TRIANGULATING POINT SETS

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Universitat Politècnica de Catalunya

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DEFINITION

A triangulation of A set P of n points in the plane is a graph having P as set of vertices which is rectilinear, planar, and maximal in the number of edges.

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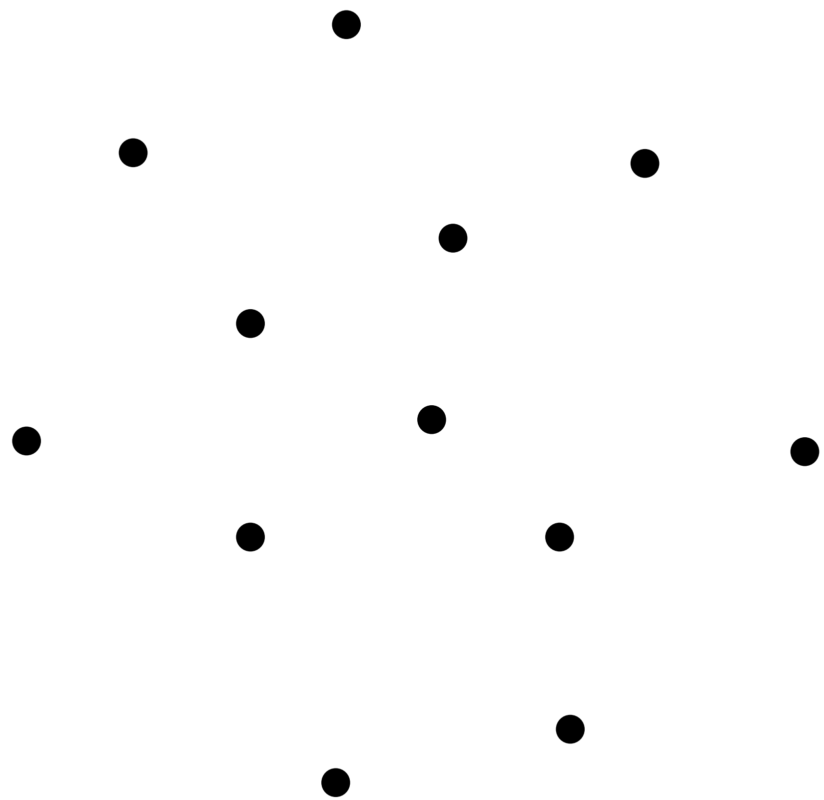

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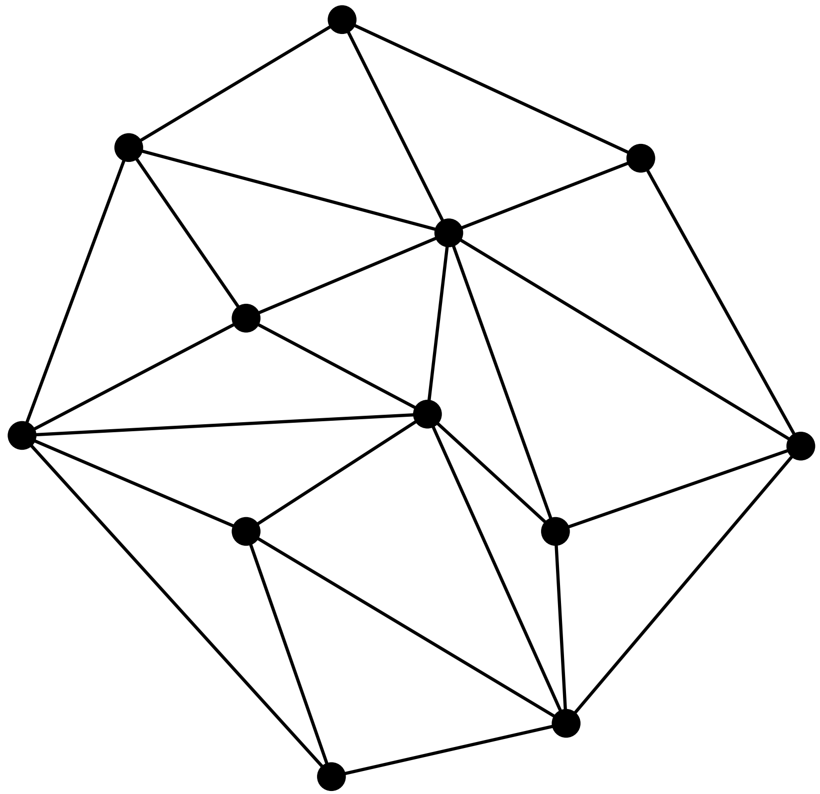

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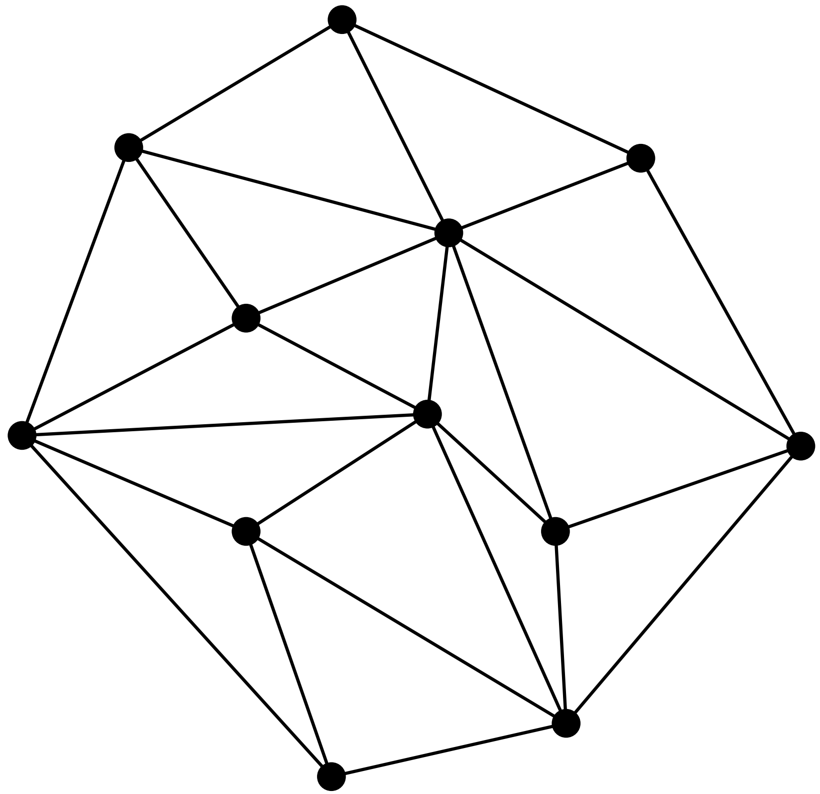
Corollary. All the faces of such a graph are triangles, except for the unbounded one, which is the exterior of the convex hull of P.



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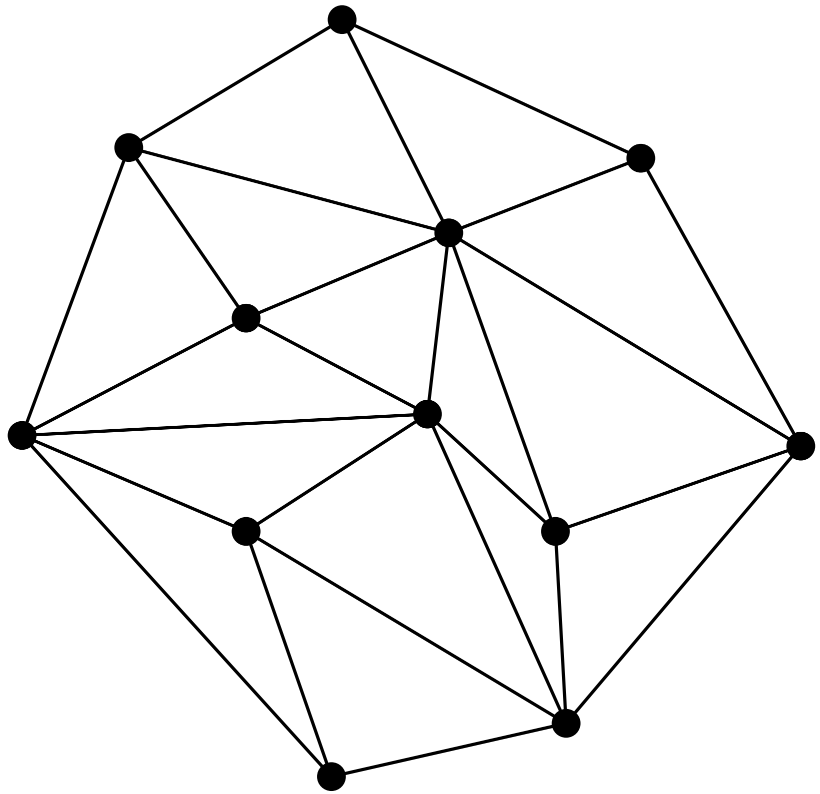
COMPLEXITY

Every triangulation of any set P of n points has:

$2n - h - 2$ triangles

$3n - h - 3$ edges

where h is the number of vertices of $ch(P)$.



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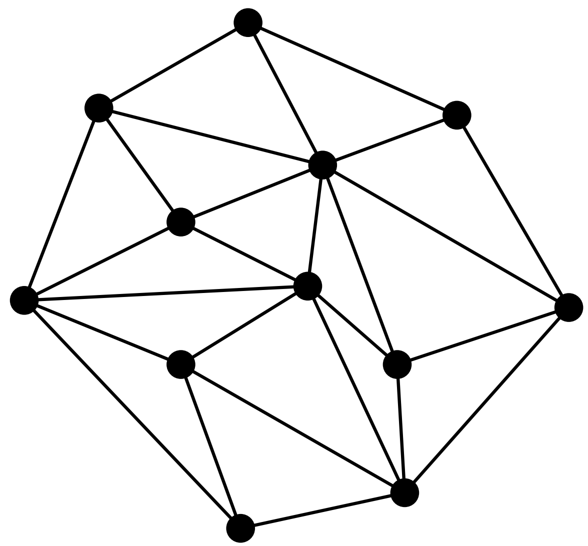

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Proof. Each triangle has exactly 3 edges. Each internal edge belongs to exactly 2 triangles. Each external edge belongs to exactly 1 triangle. Therefore, $3t = 2(e - h) + h = 2e - h$.
According to Euler's formula: $n + (t + 1) = v + f = e + 2$.
Combining both equations:

$$e = n + t - 1 \Rightarrow 3e = 3n + 3t - 3 = 3n + 2e - h - 3 \Rightarrow e = 3n - h - 3$$

$$3t = 2e - h = 6n - 2h - 6 - h = 6n - 3h - 6 \Rightarrow t = 2n - h - 2$$

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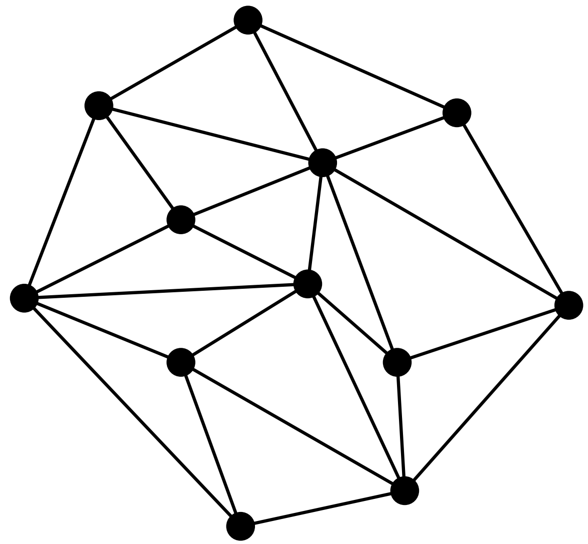

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DEGENERACIES

As you may have noticed, we are assuming that the set P does not contain three or more points on a line. The assumption is hold along the entire chapter.

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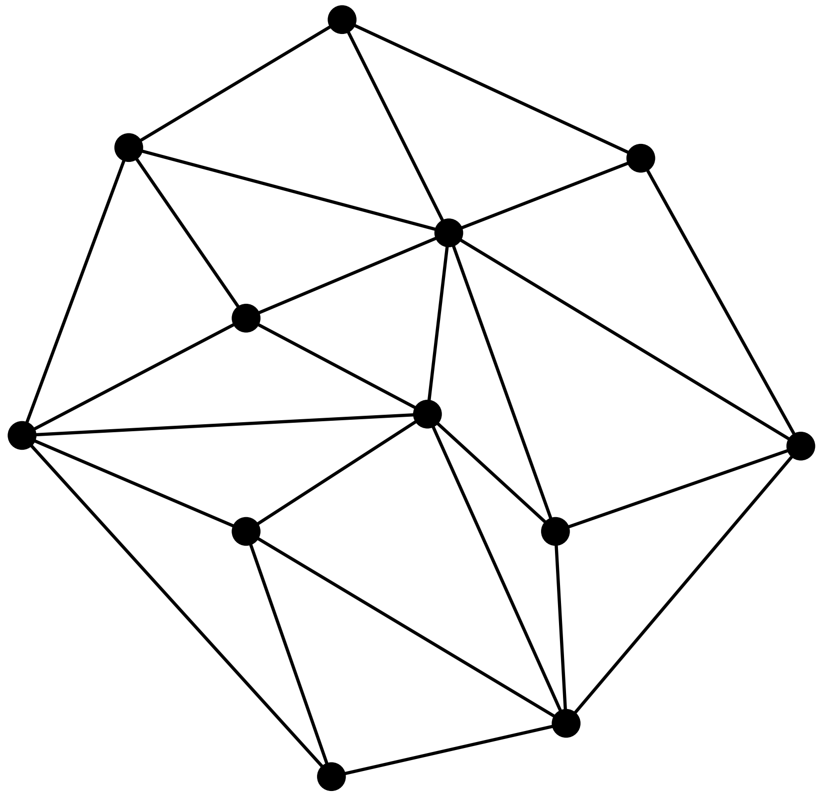
DATA STRUCTURE

We want to answer the most usual questions for any decomposition of the plane:

- For any given triangle, report its edges/vertices.
- For any given vertex, report the sorted list of edges/triangles incident to it.
- For any given edge, report its endpoints and its adjacent triangles.



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The appropiate structure is, once again, a **DCEL**:

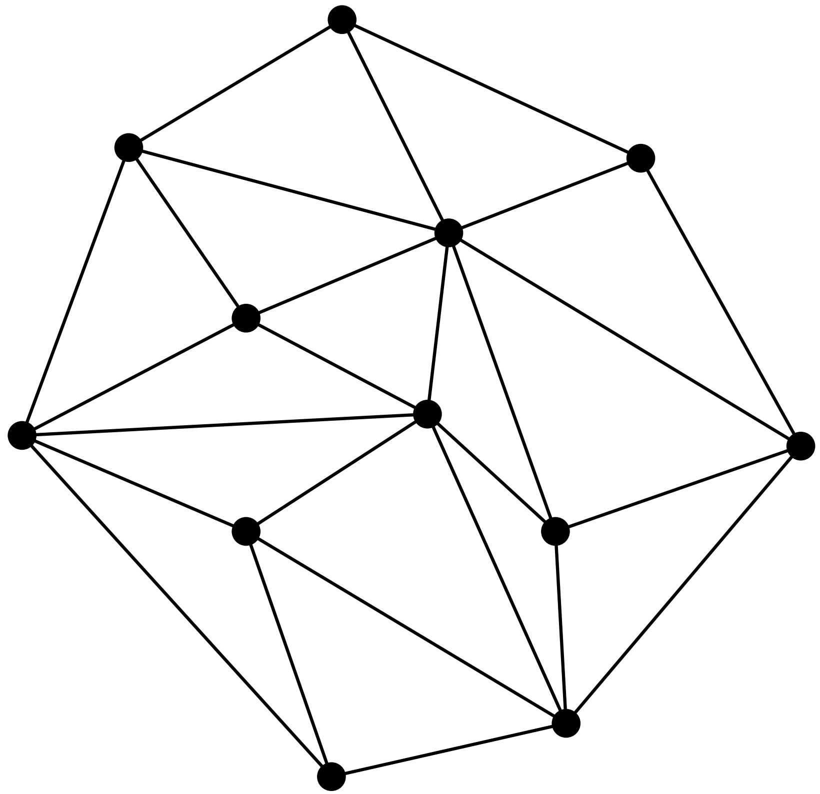

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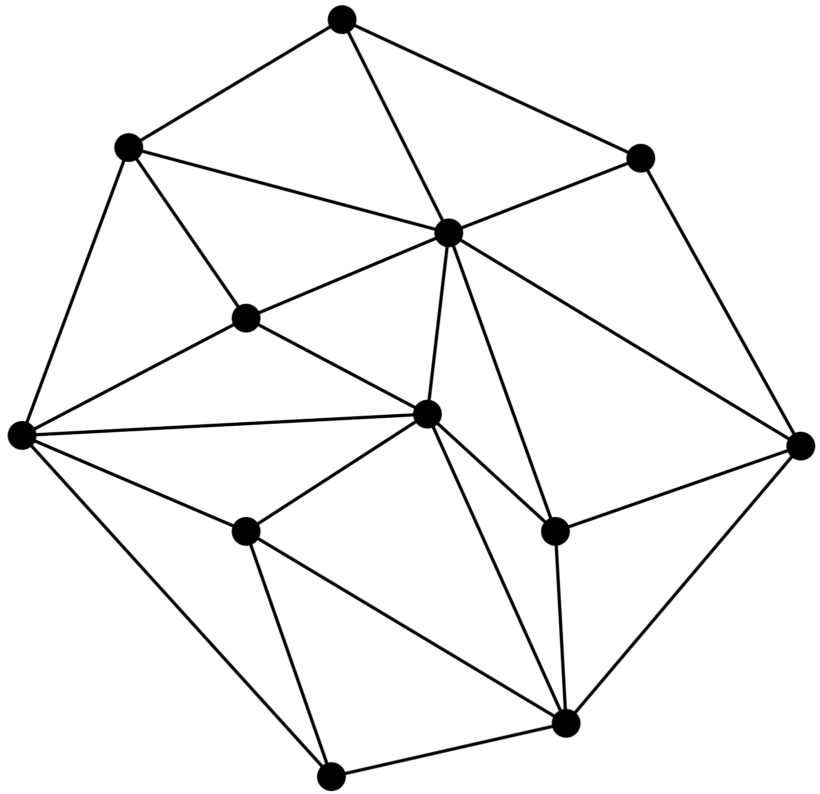
Table of vertices

v	x	y	e



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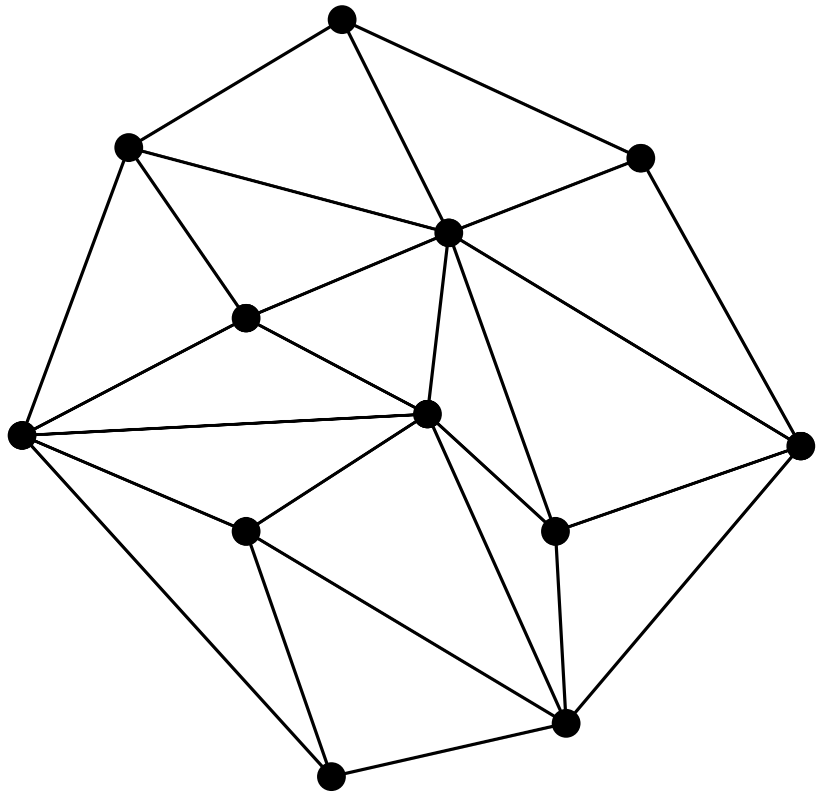

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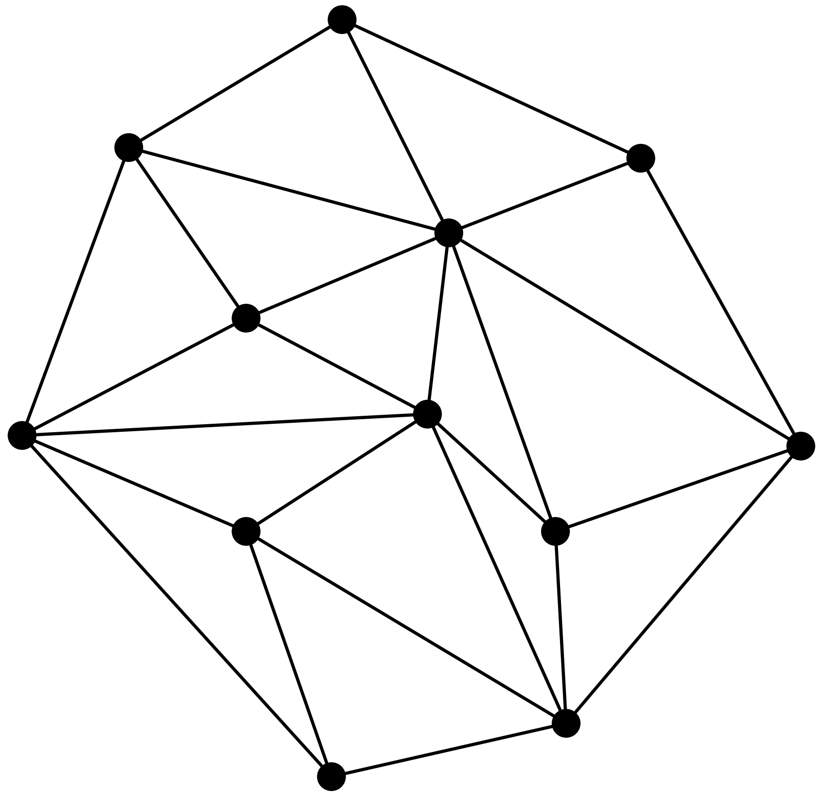

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DCEL

e	v_B	v_E	f_L	f_R	e_P	e_N

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1.1. Without sorting

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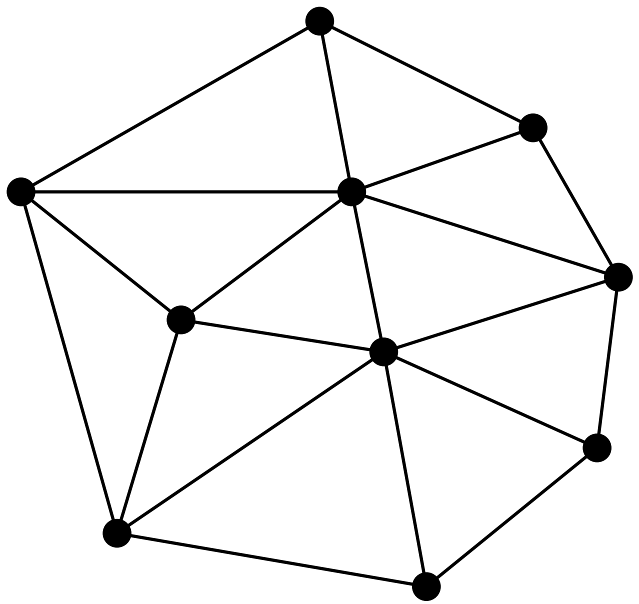

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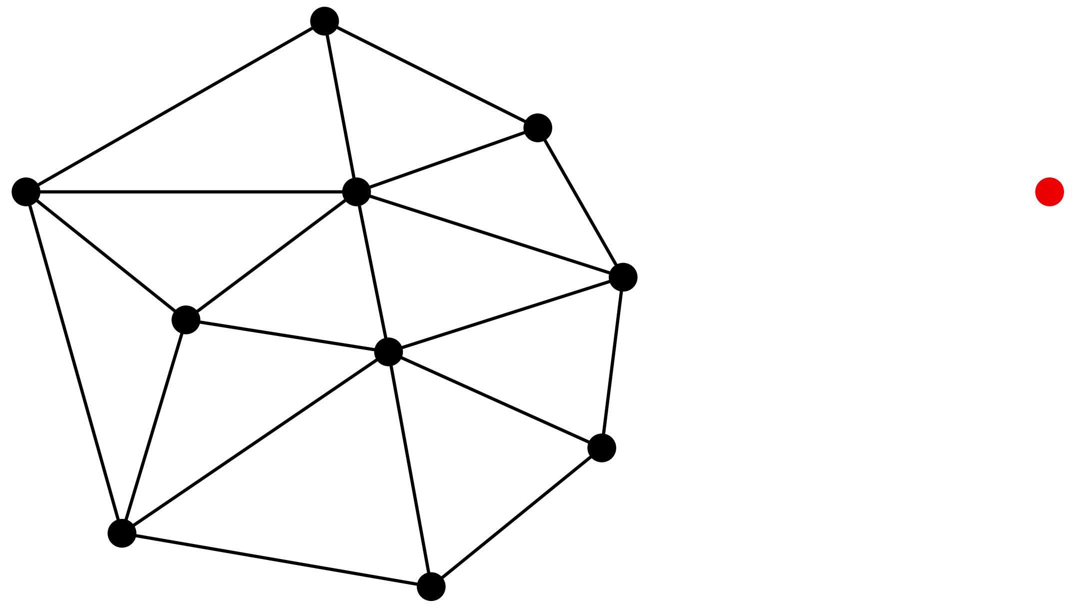

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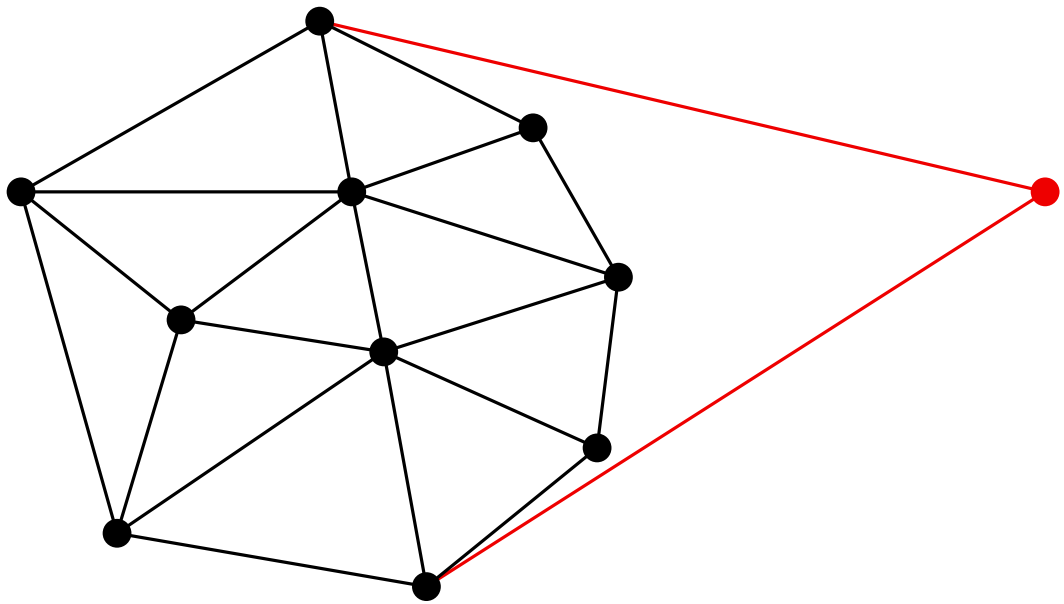

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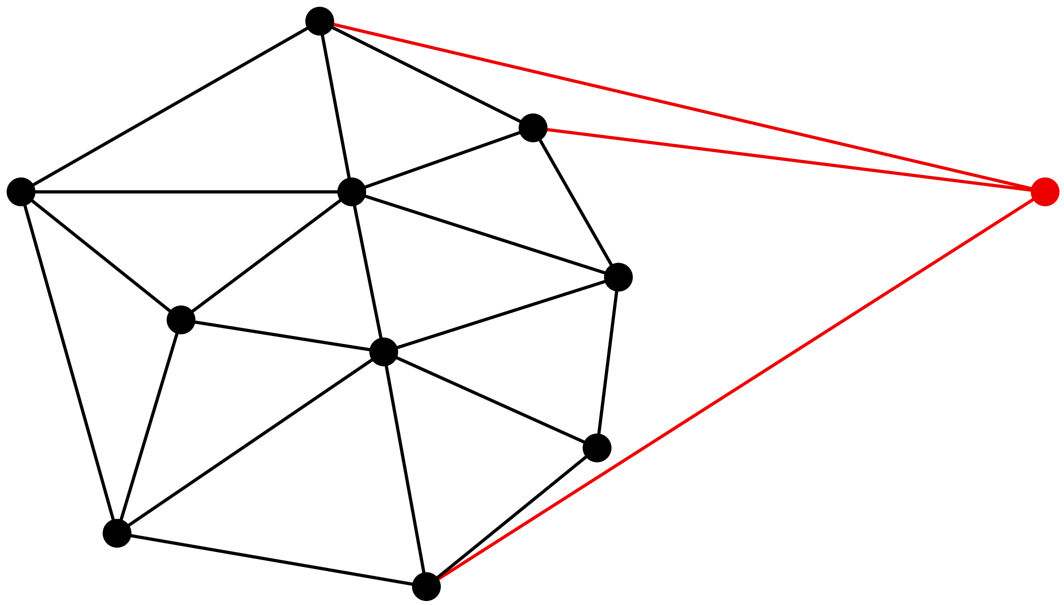

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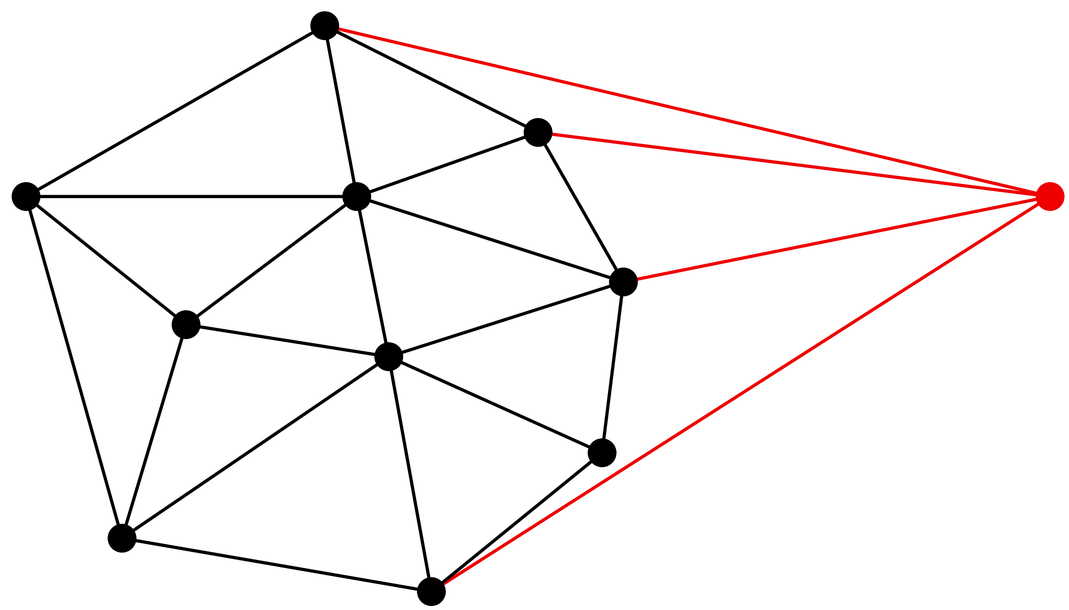

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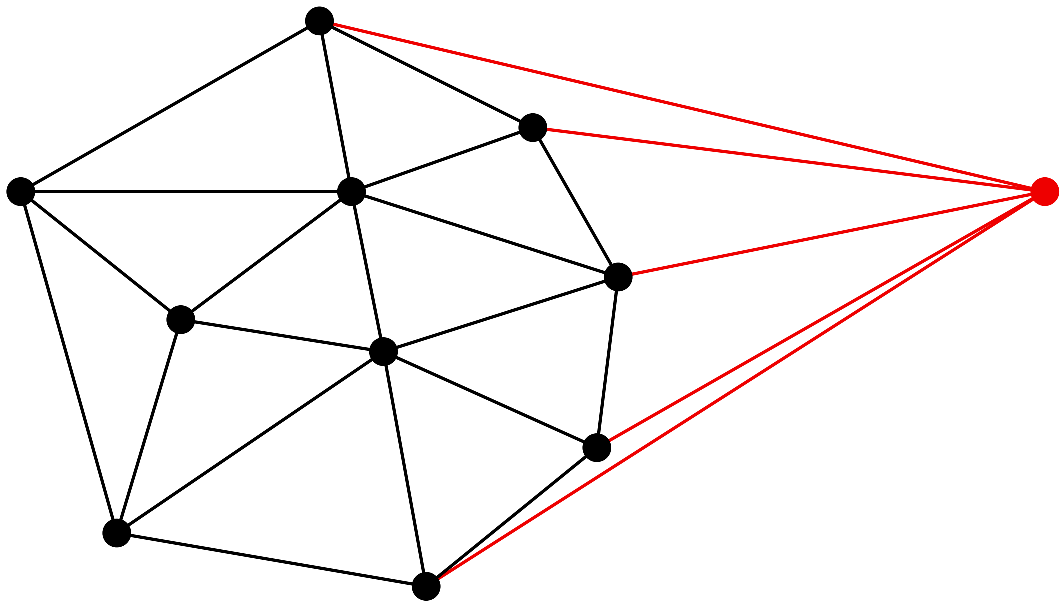

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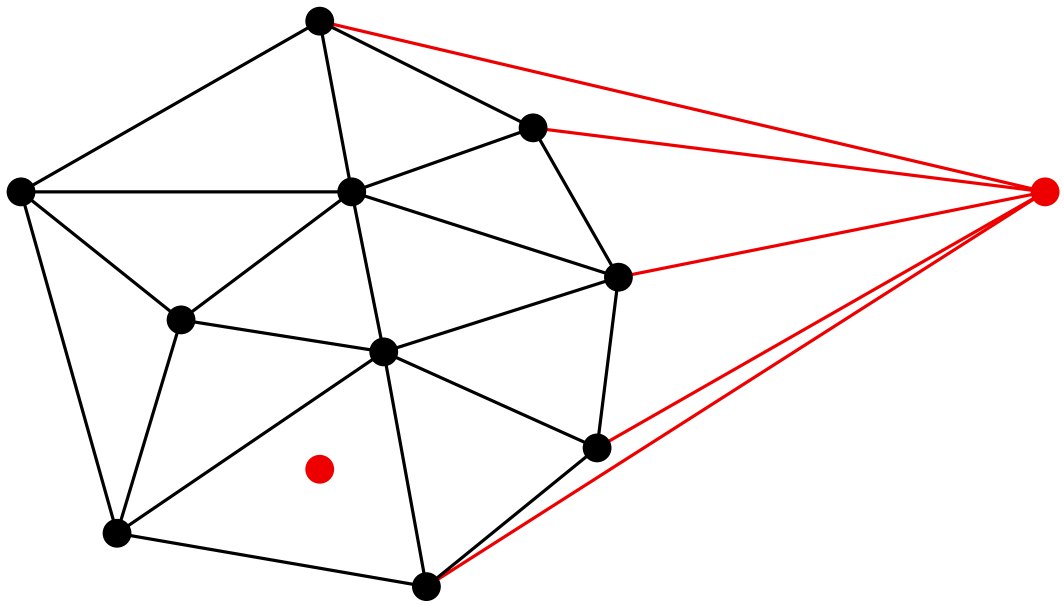

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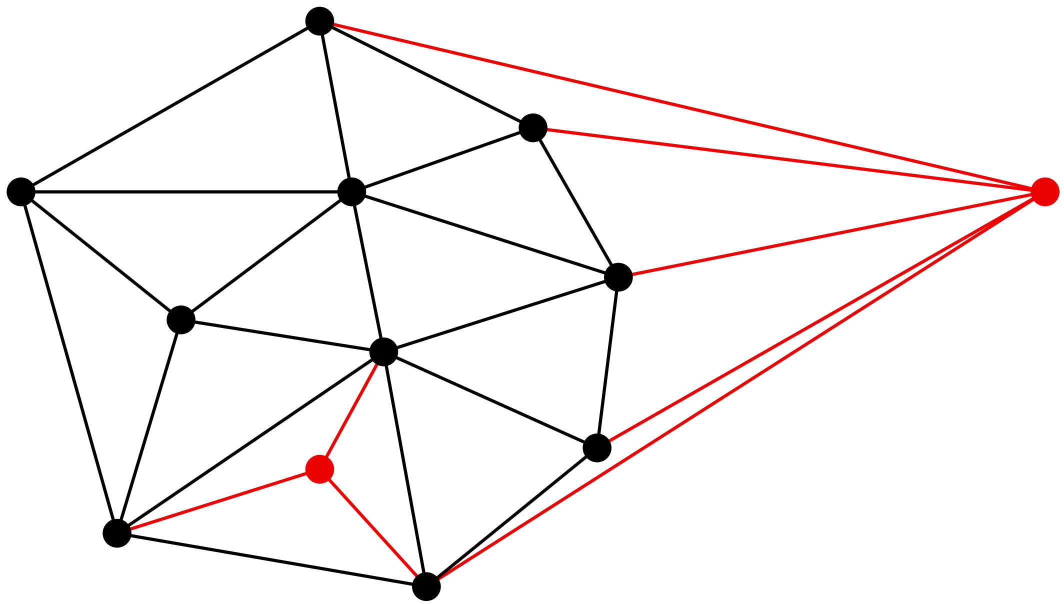

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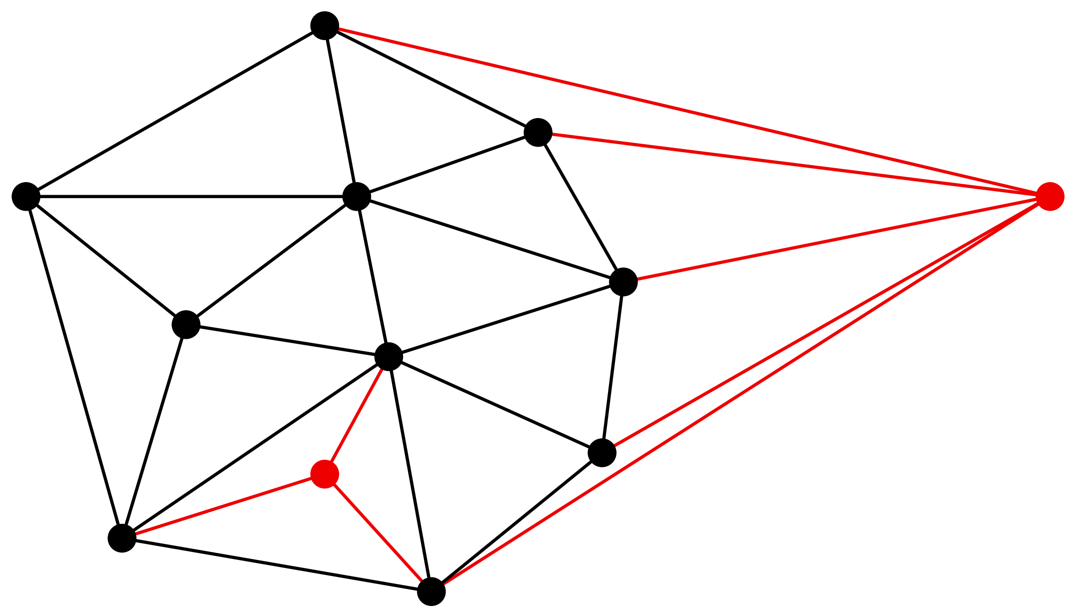

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   1.1. Without sorting
   1.2. With sorting

Running time: $O(n^2)$

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Start by sorting the points in lexicographical order in $O(n \log n)$ time. The information of the sorted order of the points allows to add the i diagonals in $O(i)$ time, so that the amortized cost of the insertion of all diagonals is done in $O(n)$ time.

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- **1.1.** Without sorting — **Running time:** $O(n^2)$
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- **1.3.** With hierarchical structure

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Using an auxiliary enclosing triangle and a hierarchy of triangles: each time a new point is added, a triangle gets subdivided into three children.

