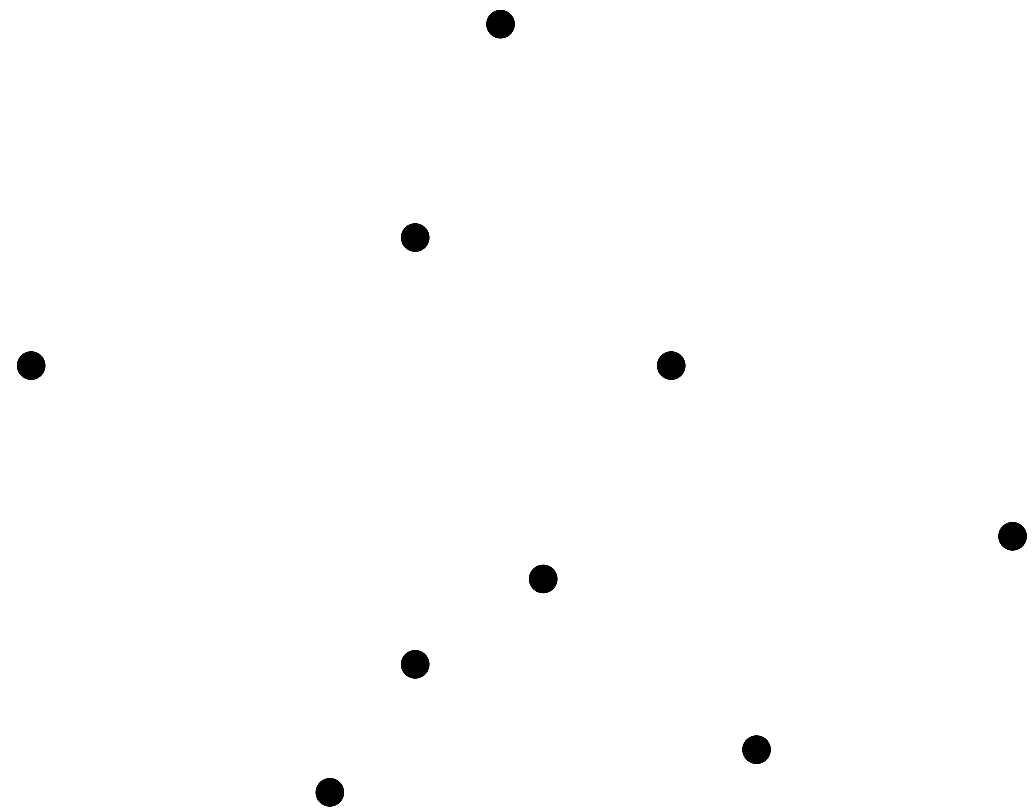


APPLICATIONS
- Sphere: study of global Earth problems
- Cylinder: study of problems with cyclic periodicity in time

GENERALIZATIONS OF THE VORONOI DIAGRAM

The Voronoi diagram of a finite set P of points in the plane is a partition of the plane such that each region is the locus of the points that lie ~~closer~~ to one element of P than to the remaining ones.

other metrics



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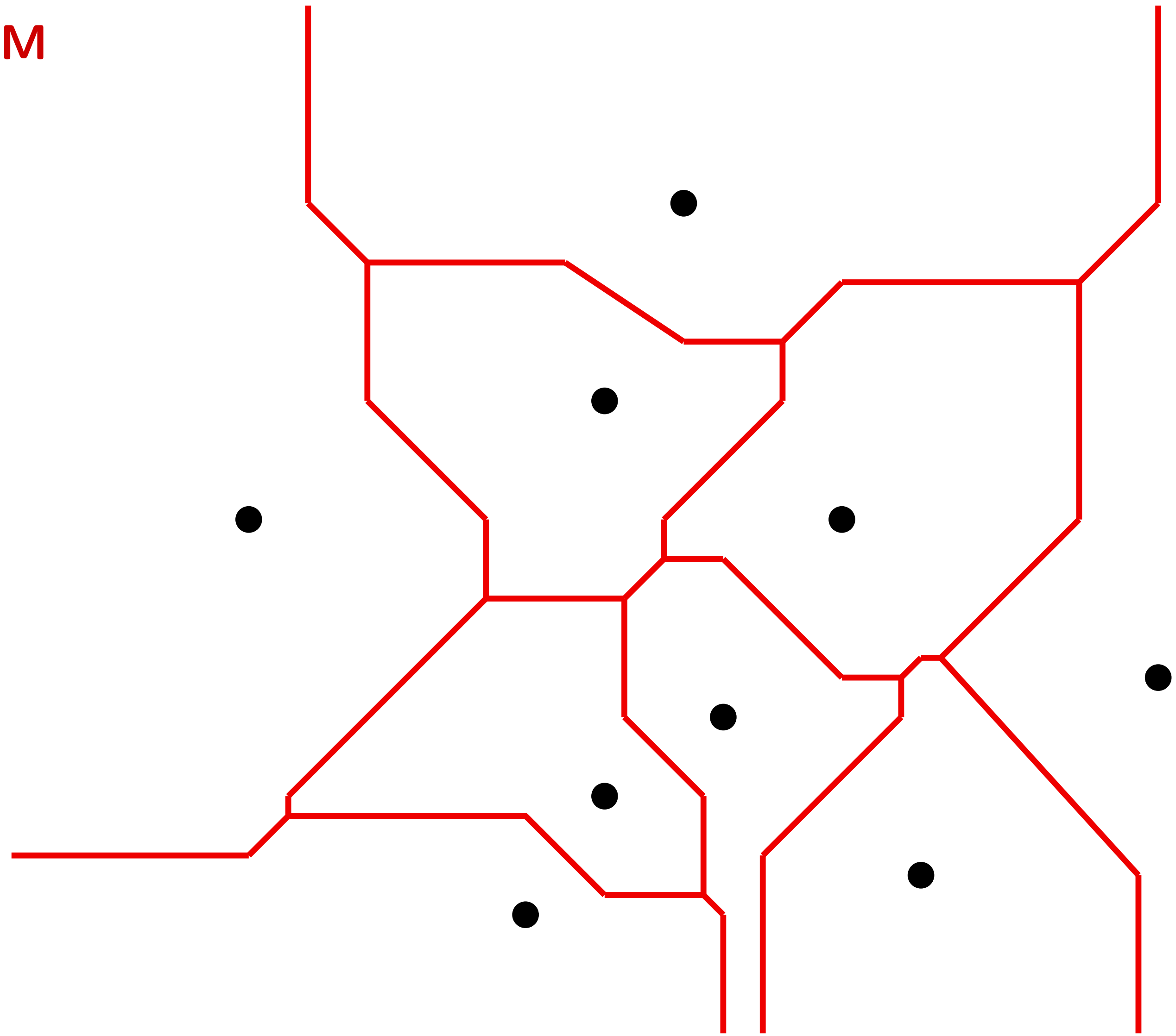

