

ARRANGEMENTS OF LINES

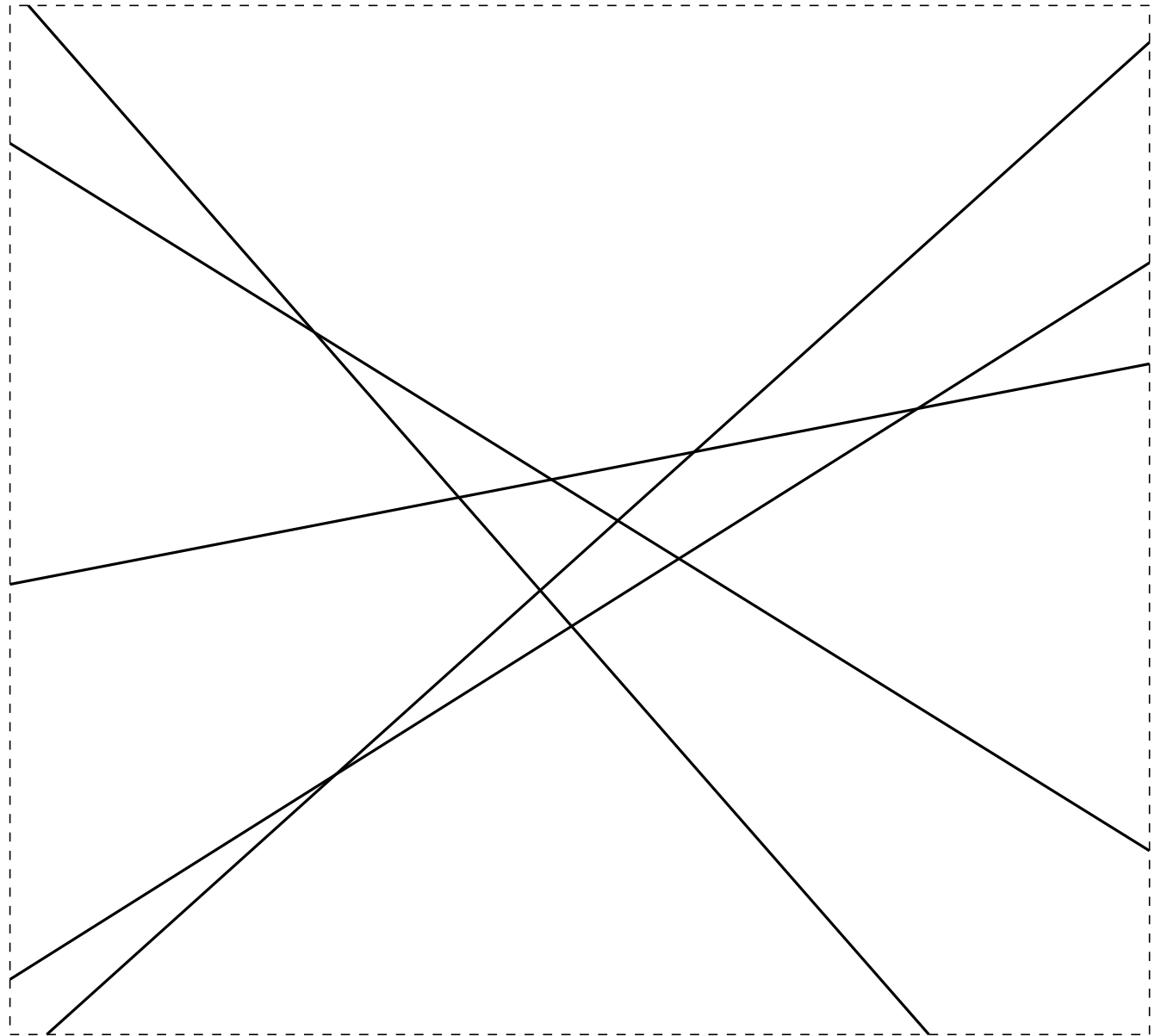
Vera Sacristán
Rodrigo Silveira

Research Group on
Discrete, Combinatorial and Computational Geometry
Universitat Politècnica de Catalunya

ARRANGEMENTS

DEFINITION

Let $L = \{l_1, \dots, l_n\}$ be a finite set of lines in the plane.

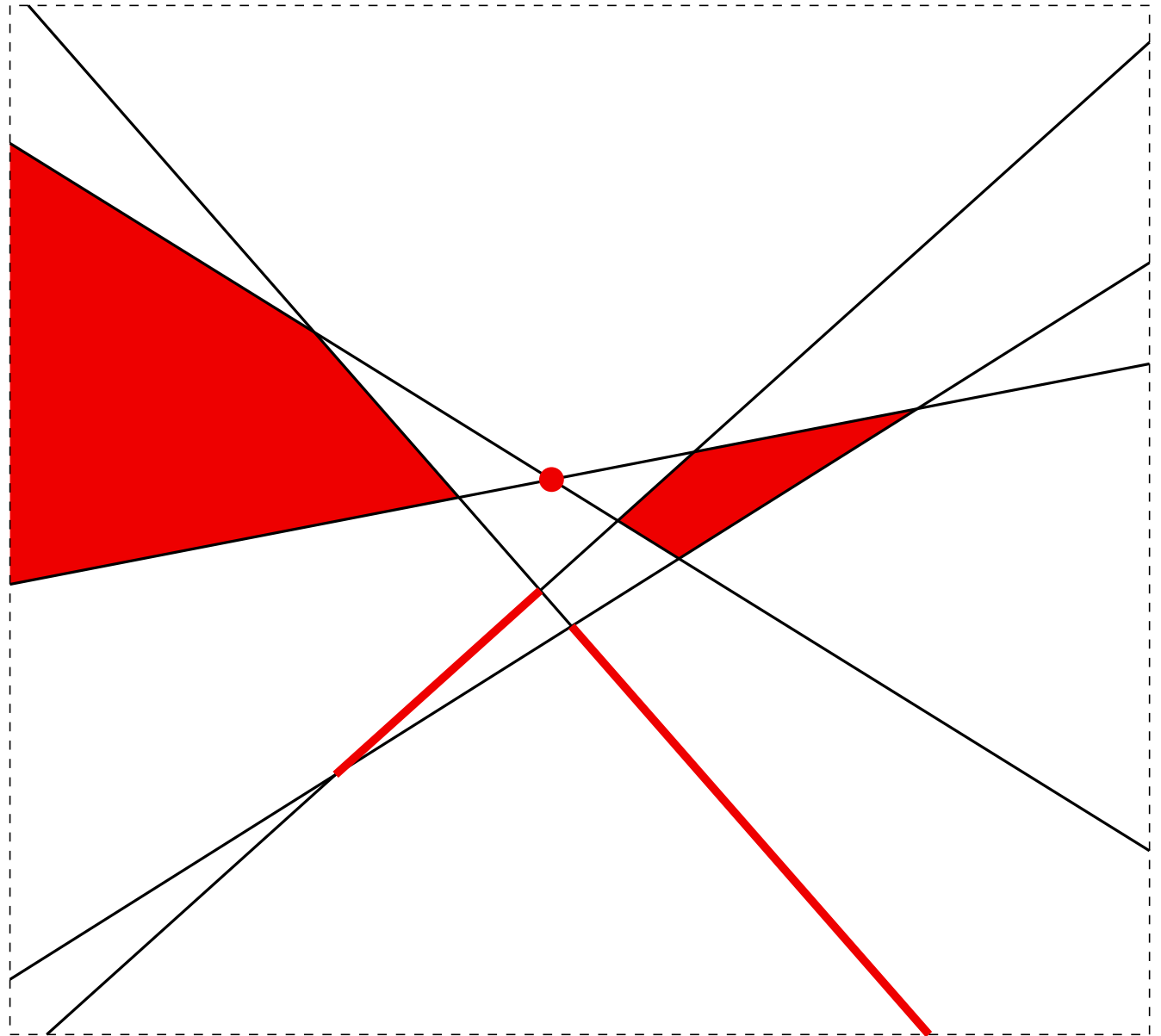


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The *arrangement* $\mathcal{A}(L)$ is the decomposition of the plane into *faces*, *edges* and *vertices* produced by L .



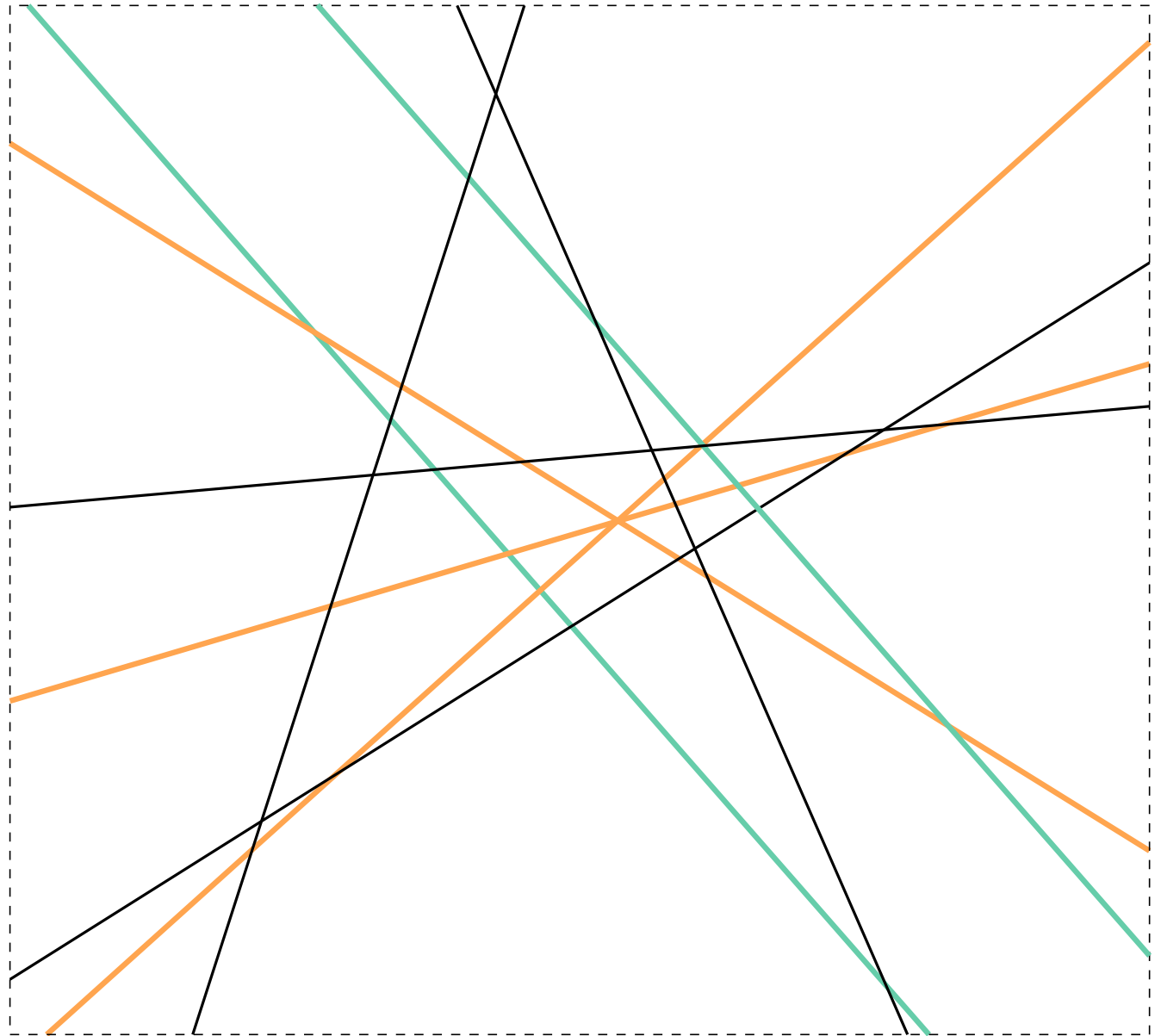
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The *arrangement* $\mathcal{A}(L)$ is the decomposition of the plane into *faces*, *edges* and *vertices* produced by L .

The arrangement $\mathcal{A}(L)$ is said to be *simple* if L does not contain any two parallel lines nor any three lines through the same point.



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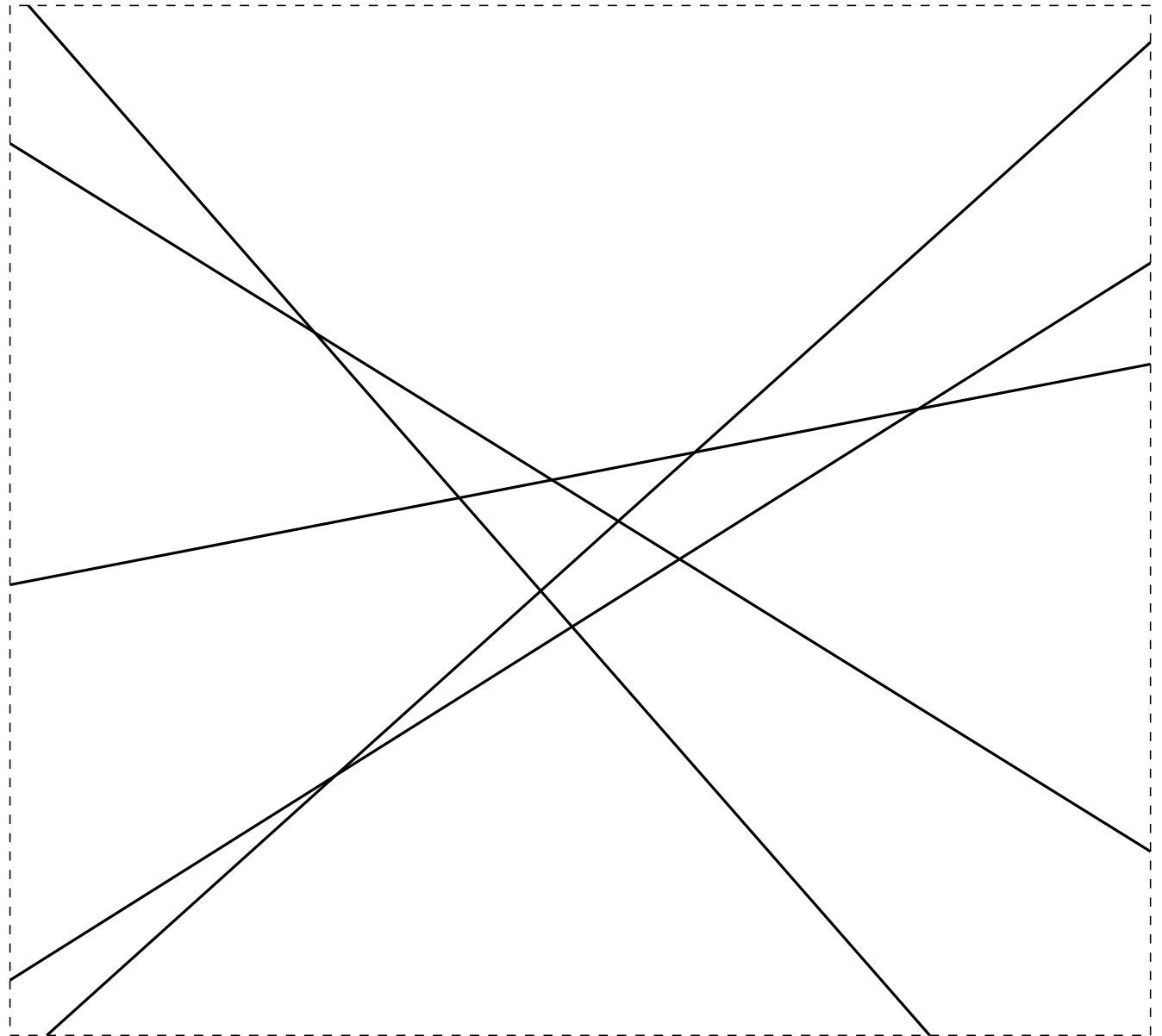
COMPLEXITY

The (*combinatorial*) complexity of an arrangement $\mathcal{A}(L)$ is its number of faces, edges and vertices.

The complexity of $\mathcal{A}(L)$ is $O(n^2)$, where $n = \#L$. Specifically:

- $v \leq \binom{n}{2} = \frac{n(n-1)}{2}$
- $e \leq n^2$
- $f \leq \frac{n^2}{2} + \frac{n}{2} + 1$

These bounds are achieved when the arrangement is simple!



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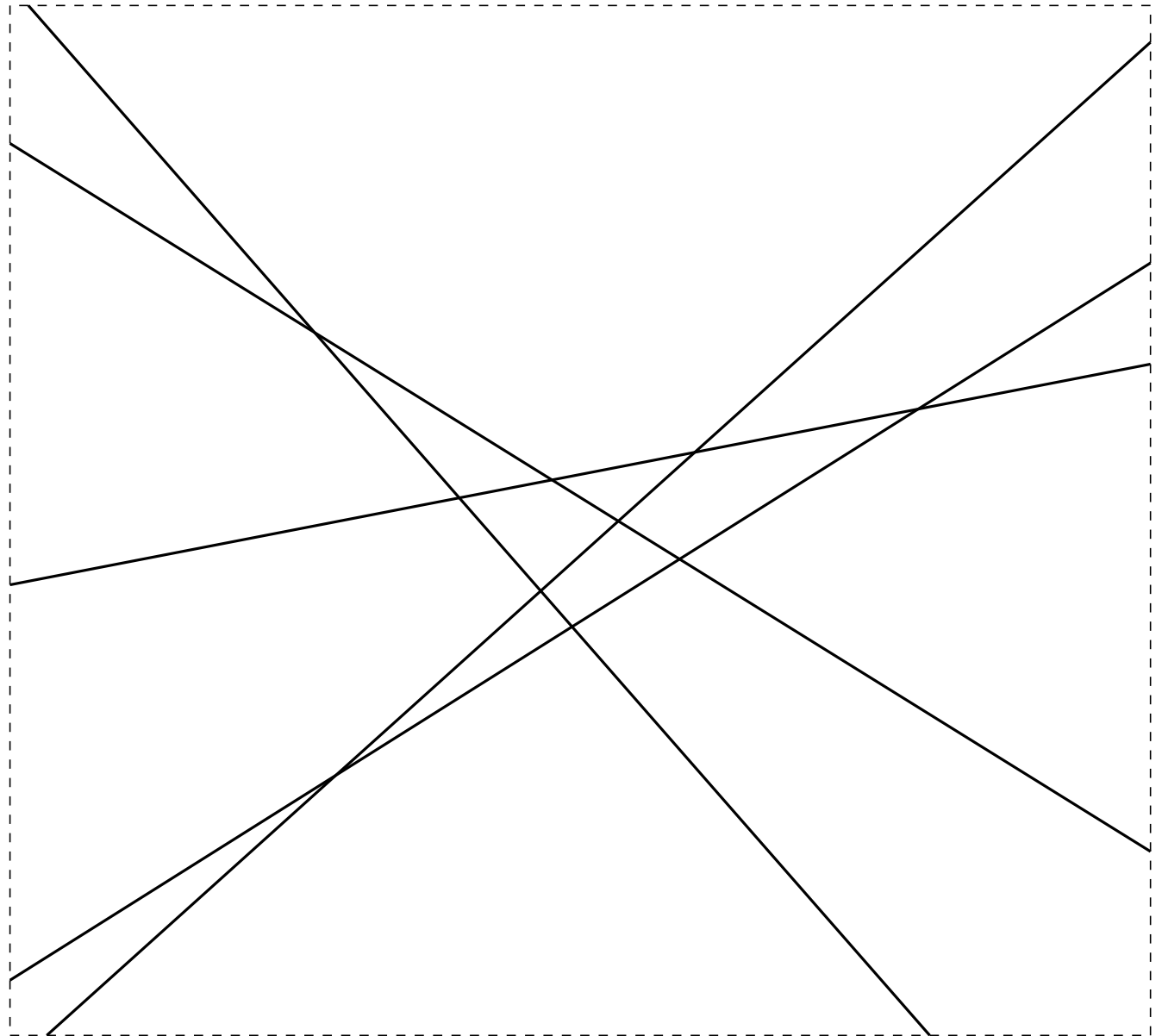
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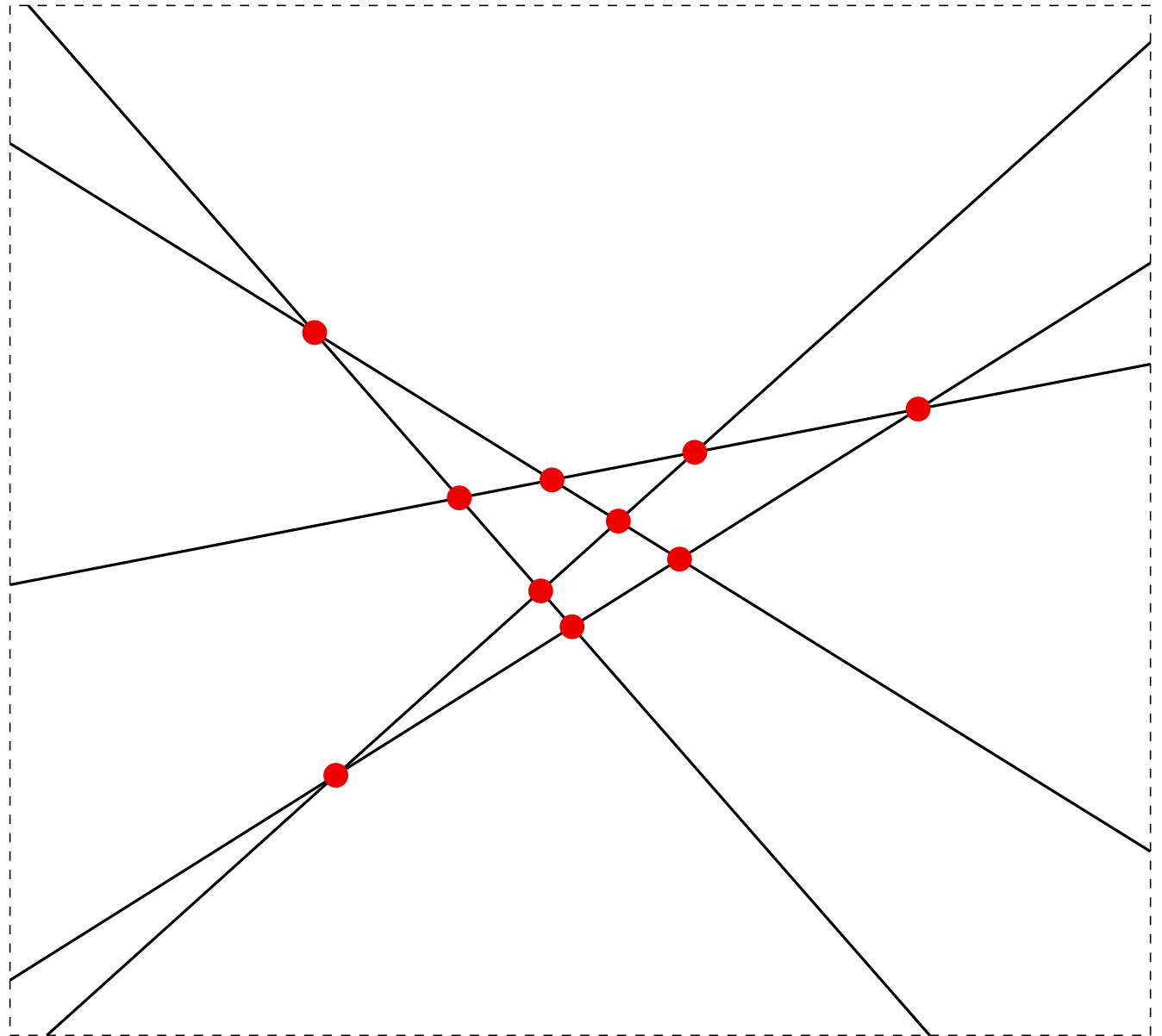
$$n = 5$$

- $v \leq \binom{n}{2} = \frac{n(n-1)}{2} \quad v = 10$

- $e \leq n^2$

- $f \leq \frac{n^2}{2} + \frac{n}{2} + 1$

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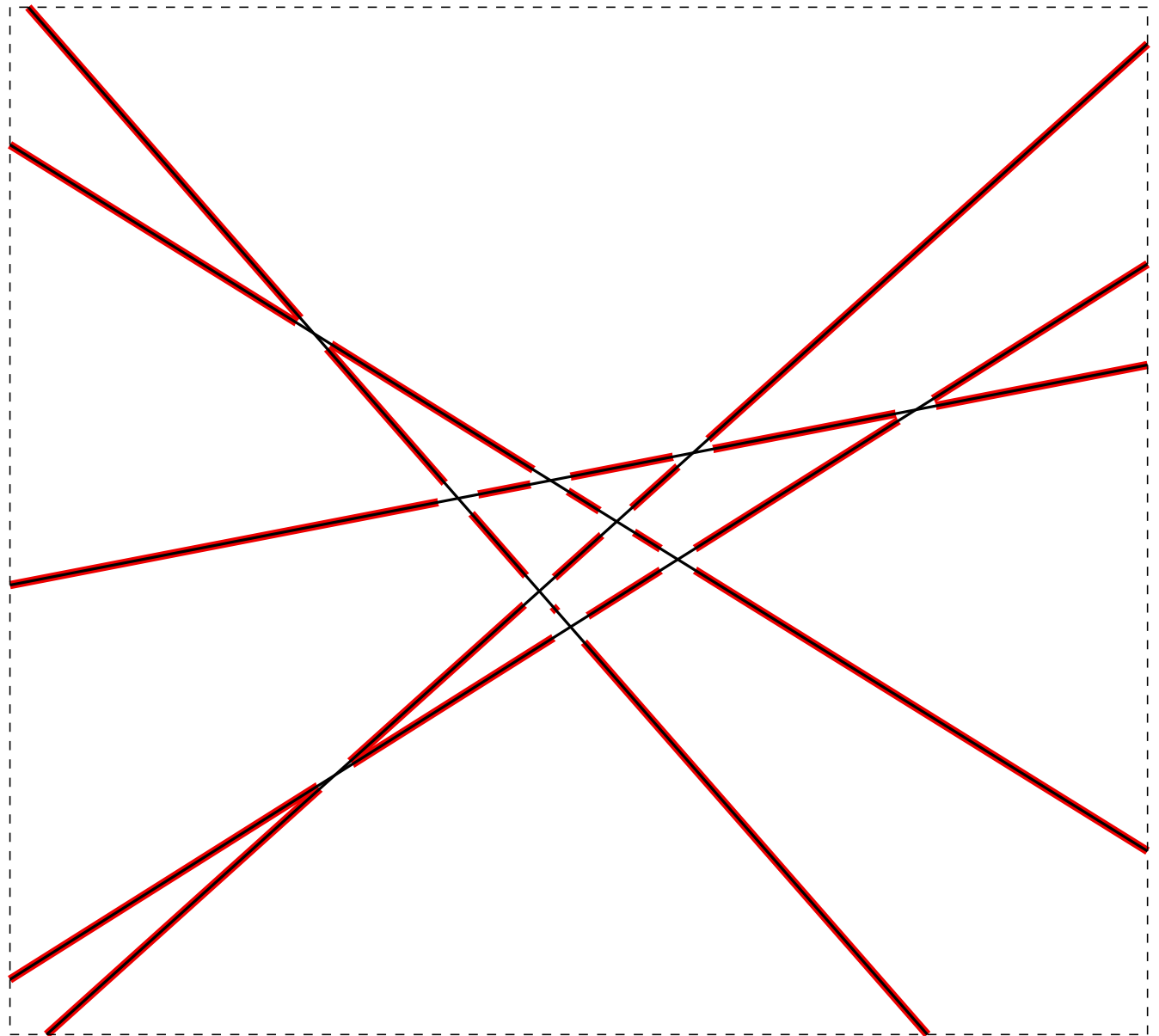
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- $v \leq \binom{n}{2} = \frac{n(n-1)}{2}$ $v = 10$

- $e \leq n^2$ $e = 25$

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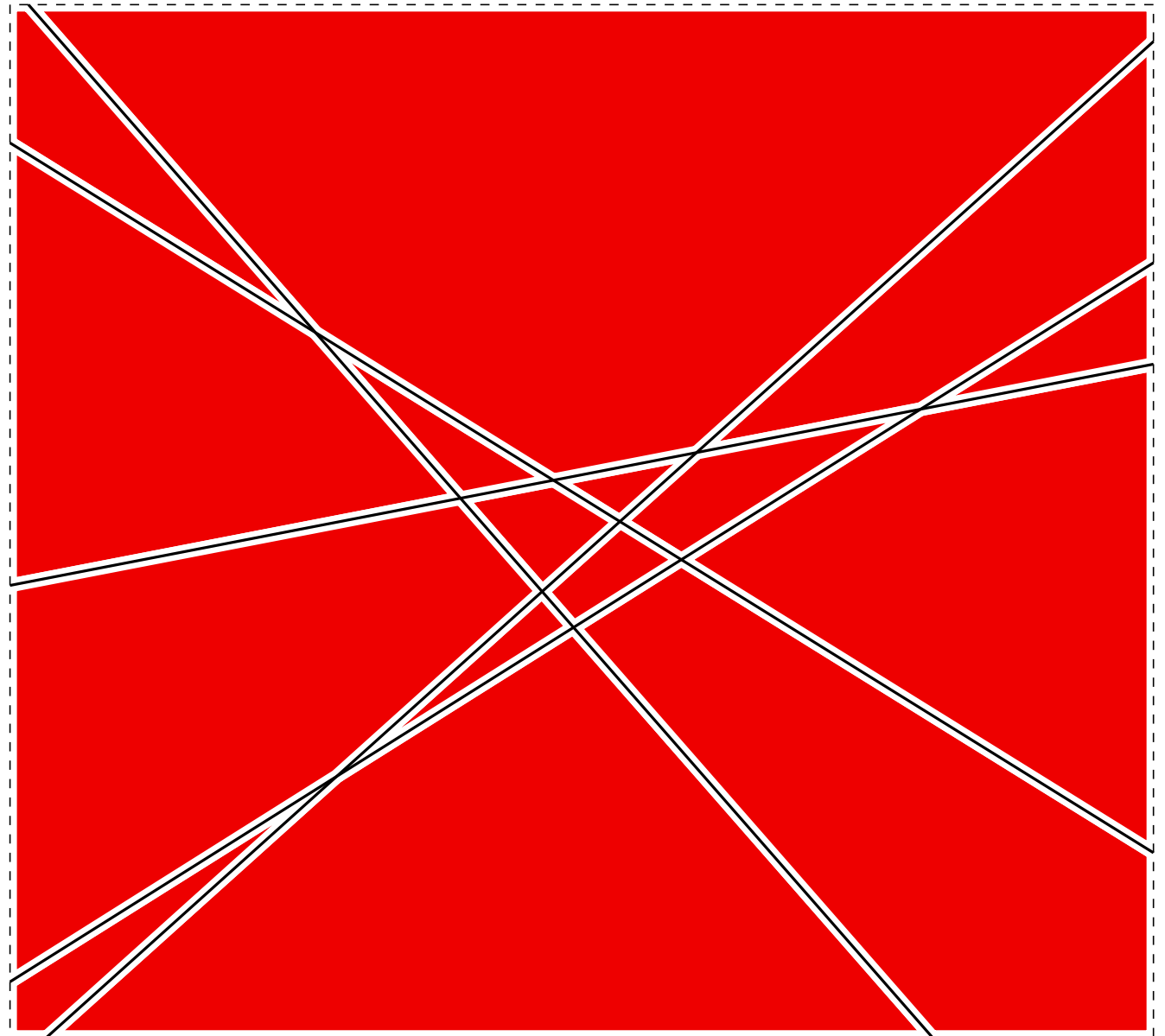
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$$\bullet e \leq n^2 \quad e = 25$$

$$\bullet f \leq \frac{n^2}{2} + \frac{n}{2} + 1 \quad f = 16$$

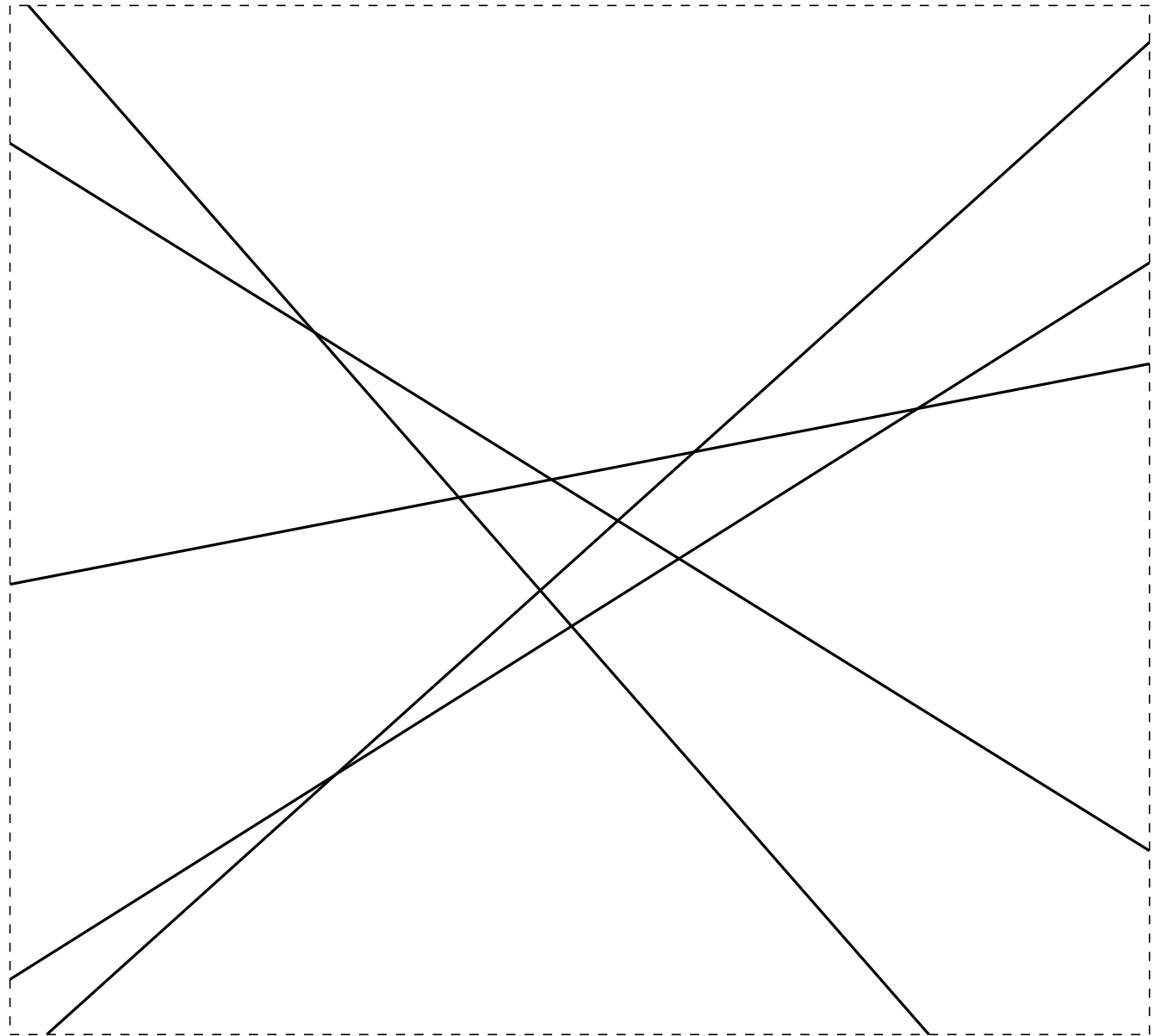
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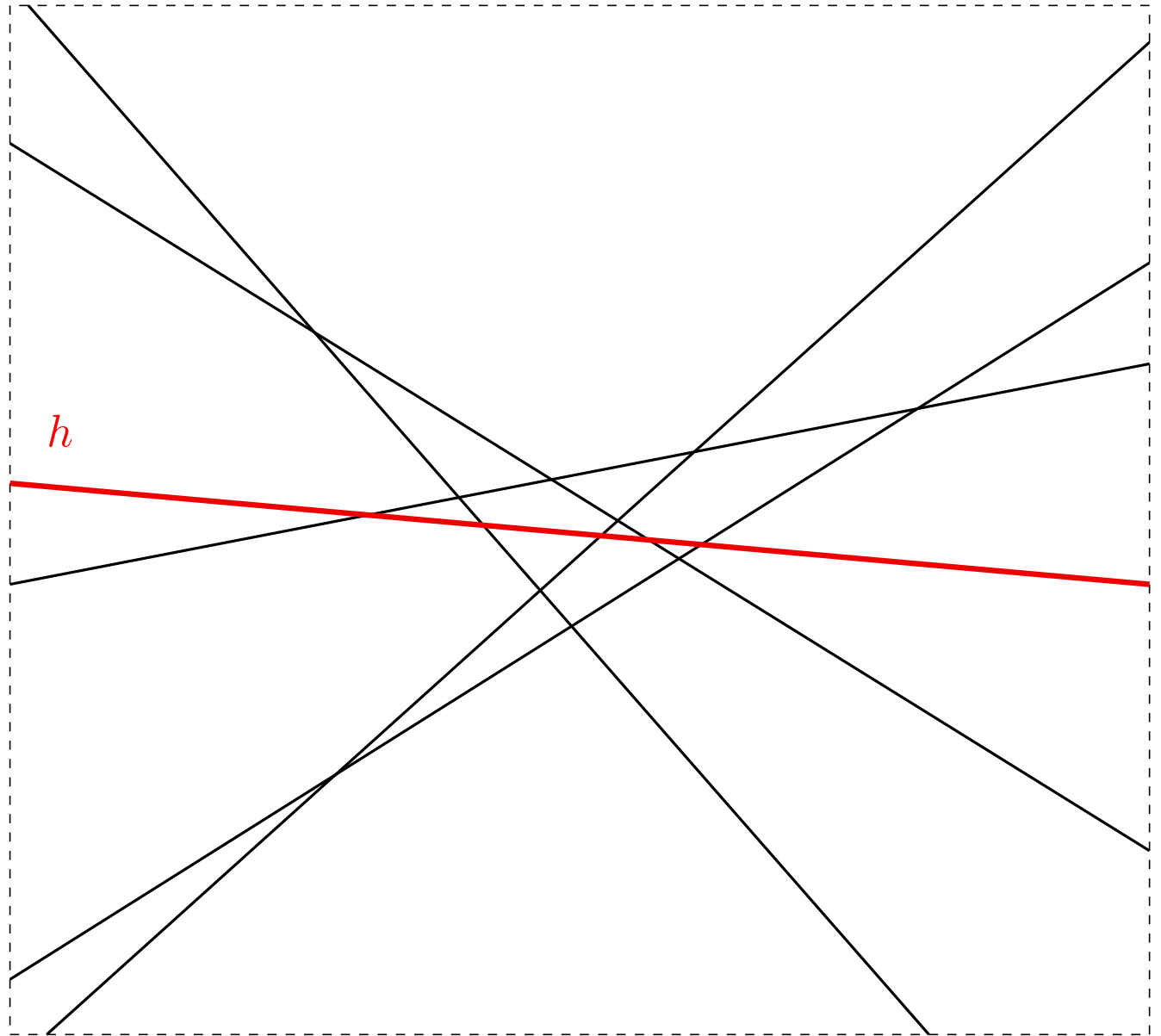
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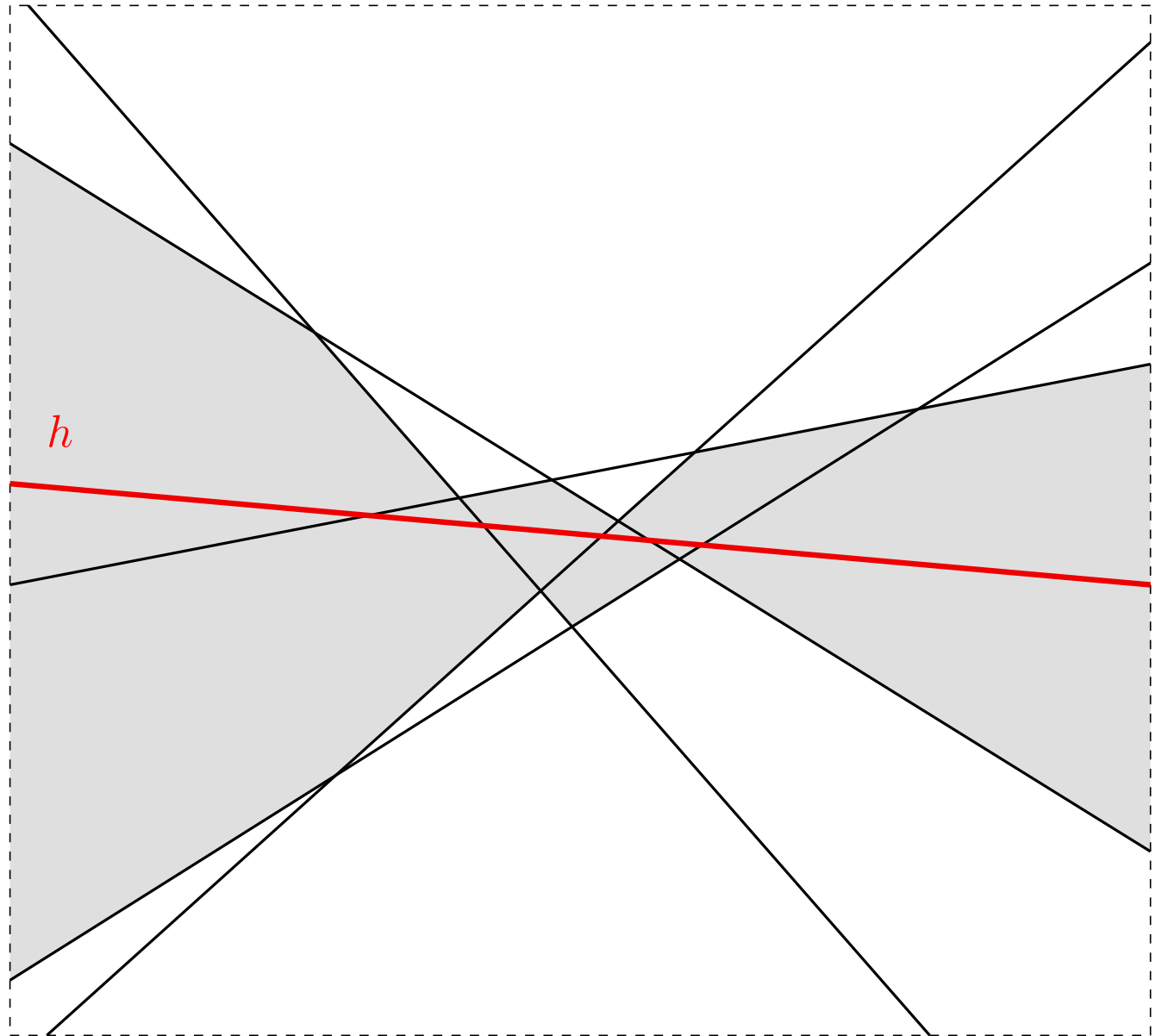
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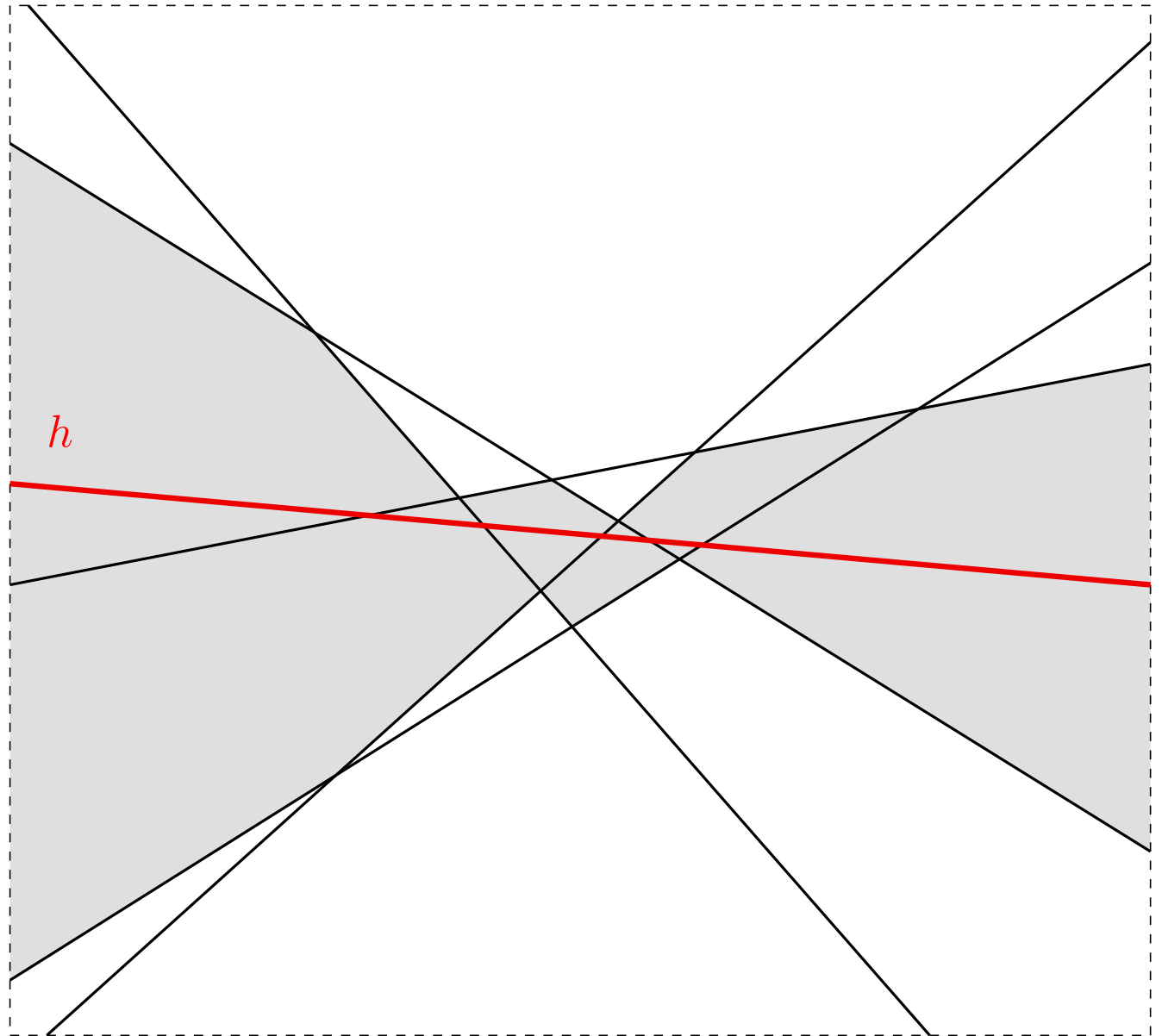


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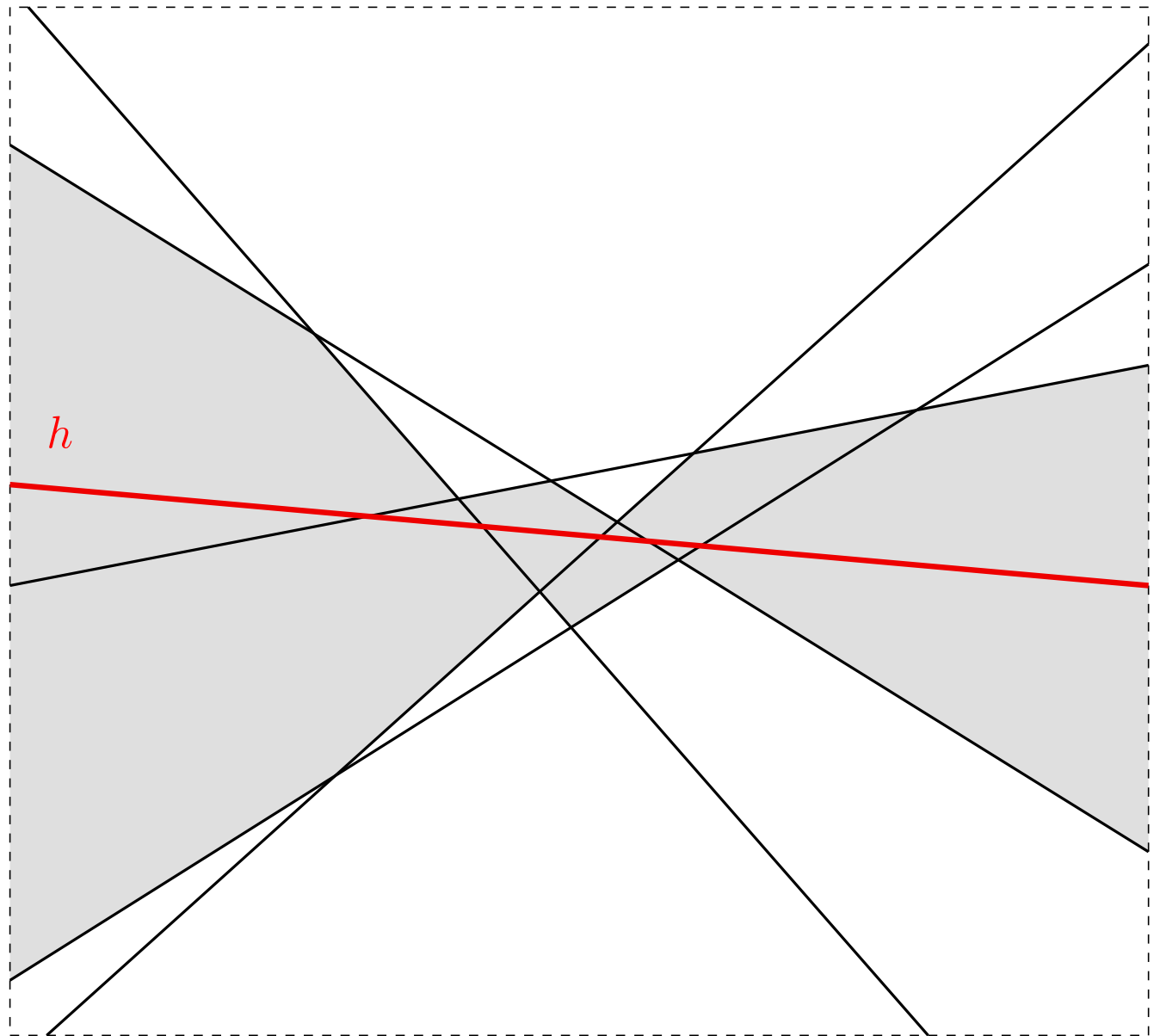
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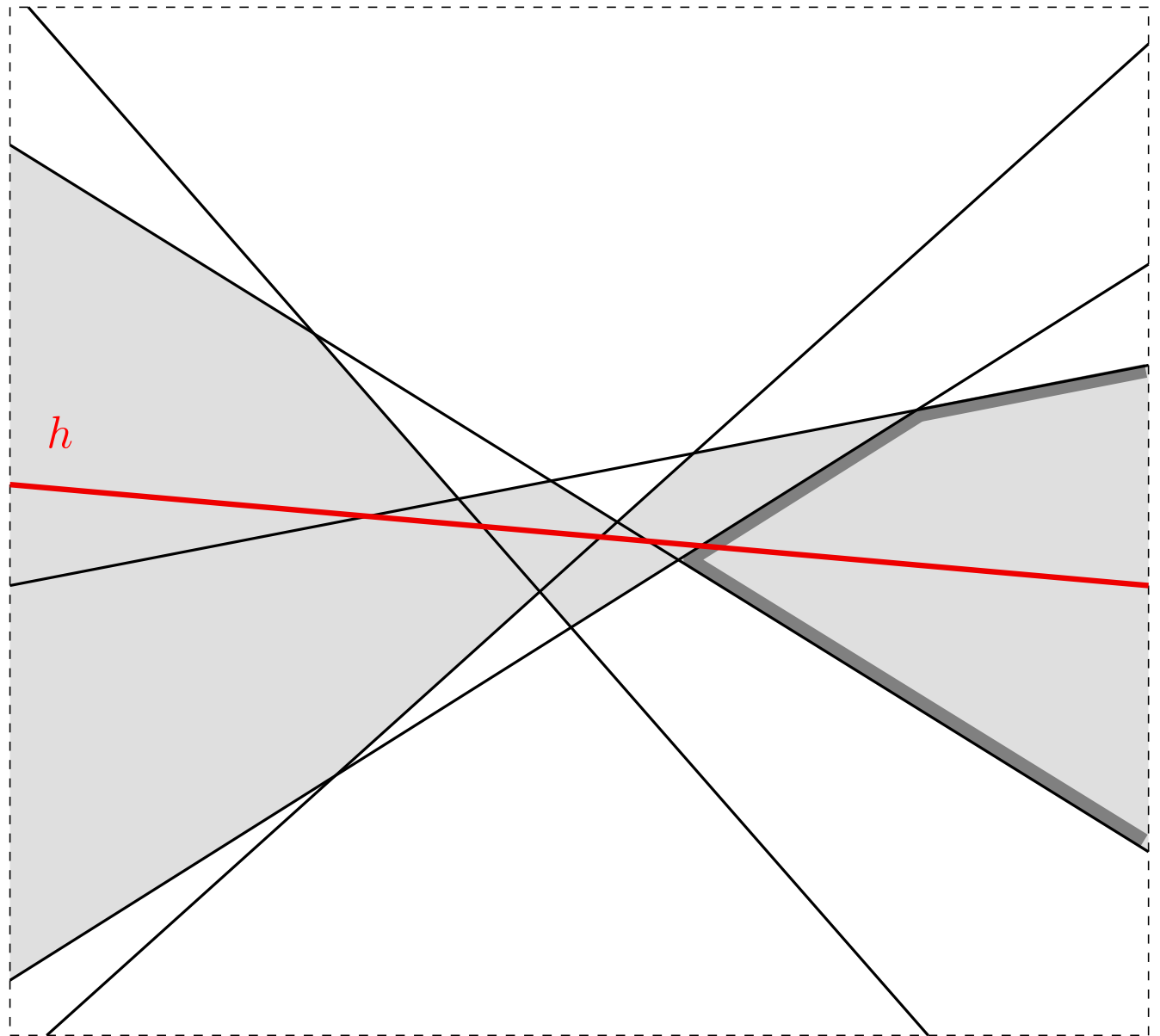
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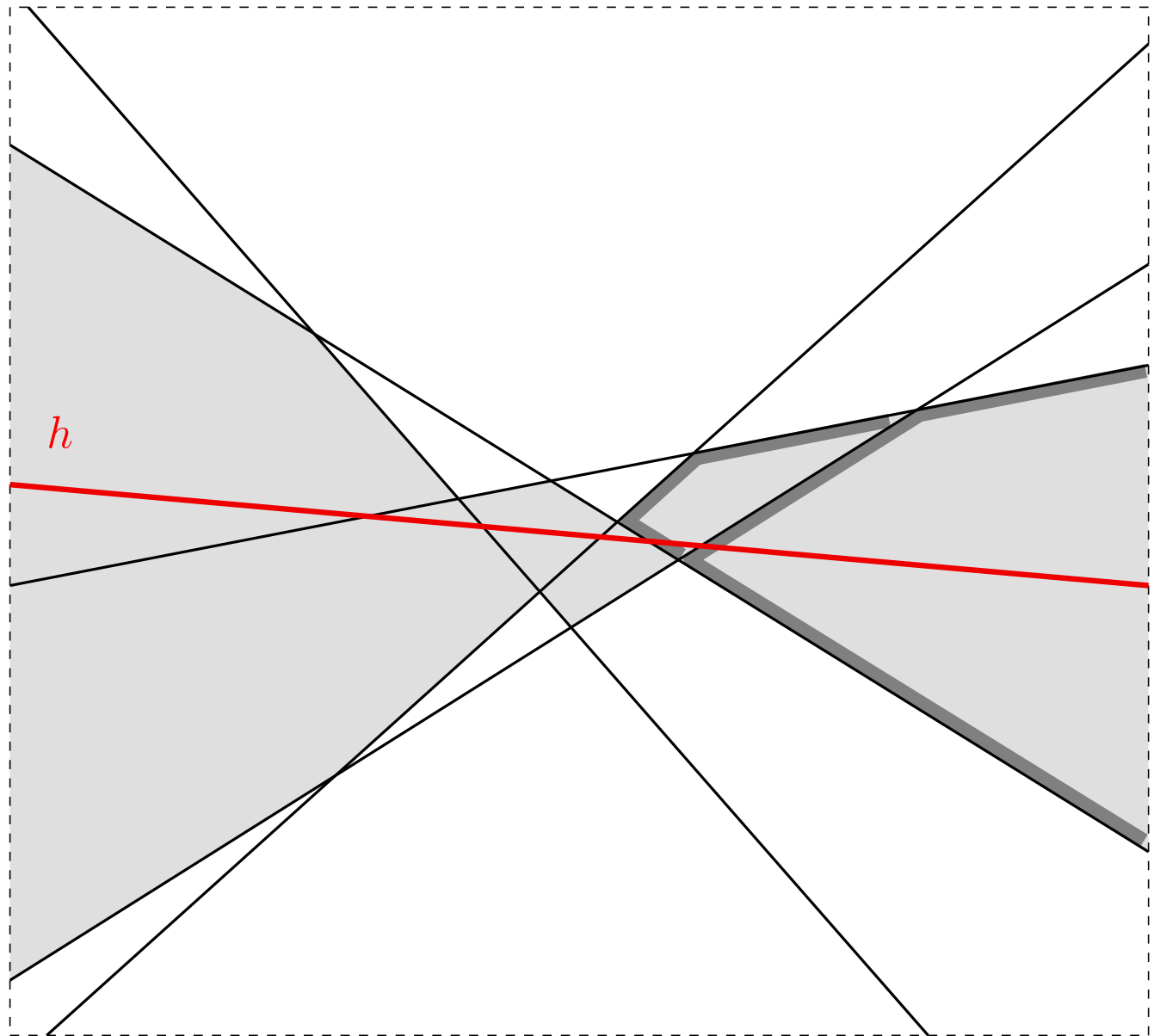
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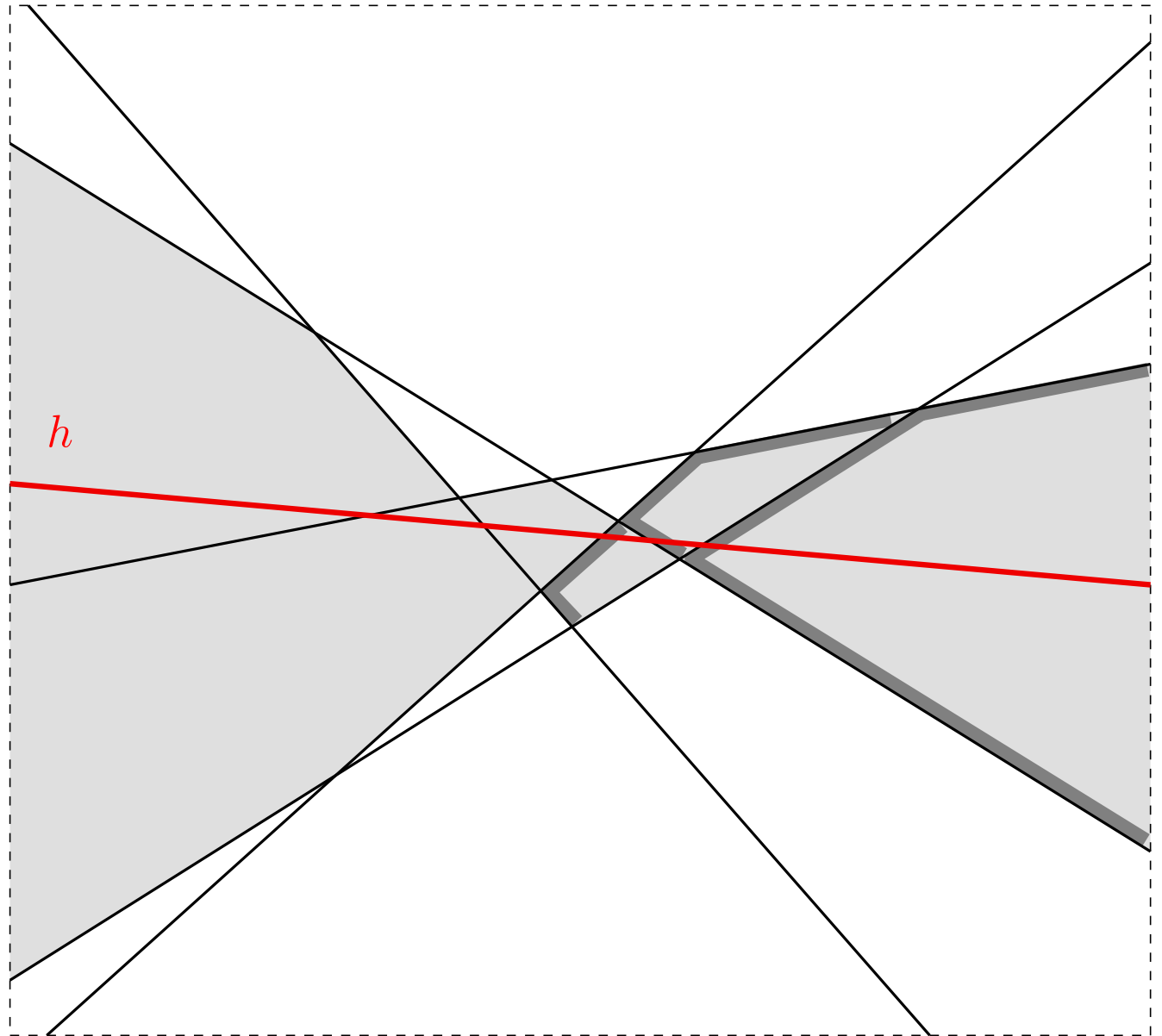
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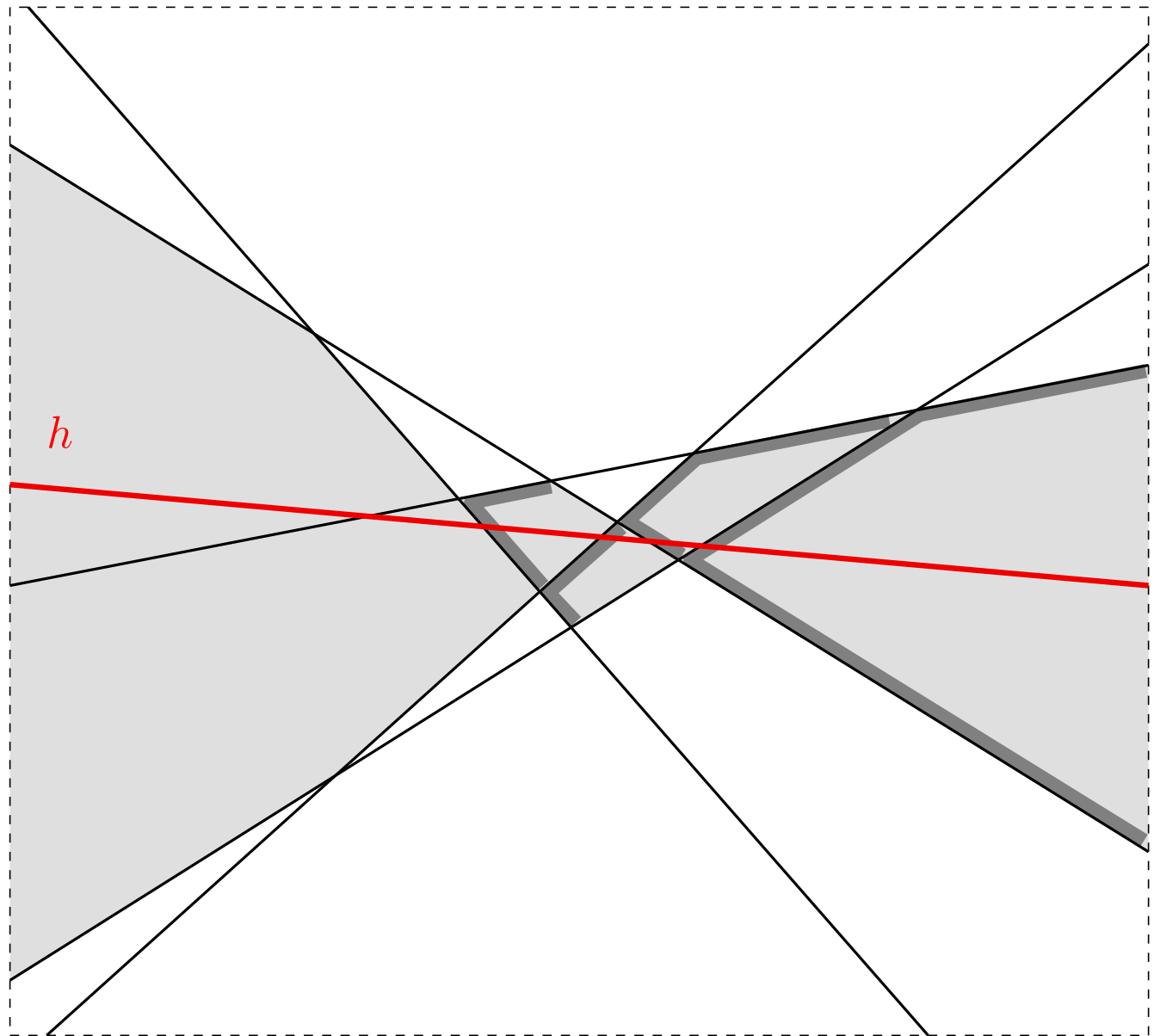
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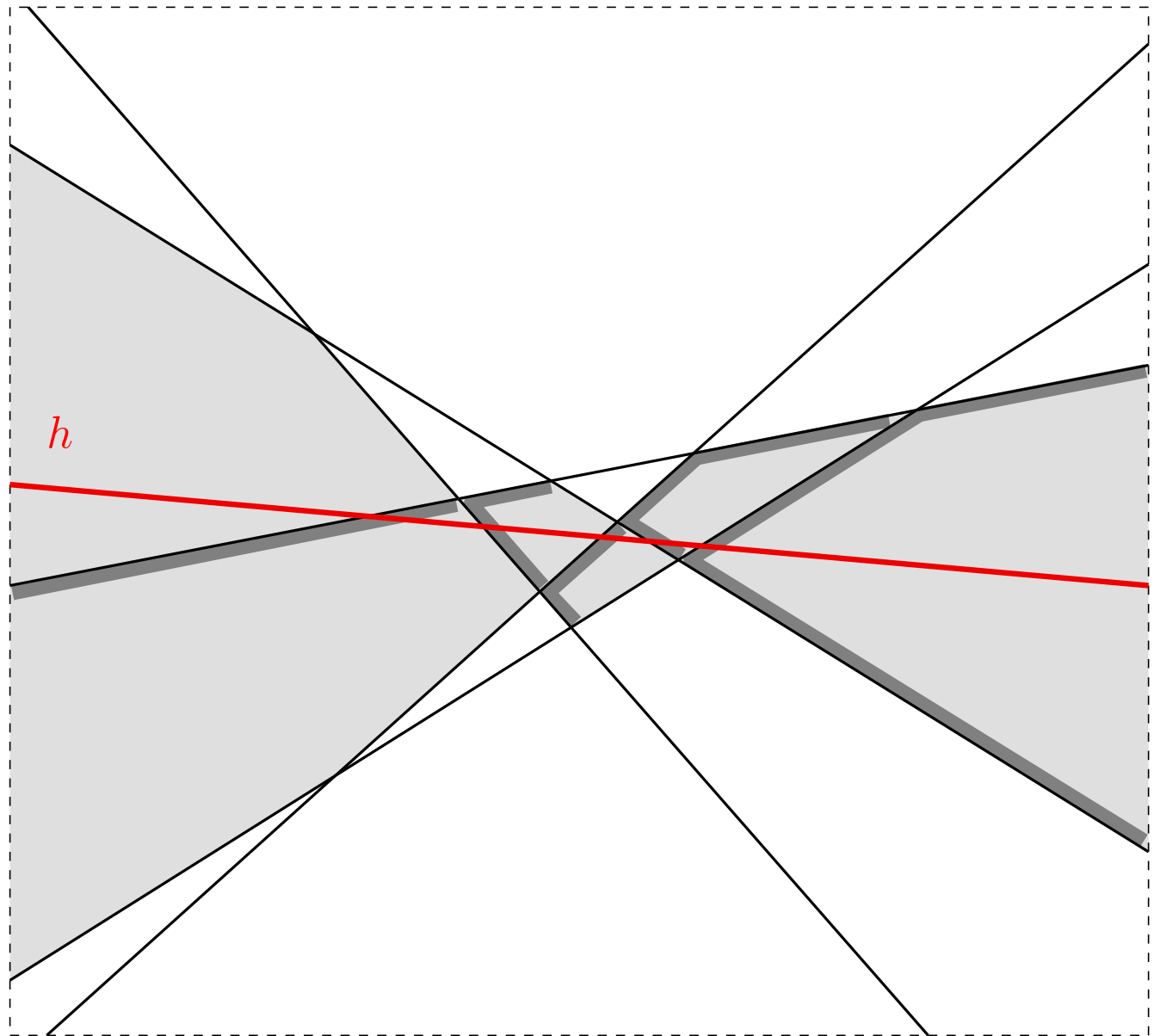
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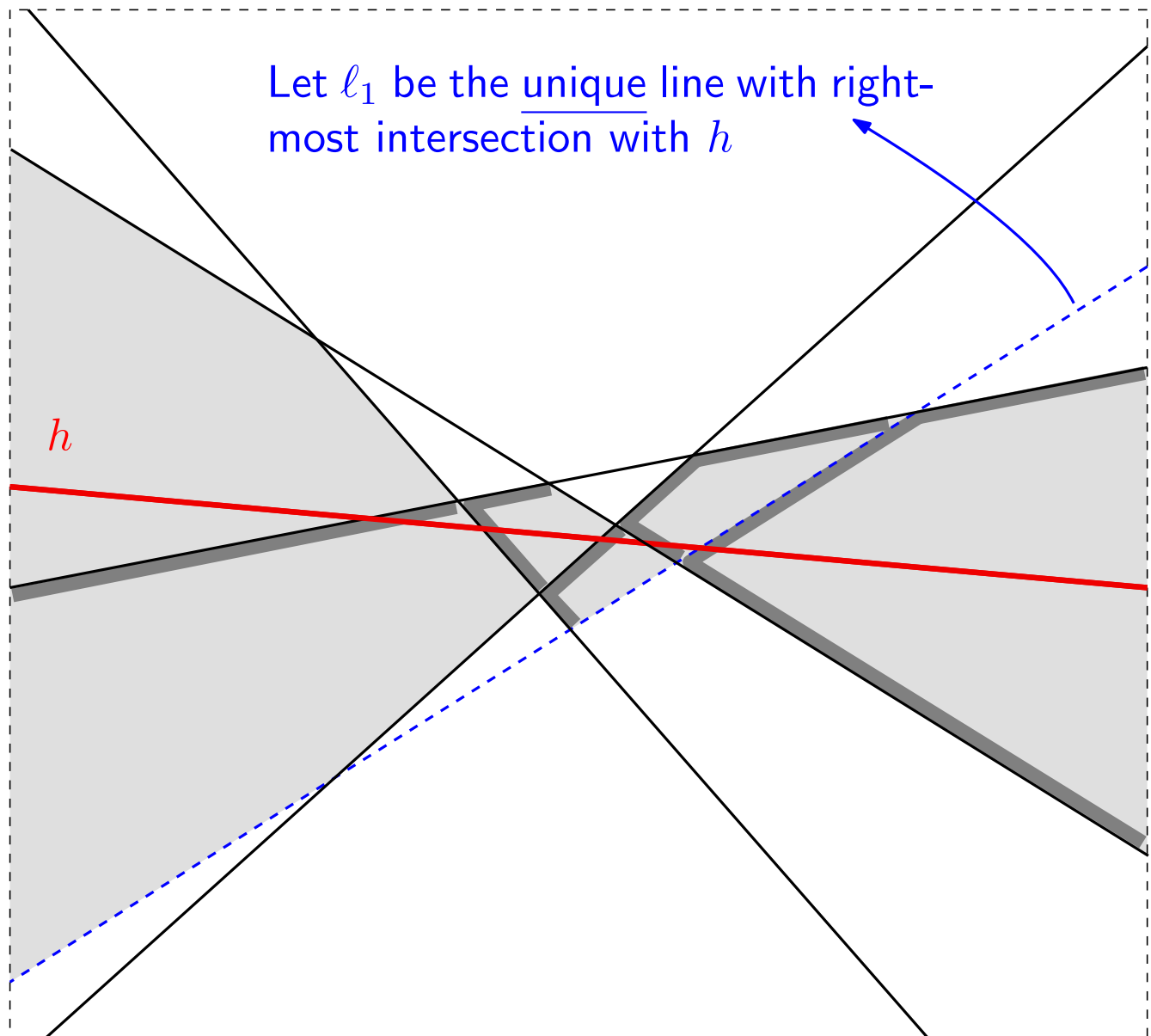
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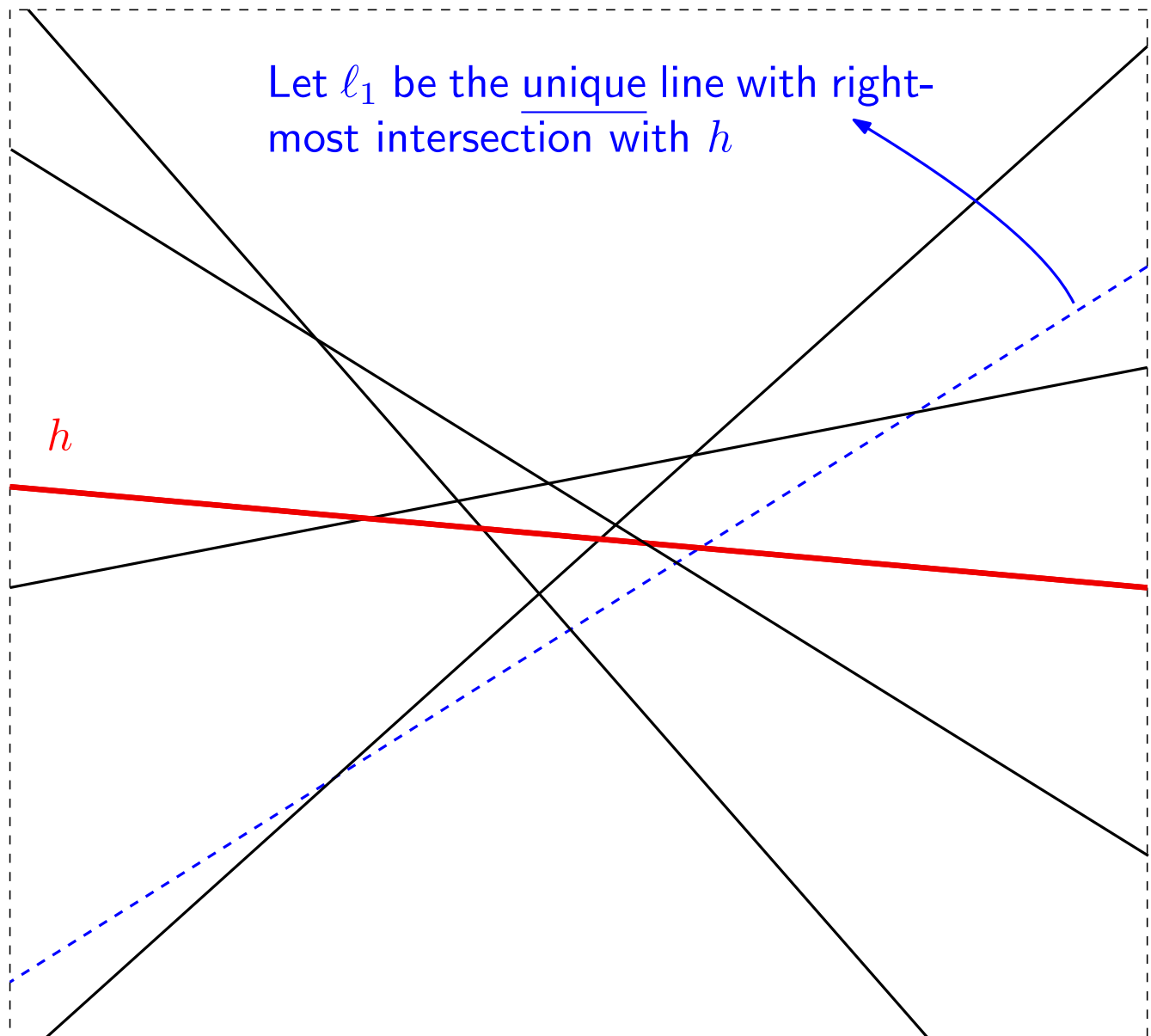
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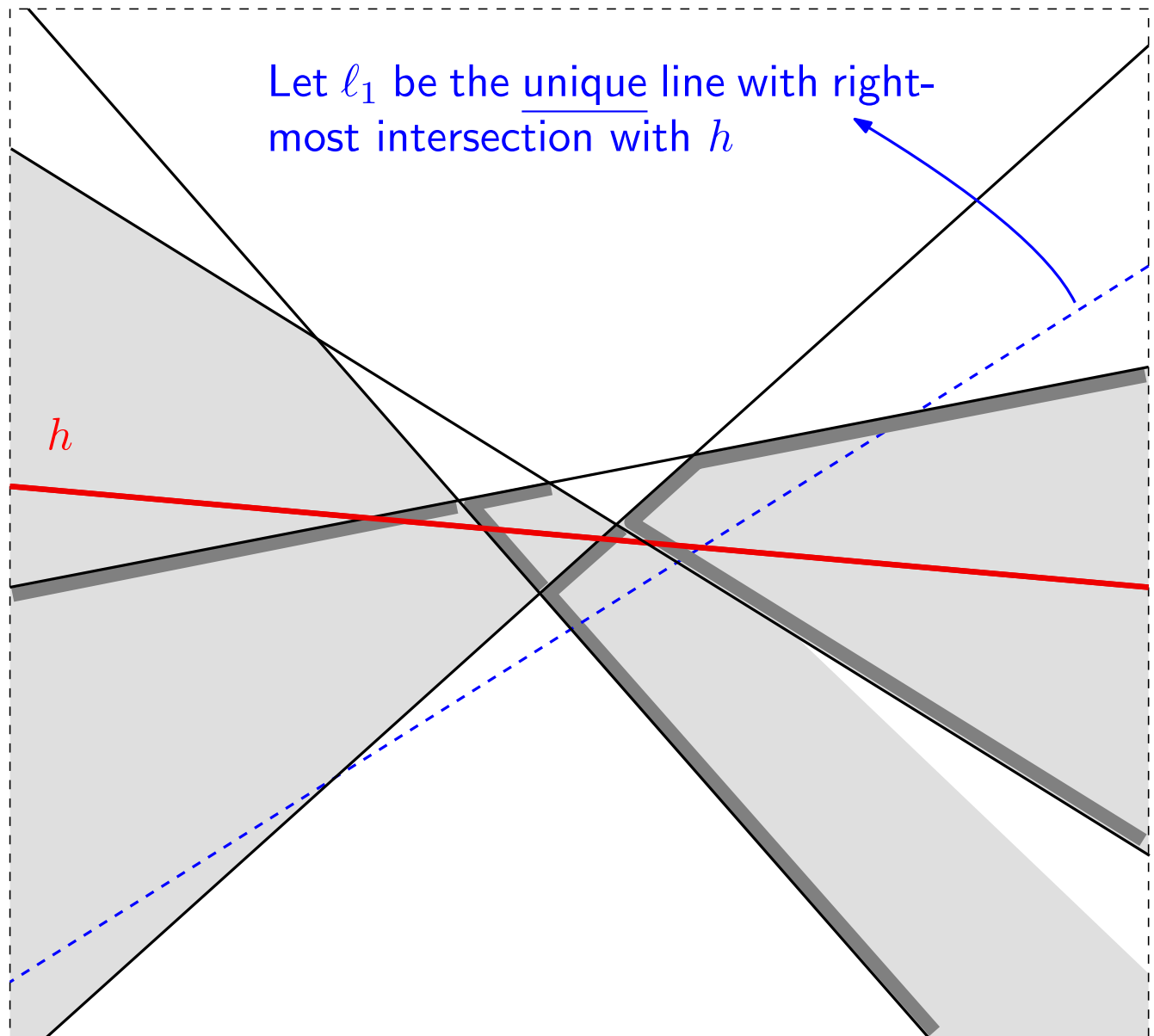
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Zone of h after removing ℓ_1 (we can apply inductive hypothesis)