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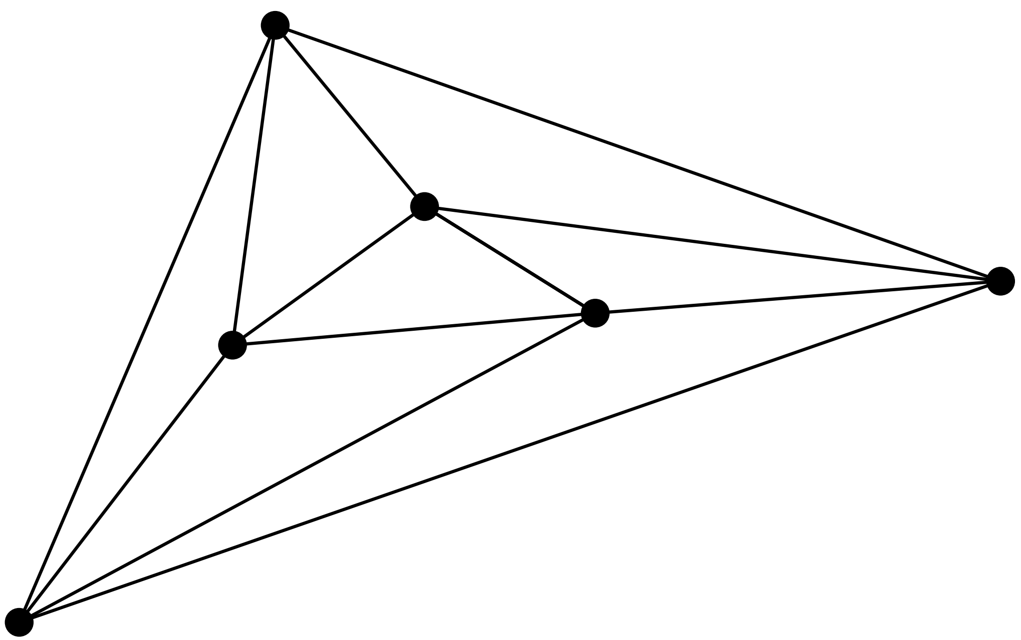

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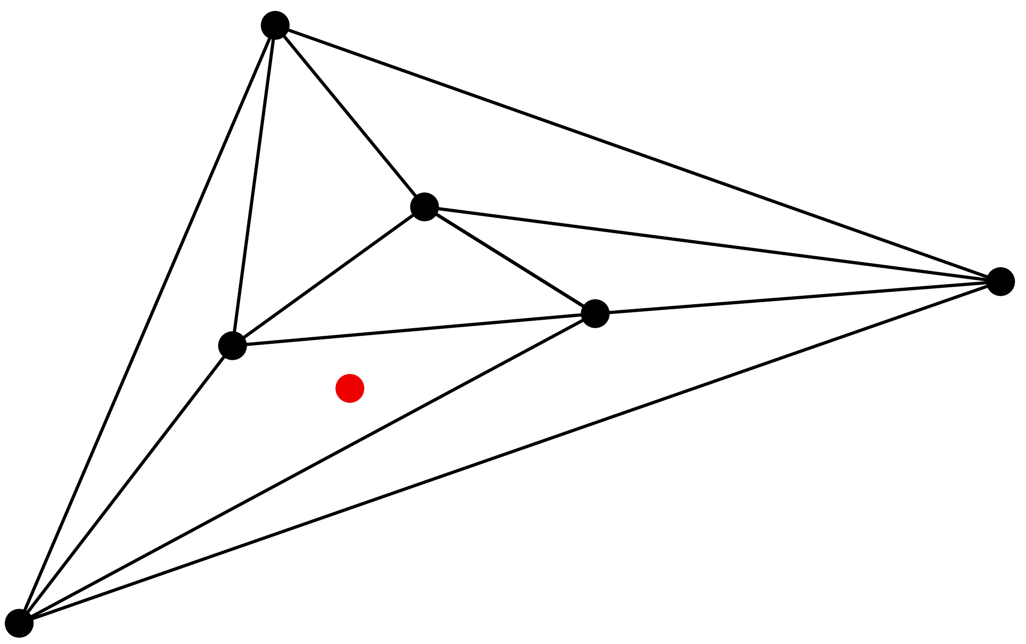

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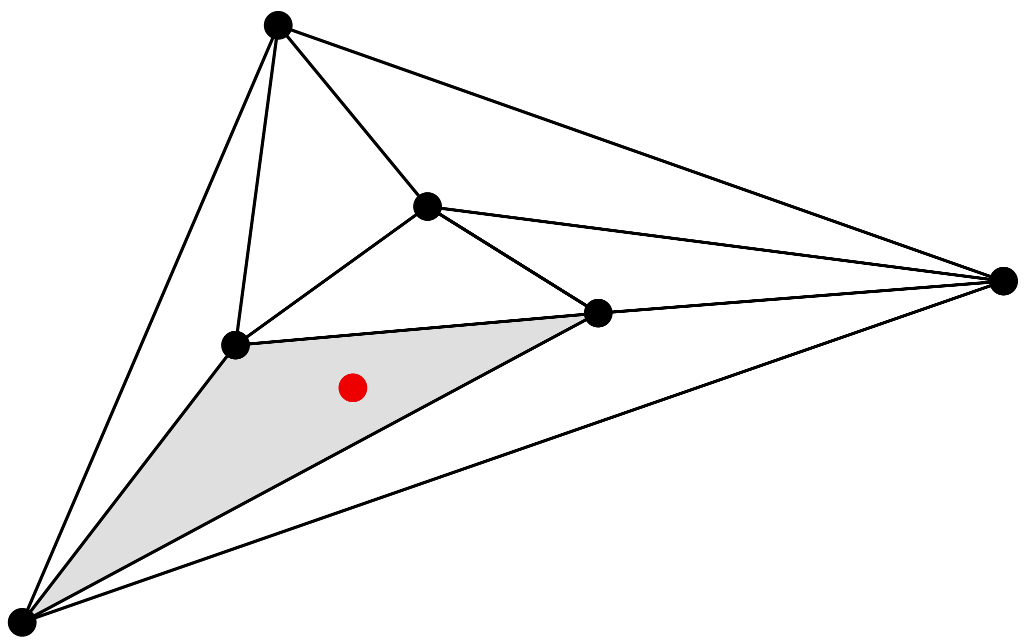

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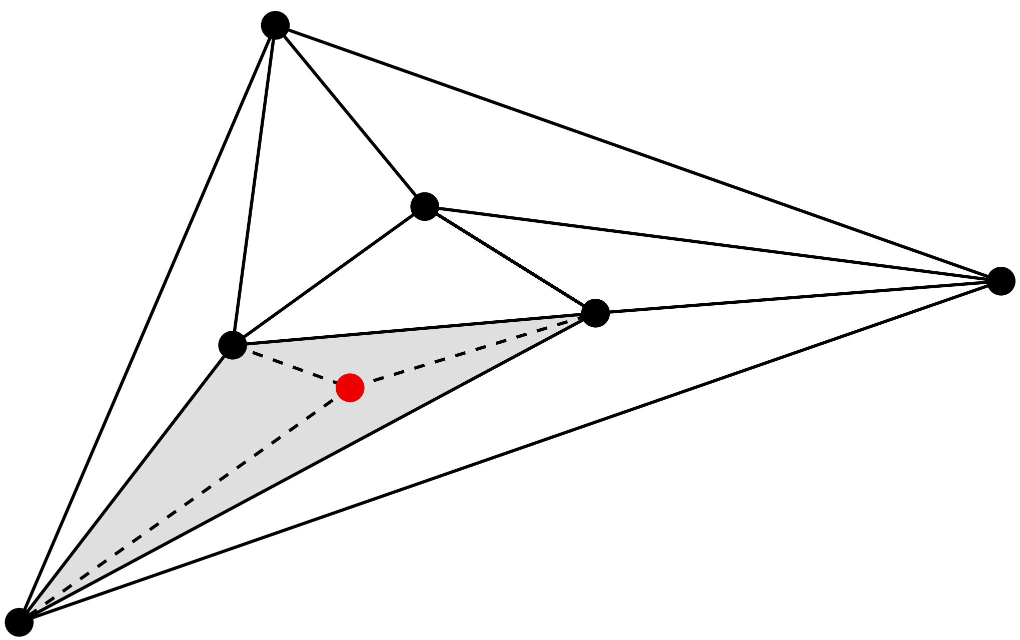

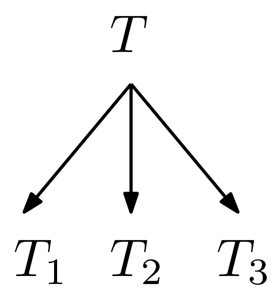

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1.2. With sorting — **Running time:** $O(n \log n)$

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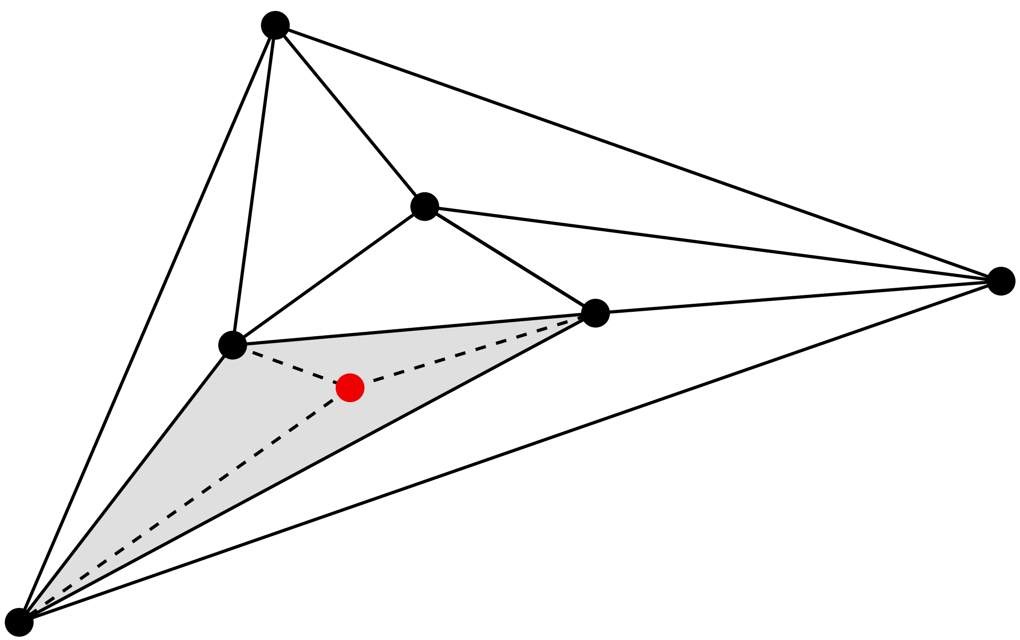

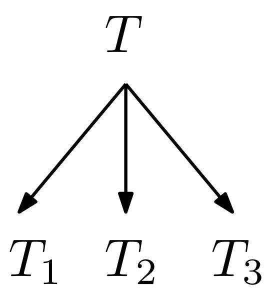

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   5. **1.5.** With auxiliary point(s)

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1.5. With auxiliary point(s)

A fixed point p is used as a reference, and $P \cup \{p\}$ is enclosed in an auxiliary triangle.
When inserting each point p_i:

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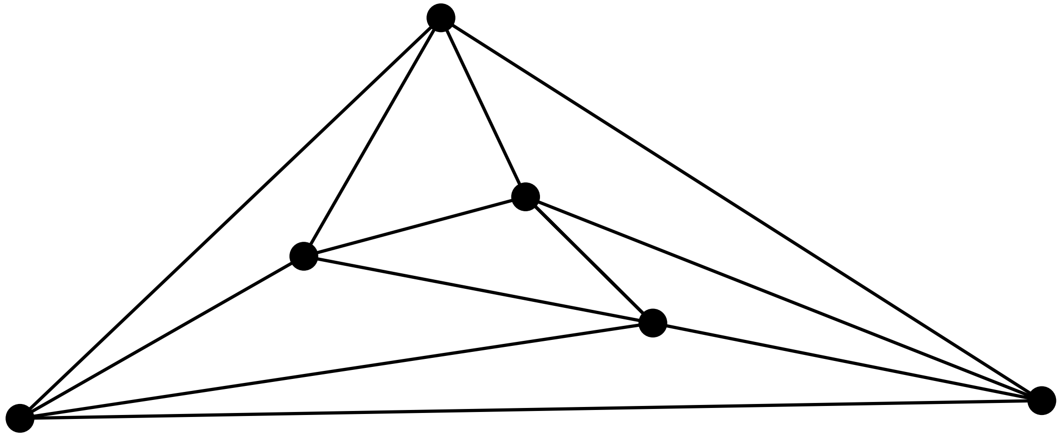

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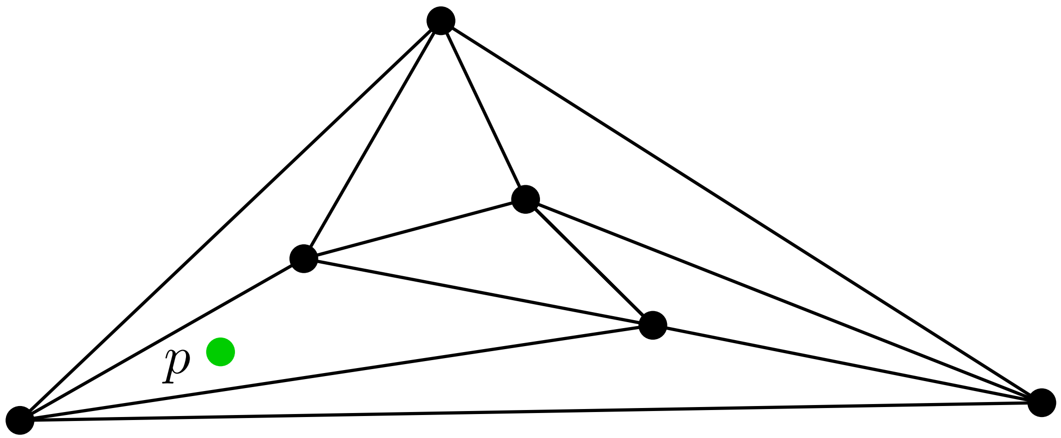

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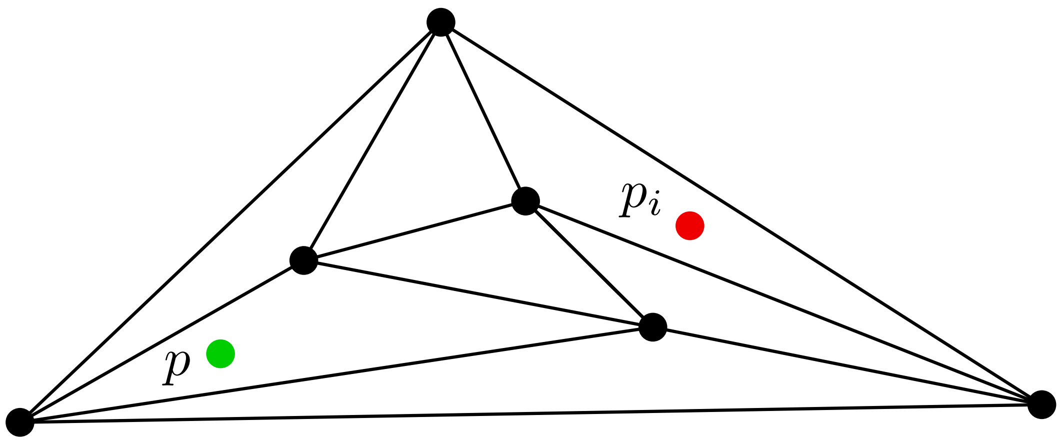

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1.1. Without sorting — **Running time:** $O(n^2)$

1.2. With sorting — **Running time:** $O(n \log n)$

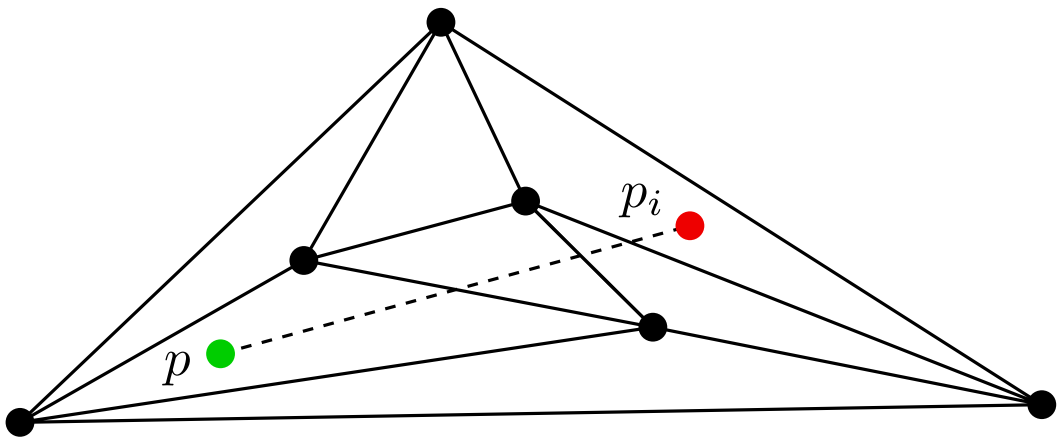
1.3. With hierarchical structure — **Running time:** $O(n^2)$ worst case, $O(n \log n)$ if balanced

1.4. Randomized — **Running time:** $O(n \log n)$ expected

1.5. With auxiliary point(s)

A fixed point p is used as a reference, and $P \cup \{p\}$ is enclosed in an auxiliary triangle.
When inserting each point p_i:

- Scan the triangles stabbed by the segment $\overline{pp_i}$.
- Update, if necessary, the information of the triangle containing p.



TRIANGULATING POINT SETS

ALGORITHMS

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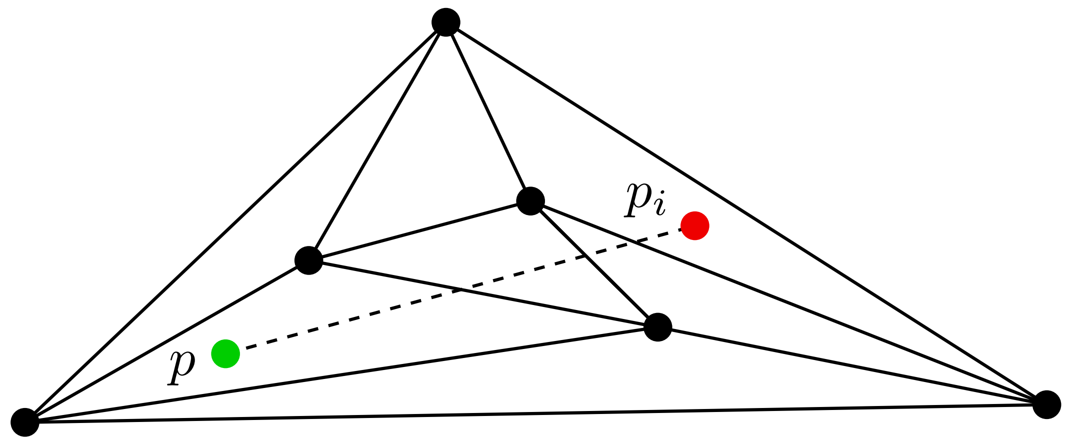

ALGORITHMS

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2. Graham's algorithm

TRIANGULATING POINT SETS

ALGORITHMS

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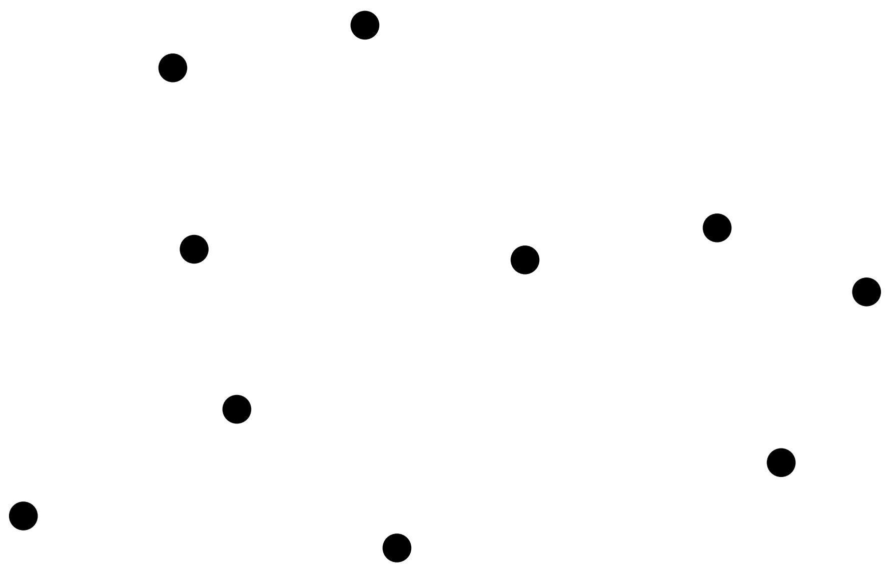

TRIANGULATING POINT SETS

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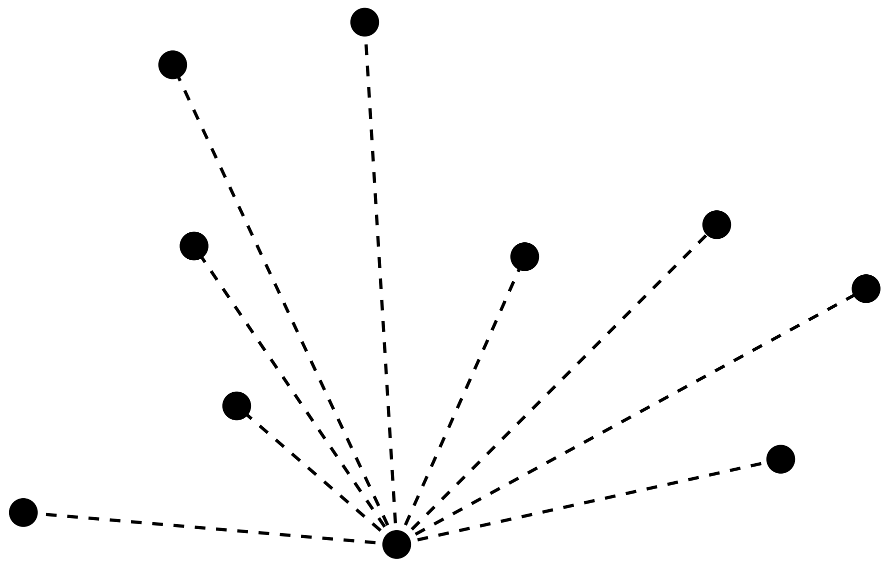

TRIANGULATING POINT SETS

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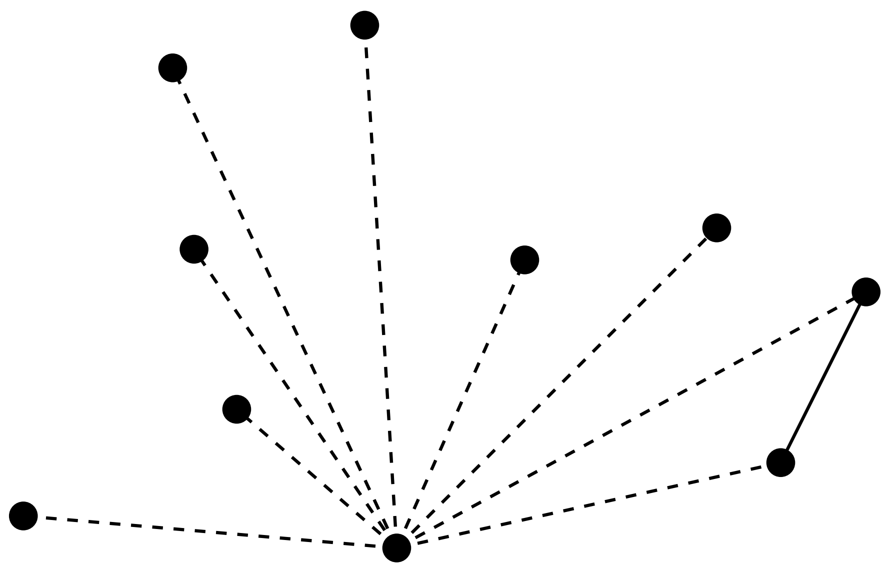

TRIANGULATING POINT SETS

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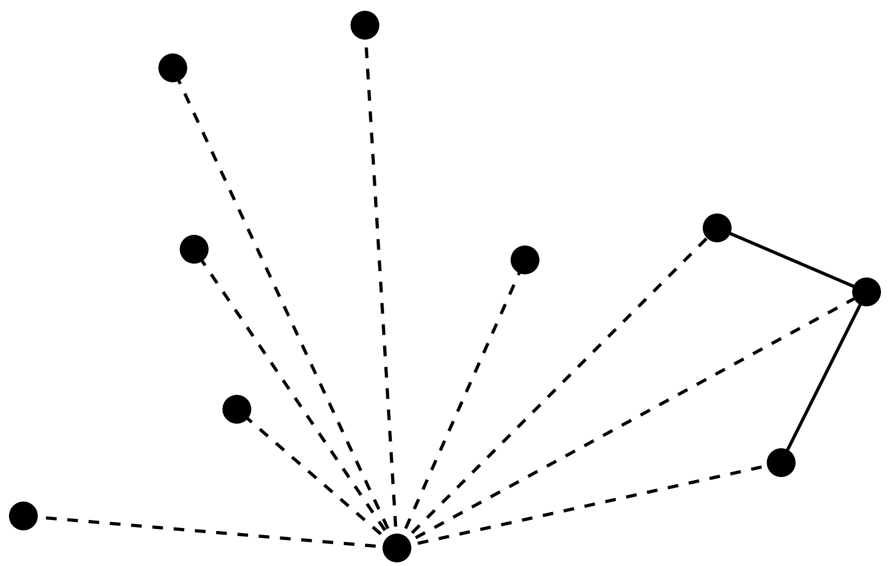

TRIANGULATING POINT SETS

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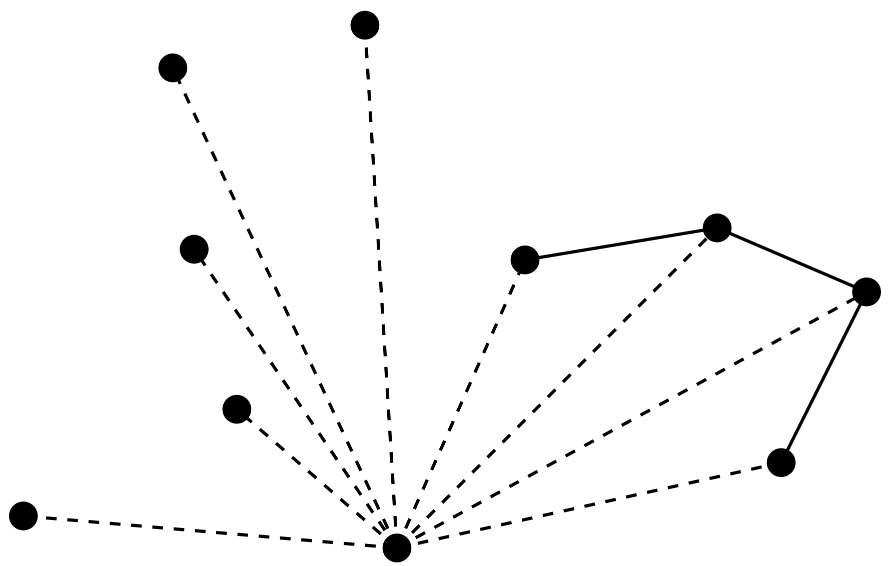

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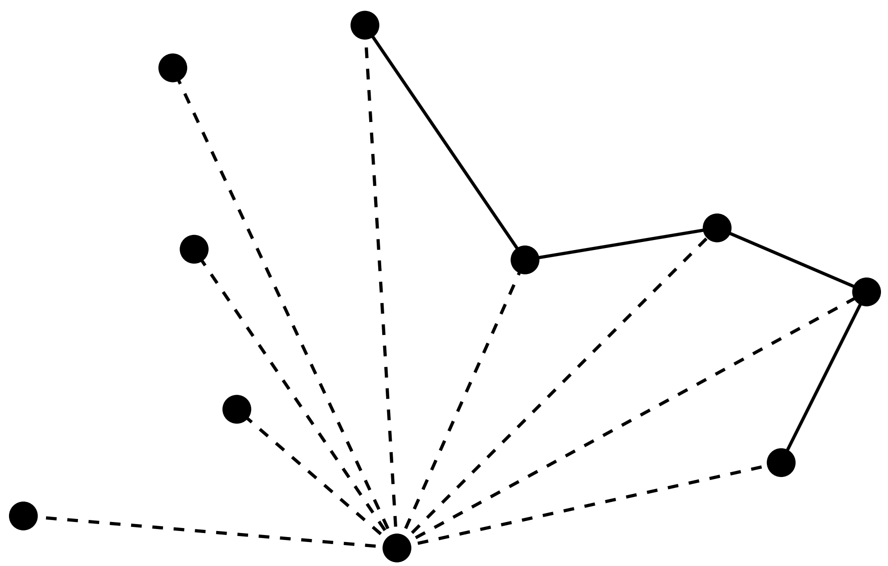

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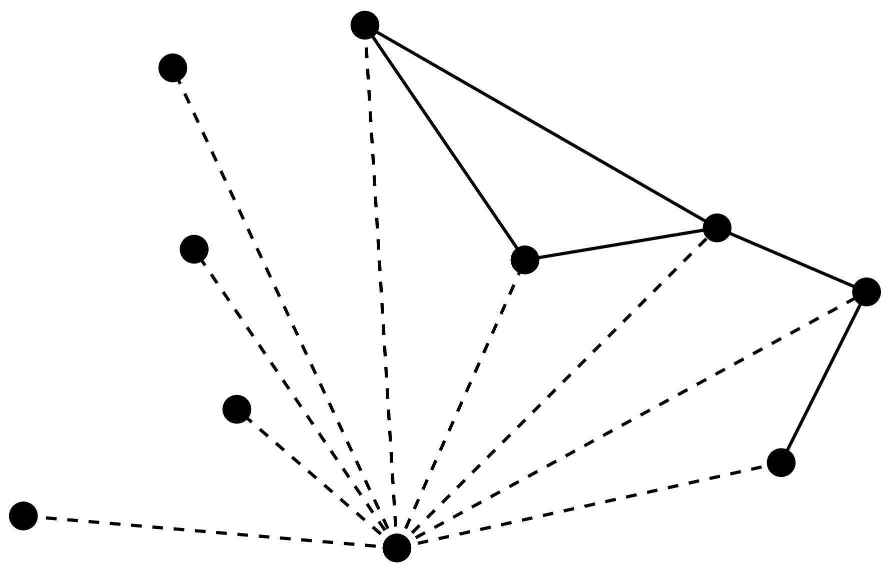

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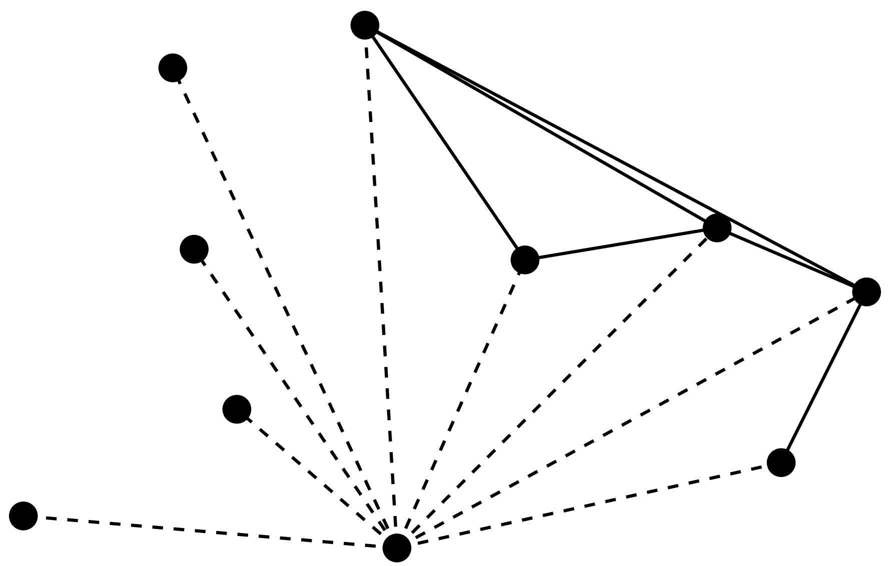

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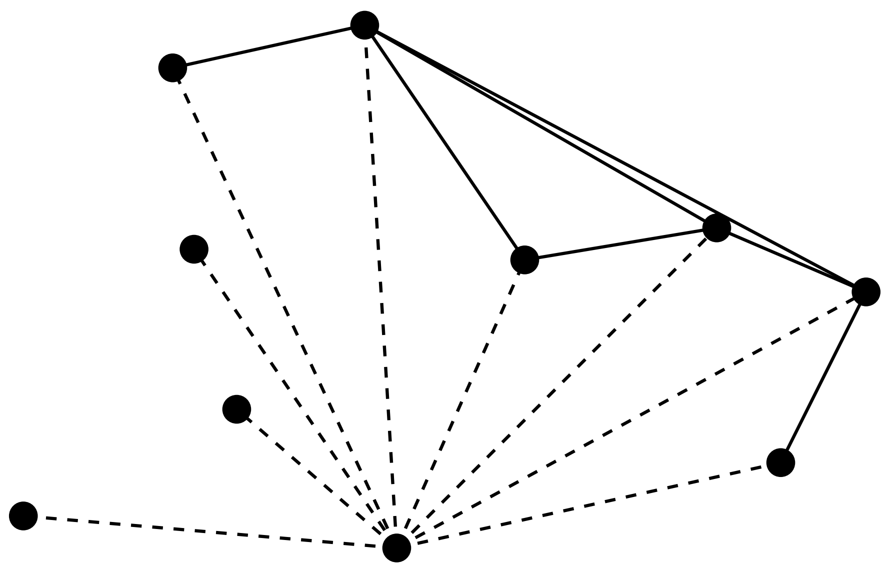

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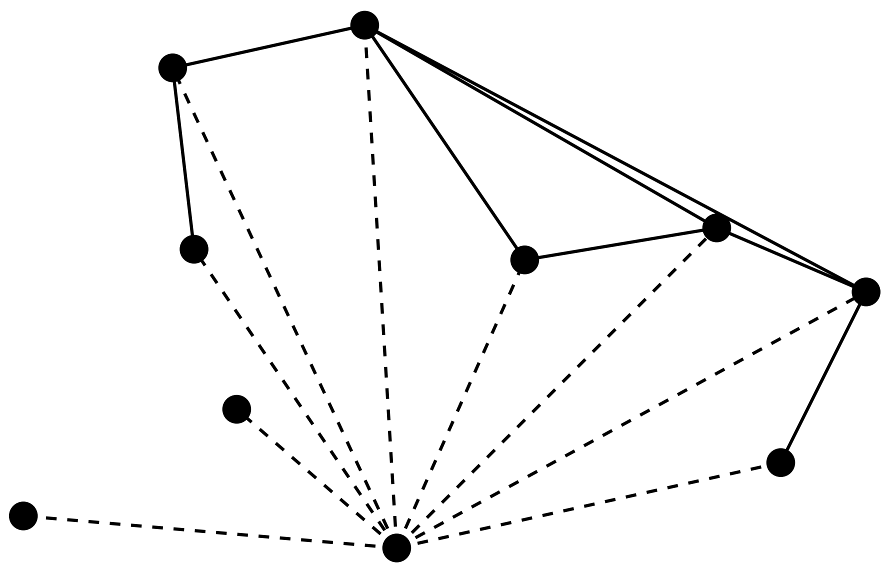

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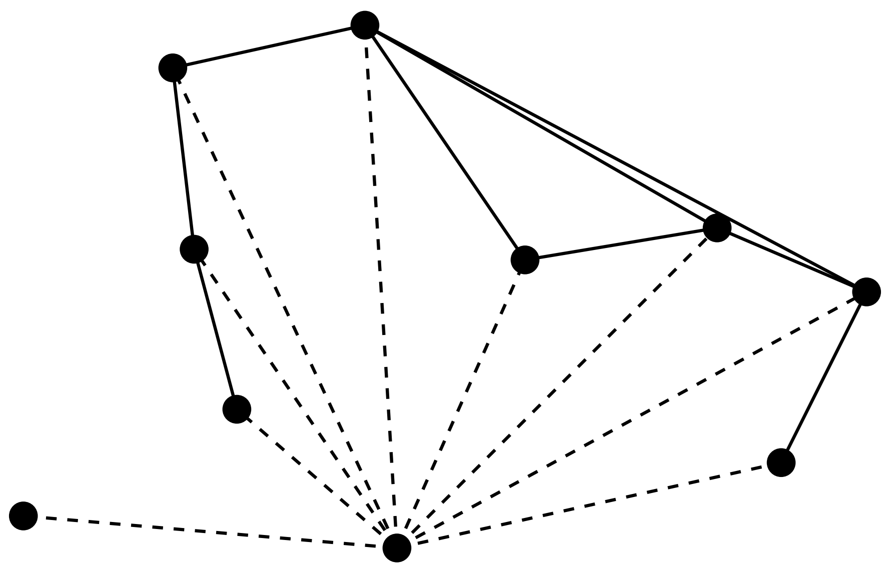

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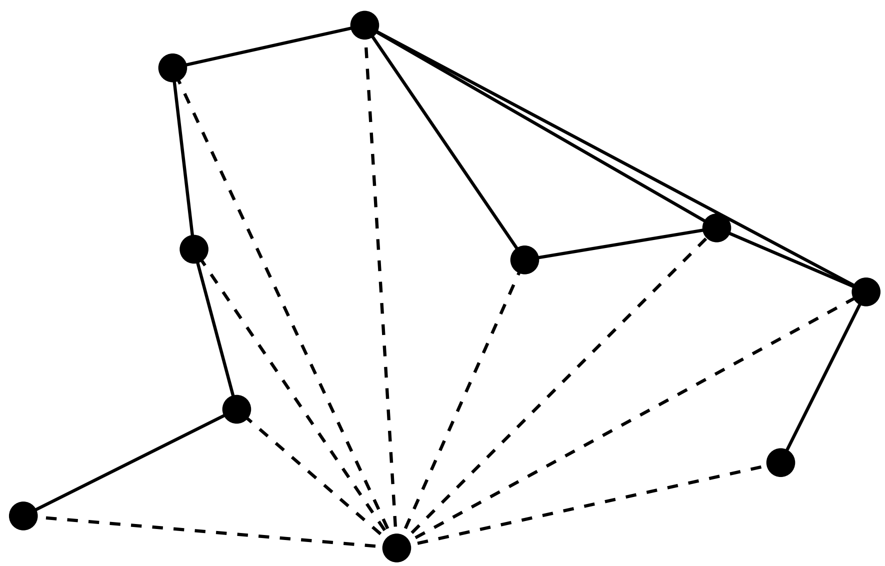

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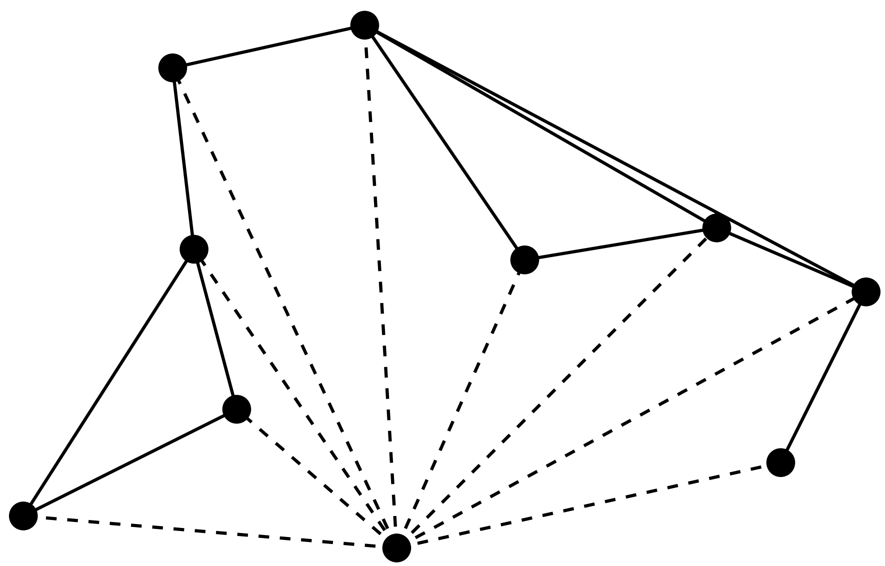

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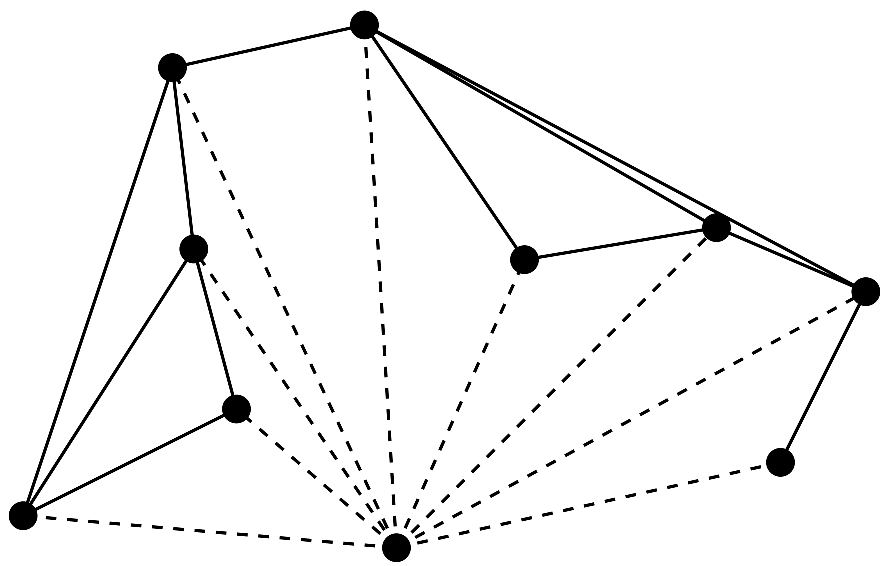

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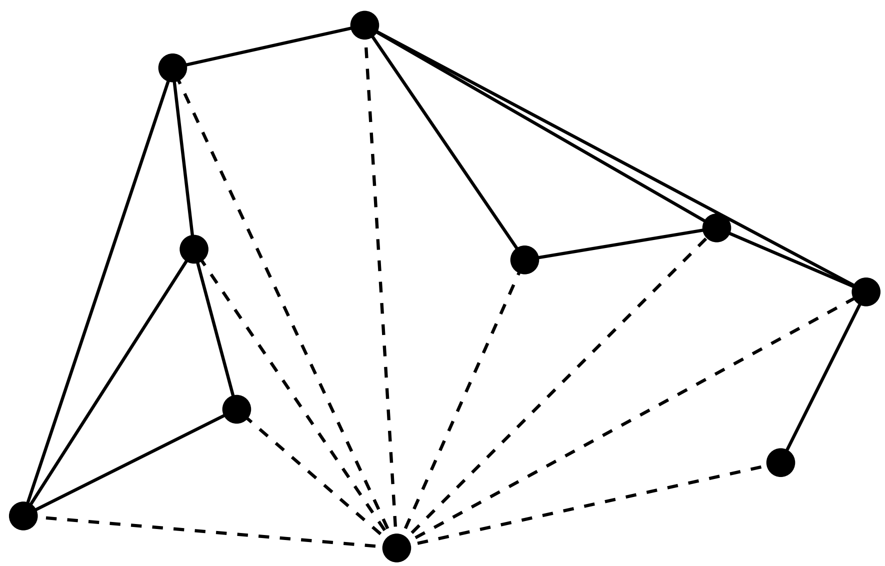

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3. Dividide and conquer

Initialization

Sort the points by abscissa

Advance

- Partition: divide the points into roughly two vertically separated halfs
- Recursion: recursively triangulate each half
- Fusion: compute the external common tangents and triangulate the intermediate space

TRIANGULATING POINT SETS

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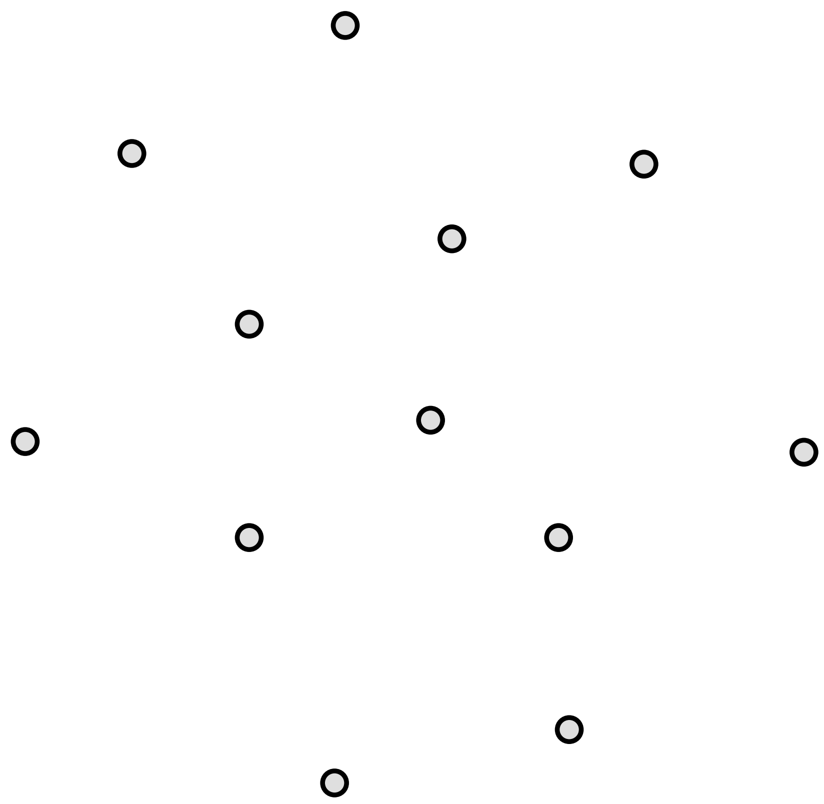
3. Dividide and conquer — **Running time:** $O(n \log n)$

LOWER BOUND

This problem has an $\Omega(n \log n)$ lower bound, since the convex hull of the set of points can be trivially obtained in $O(n)$ time from the triangulation.