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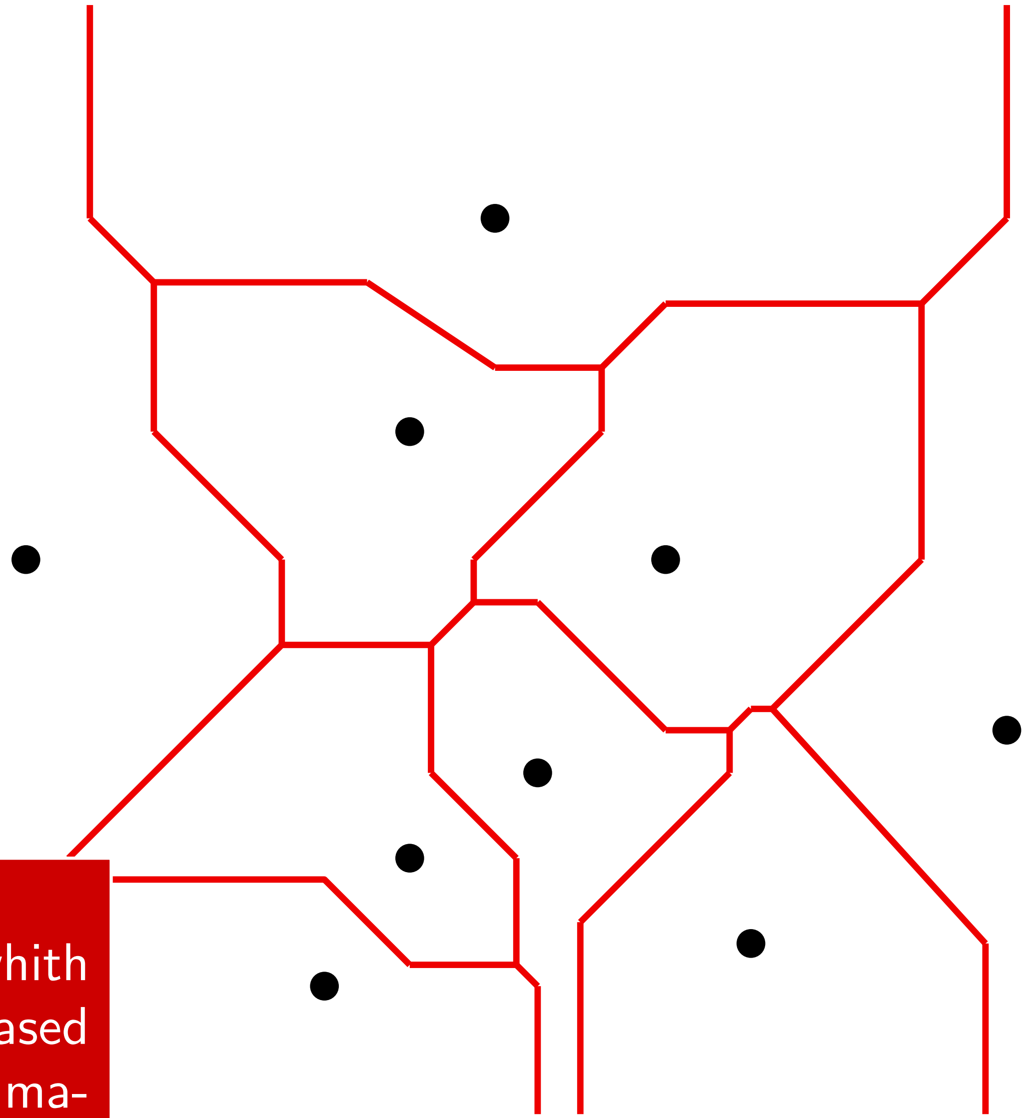

APPLICATION
Massive storage system whith a reading/writing head based on the Manhattan or the maximum metrics