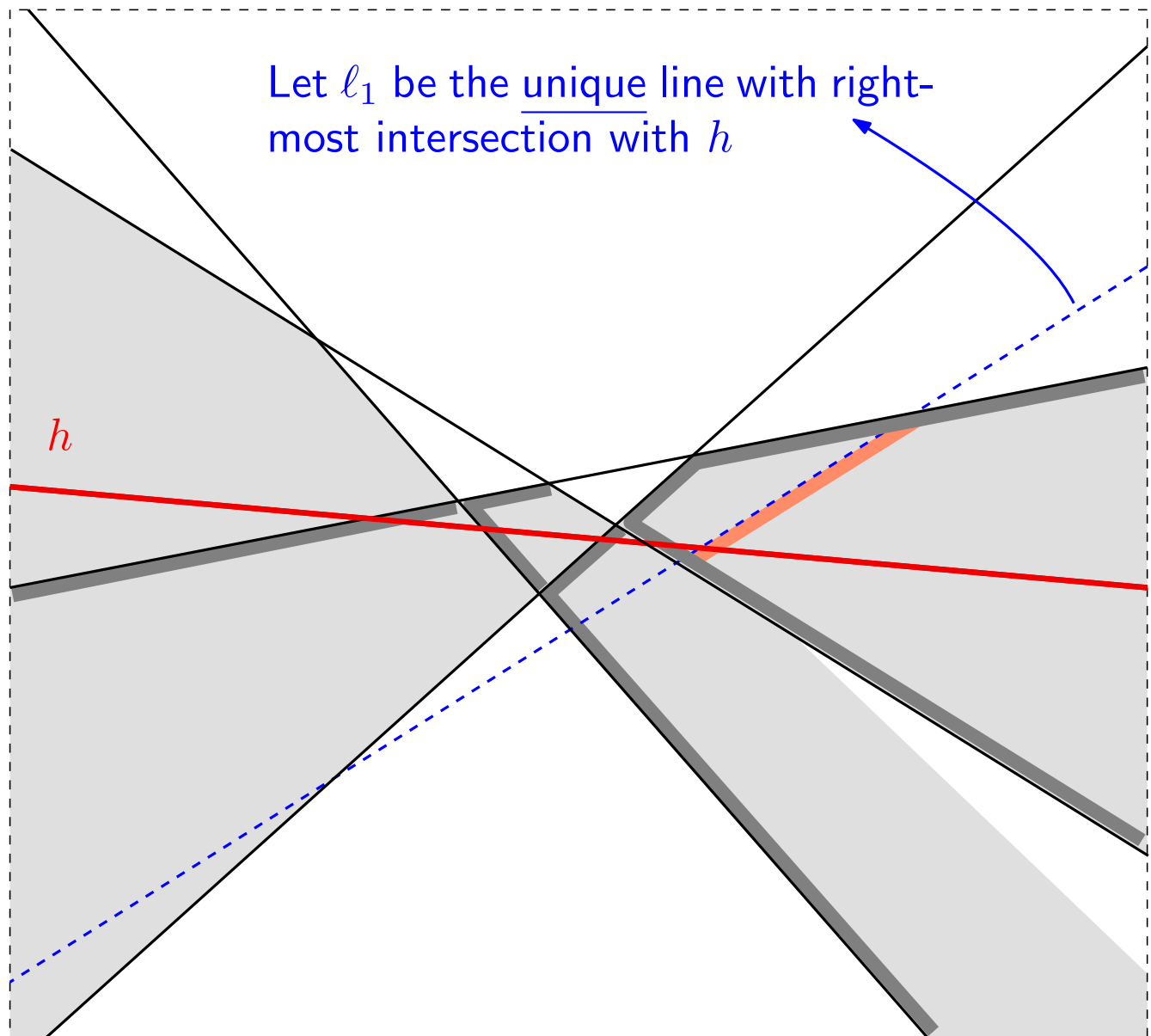
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I.H.: we have a total of $5(m - 1)$ left edges. Now we add ℓ_1 .

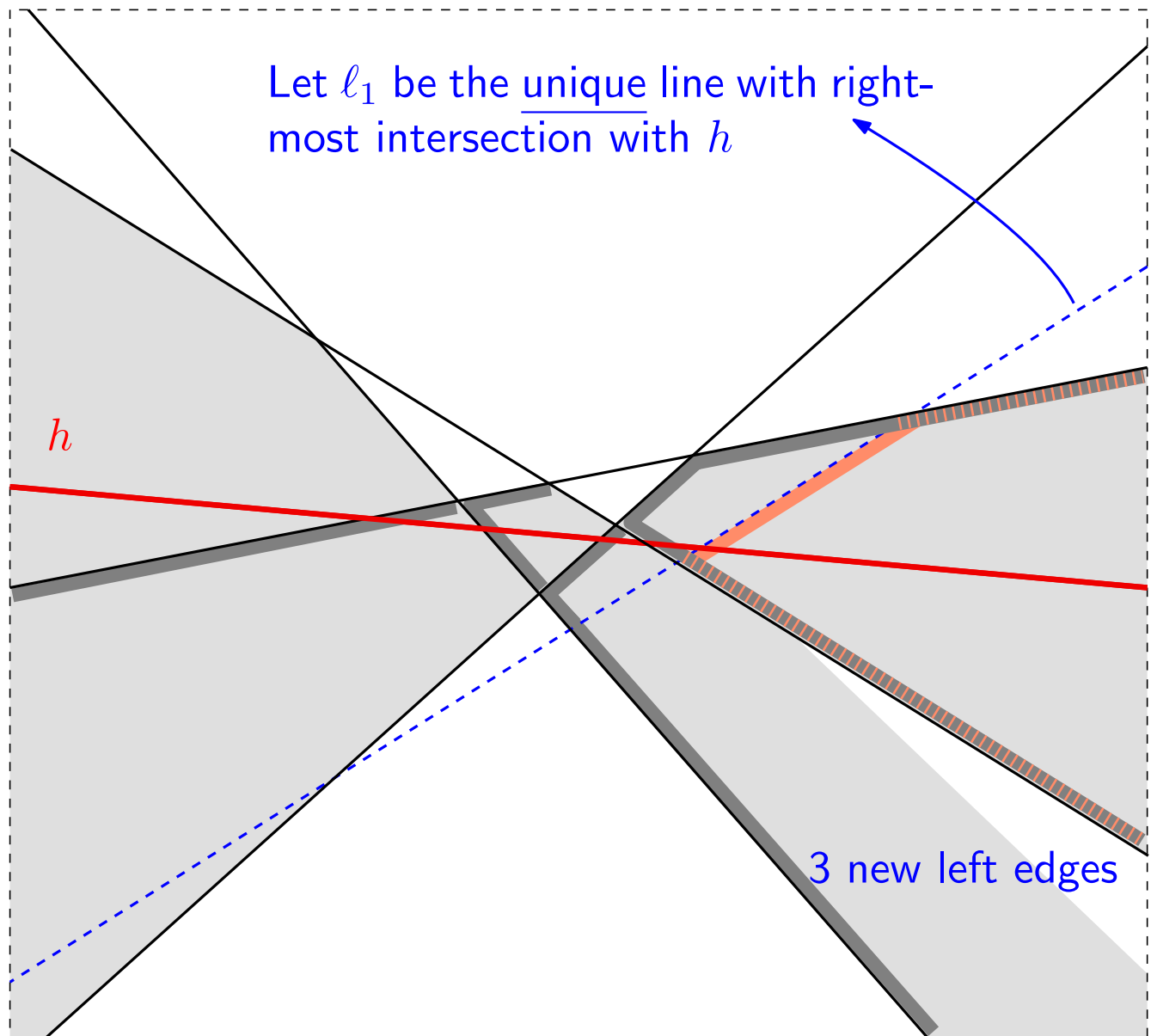
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Then in total we have $5(m - 1) + 3 \leq 5m$ left edges

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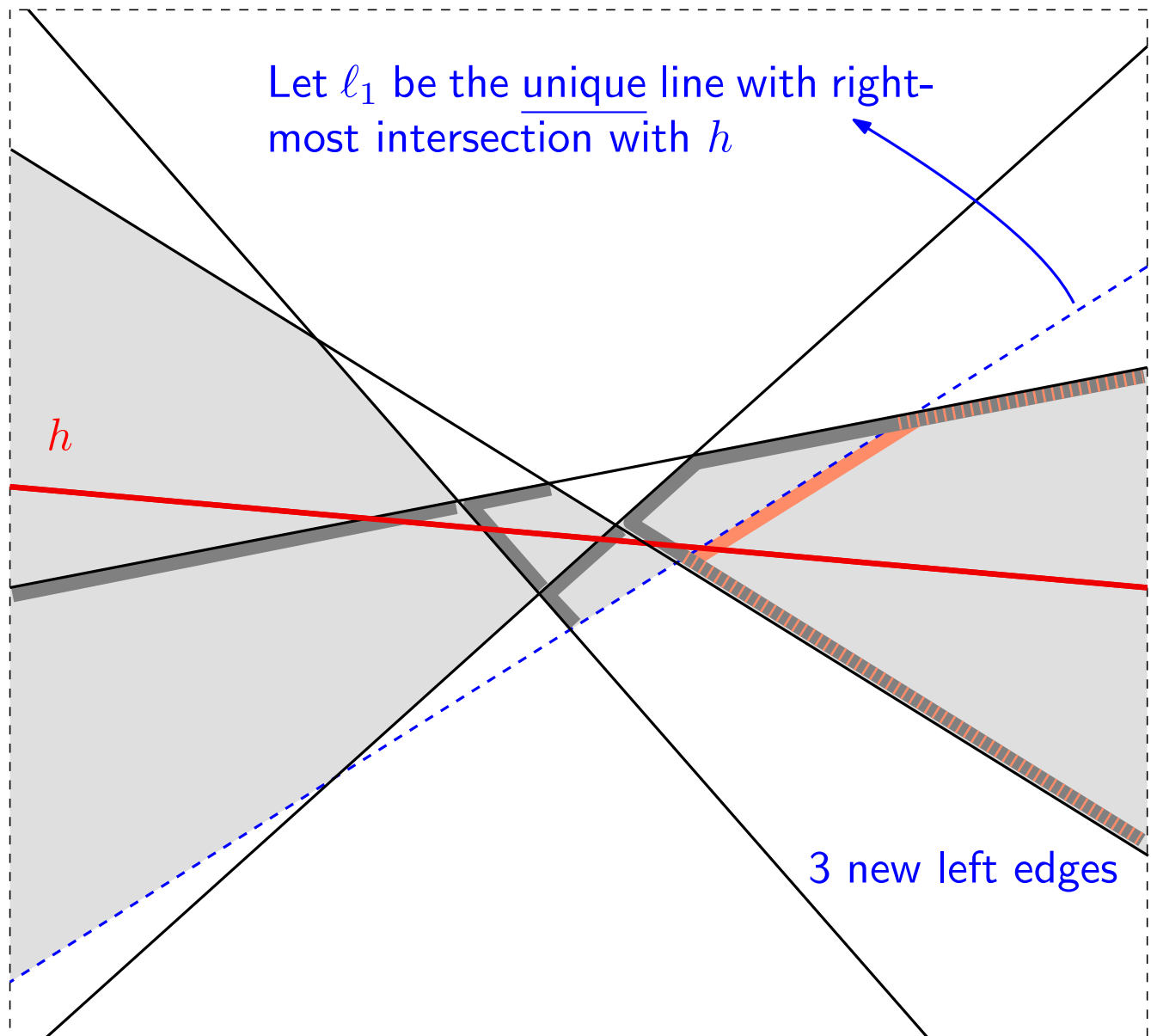
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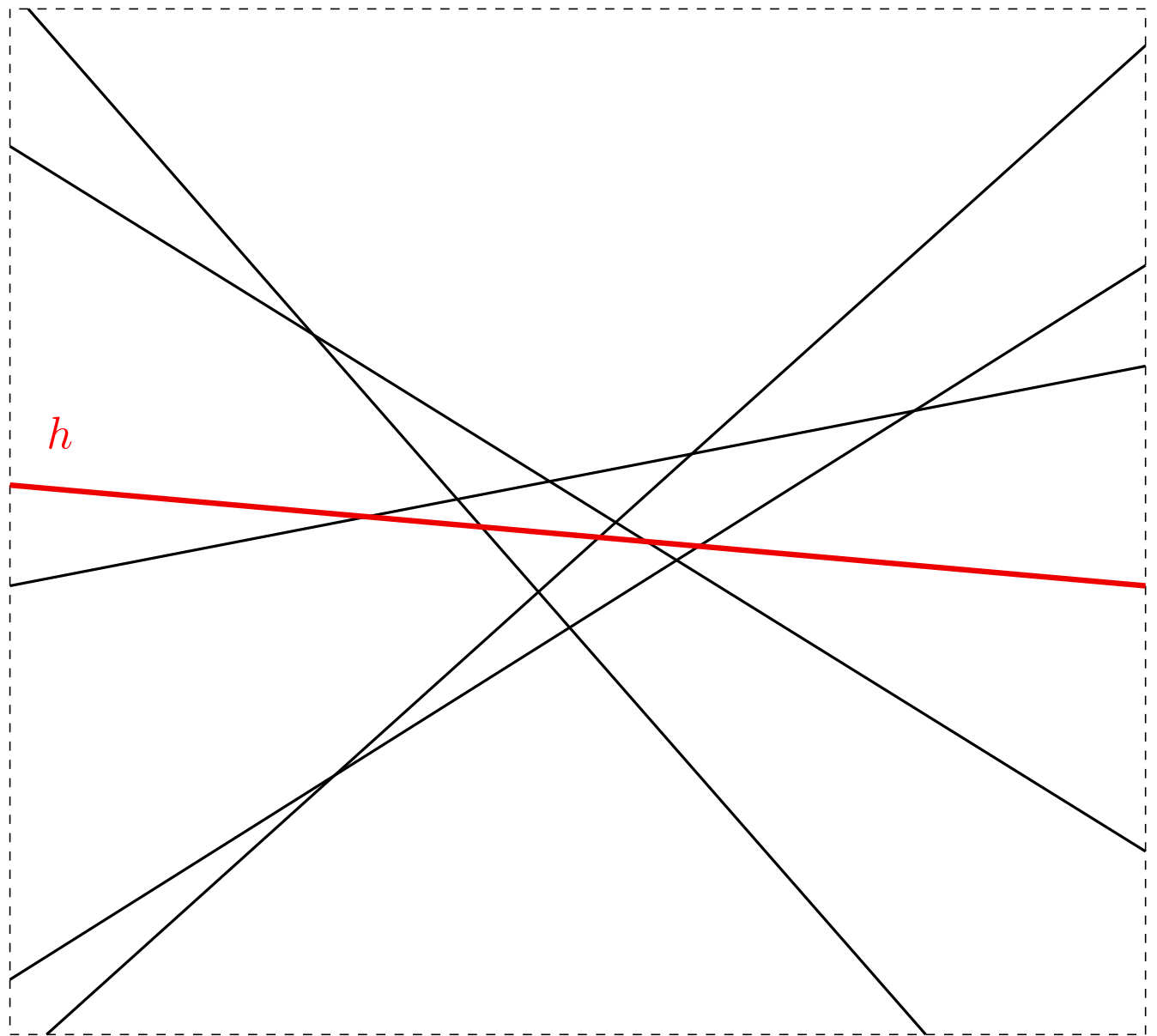
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(It is important to consider the rightmost intersecting line).



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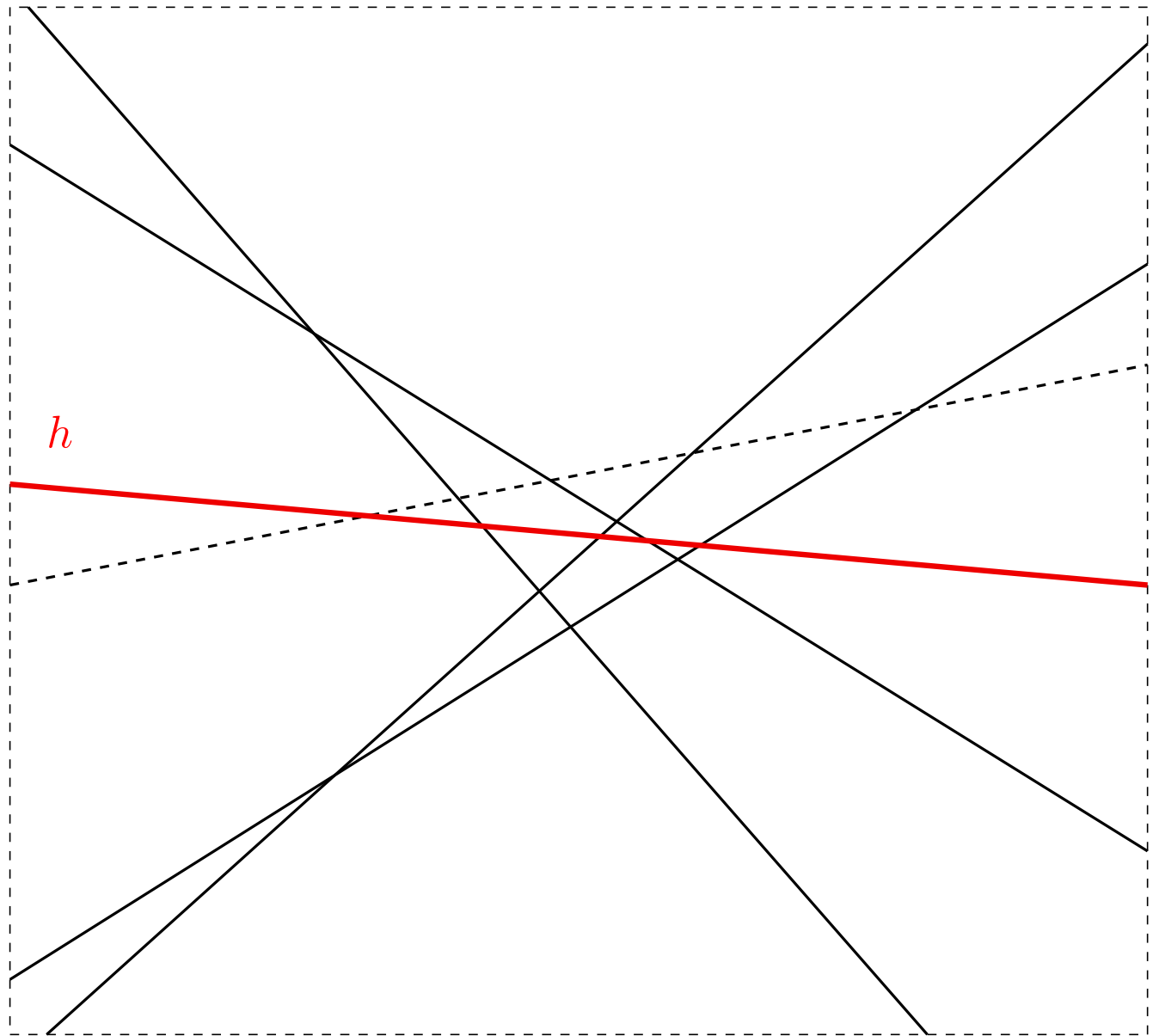
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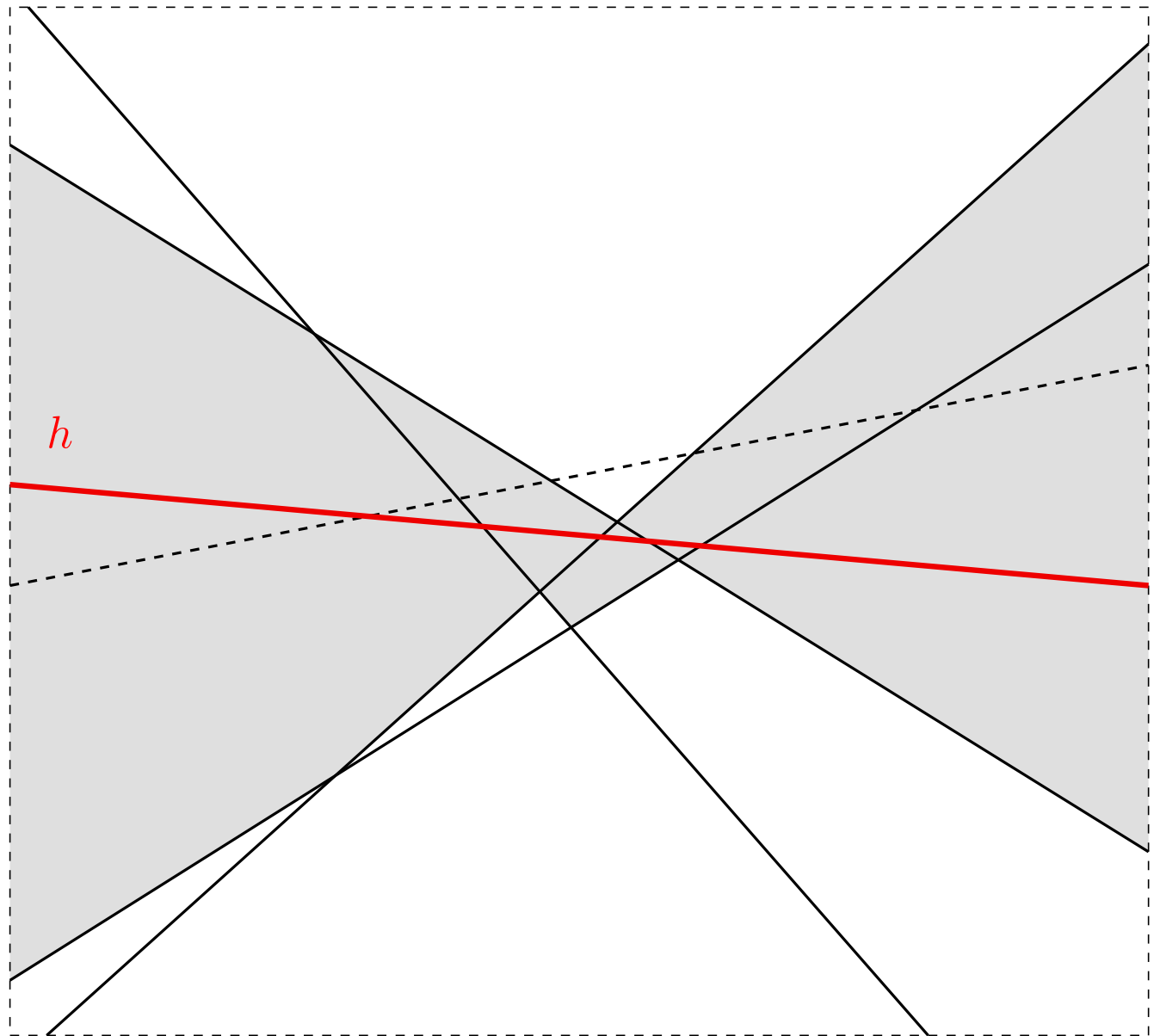
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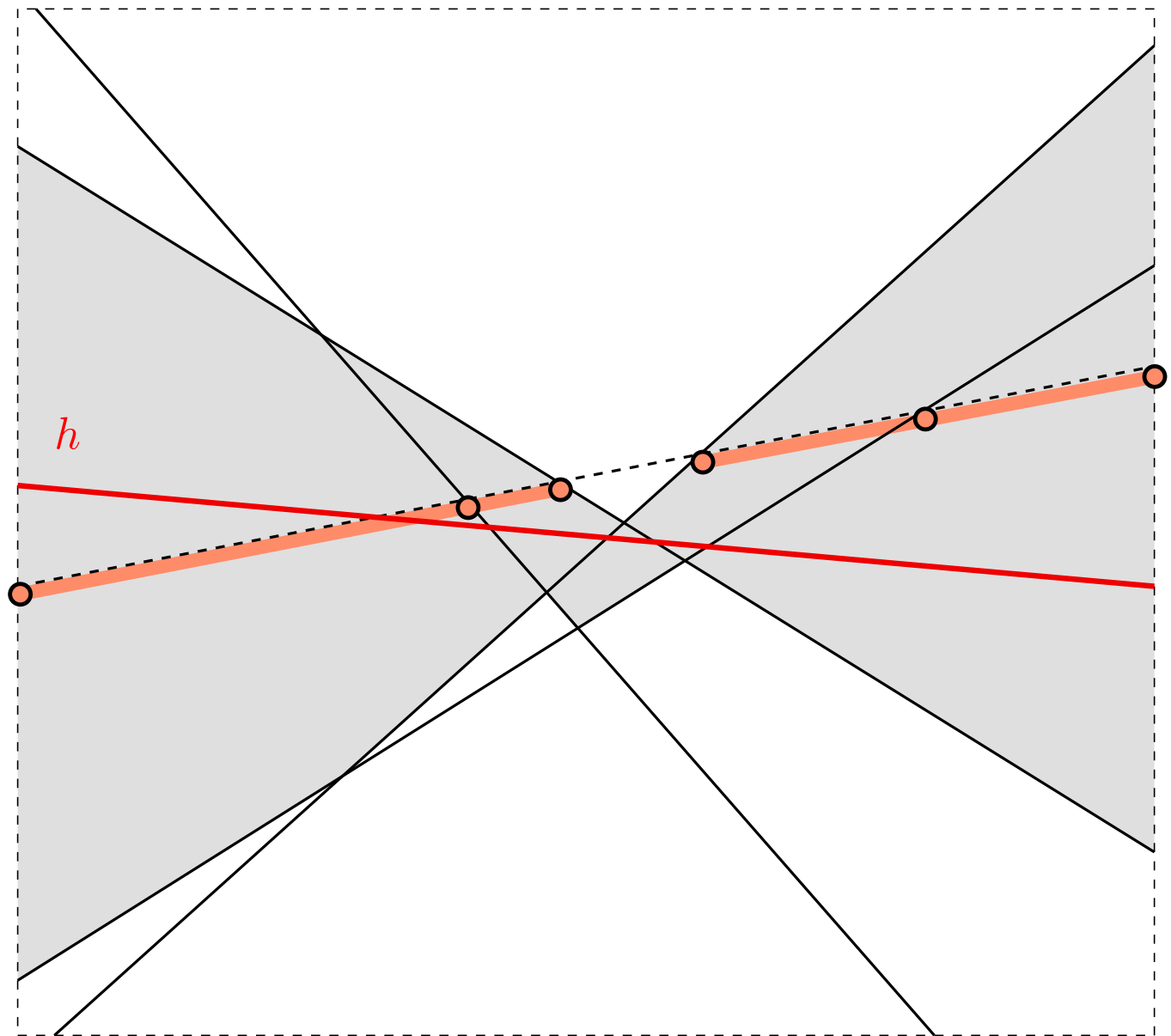
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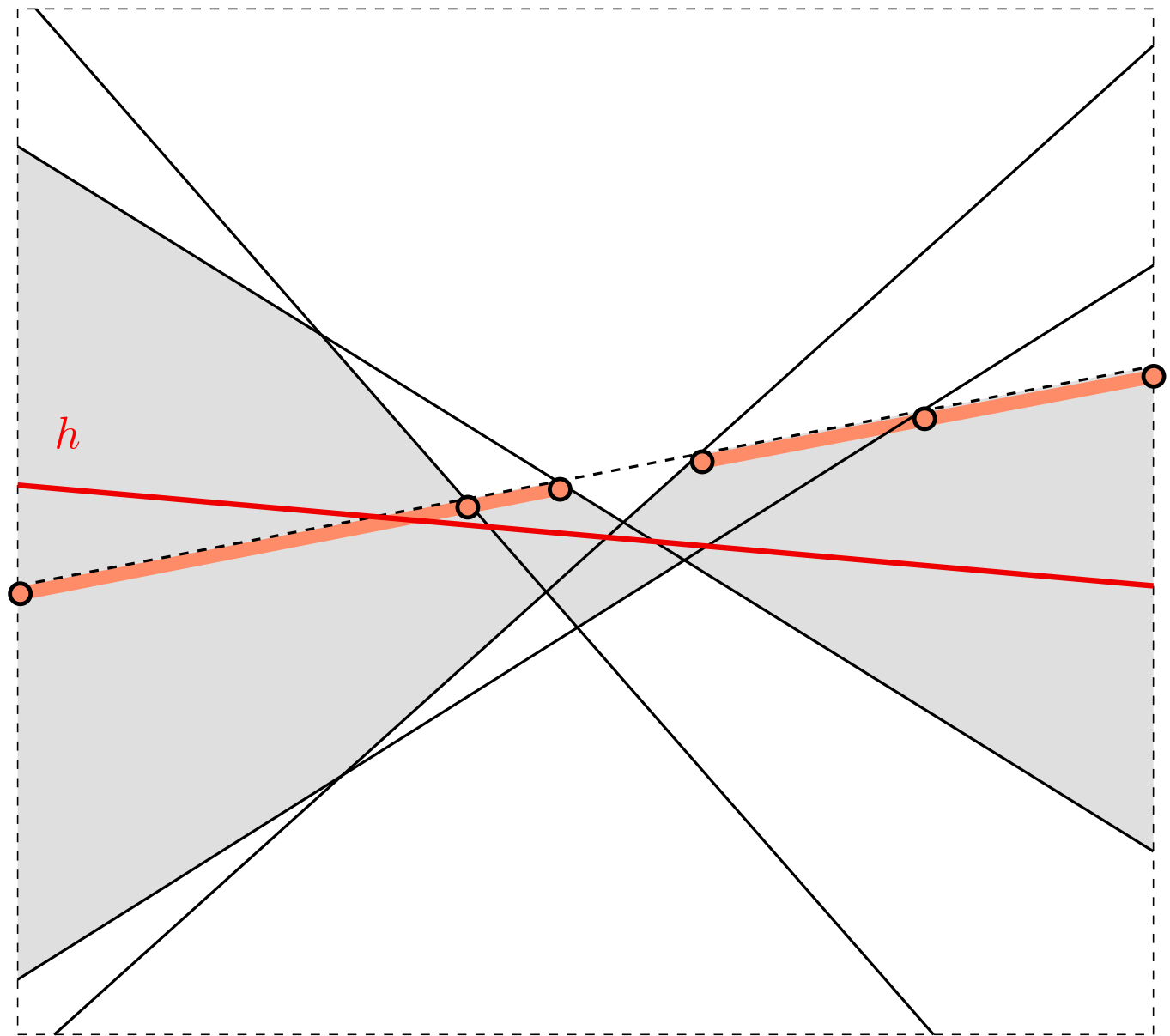
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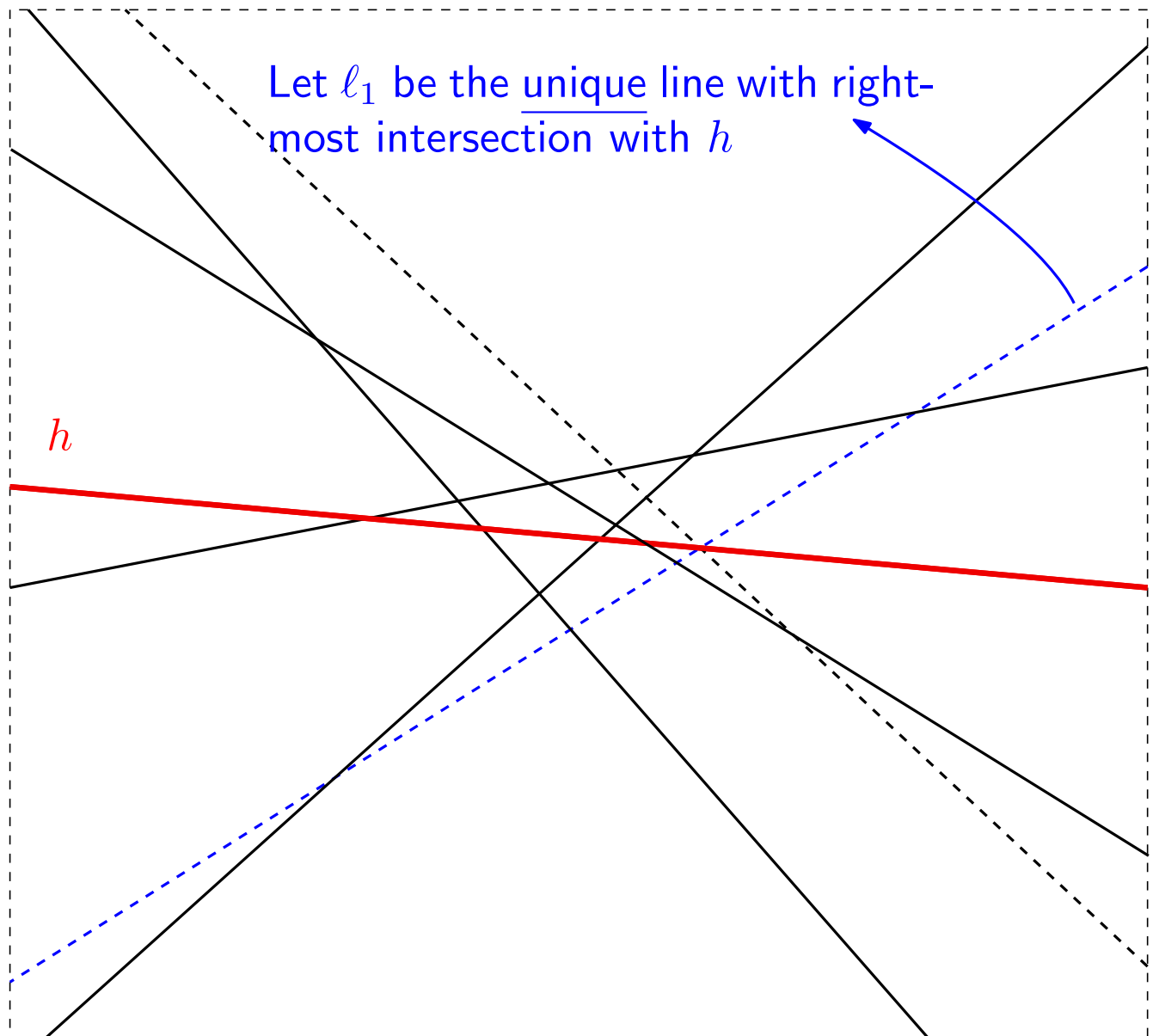
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Before we were assuming ℓ_1 was unique! If not, up to 5 new edges

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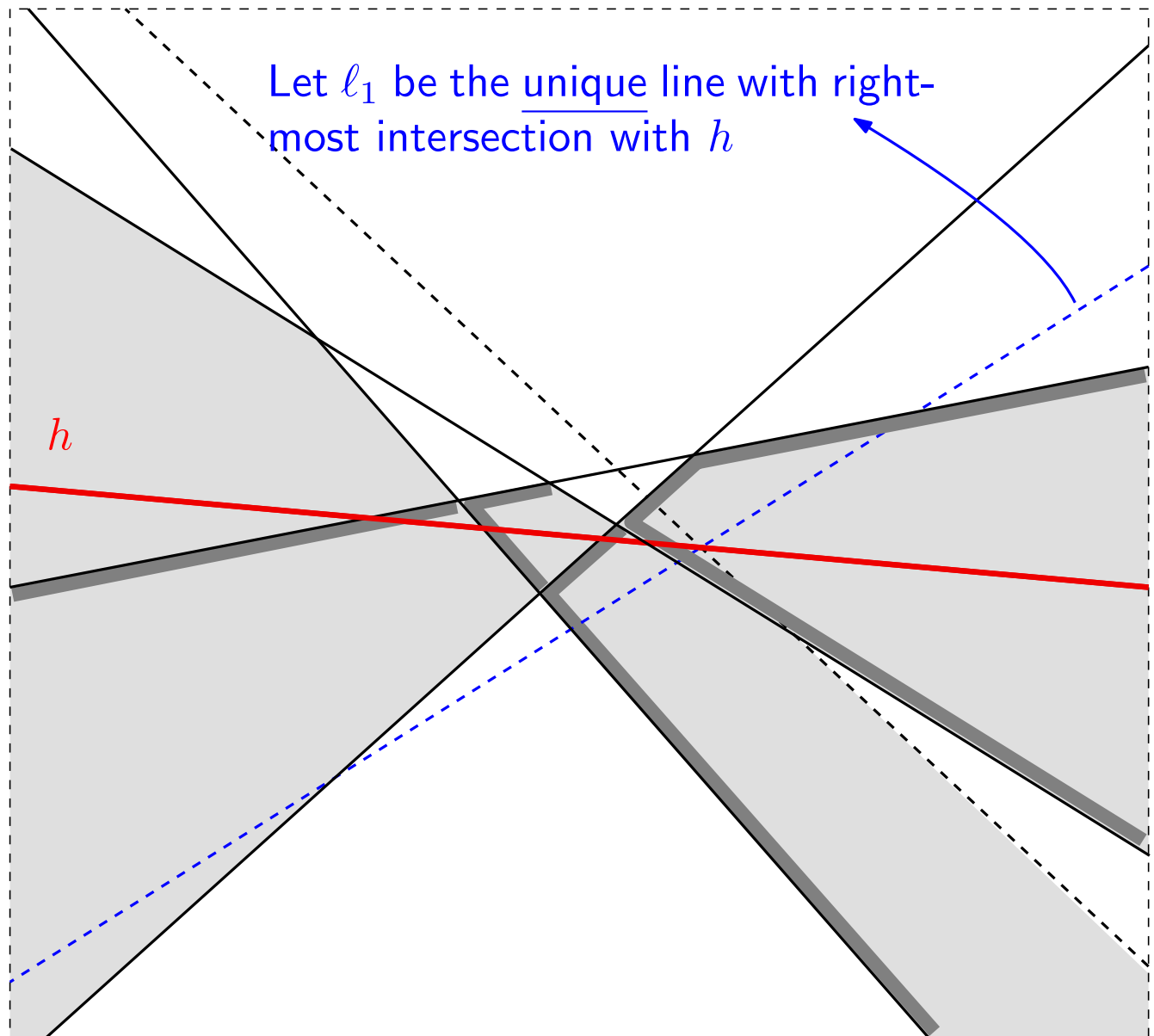
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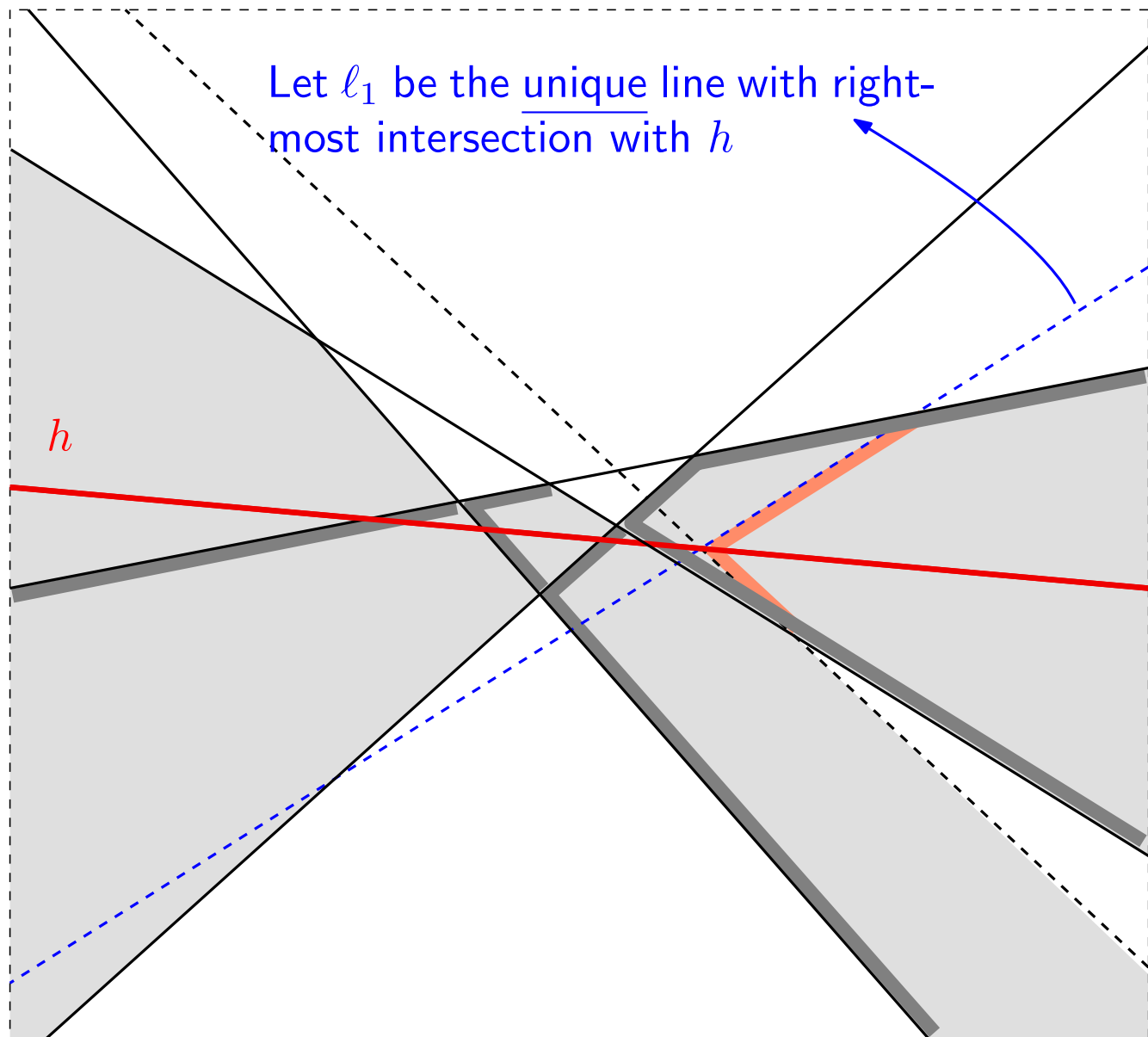
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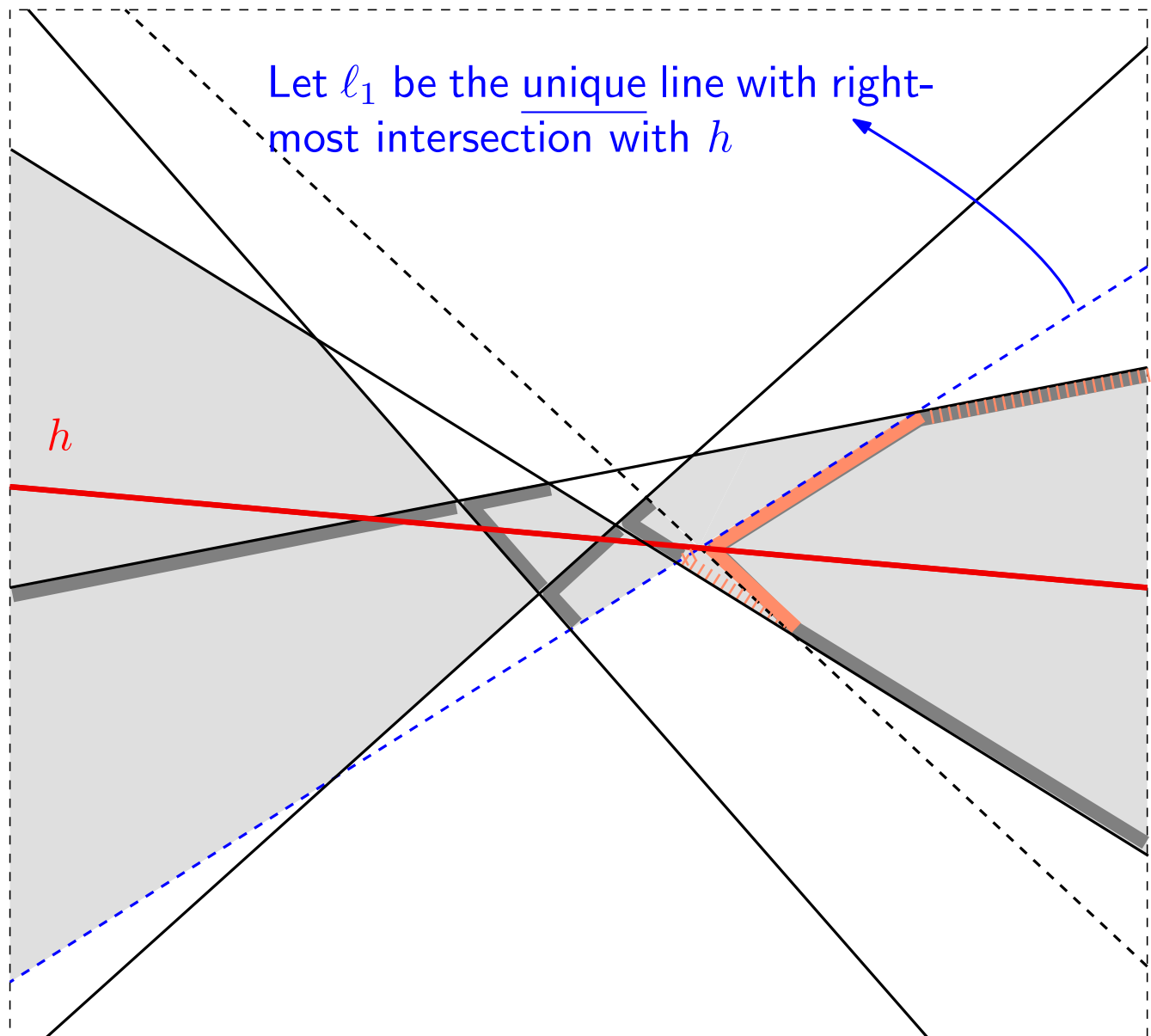
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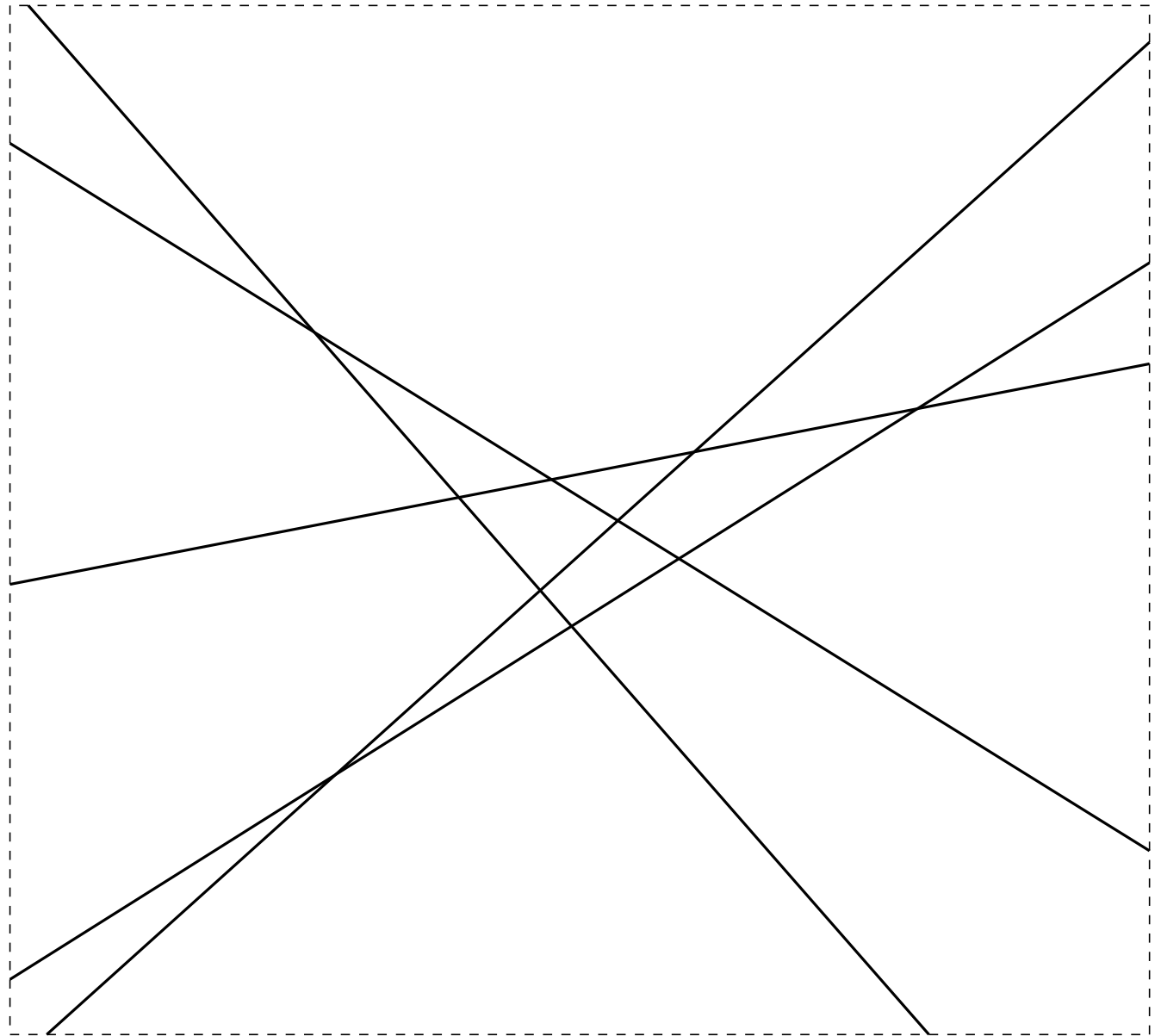
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Definition: The k th *level* of an arrangement $\mathcal{A}(L)$ without vertical lines is the set of points p of the plane such that:

- The number of lines above p is $\leq k - 1$.
- The number of lines below p is $\leq n - k$.