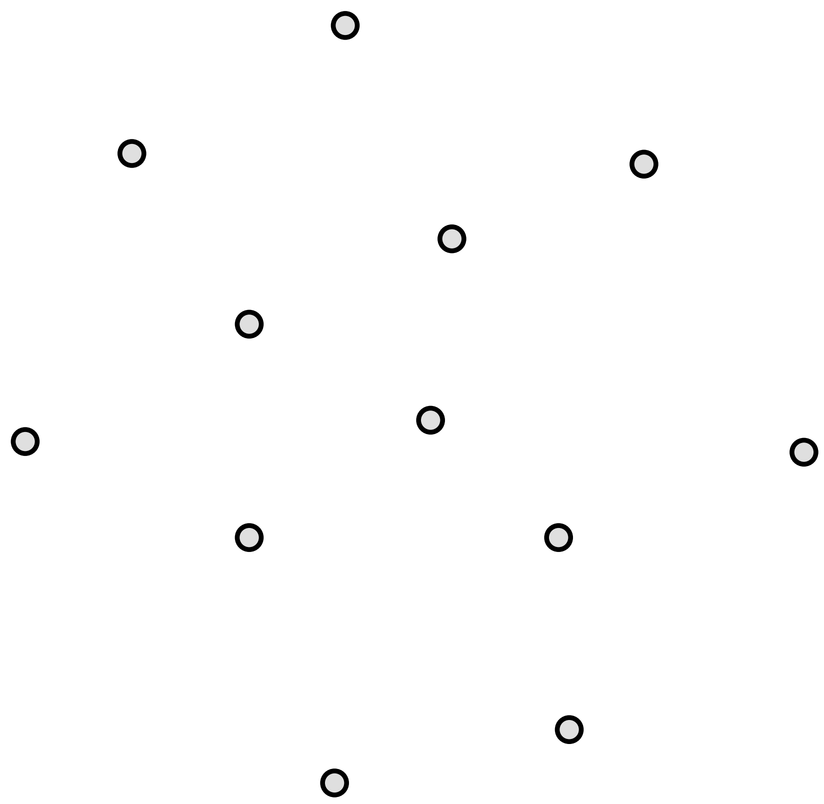
Quality of a triangulation

Quality of a triangulation



Quality of a triangulation

Incremental, without sorting



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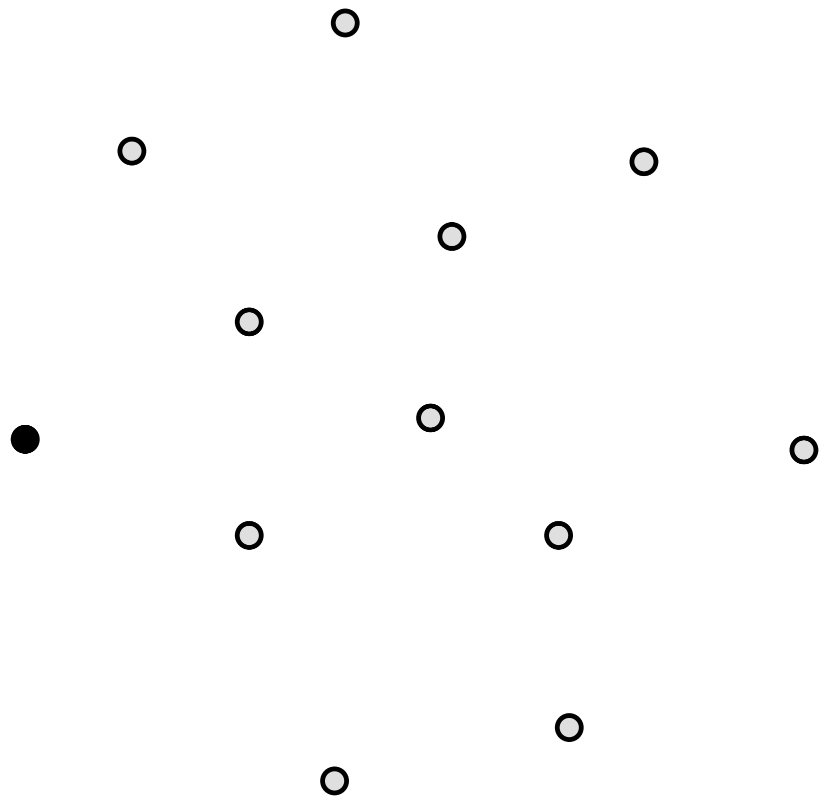

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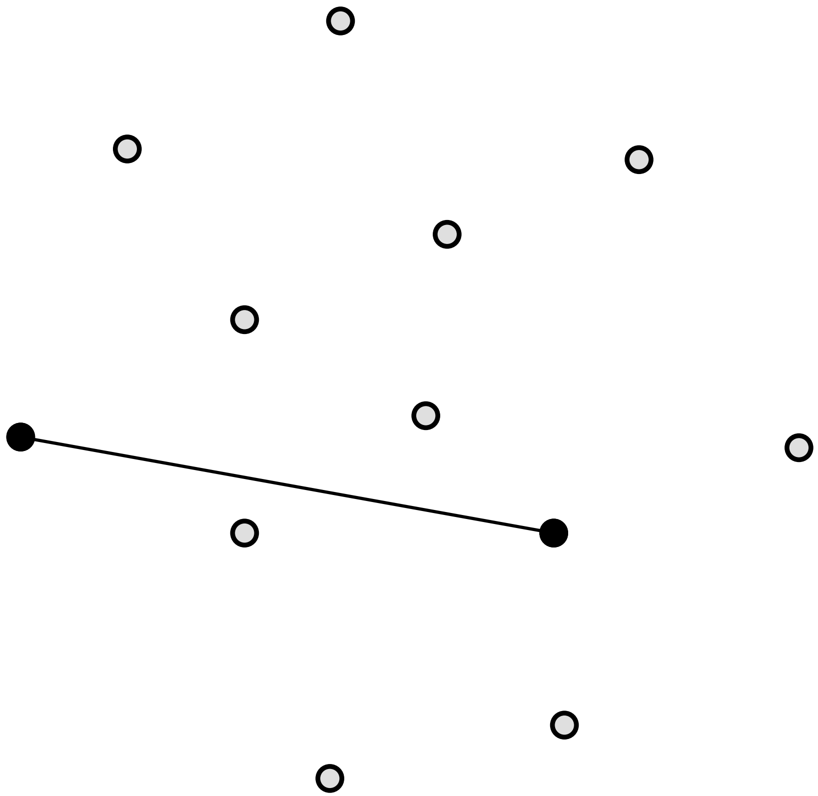

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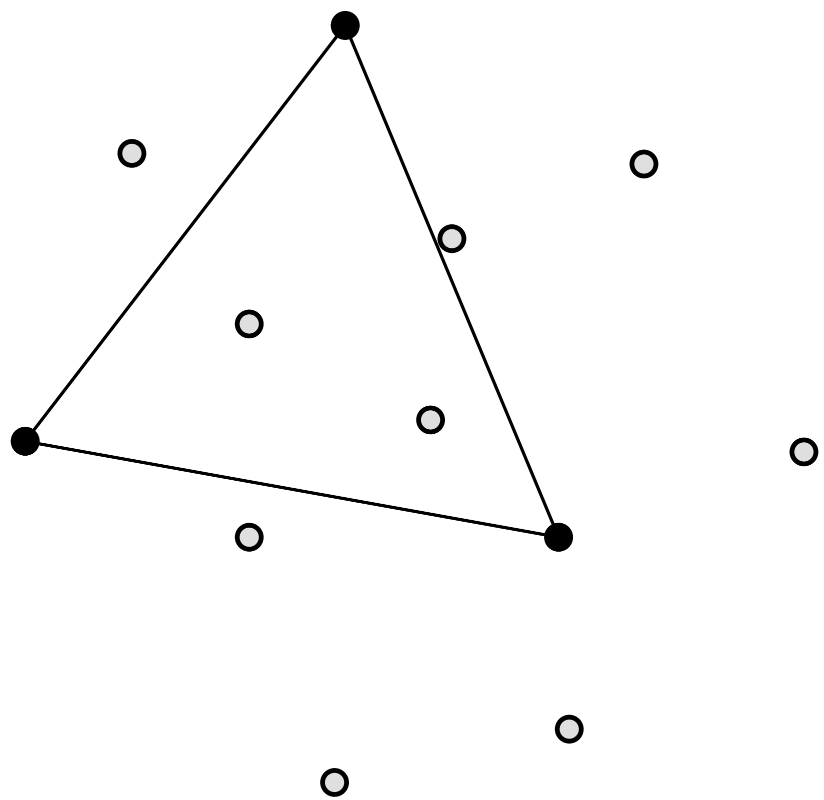

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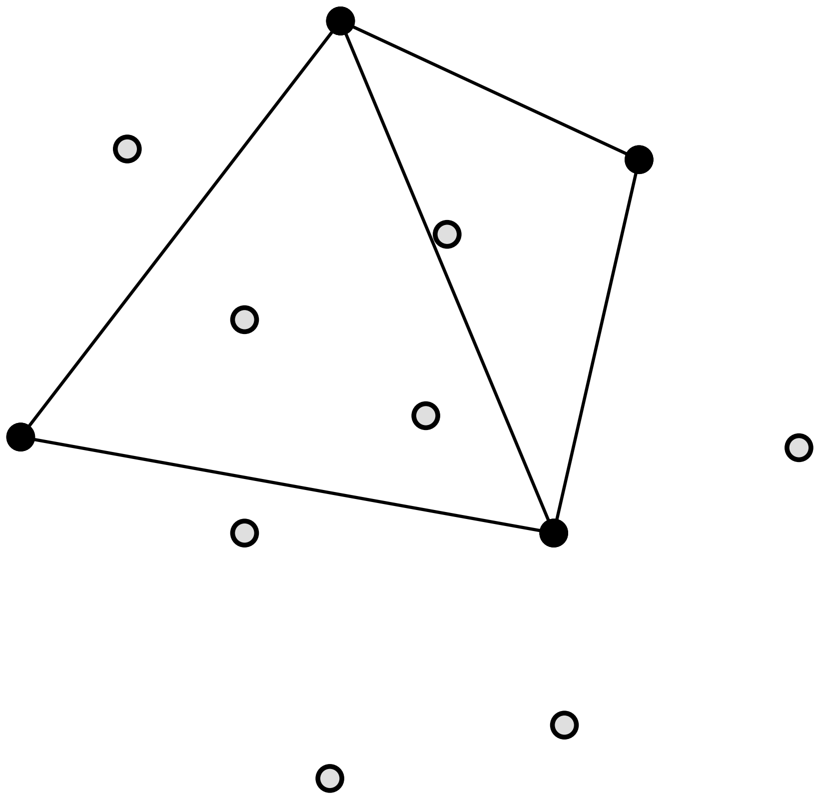

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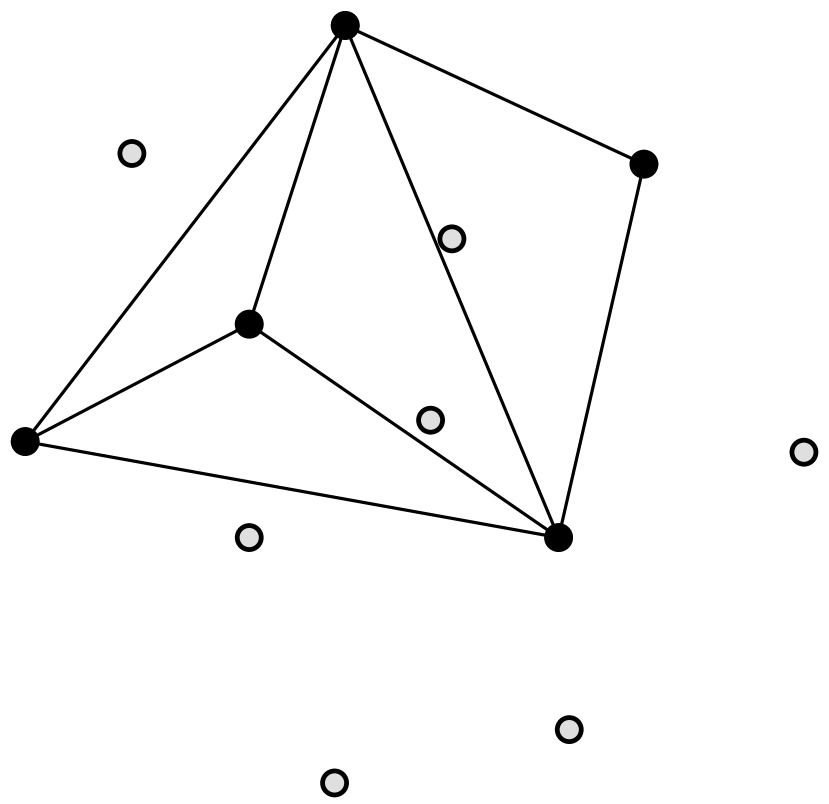

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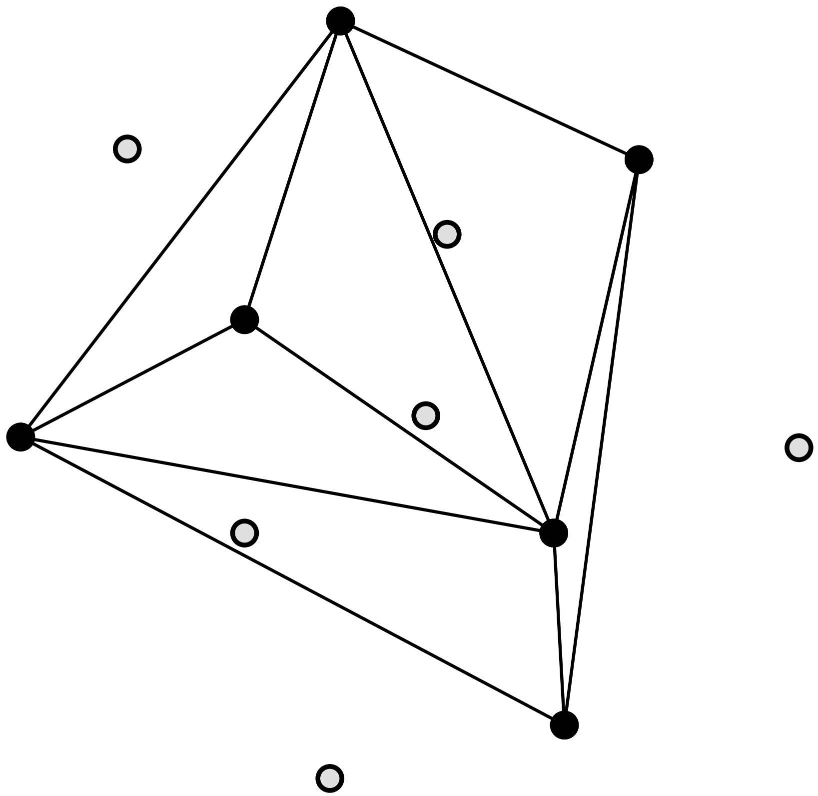

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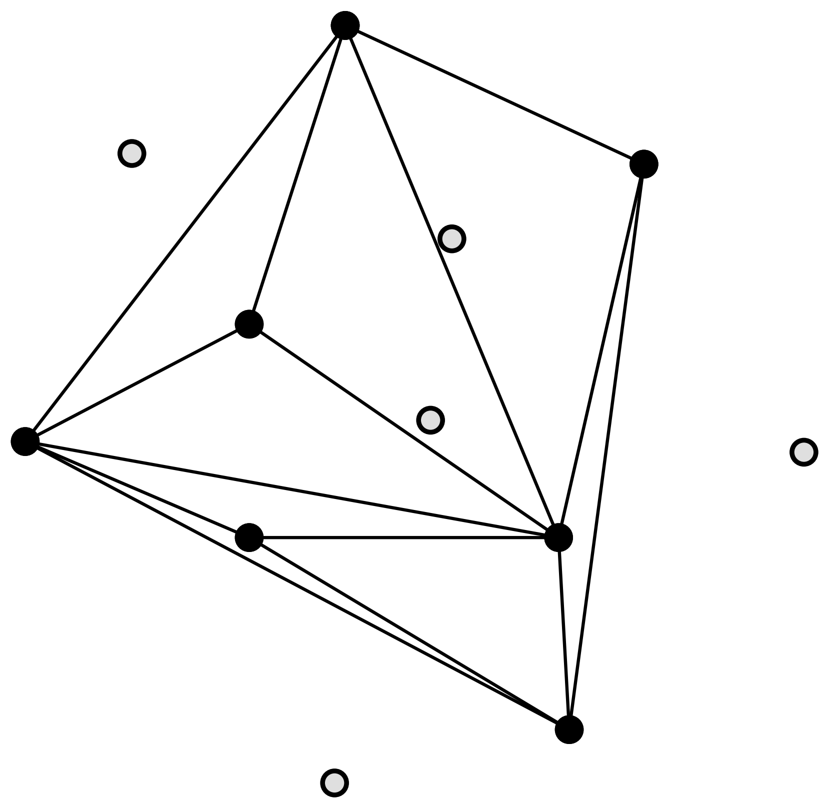

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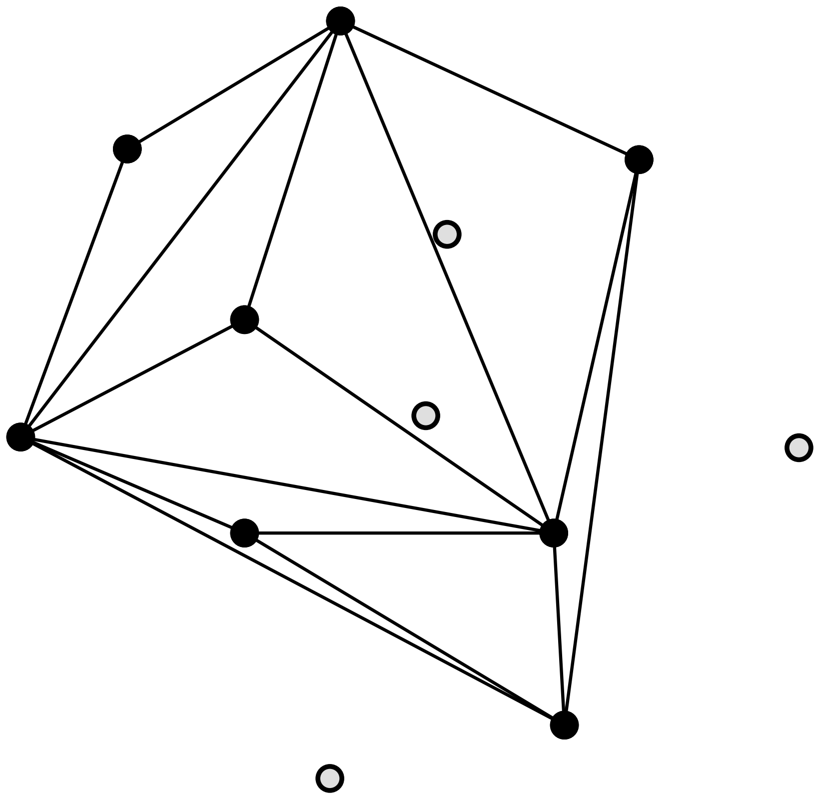

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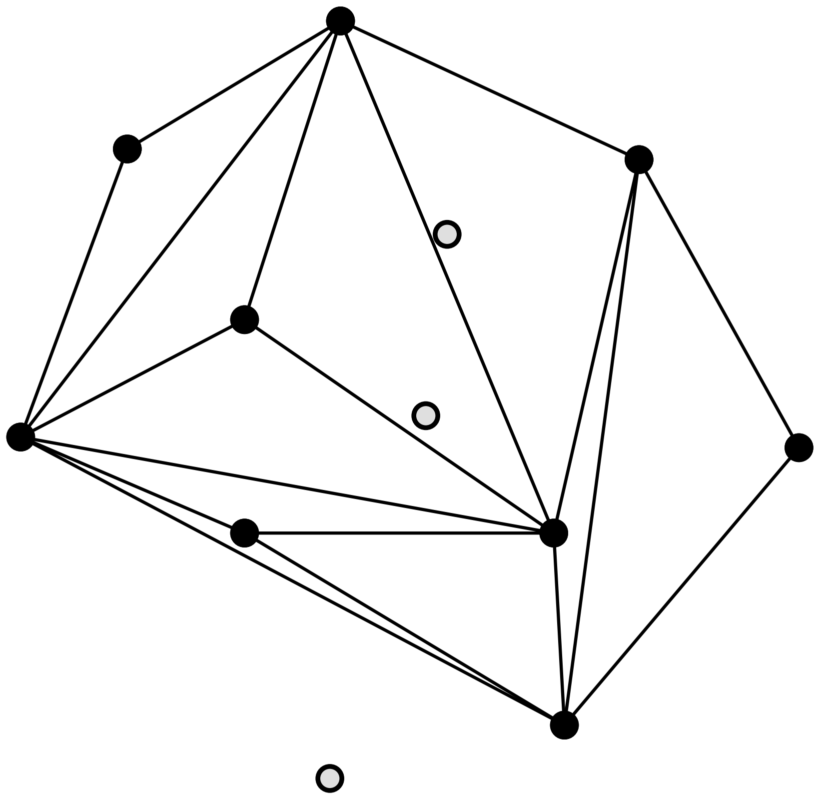

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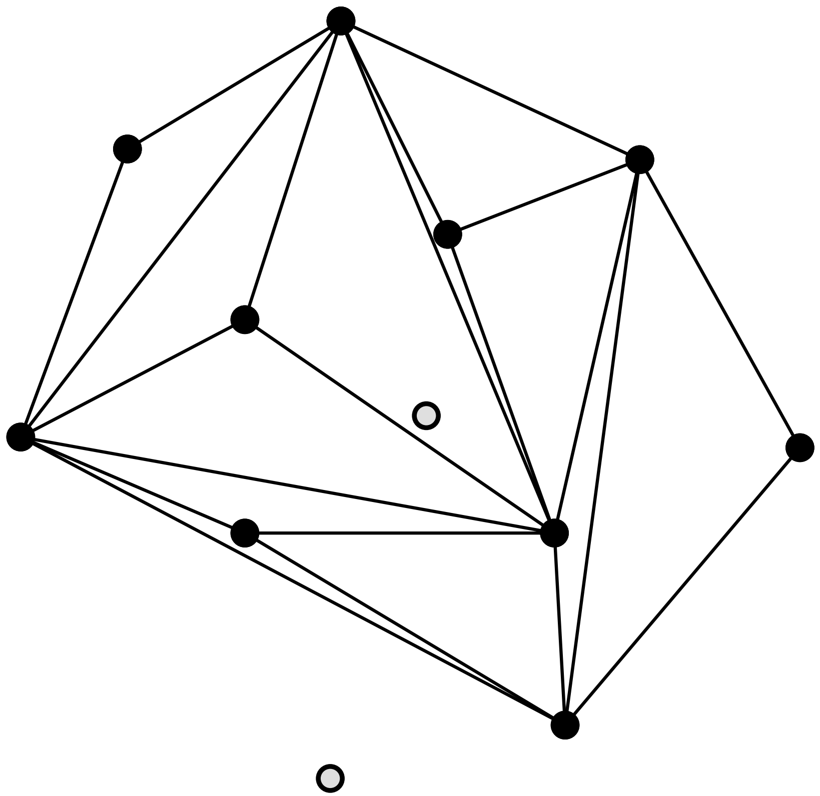

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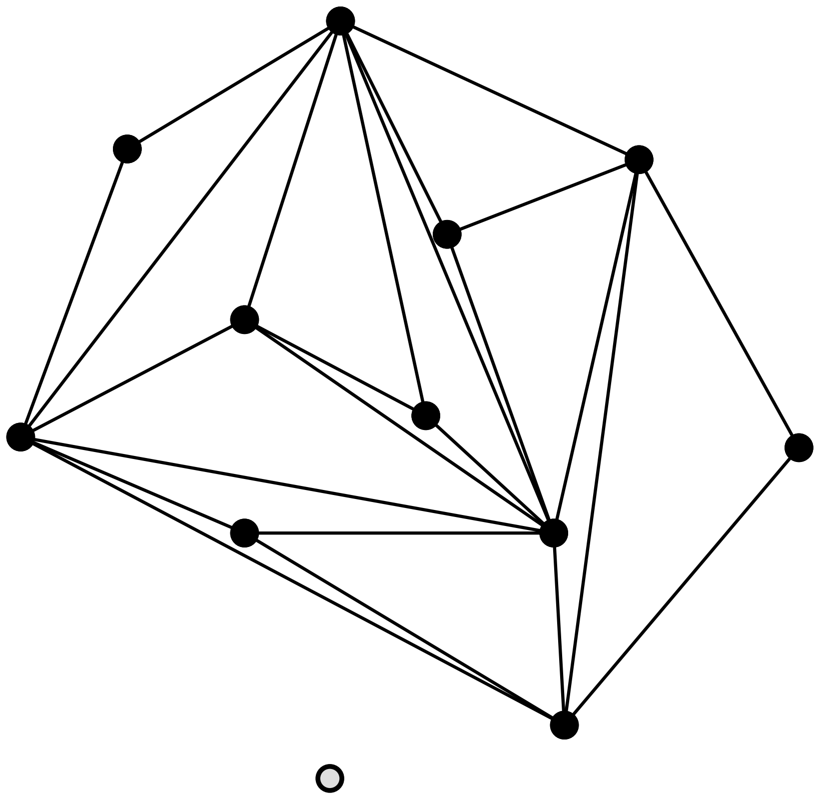

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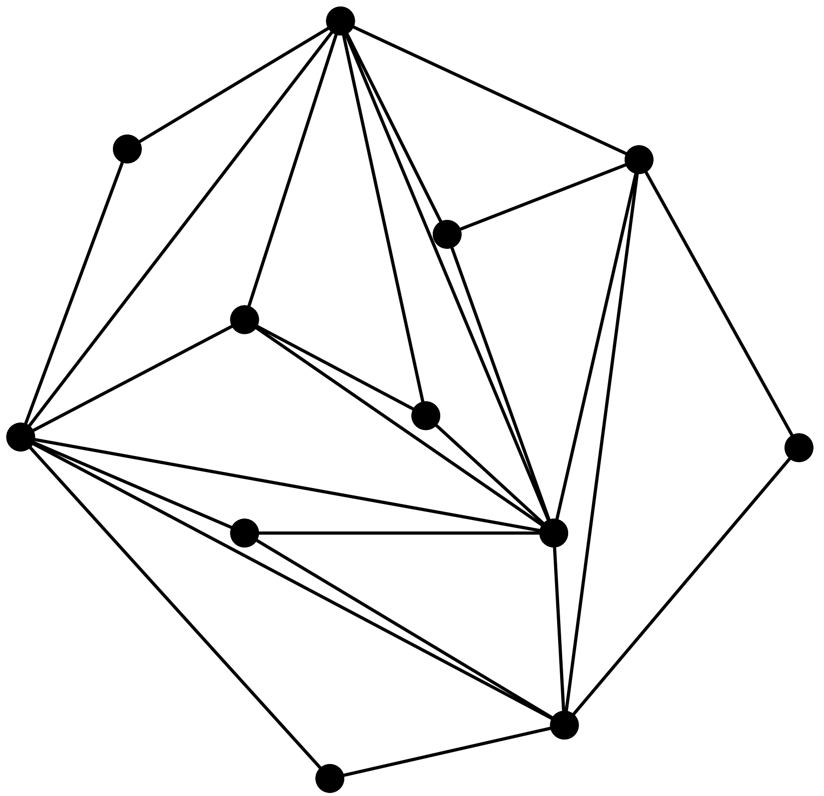

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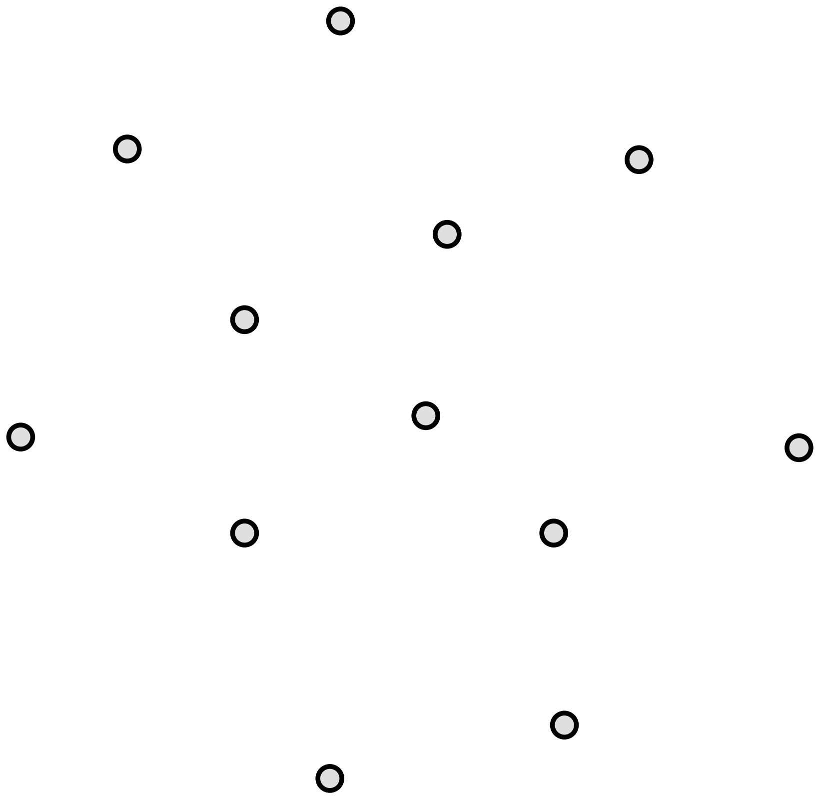

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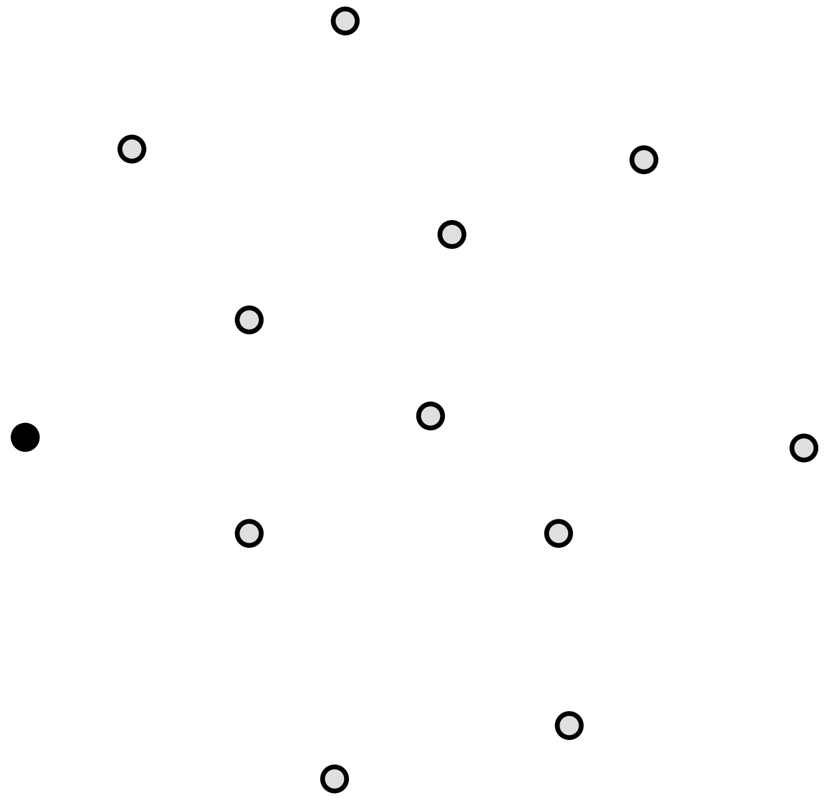

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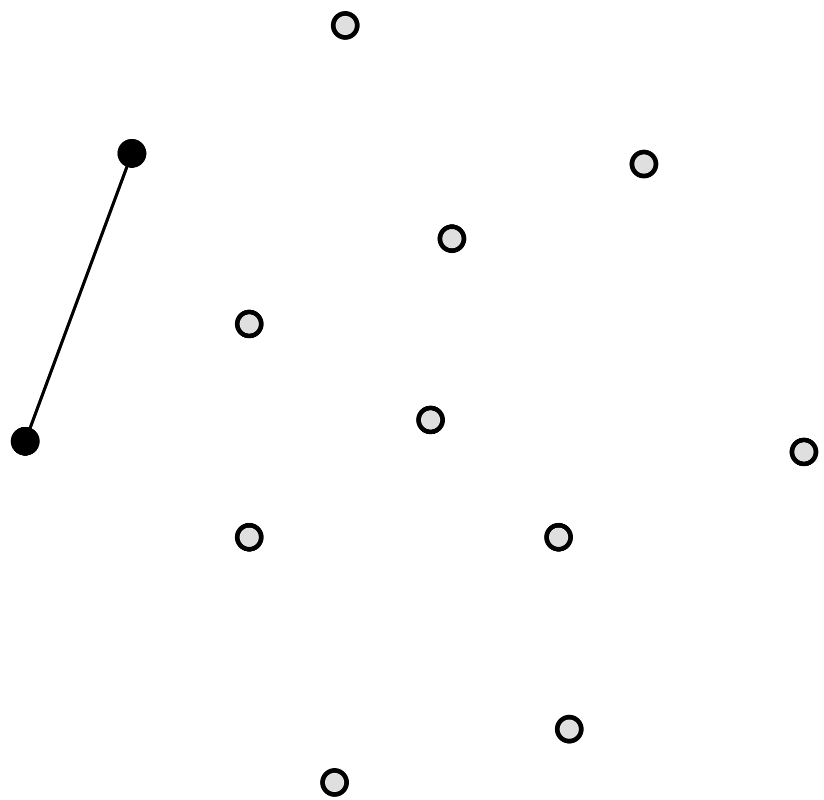

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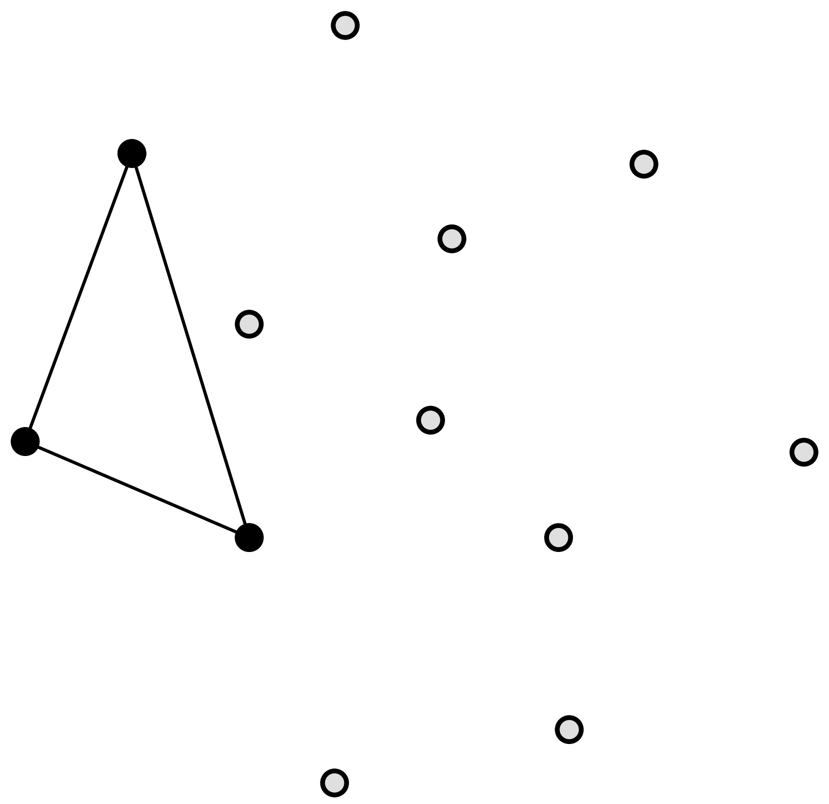