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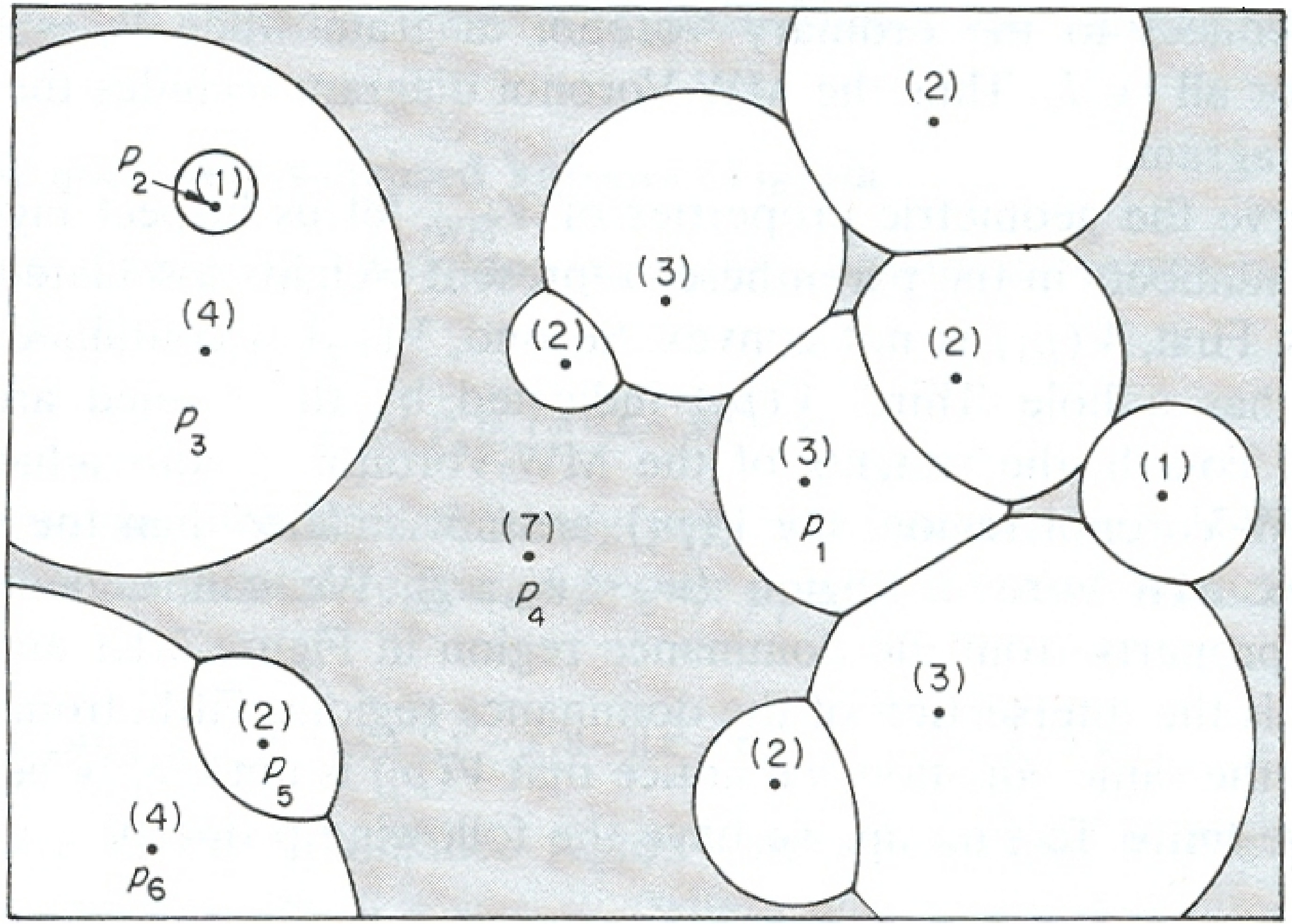

Numbers indicate multiplicative weights

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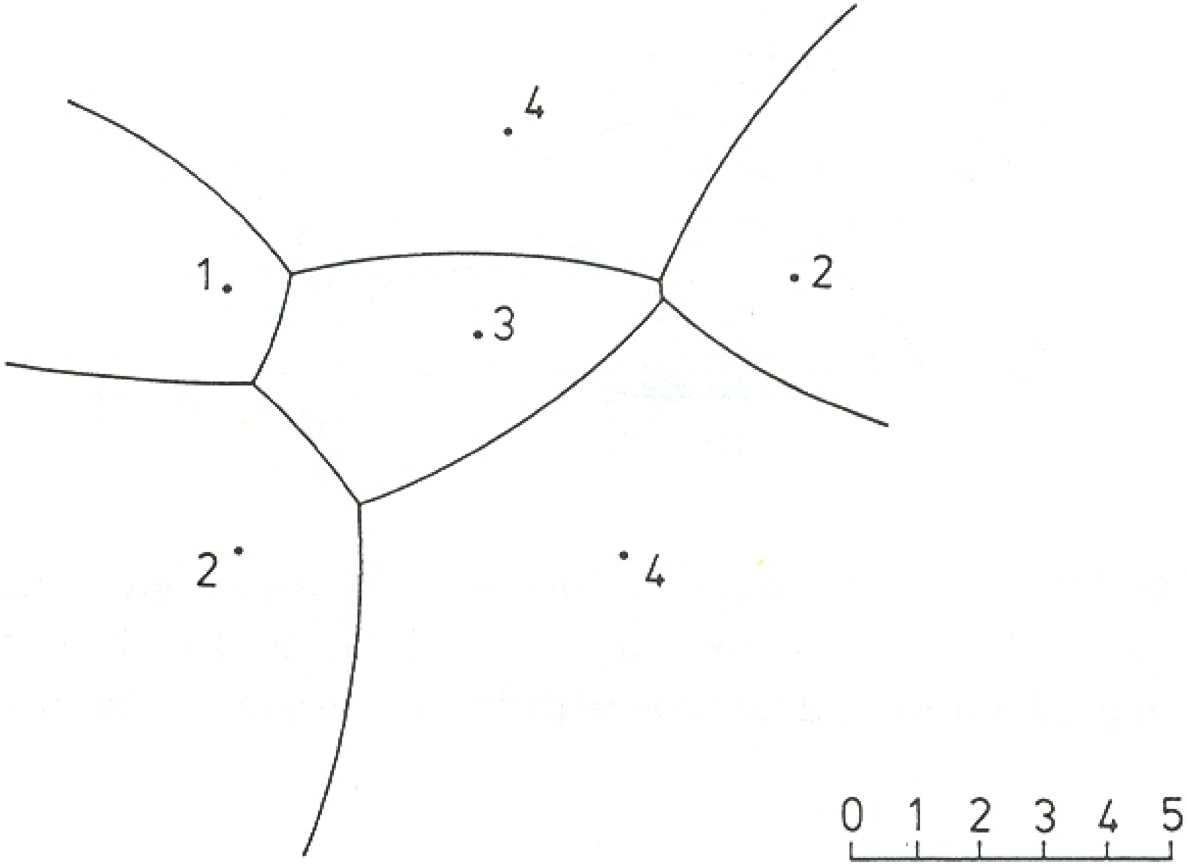