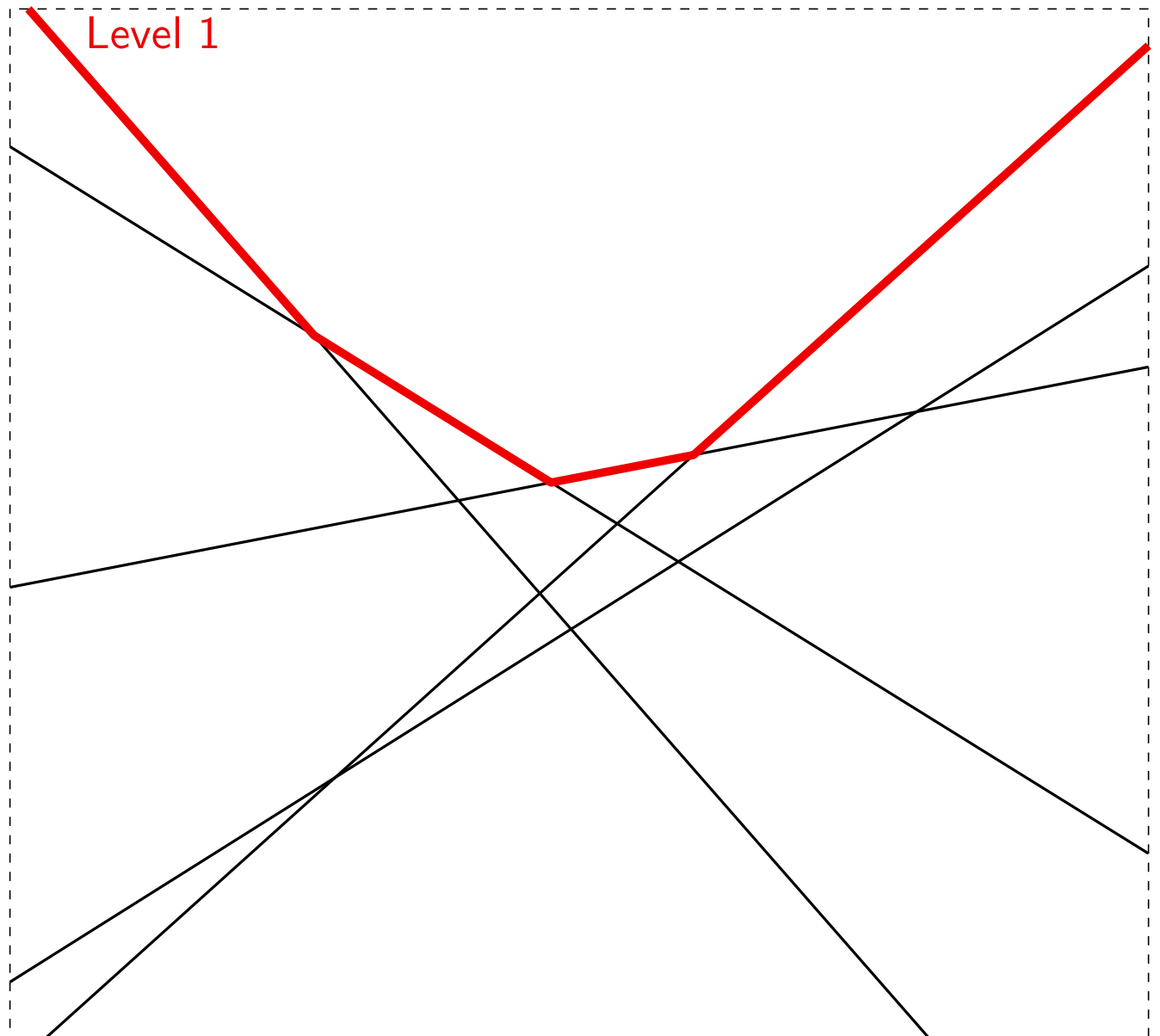


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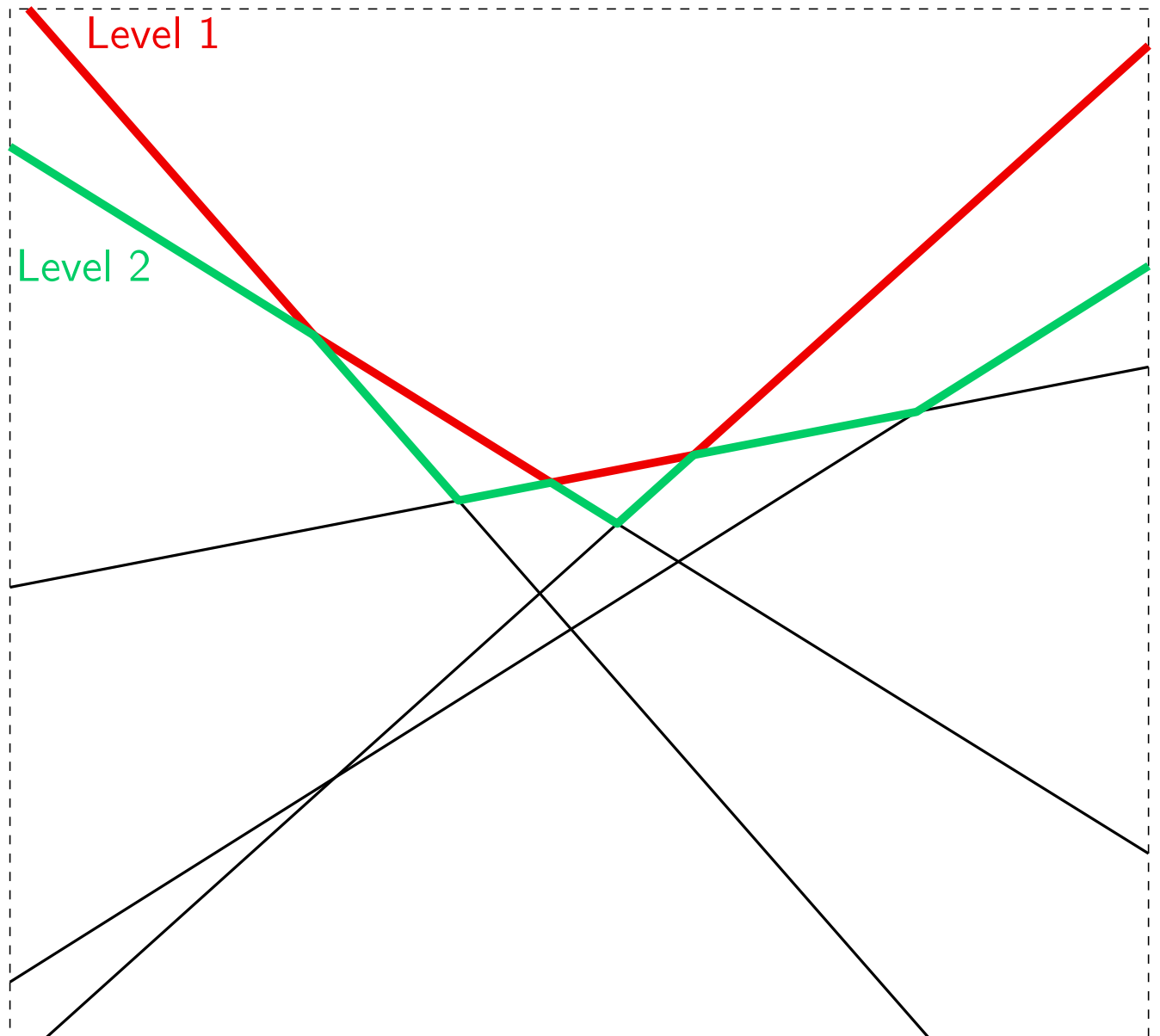


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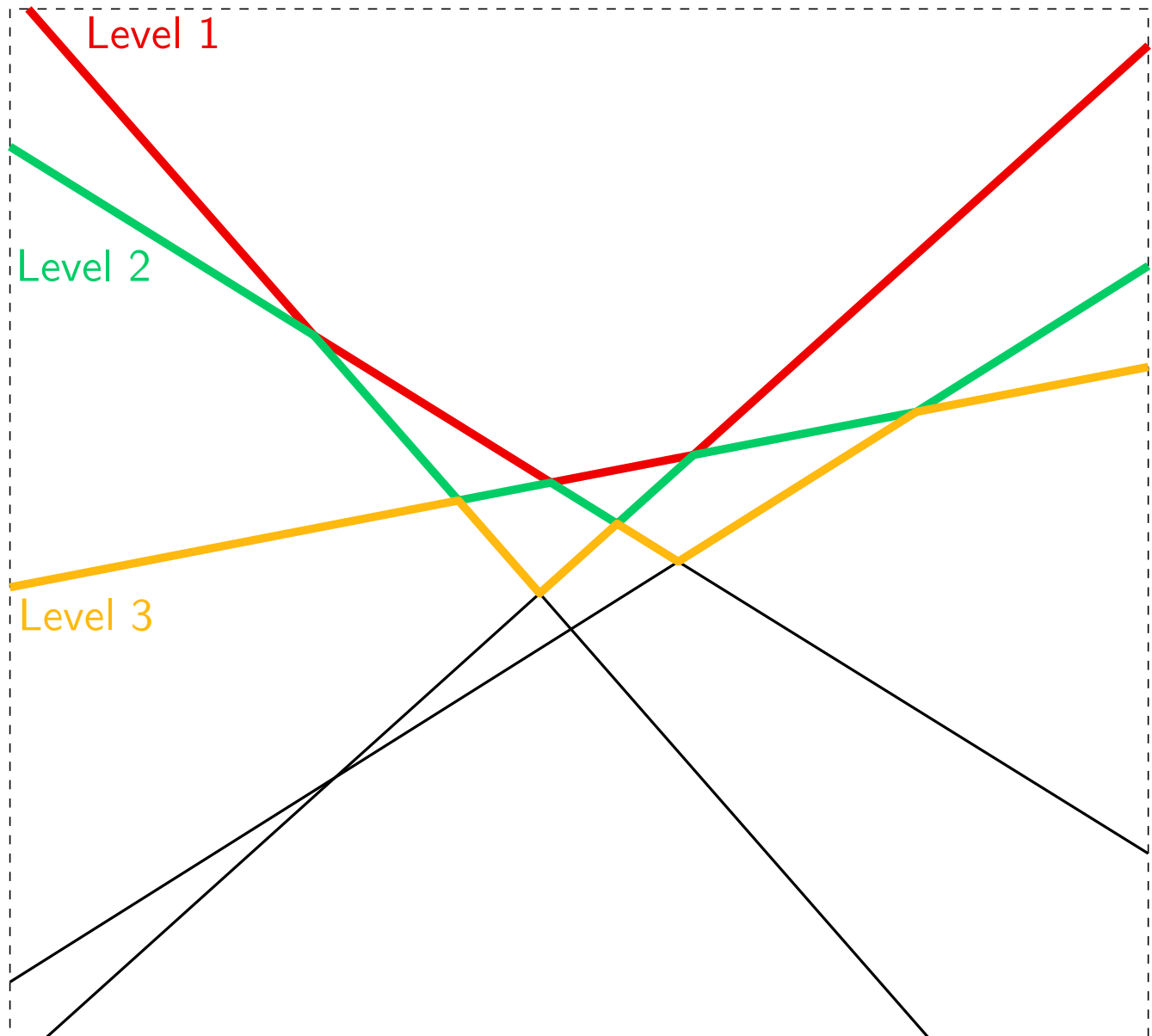


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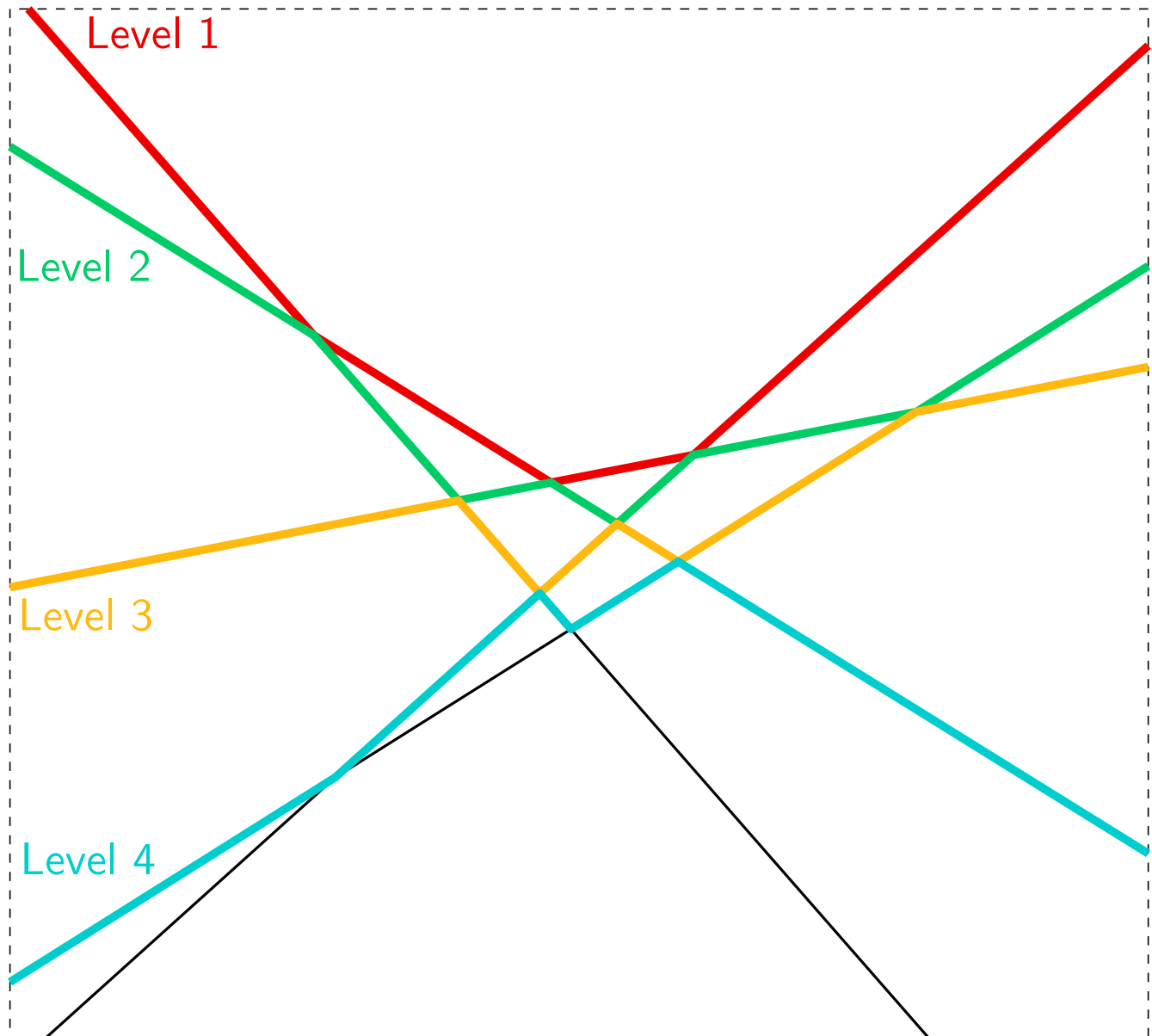


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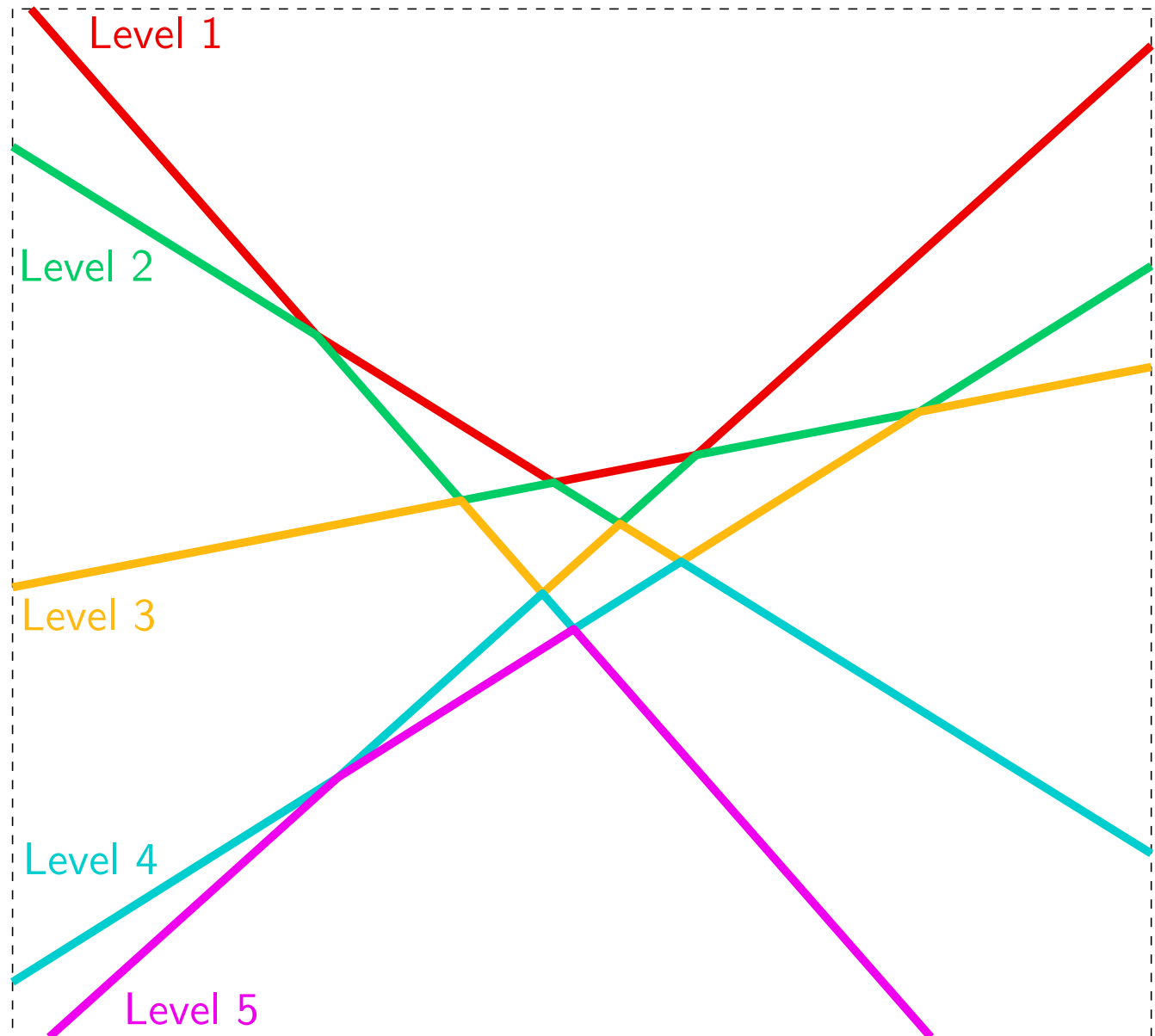


ARRANGEMENTS

LEVELS

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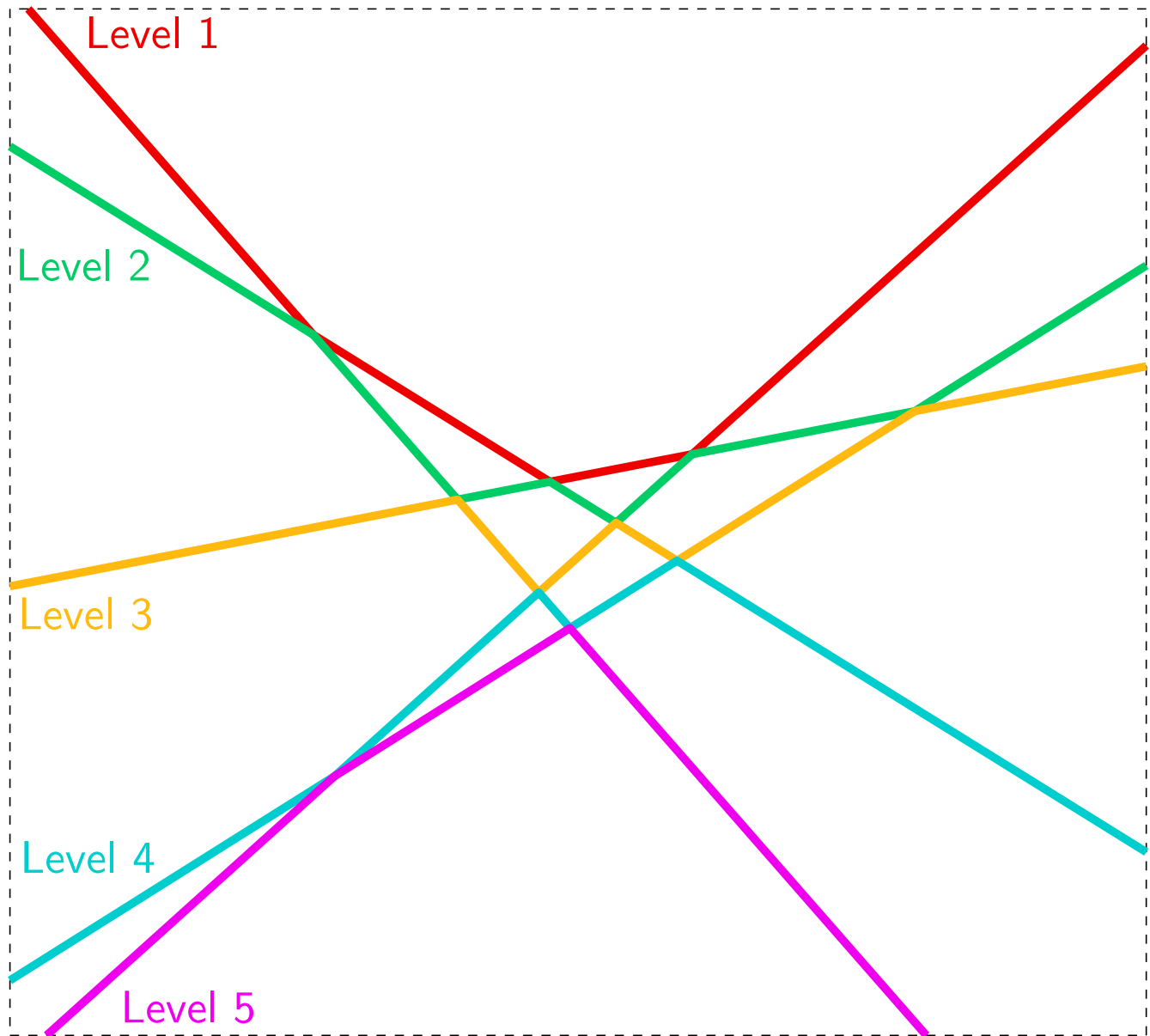


ARRANGEMENTS

LEVELS

Properties

1. A point p belongs to one (more than one) level if and only if it belongs to one (more than one) of the lines.
2. Any vertical line intersects each level in a single point.
3. If a vertical line intersects two levels, $i \leq j$ in two points p_i and p_j , then p_i is above p_j . Consequently, the levels are totally ordered.



ARRANGEMENTS

ARRANGEMENTS AND ORDER- k VORONOI DIAGRAMS

All previous definitions and properties extend to arrangements of hyperplanes in \mathbb{R}^d (simplicity, combinatorics, levels,...), but the complexity of the arrangement depends on the dimension.

ARRANGEMENTS

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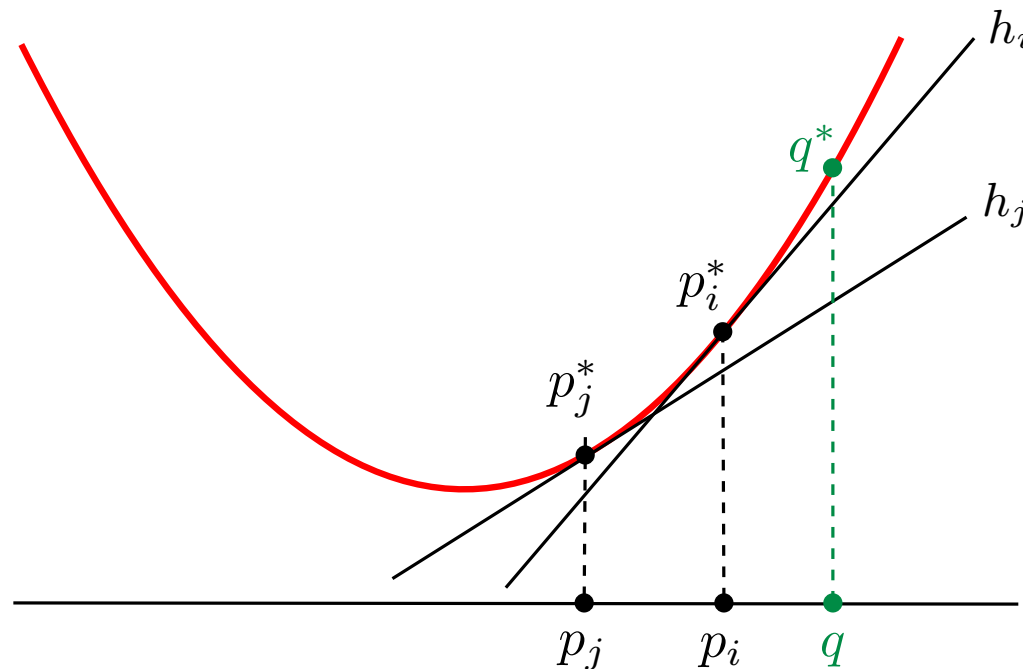
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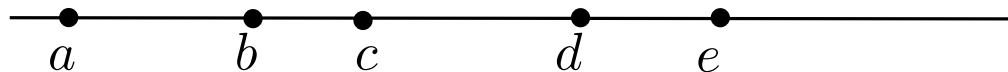
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ARRANGEMENTS

ARRANGEMENTS AND ORDER- k VORONOI DIAGRAMS

Example in 1D, for $k = 2$



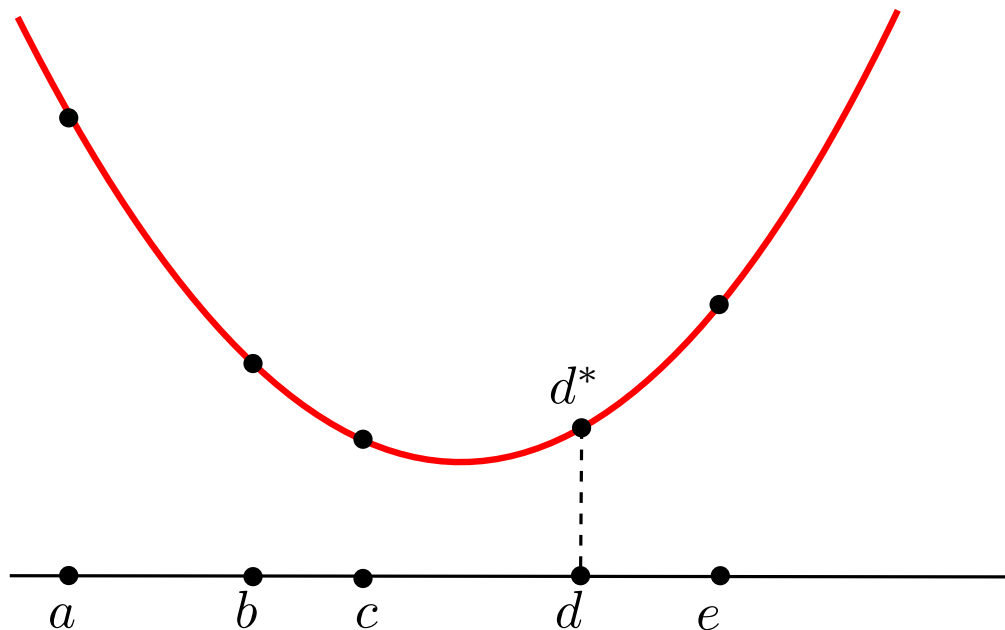
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ARRANGEMENTS

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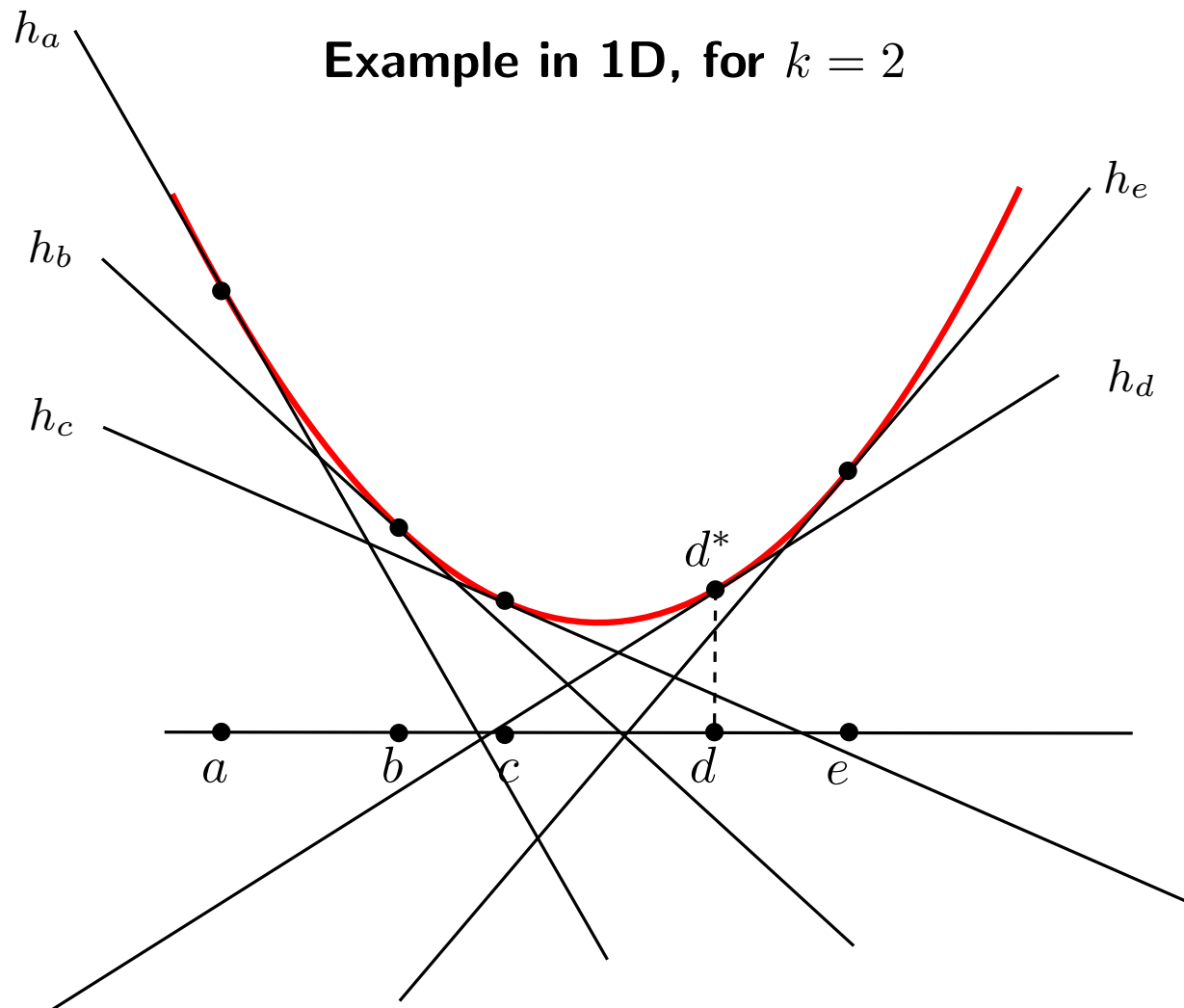


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ARRANGEMENTS

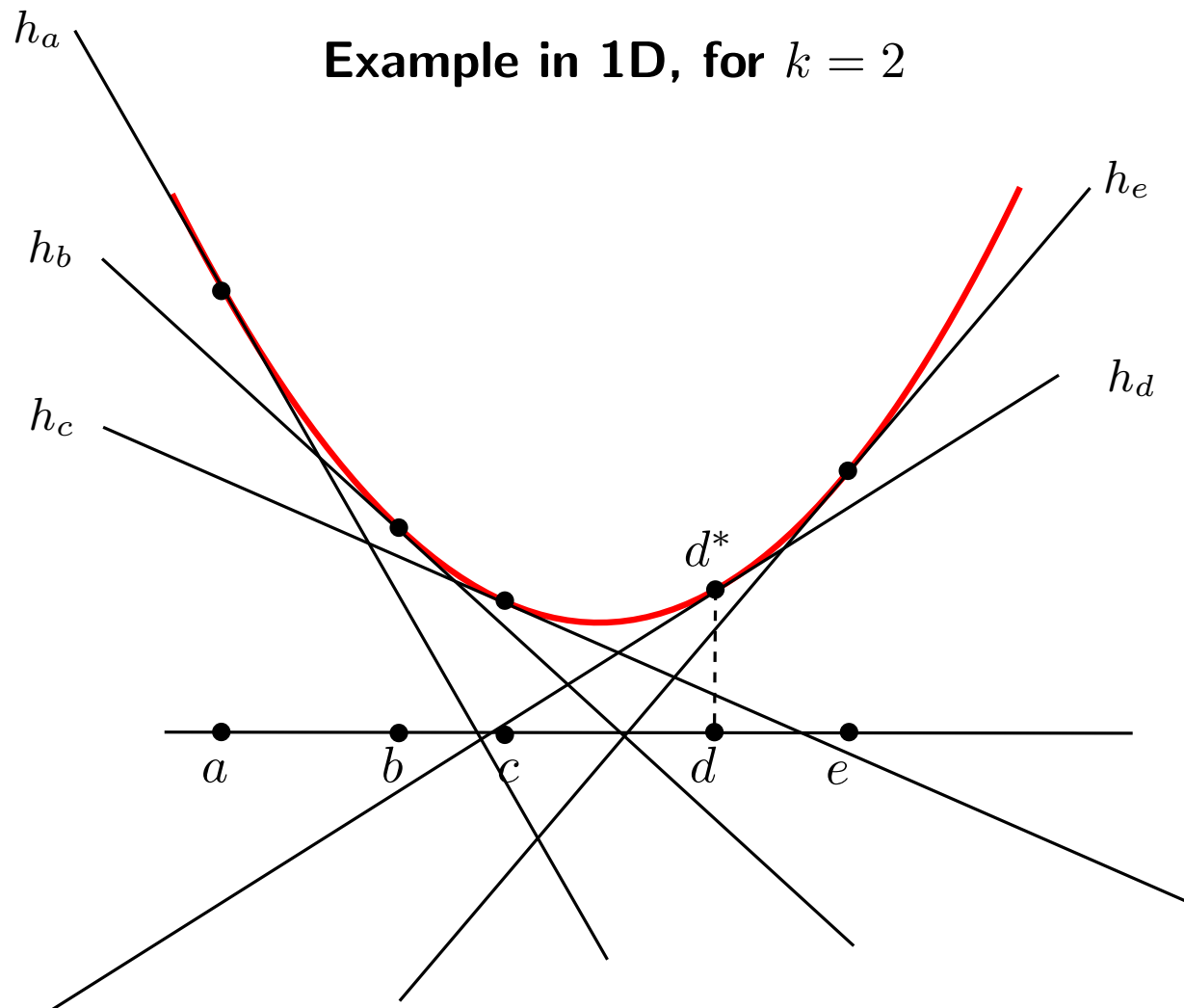
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ARRANGEMENTS

ARRANGEMENTS AND ORDER- k VORONOI DIAGRAMS

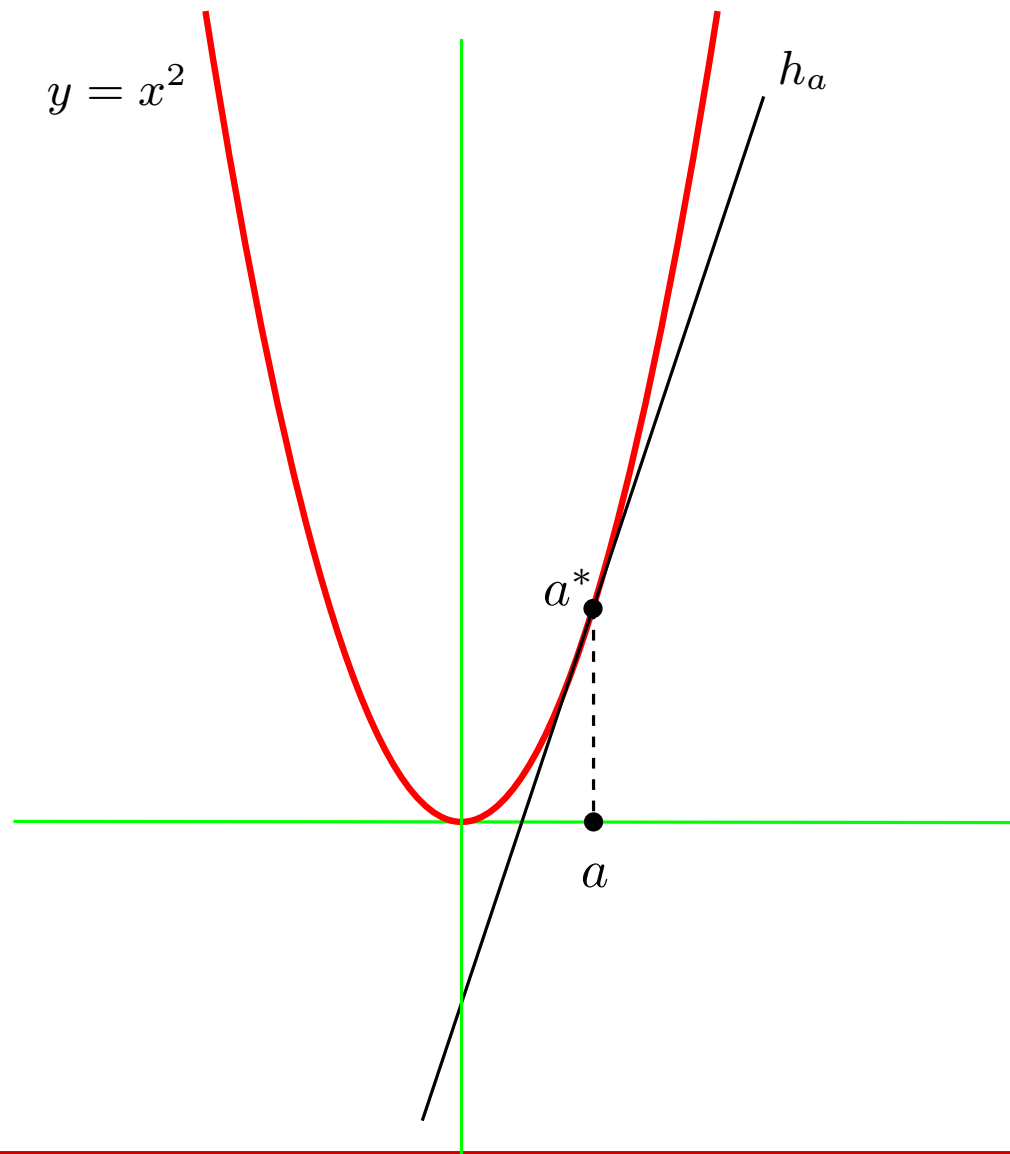


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Why does this work?

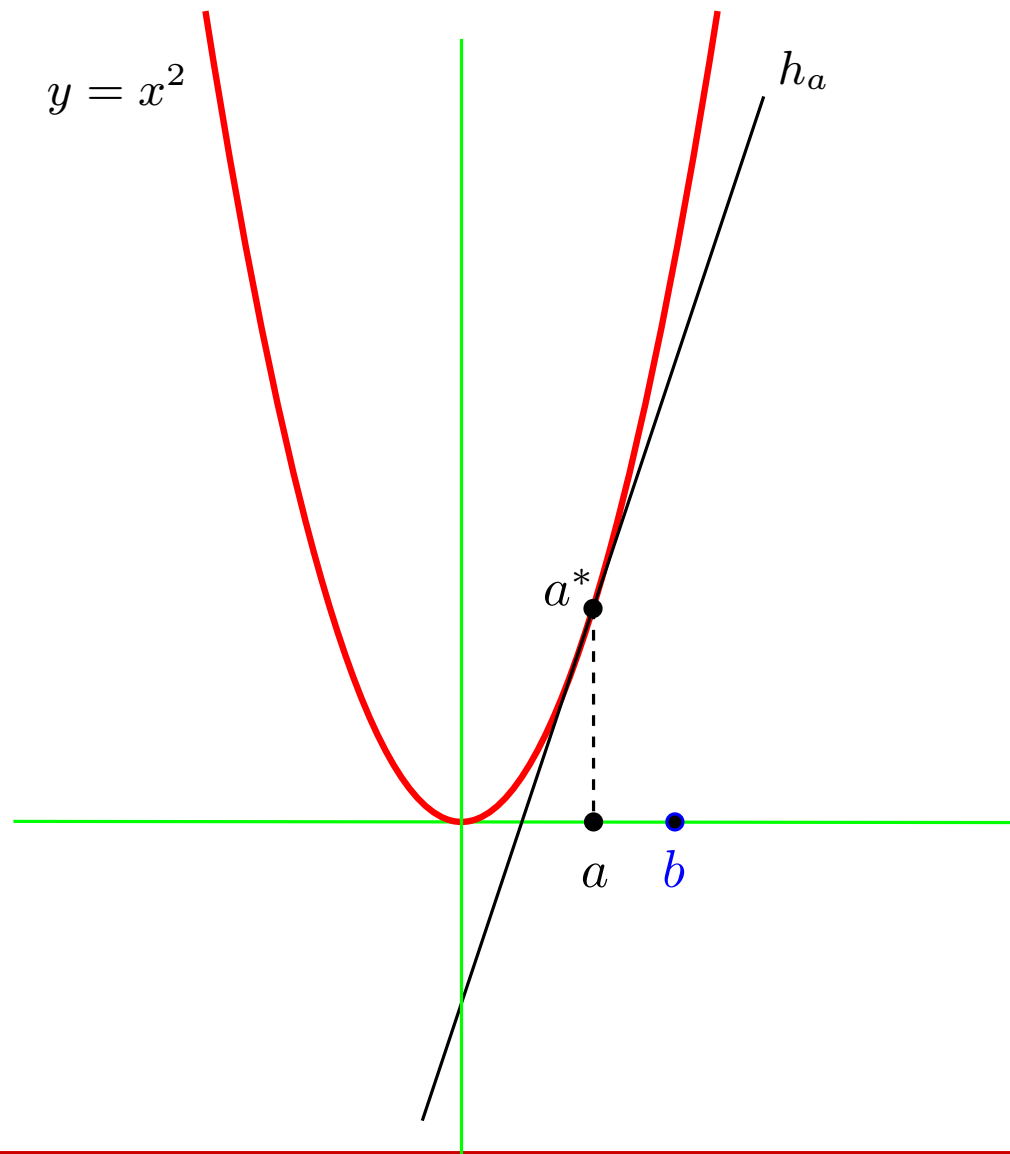
ARRANGEMENTS

Relation between lines and distances (1D)



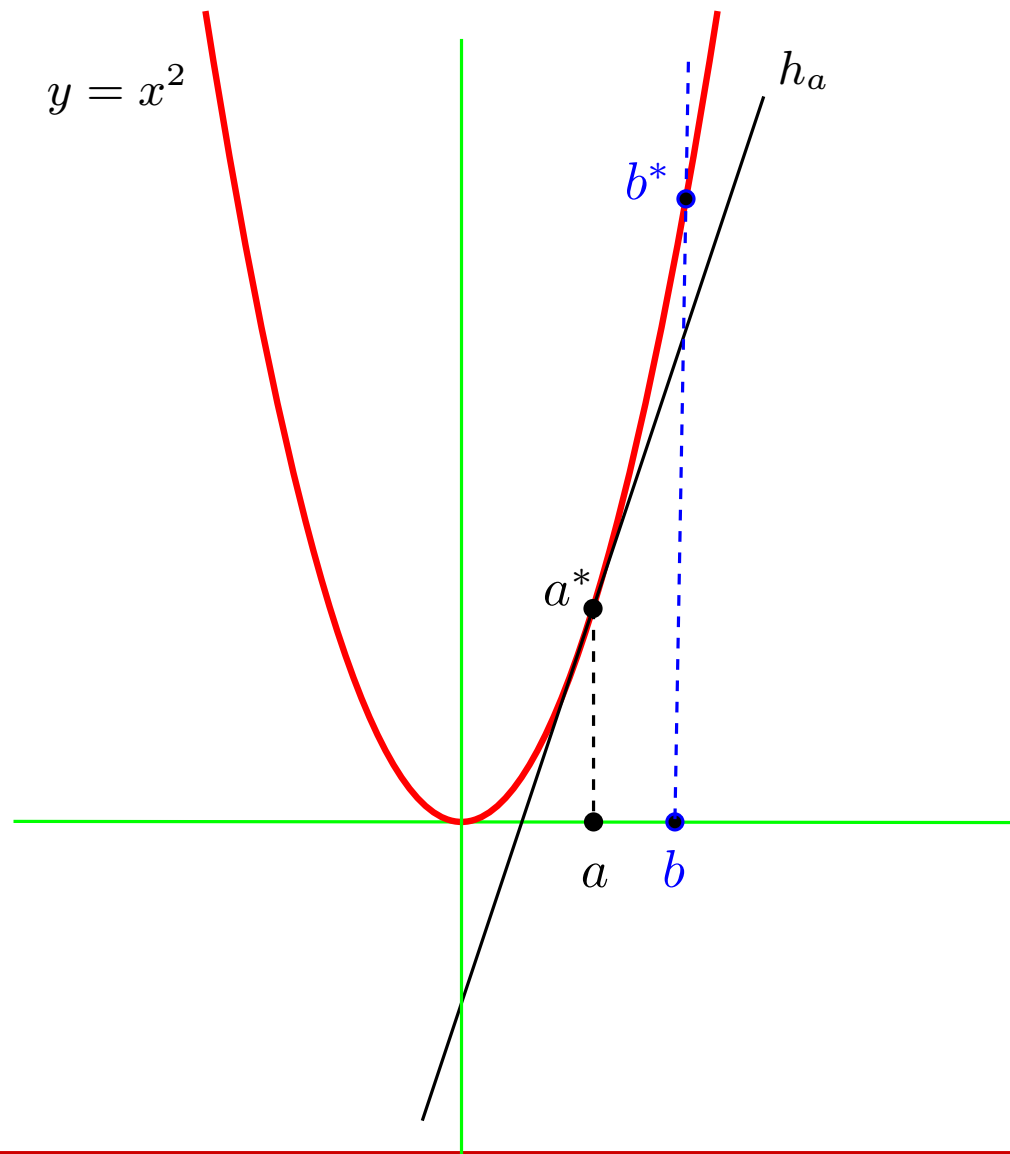
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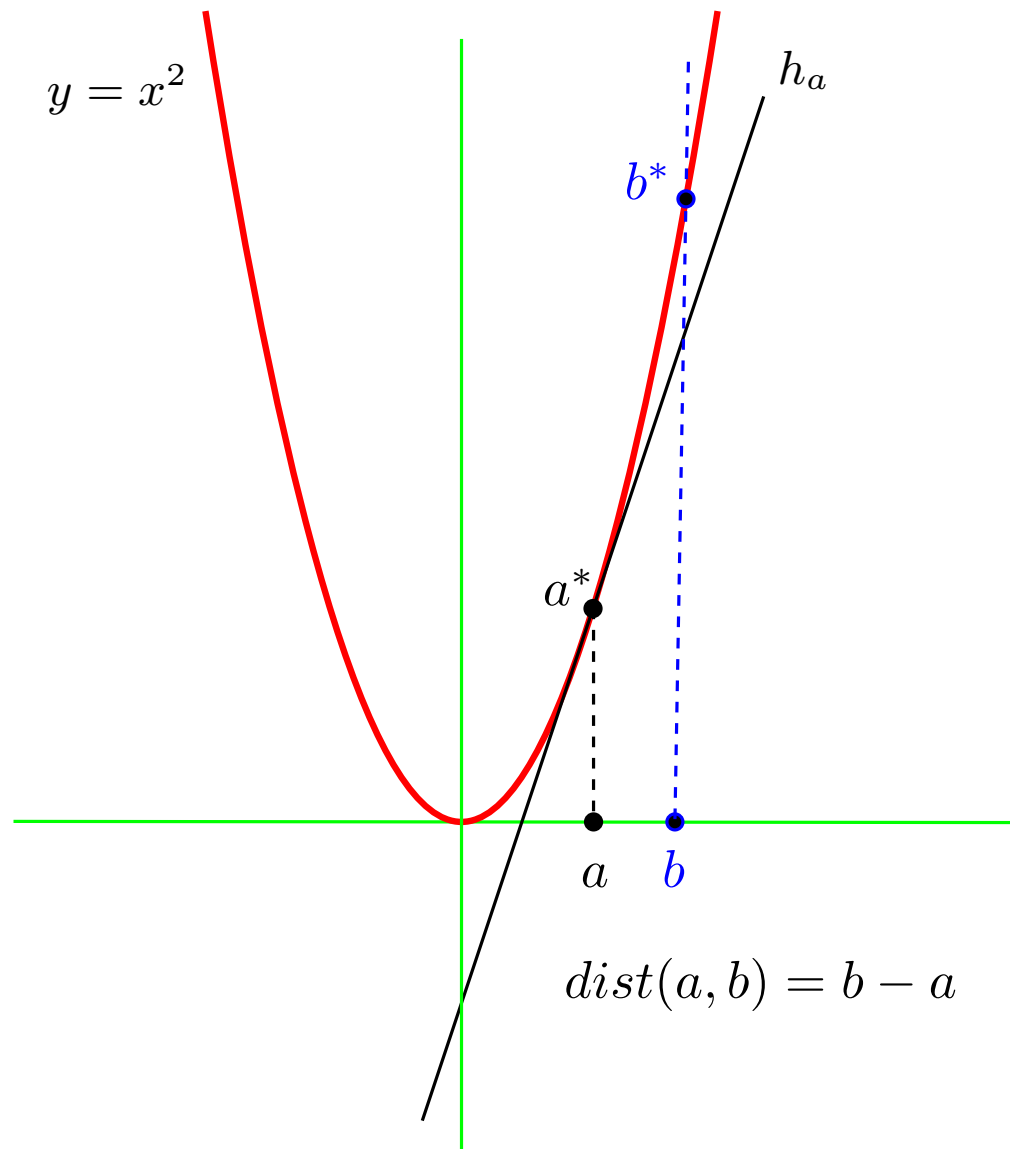
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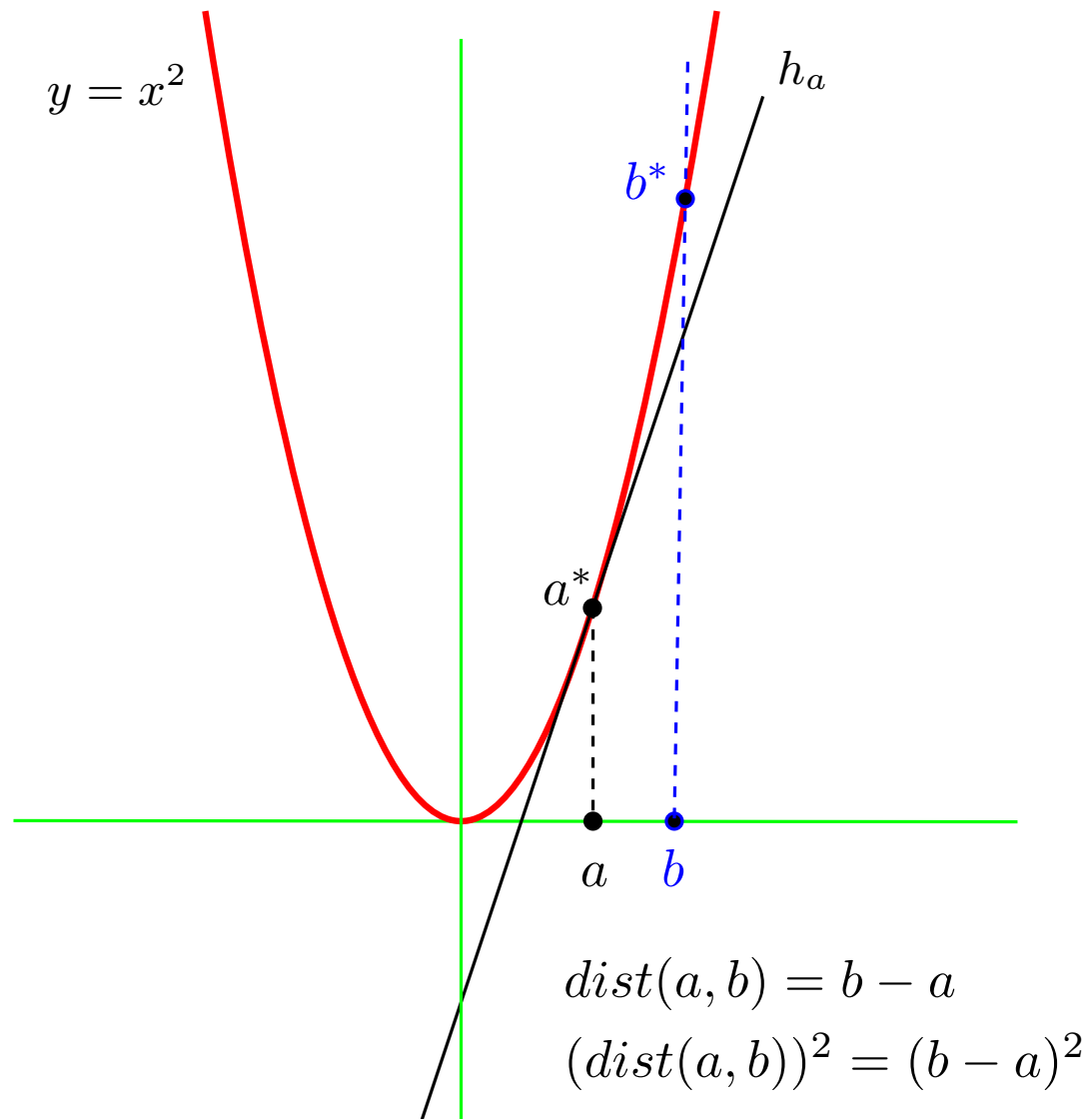
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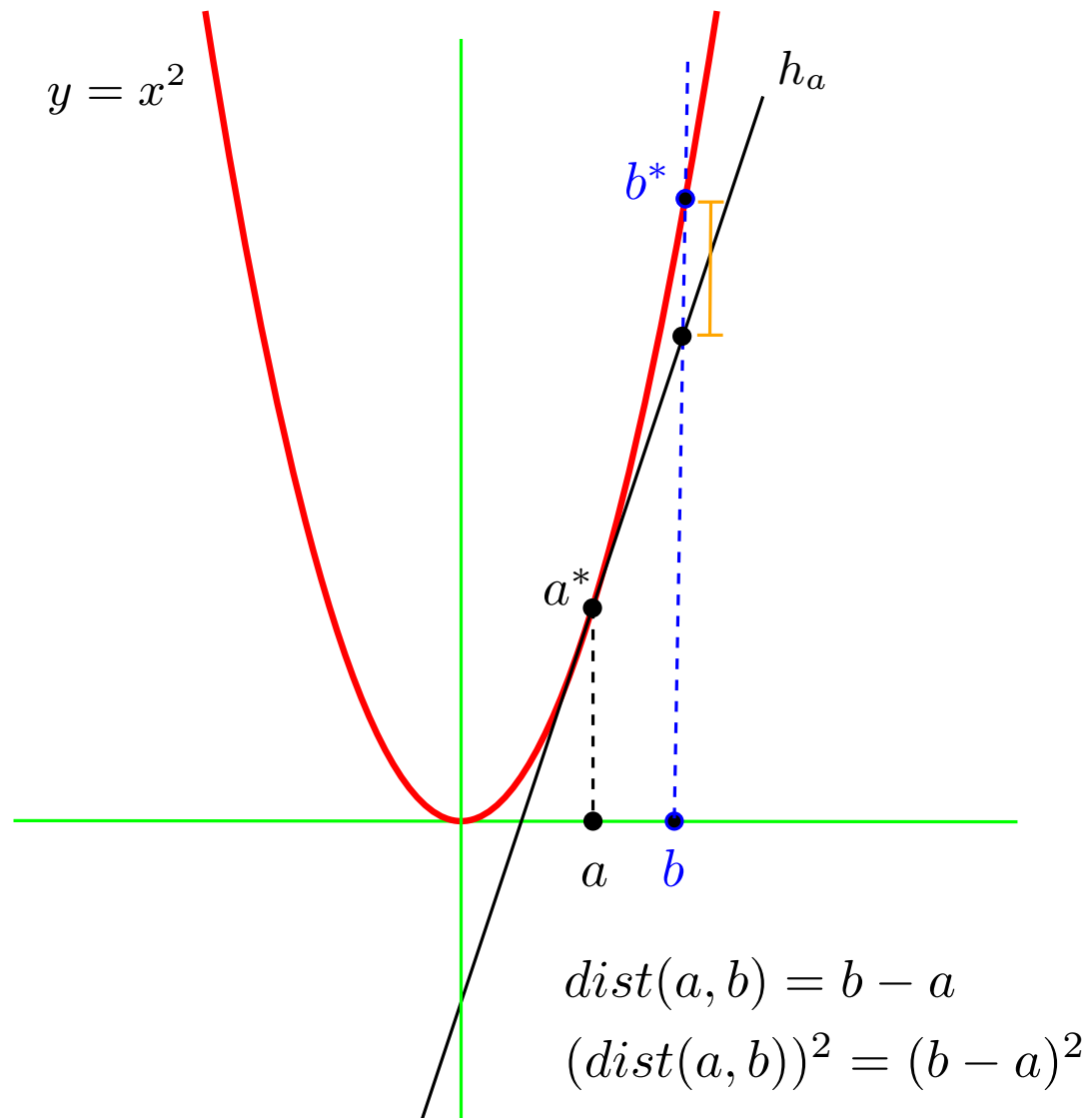
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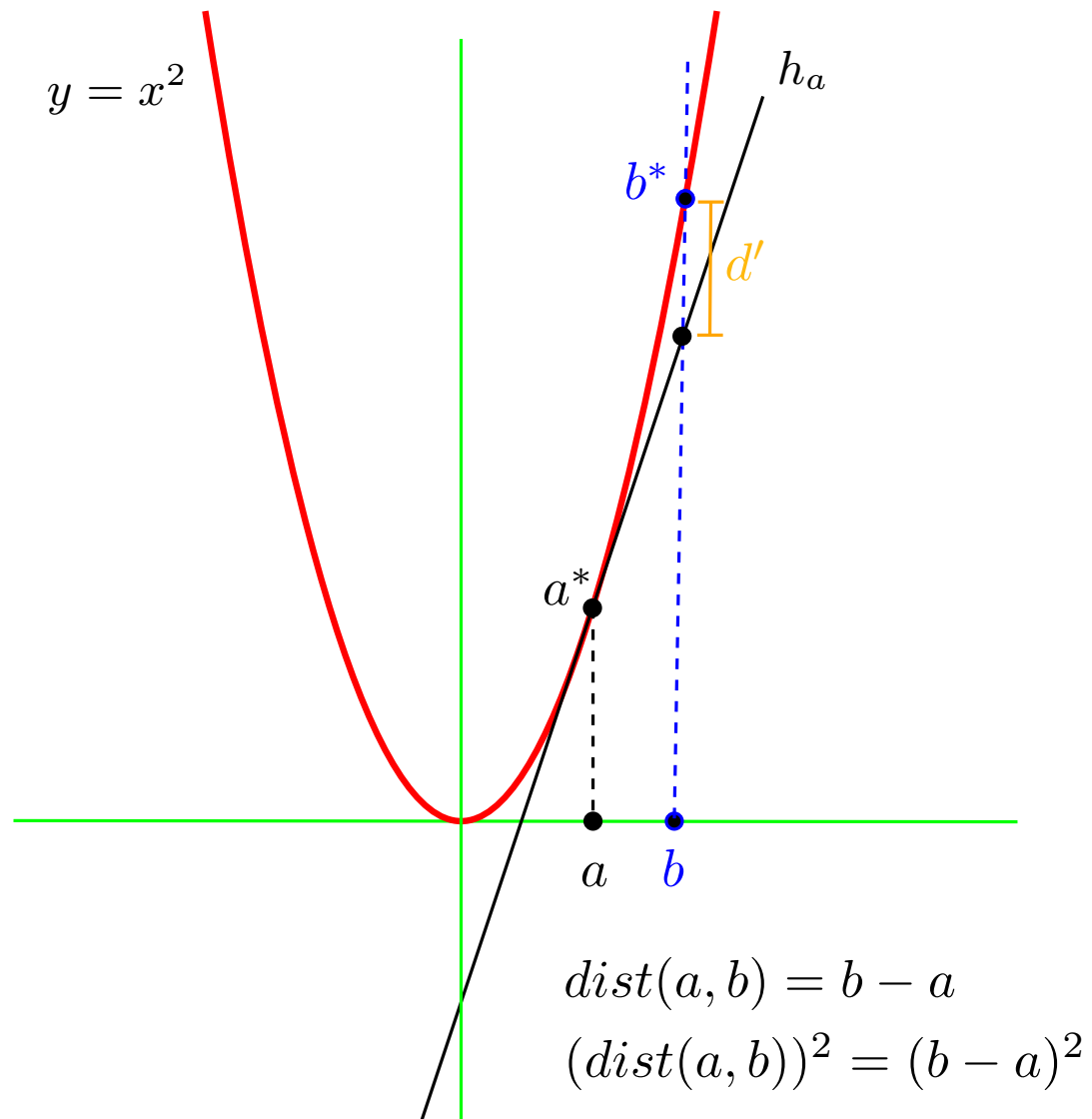
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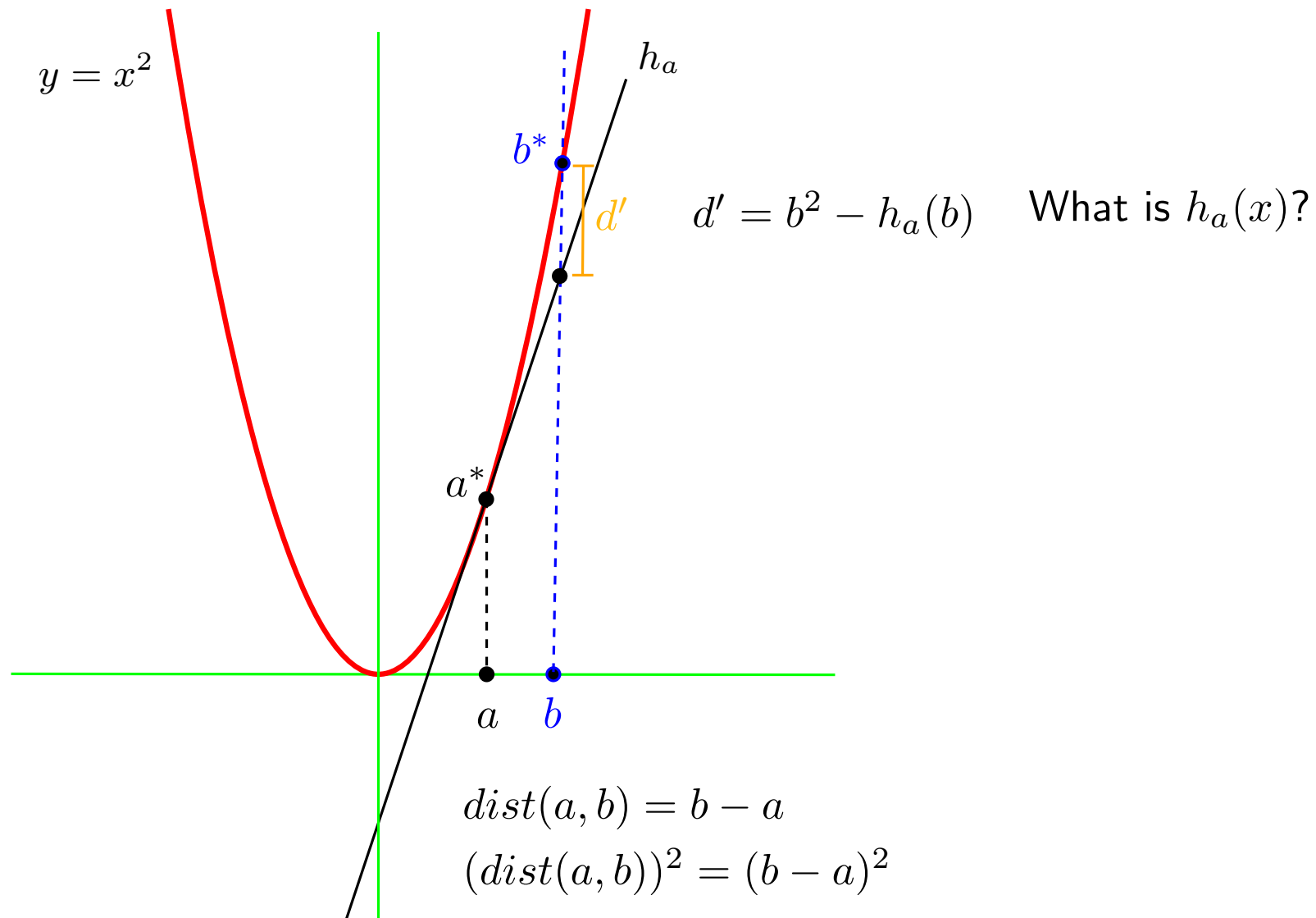
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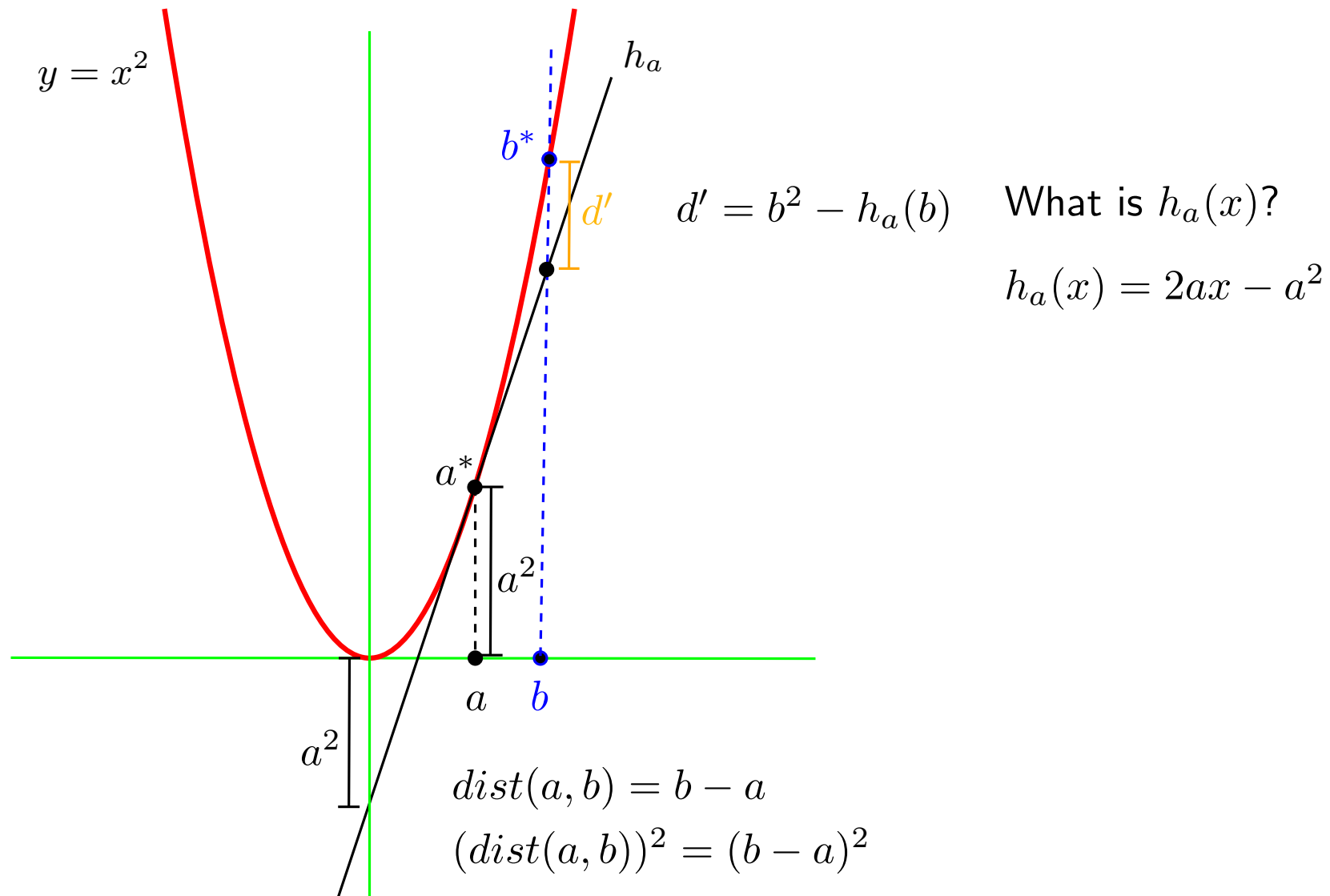
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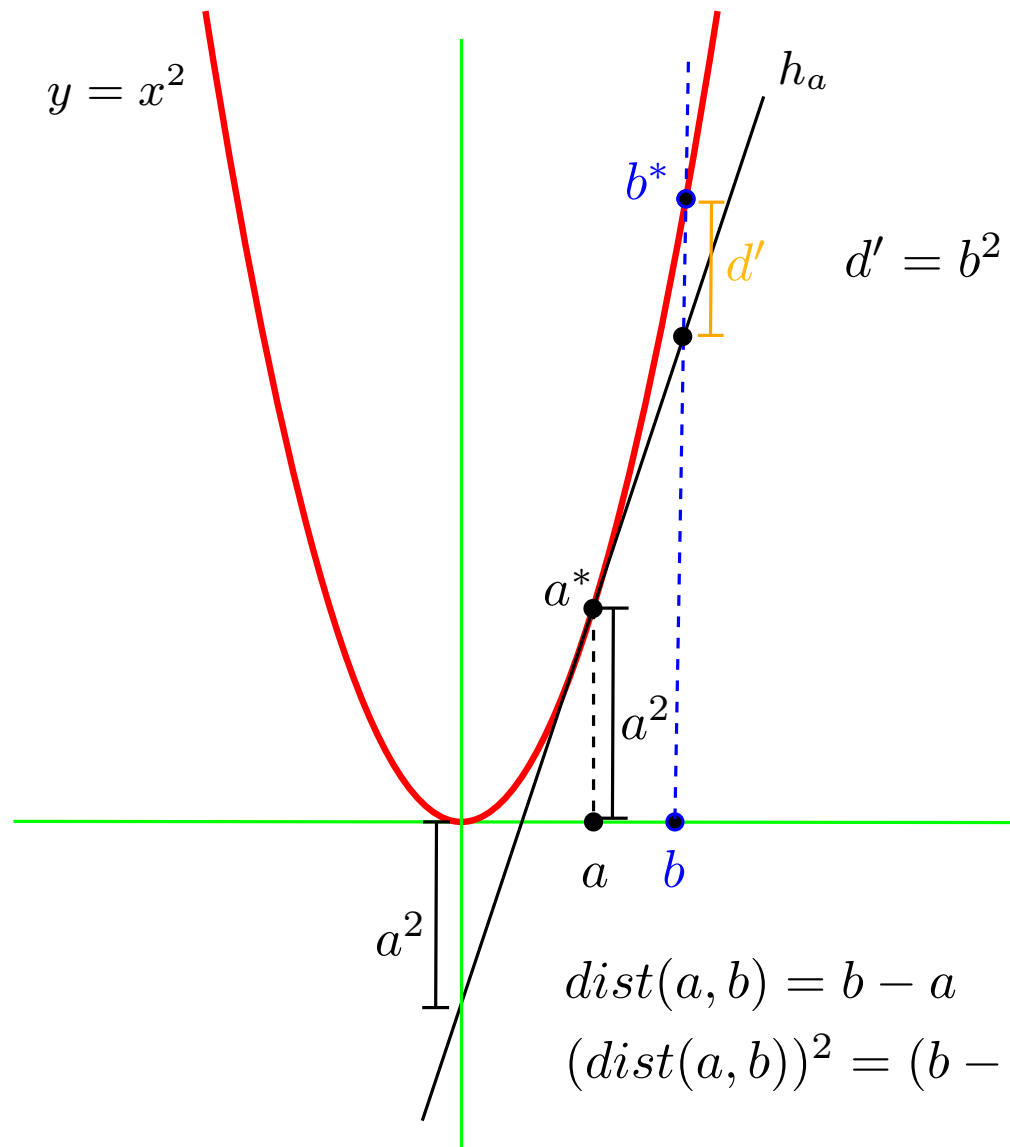
ARRANGEMENTS