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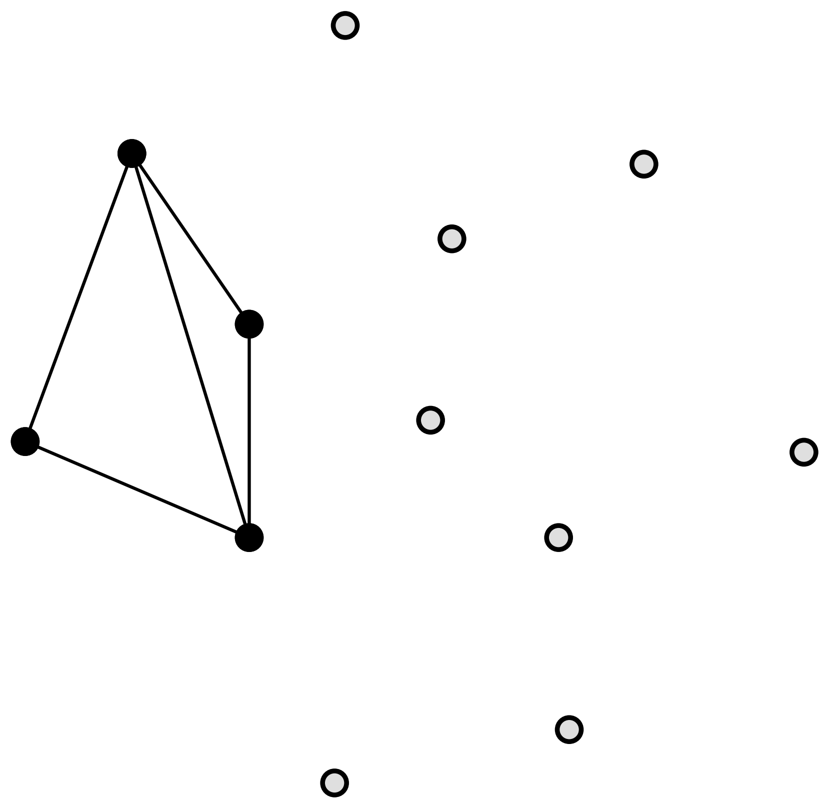
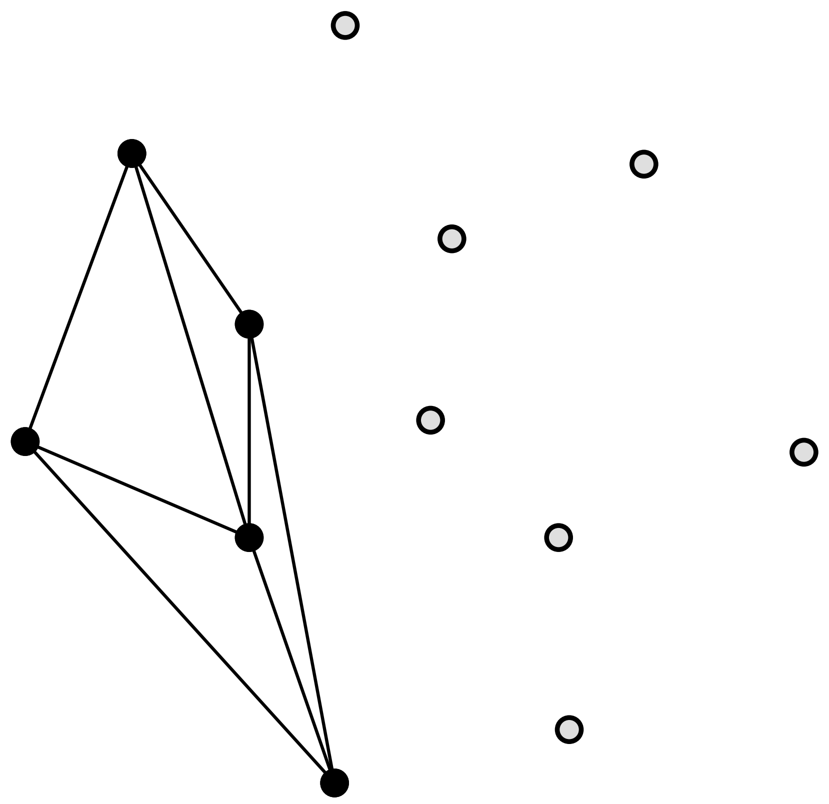


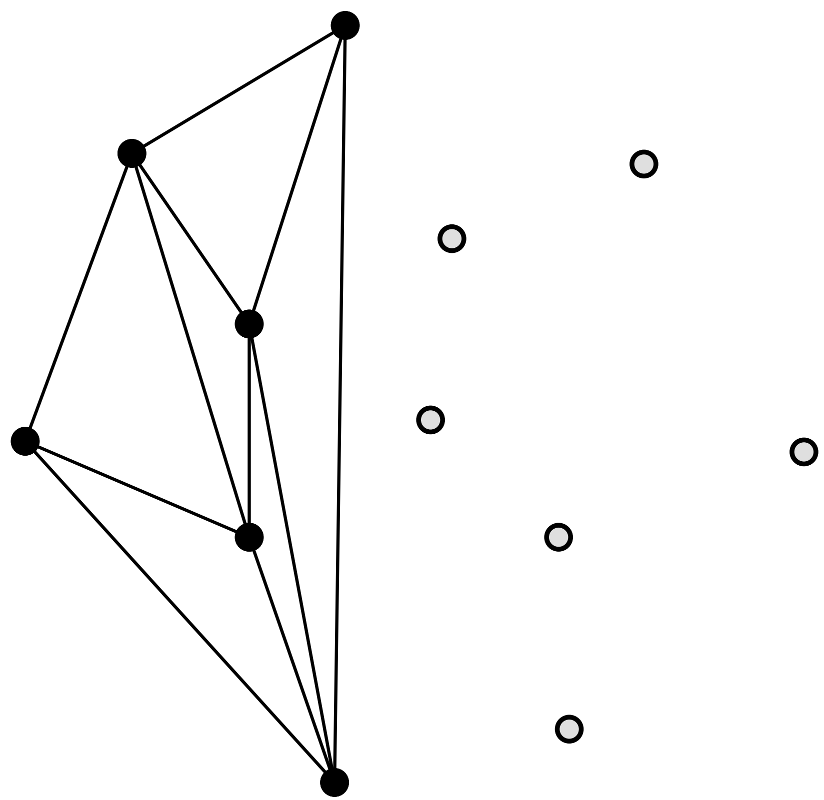
Quality of a triangulation

Incremental, sorting



Quality of a triangulation

Incremental, sorting



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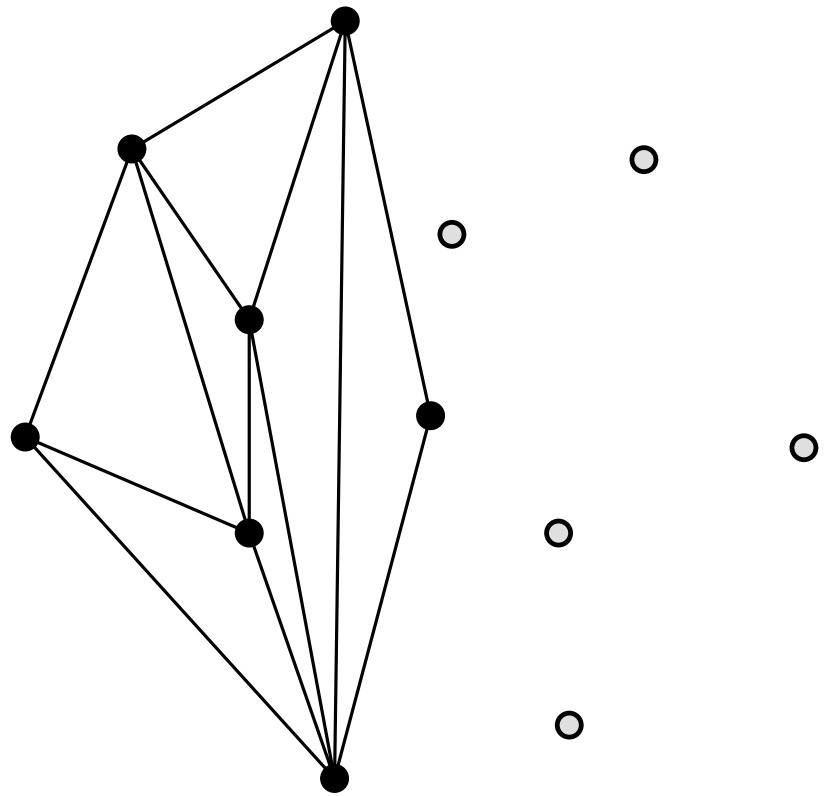

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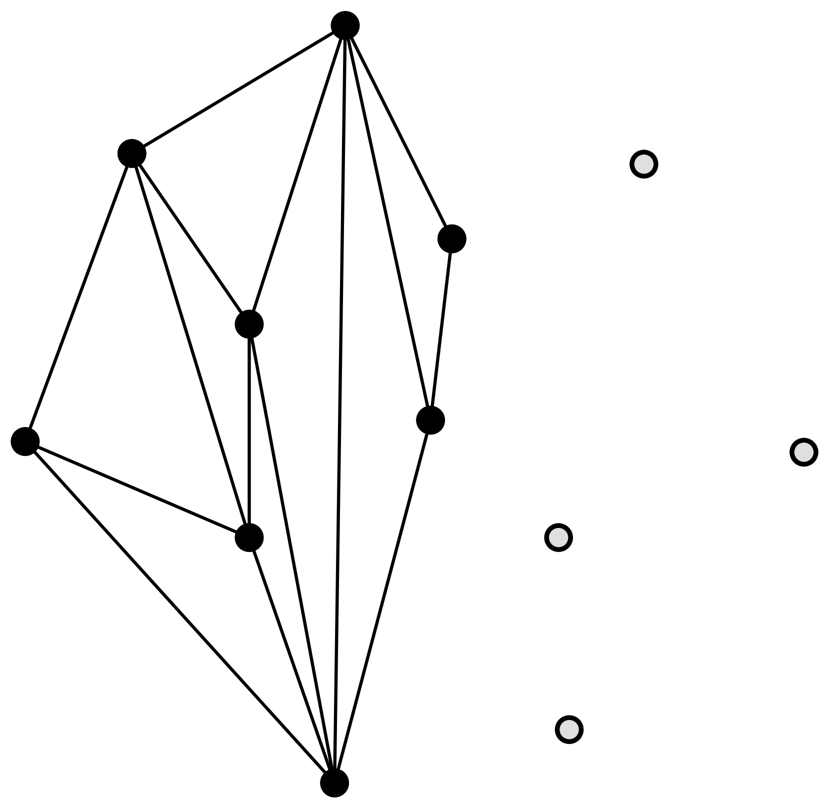

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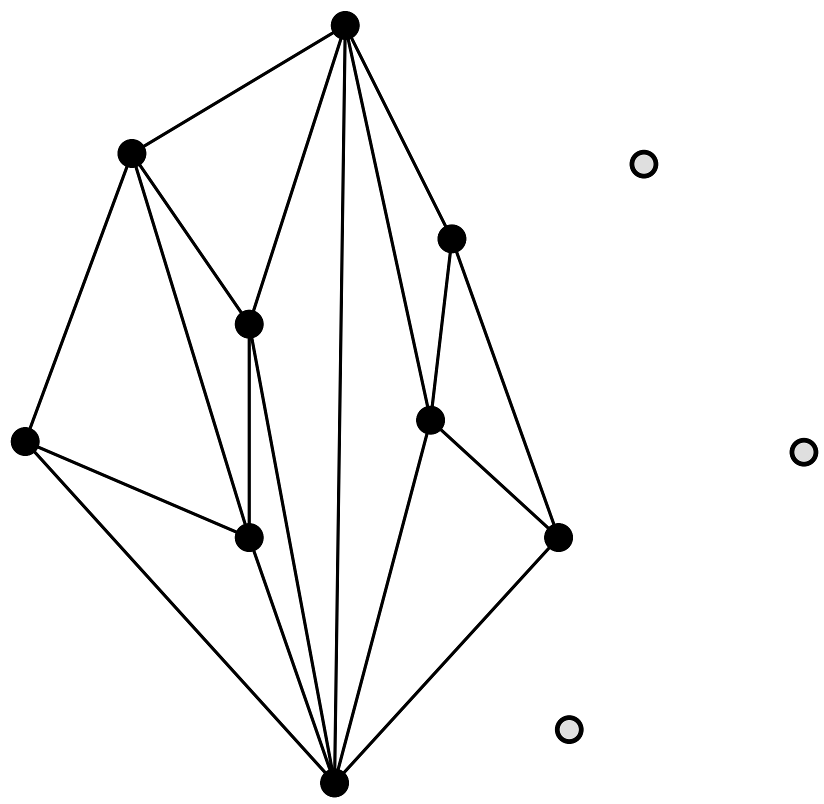

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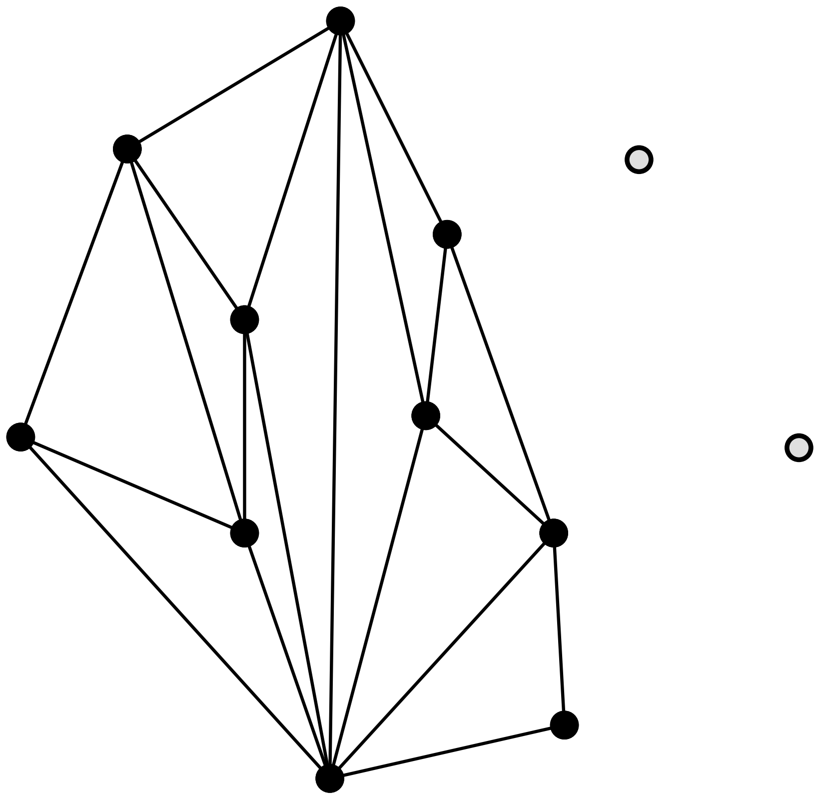

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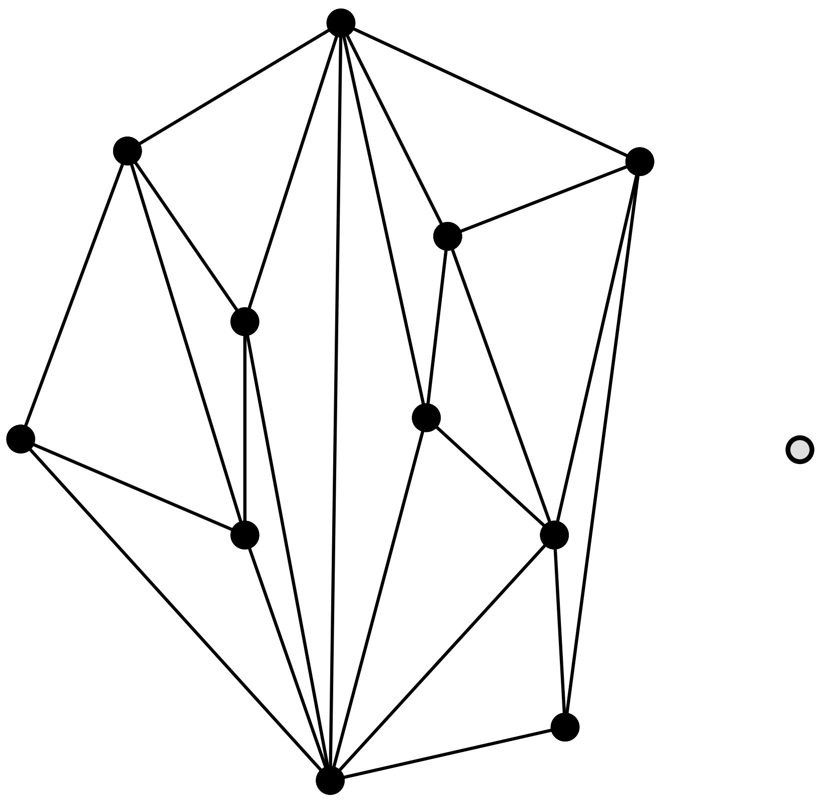

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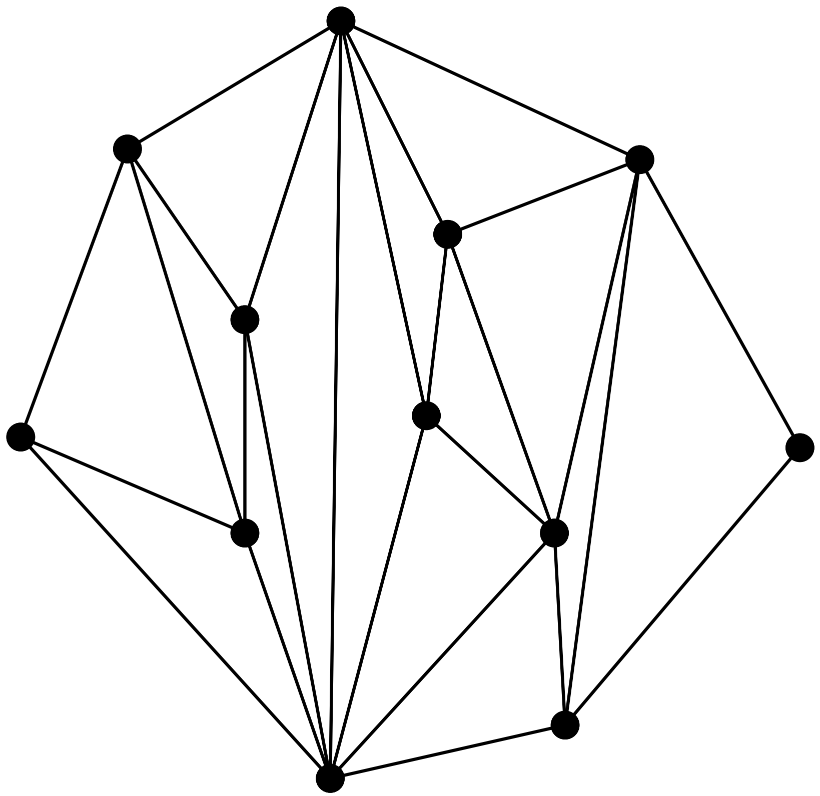
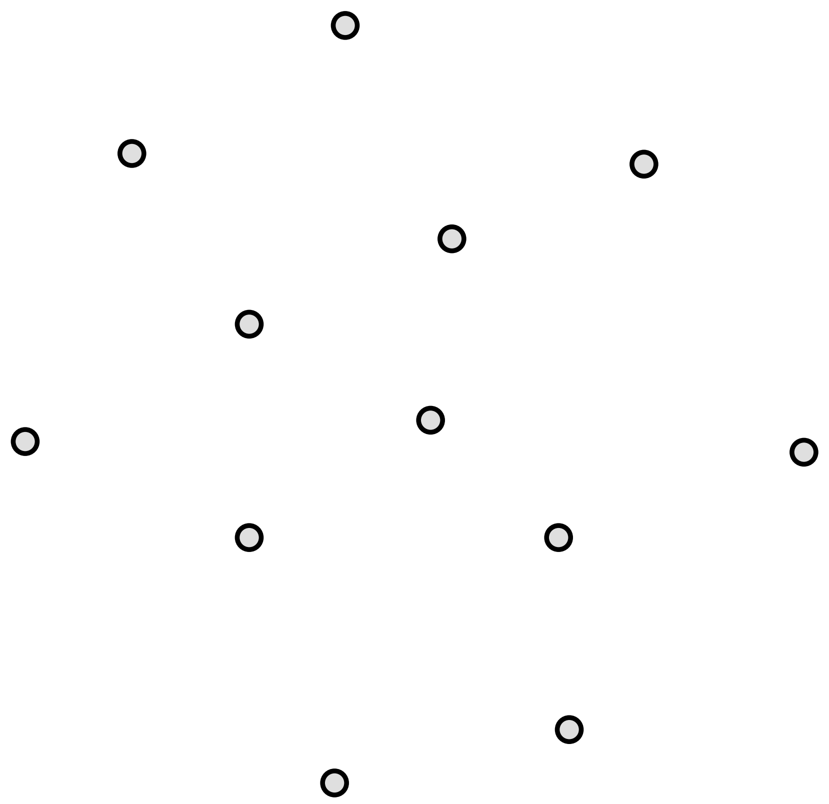


Quality of a triangulation

Graham's



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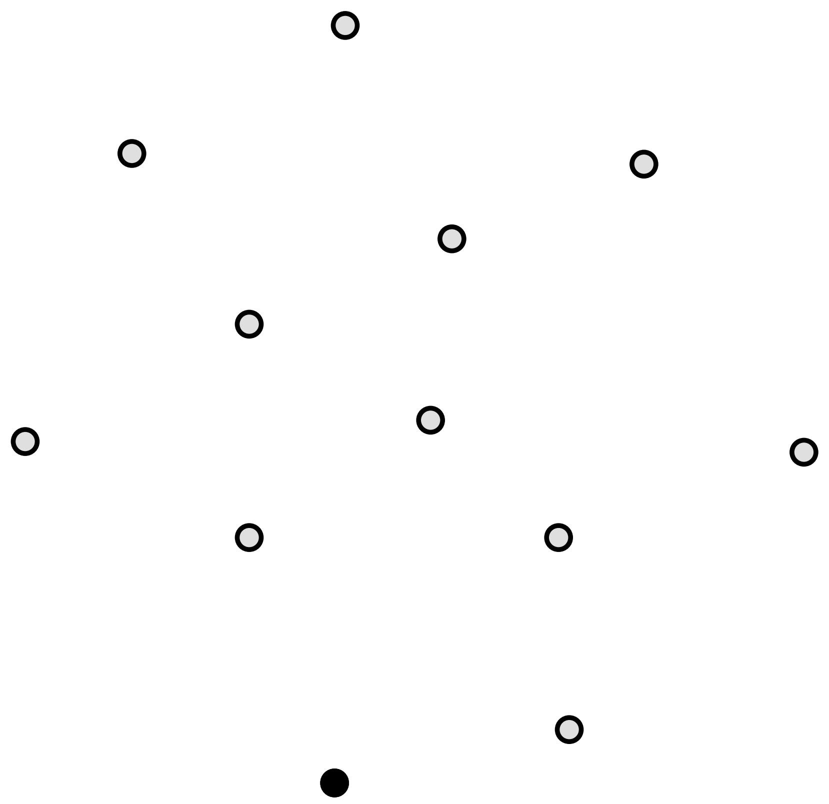

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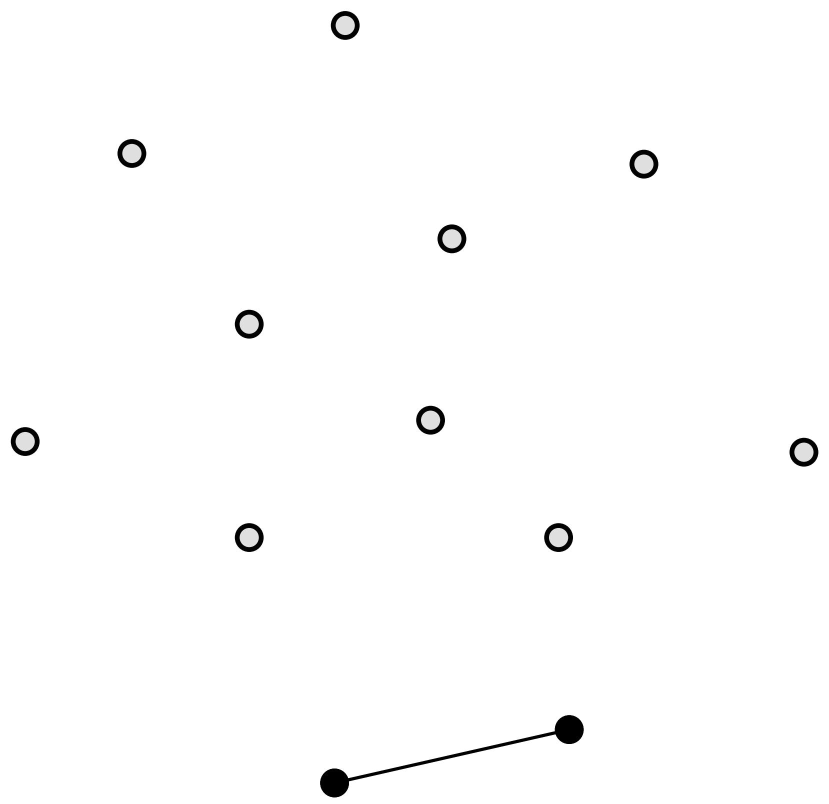

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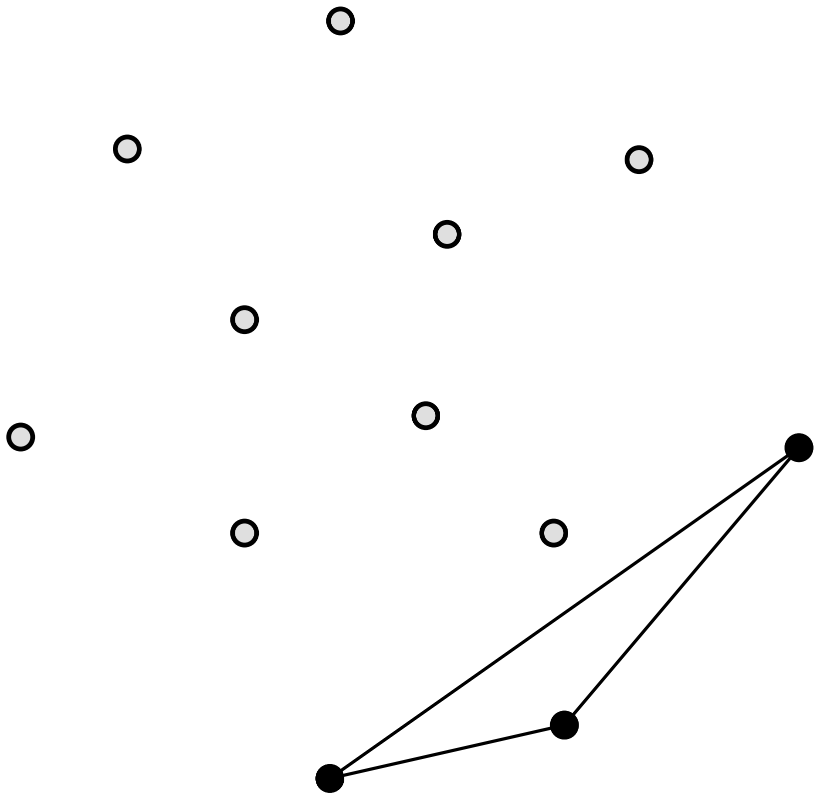

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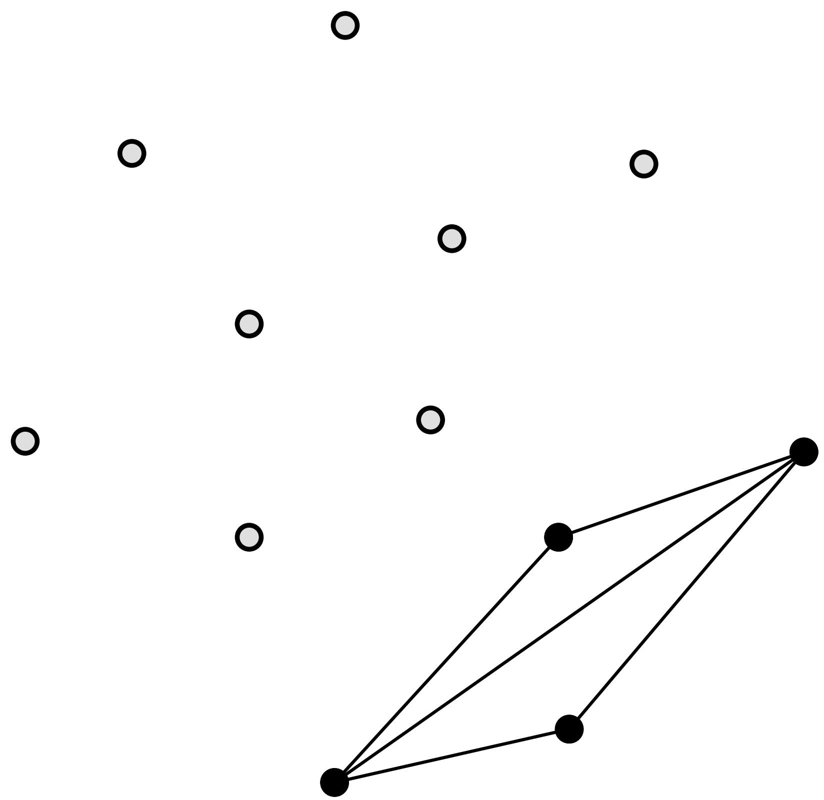

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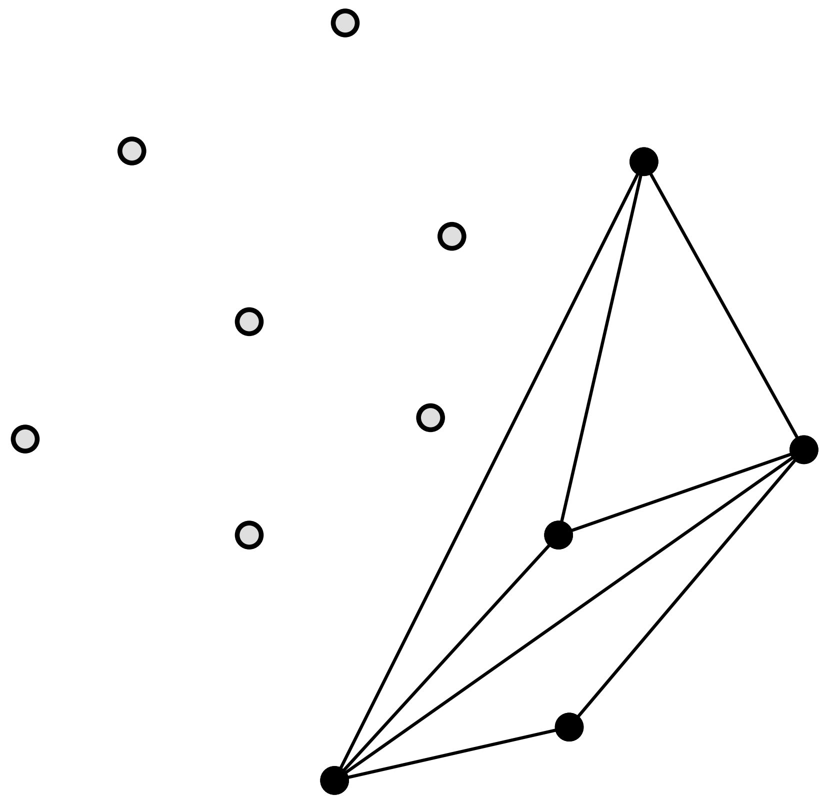

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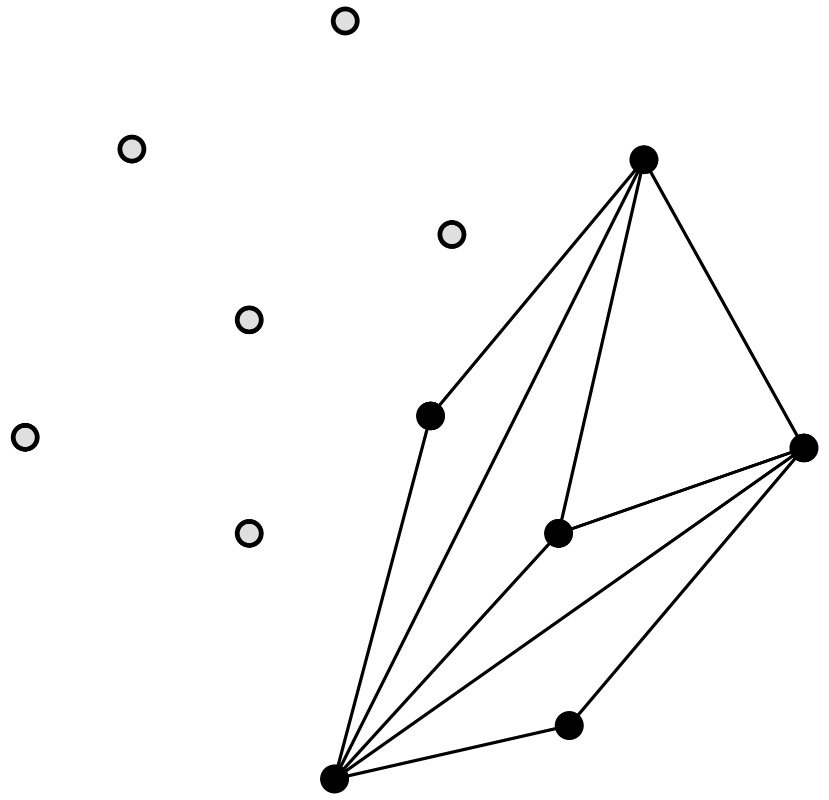

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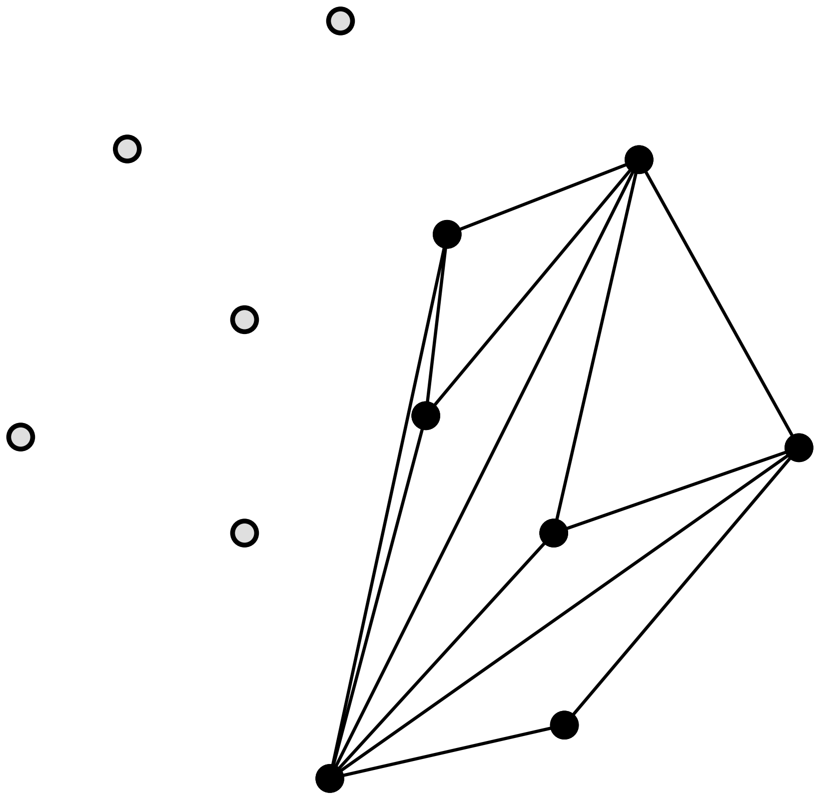

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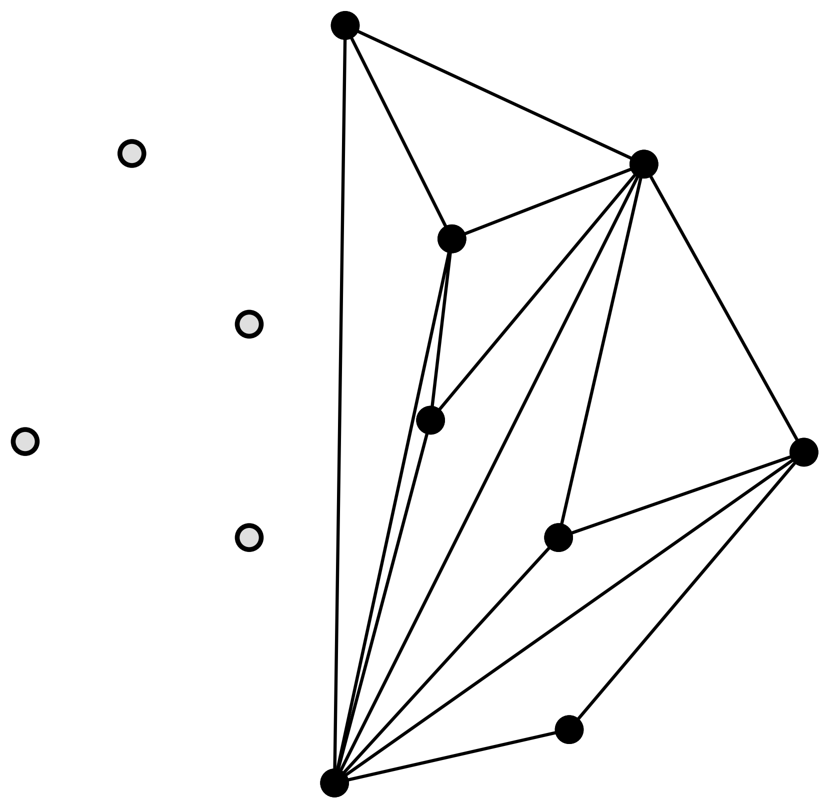

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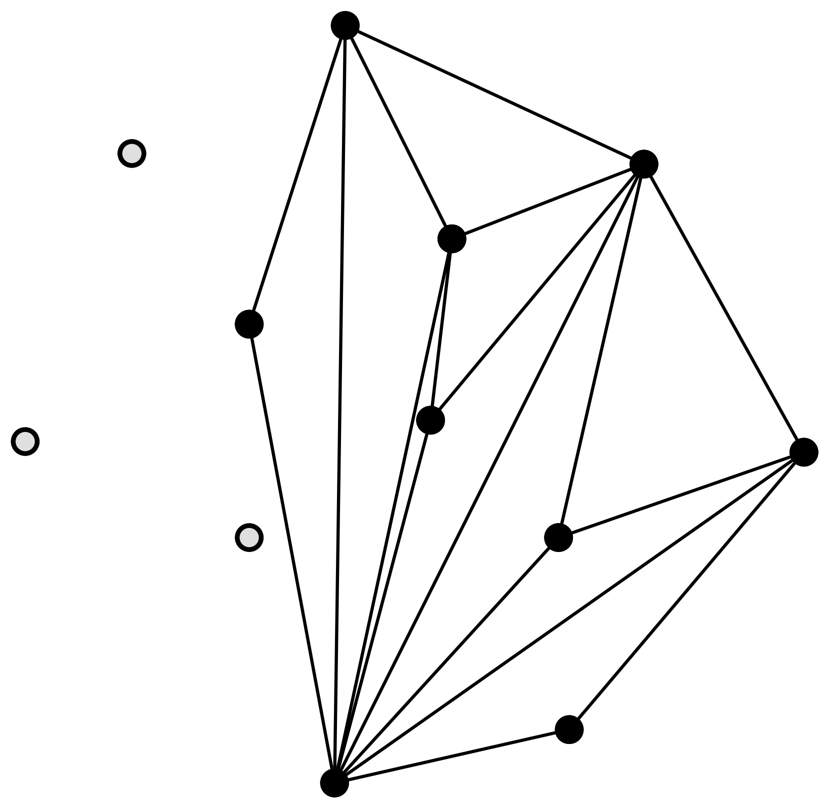