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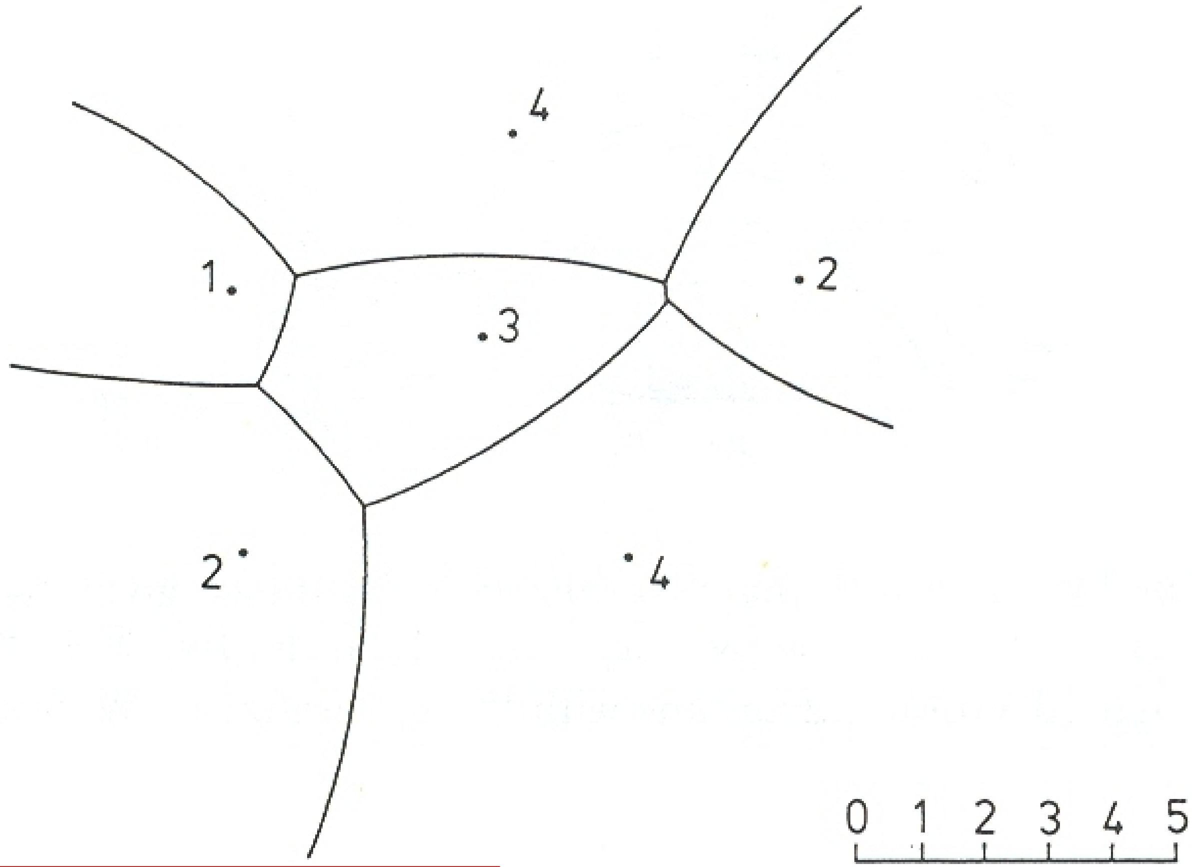

Numbers indicate additive weights

APPLICATIONS
- Population of each city
- Amount of emissions of some polluting product
- Size of each atom in a crystalline structure
- Combination of transportation prices and costs

GENERALIZATIONS OF THE VORONOI DIAGRAM

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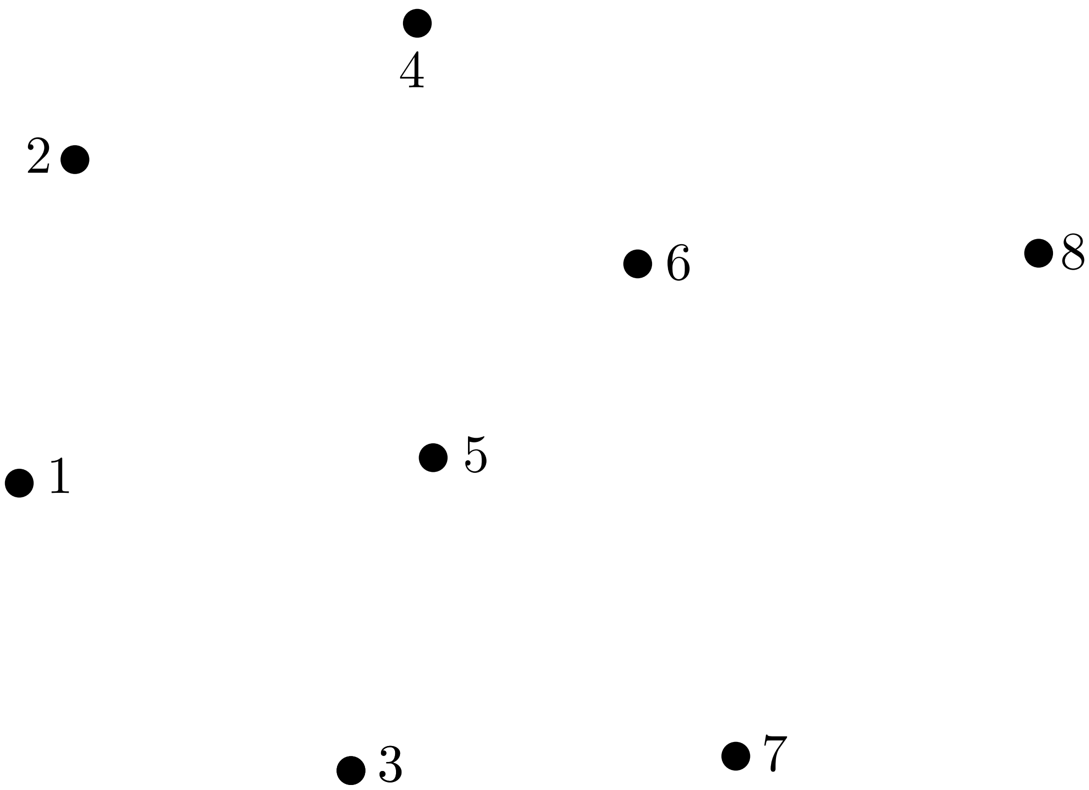
k