Relation between lines and distances (1D)



ARRANGEMENTS

Relation between lines and distances (1D)



$$d' = b^2 - h_a(b)$$

What is $h_a(x)$?

$$h_a(x) = 2ax - a^2$$

Then we have

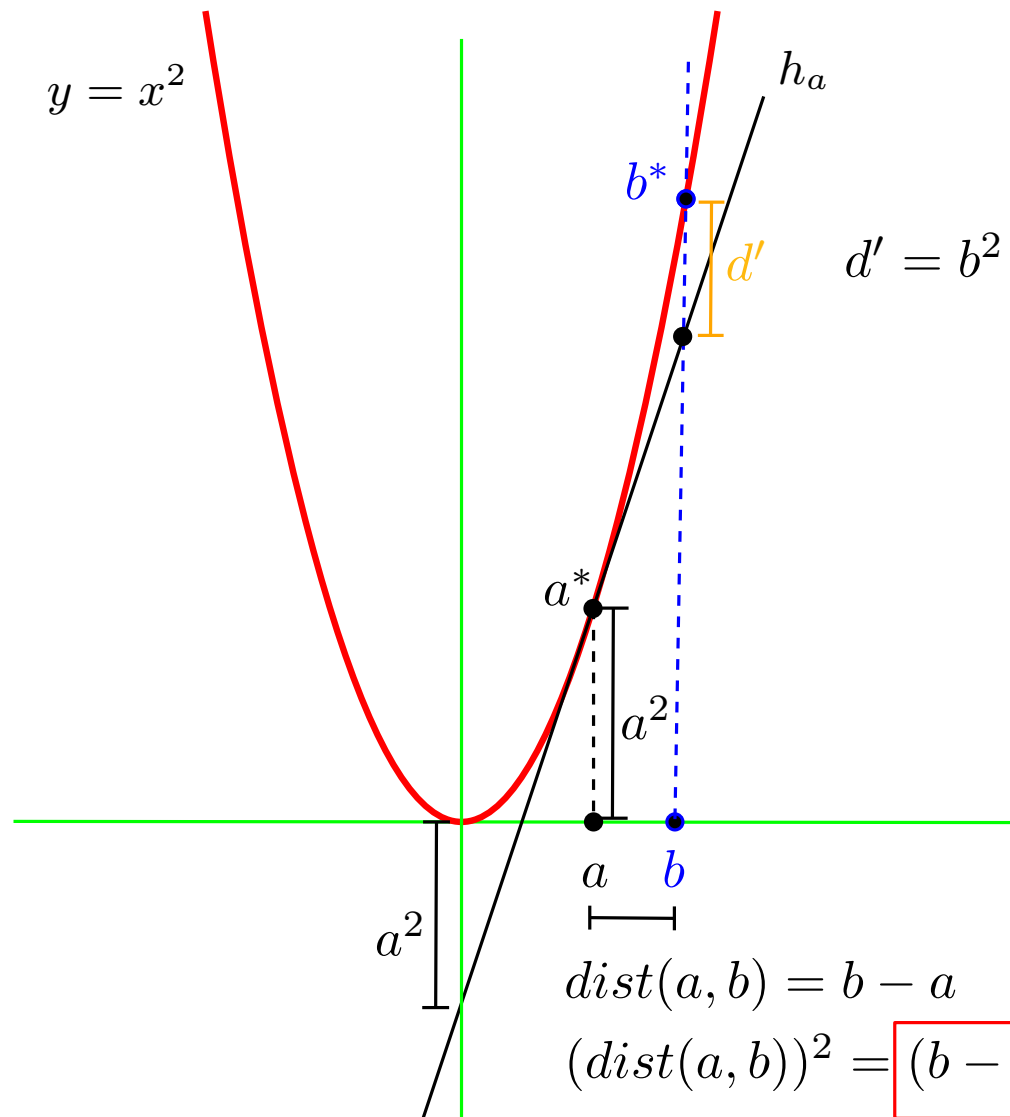
$$\begin{aligned} d' &= b^2 - (2ab - a^2) \\ &= b^2 - 2ab + a^2 \\ &= (b - a)^2 \end{aligned}$$

$$\text{dist}(a, b) = b - a$$

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ARRANGEMENTS

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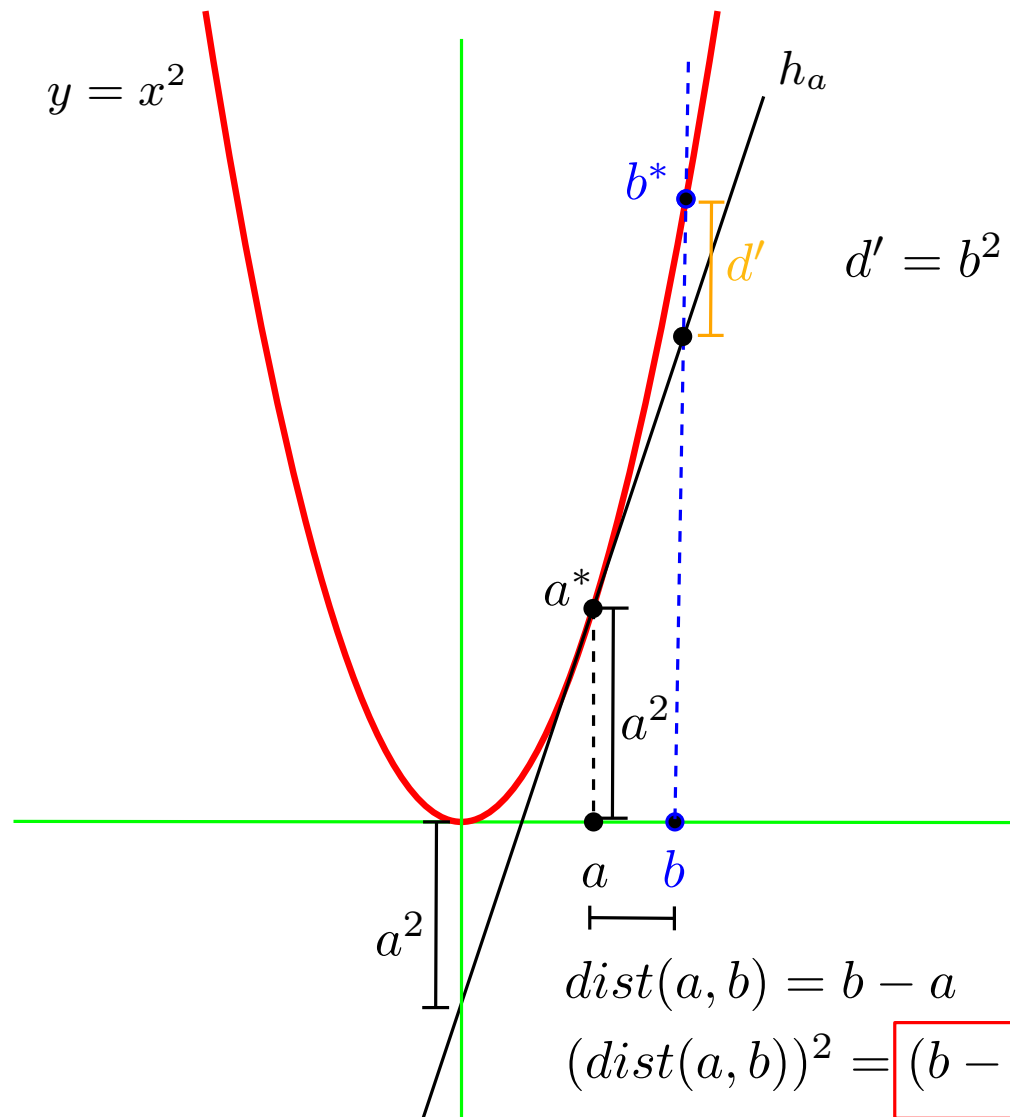
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ARRANGEMENTS

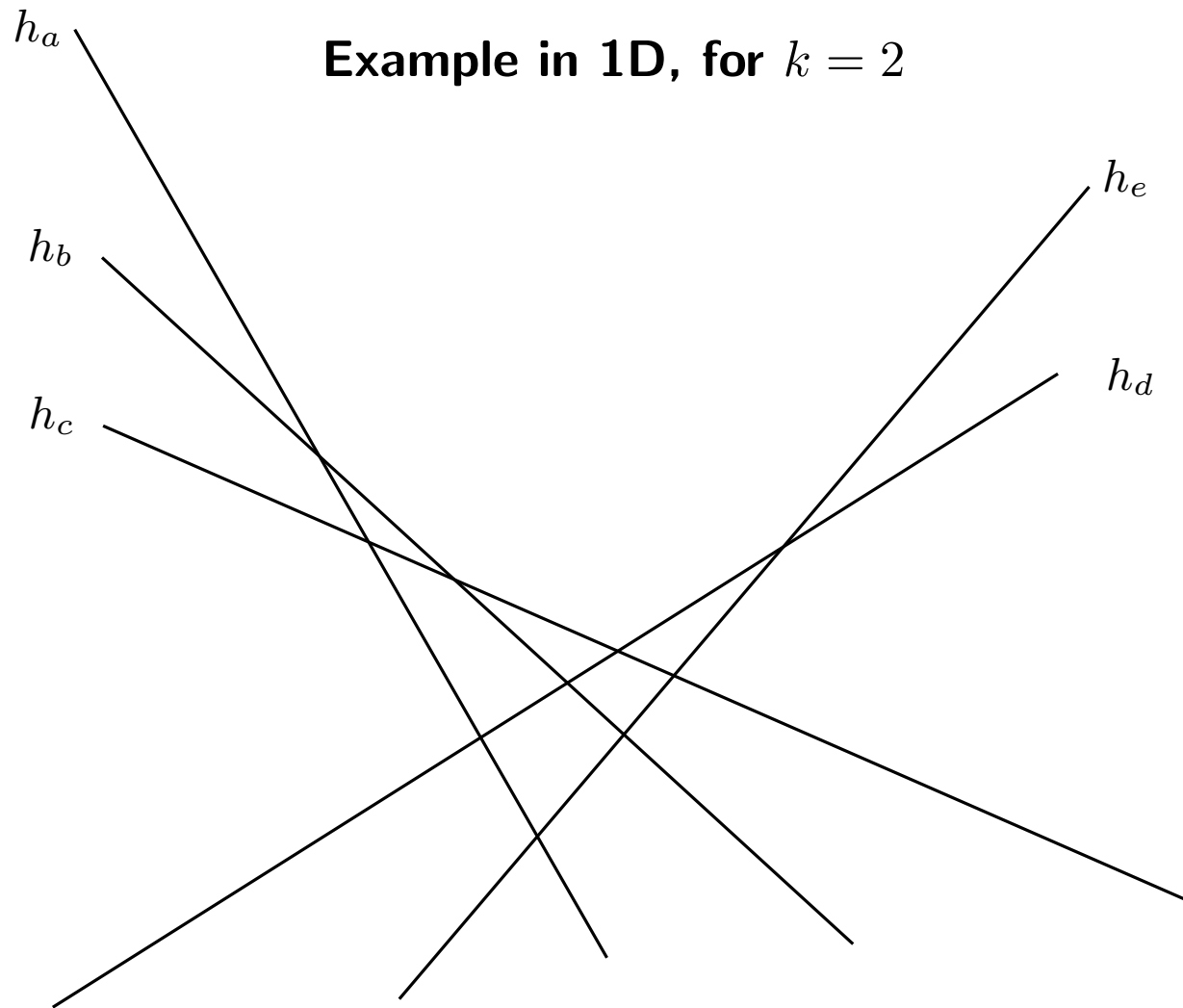
Relation between lines and distances (1D)



The smaller d' ,
the closer a is to b

ARRANGEMENTS

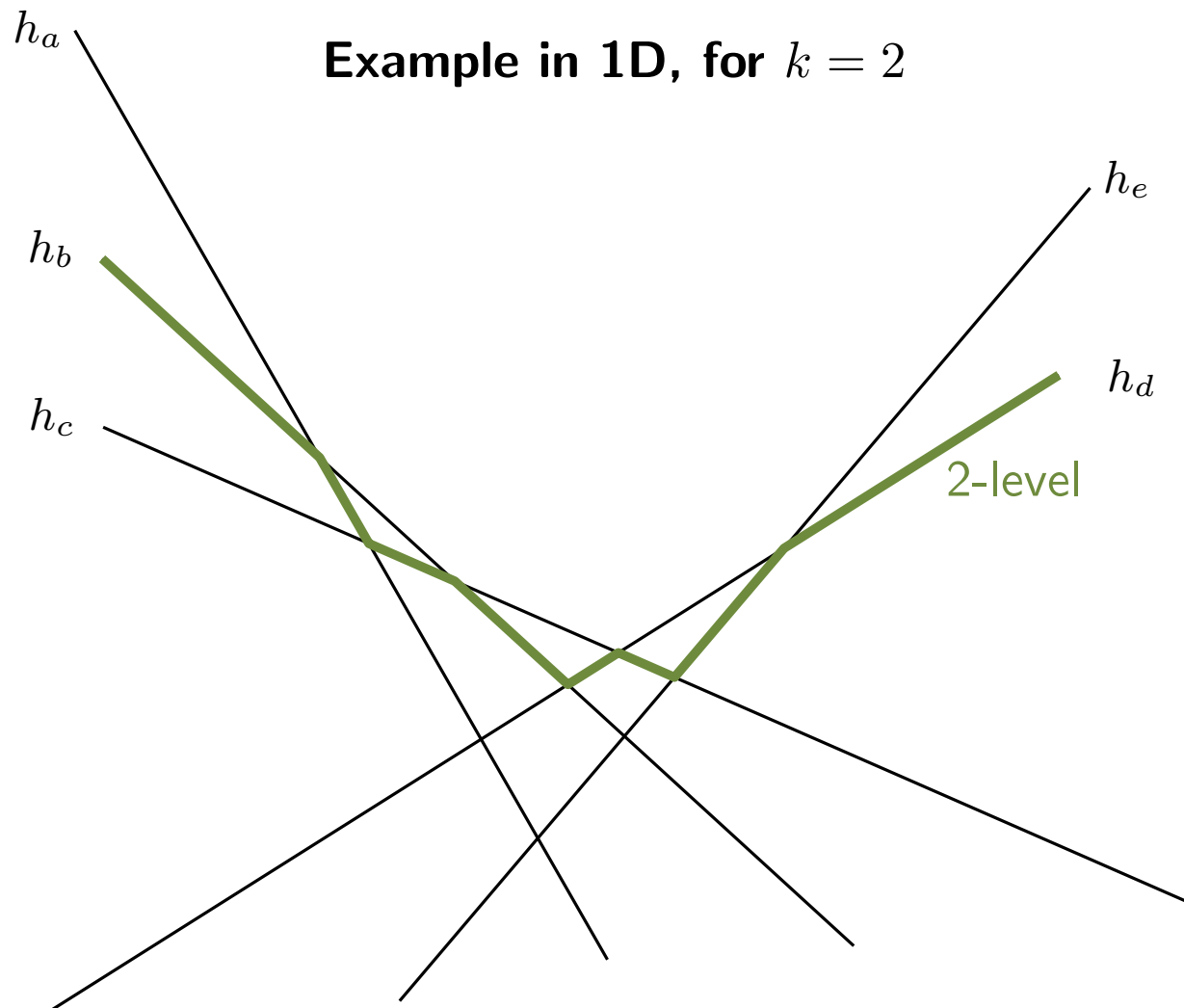
ARRANGEMENTS AND ORDER- k VORONOI DIAGRAMS



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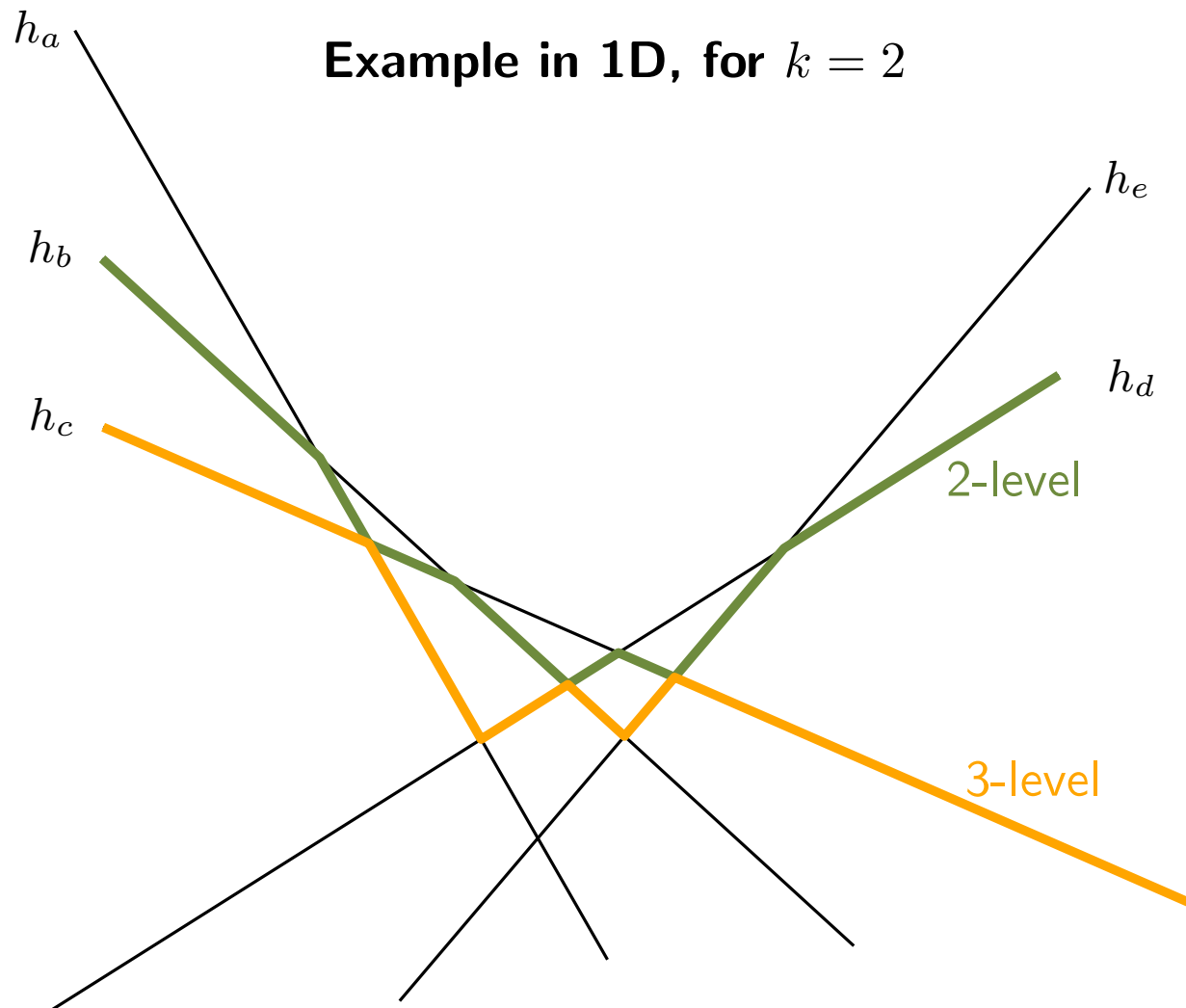
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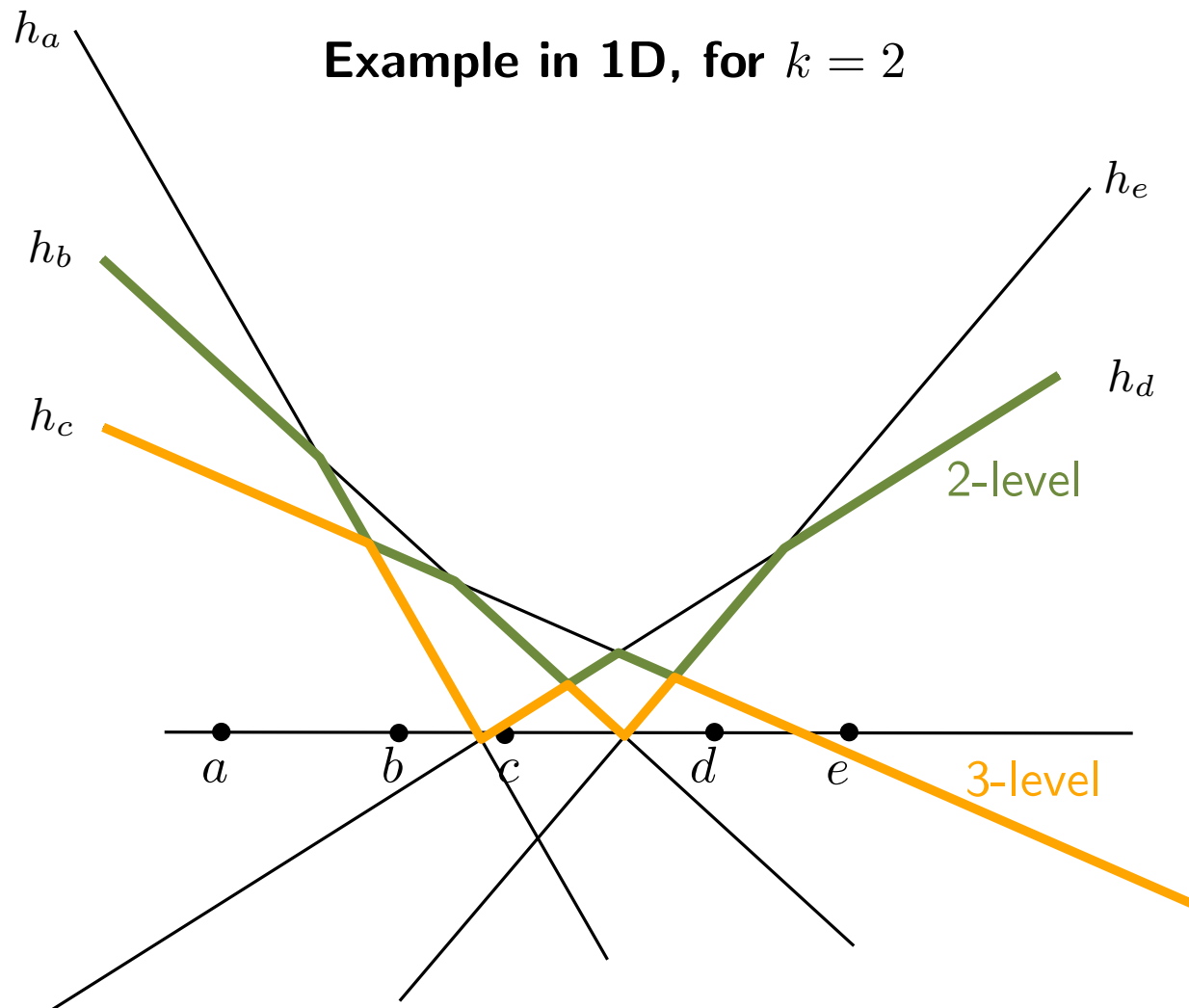
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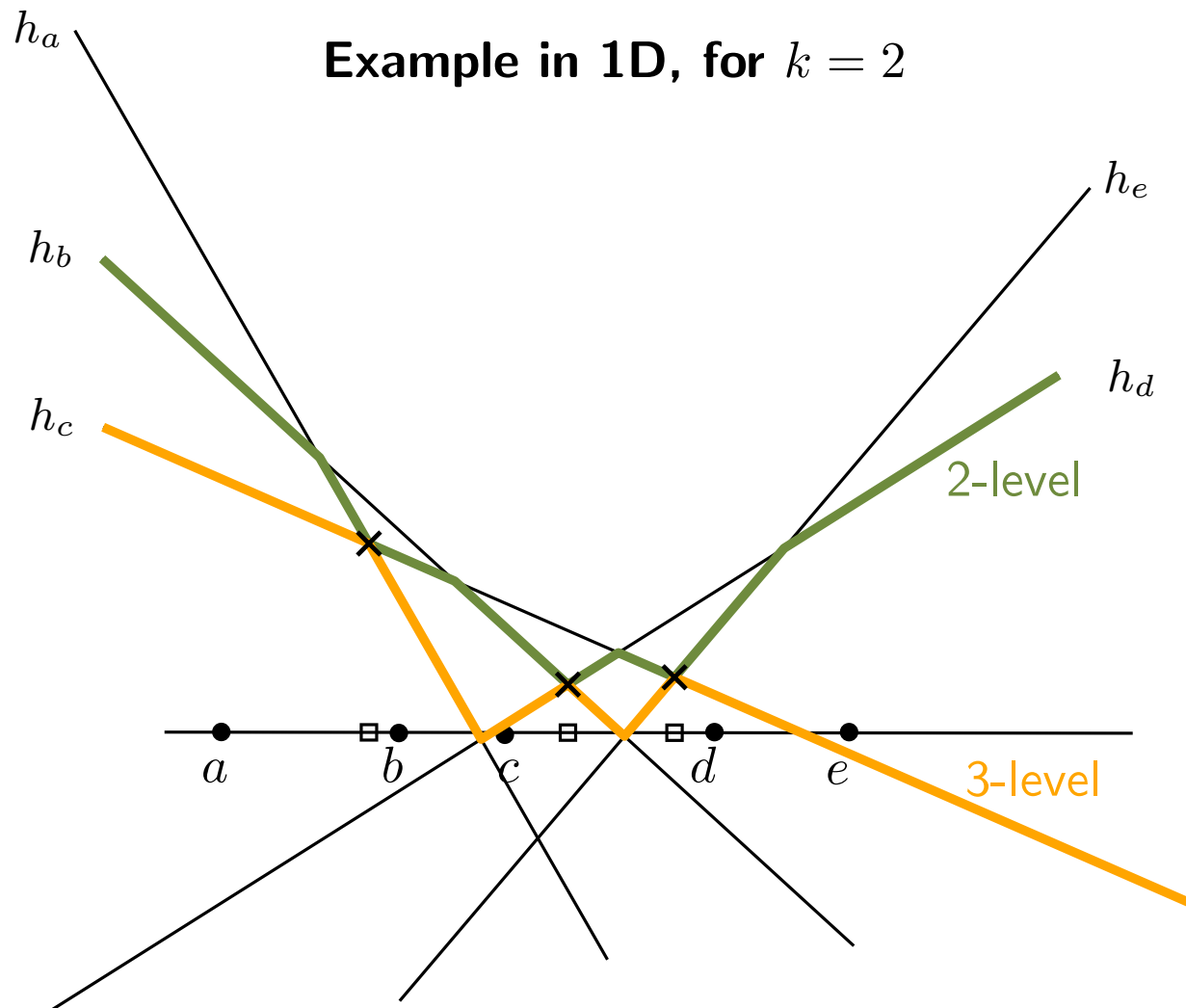
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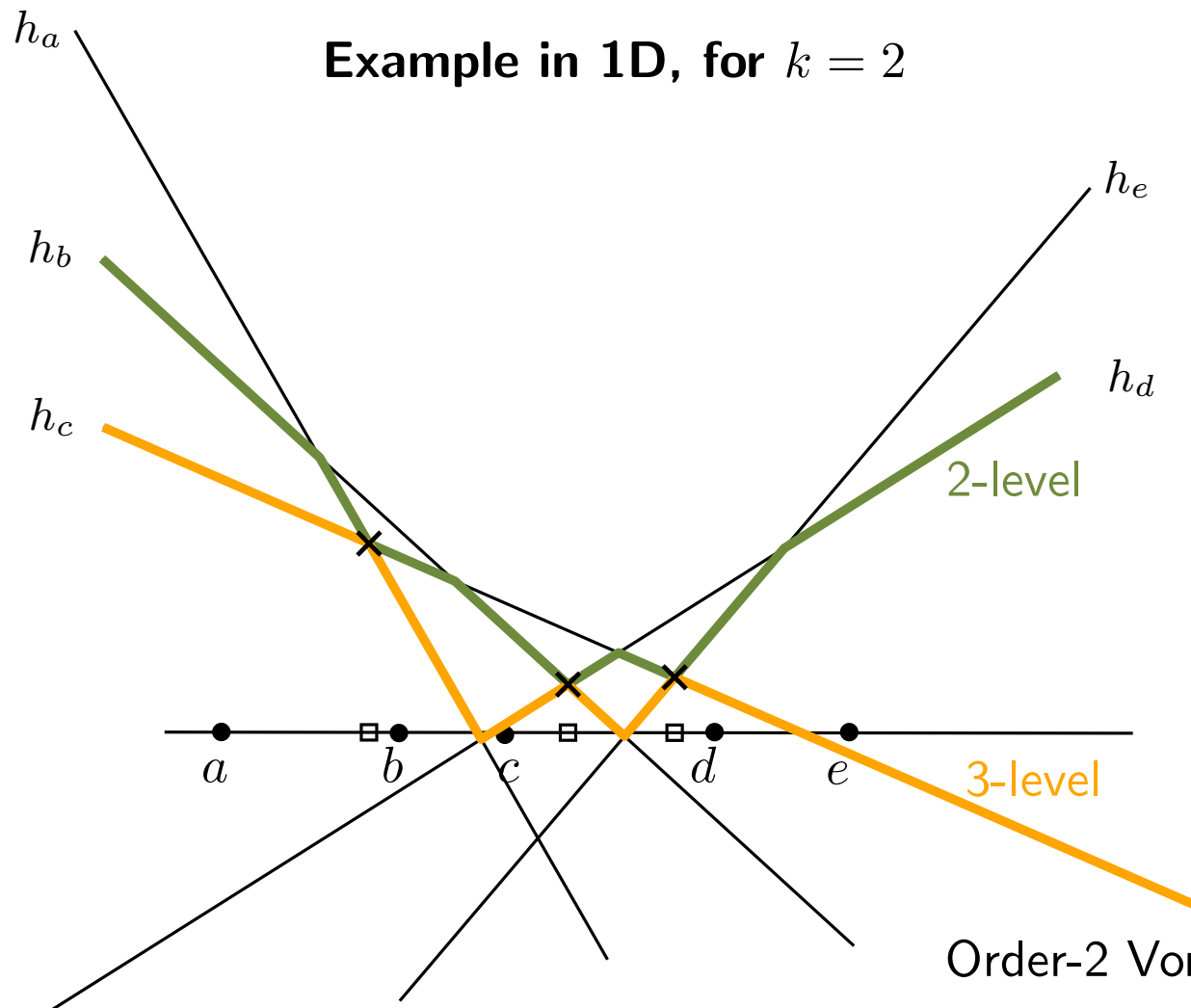
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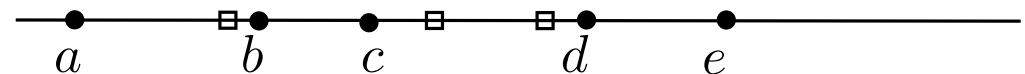
ARRANGEMENTS

ARRANGEMENTS AND ORDER- k VORONOI DIAGRAMS



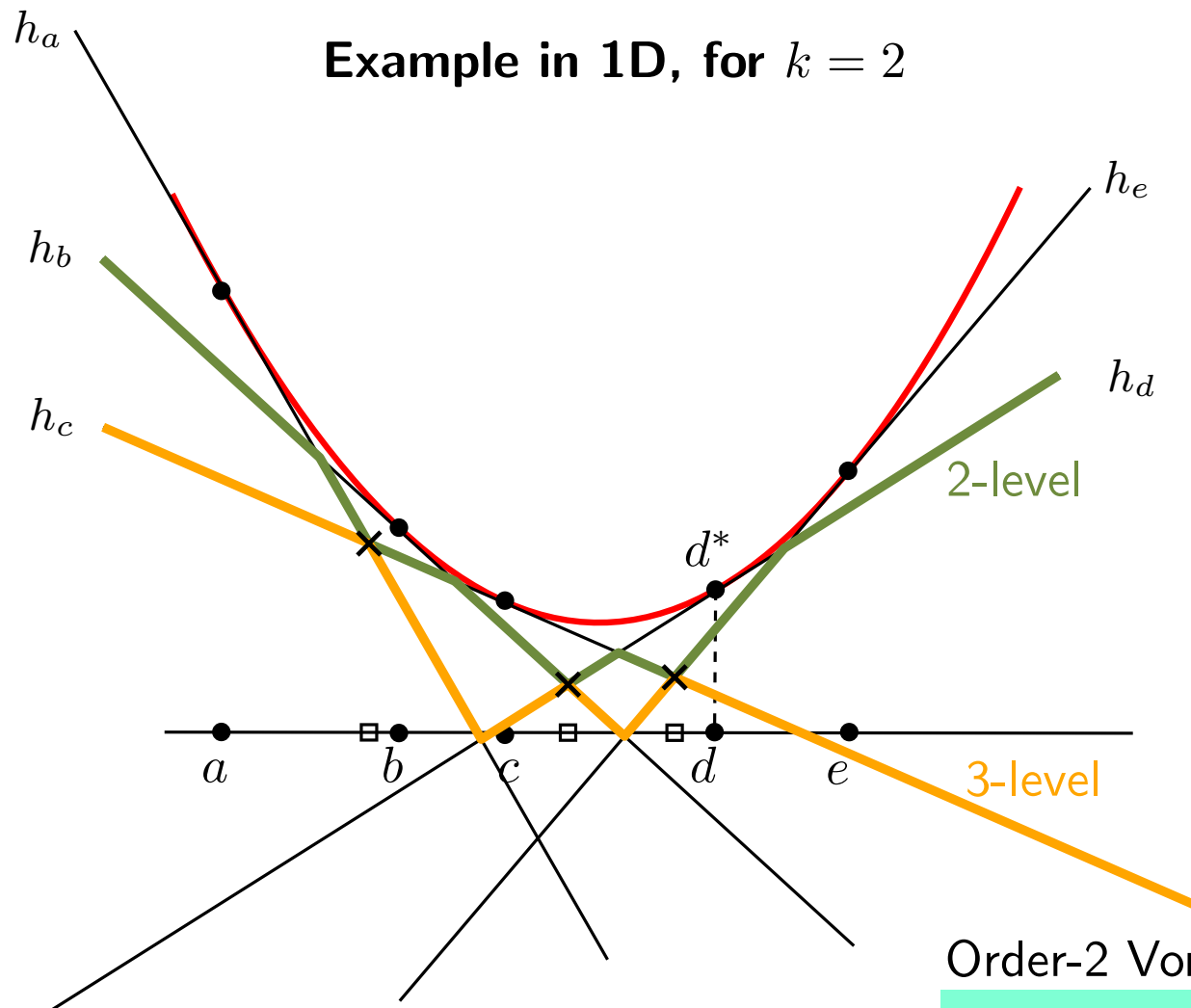
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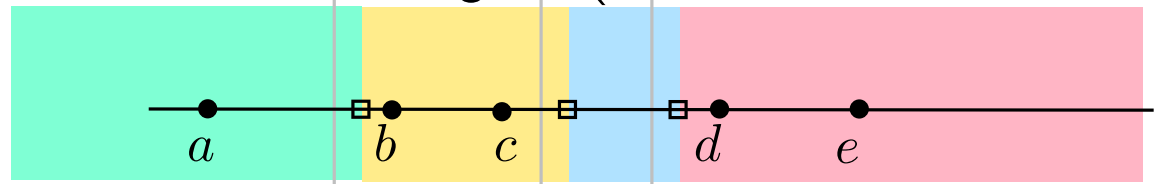
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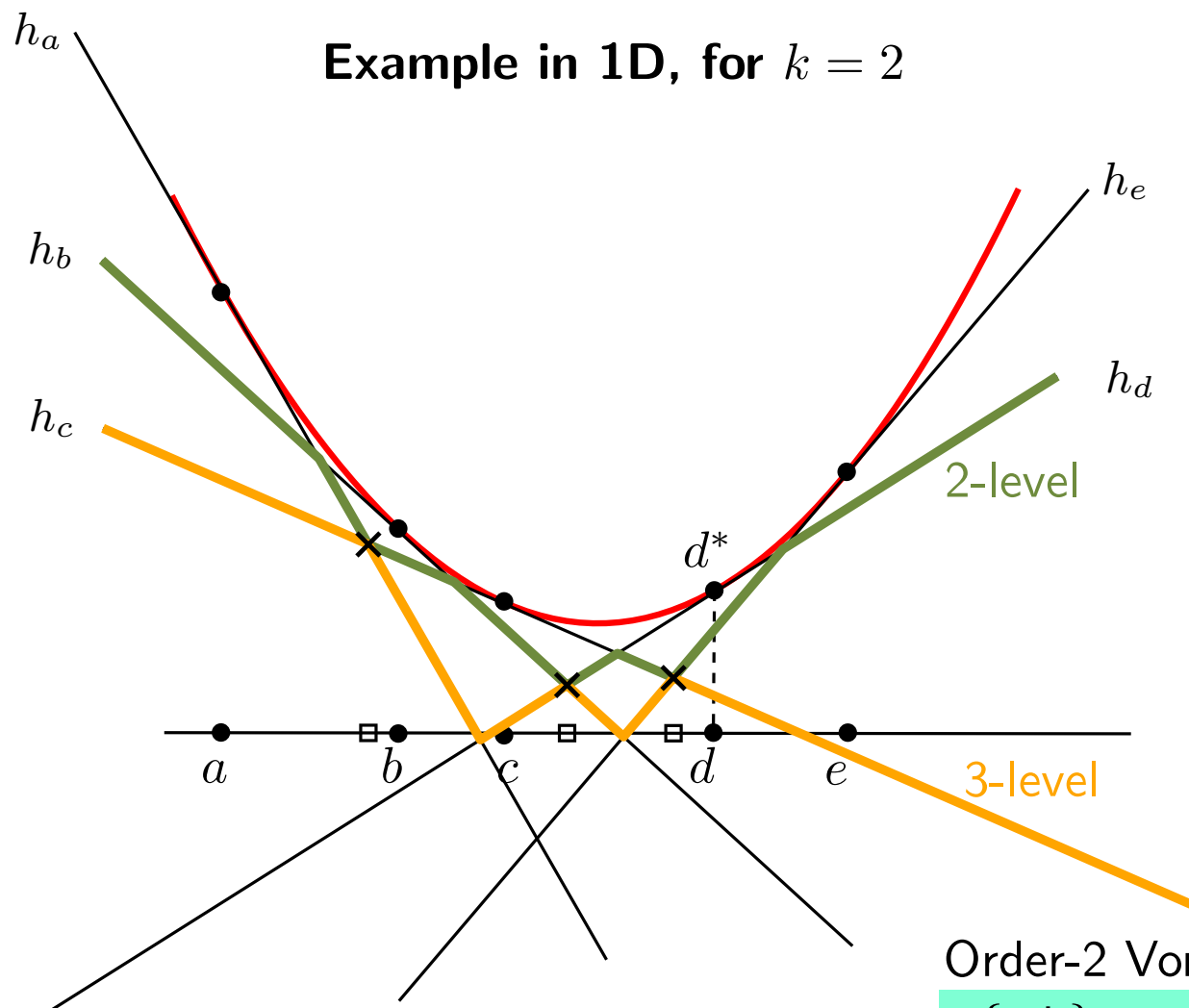
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Order-2 Voronoi diagram (Note it should be in 1D!)



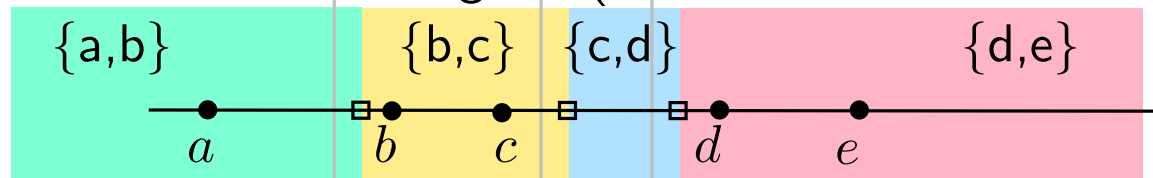
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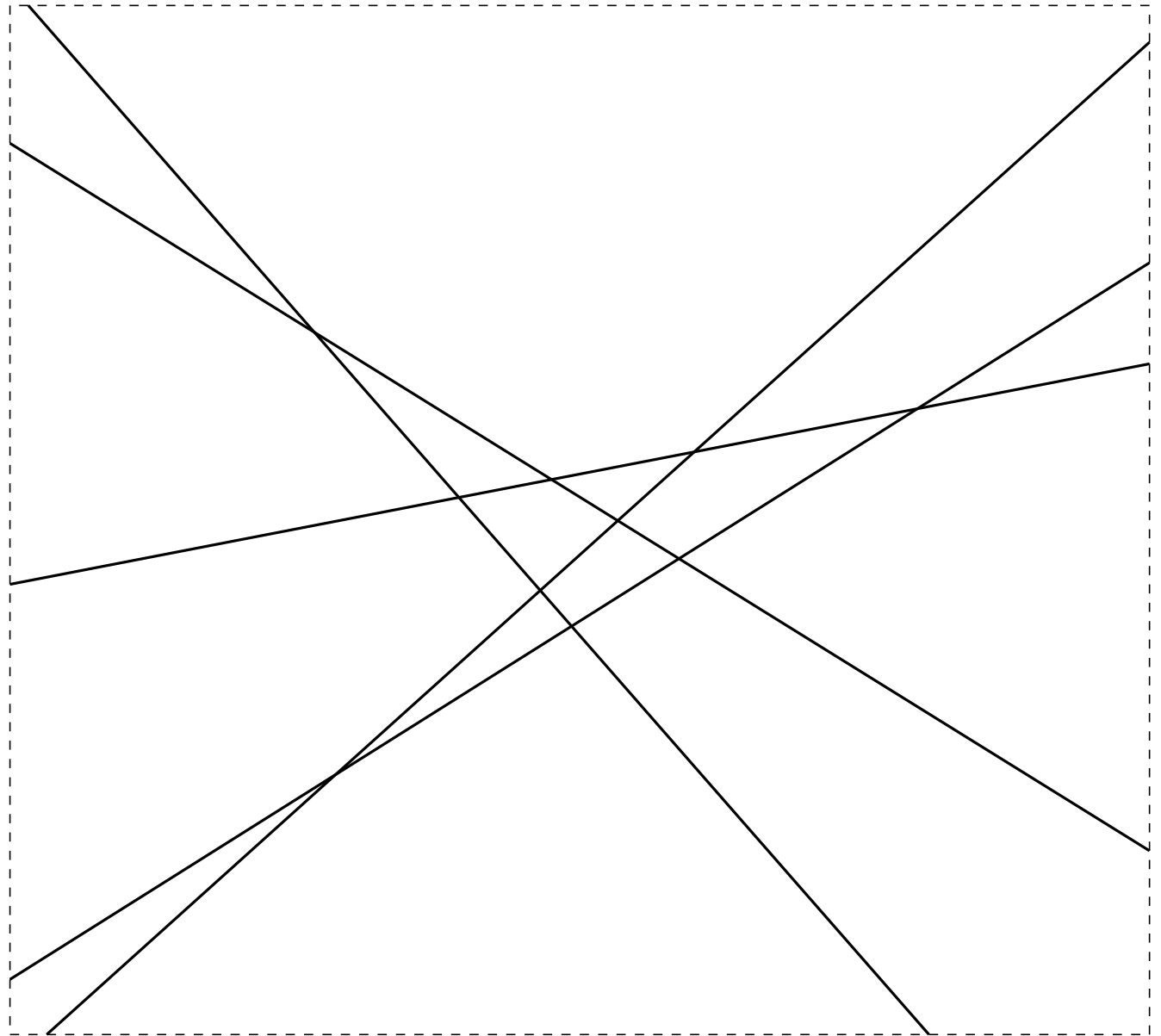
Order-2 Voronoi diagram (Note it should be in 1D!)



ARRANGEMENTS

STORAGE

As an arrangement of lines is a decomposition of the plane, the structure usually used to store it is a DCEL.

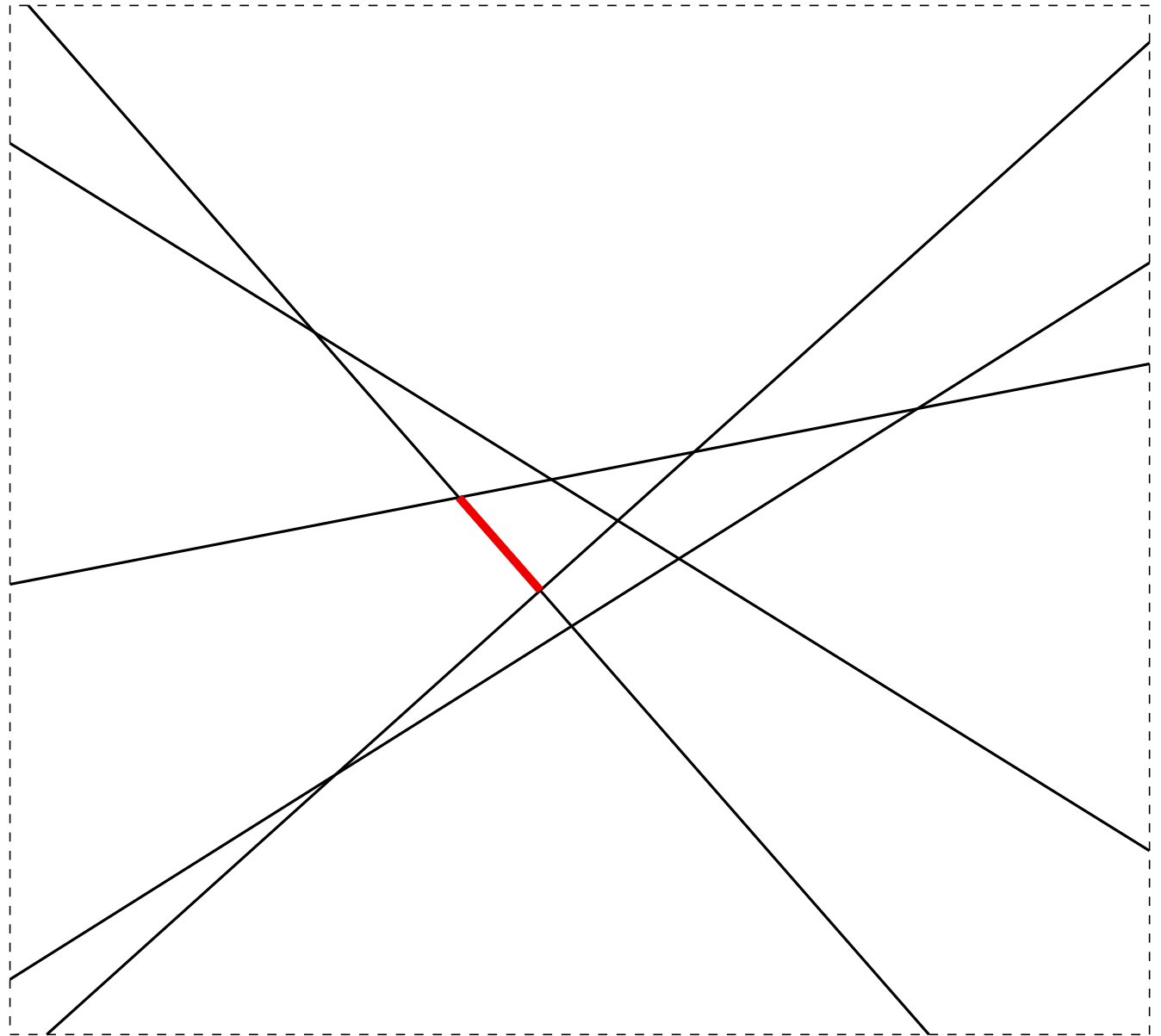


ARRANGEMENTS

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e

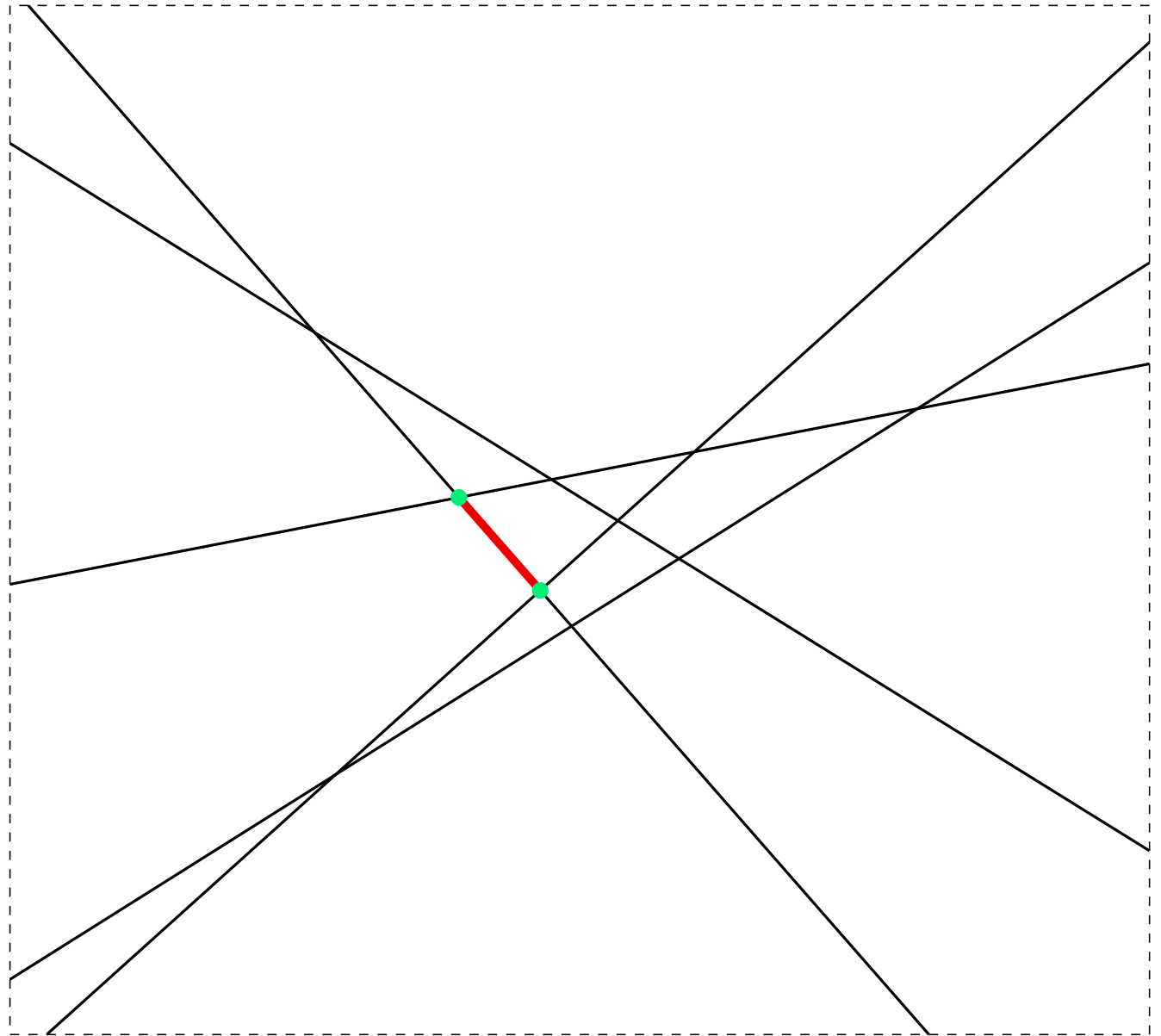


ARRANGEMENTS

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$$e \rightarrow v_B v_E$$

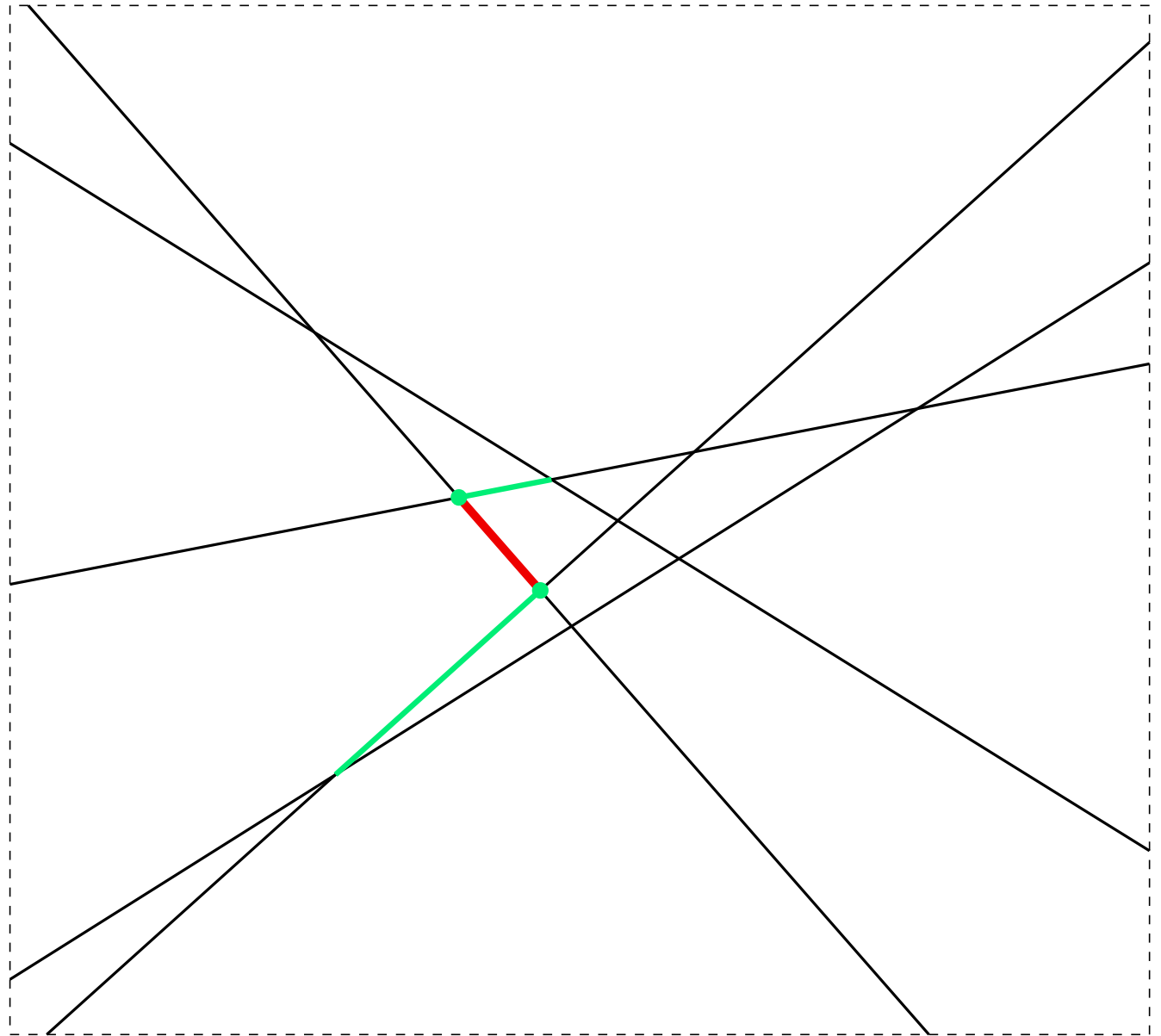


ARRANGEMENTS

STORAGE

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$$e \rightarrow v_B v_E e_P e_N$$