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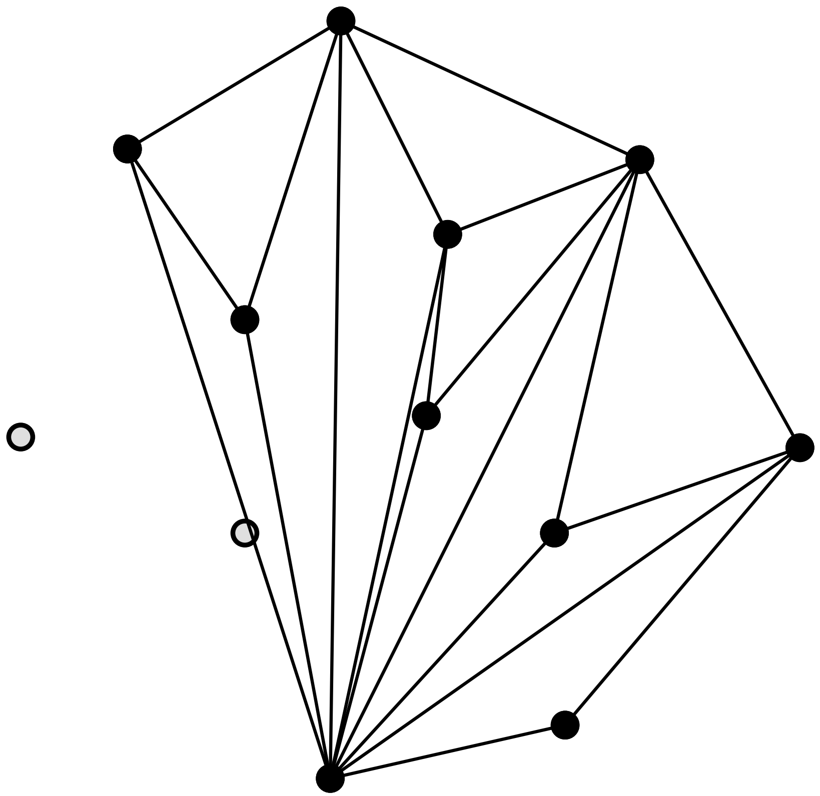

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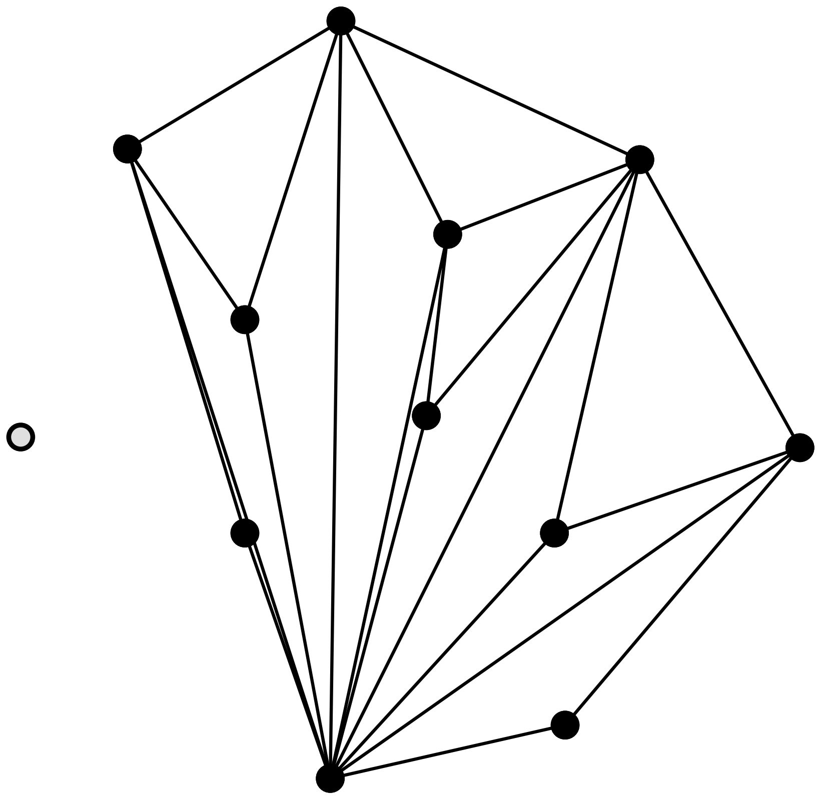

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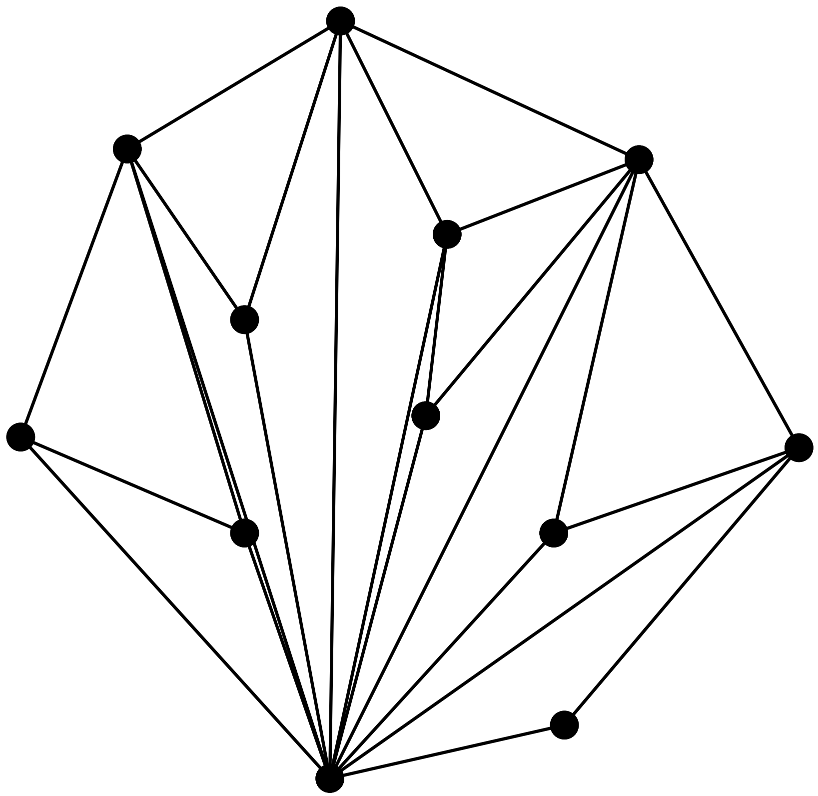
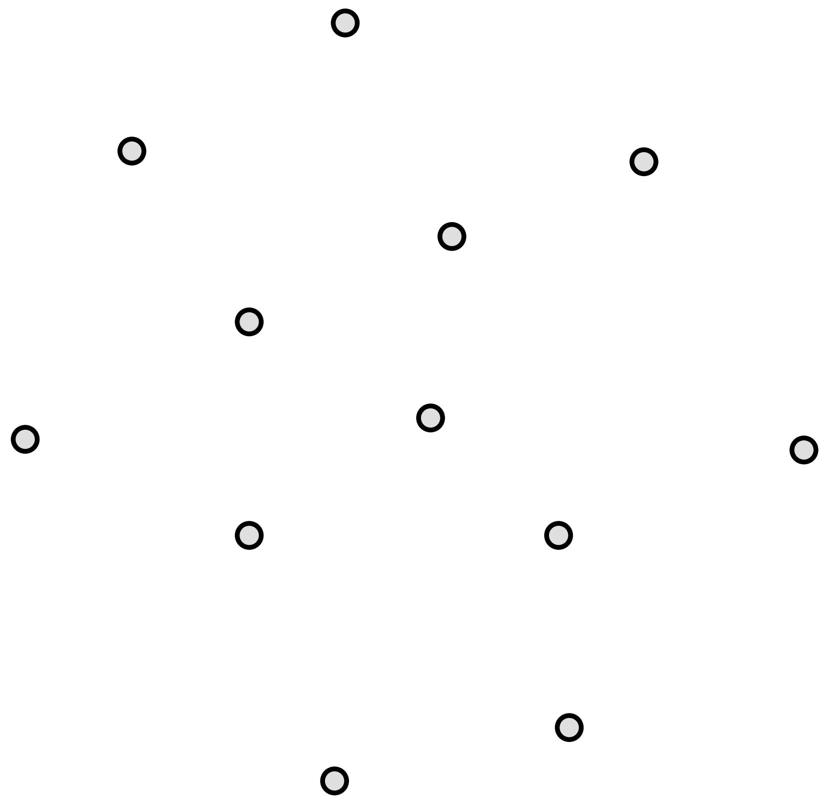


Quality of a triangulation

Divide and conquer



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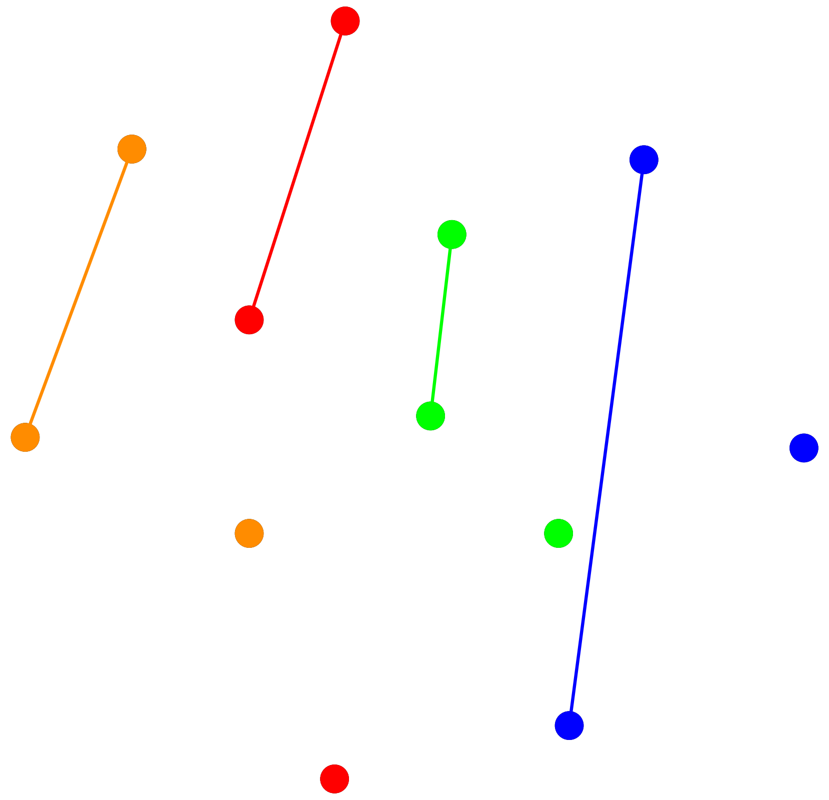

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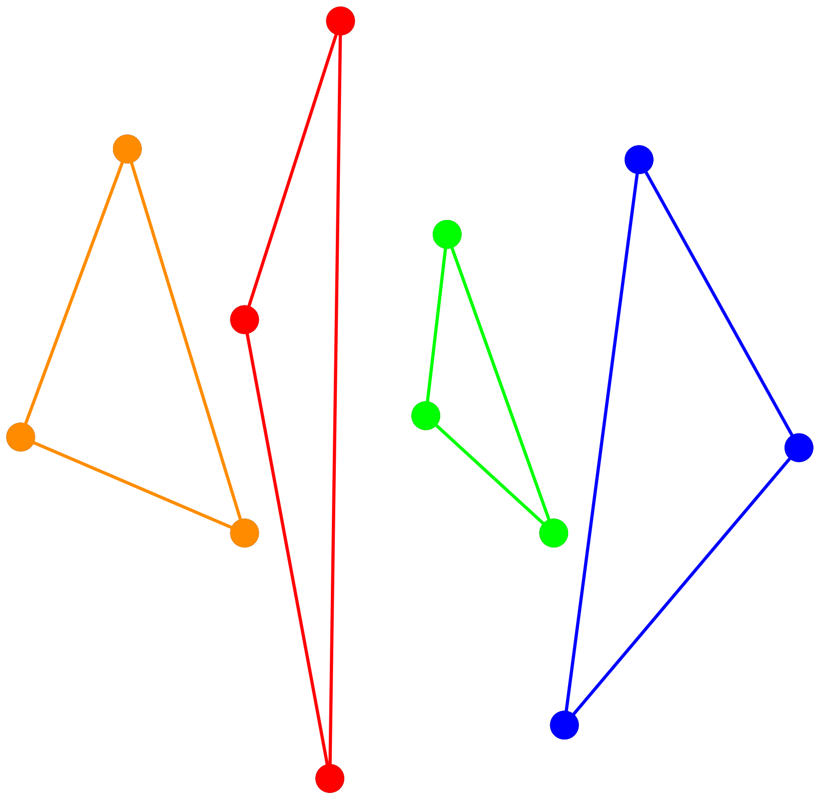

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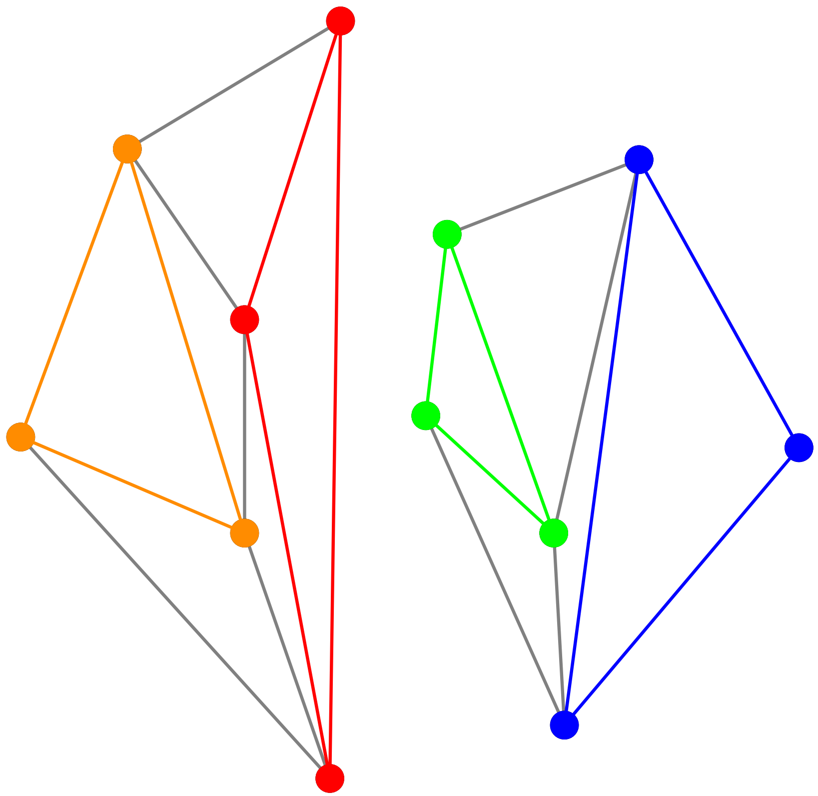

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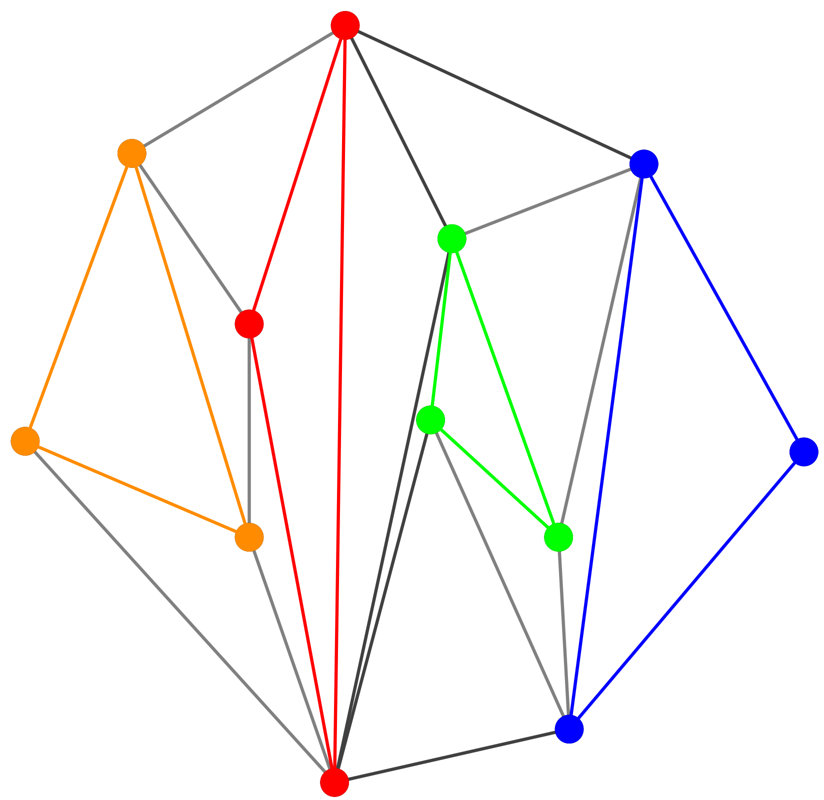

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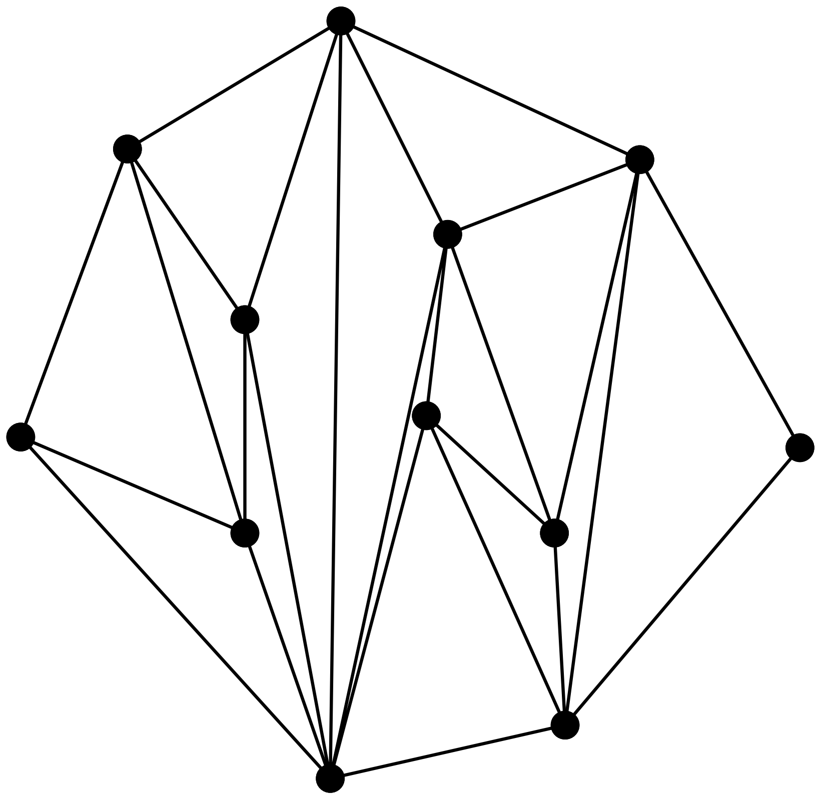
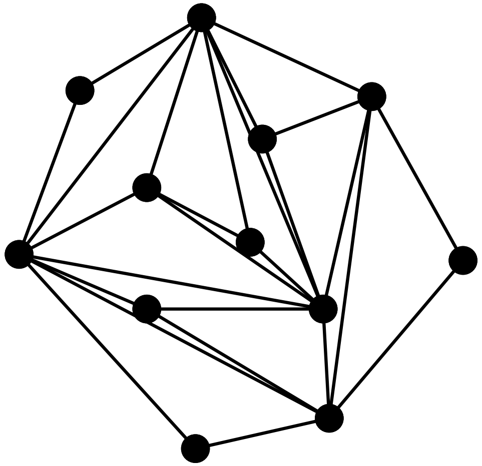
Divide and conquer

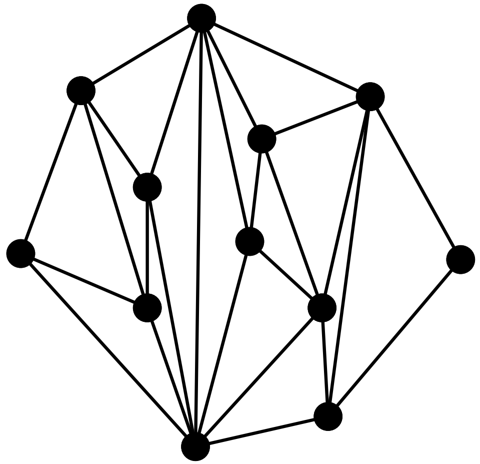


TRIANGULATING POINT SETS

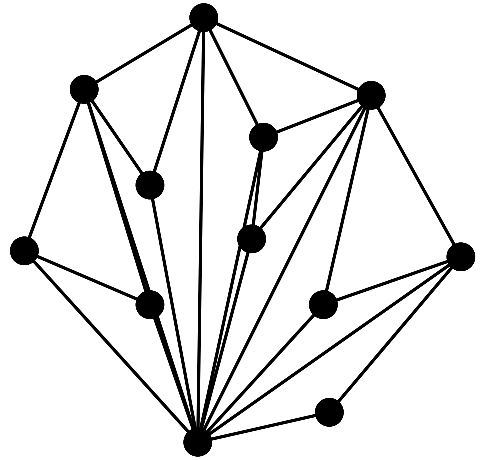
Quality of a triangulation



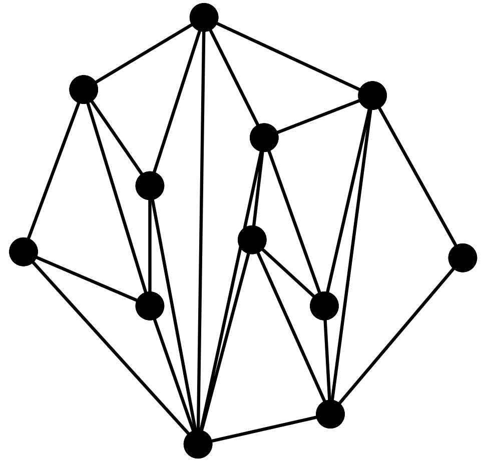
Incremental, unsorted



Incremental, sorted



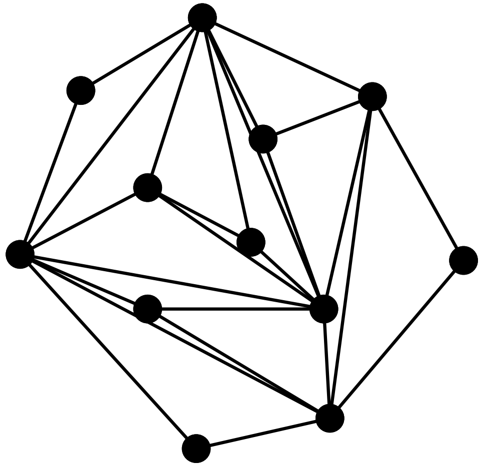
Graham's scan



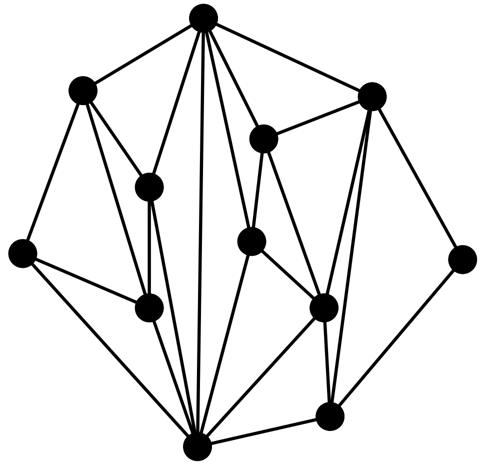
Divide and conquer

TRIANGULATING POINT SETS

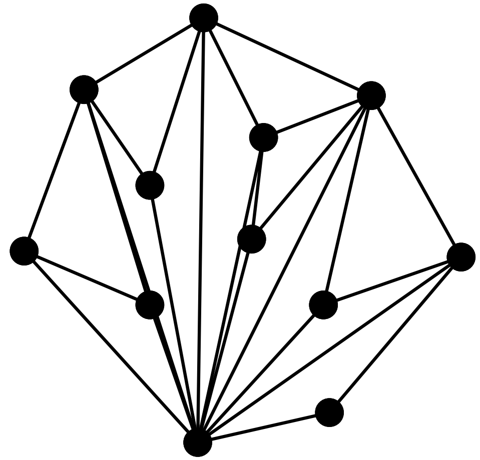
Quality of a triangulation



Incremental, unsorted



Incremental, sorted



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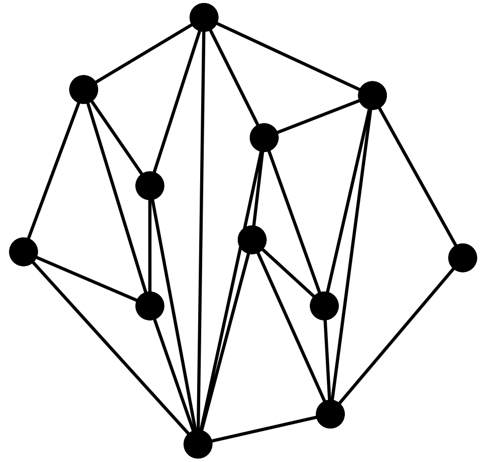
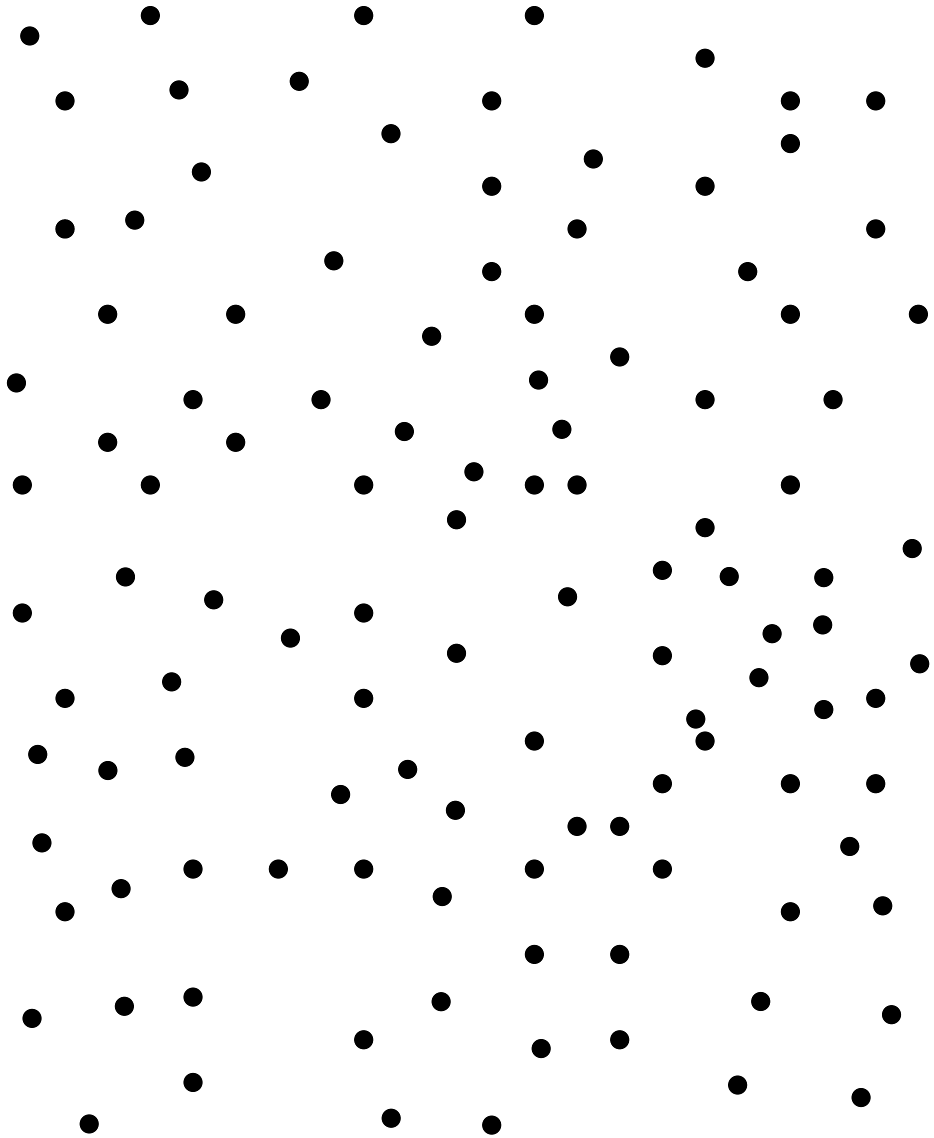


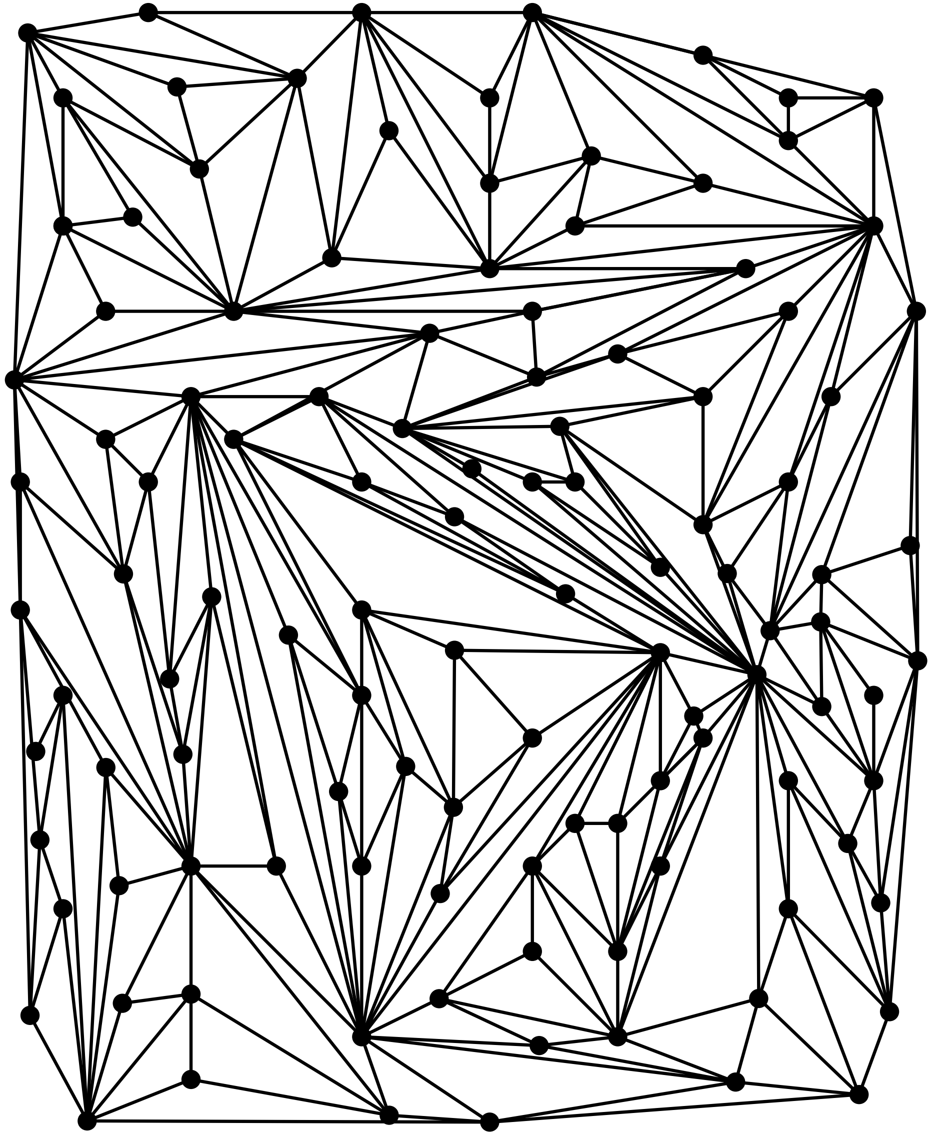
Divide and conquer

Not *all* triangulations are the same

TRIANGULATING POINT SETS



TRIANGULATING POINT SETS



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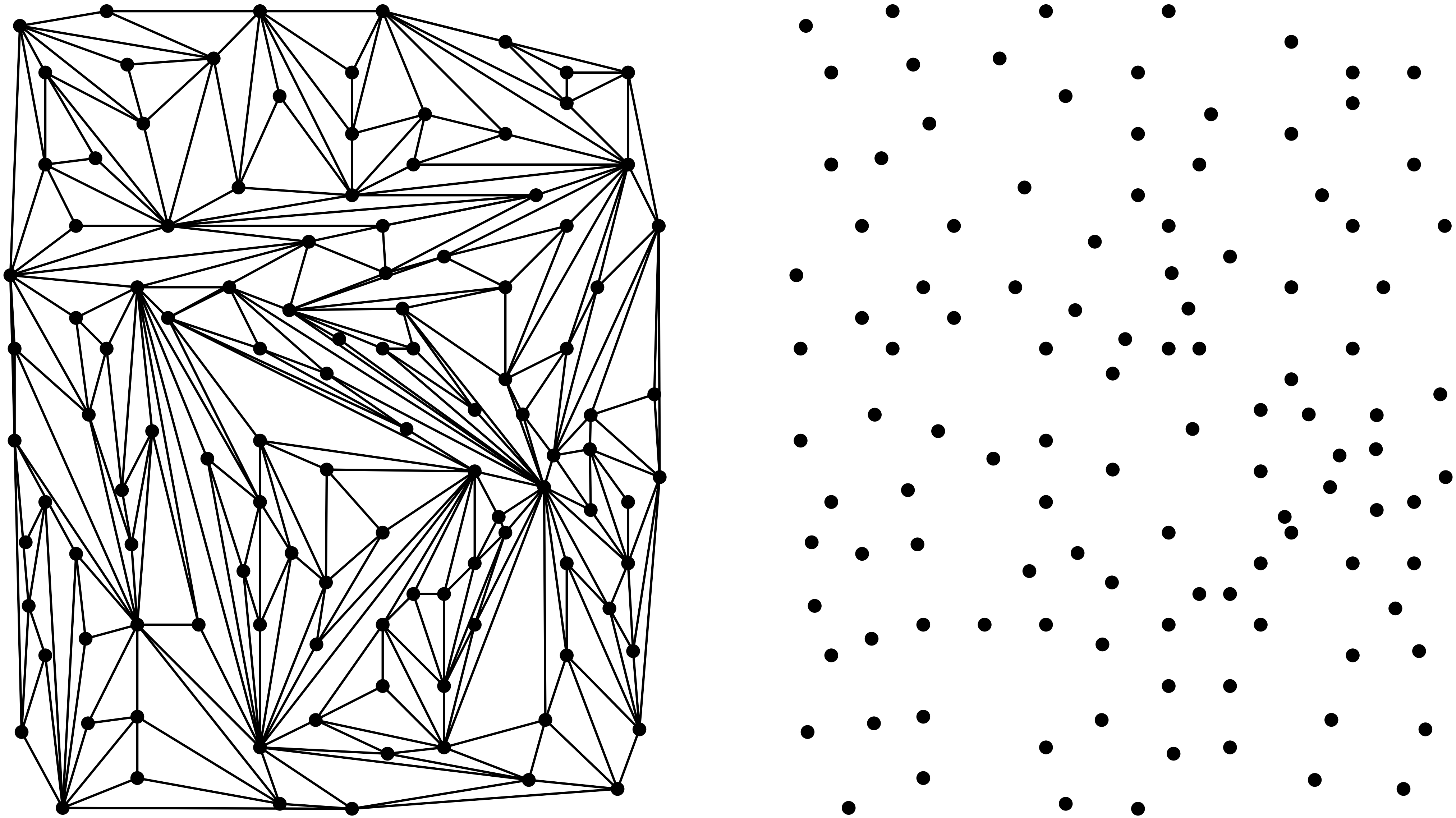