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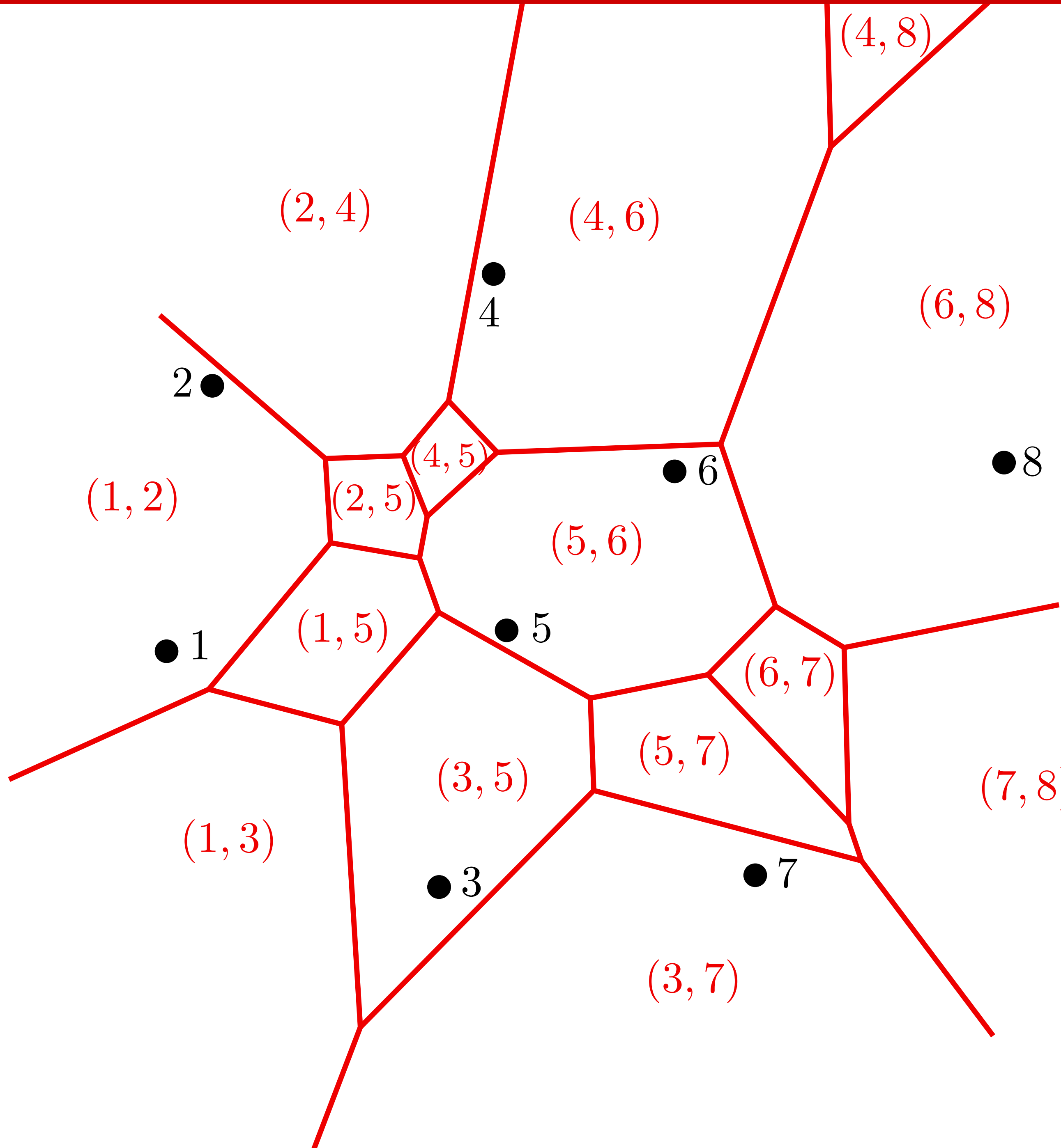