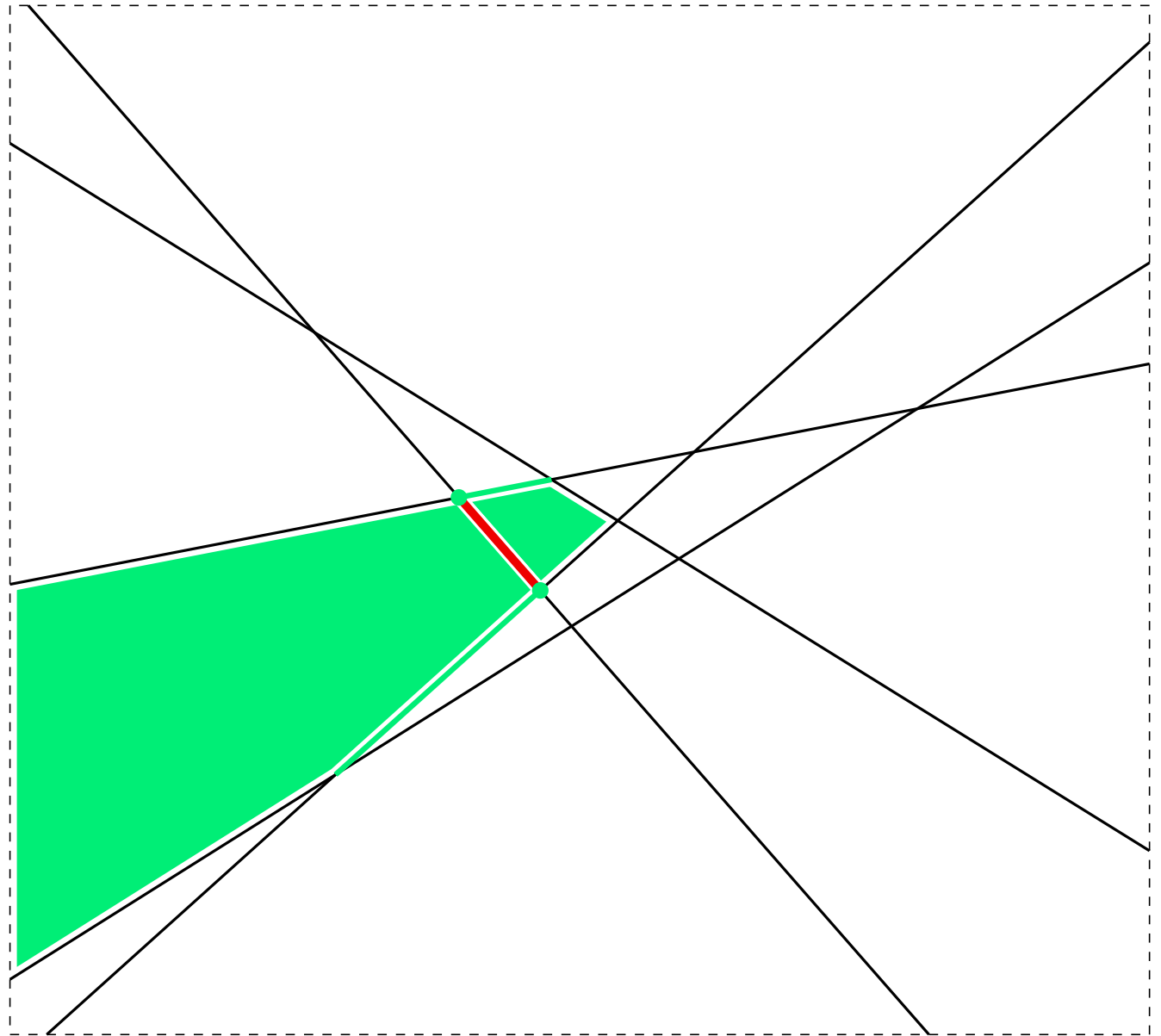


ARRANGEMENTS

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ARRANGEMENTS

STORAGE

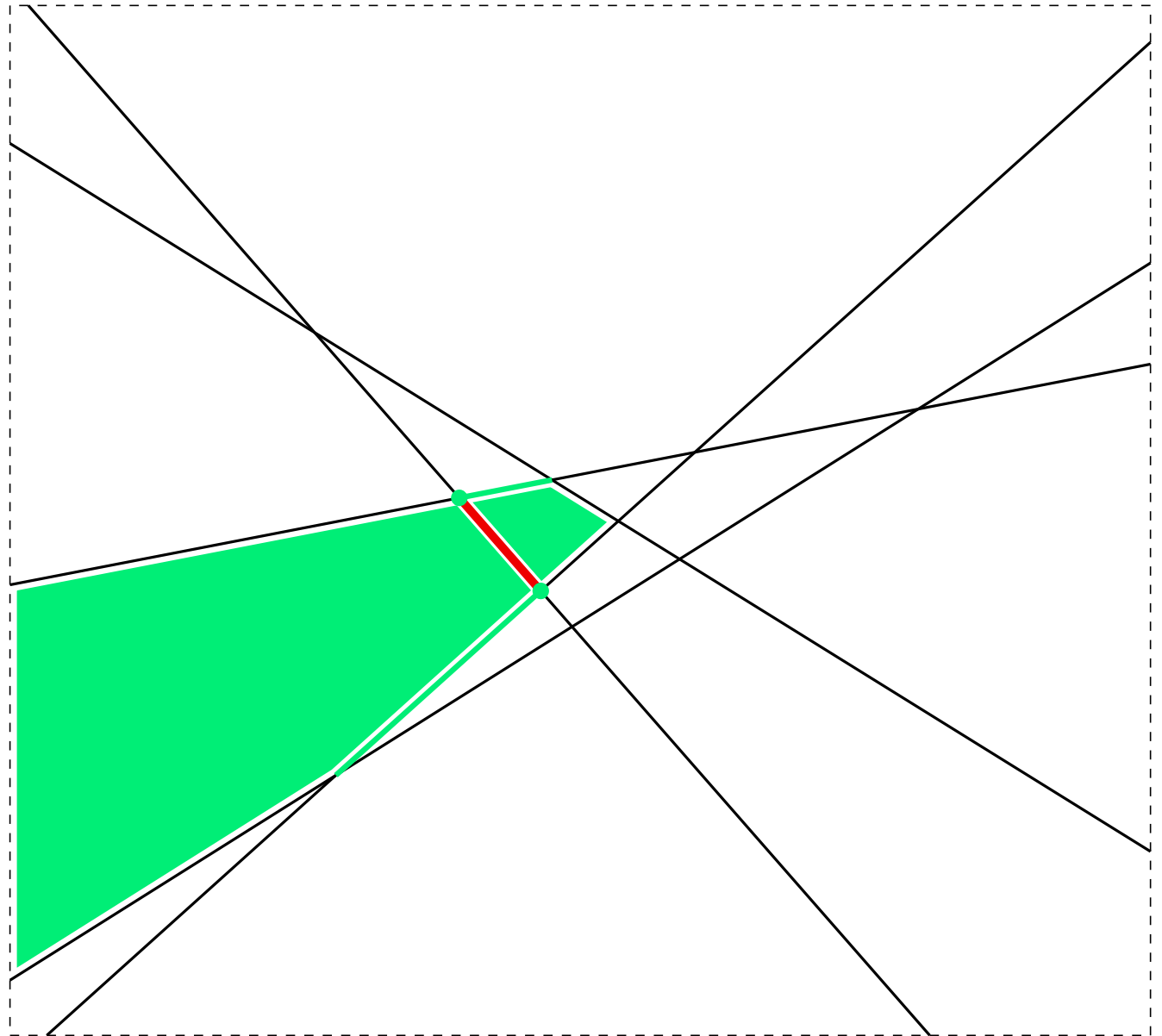
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$$e \rightarrow v_B v_E e_P e_N f_L f_R$$

$$v_i \rightarrow x_i, y_i, e$$

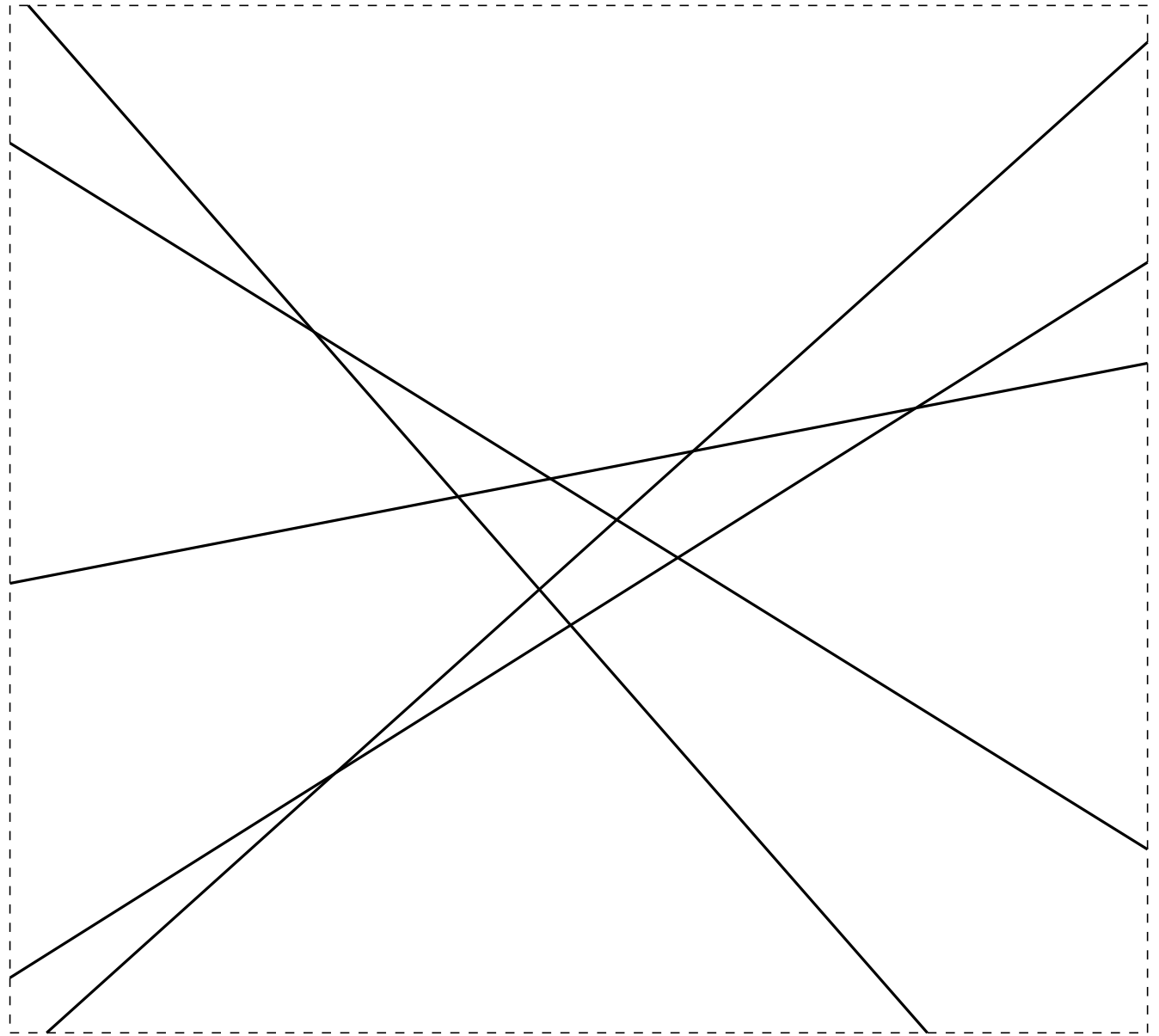
$$f_j \rightarrow e$$

(Recall that halflines require a specific treatment, for instance, by adding a large enough bounding box)



ARRANGEMENTS

COMPUTATION



ARRANGEMENTS

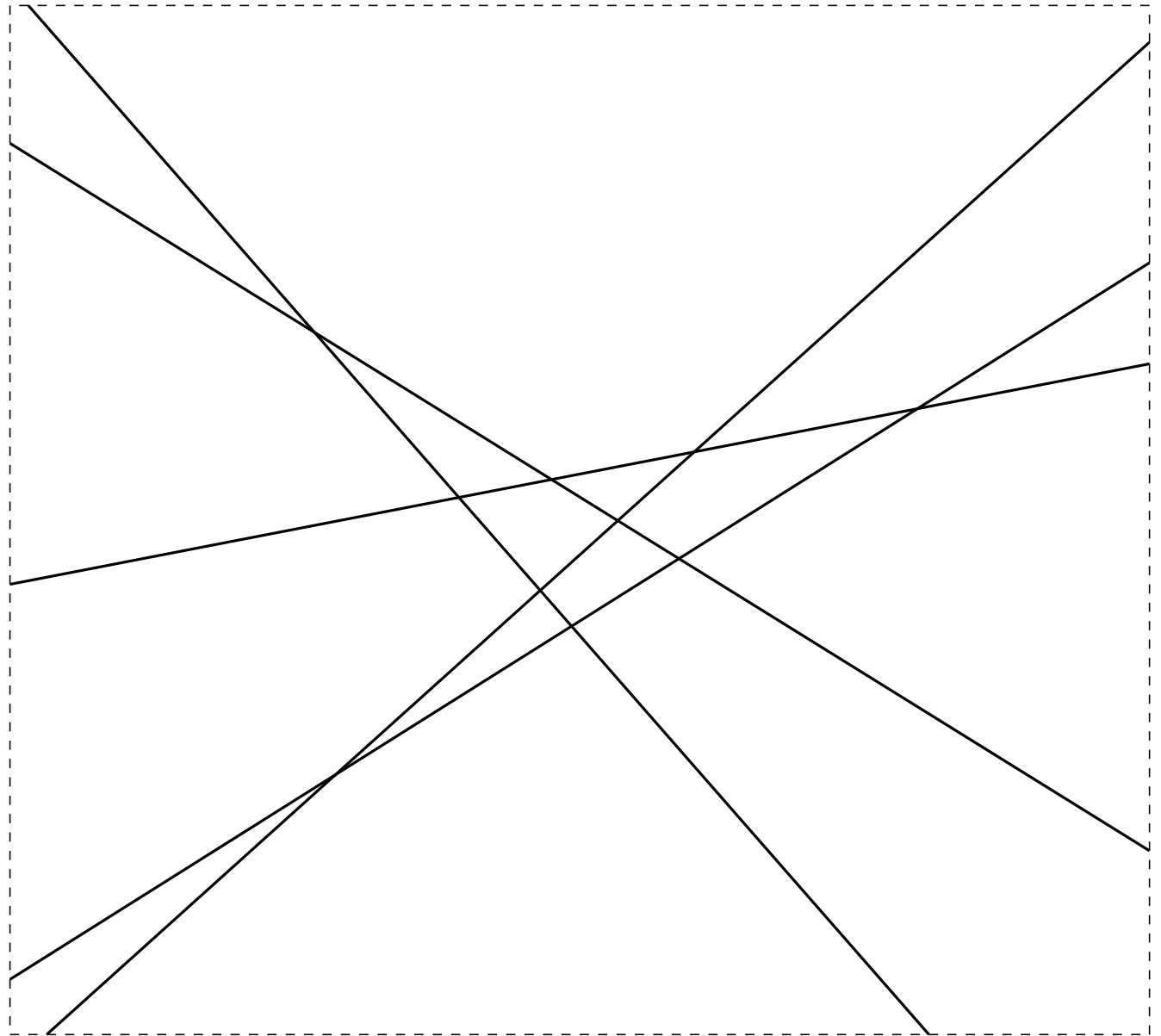
COMPUTATION

Input

A set of n lines $L = \{l_1, \dots, l_n\}$

Output

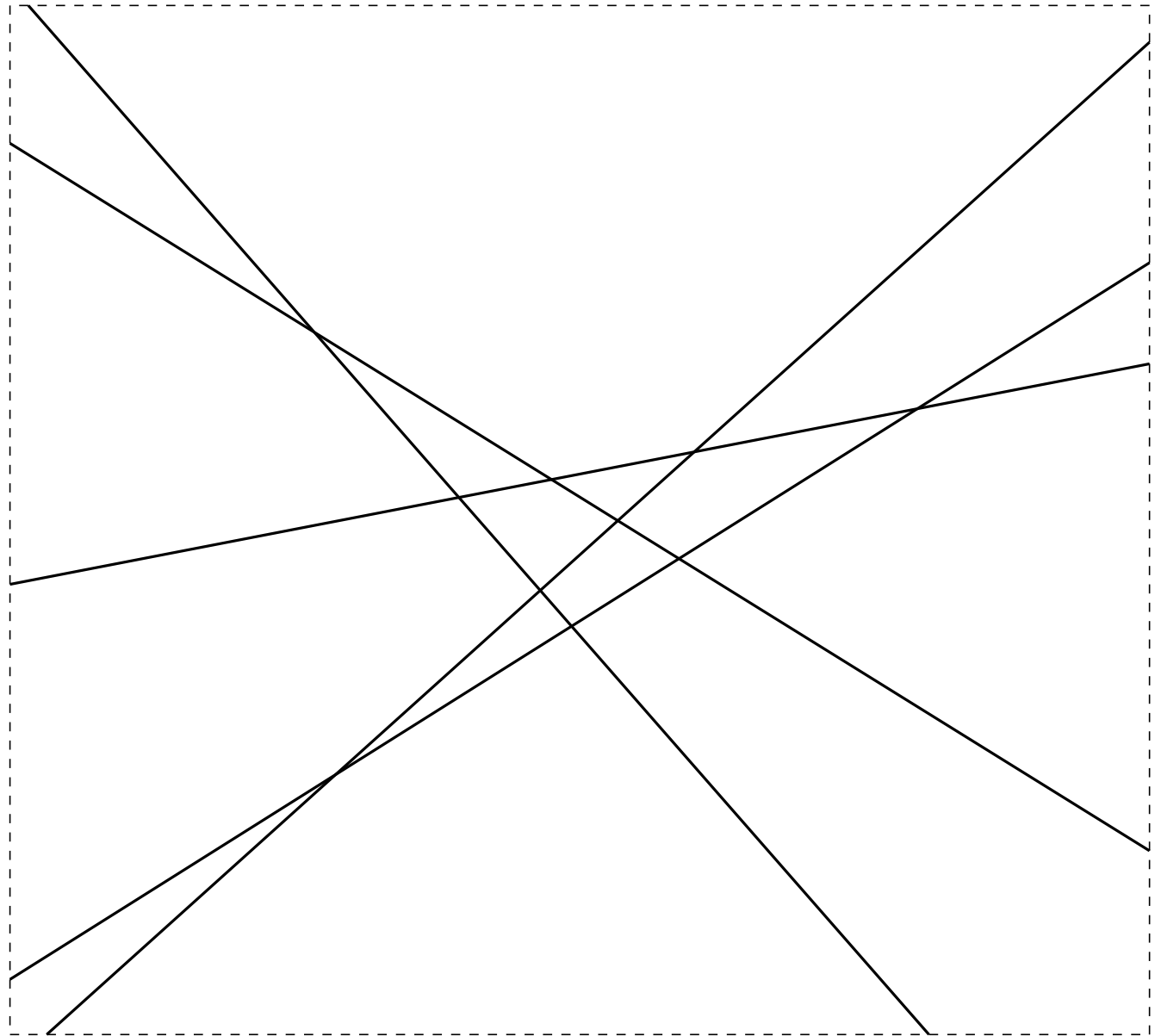
The arrangement $\mathcal{A}(L)$ i.e.,
the DCEL of the arrangement.



ARRANGEMENTS

COMPUTATION

Sweep line algorithm



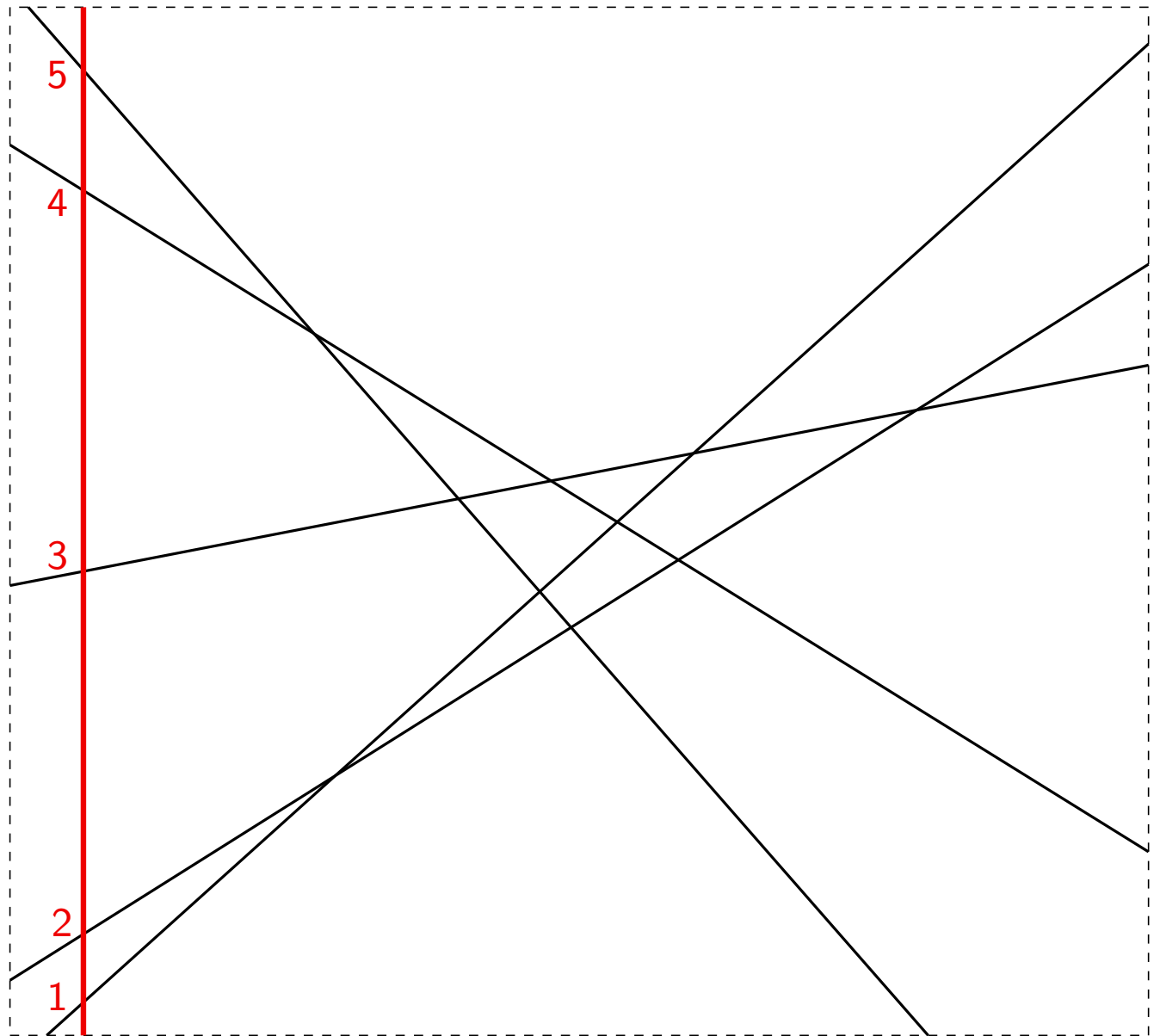
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Sweep line

Sorted list of the stabbed lines, in vertical order.



ARRANGEMENTS

COMPUTATION

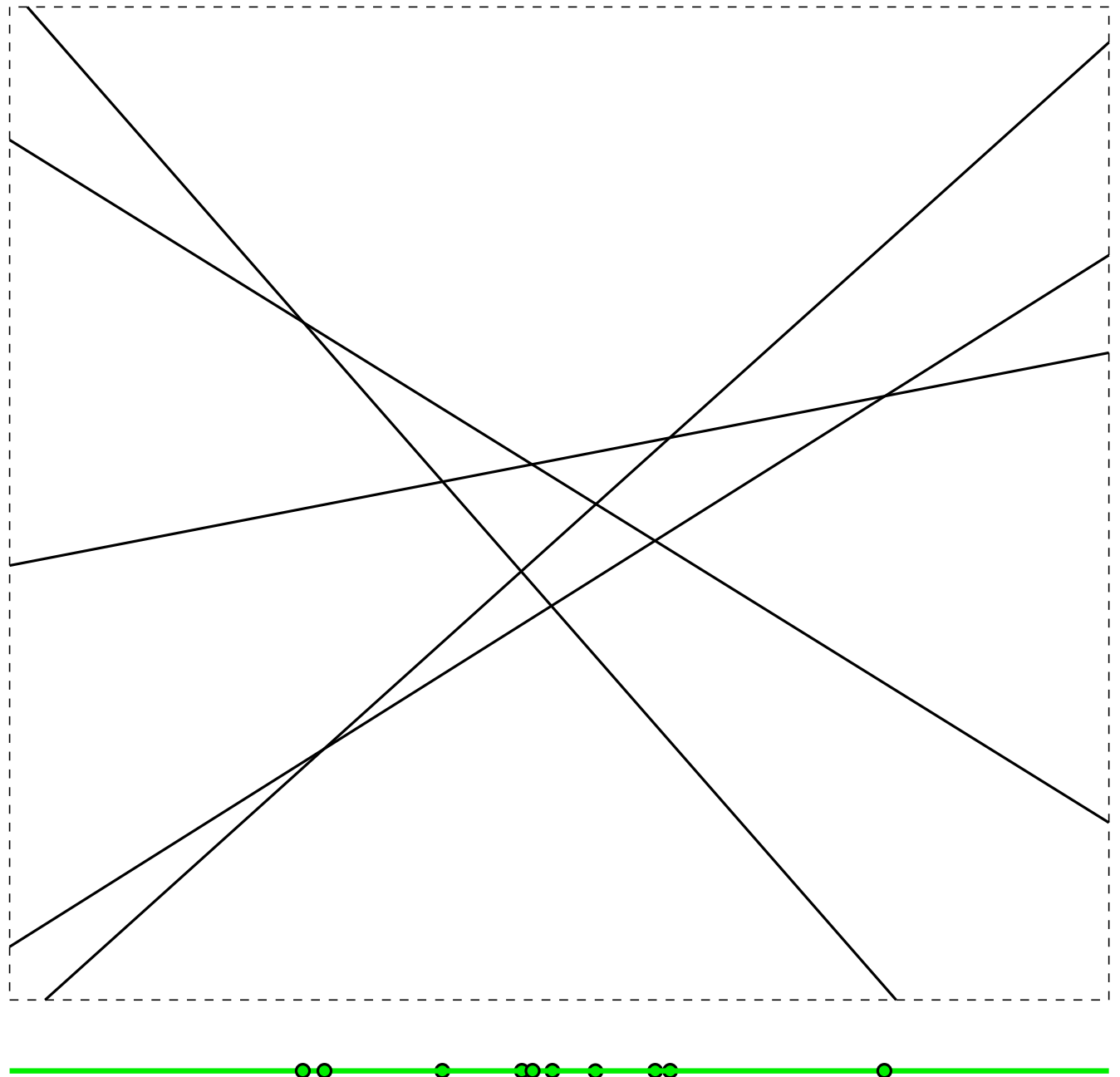
Sweep line algorithm

Sweep line

Sorted list of the stabbed lines, in vertical order.

Events queue

Sorted list of the intersection points, in horizontal order. Not known a priori, they are found on the fly.



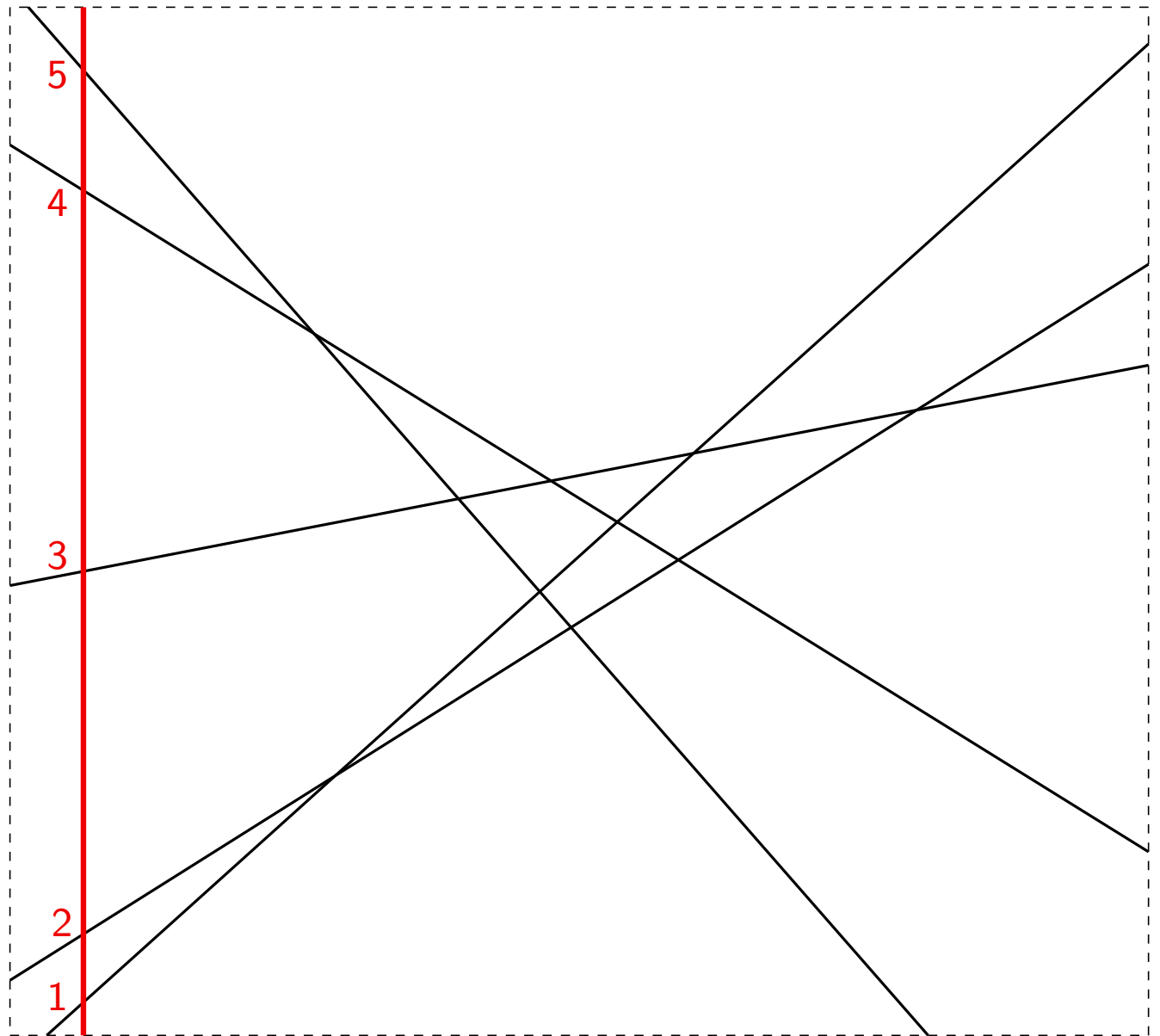
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Initialization:

- Sort the lines by slope and store them in the sweep line.



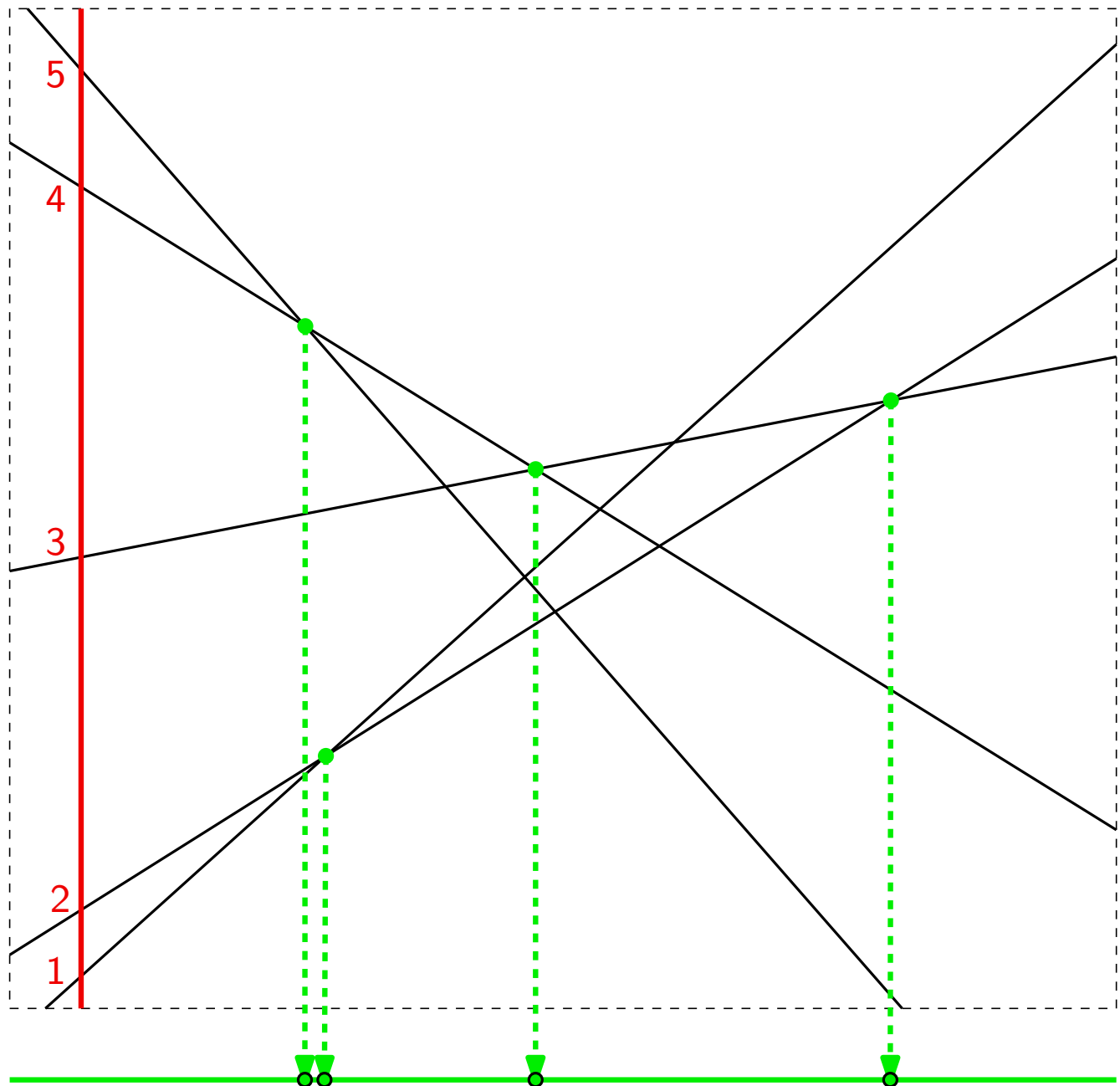
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Initialization:

- Sort the lines by slope and store them in the sweep line.
- Detect the intersection point of each pair of consecutive lines, sort them by abscissa and store them in the events queue.

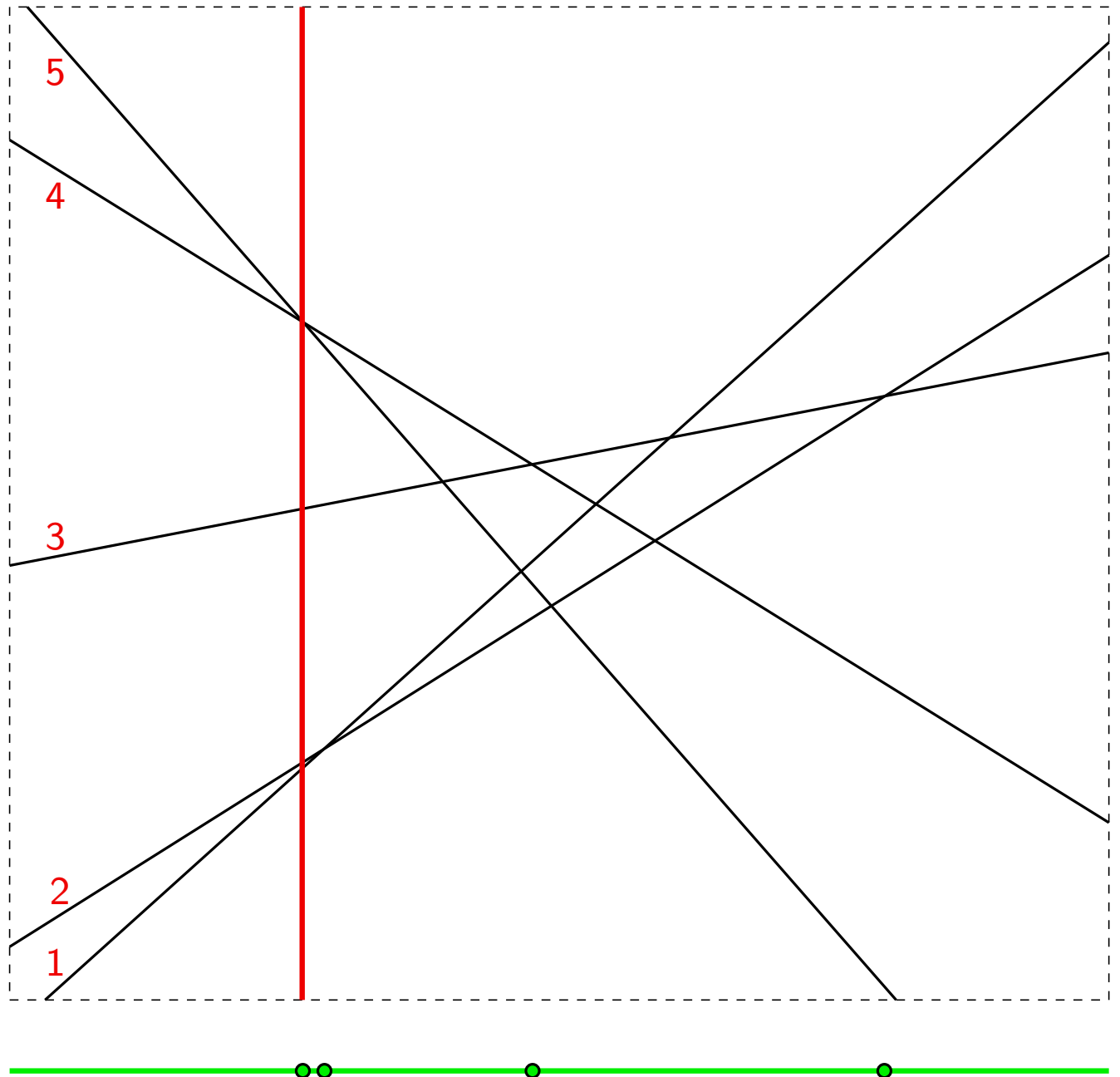


ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Advance: at each event,



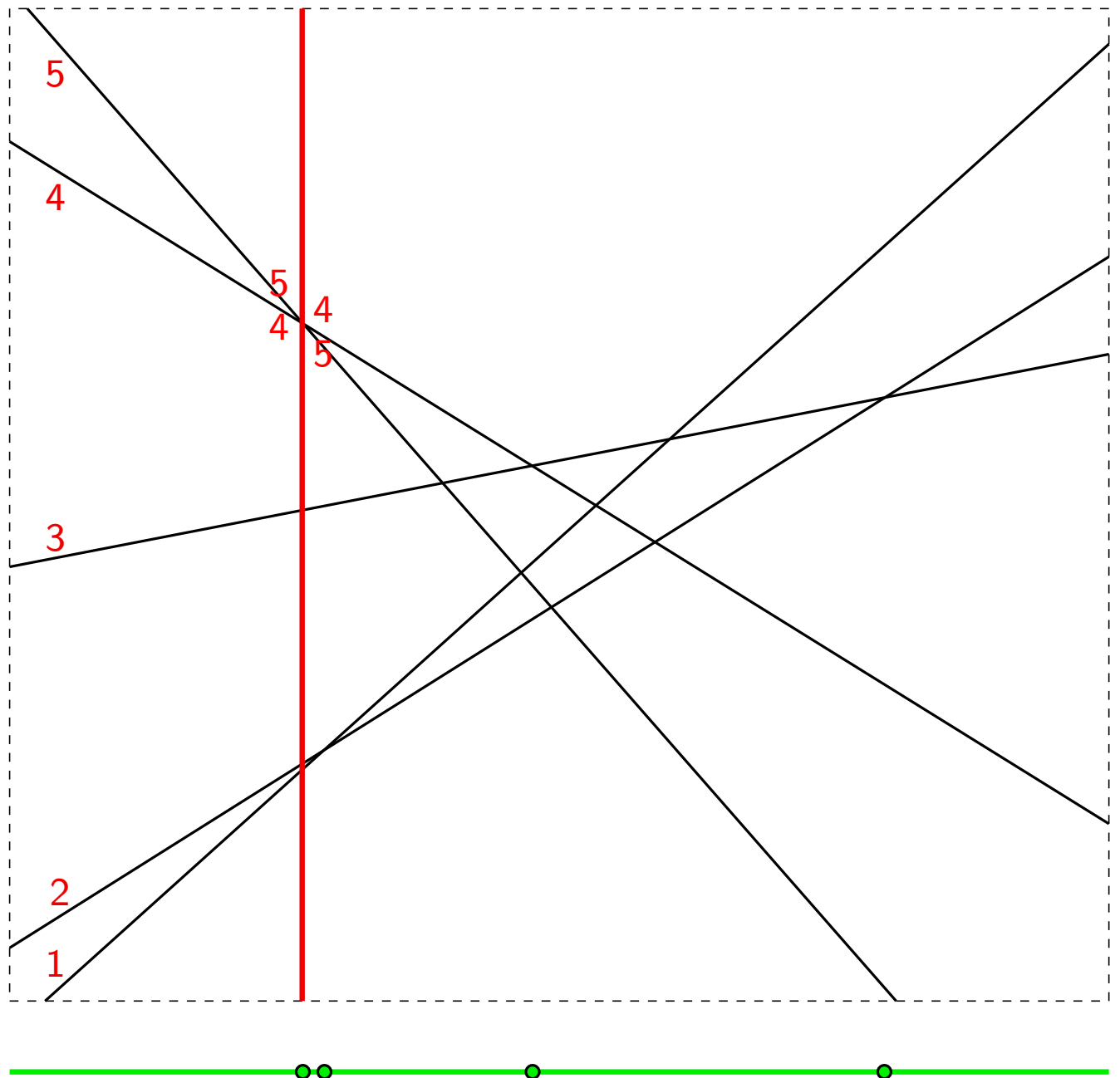
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Advance: at each event,

1. Swap the order of the two intersecting lines in the sweep line.



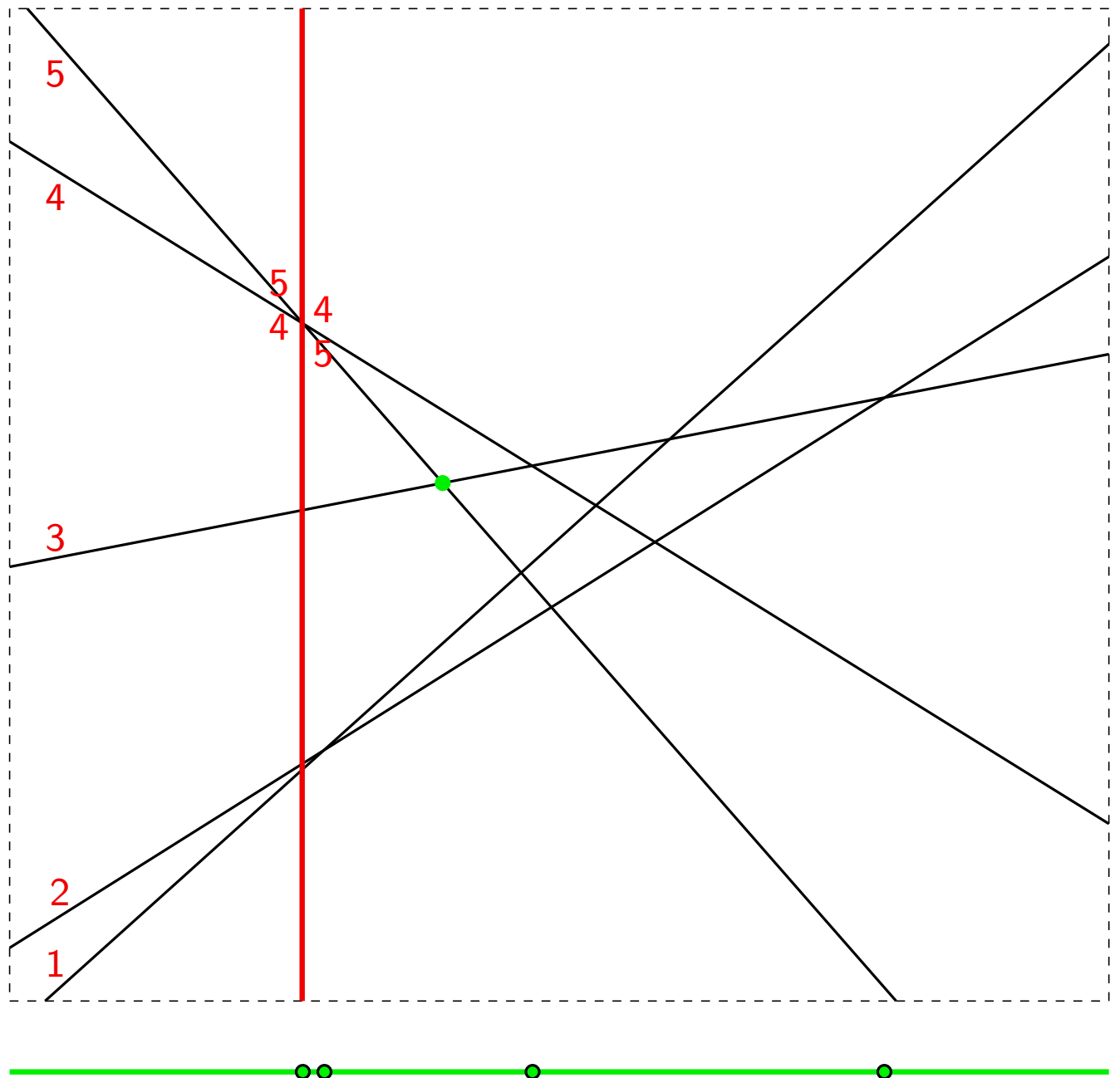
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Advance: at each event,

1. Swap the order of the two intersecting lines in the sweep line.
2. Compute the intersection points with the two new neighboring lines.



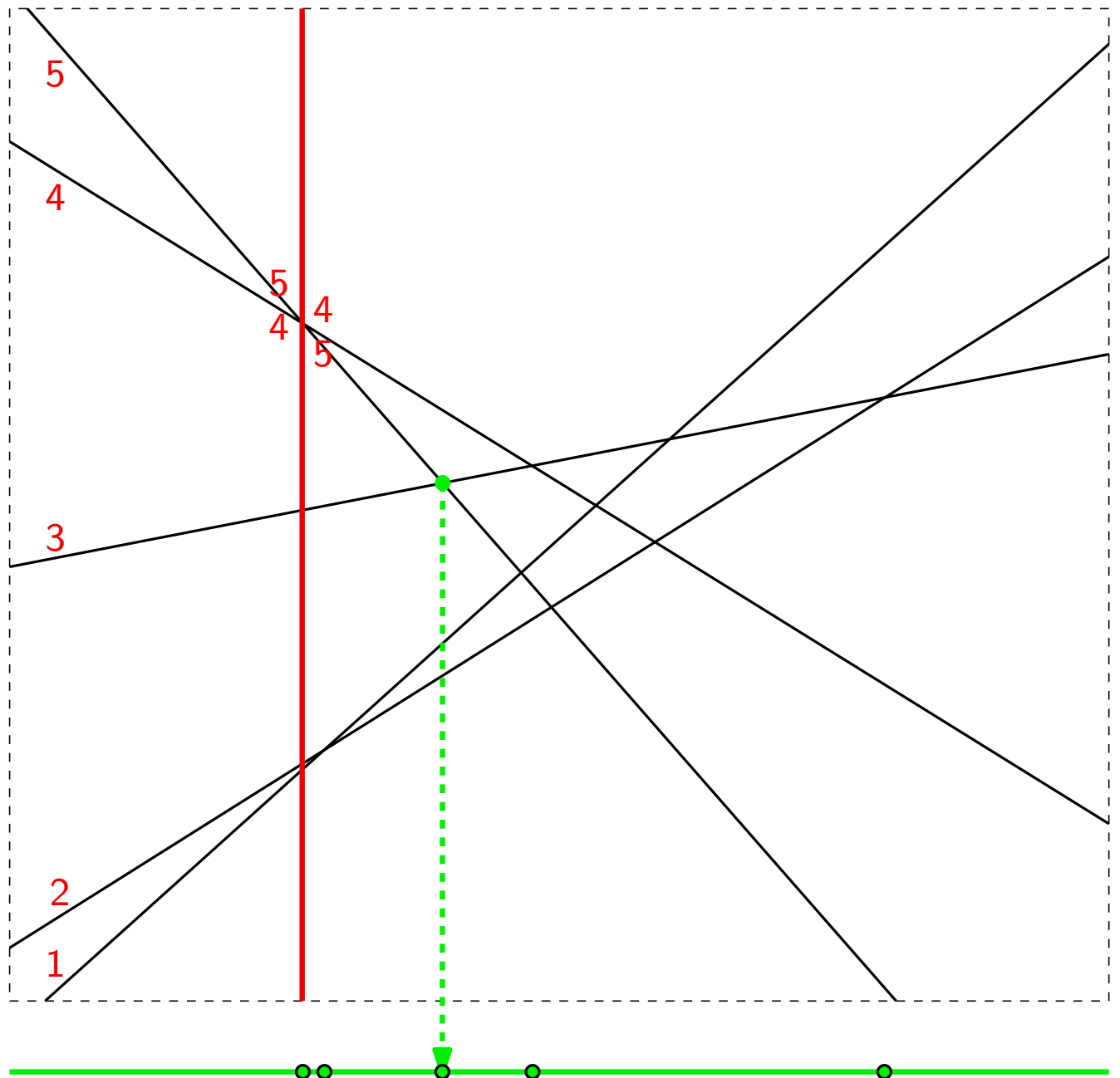
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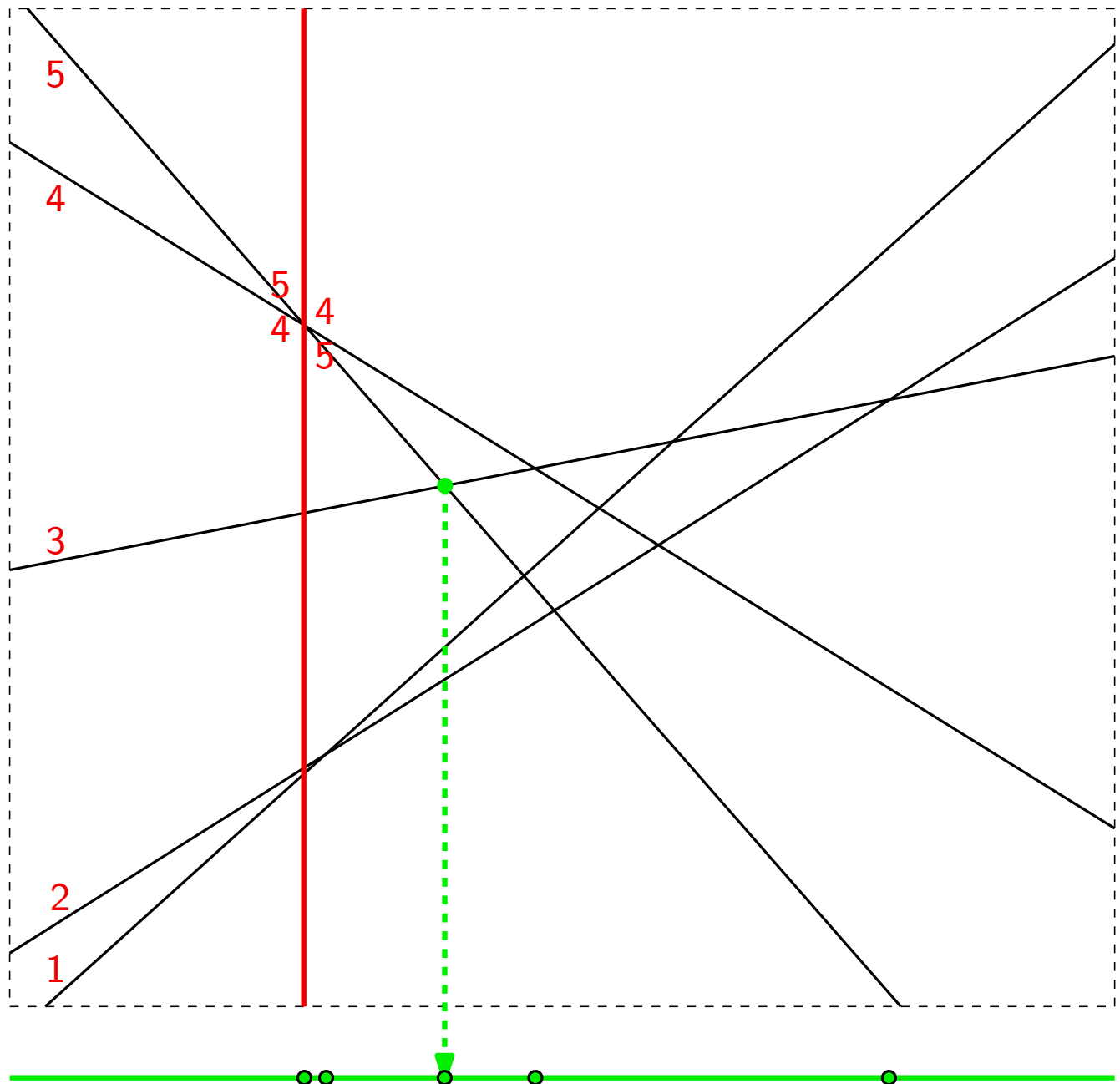
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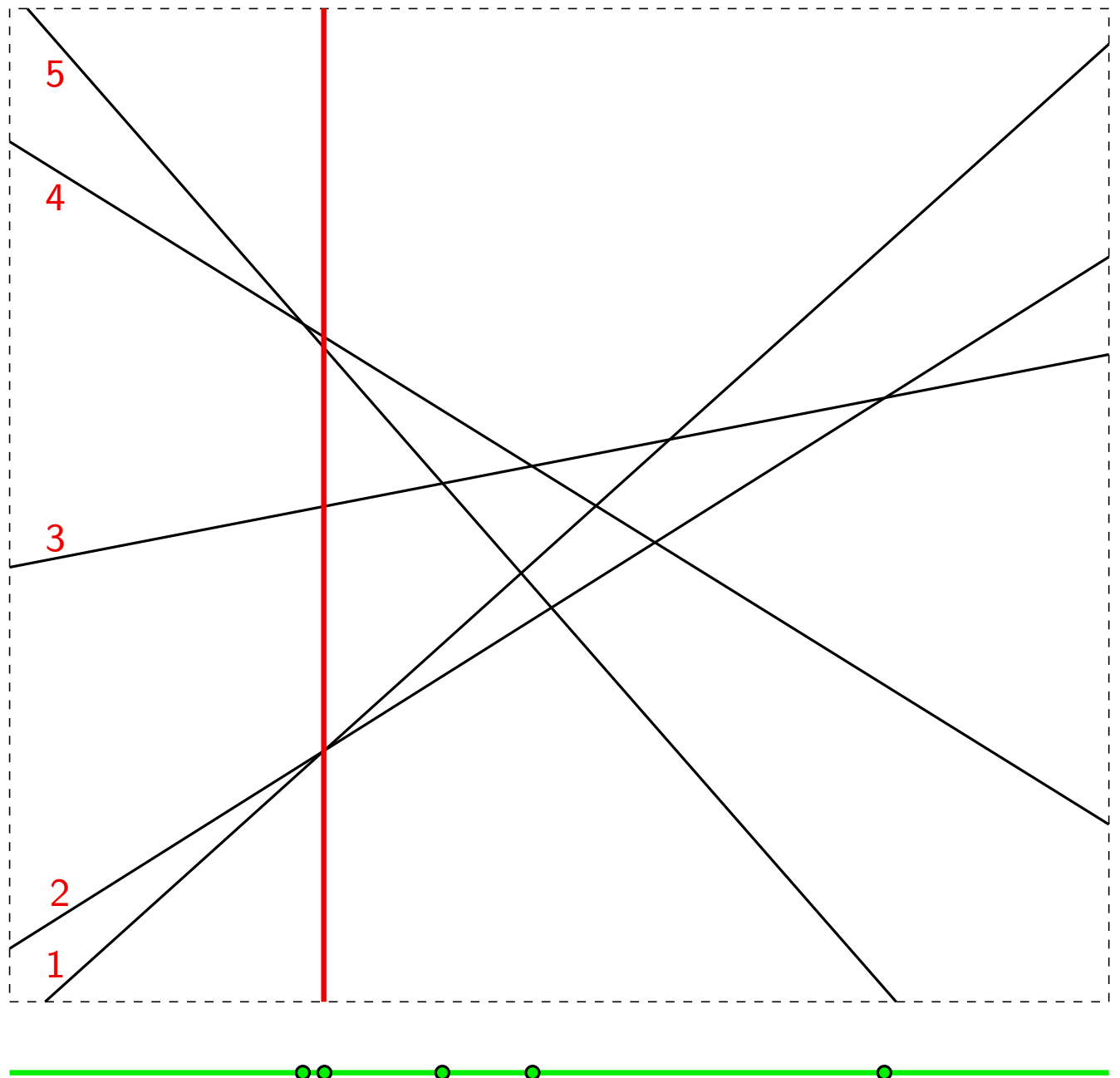
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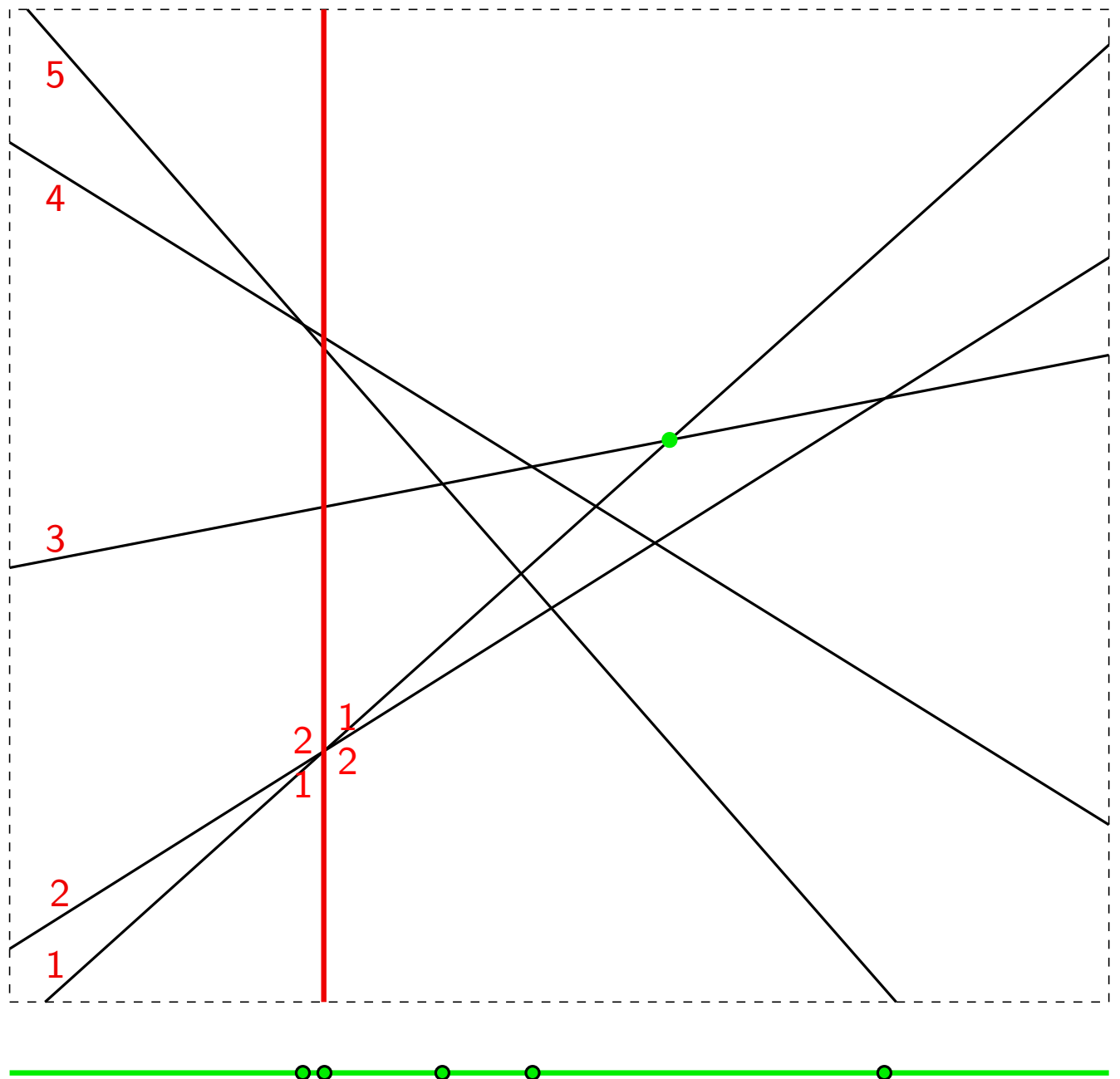
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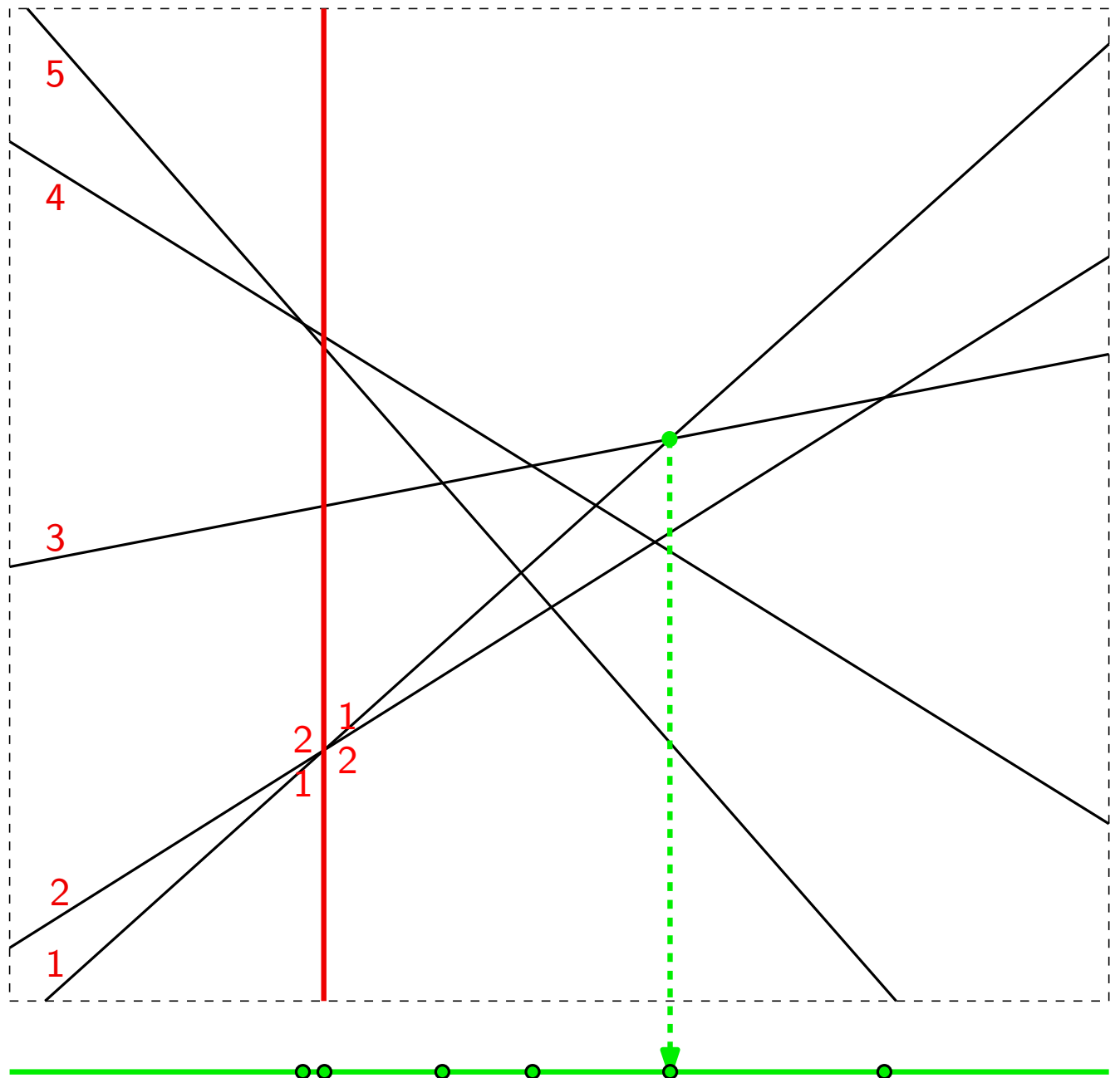
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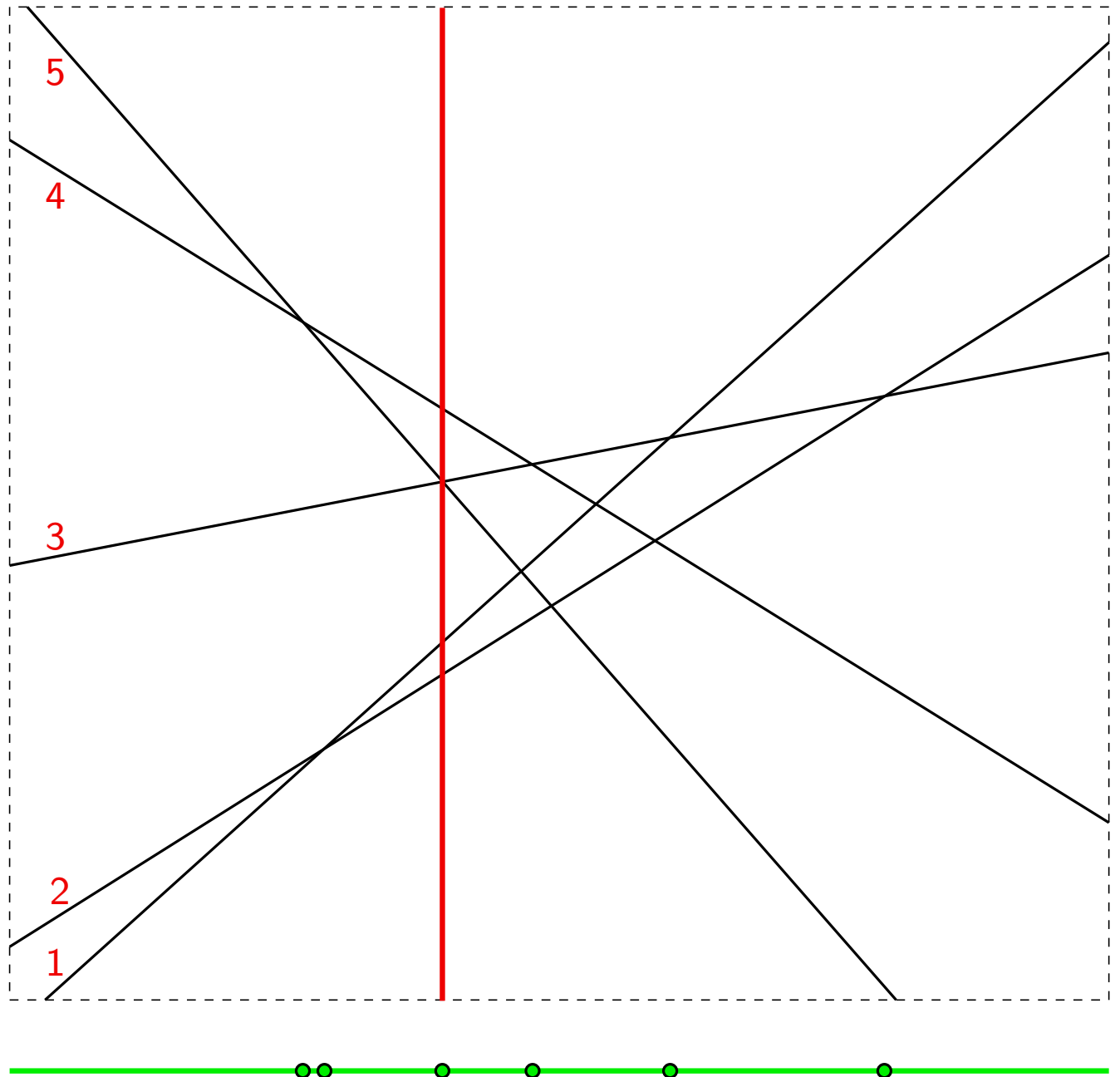
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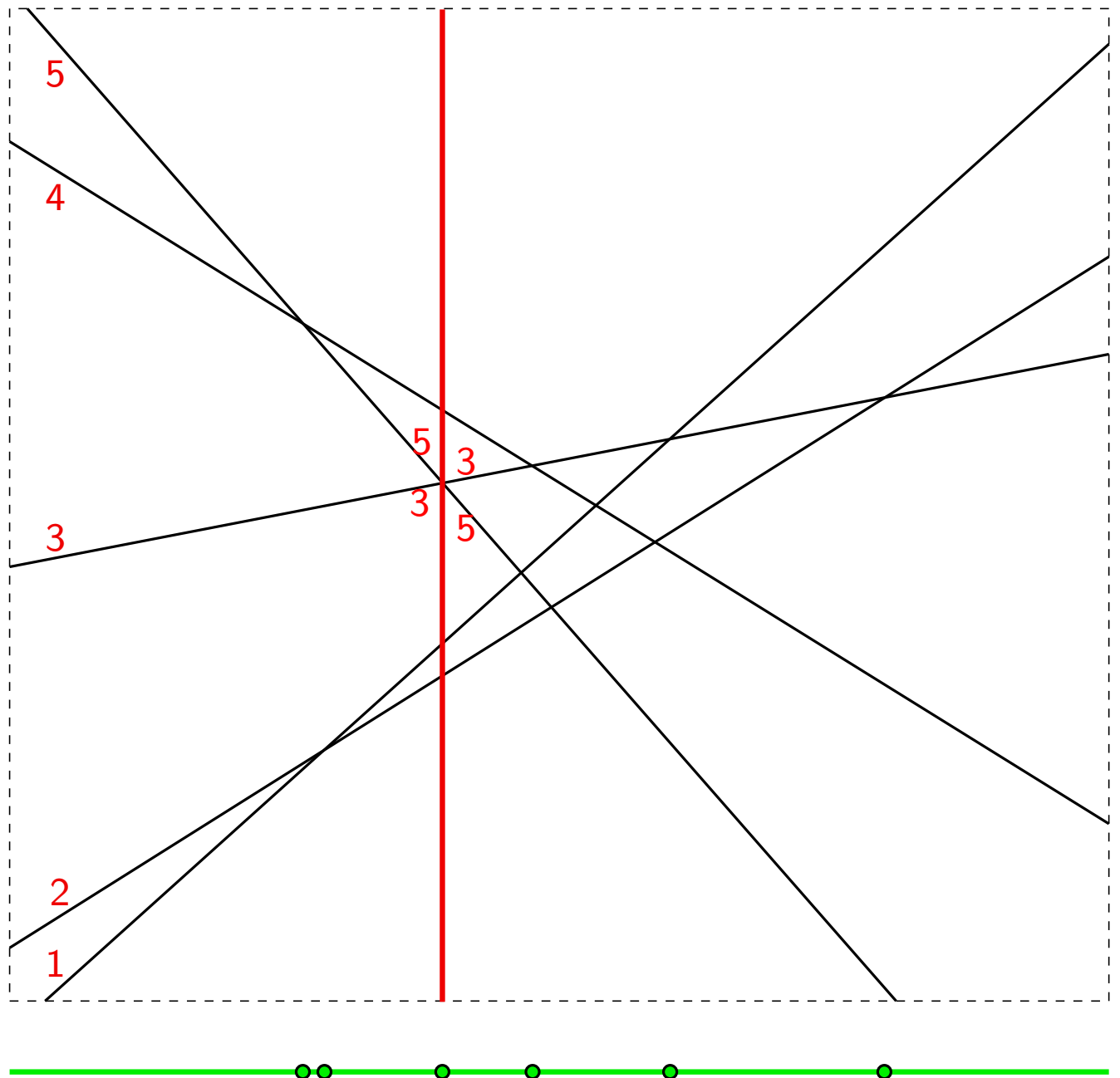
ARRANGEMENTS