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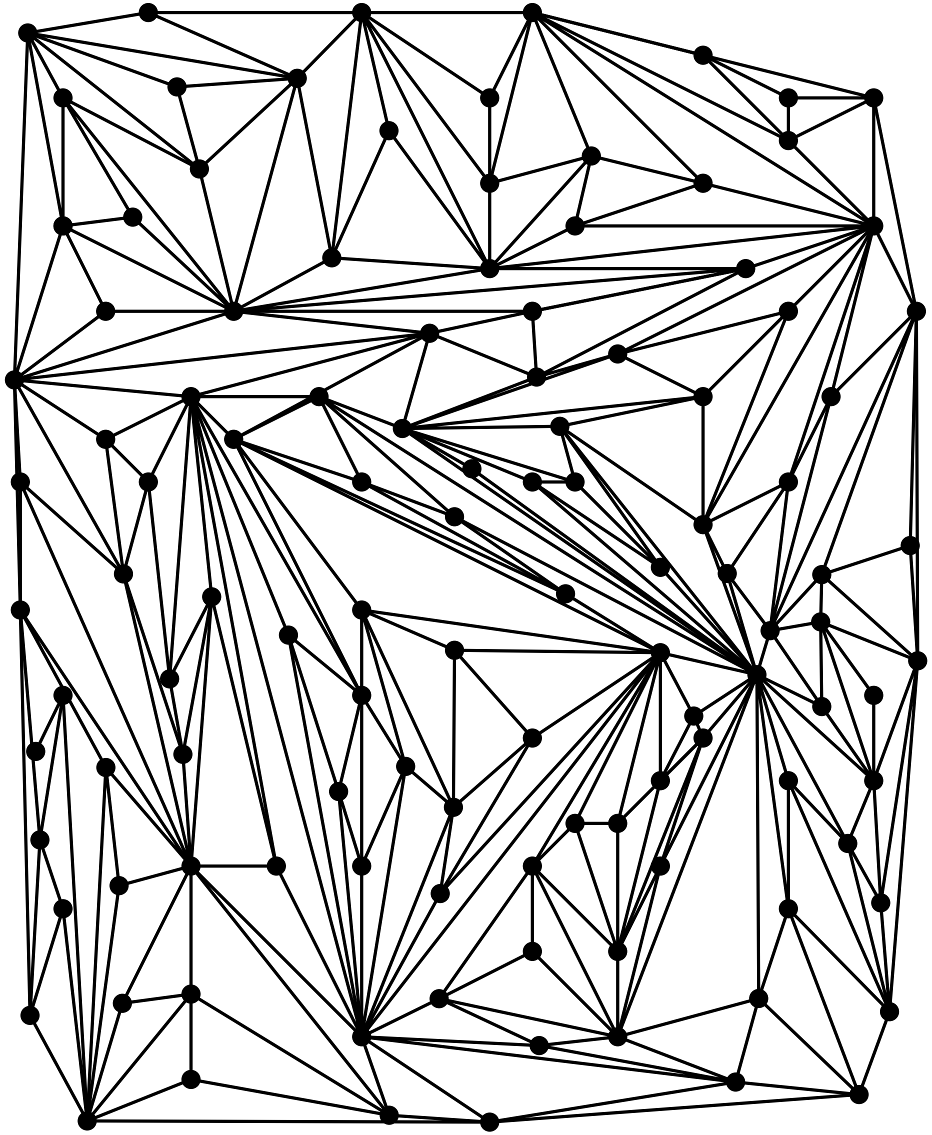
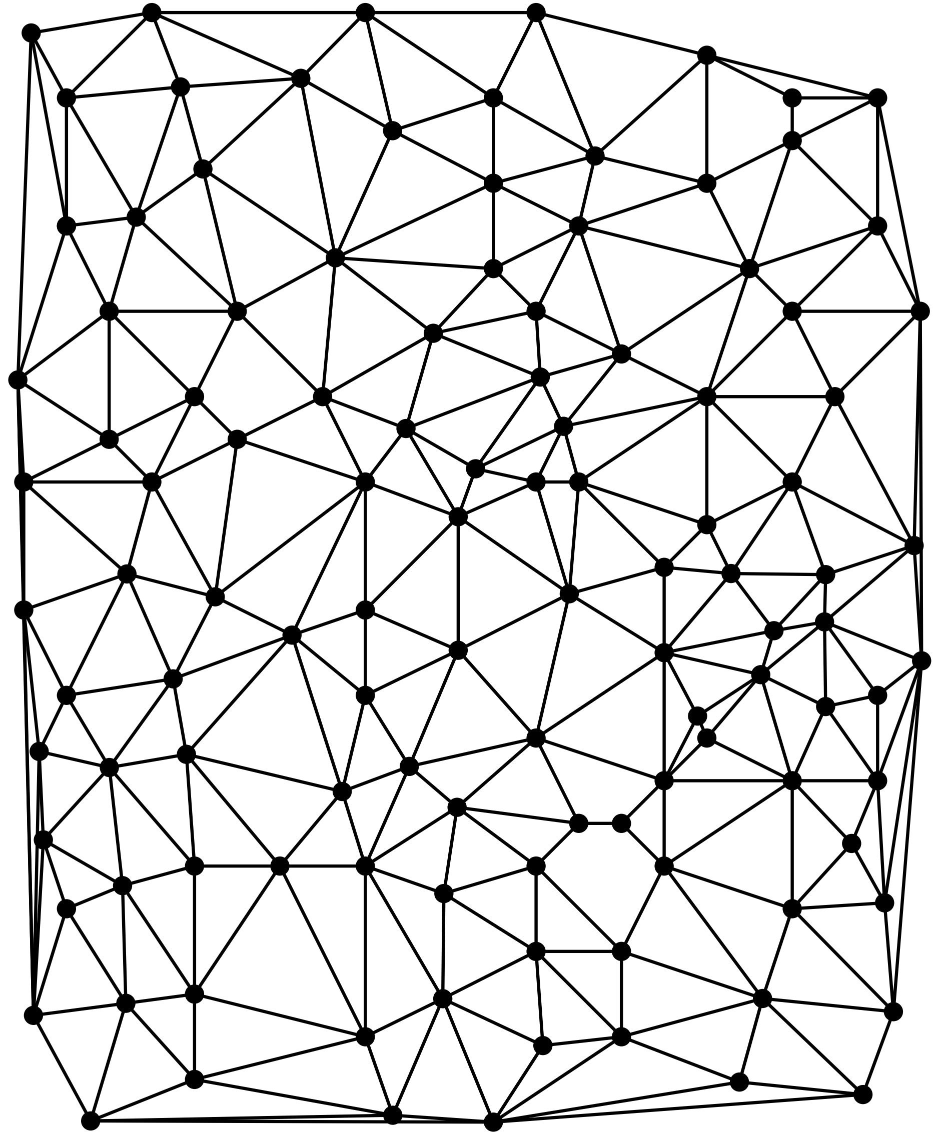
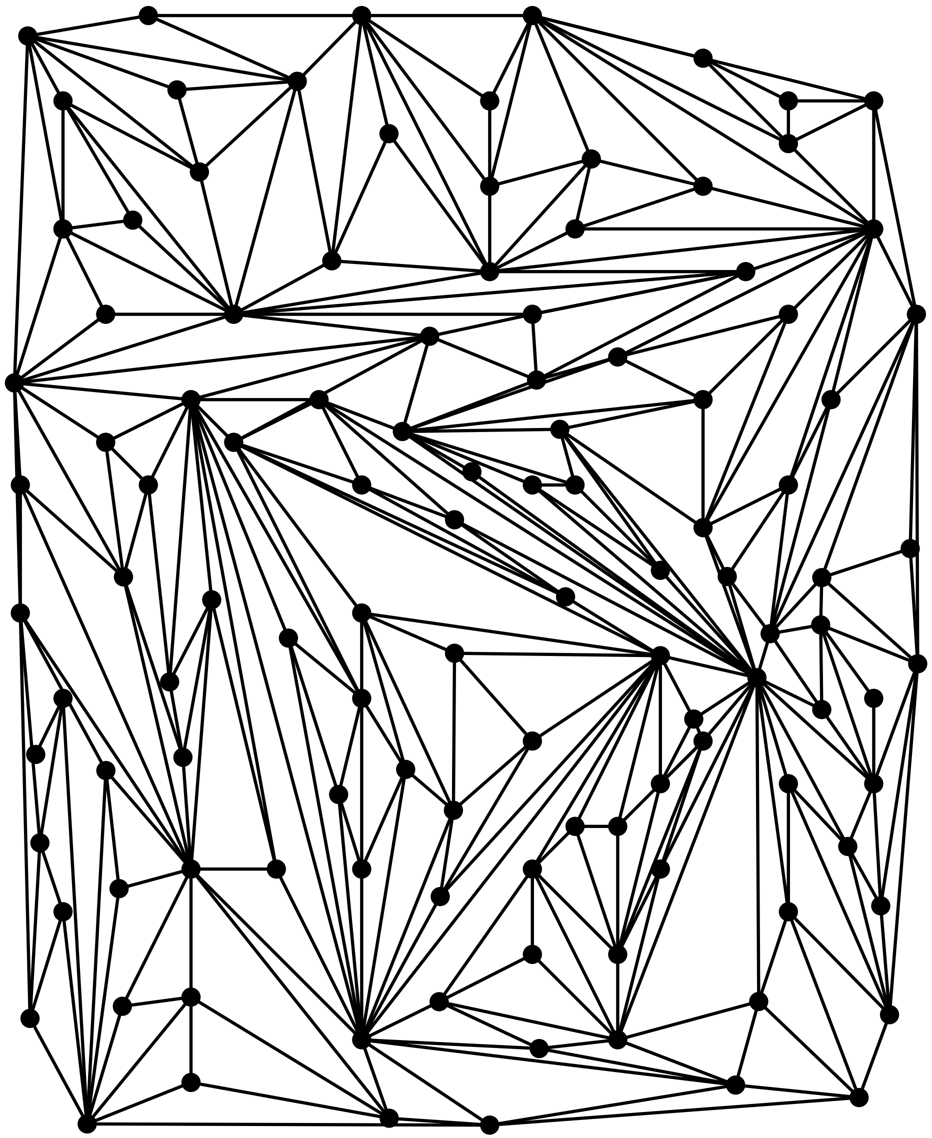
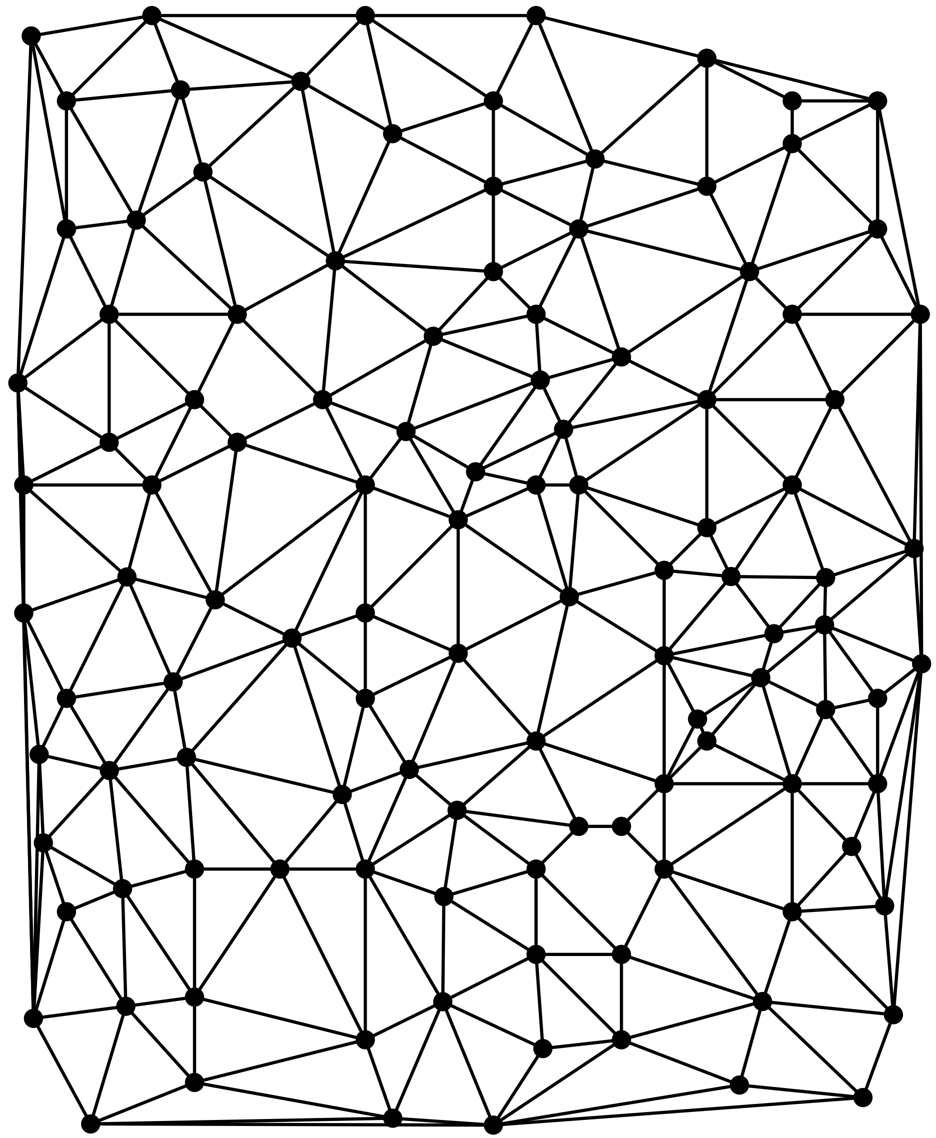




TRIANGULATING POINT SETS





Delaunay triangulation

Delaunay Triangulation

DELAUNAY TRIANGULATION

A TOOL FOR INTERPOLATION

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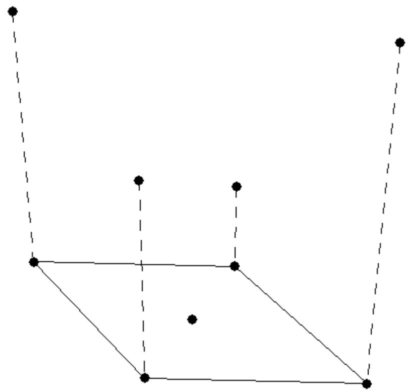
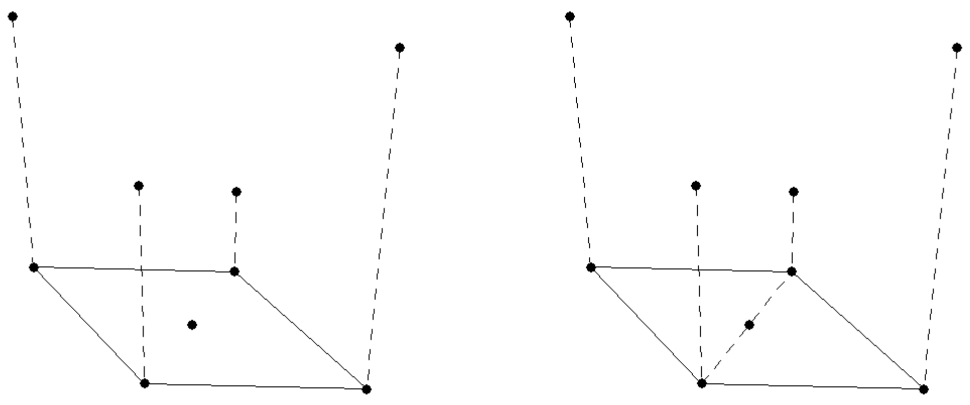
A TOOL FOR INTERPOLATION



DELAUNAY TRIANGULATION

A TOOL FOR INTERPOLATION



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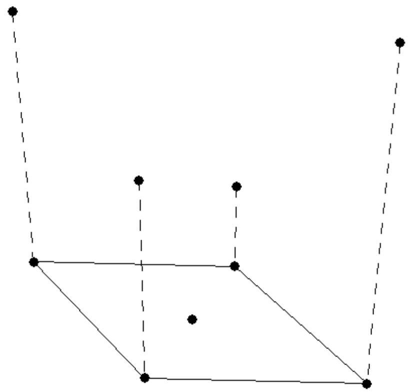

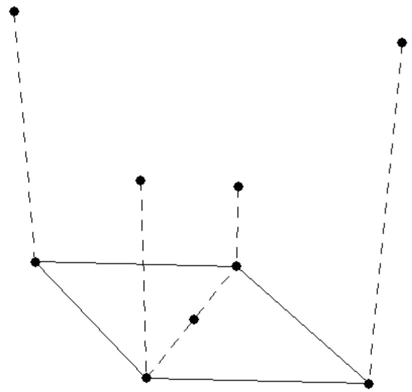

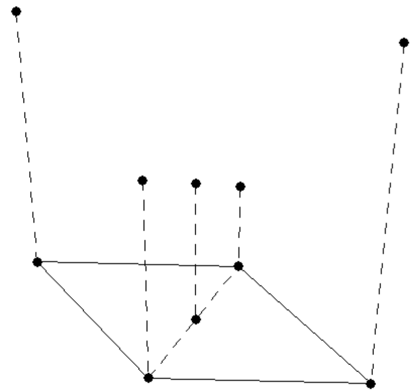

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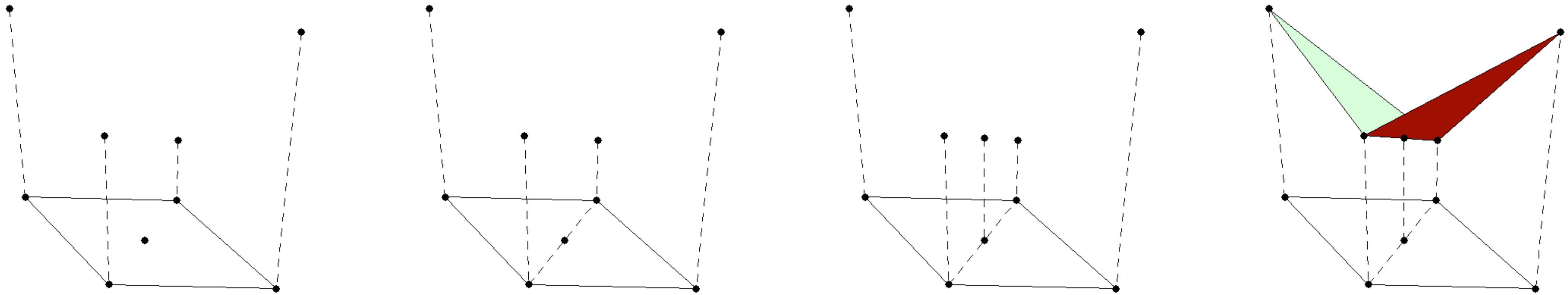

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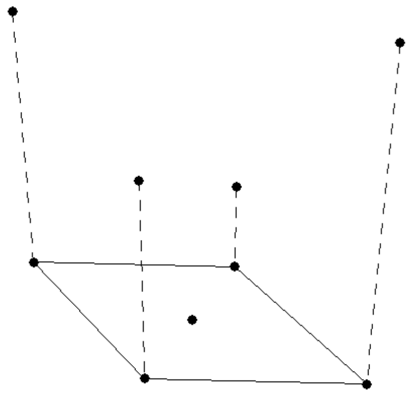
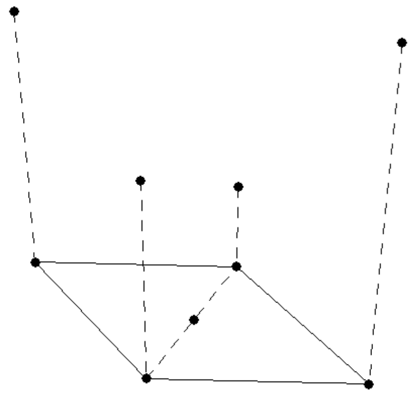
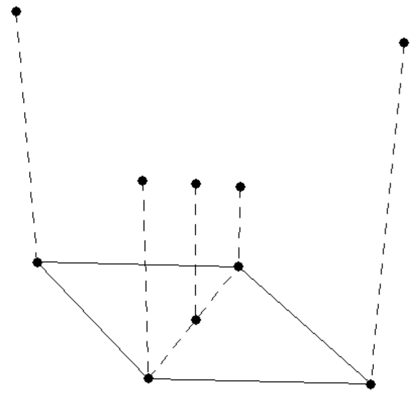
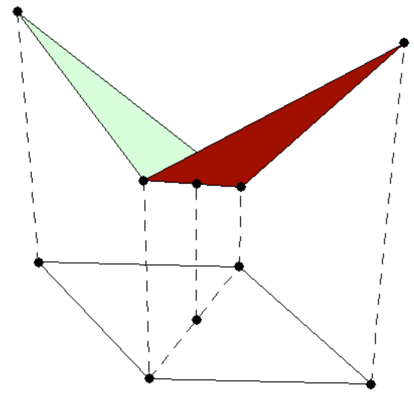
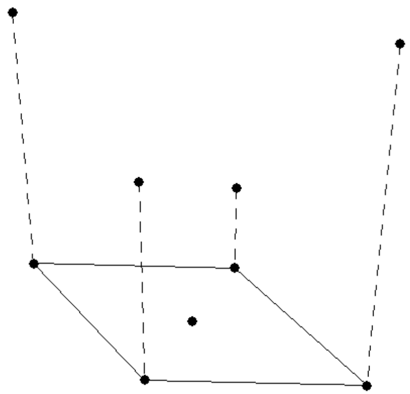

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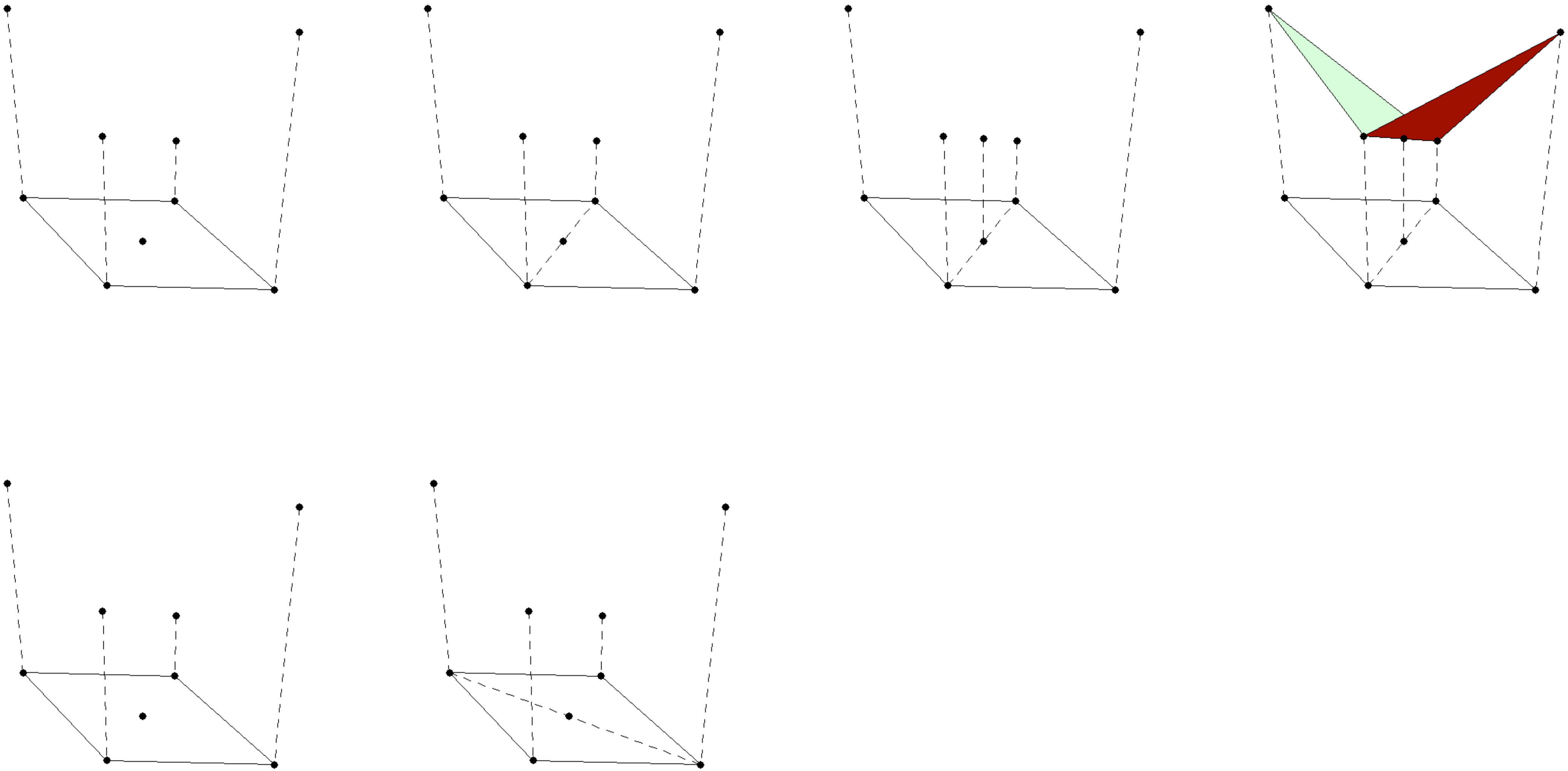

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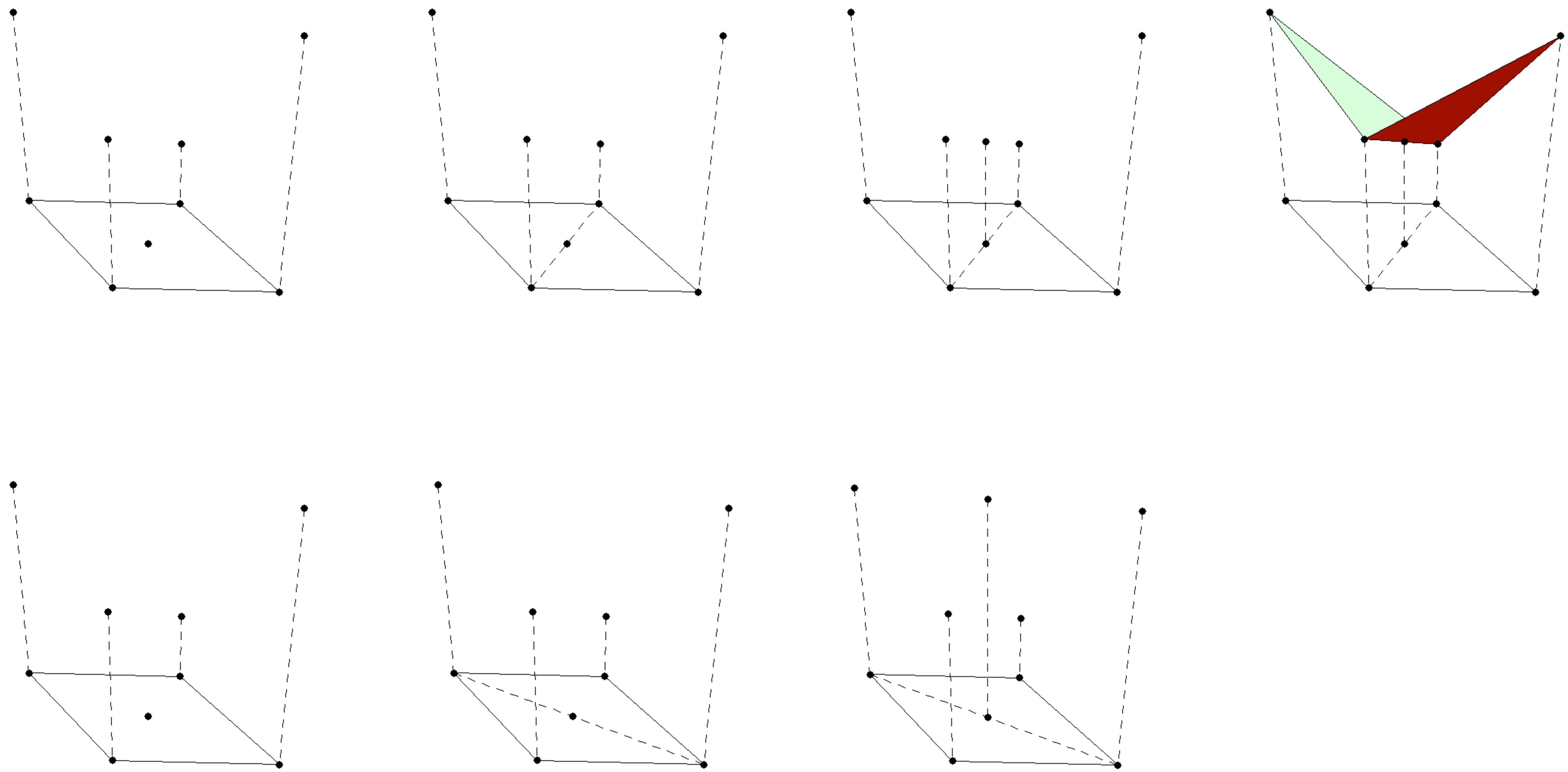

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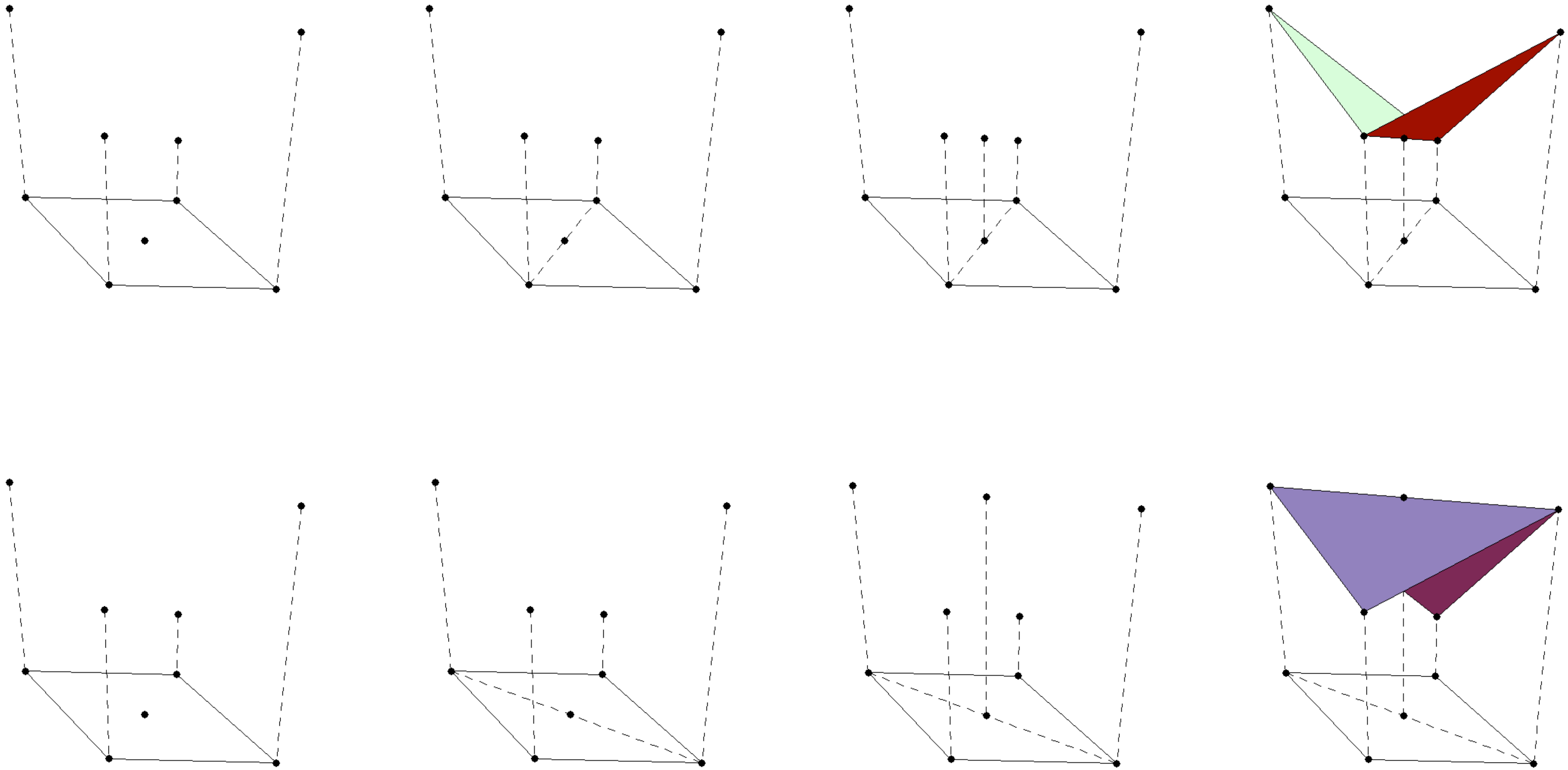


DELAUNAY TRIANGULATION

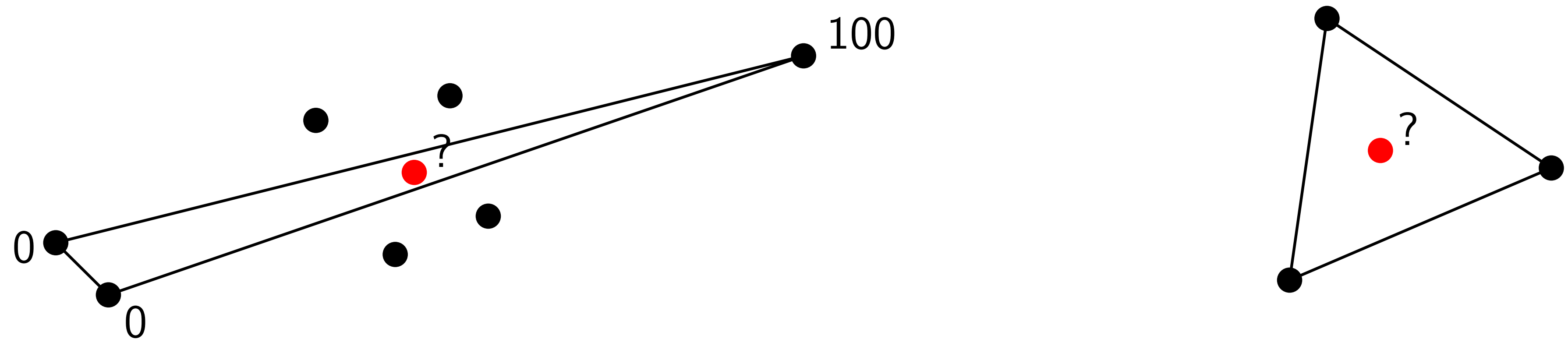
A TOOL FOR INTERPOLATION

Motivation

For many interpolation applications, it makes more sense to interpolate from points that are nearby than from points that are far away

To do that, the shape of the triangles matters:

- Long and skinny triangles produce interpolation from points that are far
- Equilateral-like triangles tend to give interpolations from points closer and well distributed



DELAUNAY TRIANGULATION

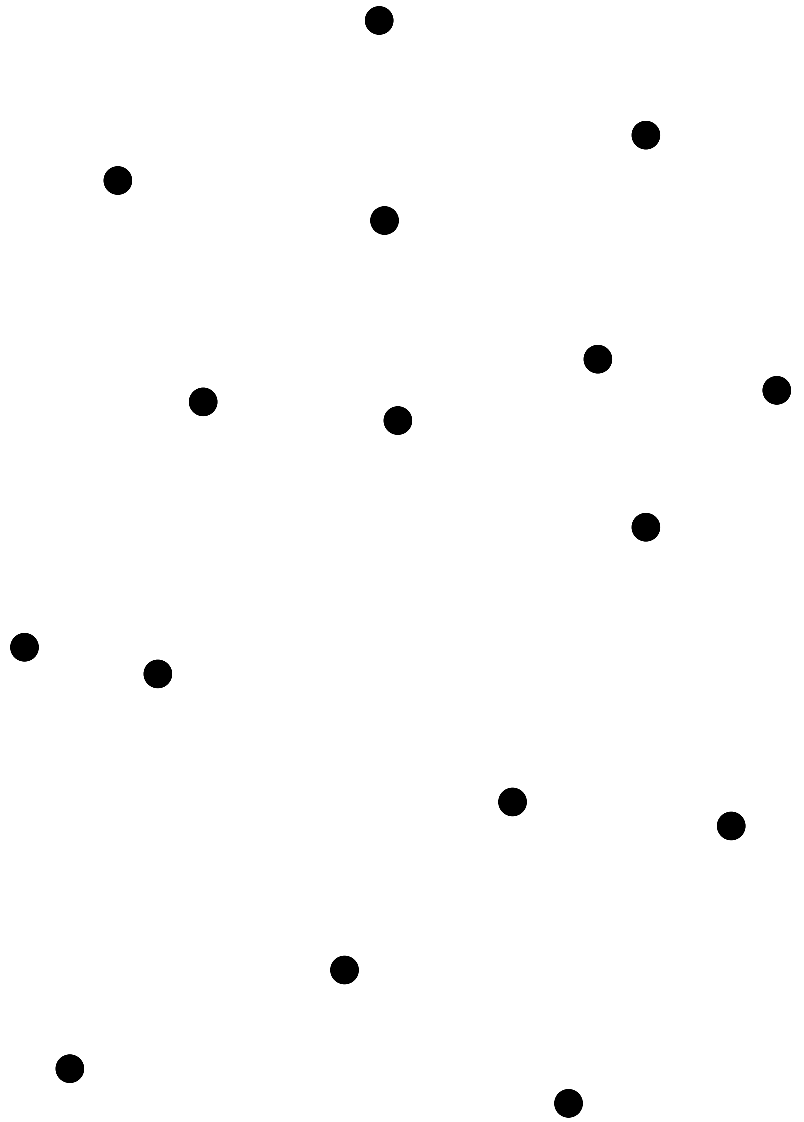
DEFINITION AND PROPERTIES

DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition (triangle-based)

Given a set P with n points in the plane...



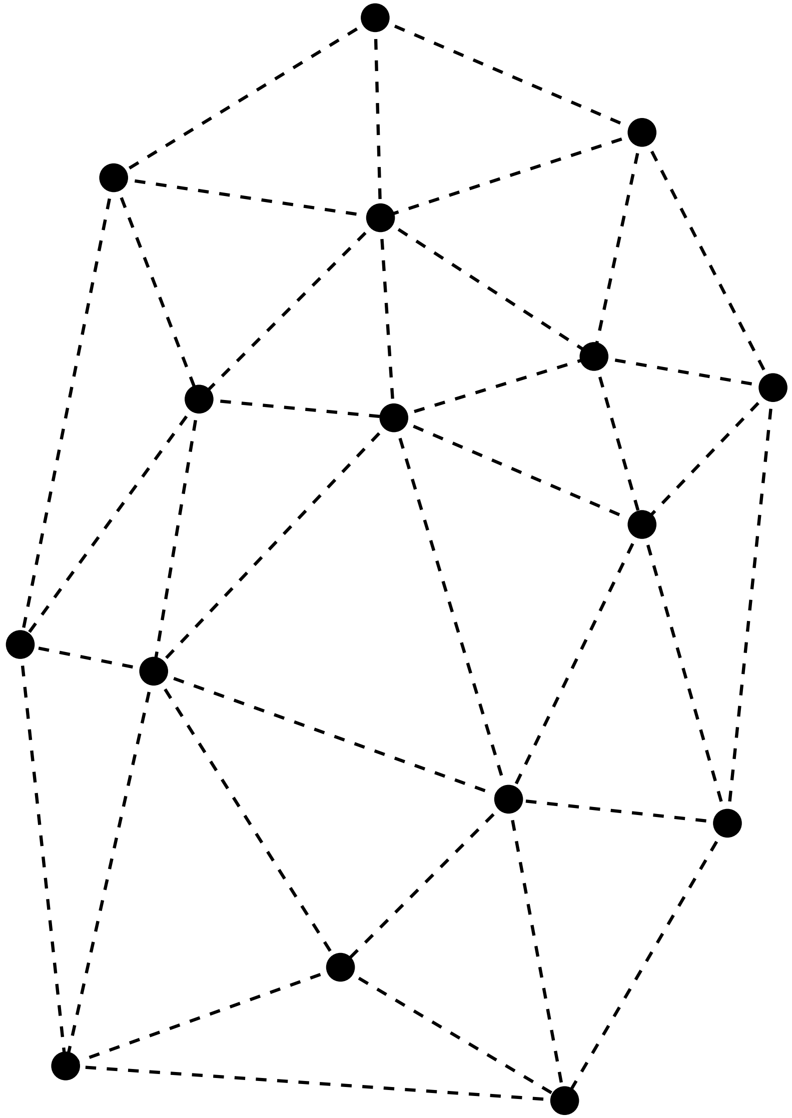
DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition (triangle-based)

Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is a triangulation where all triangles are *Delaunay triangles*.

Three points p_i, p_j, p_k form a *Delaunay triangle* (in general, are vertices of a face) if and only if the circle through them does not contain any point of P in its interior.



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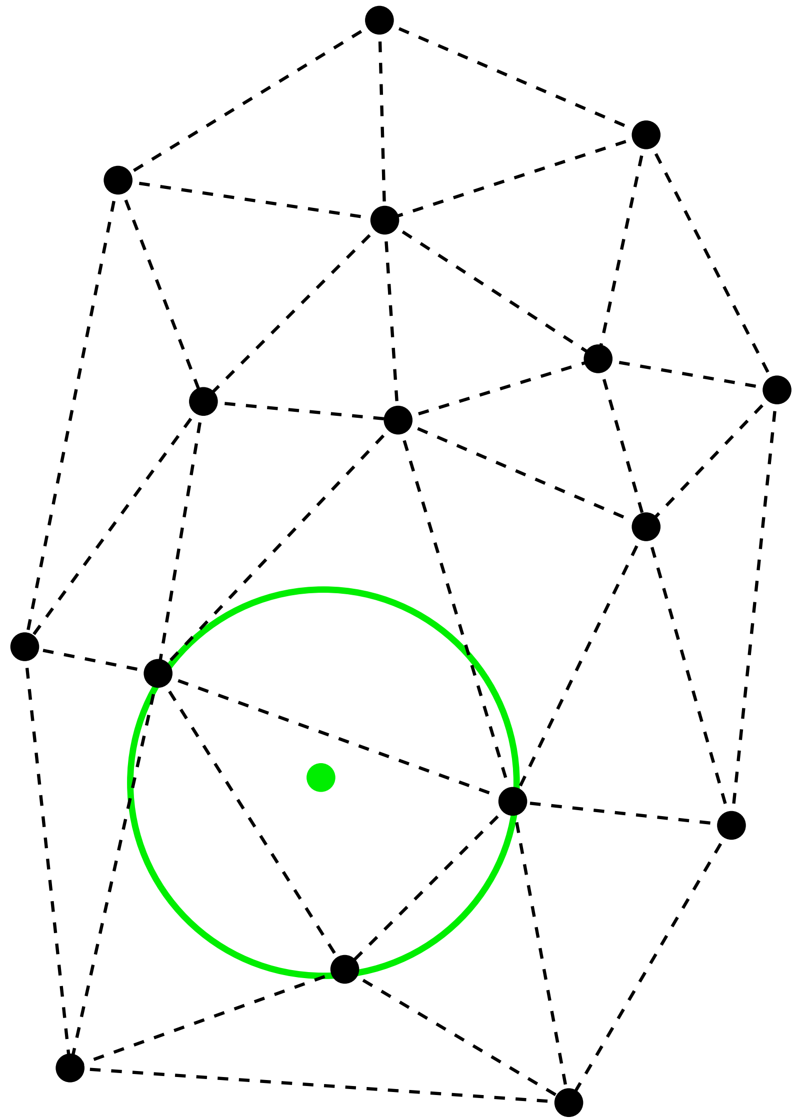

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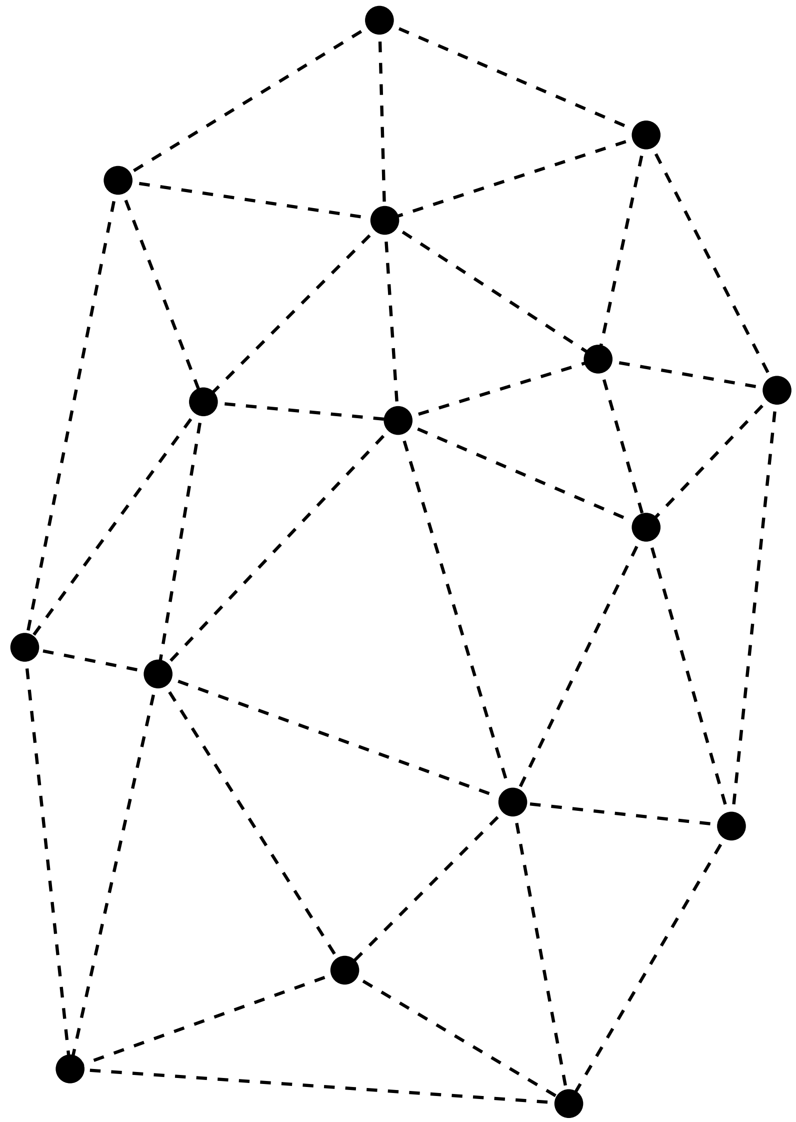
Equivalent characterization (edge-based)

- Two points $p_i, p_j \in P$ form a Delaunay edge if and only if there exists a circle through p_i and p_j that does not contain any point of P in its interior.