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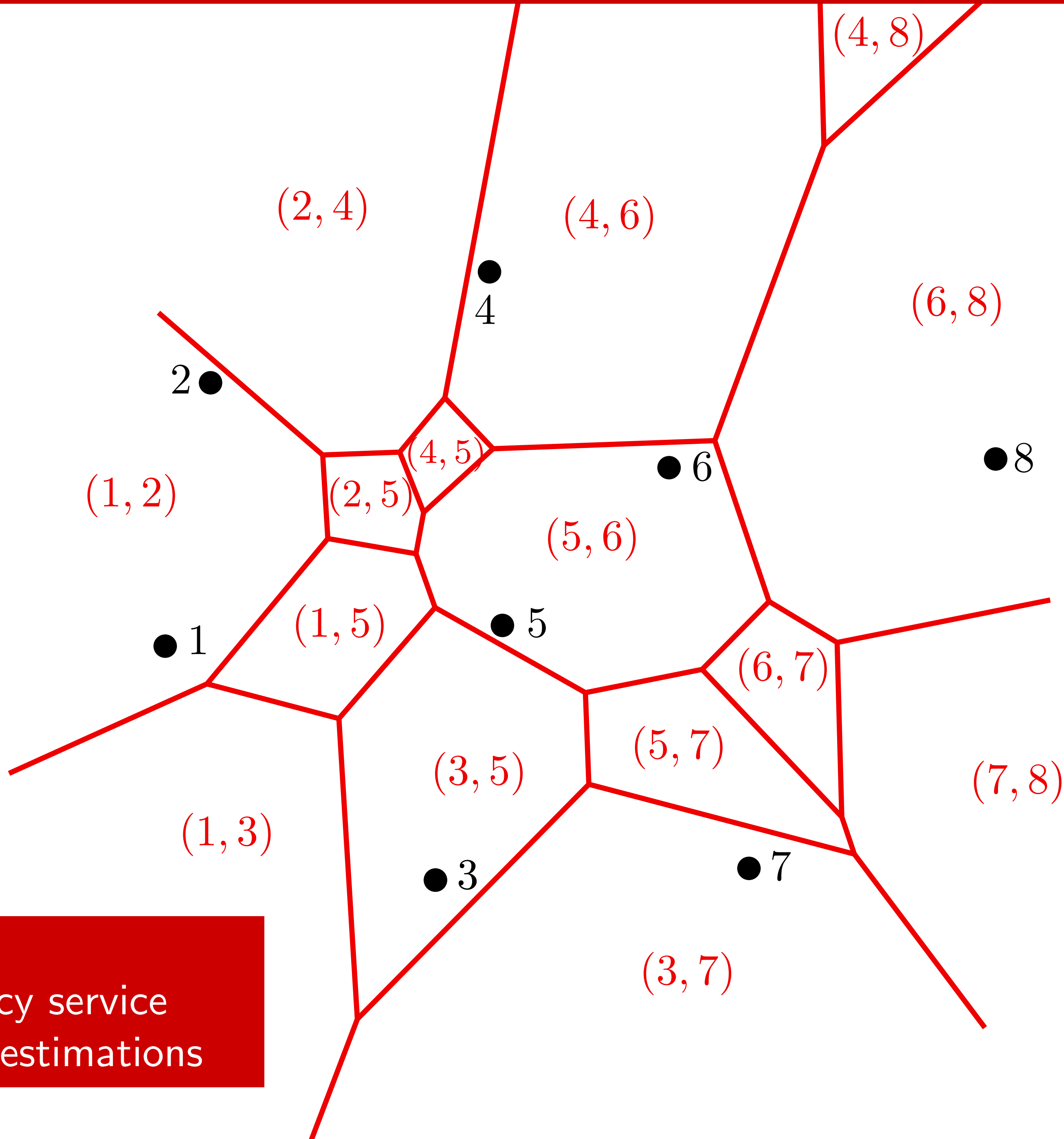

APPLICATIONS
- Finding the k-th closest emergency service
- Spatial interpolation in statistics estimations

GENERALIZATIONS OF THE VORONOI DIAGRAM

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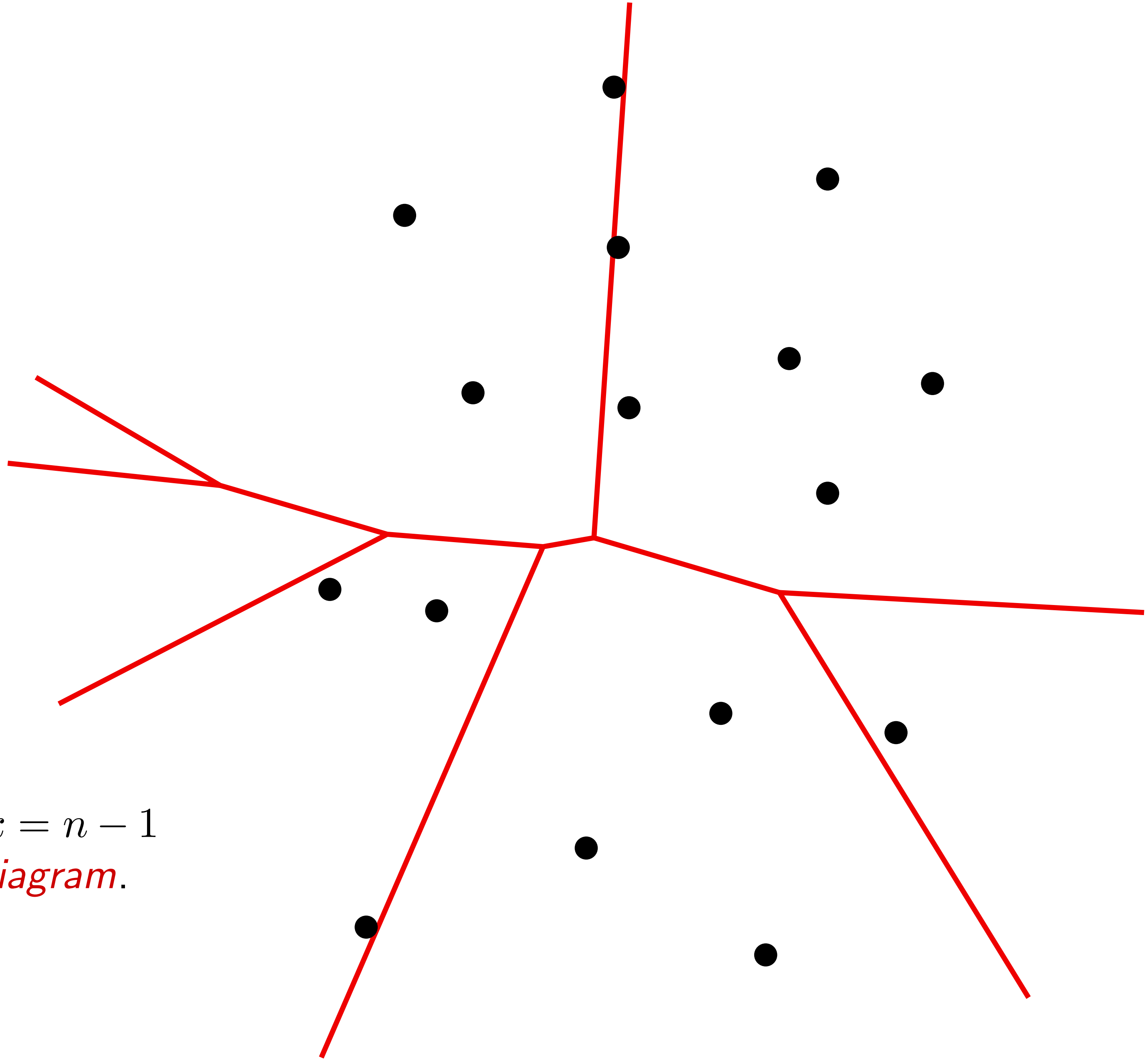

The Voronoi diagram of order $k = n - 1$ is the *Furthest Point Voronoi diagram*.