COMPUTATION

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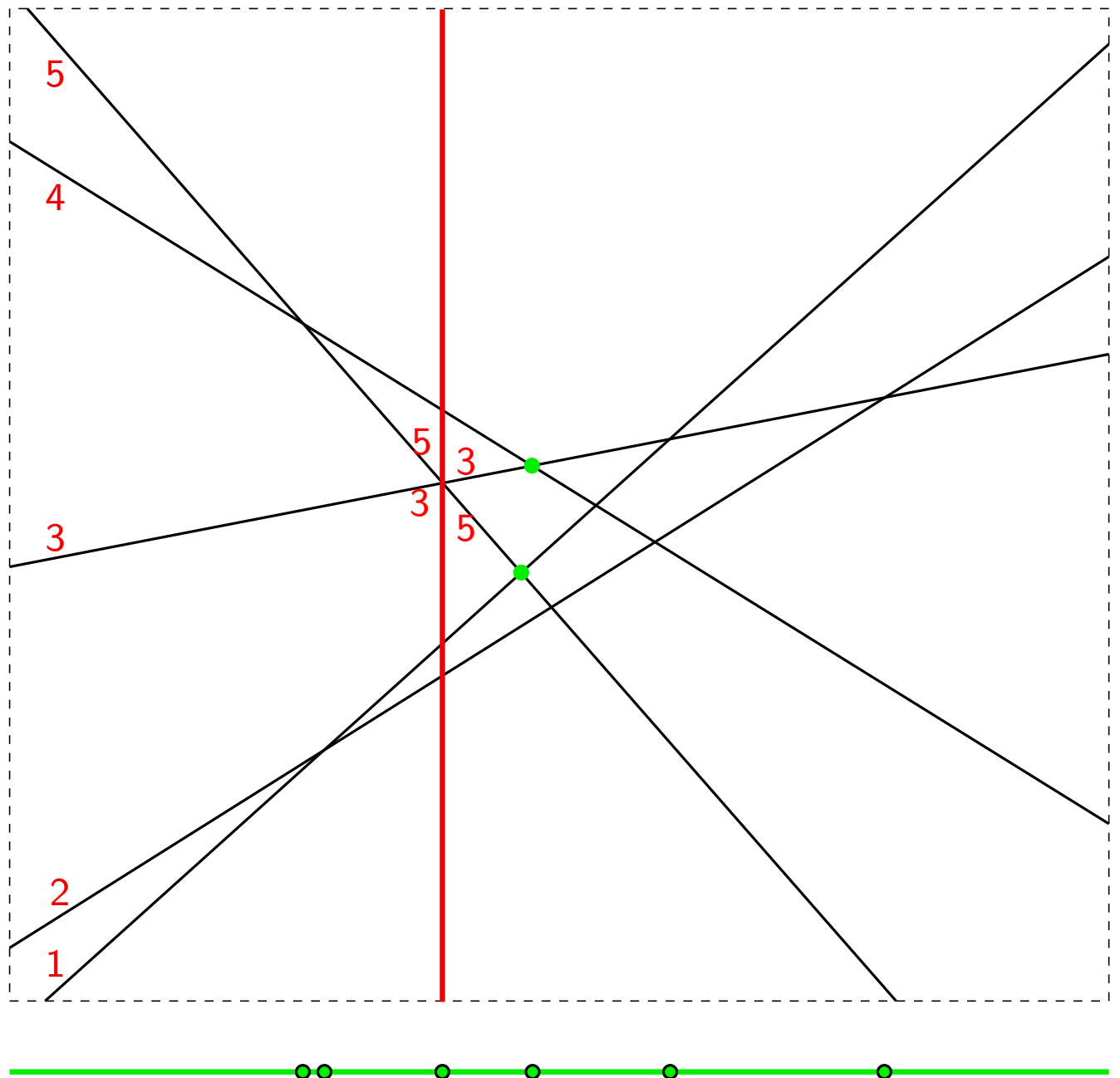
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COMPUTATION

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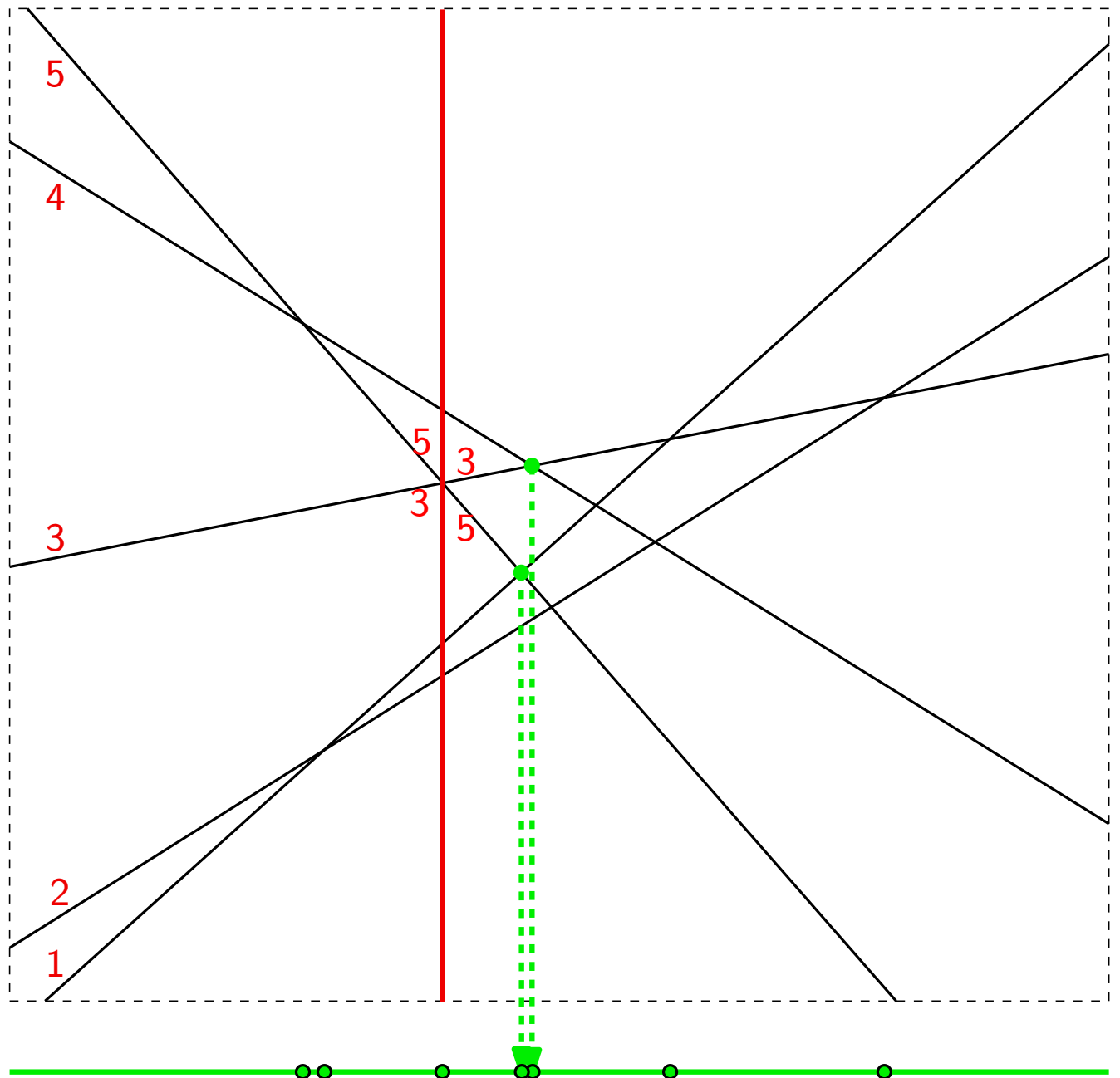
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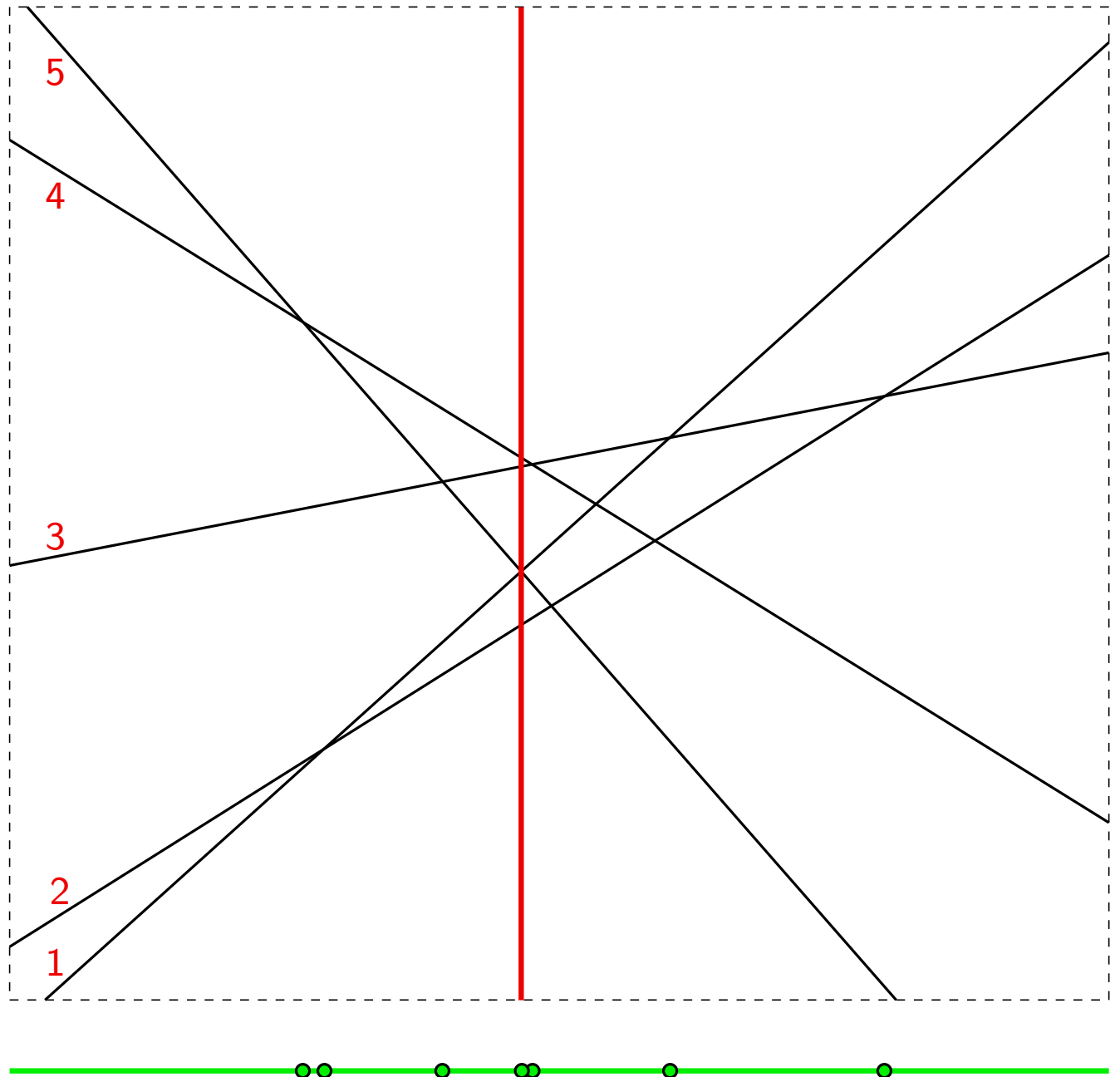
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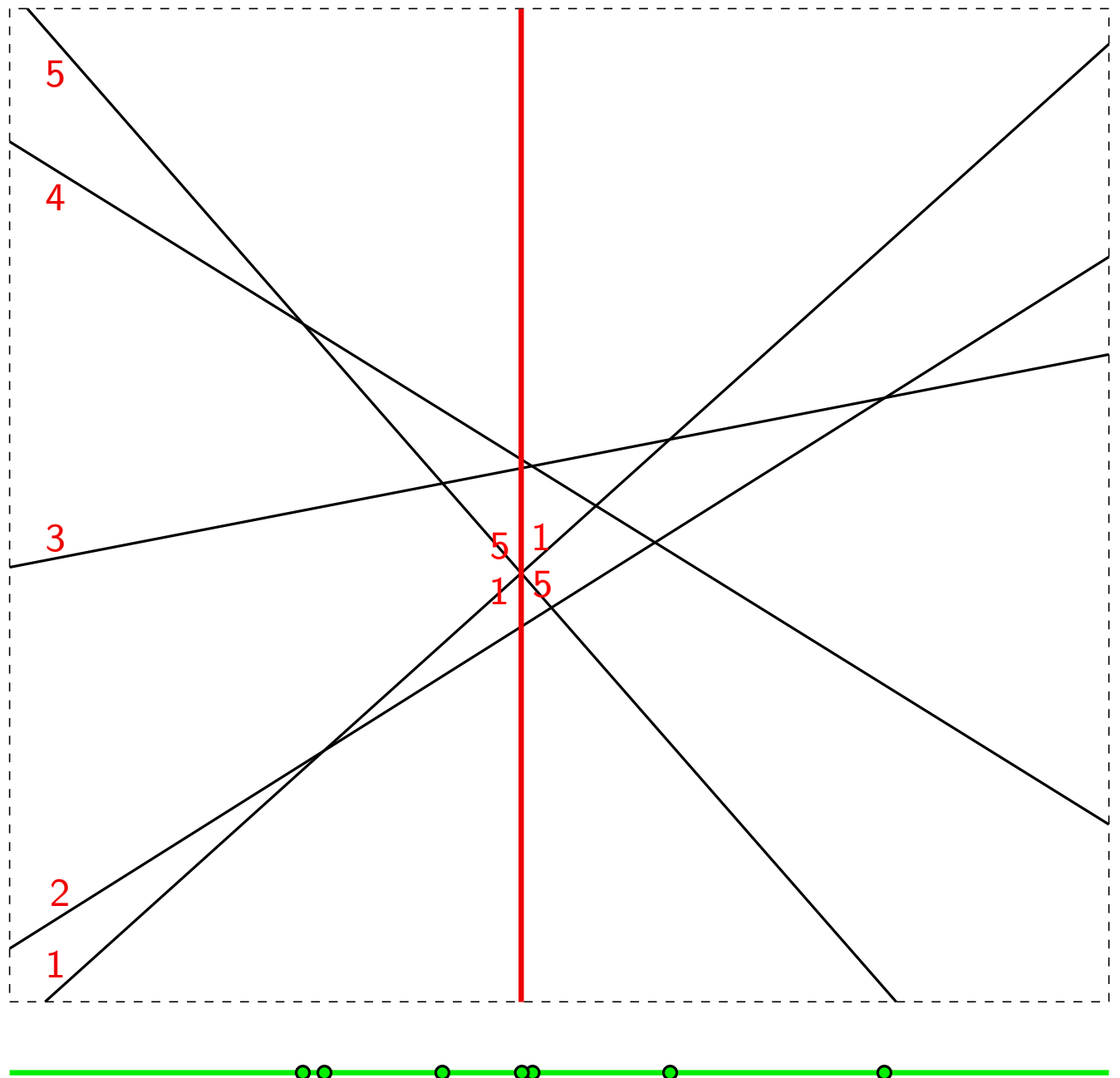
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COMPUTATION

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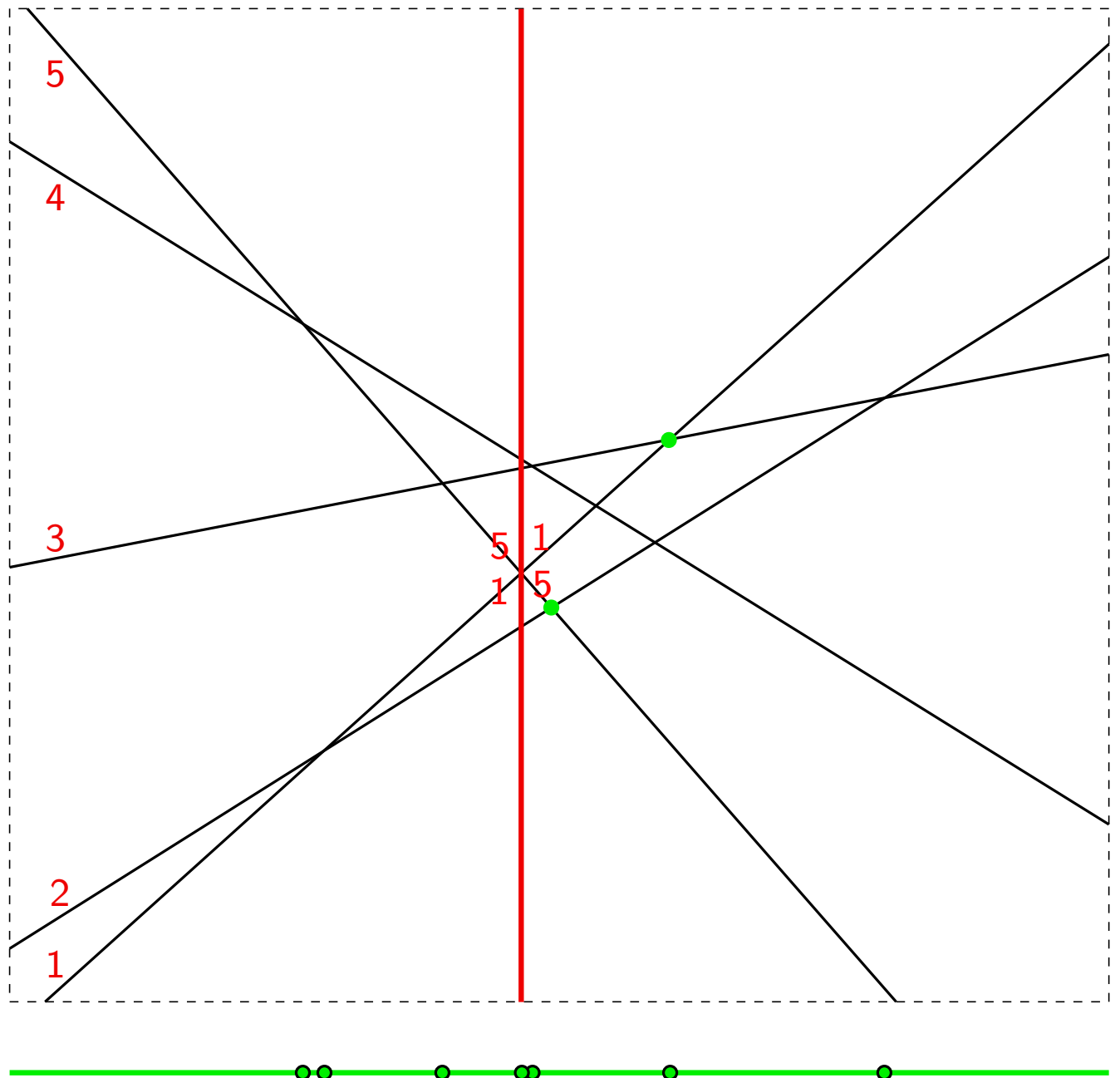
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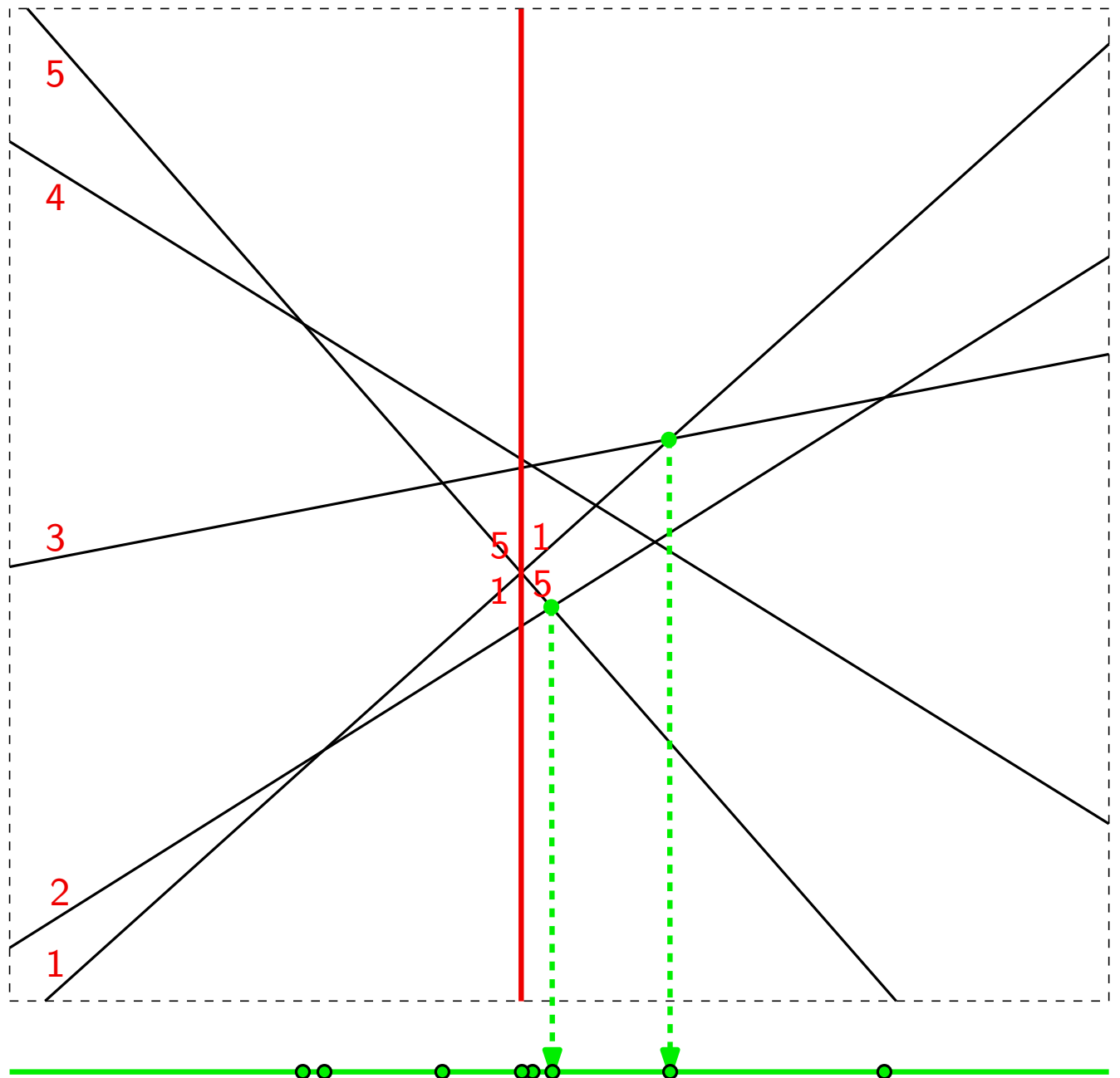
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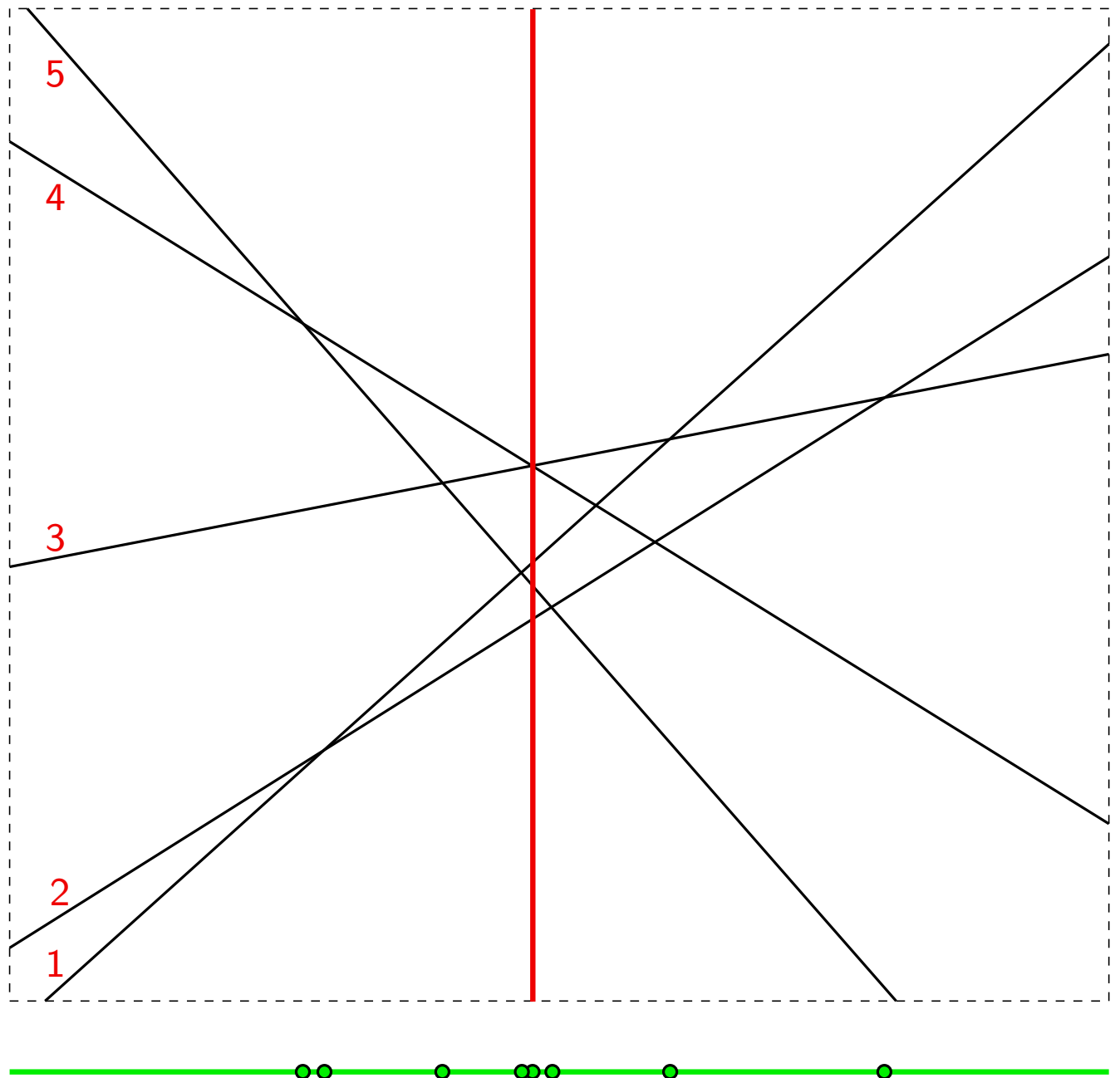
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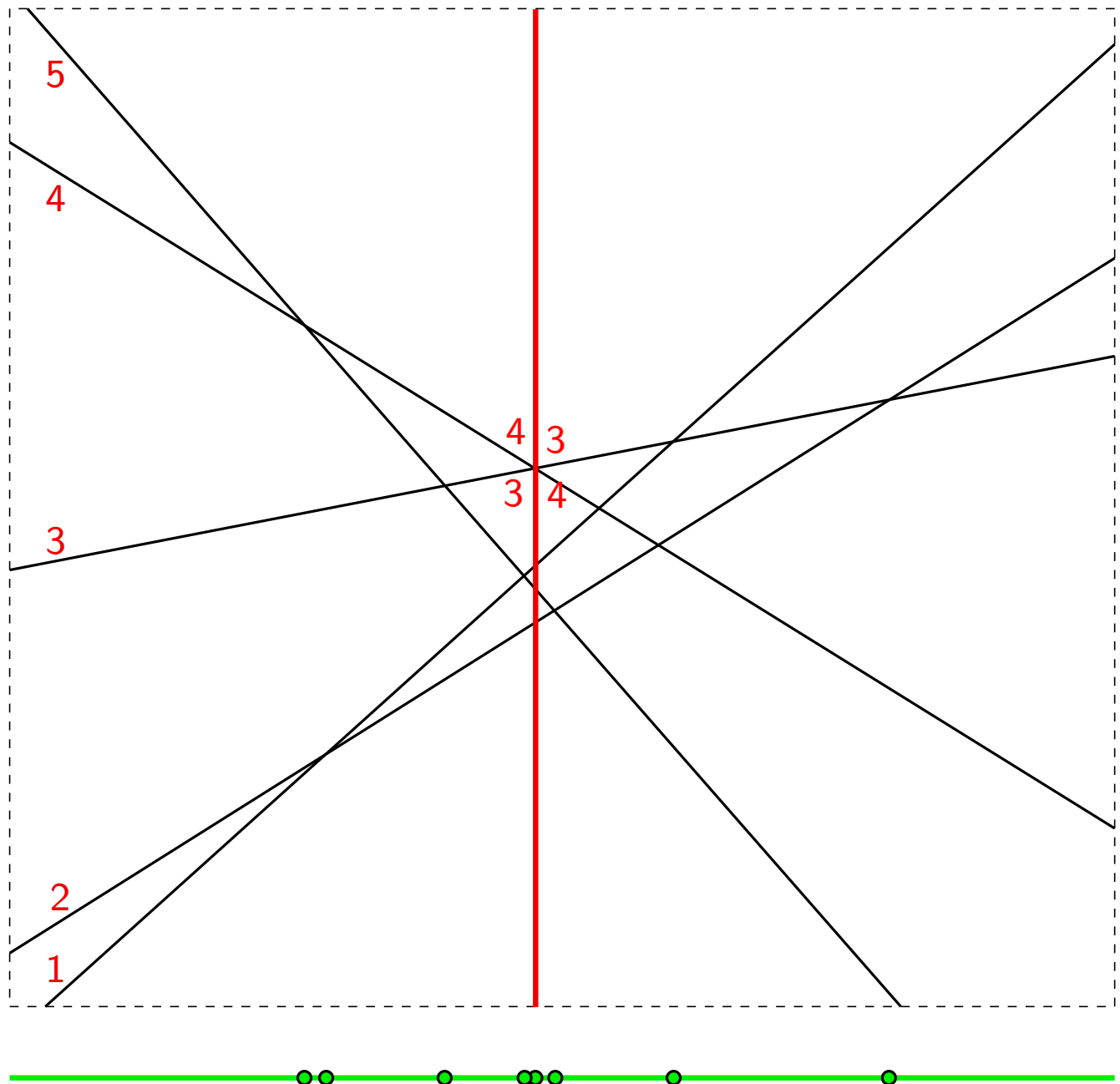
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COMPUTATION