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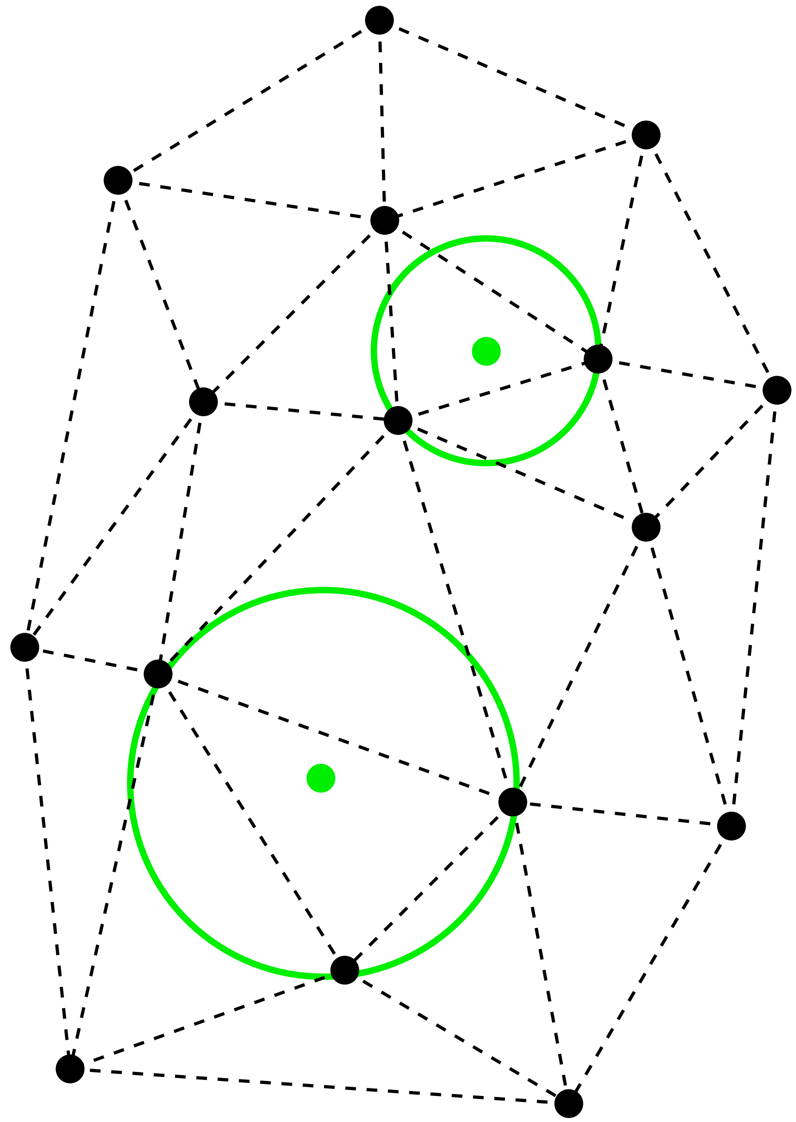

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DEFINITION AND PROPERTIES

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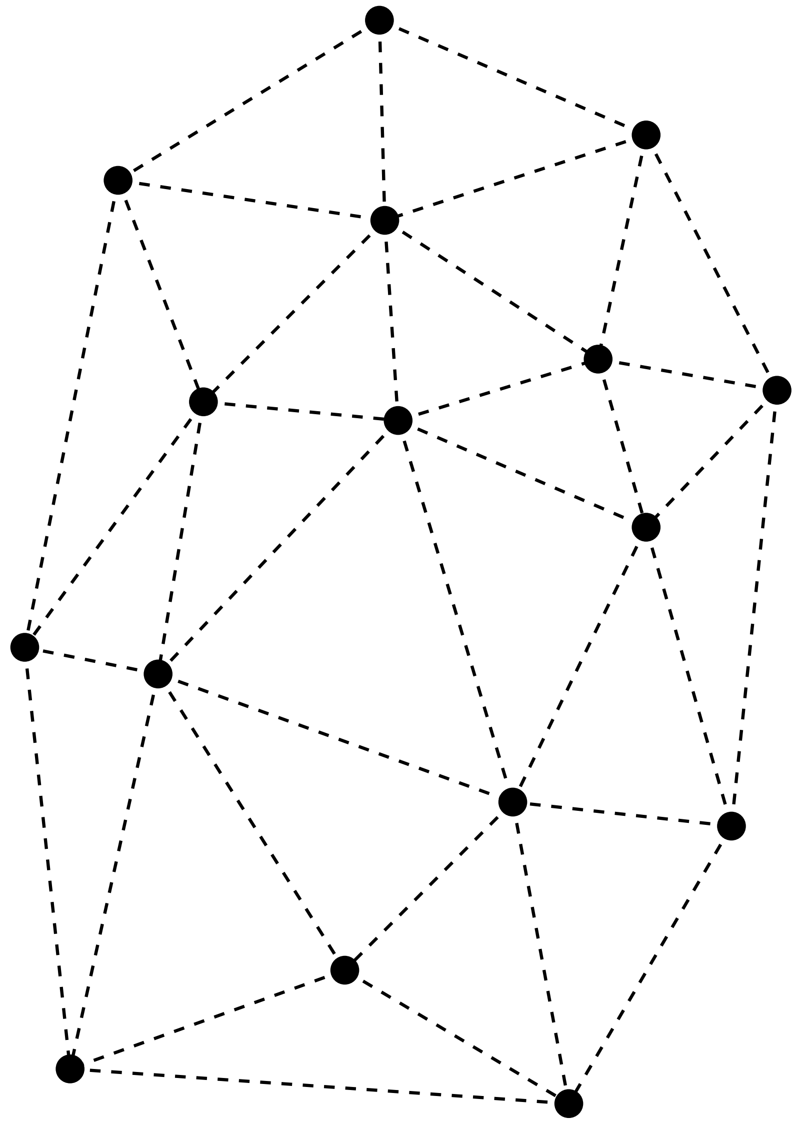
Properties of edge-based definition

$Del(P)$ is a plane graph



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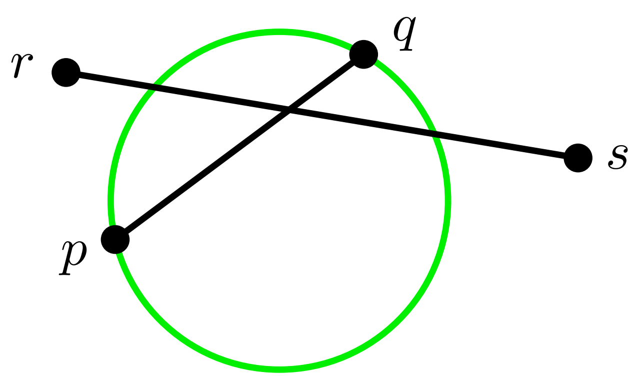
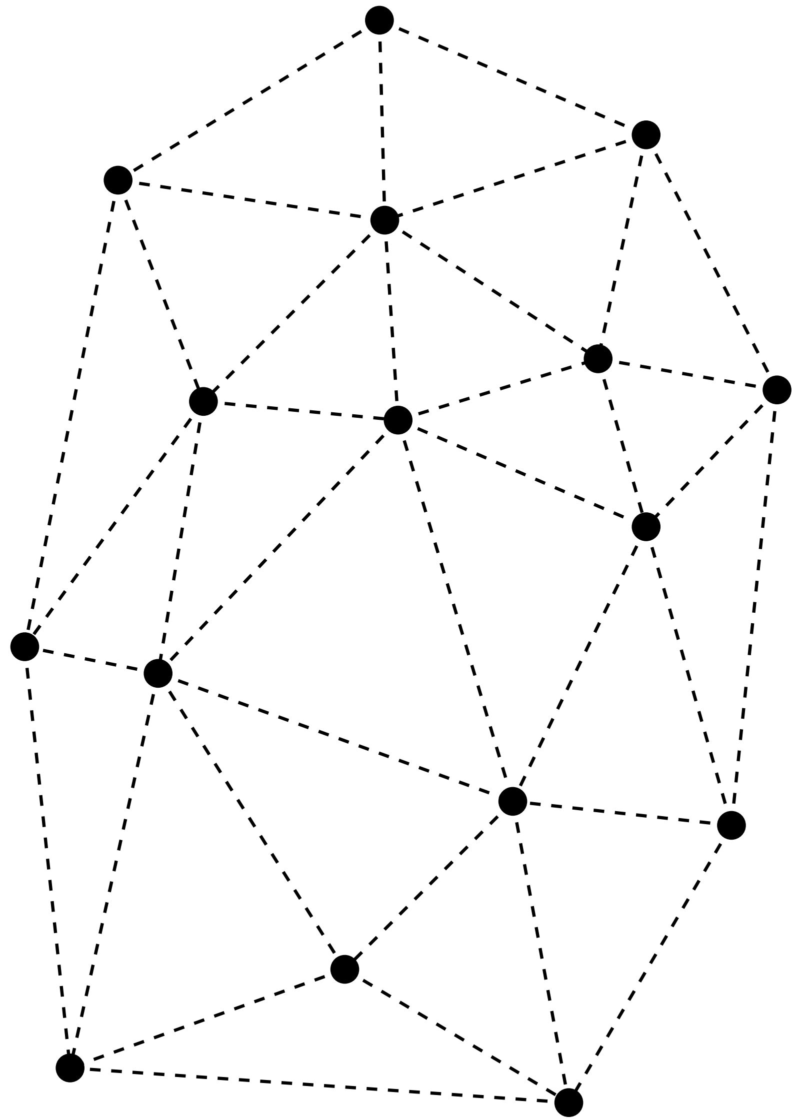
If $\overline{pq}$ is a Delaunay edge, there exists an empty circle through p and q. If a segment $\overline{rs}$ intersects $\overline{pq}$, then every circle through r and s contains at least one of p or q.





DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition (triangle-based)

Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is a triangulation where all triangles are *Delaunay triangles*.

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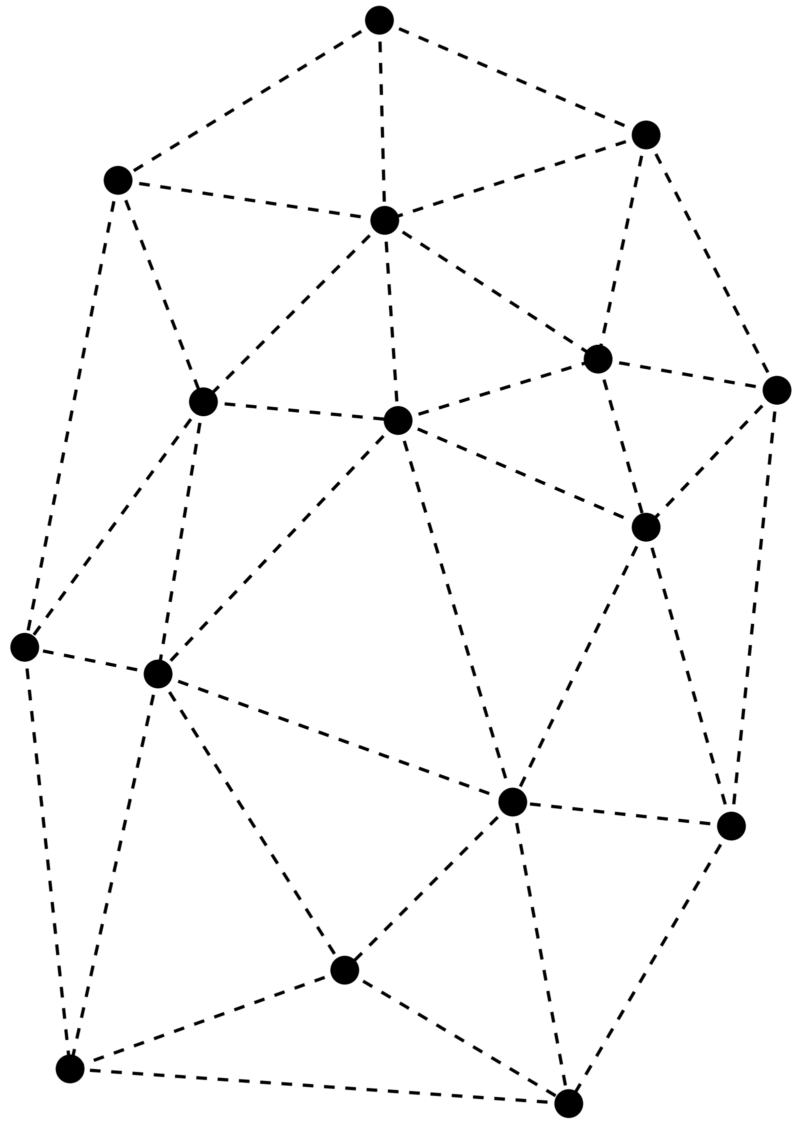
Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



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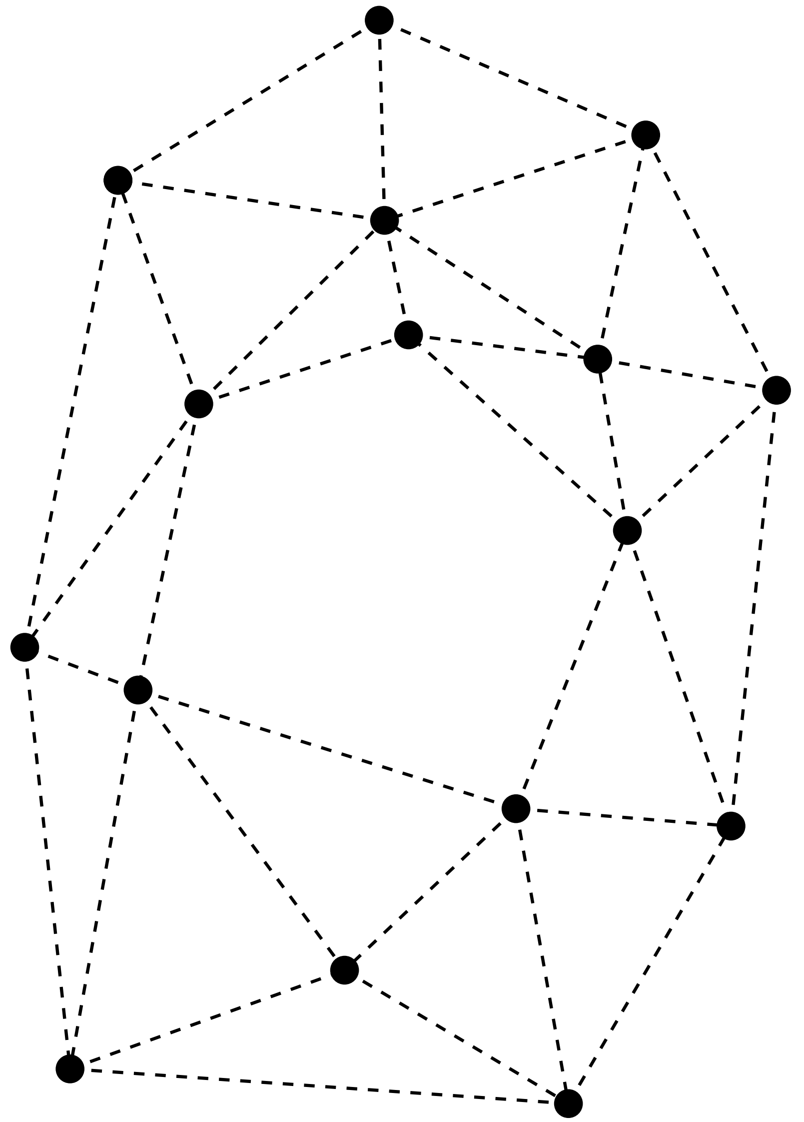

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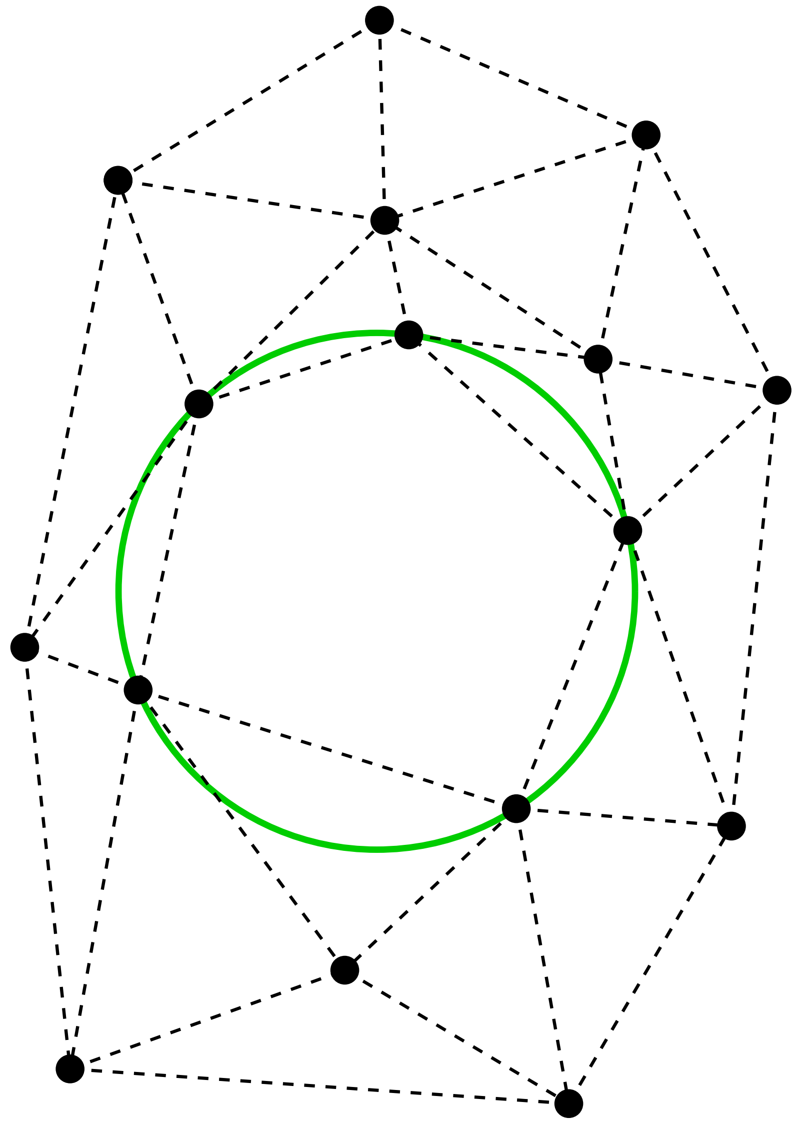

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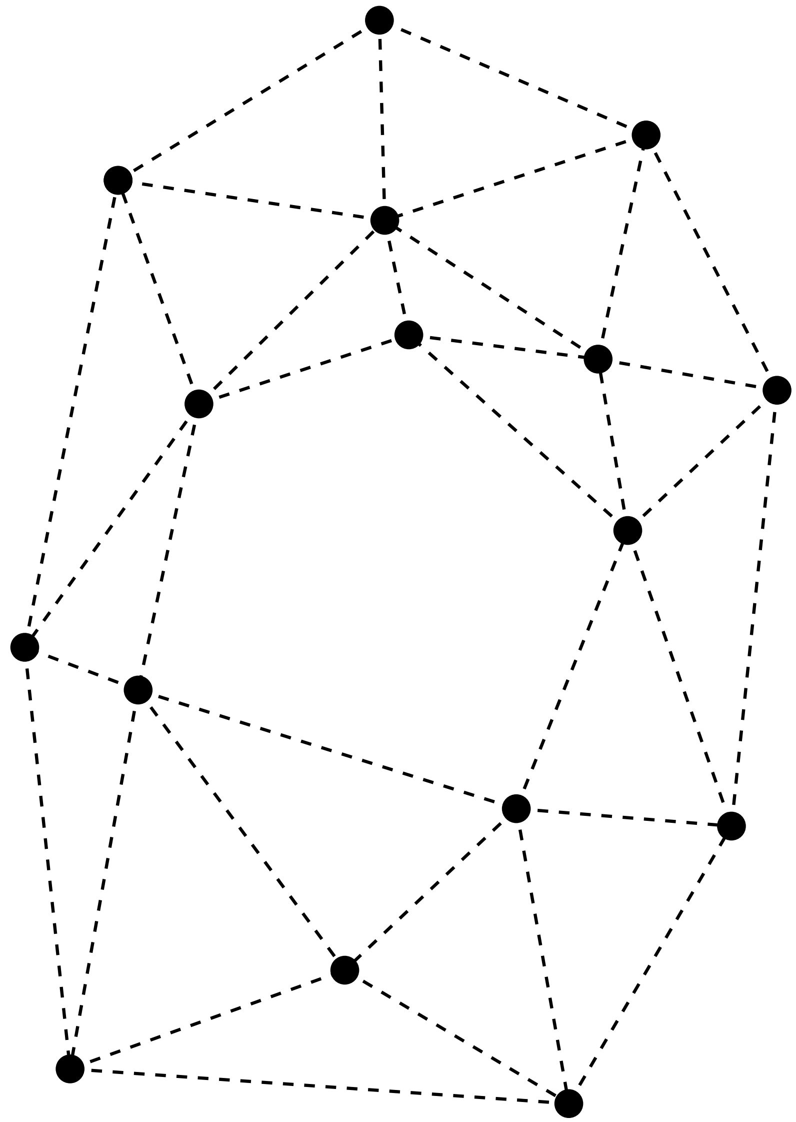

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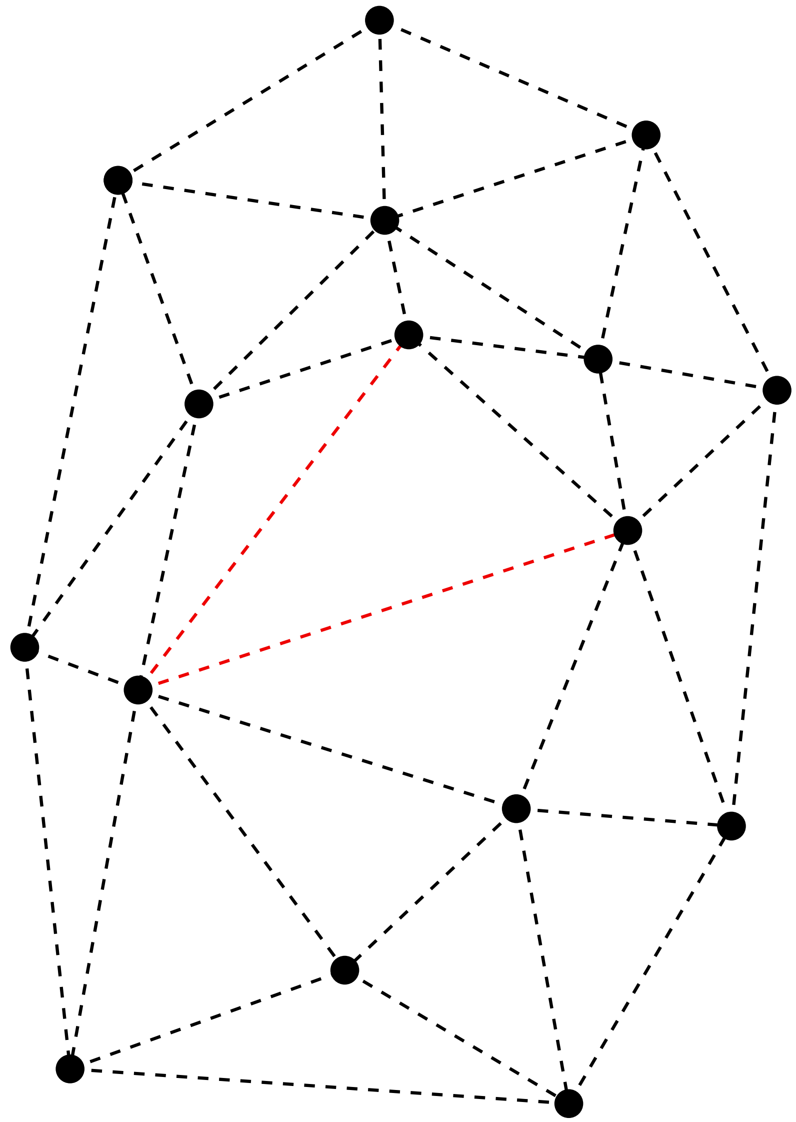

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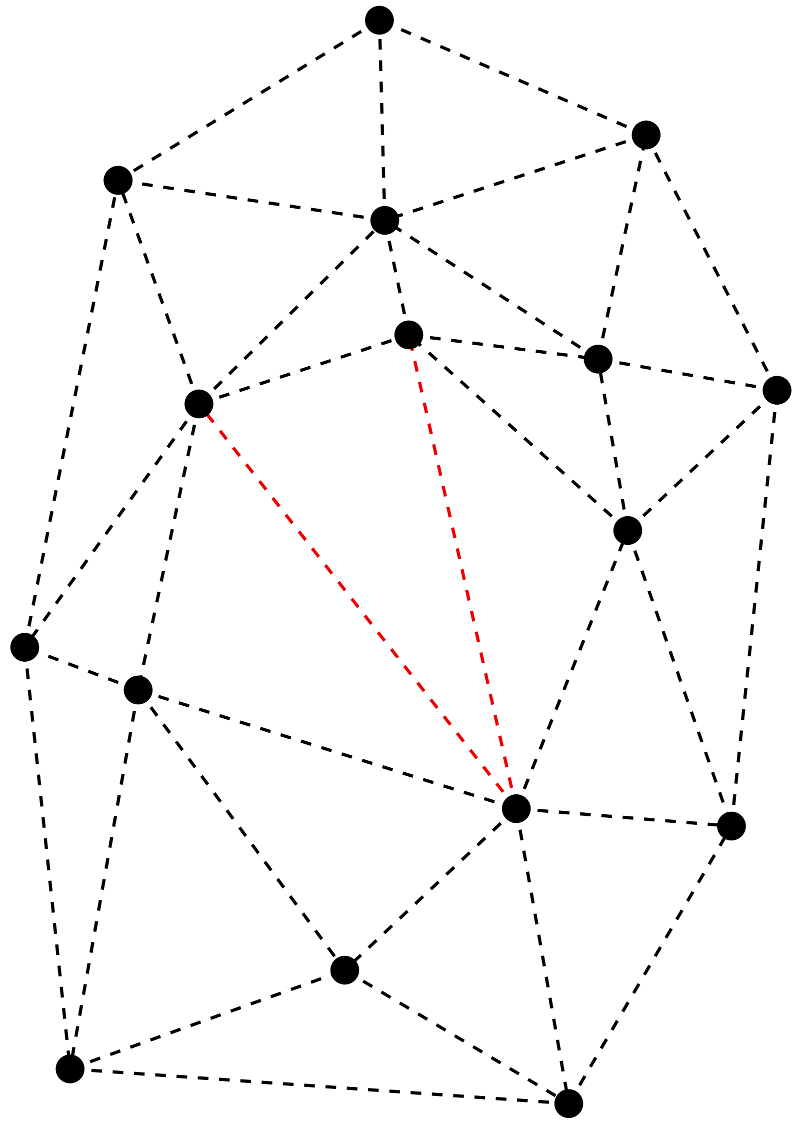

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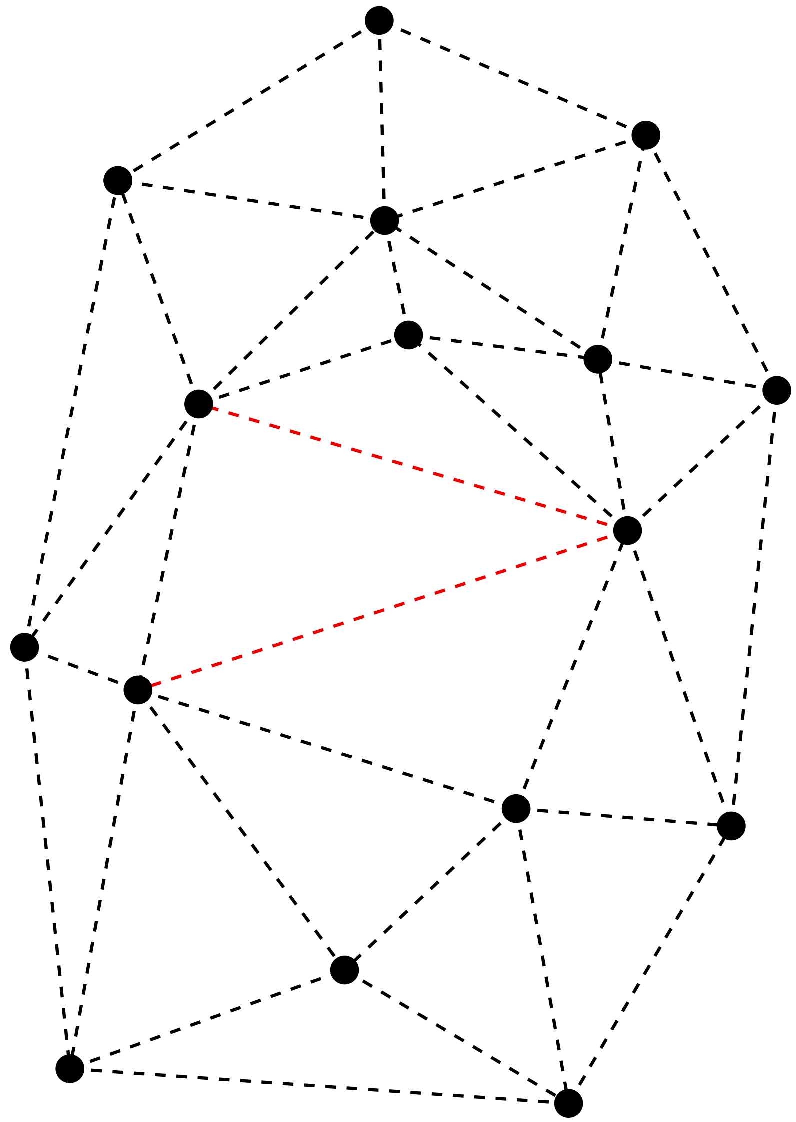

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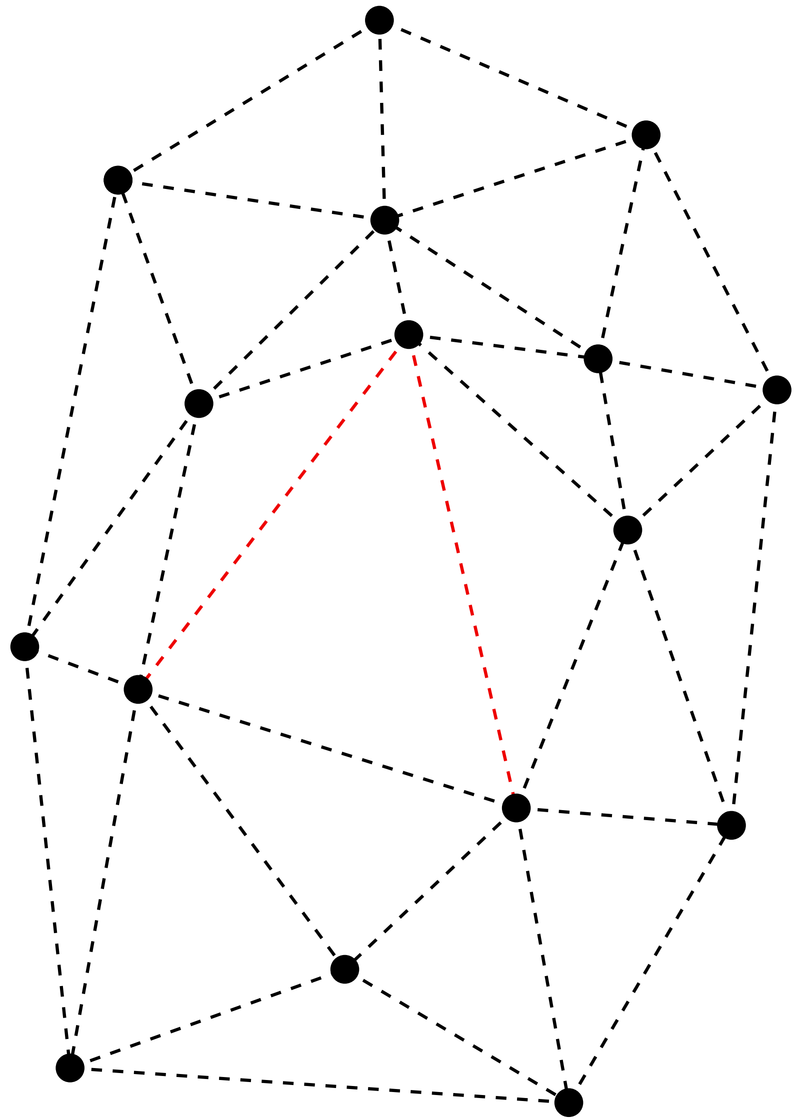

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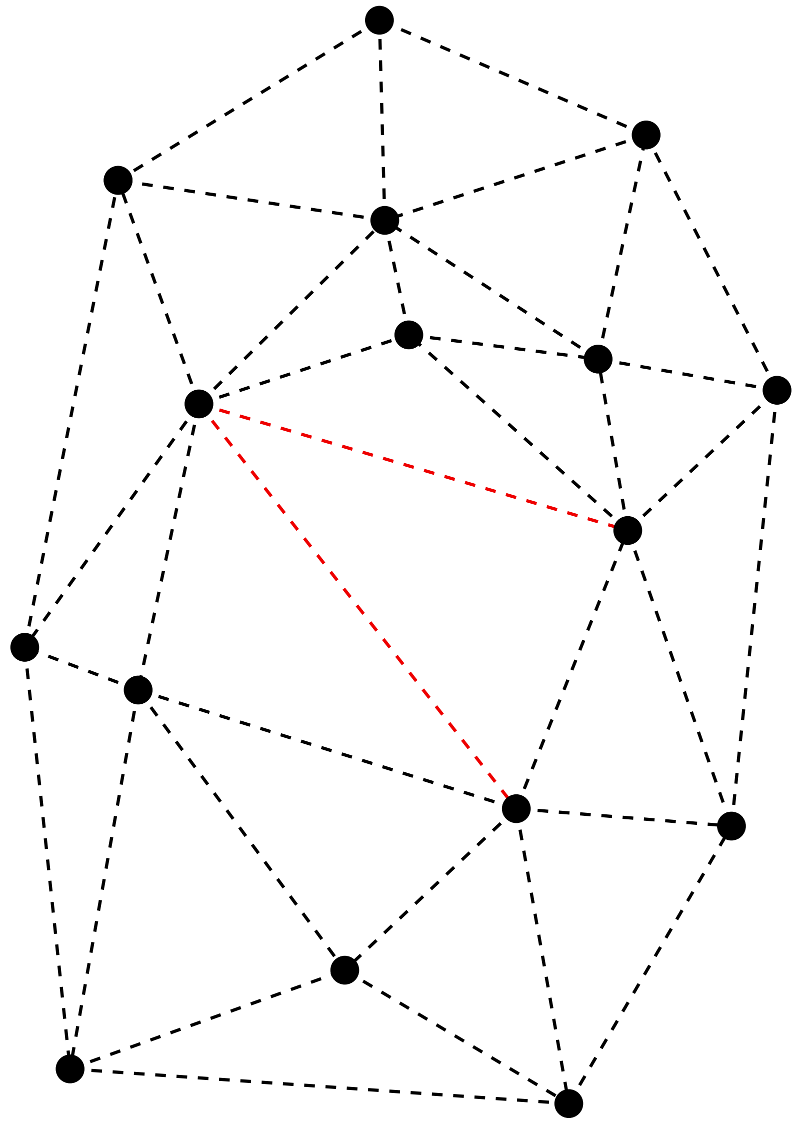

DELAUNAY TRIANGULATION AND 3D CONVEX HULL

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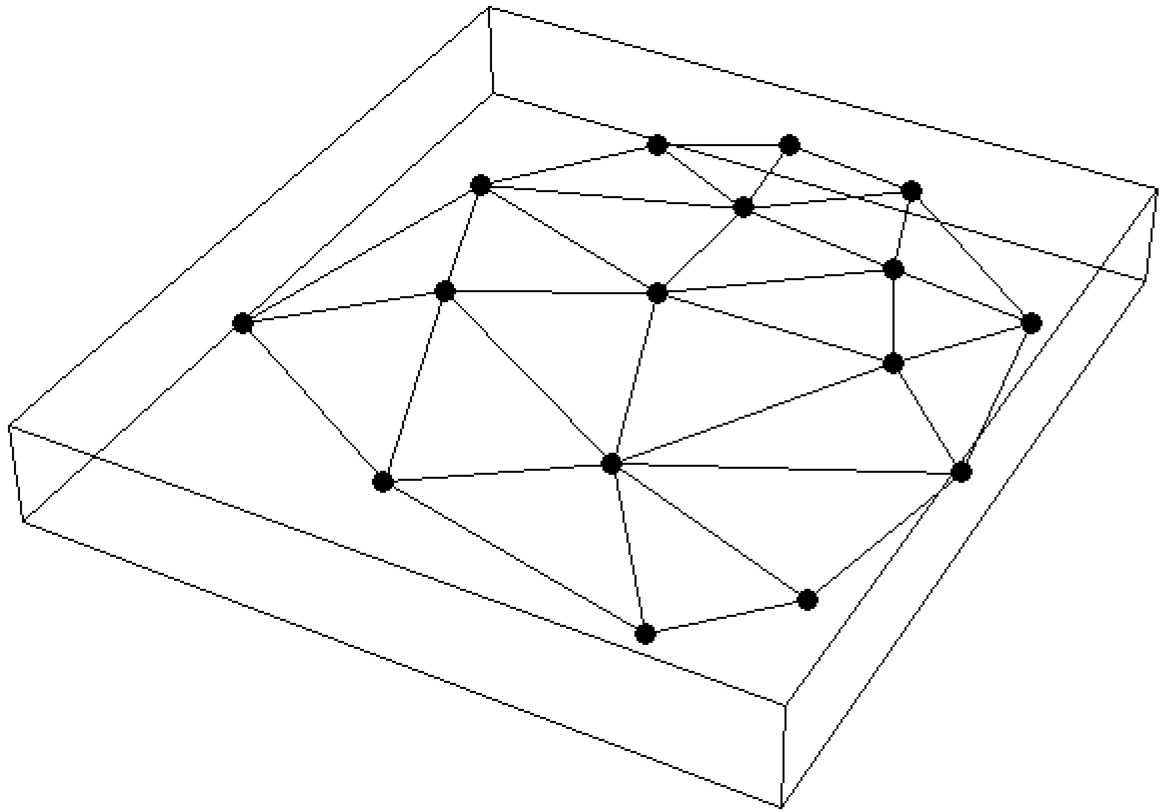