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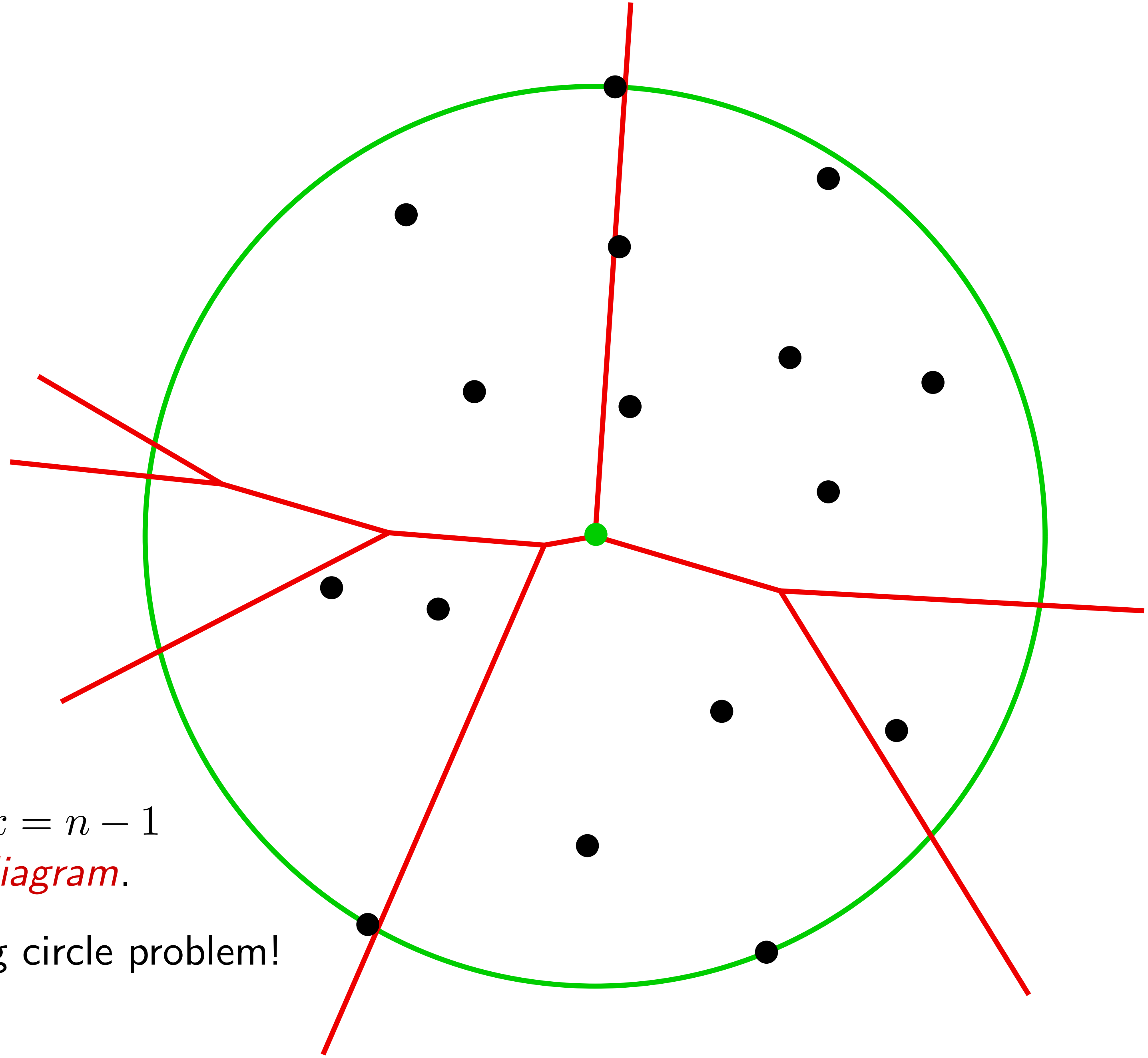

The Voronoi diagram of order $k = n - 1$ is the *Furthest Point Voronoi diagram*.

It solves the minimum spanning circle problem!

VORONOI DIAGRAM AND INTERSECTION OF HALFSPACES IN 3D

Consider the point set P as being embedded in the plane $z = 0$.

Consider the paraboloid $z = x^2 + y^2$.

For each point $p_i \in P$ let p_i^* be its vertical projection of p_i onto the paraboloid, i.e.:

$$\text{if } p_i = (a_i, b_i, 0), \text{ then } p_i^* = (a_i, b_i, a_i^2 + b_i^2).$$

For each point p_i^* consider the plane which is tangent to the paraboloid at p_i^*.