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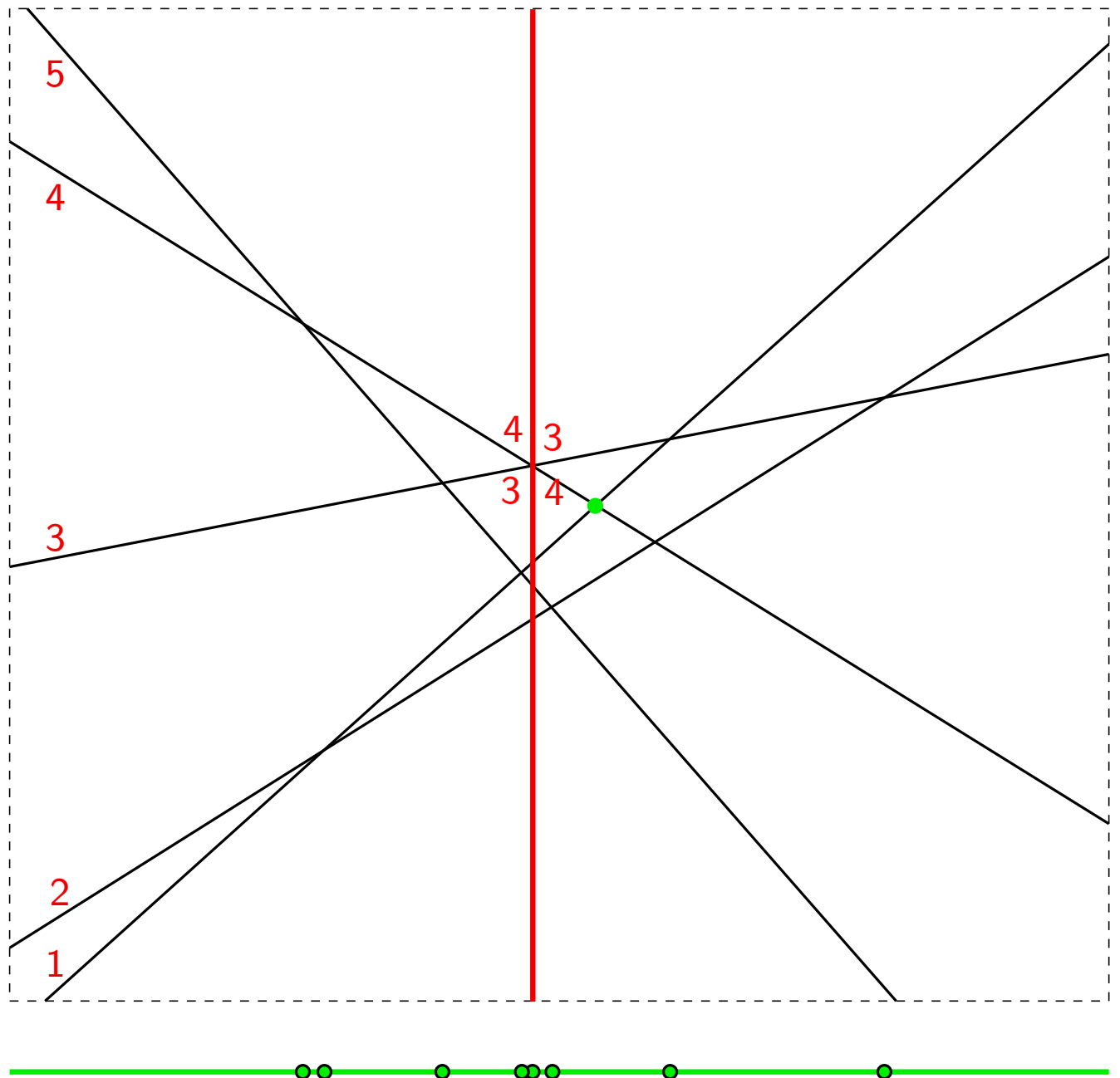
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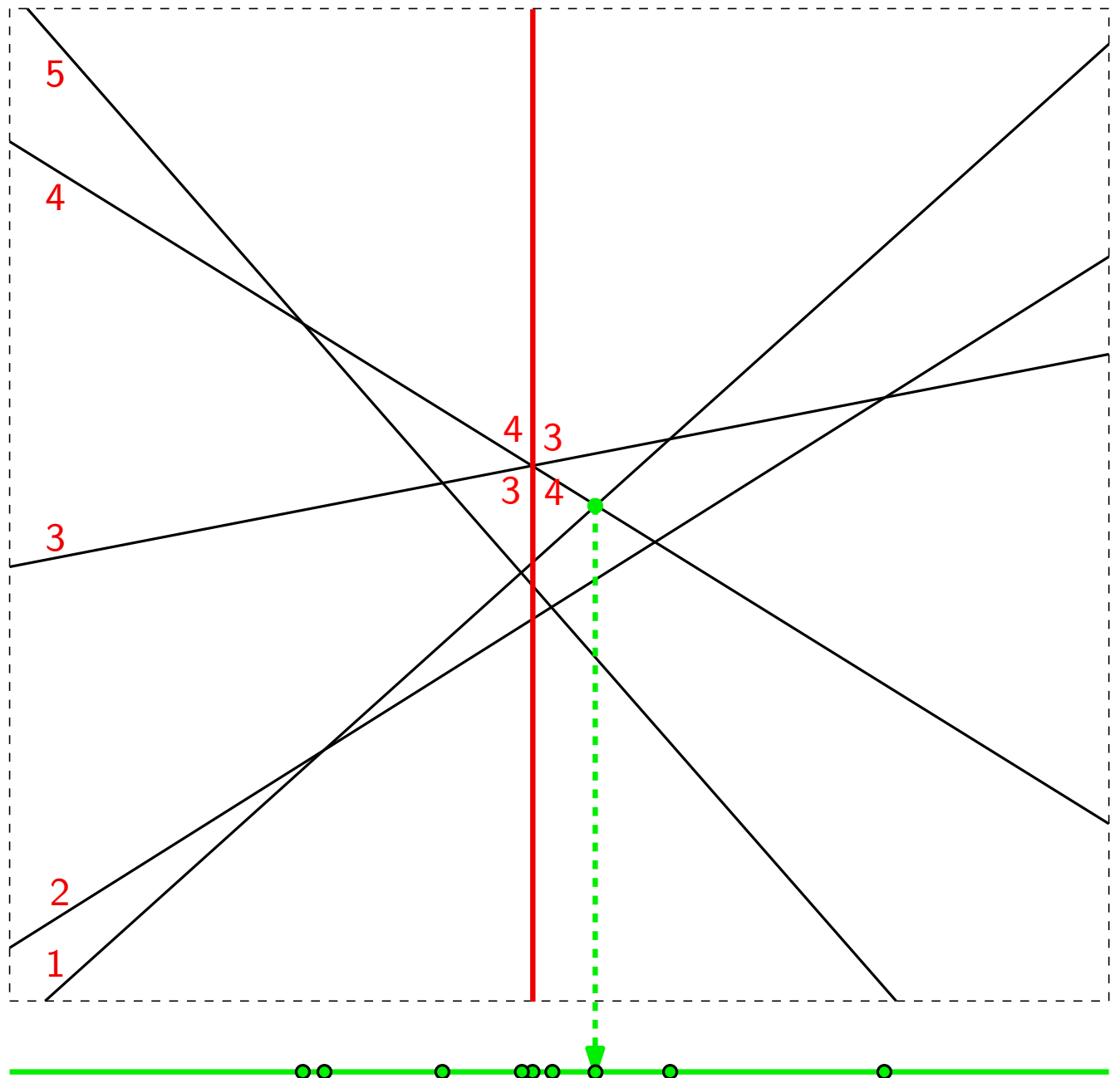
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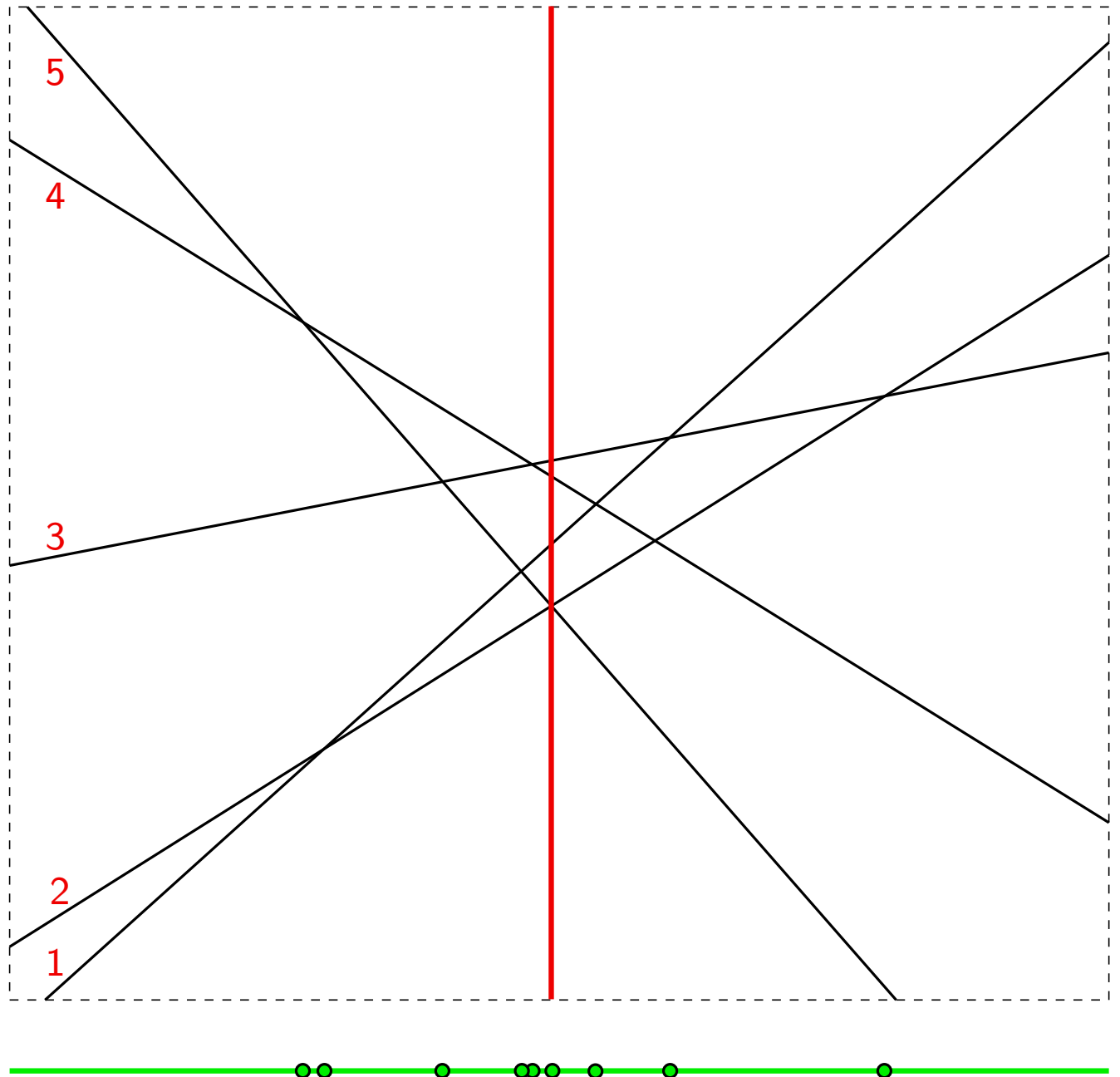
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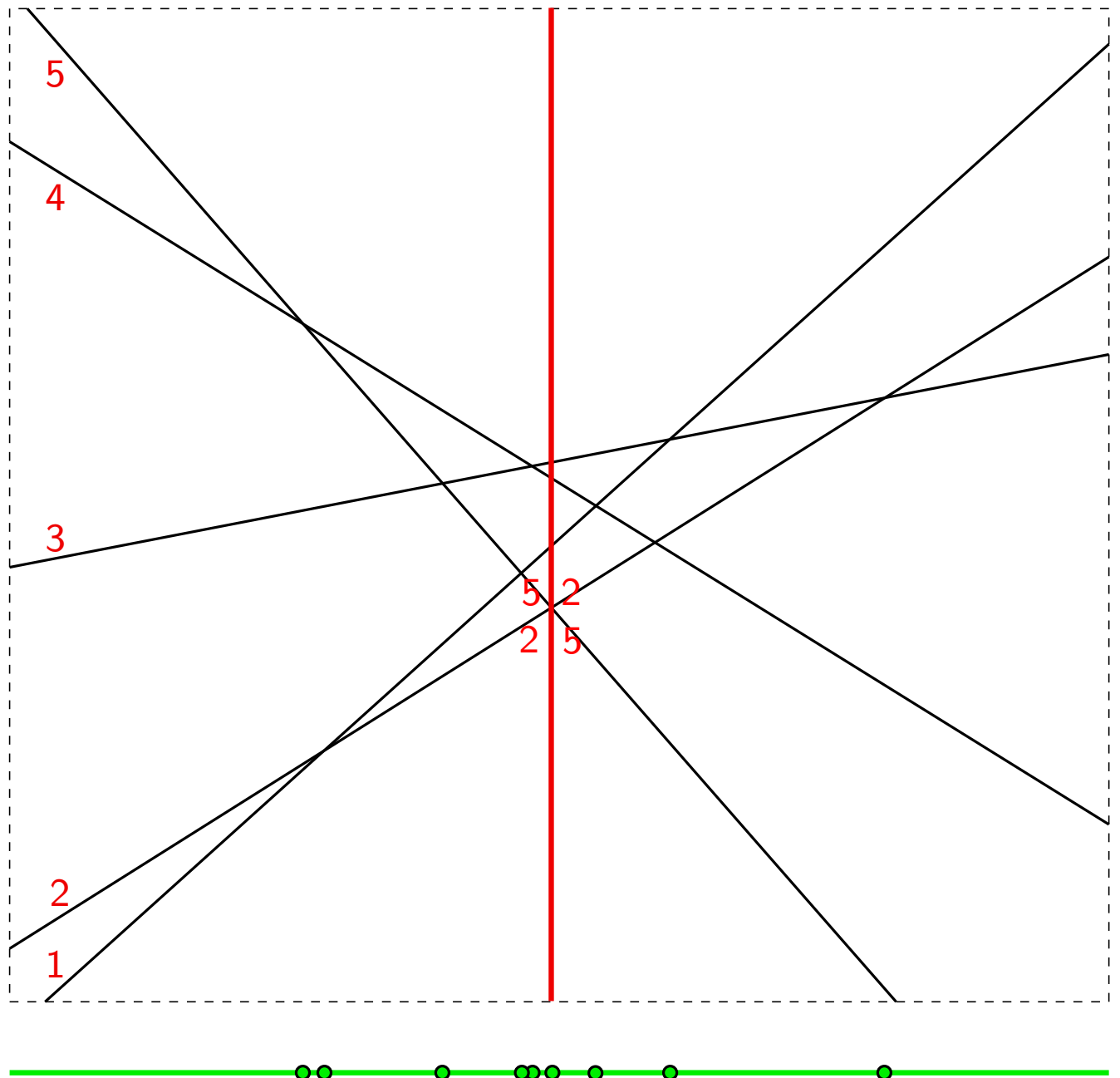
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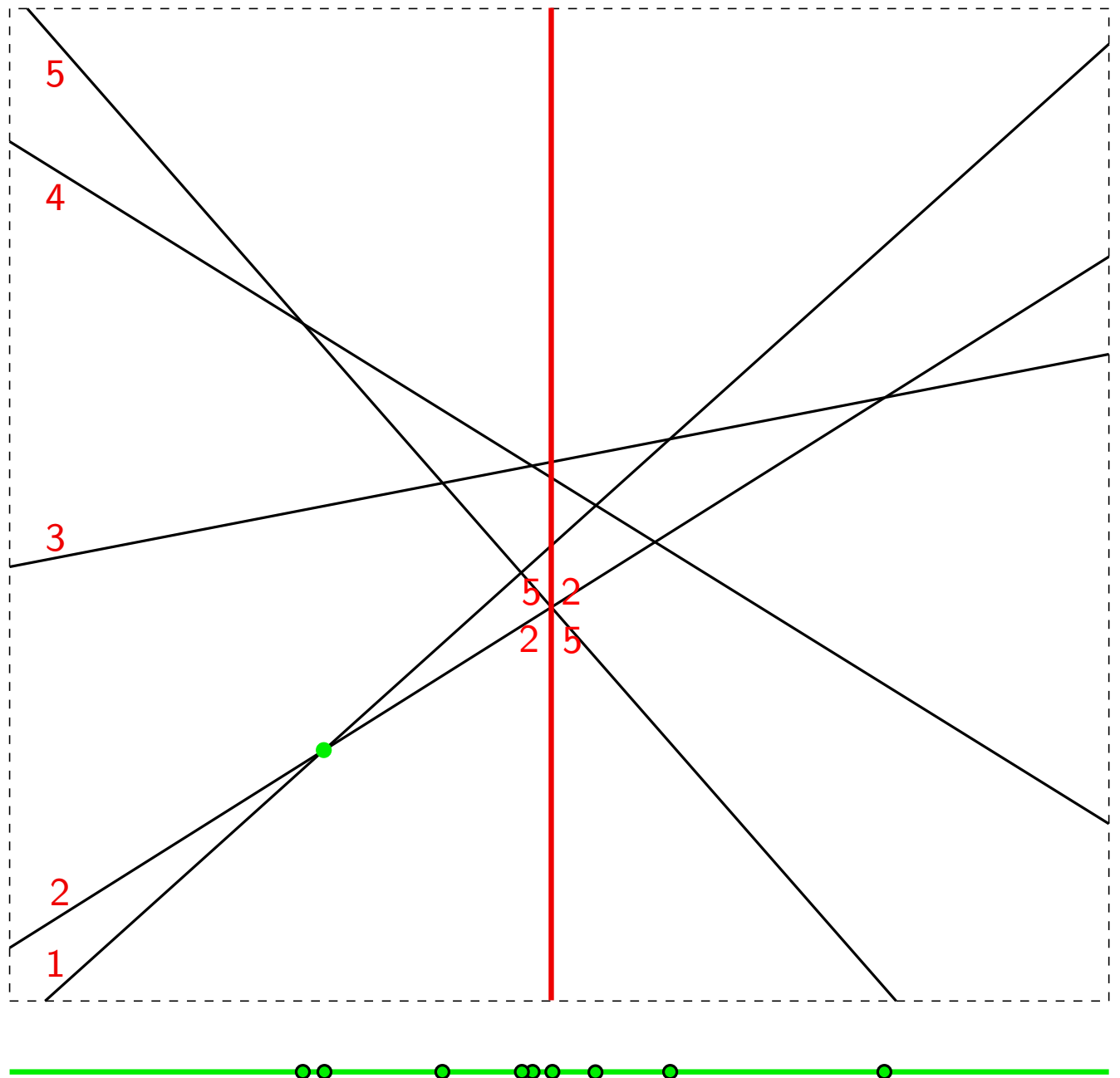
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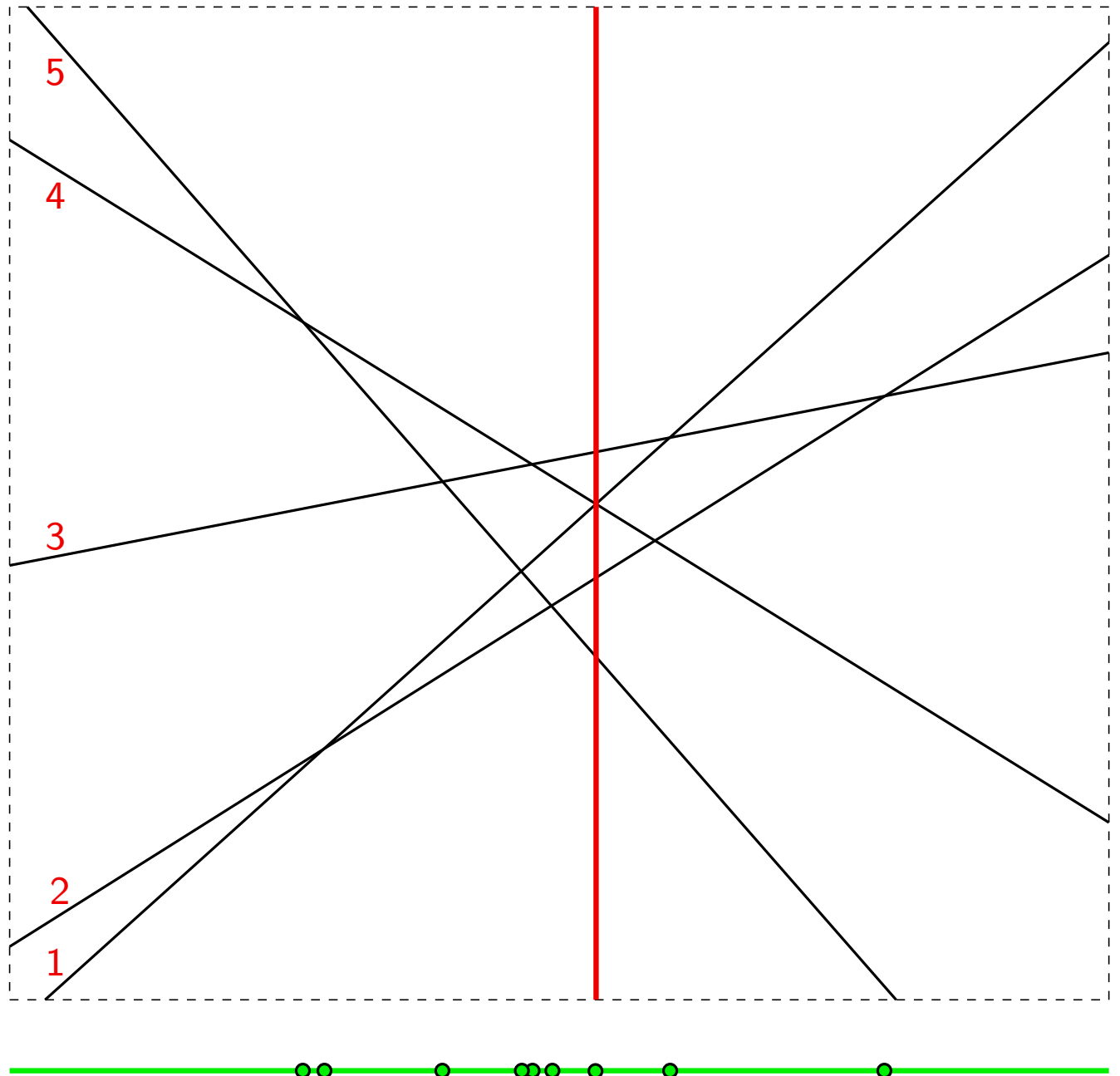
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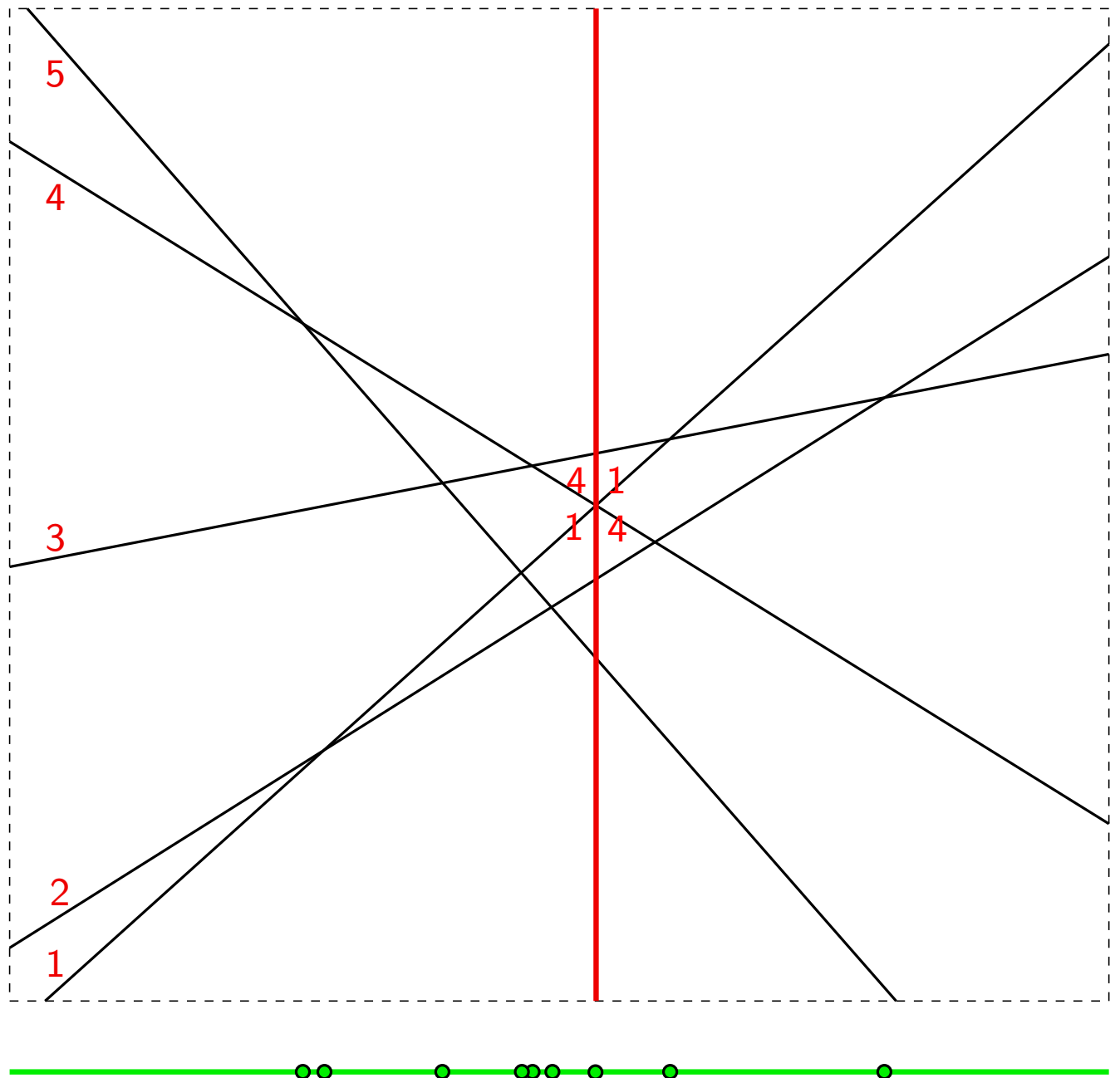
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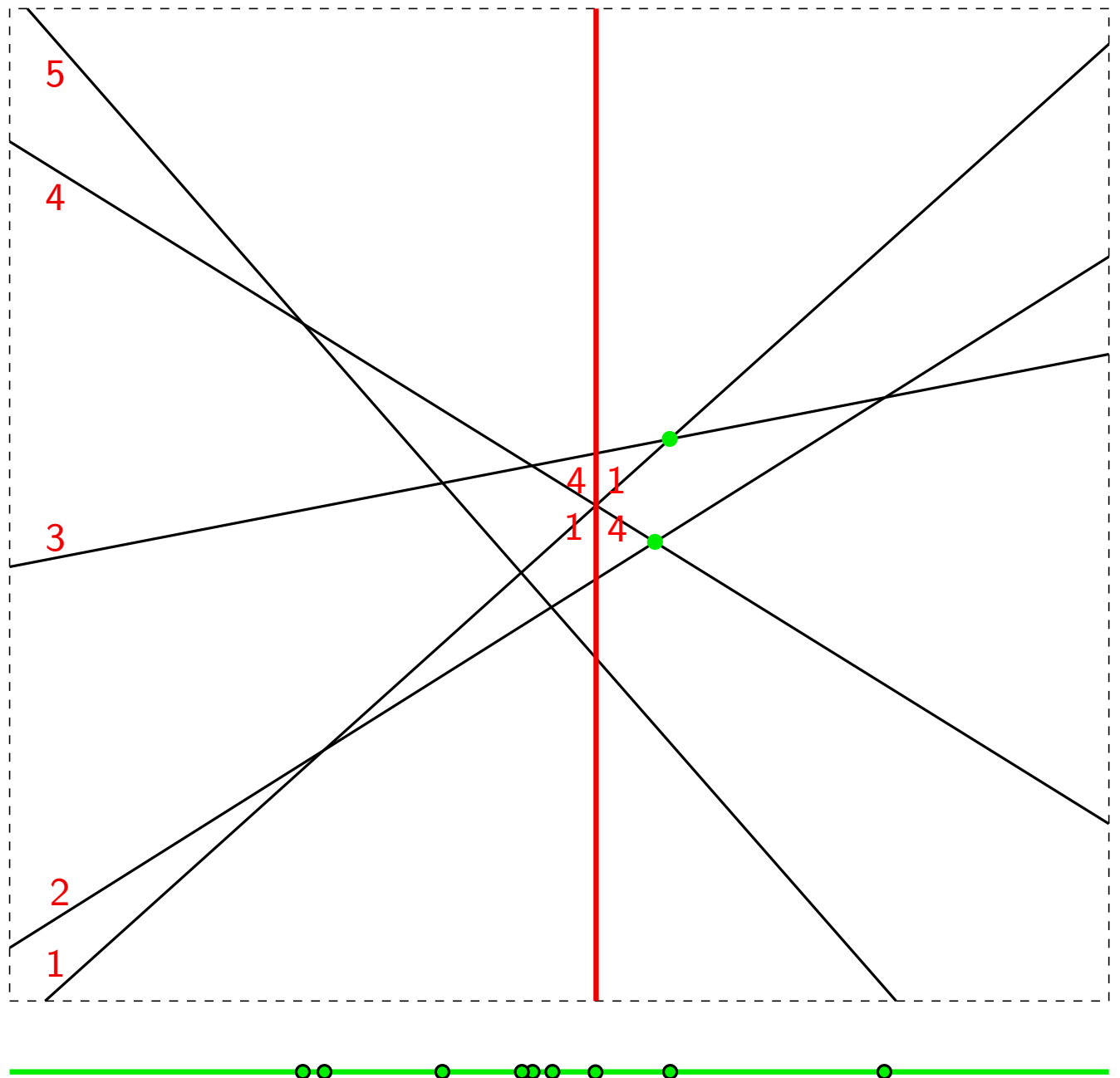
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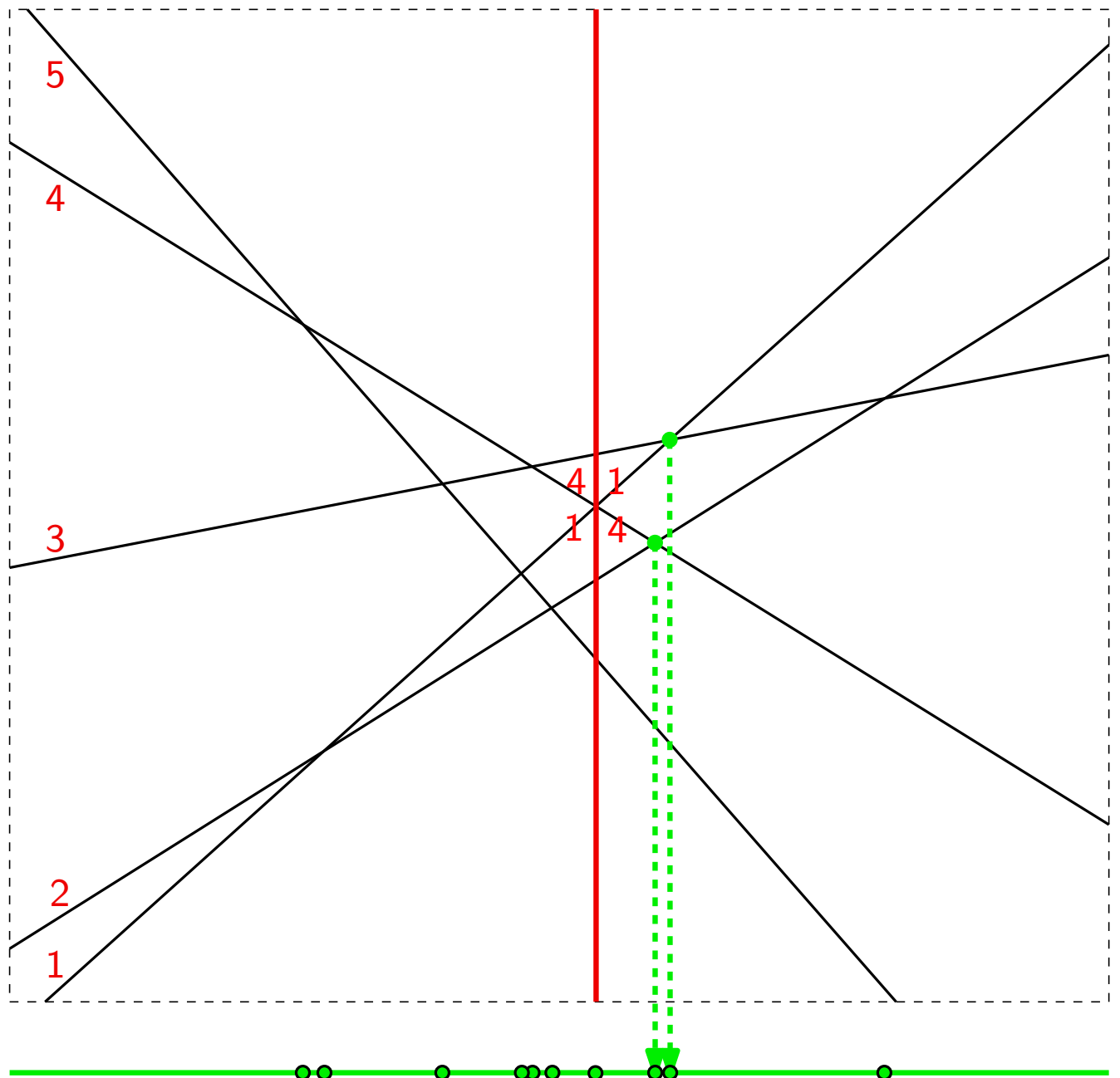
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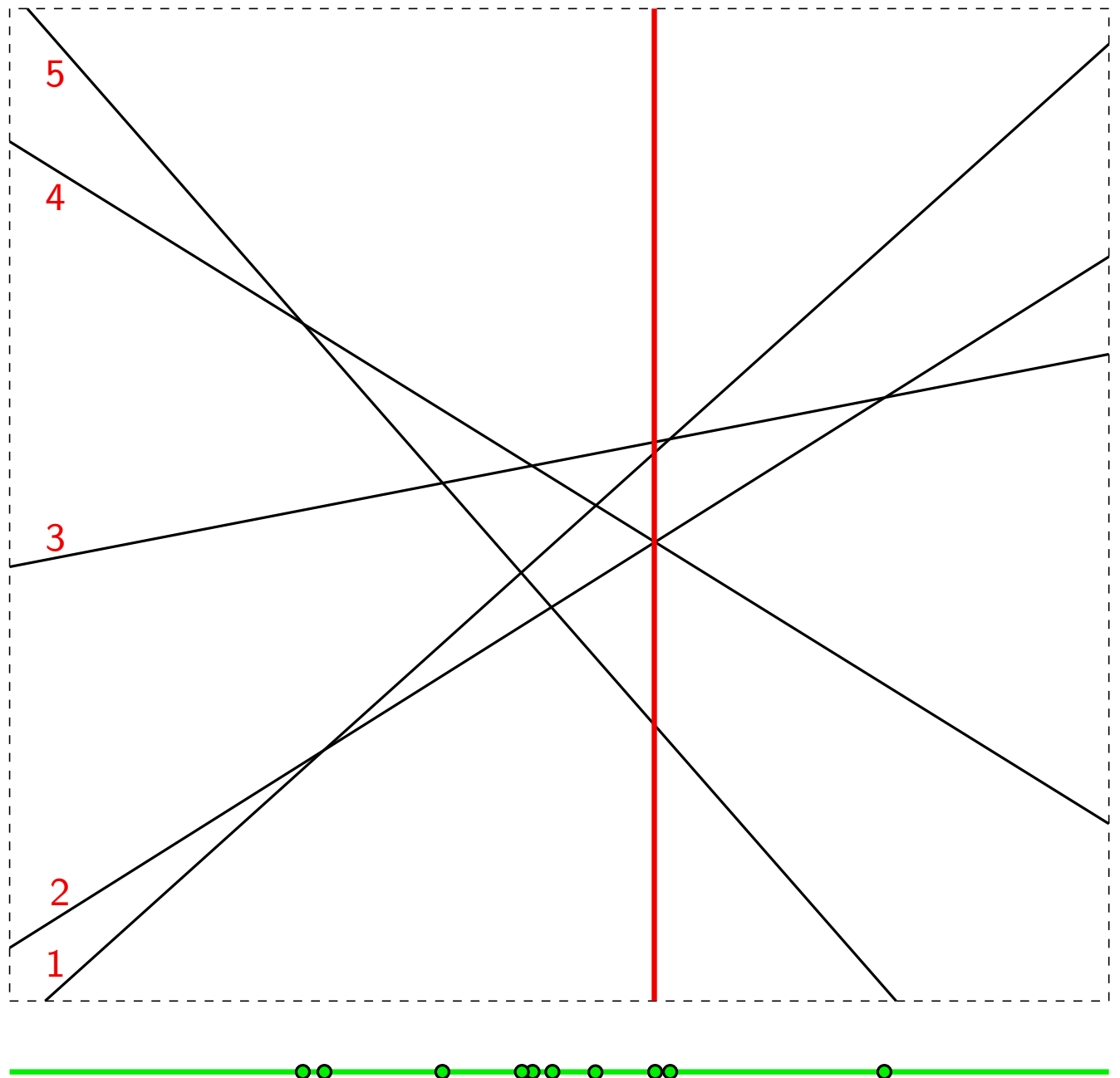
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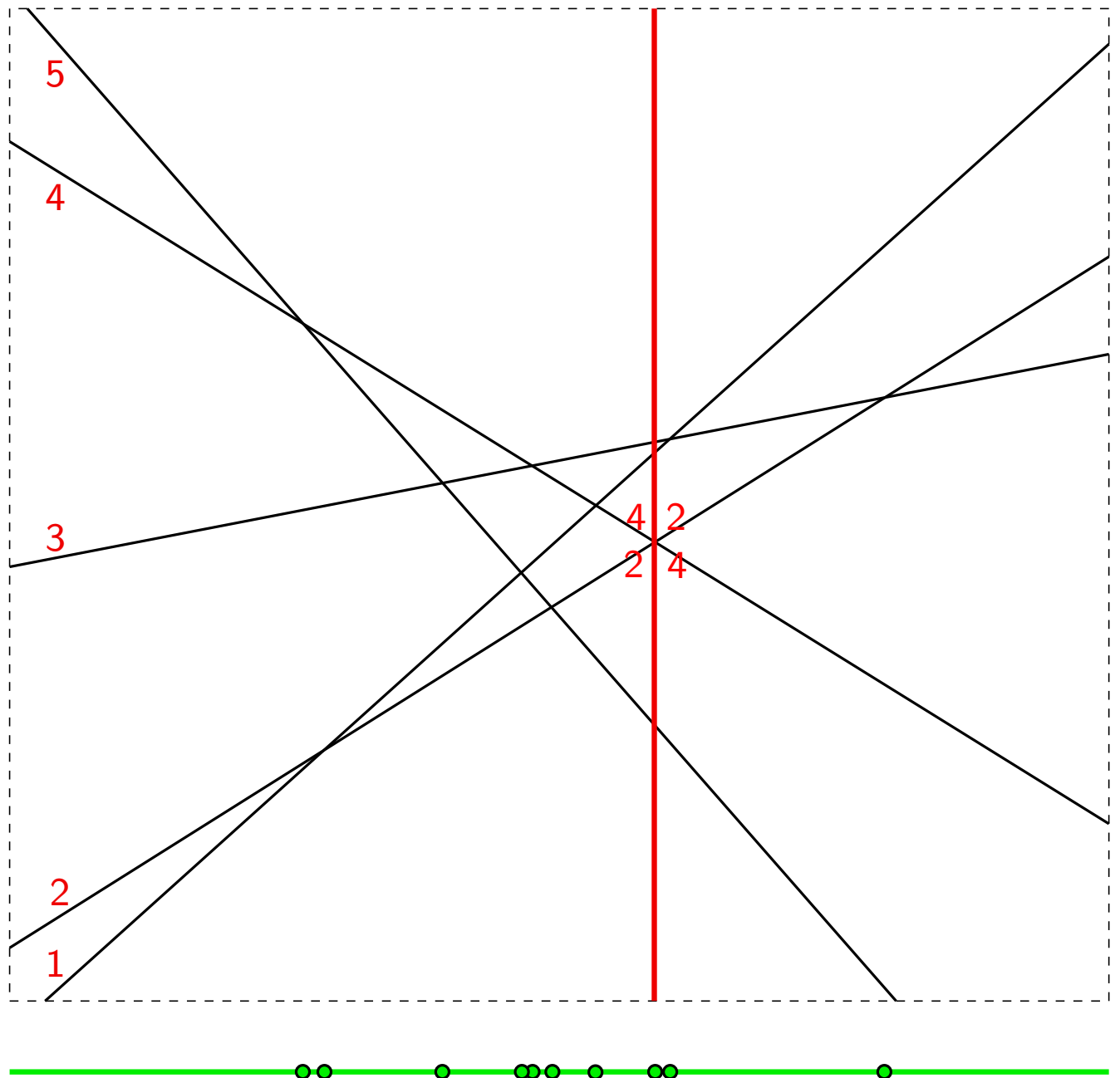
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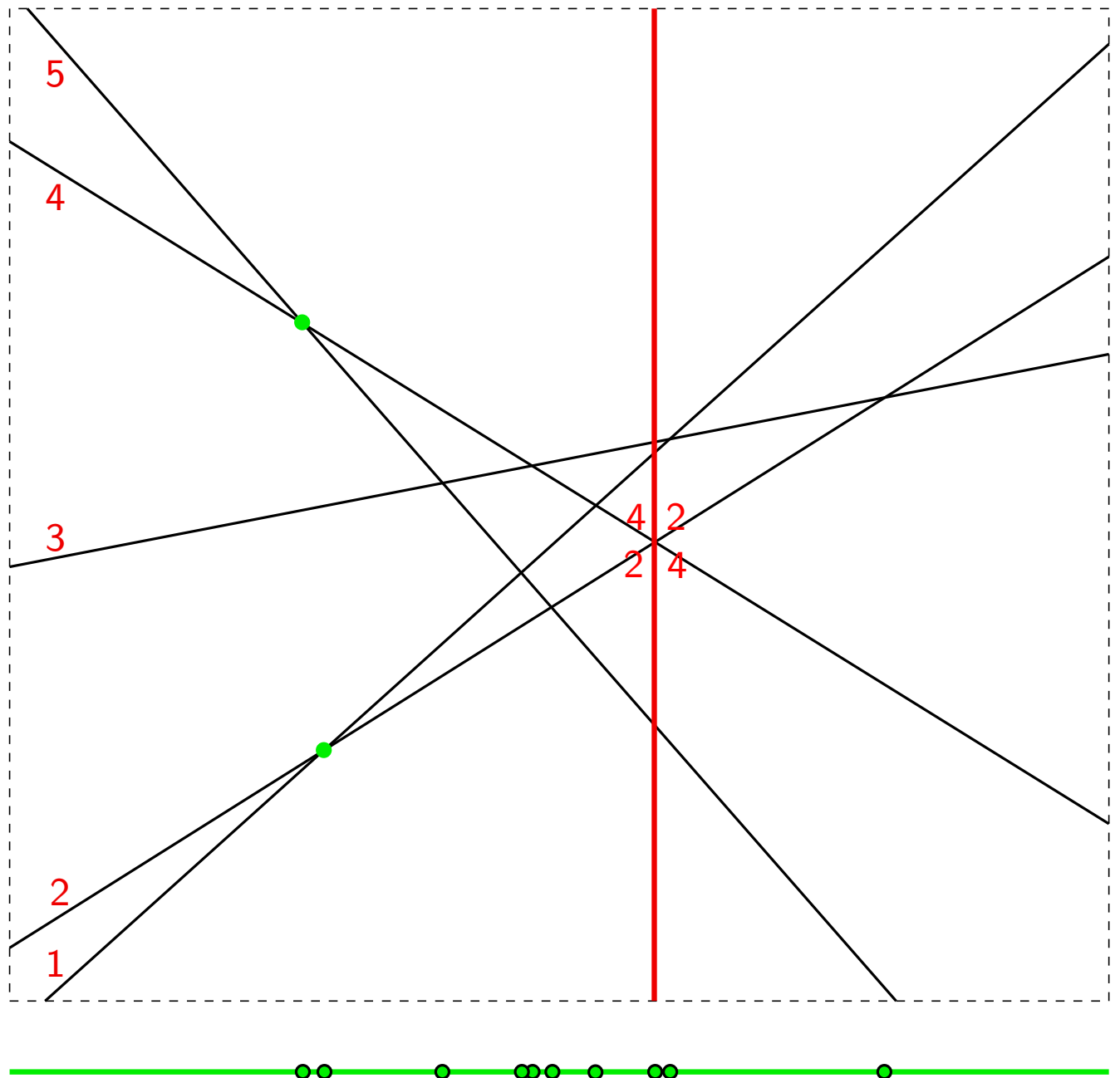
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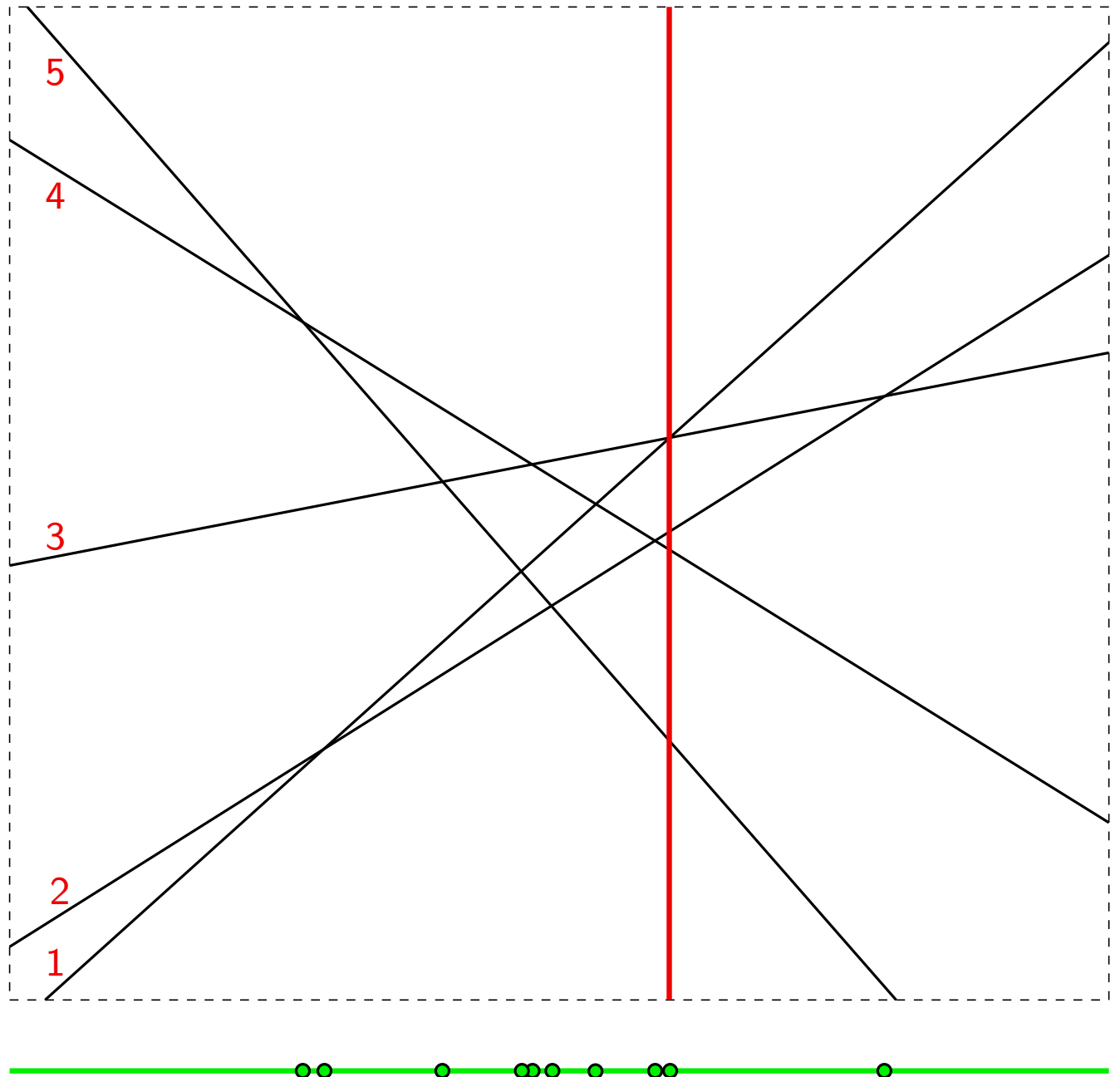
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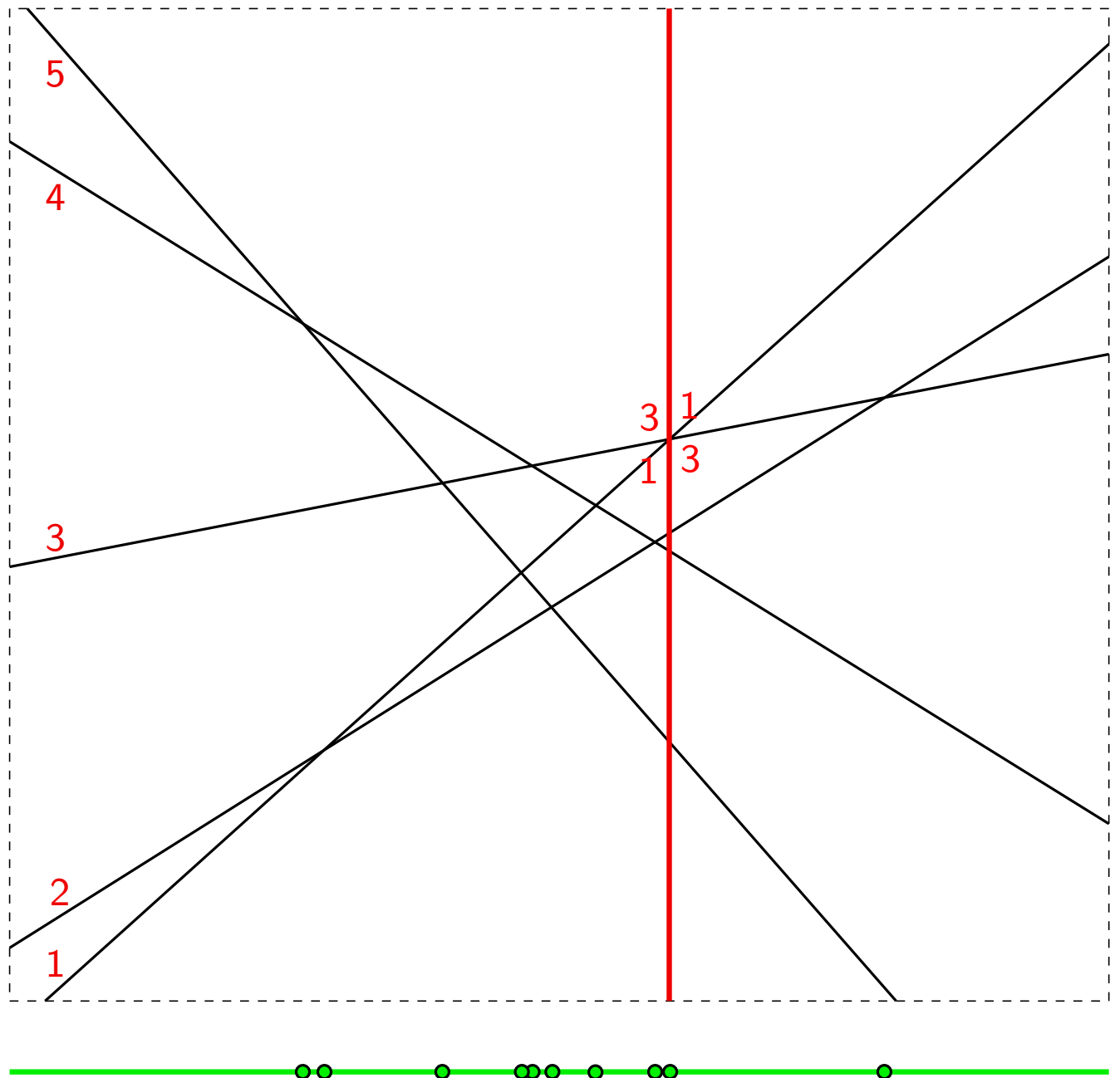
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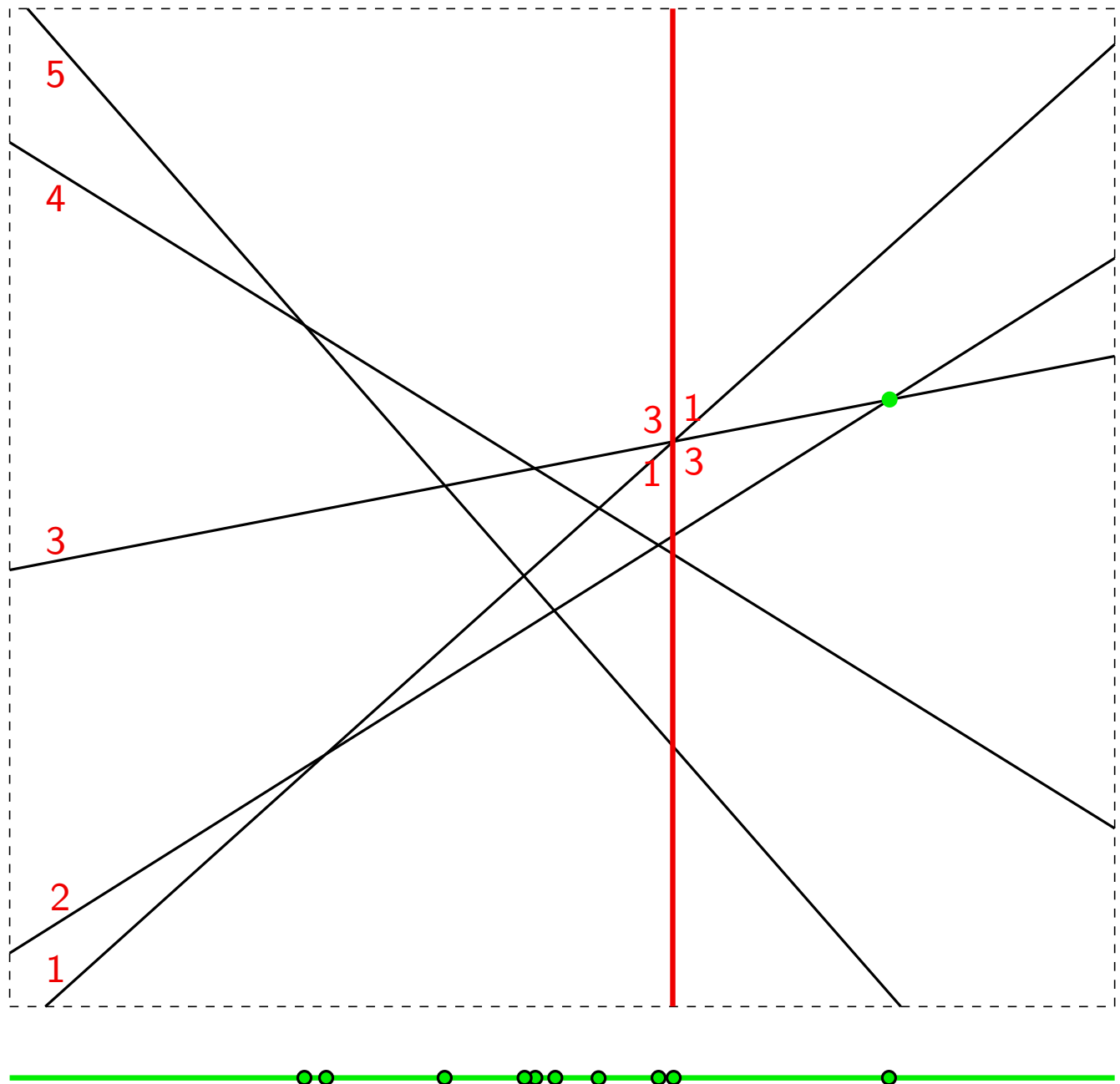
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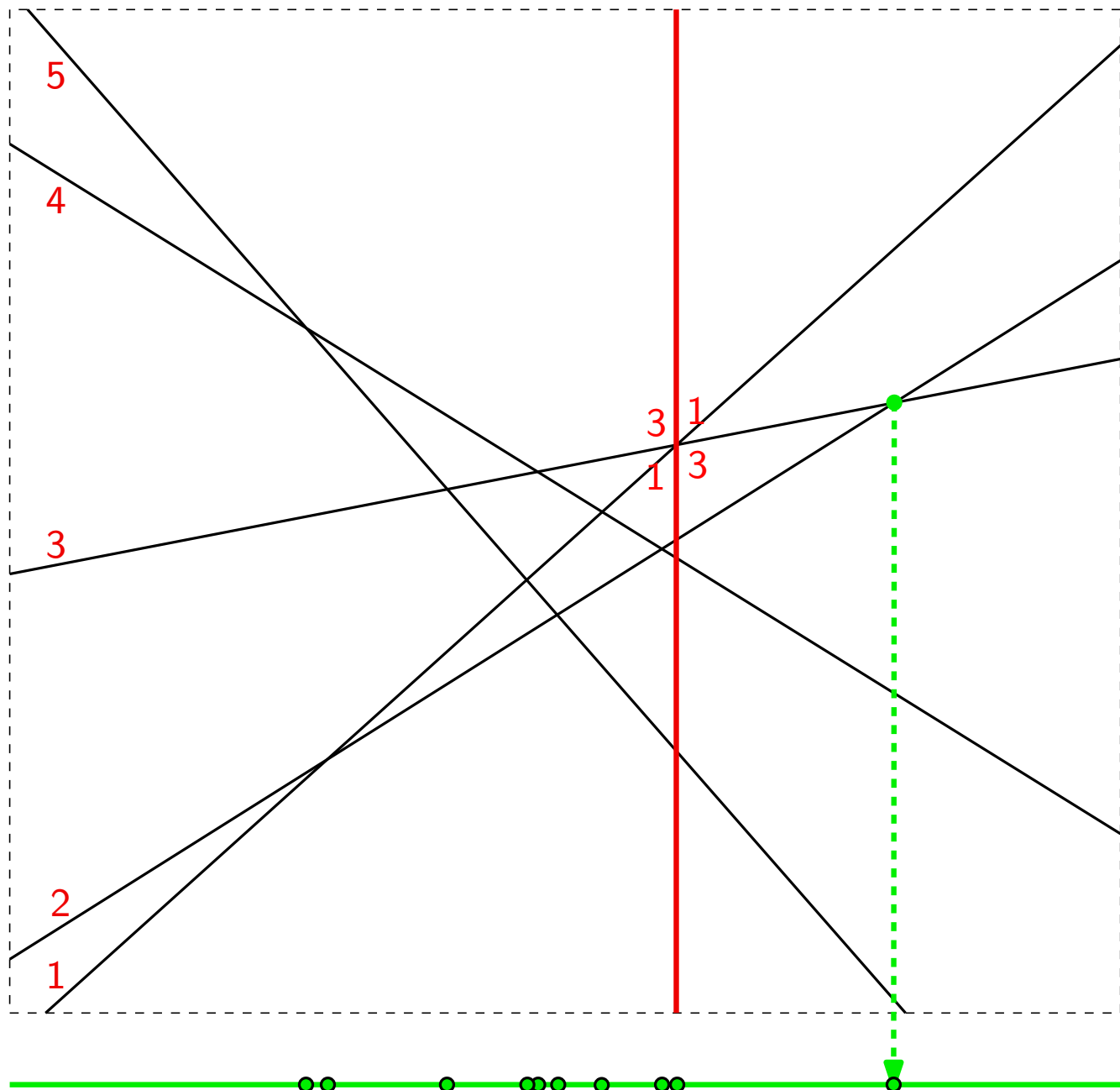
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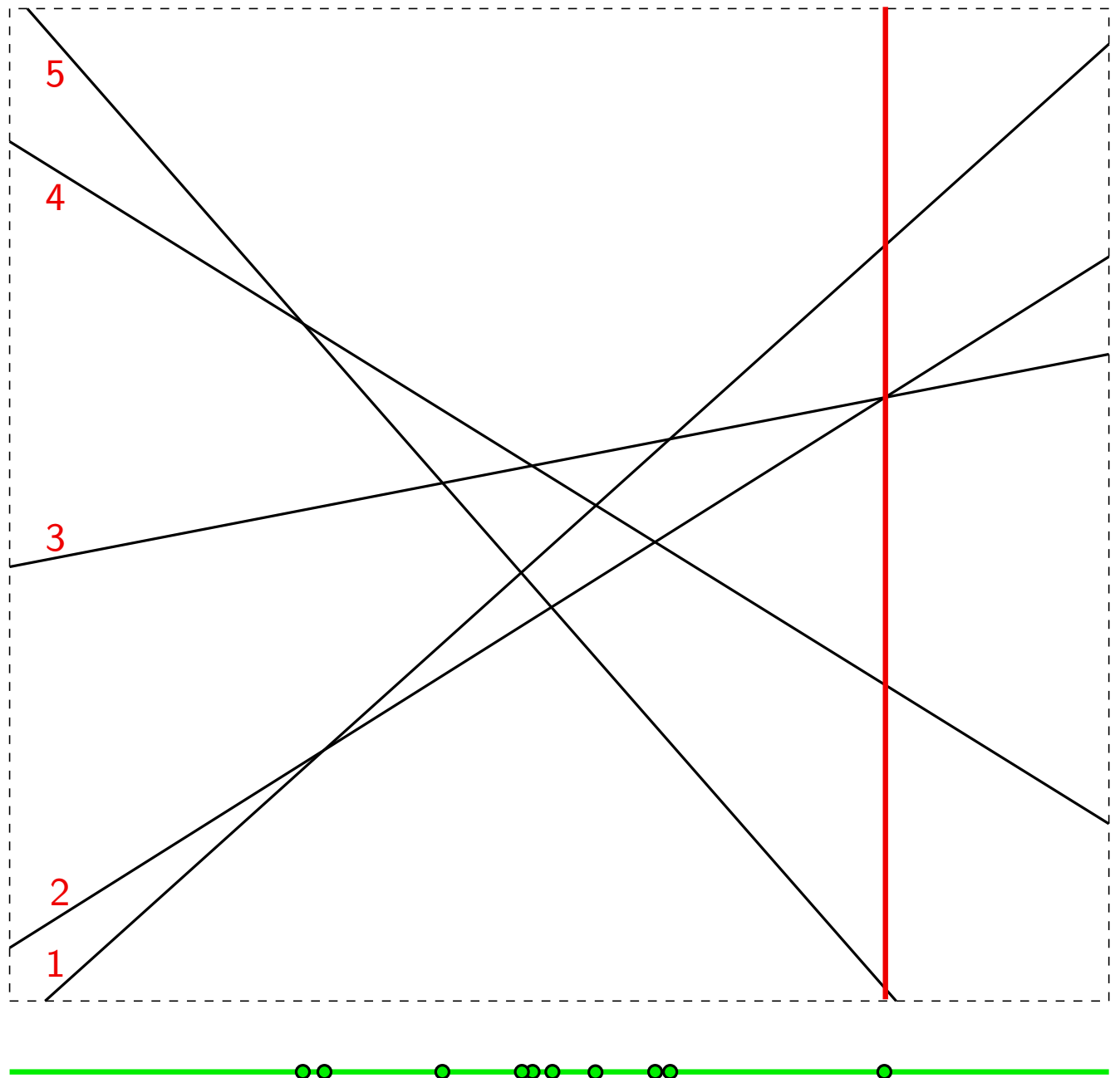
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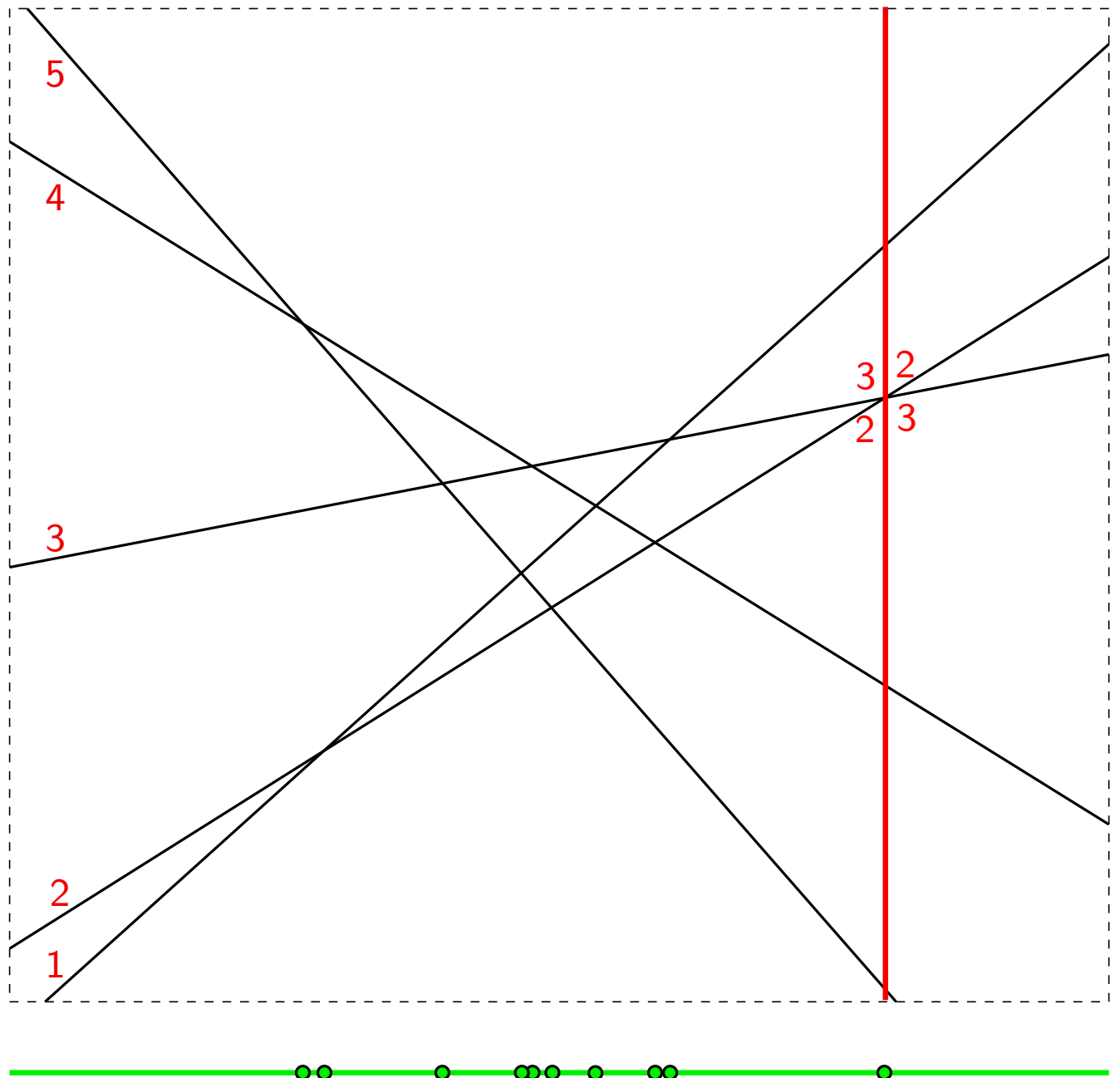
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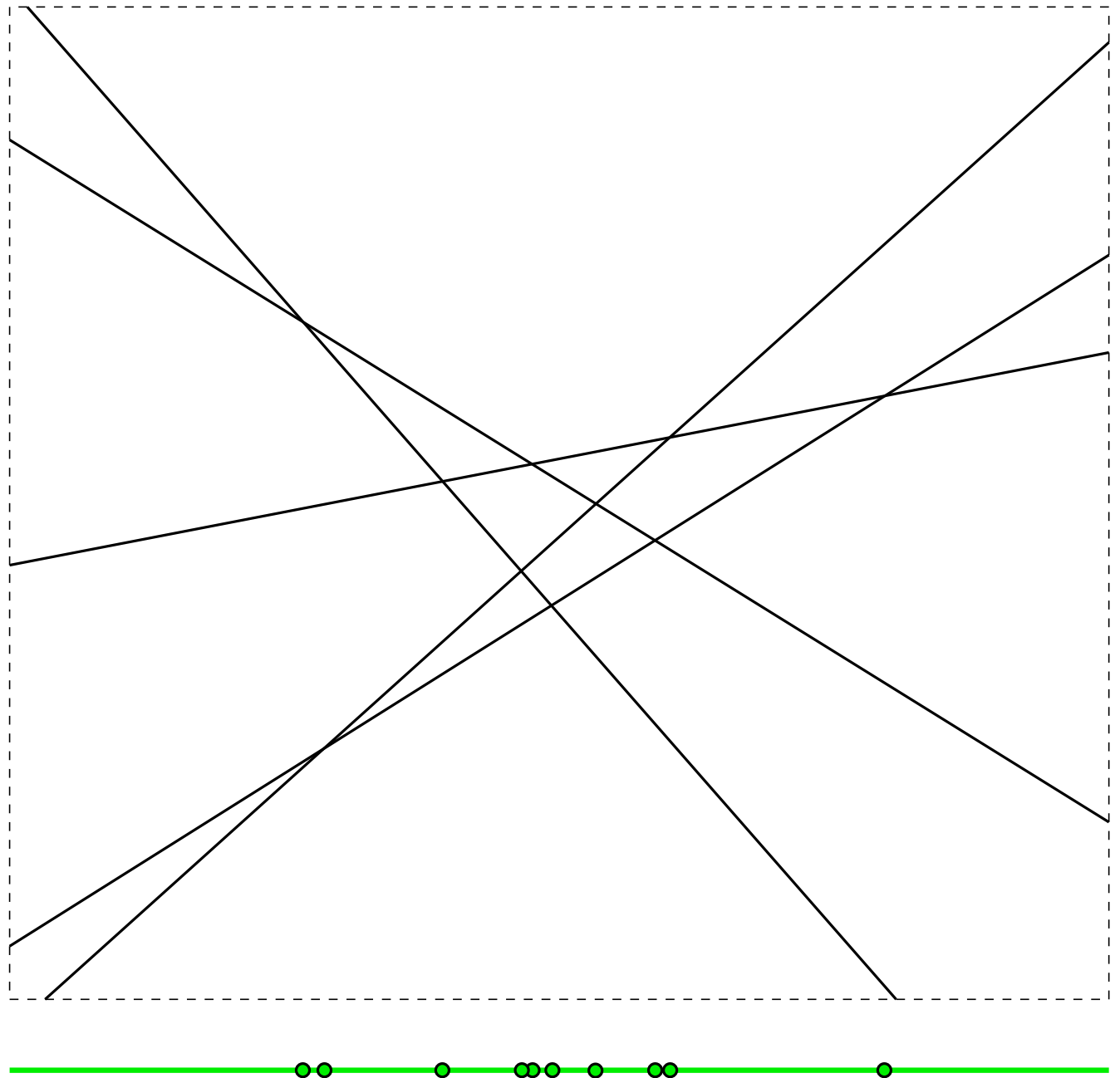
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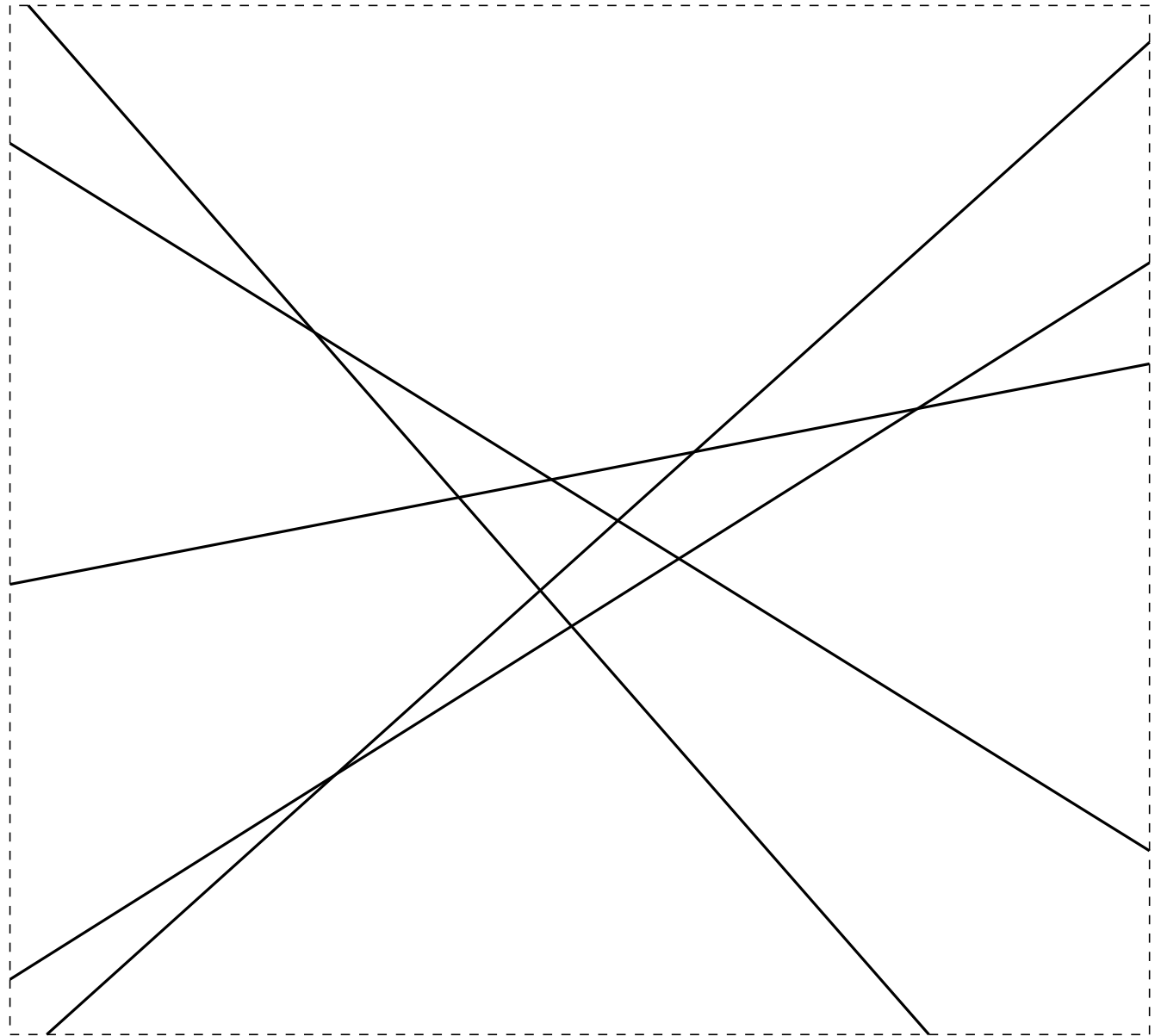


ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Updating the DCEL:

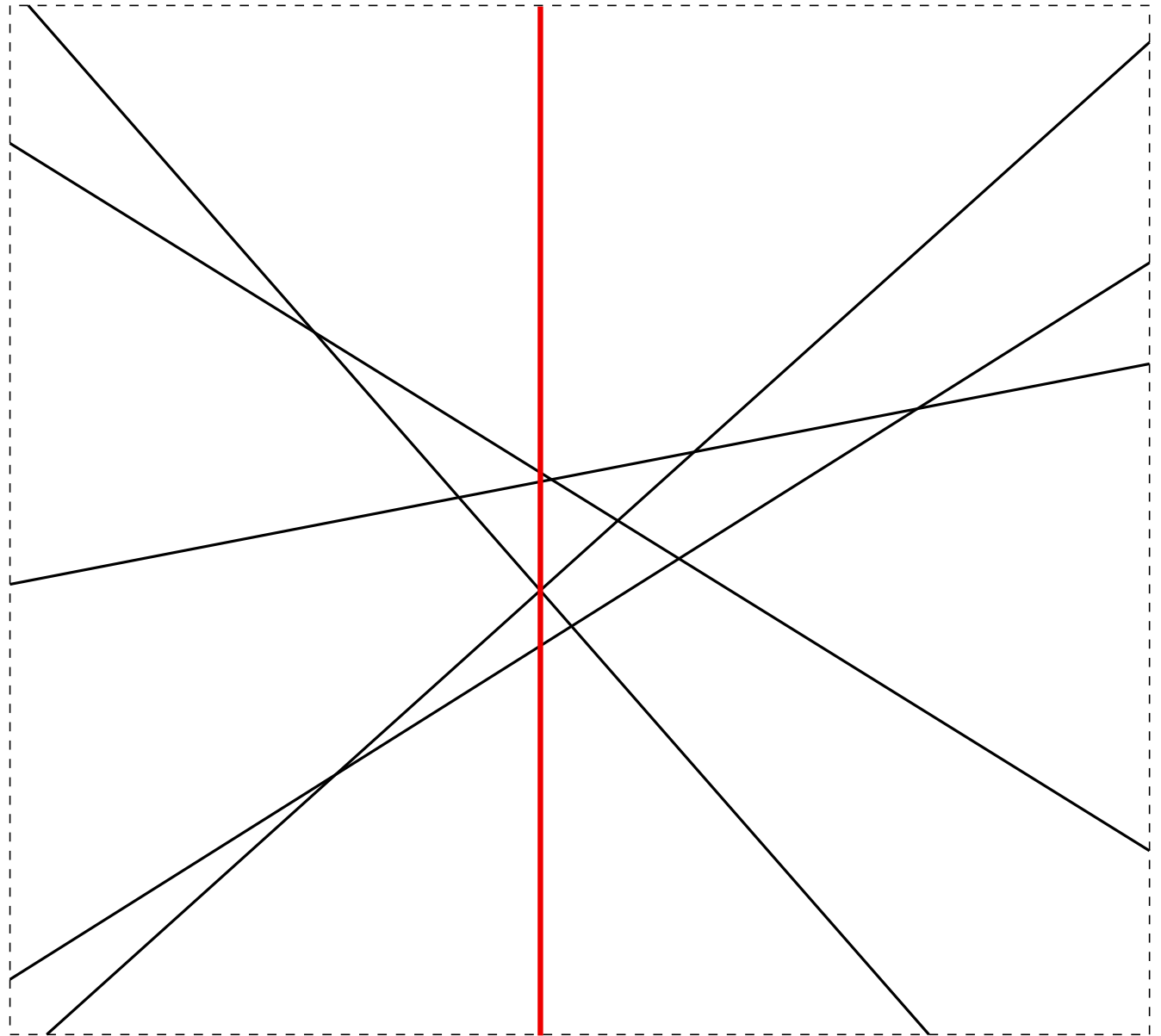


ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Updating the DCEL:



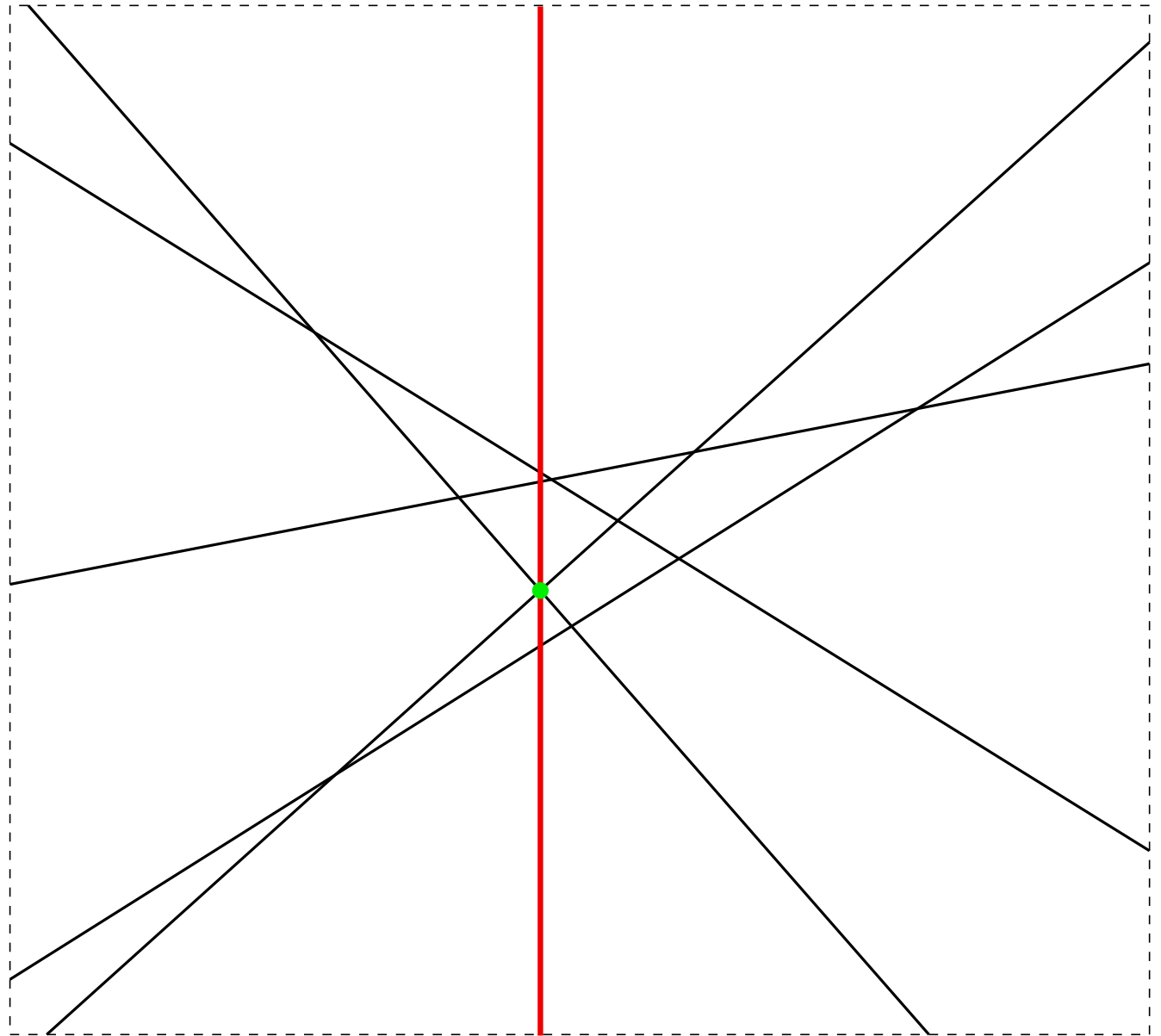
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Updating the DCEL:

- Add the vertex to the table of vertices, with its coordinates and a pointer to an edge.



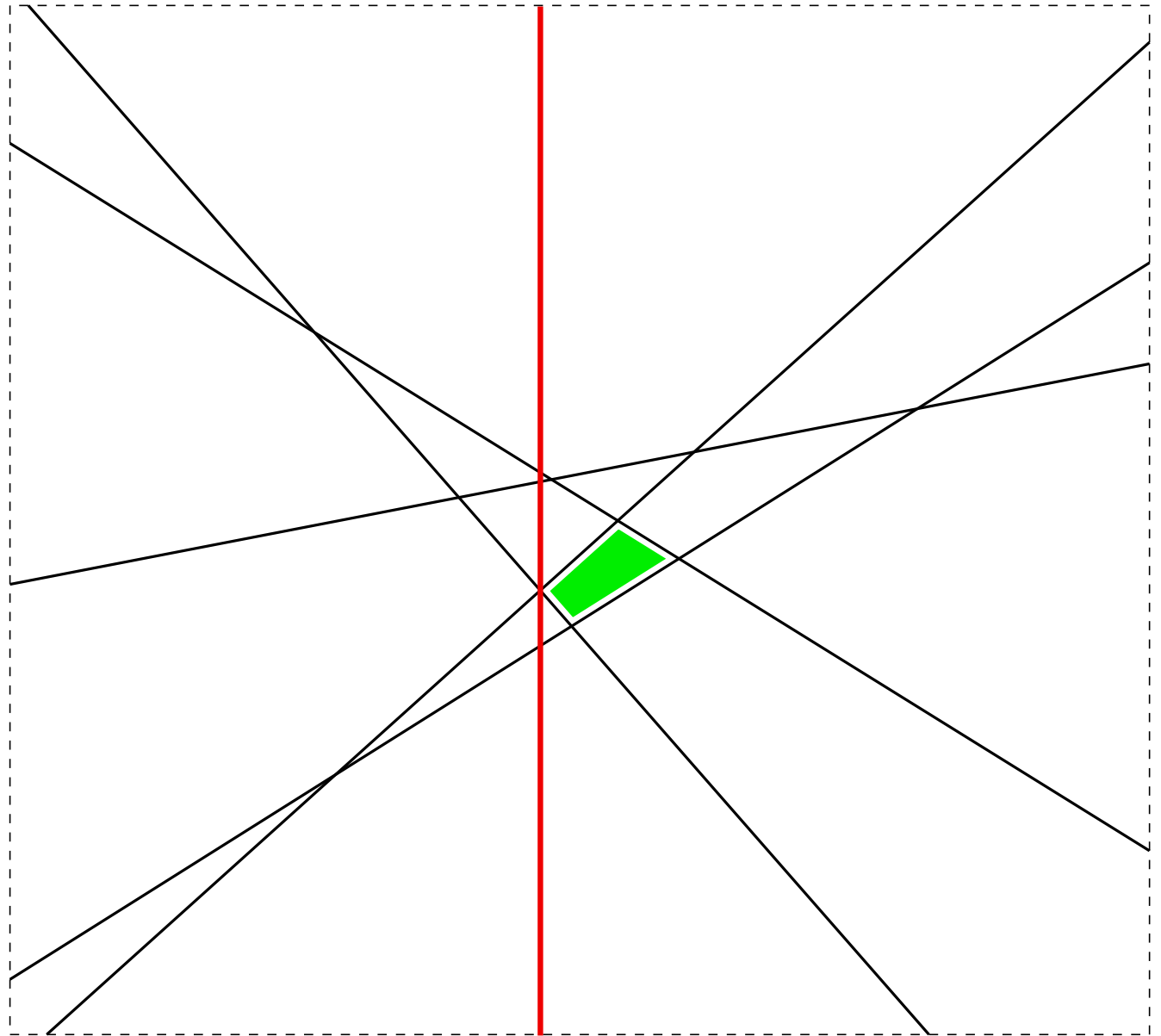
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Updating the DCEL:

- Add the vertex to the table of vertices, with its coordinates and a pointer to an edge.
- Add the starting face to the table of faces, with a pointer to an edge.



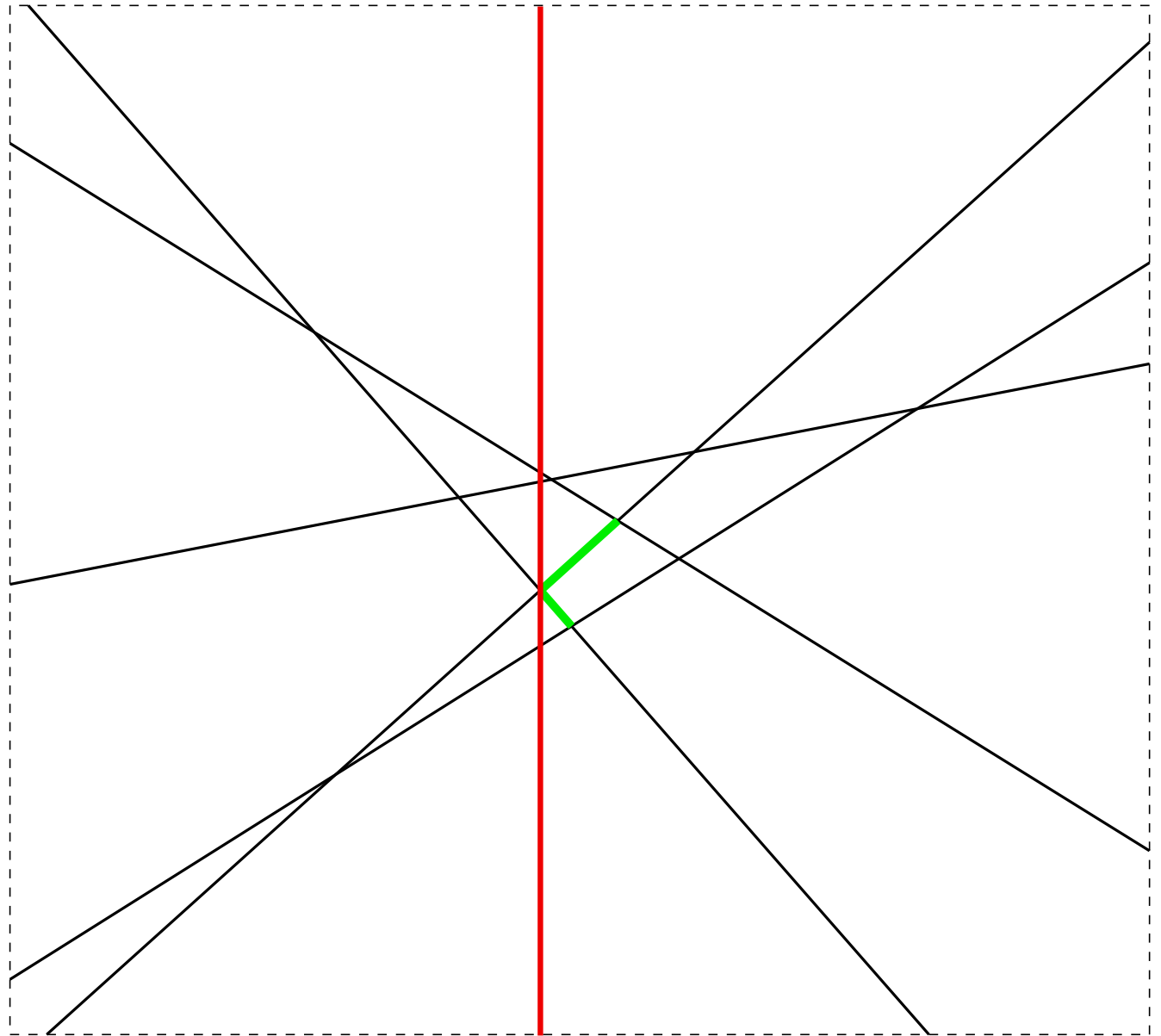
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Updating the DCEL:

- Add the vertex to the table of vertices, with its coordinates and a pointer to an edge.
- Add the starting face to the table of faces, with a pointer to an edge.
- Add to the DCEL the two starting edges, with pointers to $v_B = v$, e_P , f_L and f_R .



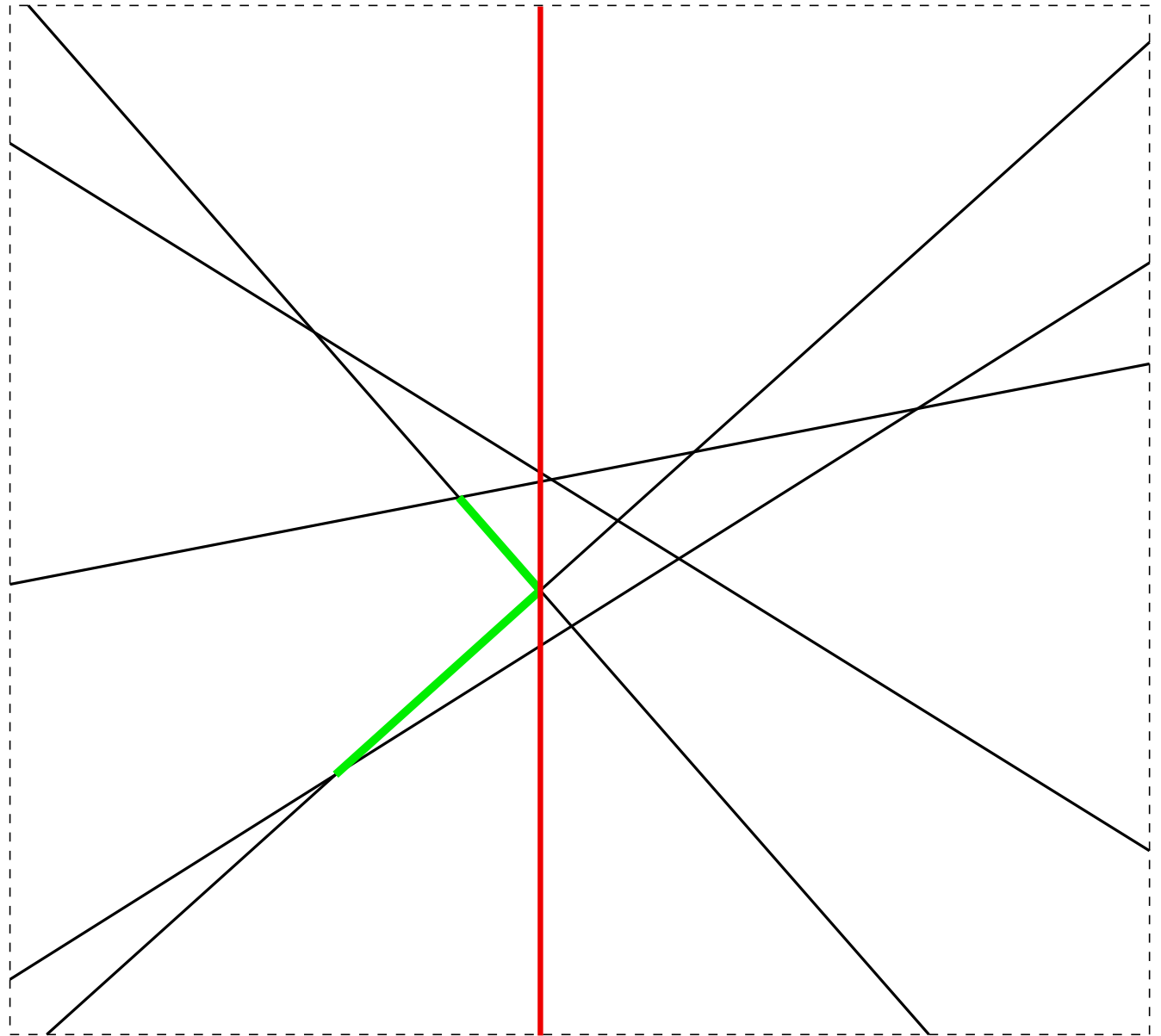
ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Updating the DCEL:

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- Add the starting face to the table of faces, with a pointer to an edge.
- Add to the DCEL the two starting edges, with pointers to $v_B = v$, e_P , f_L and f_R .
- Add to the DCEL pointers to $v_E = v$ and e_N , corresponding to the two ending edges.



ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Correctness:

All intersection points are found because:

- Any two lines must be adjacent in the sweeping line immediately before intersecting.
- Adjacent lines in the sweep line are always checked for intersection.

ARRANGEMENTS

COMPUTATION

Sweep line algorithm

Running time:

Initialization: $O(n \log n)$

At each event (there are $O(n^2)$ events):

1. Locate and swap: $O(\log n)$
2. Intersect: $O(1)$
3. Insert: $O(\log n^2) = O(\log n)$
4. Update the DCEL: $O(1)$

Total running time: $O(n^2 \log n)$

ARRANGEMENTS

COMPUTATION

Sweep line algorithm

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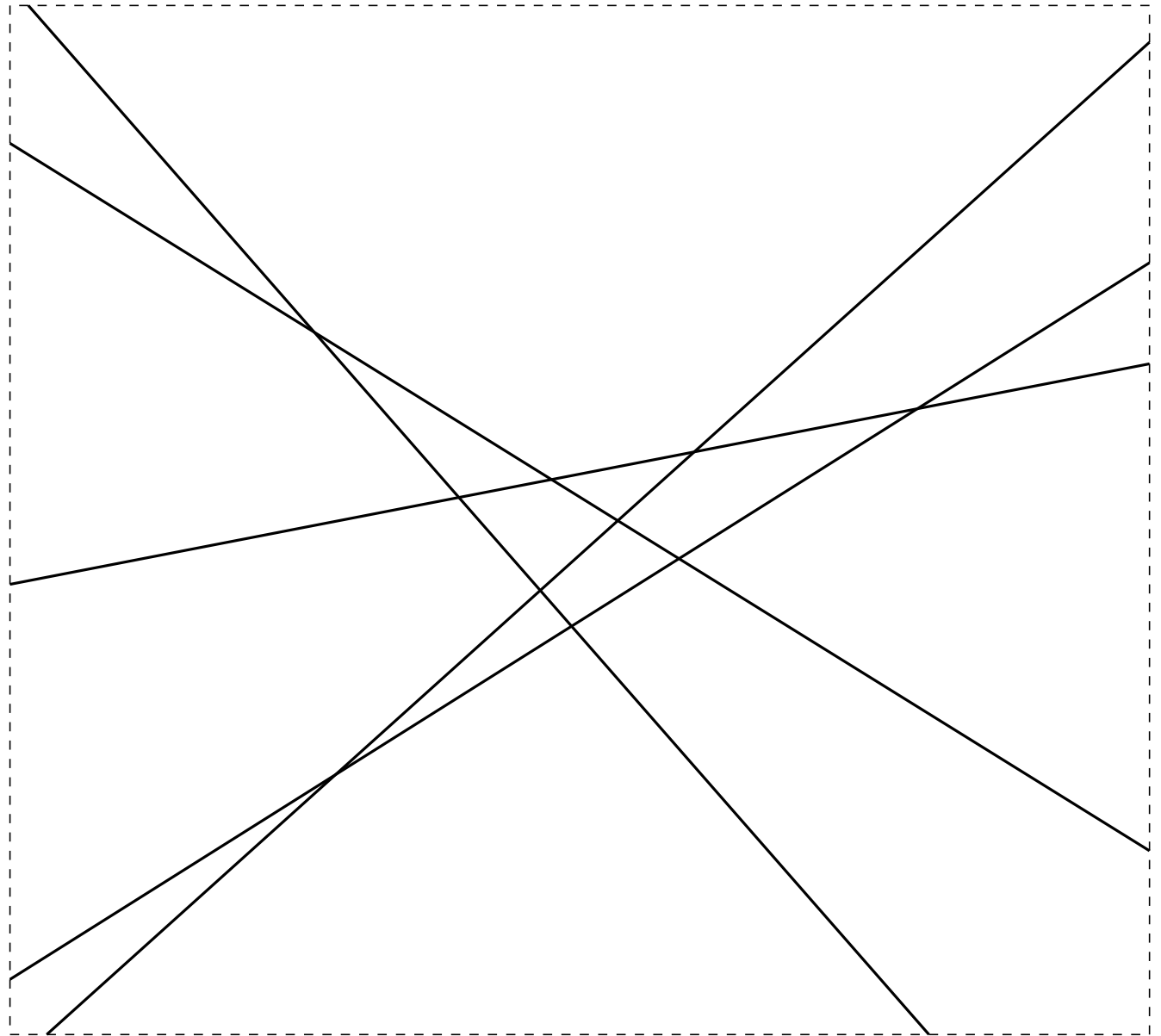
The description of the algorithm we have shown assumes that the arrangement is simple.