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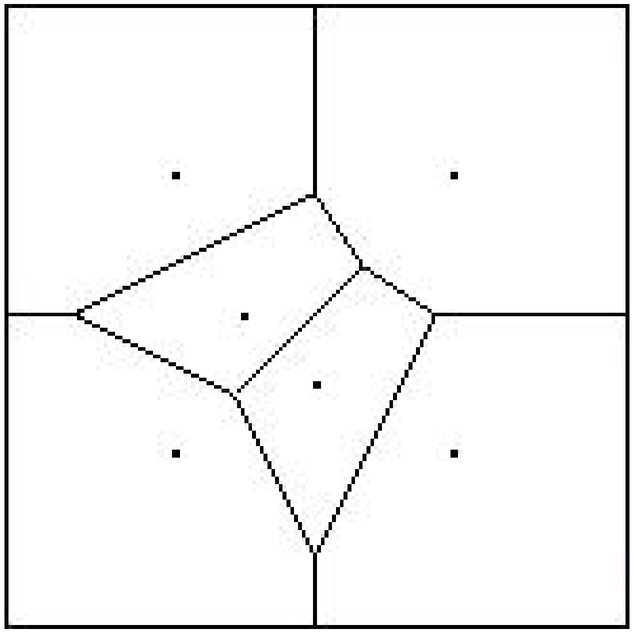
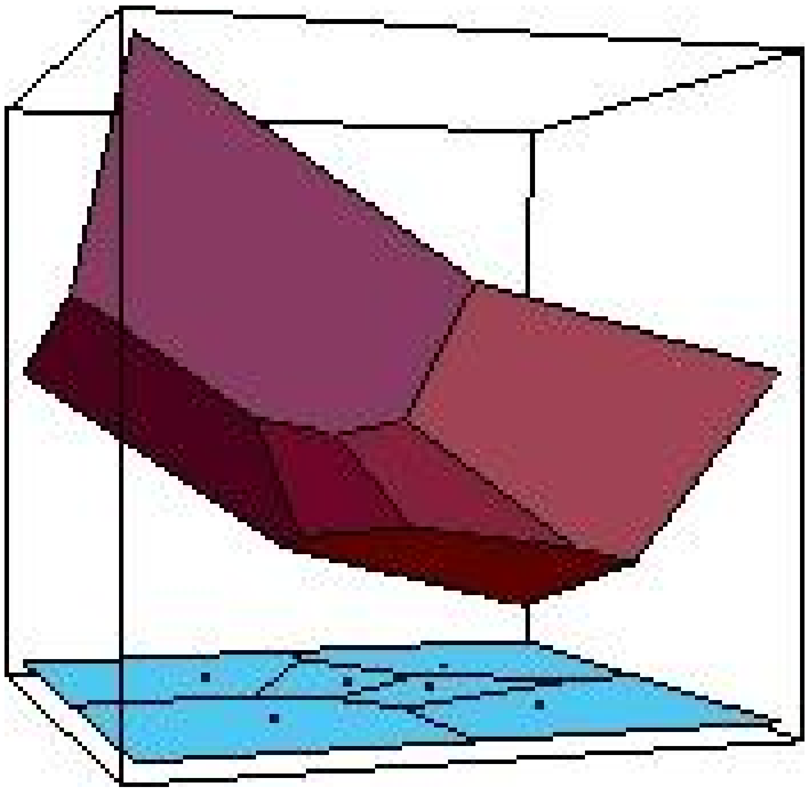
The (farthest point) Voronoi diagram of P is the orthogonal projection onto the plane $z = 0$ of the polyhedral convex region obtained when intersecting the upper (lower) halfspaces defined by these planes.

PROXIMITY





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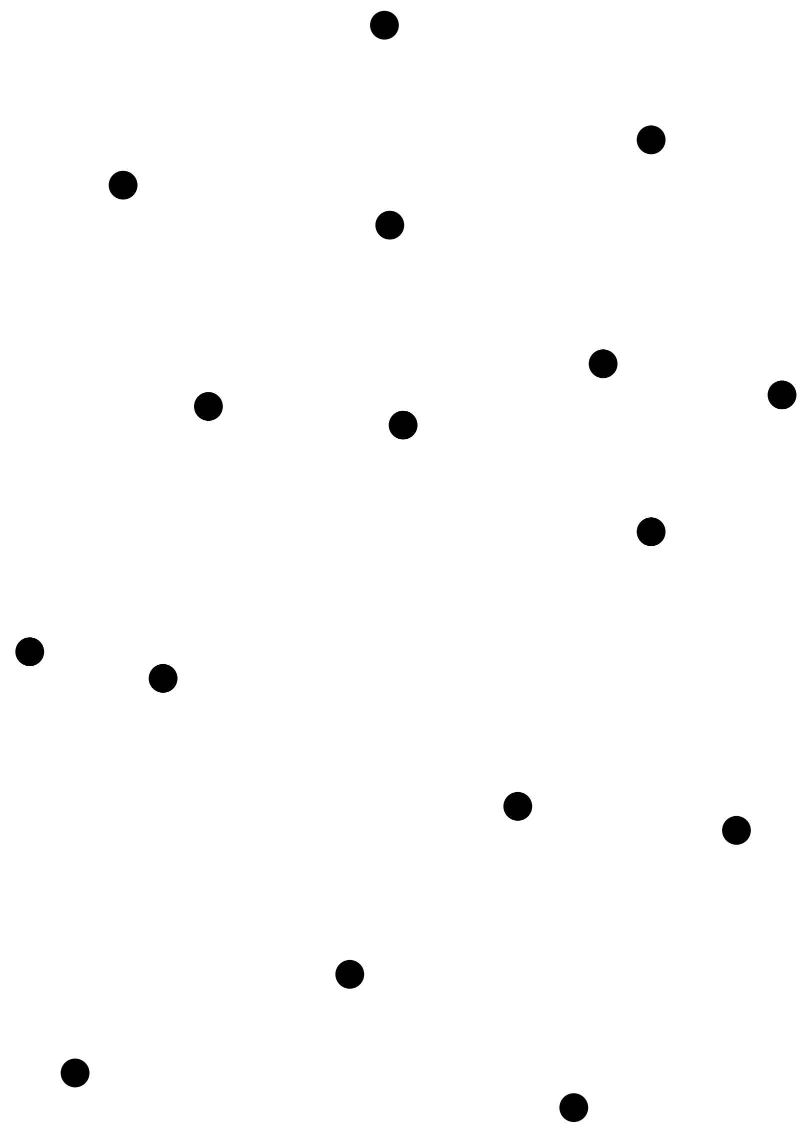
PROXIMITY

THREE BIG QUESTIONS

1. How to build the Voronoi diagram of P?
2. How to store the Voronoi diagram of P?
3. How to use the Voronoi diagram to improve the solutions of the proximity problems?



TWO BOOKS WITH MUCH MORE INFORMATION

A. Okabe, B. Boots, K. Sugihara, S. N. Chiu
Spatial Tessellations
2nd ed., J. Wiley & Sons, 2000.

F. Aurenhammer, R. Klein, D.-T. Lee
Voronoi Diagrams and Delaunay Triangulations
World Scientific, 2013.