An easy modification allows adapting it to the general case.

ARRANGEMENTS

COMPUTATION

Incremental algorithm

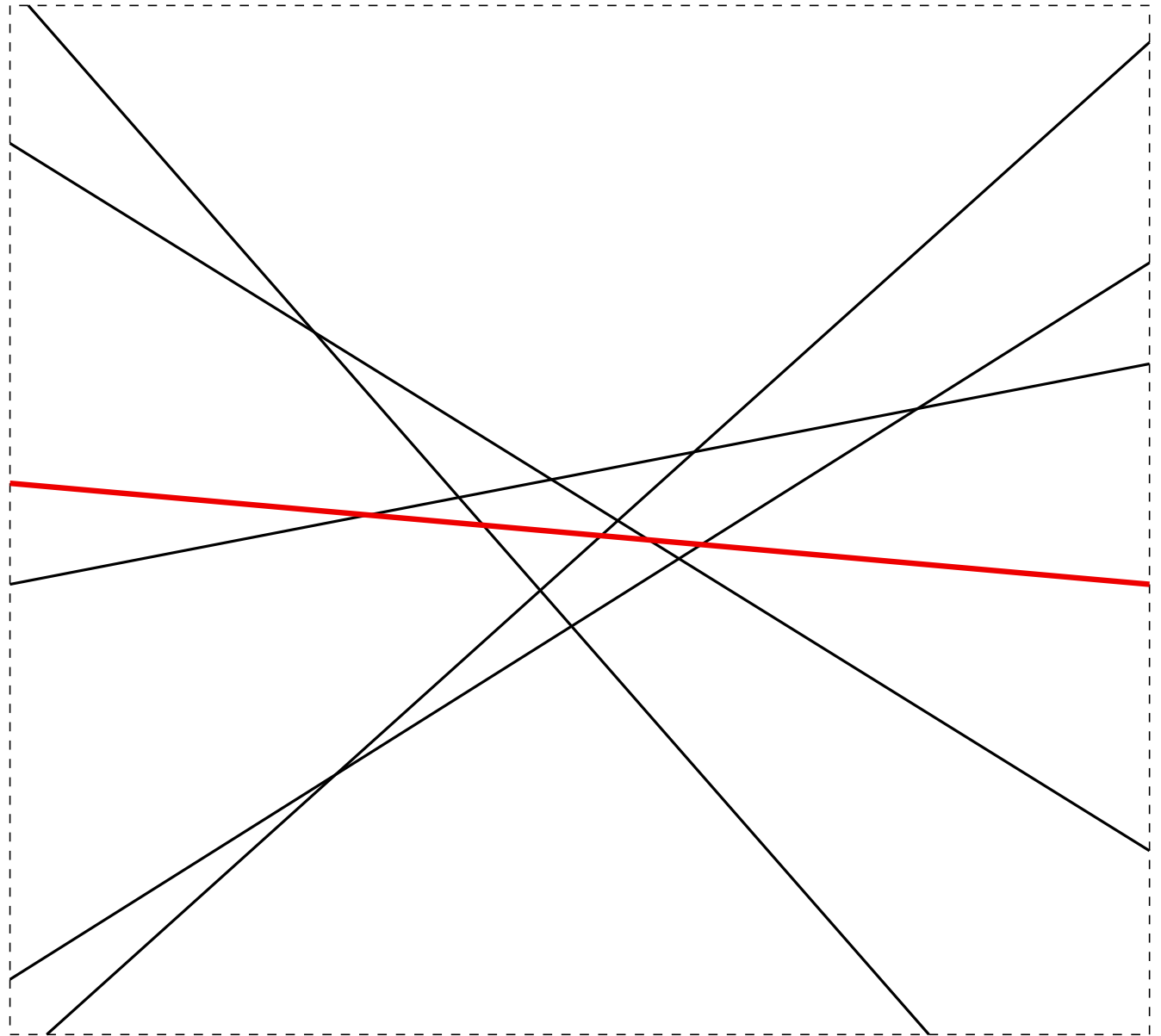


ARRANGEMENTS

COMPUTATION

Incremental algorithm

For each new line l :



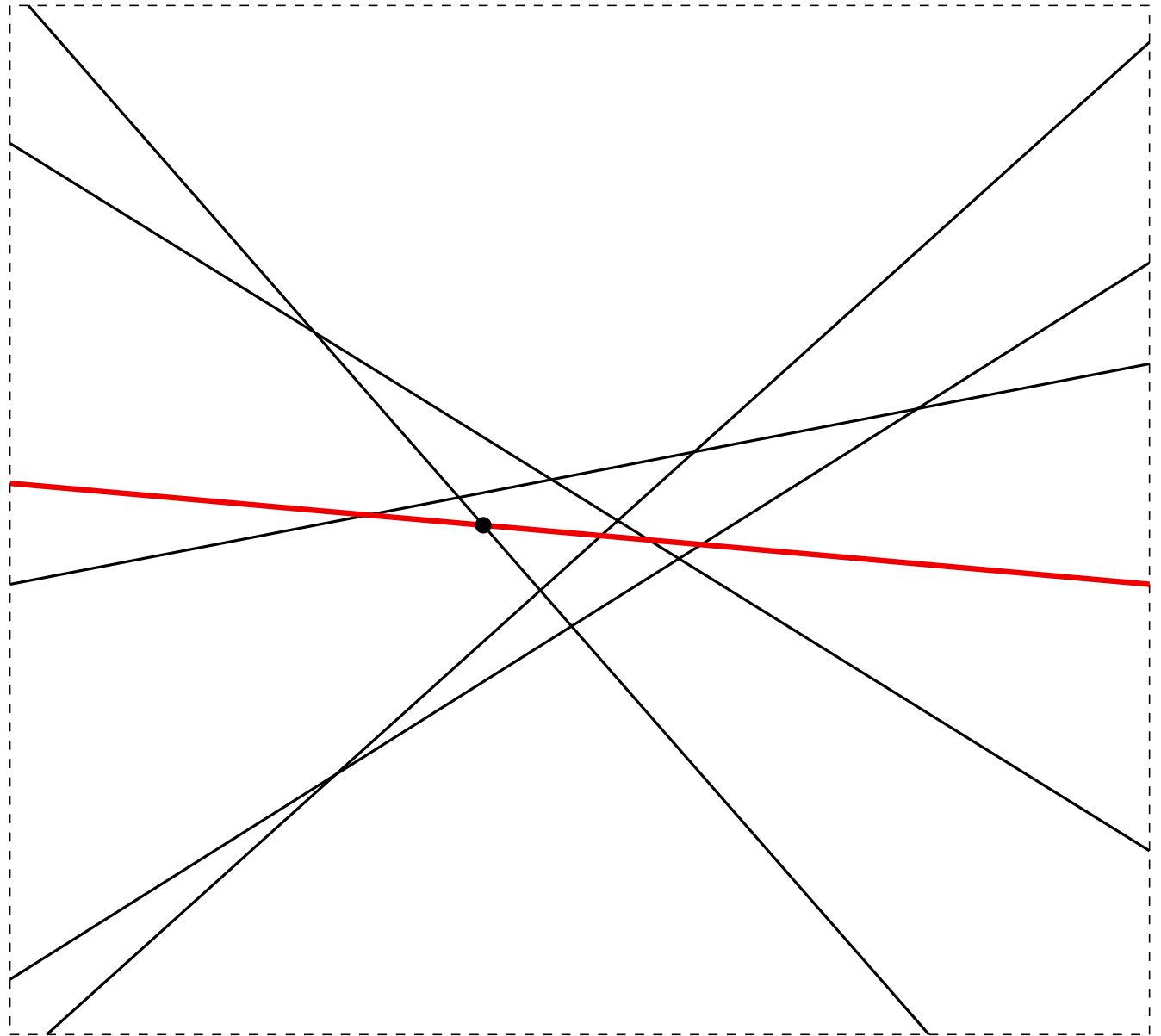
ARRANGEMENTS

COMPUTATION

Incremental algorithm

For each new line l :

1. Intersect l with any line l_i pre-existent in the arrangement.



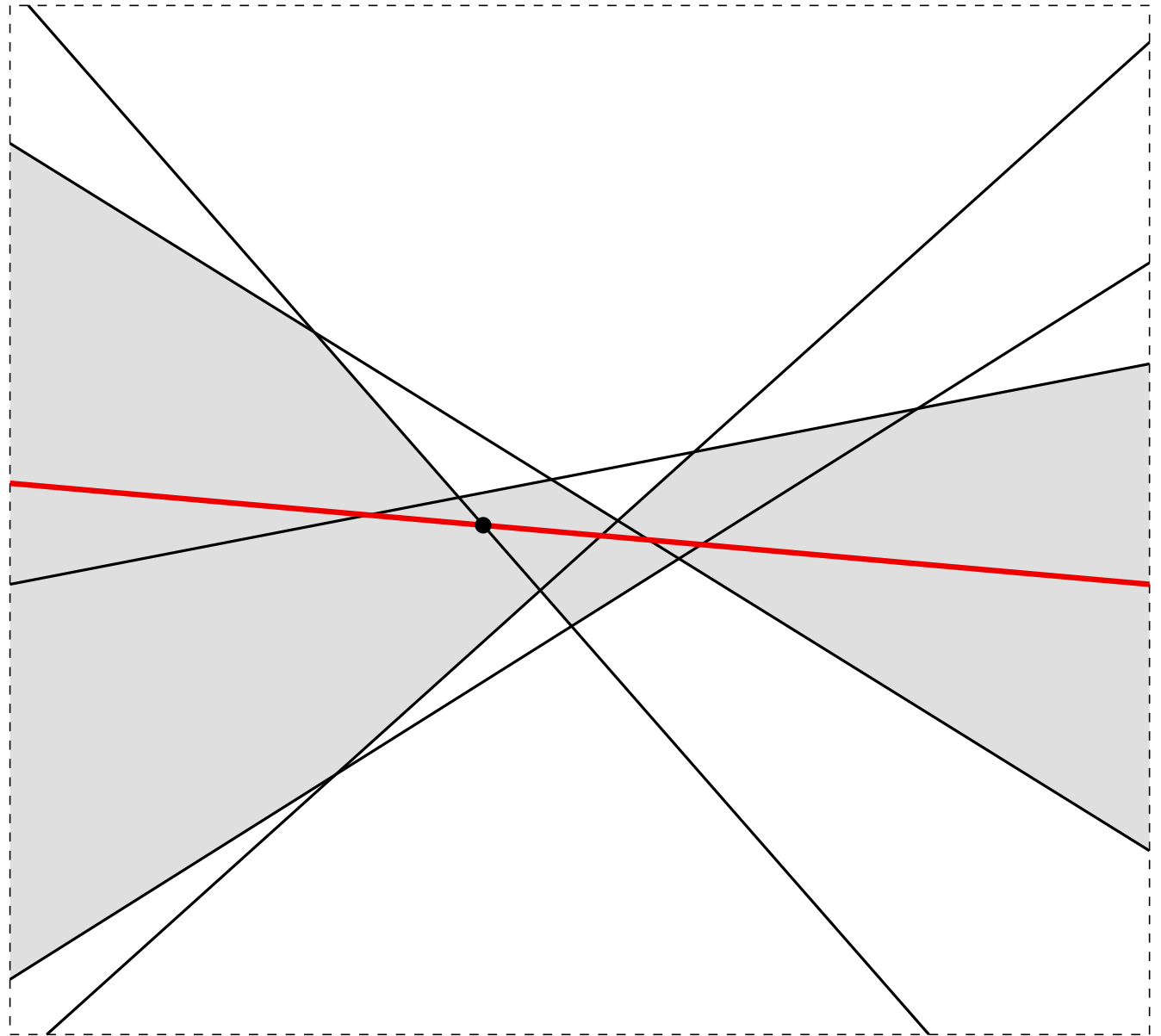
ARRANGEMENTS

COMPUTATION

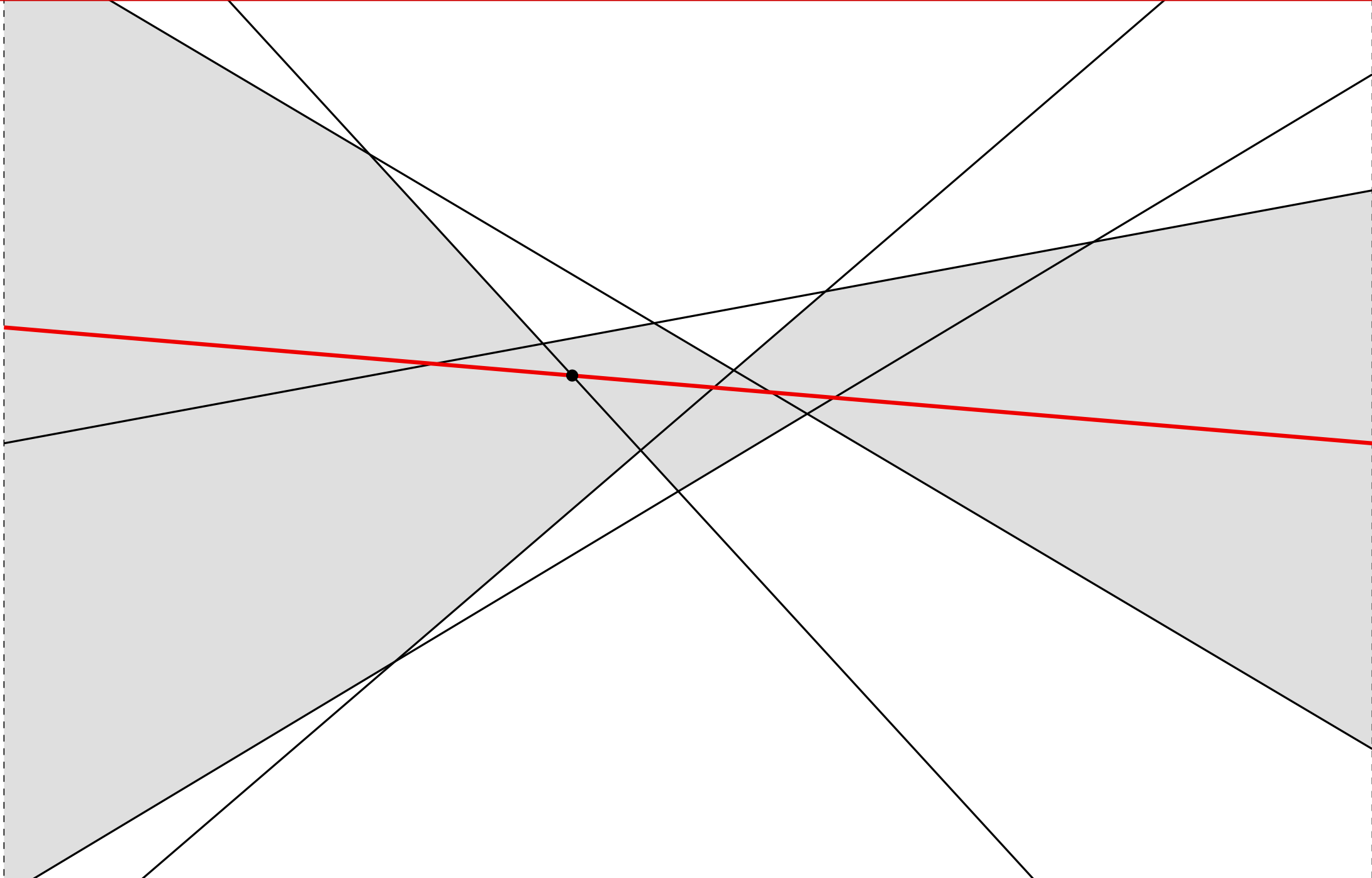
Incremental algorithm

For each new line l :

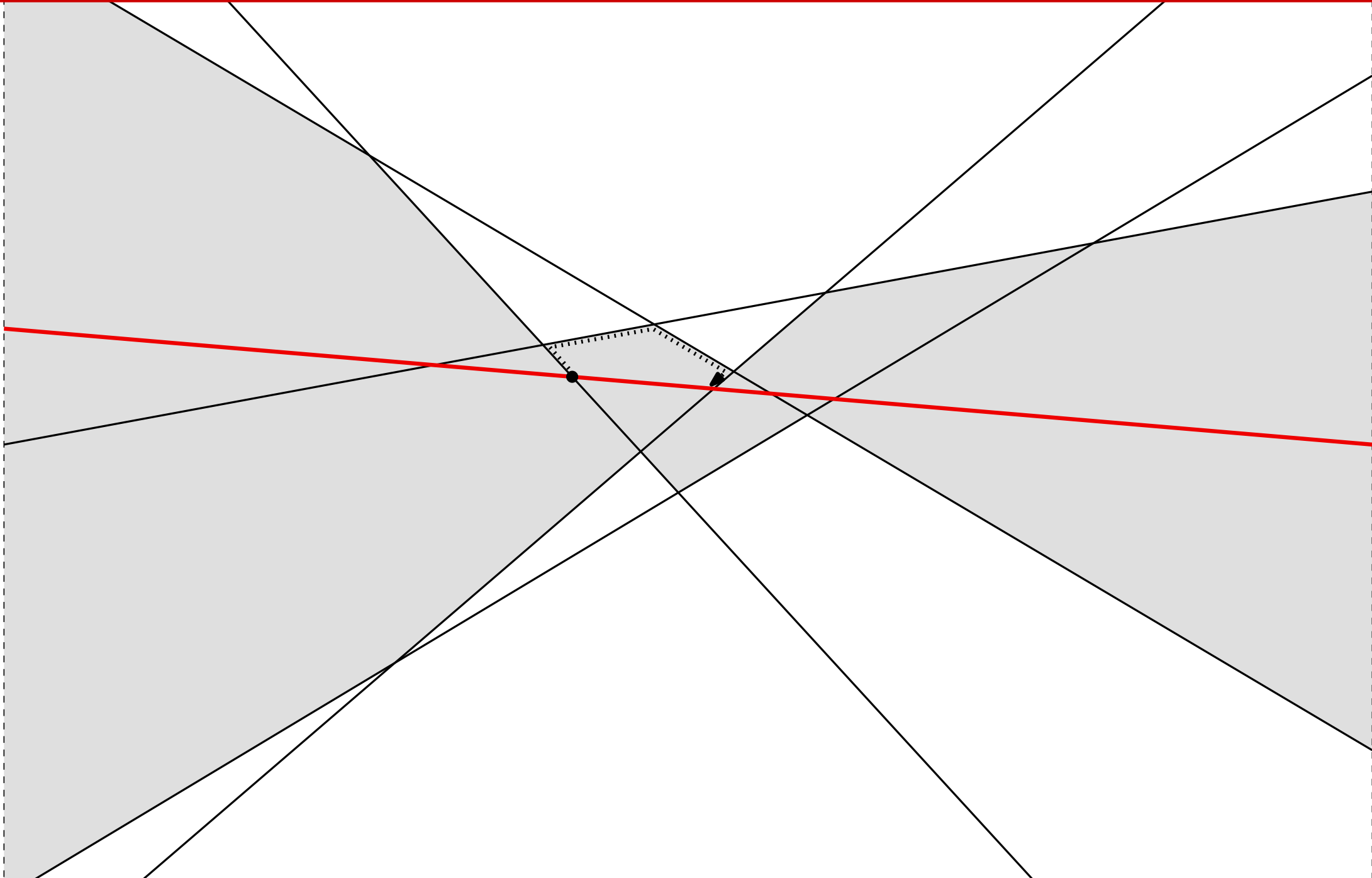
1. Intersect l with any line l_i pre-existent in the arrangement.
2. Traverse the zone of l ...



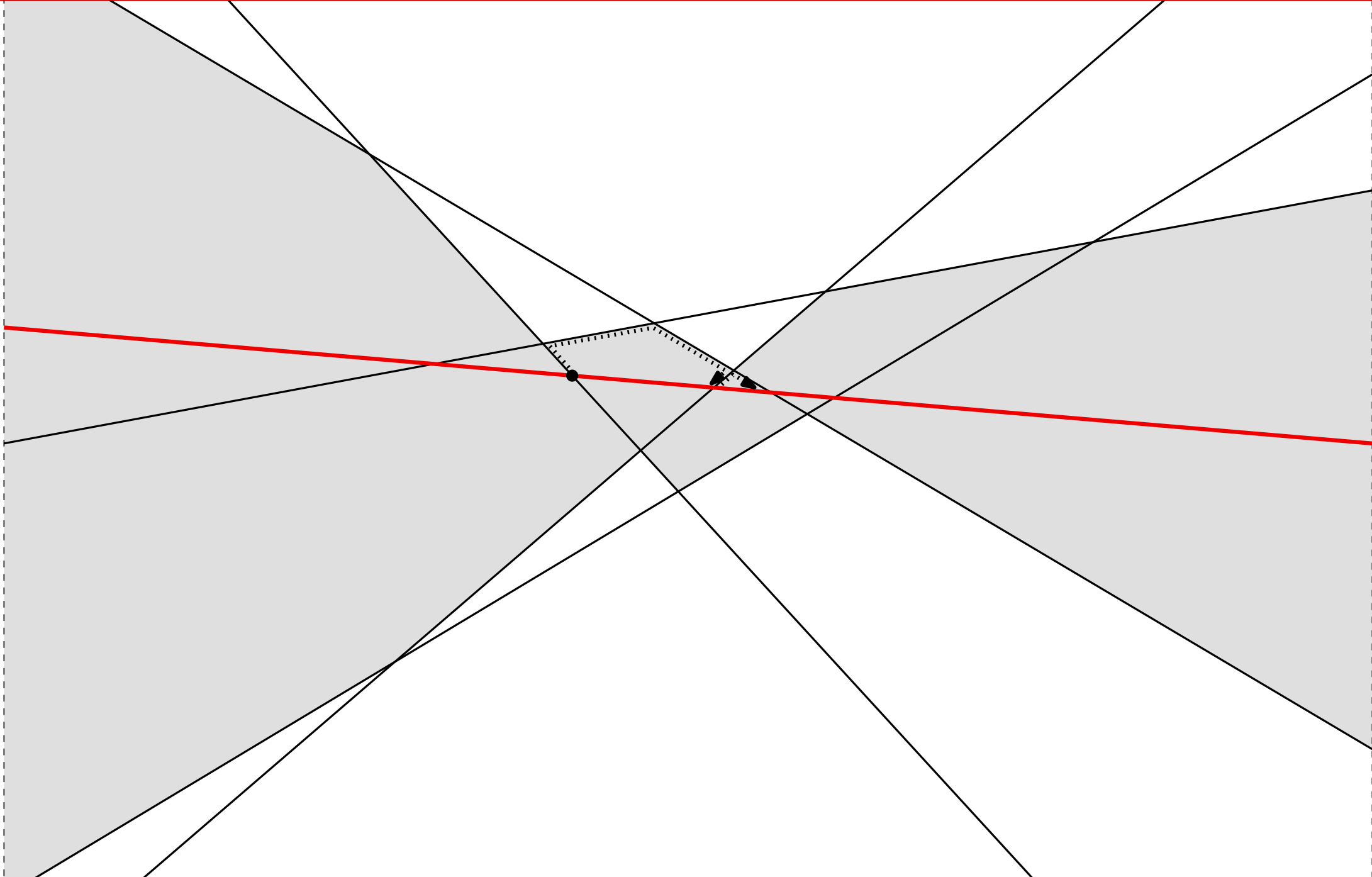
ARRANGEMENTS



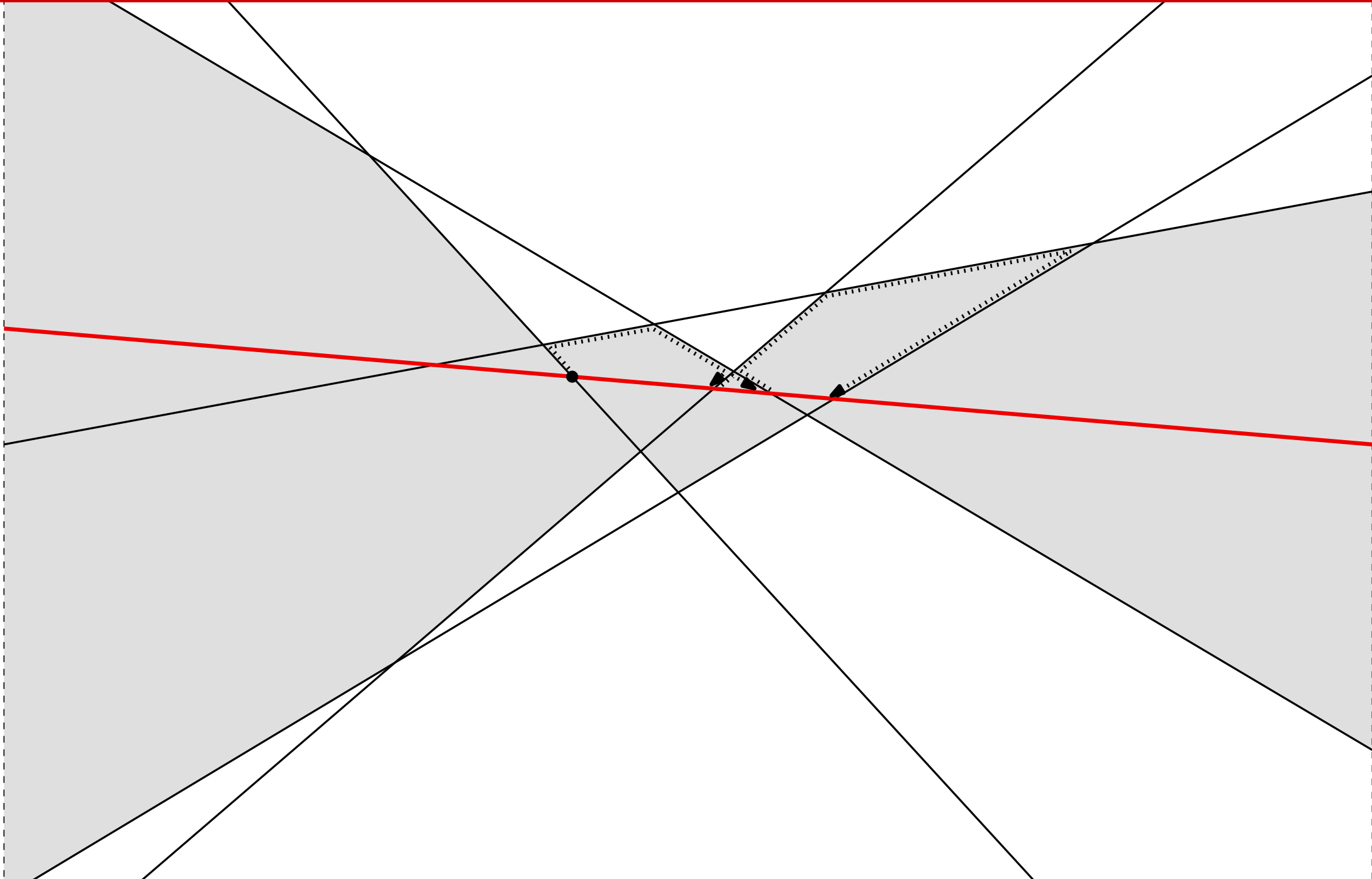
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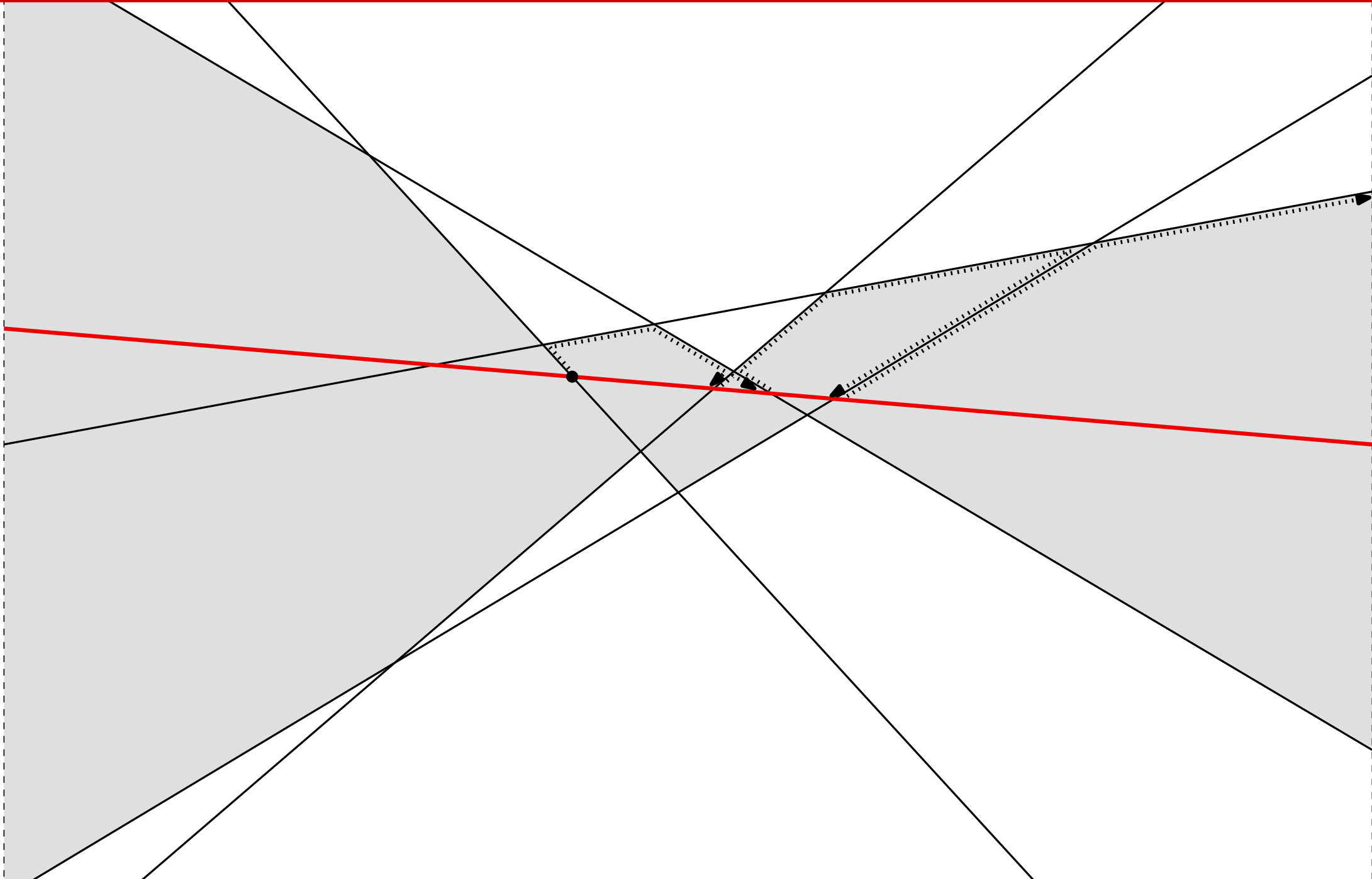
ARRANGEMENTS



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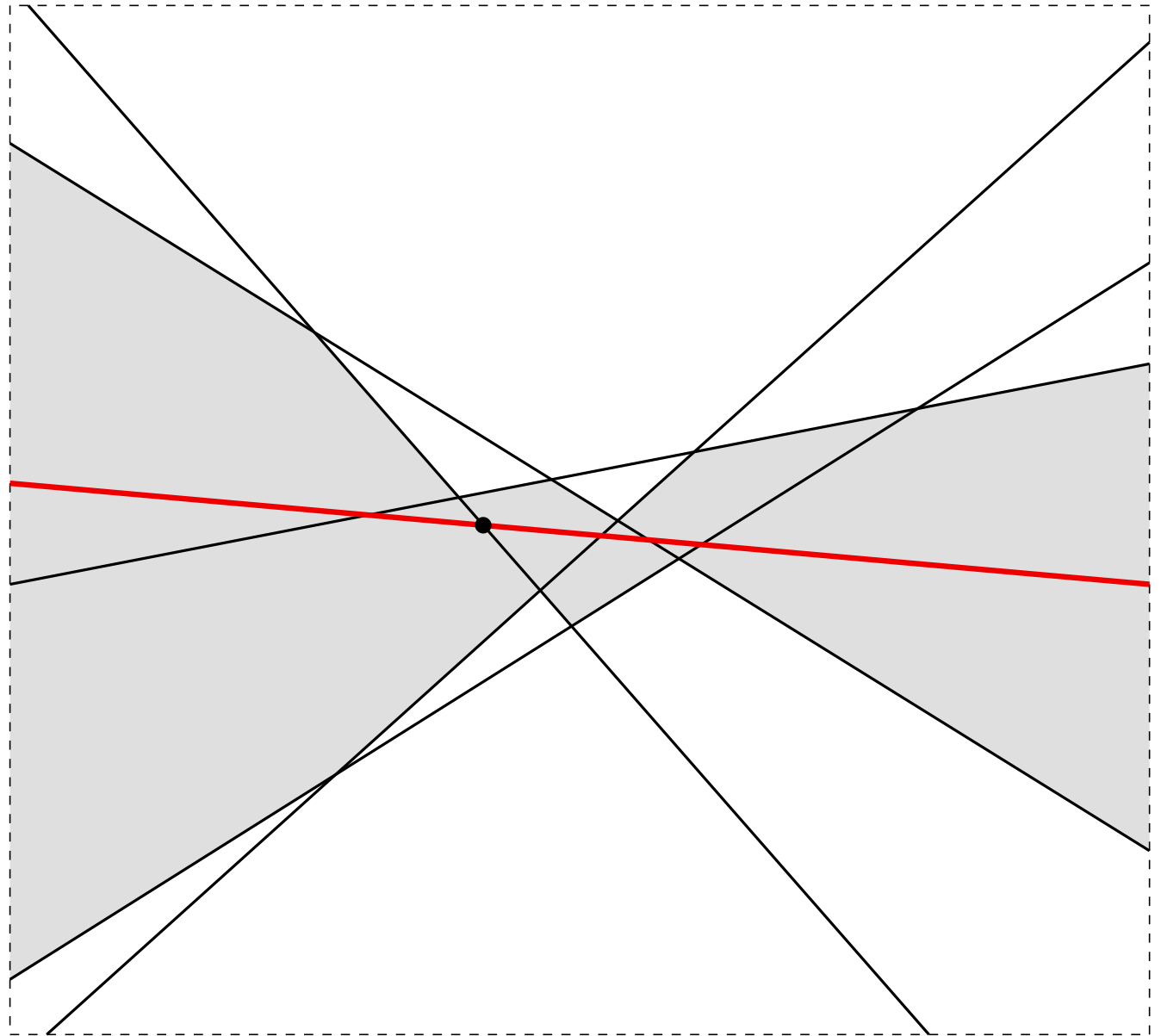
ARRANGEMENTS

COMPUTATION

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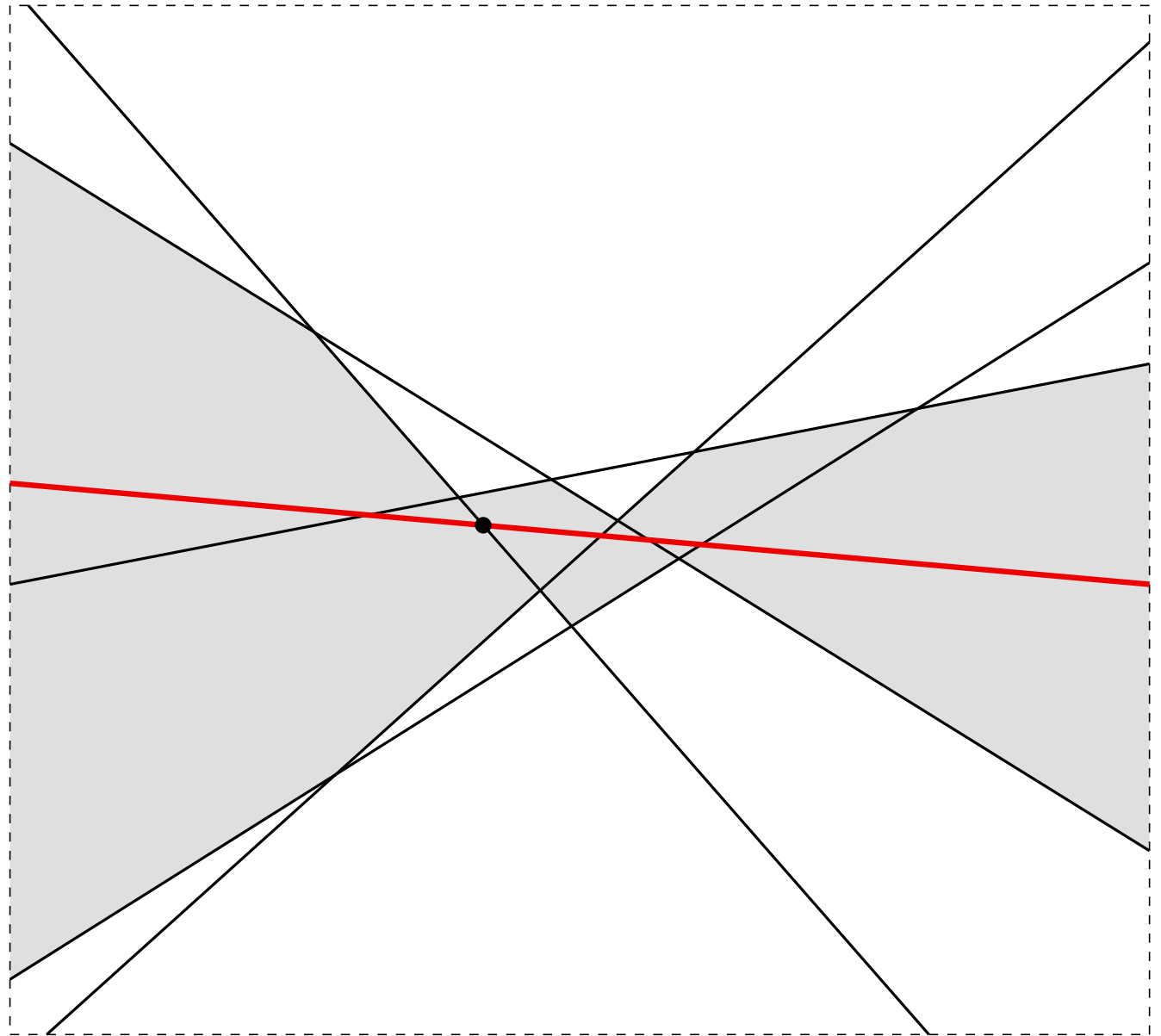
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...updating the DCEL.



ARRANGEMENTS

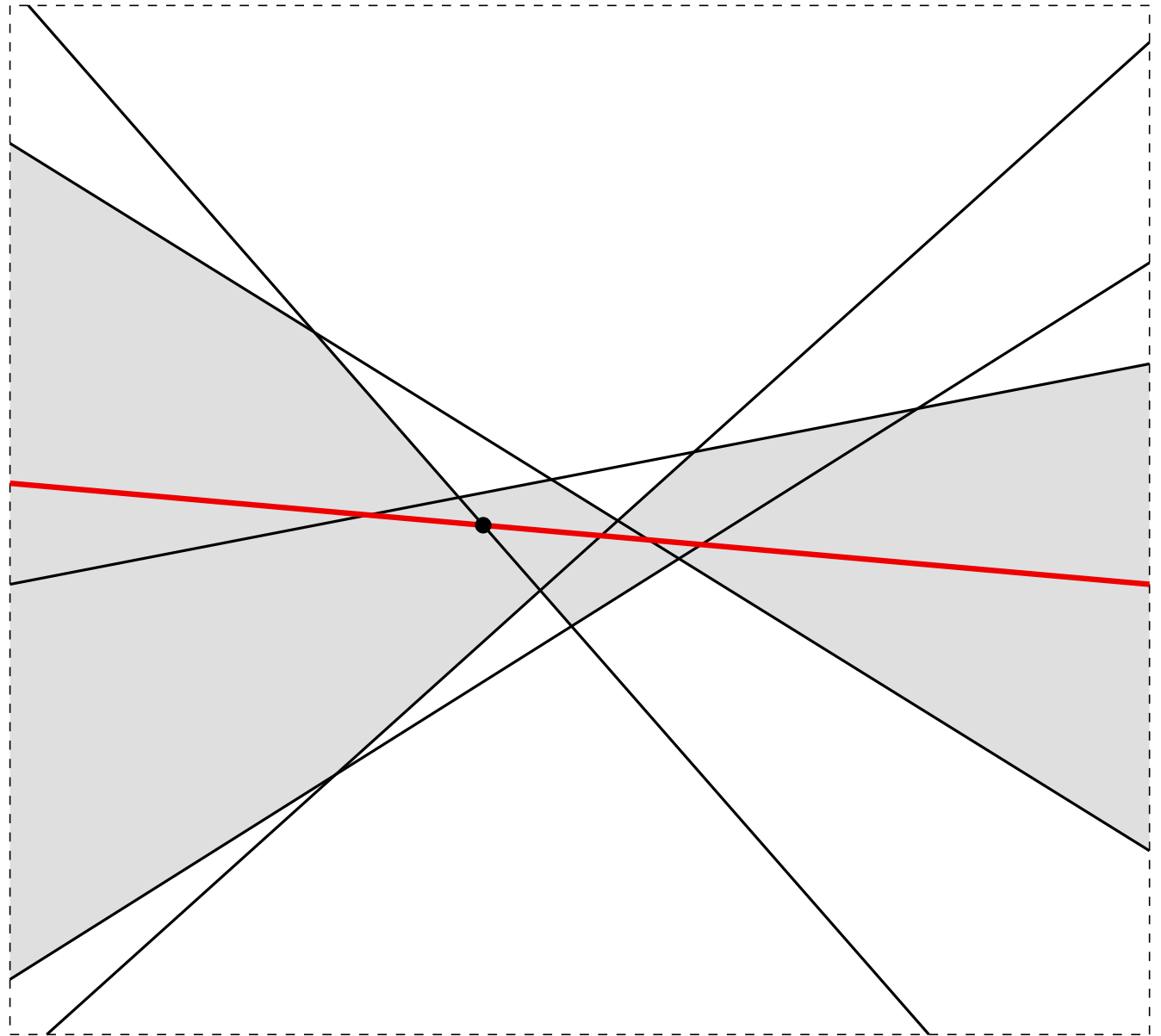
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Updating the DCEL:



ARRANGEMENTS

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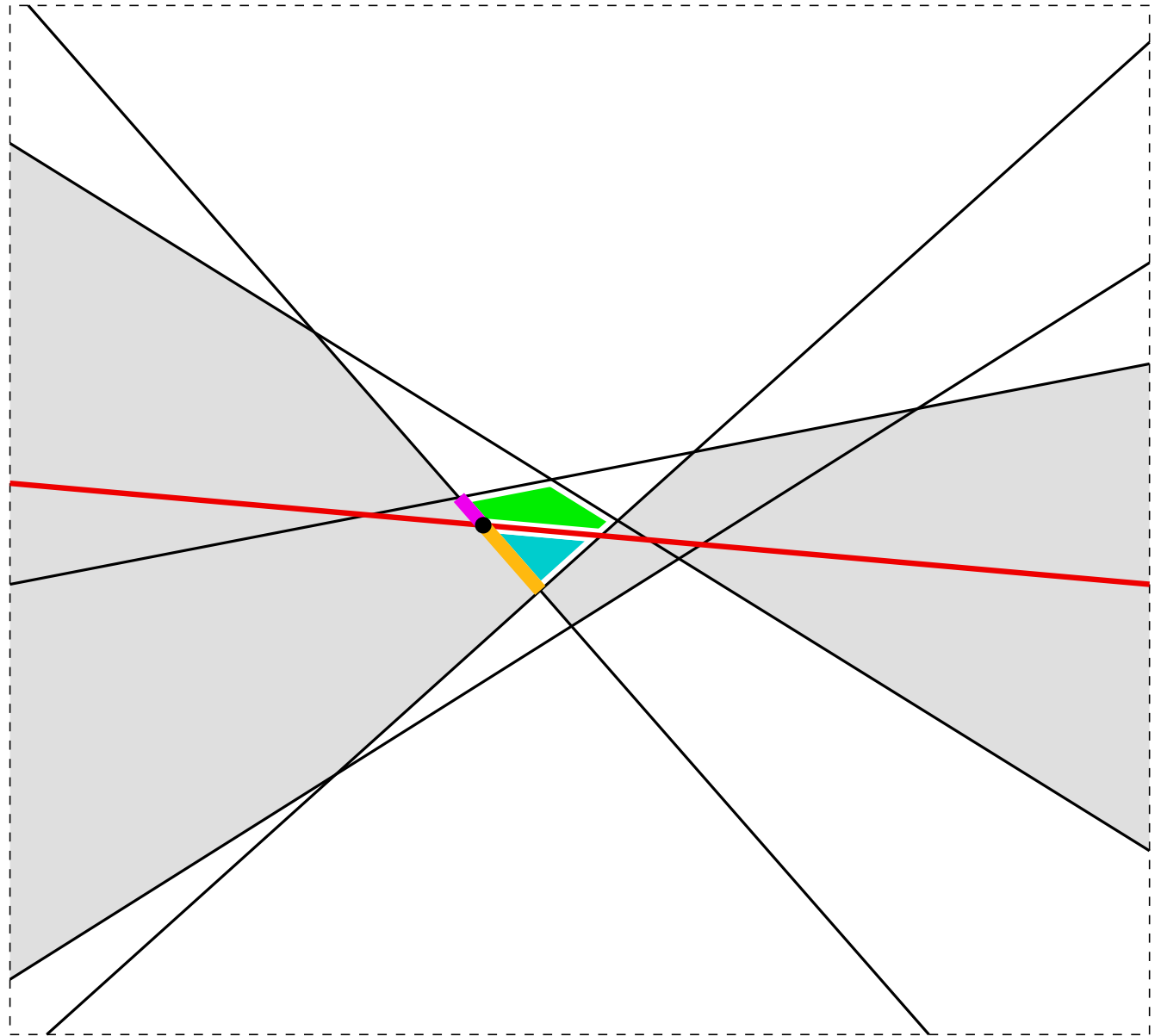
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Updating the DCEL:

At each intersection point v , the line l

- partitions an old edge e into two new edges, e_1 and e_2 ;
- a new edge b_1 ends and a new edge b_2 starts;
- partitions an old face f into two new faces, f_1 and f_2 .



ARRANGEMENTS

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Incremental algorithm

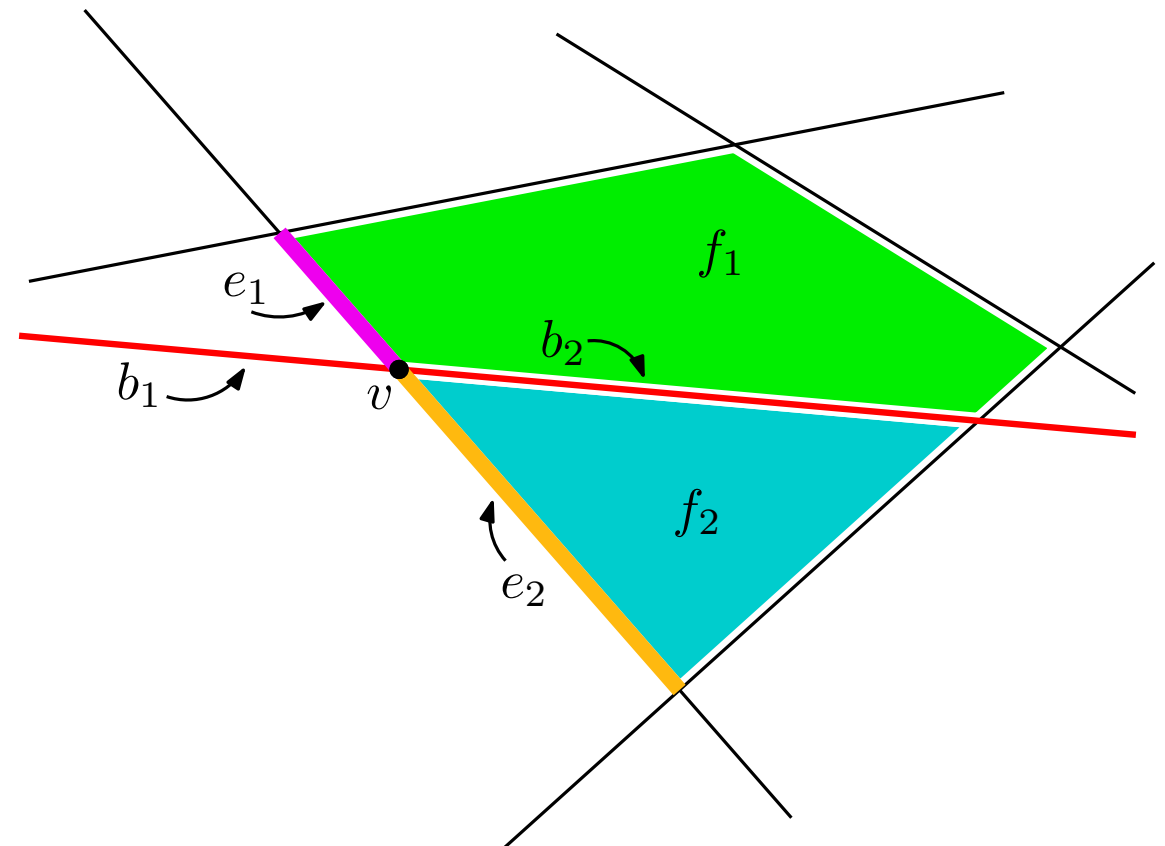
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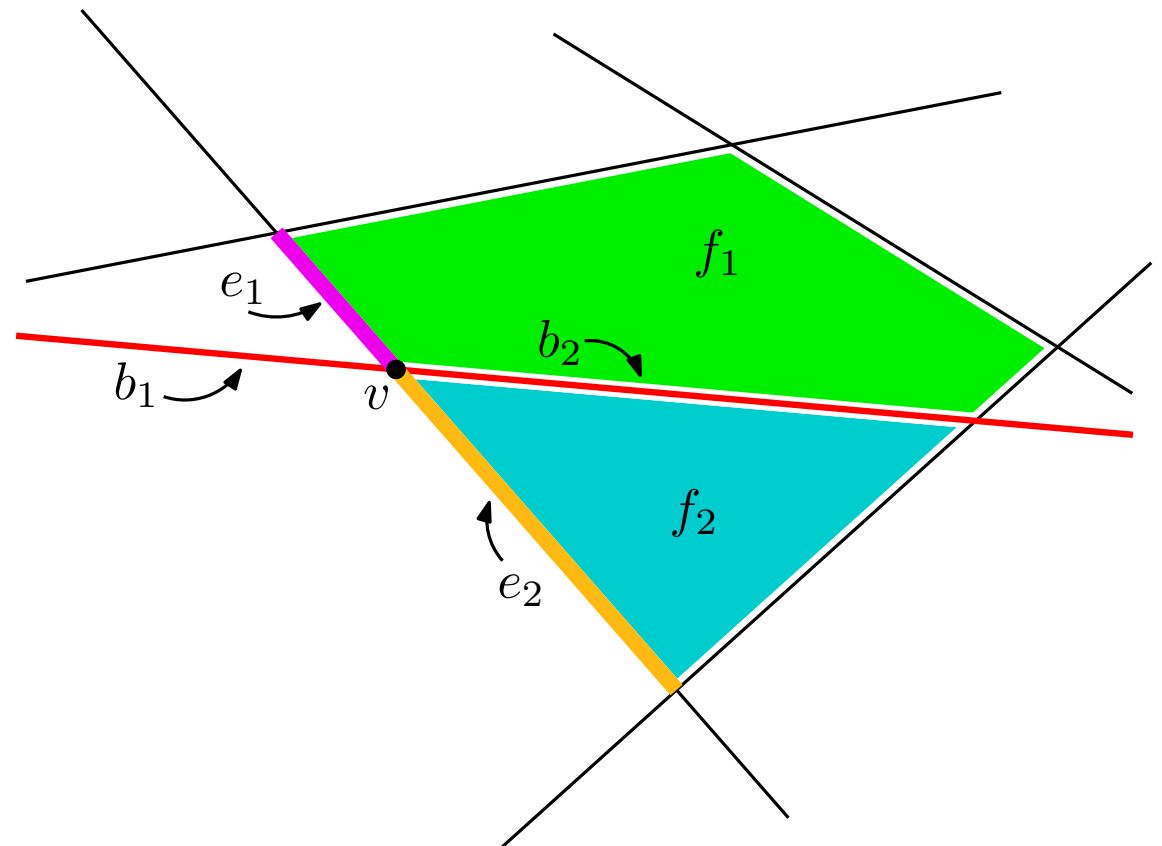
Incremental algorithm

At each intersection point v ,

- Add v to the table, pointing at b_2 .
- Delete f and add f_1 and f_2 , pointing at b_2 .
- Add v_E and e_N for b_1 .
- Add b_2 with all its pointers, excluding v_E and e_N .
- Add e_1 and e_2 with all their pointers, and delete e .

Between two intersection points,

- Modify the pointer of each visited edge to the newly created face.



ARRANGEMENTS

COMPUTATION

Incremental algorithm

Running time:

For each line (there are n lines):

1. Compute the intersection point in $O(1)$ time and locate, in $O(n)$ time, which edge of the current arrangement contains it.
2. Traverse the zone, which has complexity $O(n)$, updating the DCEL in $O(1)$ time at
 - each intersection point and
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The description of the algorithm we have shown assumes that the arrangement is simple.

An easy modification allows adapting it to the general case.

ARRANGEMENTS

EXTENSIONS

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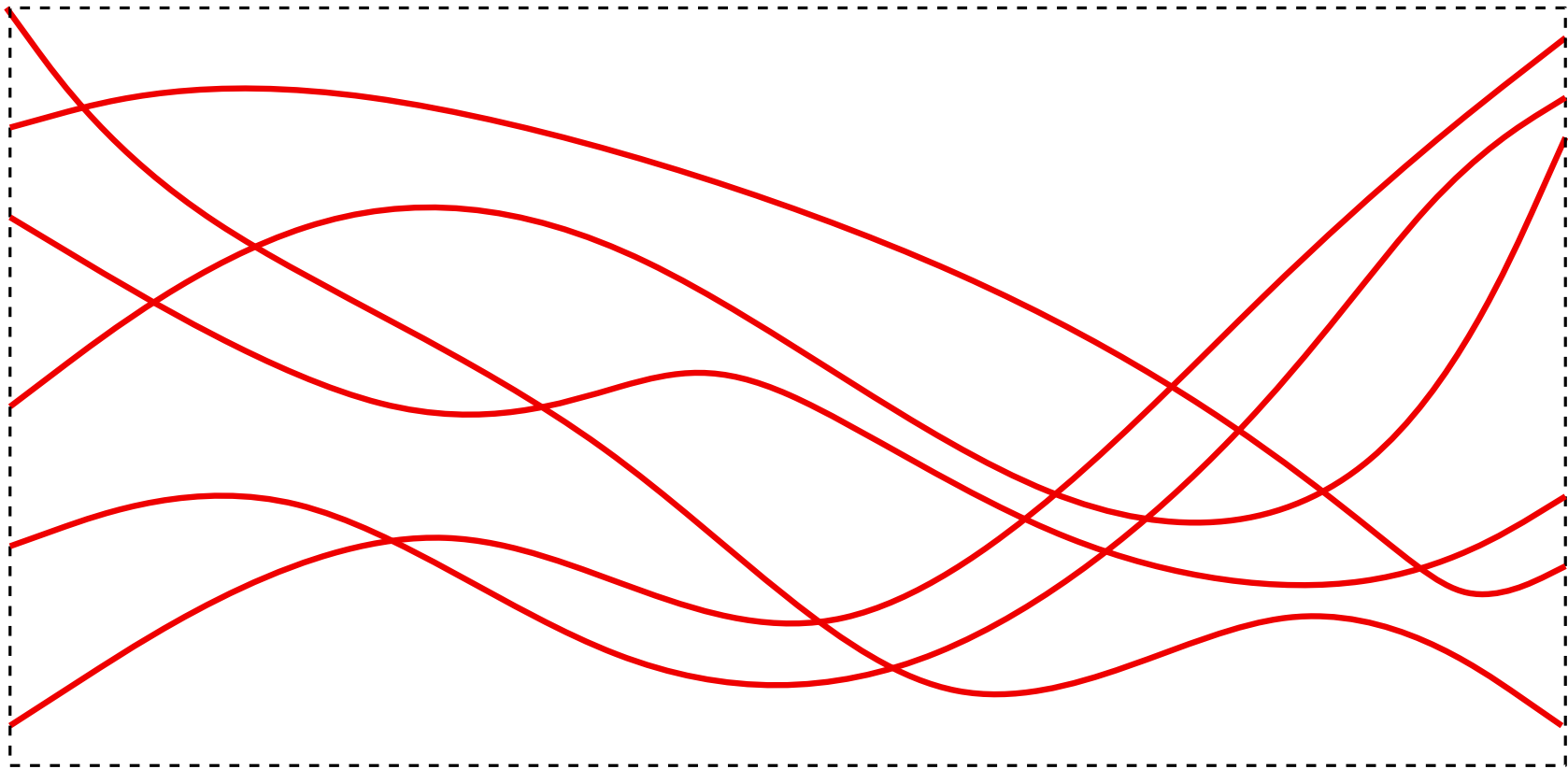
↑
curves

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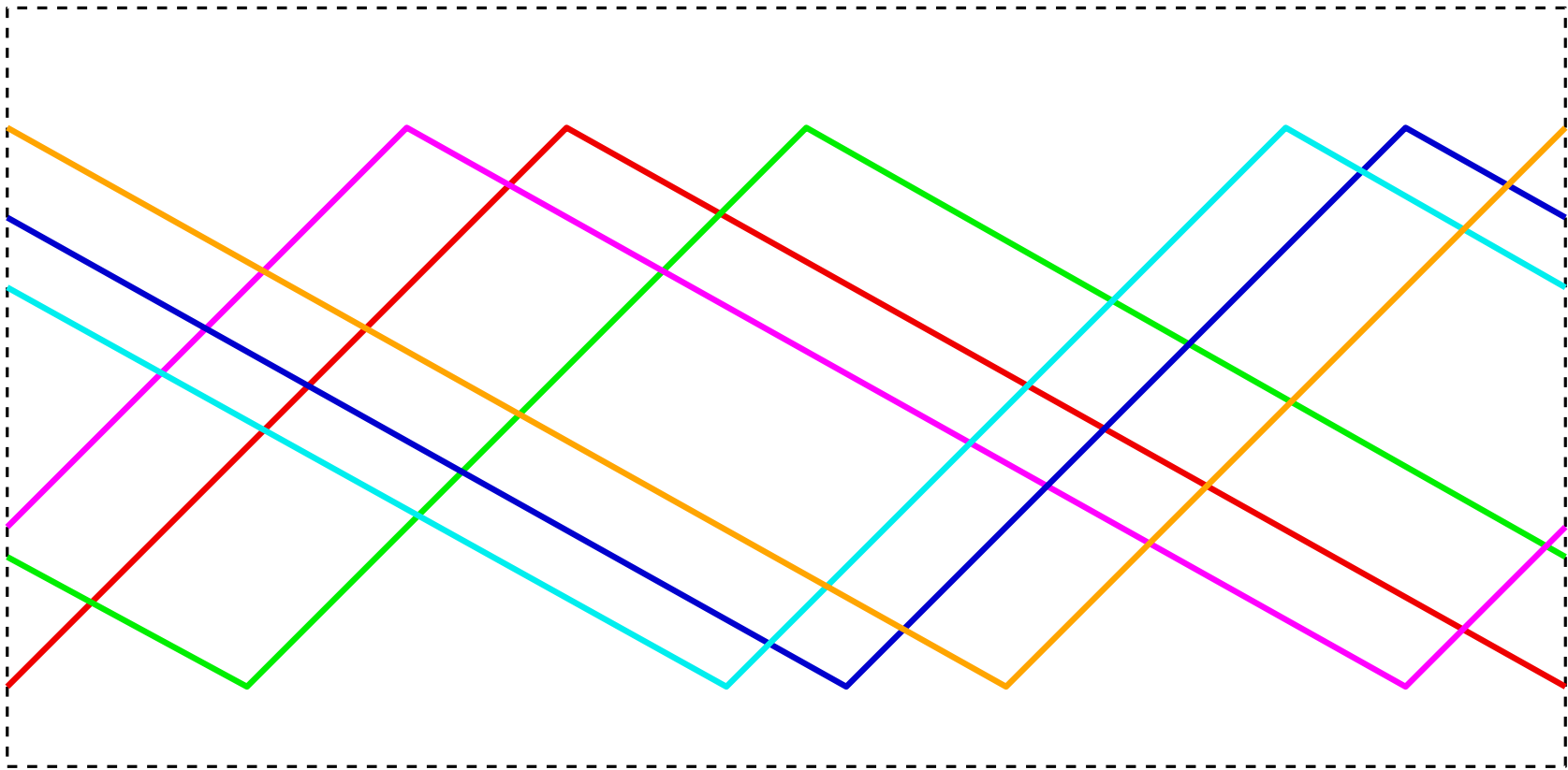


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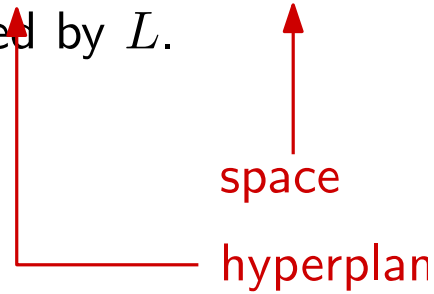
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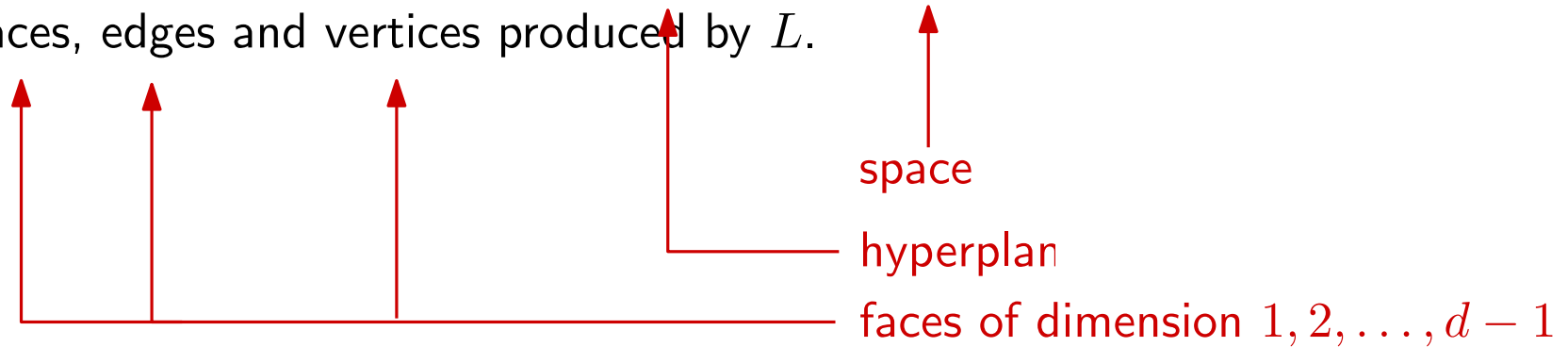
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ARRANGEMENTS

APPLICATIONS

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- Arrangements and Voronoi diagrams (already seen)
- Arrangements of lines as duals of point sets (with much more structure)
 - k -sets
 - ham-sandwich cuts
- Elimination of hidden surfaces (for n disjoint polygons in space, solved in $O(n^2)$ time by topological sweep)
- Aspect graphs
- Motion planning (visibility graphs)
- And many more! ... even for checking whether there exist 3 or more colinear points in a set of n points (using duality!).

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How can you see in the dual when three points are colinear?

ARRANGEMENTS

MORE ABOUT IT

- J. O'Rourke: *Computational Geometry in C*, Cambridge University Press.
- M. de Berg, O. Cheong, M. van Kreveld, M. Overmars: *Computational Geometry: Algorithms and Applications*, Springer.
- H. Edelsbrunner: *Algorithms in Combinatorial Geometry*, Springer.
- L. J. Guibas, M. Sharir: Combinatorics and Algorithms of Arrangements, in *New Trends in Discrete and Computational Geometry. Algorithms and Combinatorics*, J. Pach ed., Springer.
- M. Sharir, P. K. Agarwal: *Davenport-Shinzel Sequences and Their Geometric Applications*, Cambridge University Press.