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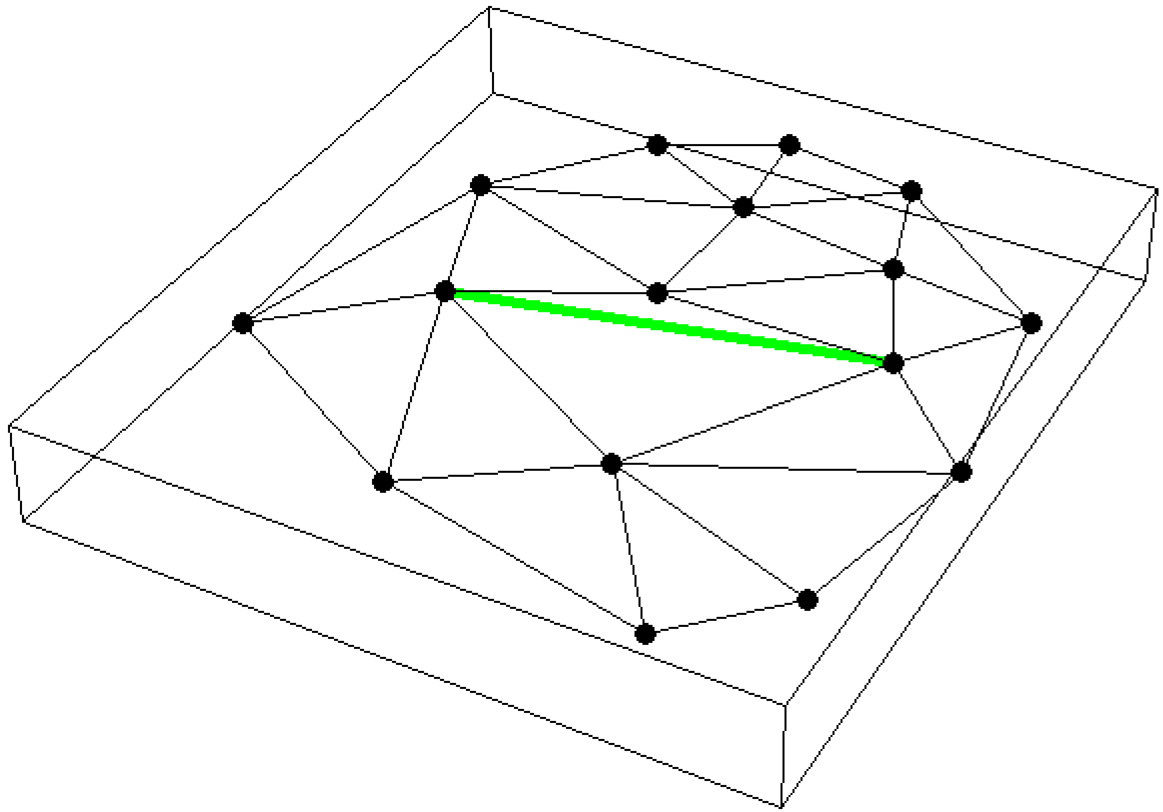

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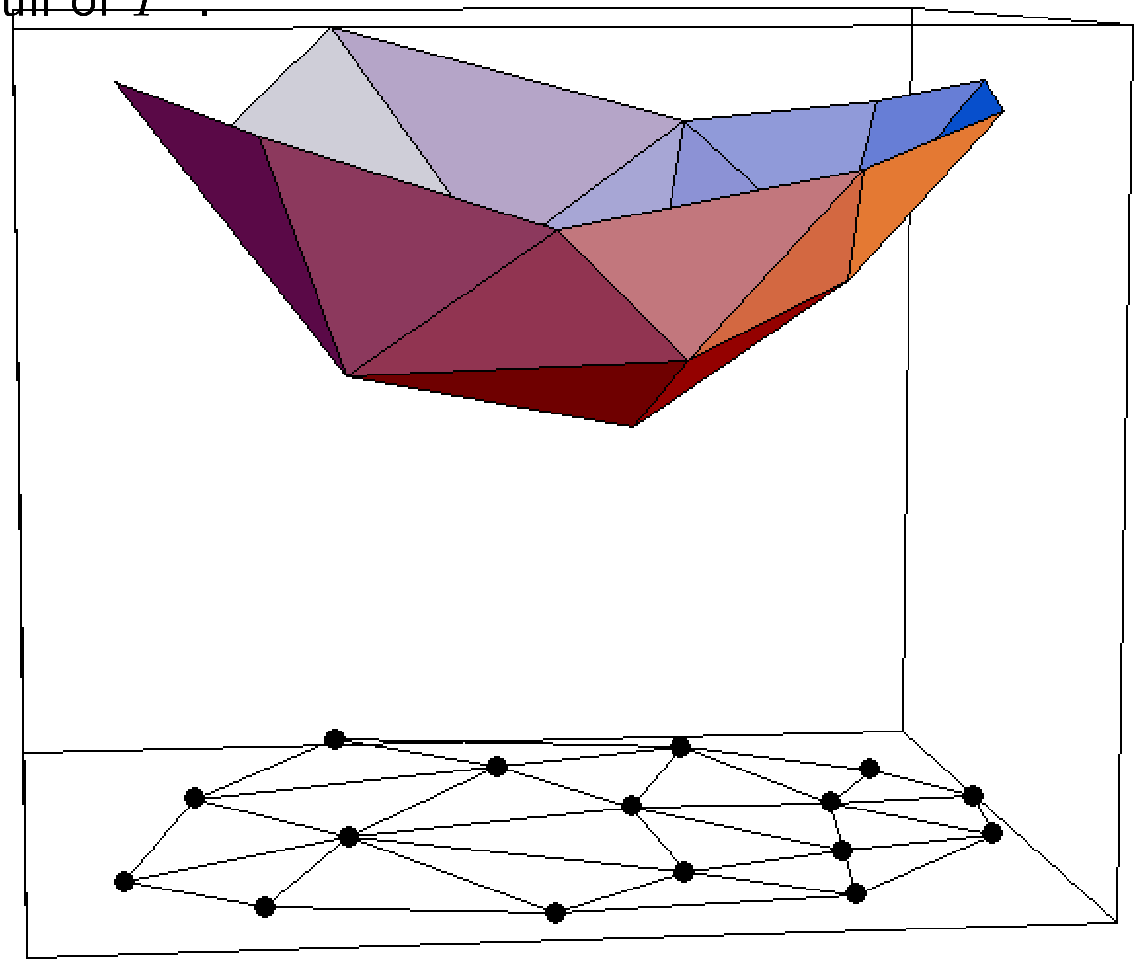

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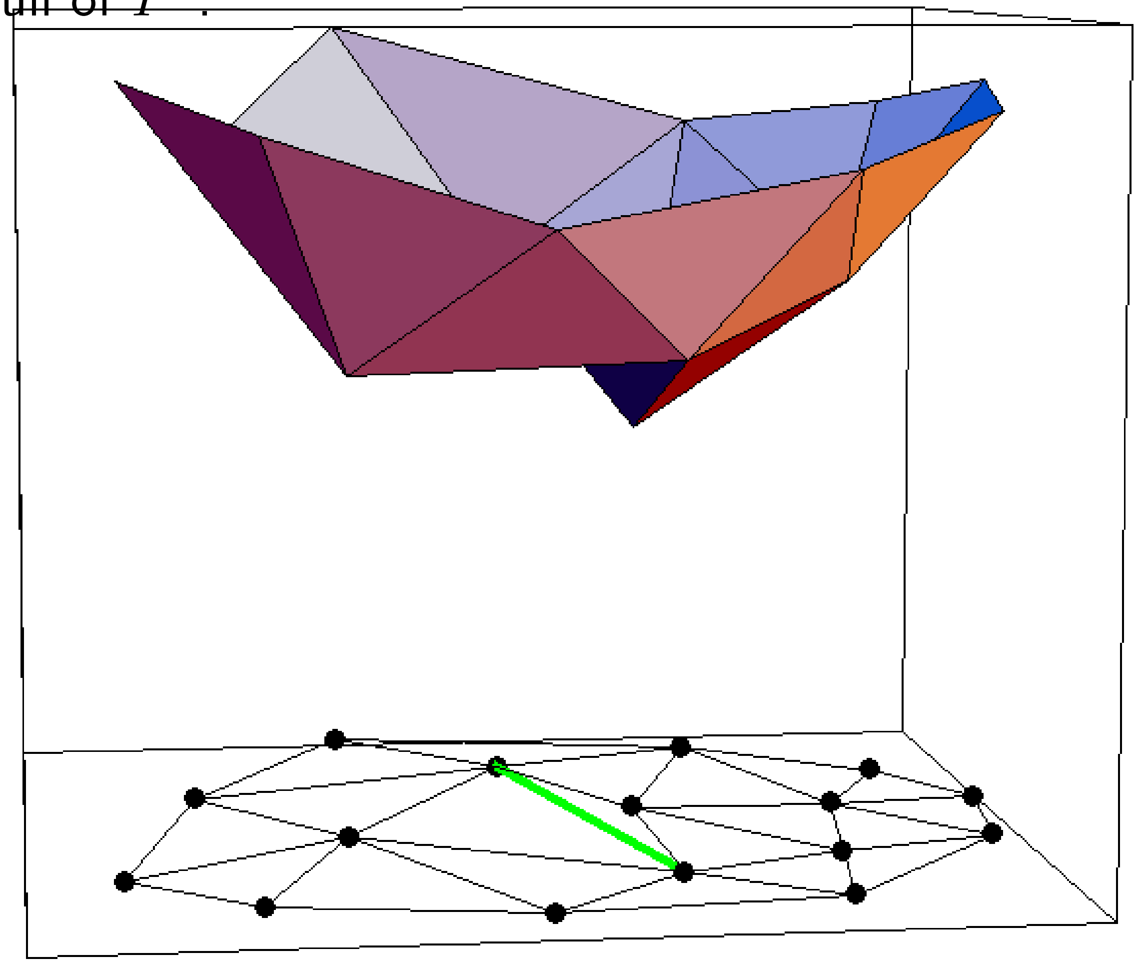

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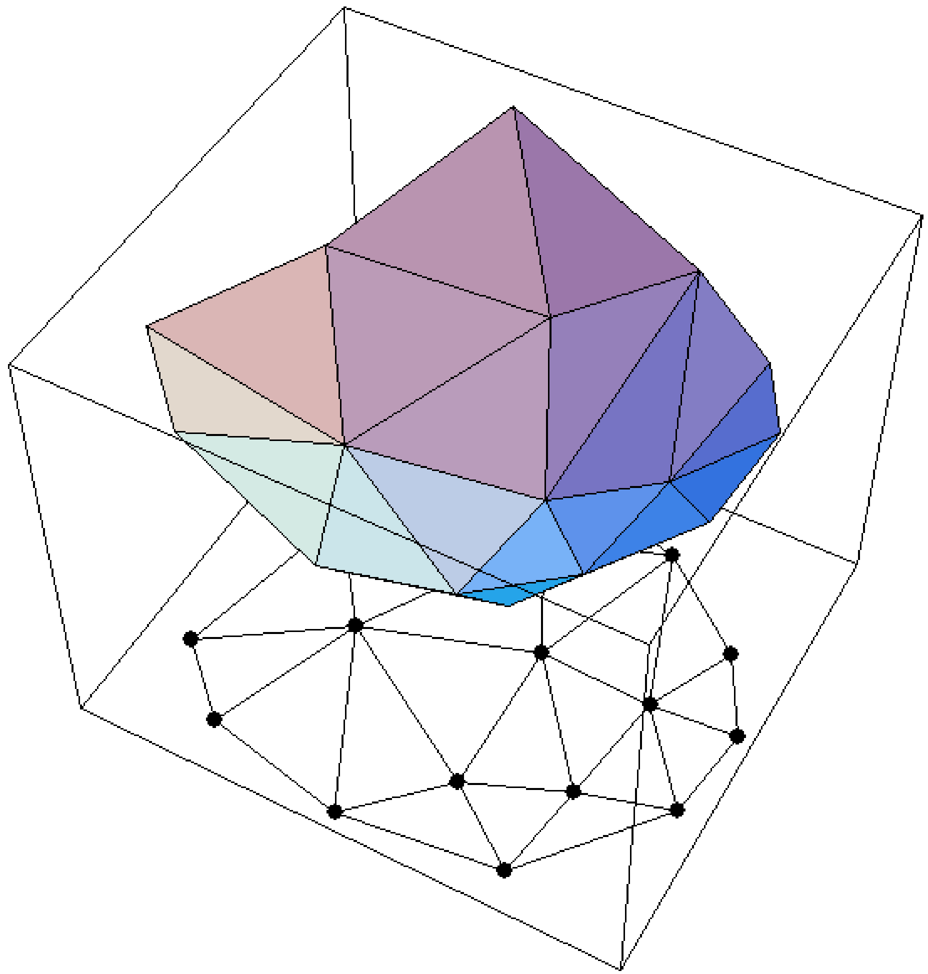

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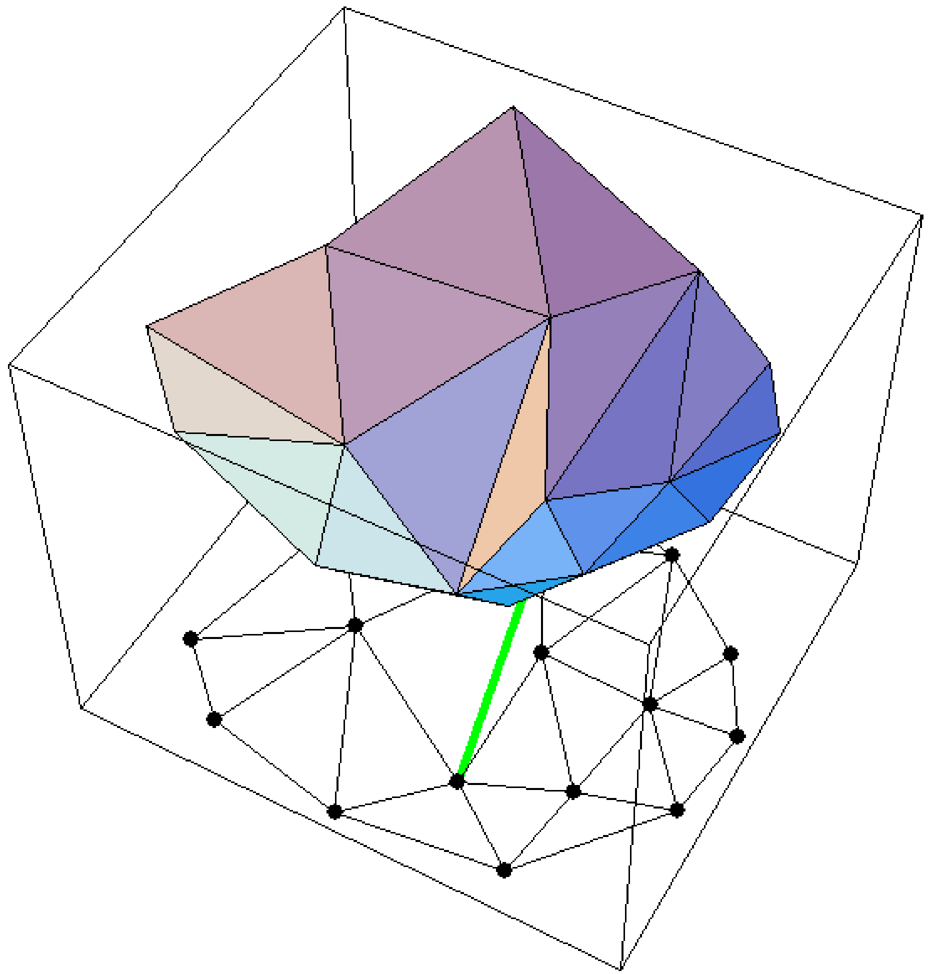

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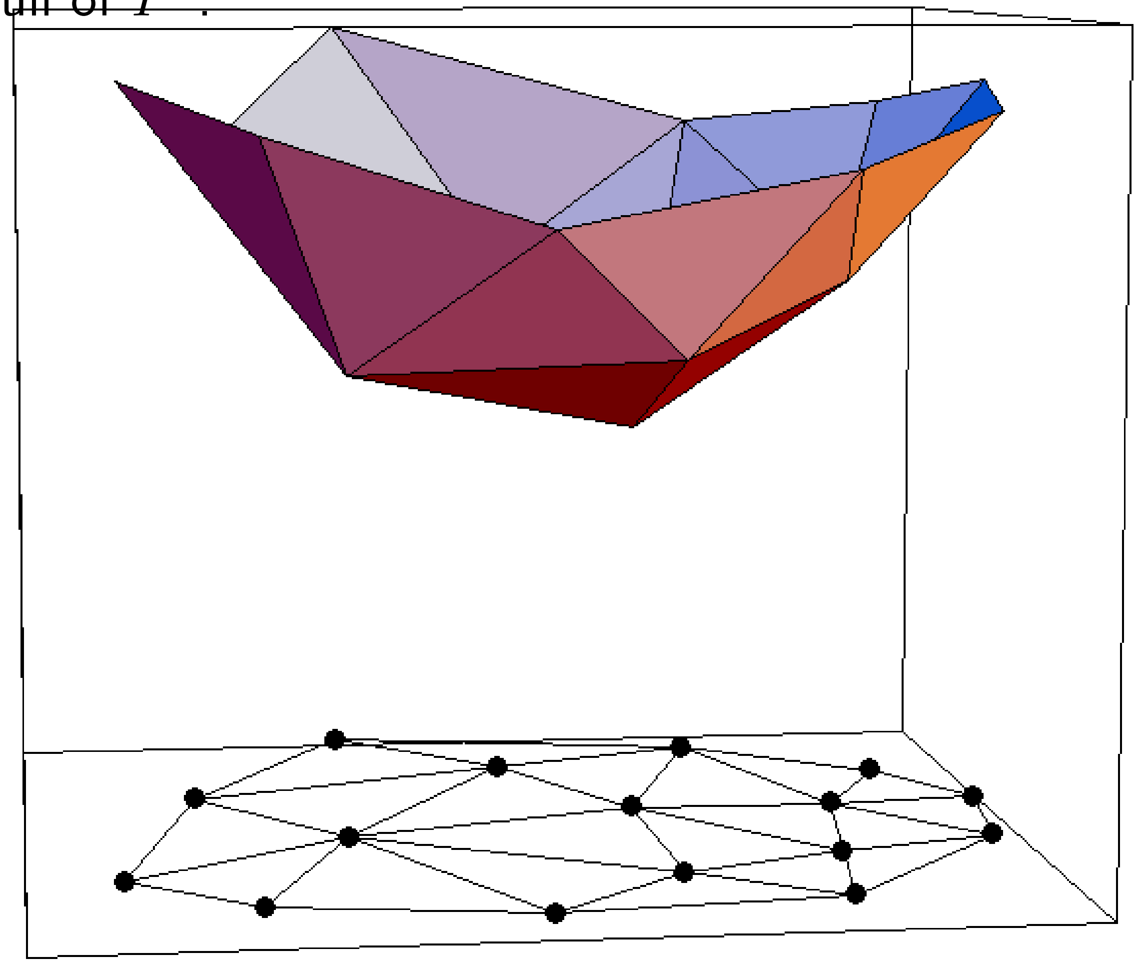

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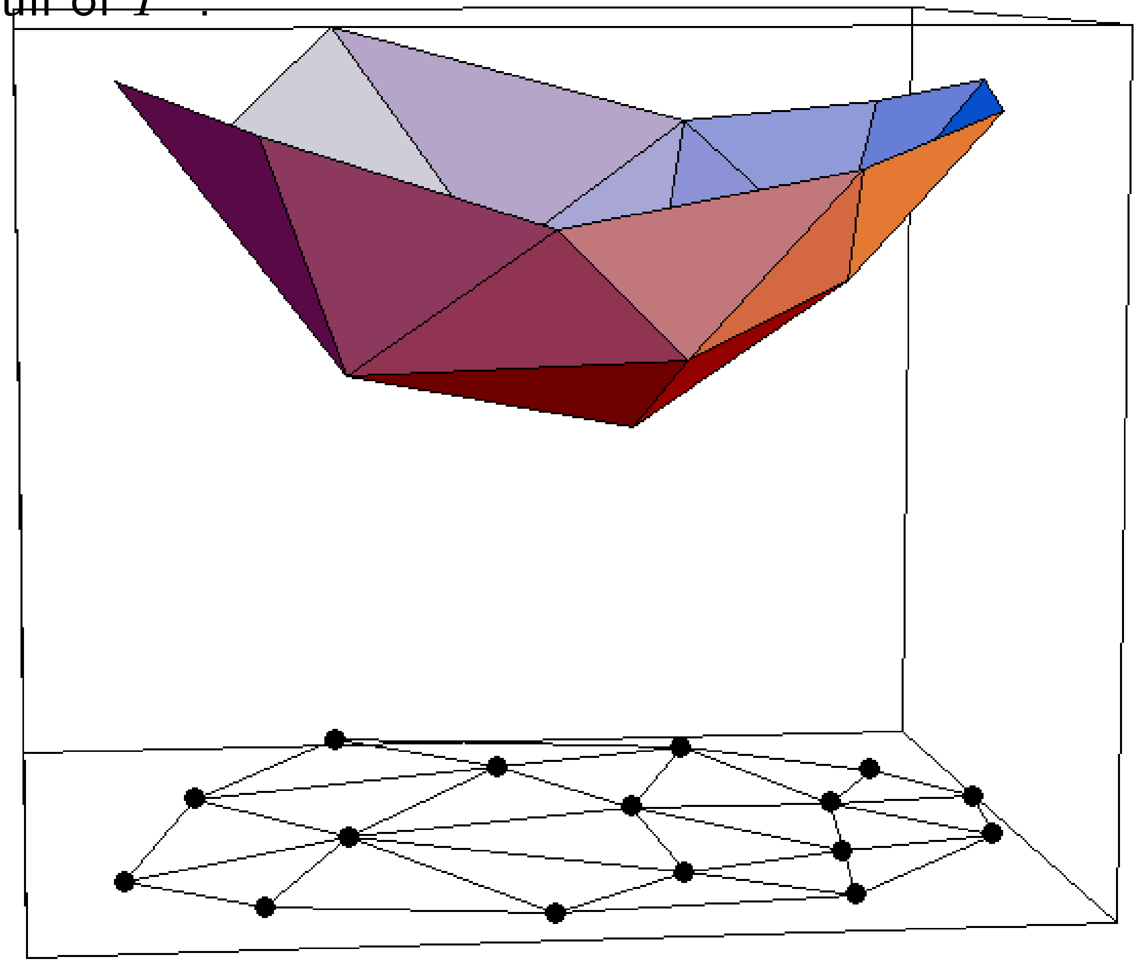
p_i^*, p_j^*, p_k^* form a (triangular) face of the lower convex hull of P^*



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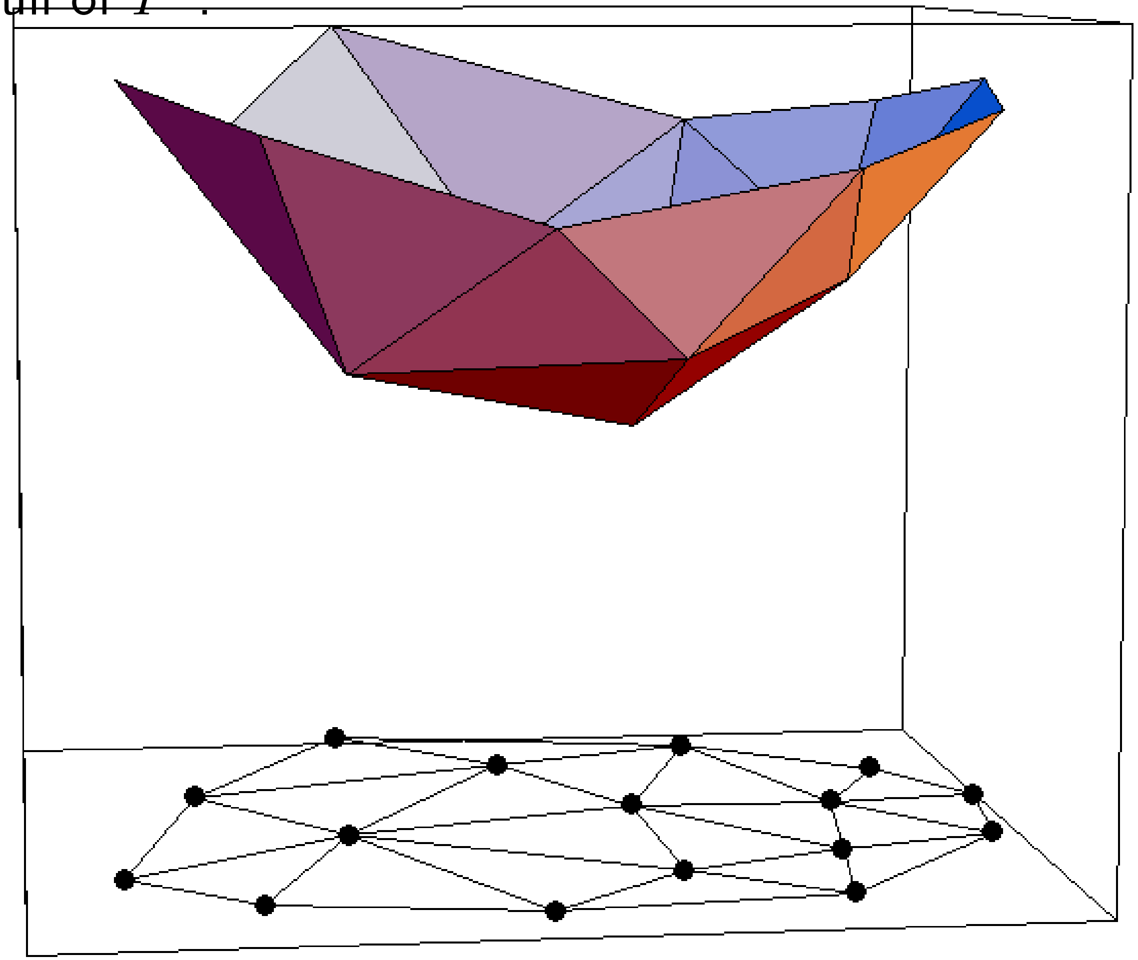
$$\Updownarrow$$

The plane through p_i^*, p_j^*, p_k^* leaves all the remaining points of P^* above it



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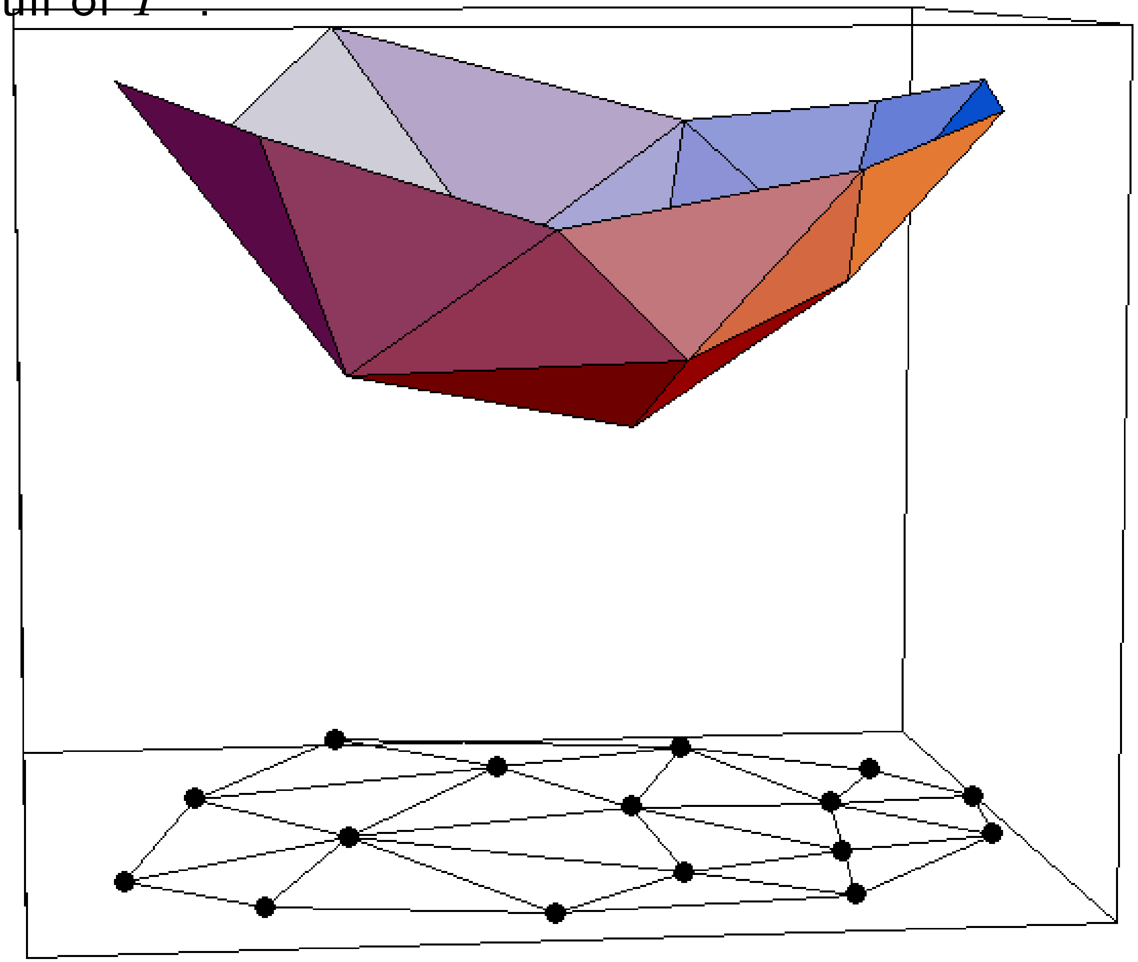
The circle through p_i, p_j, p_k leaves all the remaining points of P in its exterior



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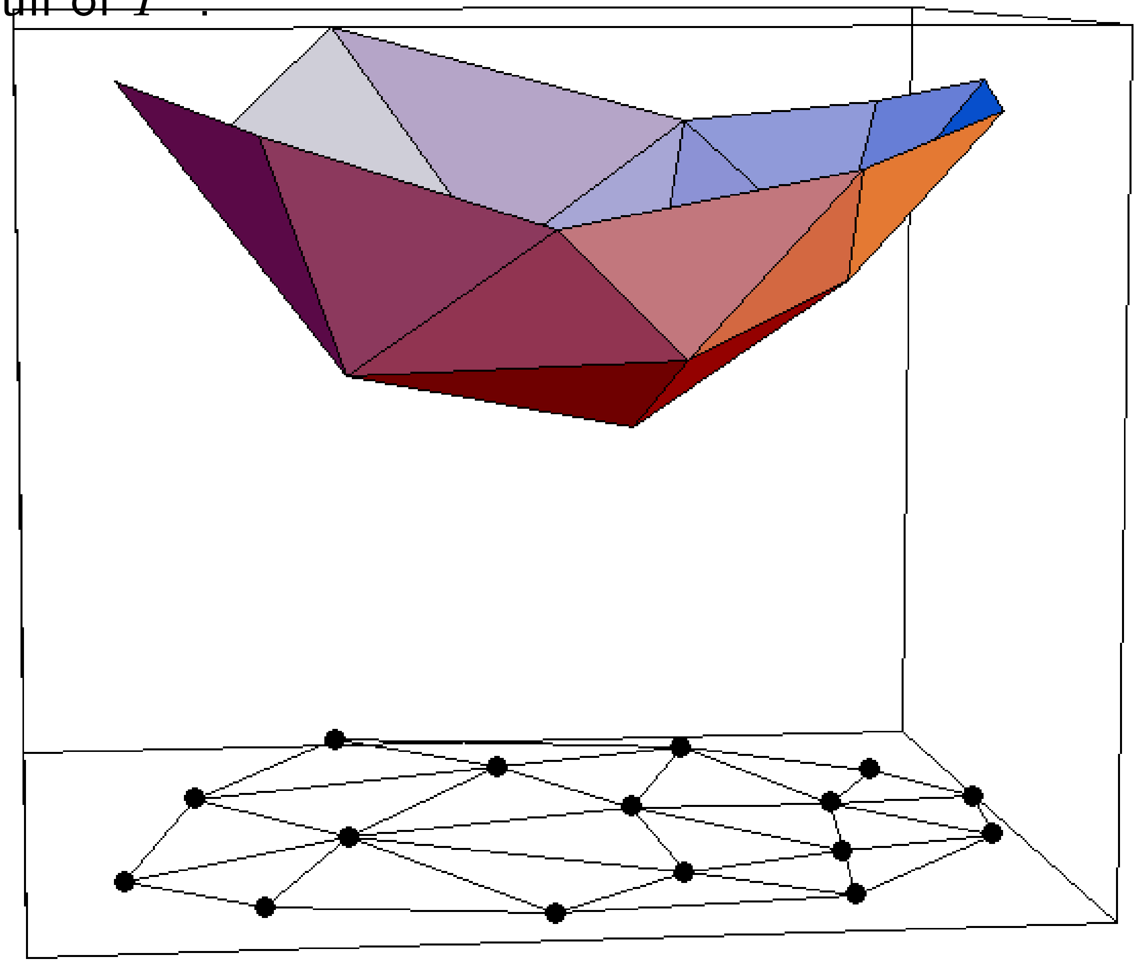
p_i, p_j, p_k form a triangle of $Del(P)$



DELAUNAY TRIANGULATION

LOCAL CHARACTERIZATION

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Definition

A triangulation $T(P)$ is locally Delaunay if each pair of triangles $p_ip_jp_k$ and $p_ip_jp_l$ sharing an edge p_ip_j satisfies $p_l \notin C_{ijk}$ and $p_k \notin C_{ijl}$.

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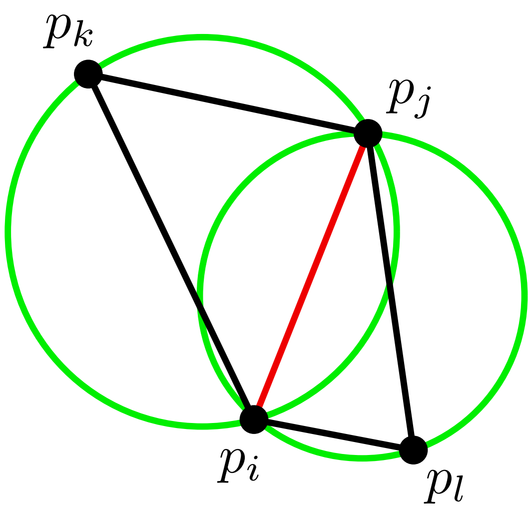

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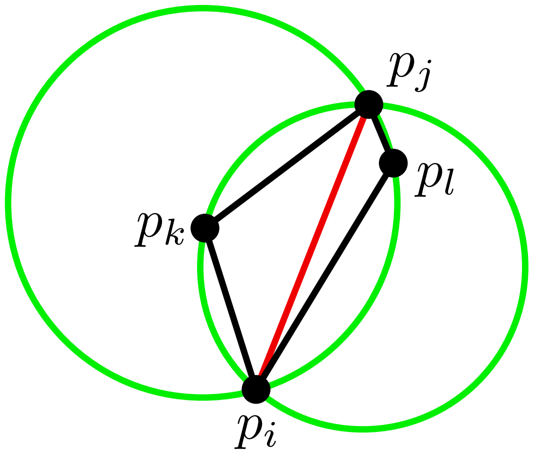
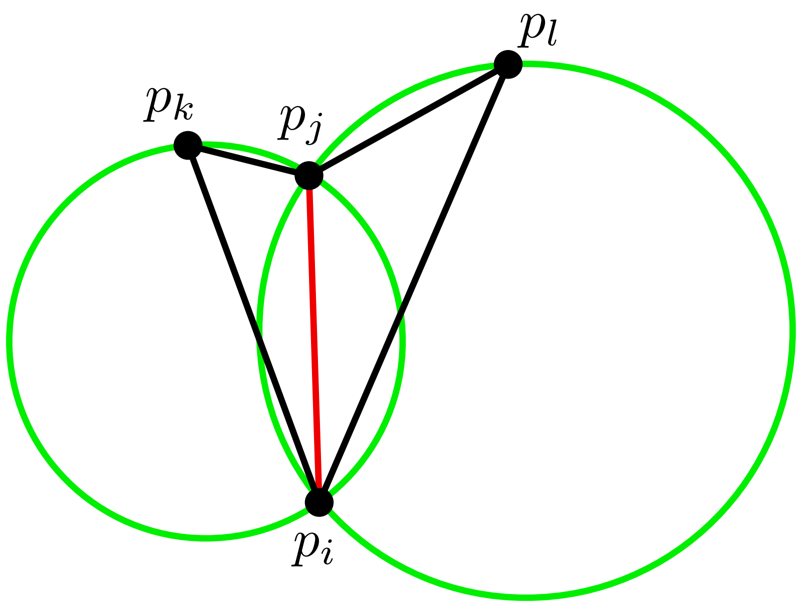


The edge $p_i p_j$ is not locally Delaunay



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Note that the quadrilateral $p_i p_l p_j p_k$ is not convex

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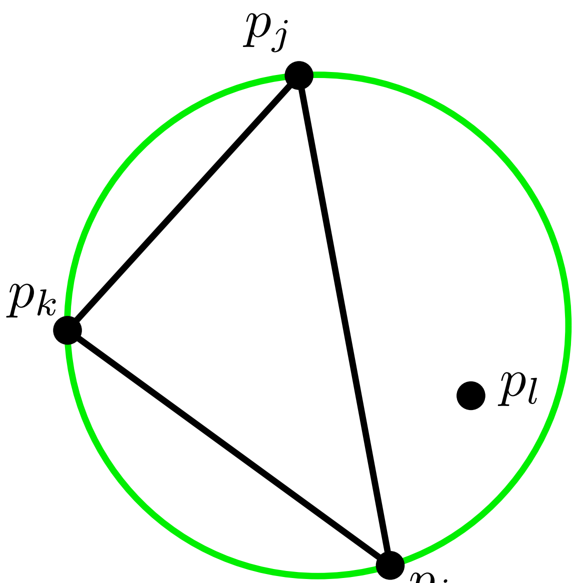

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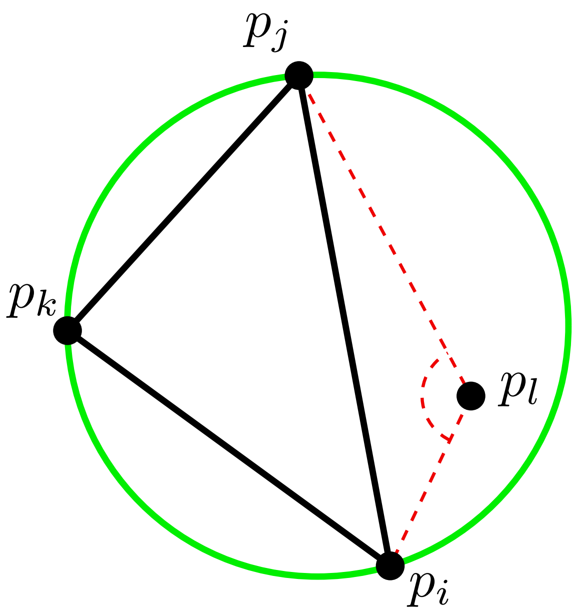

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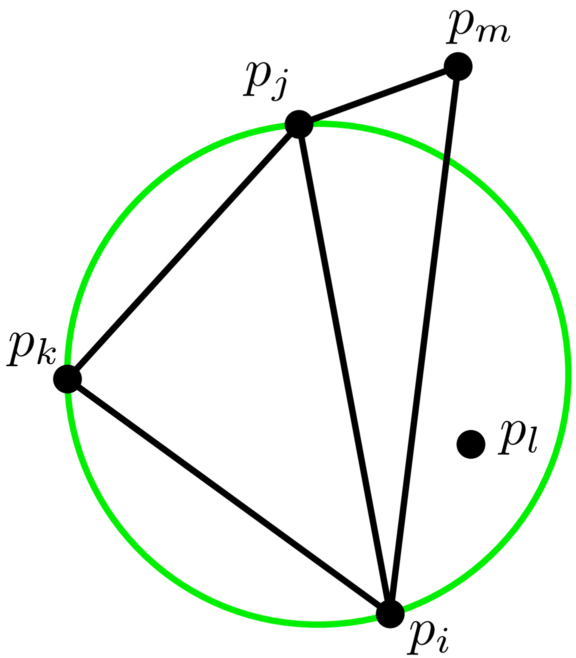

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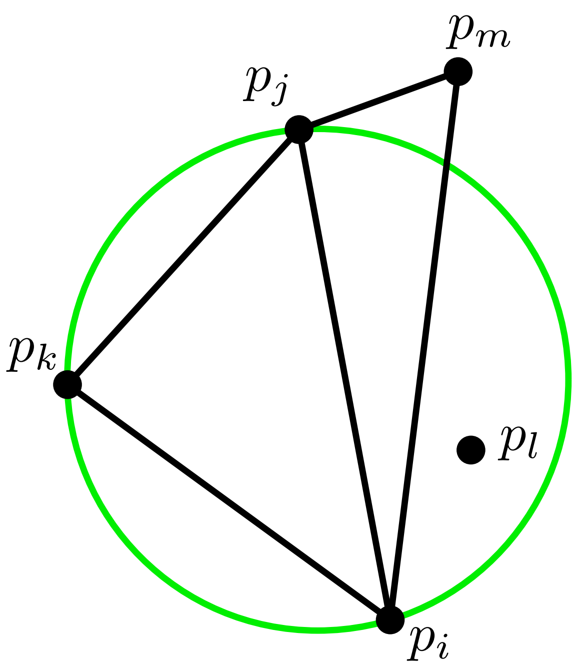

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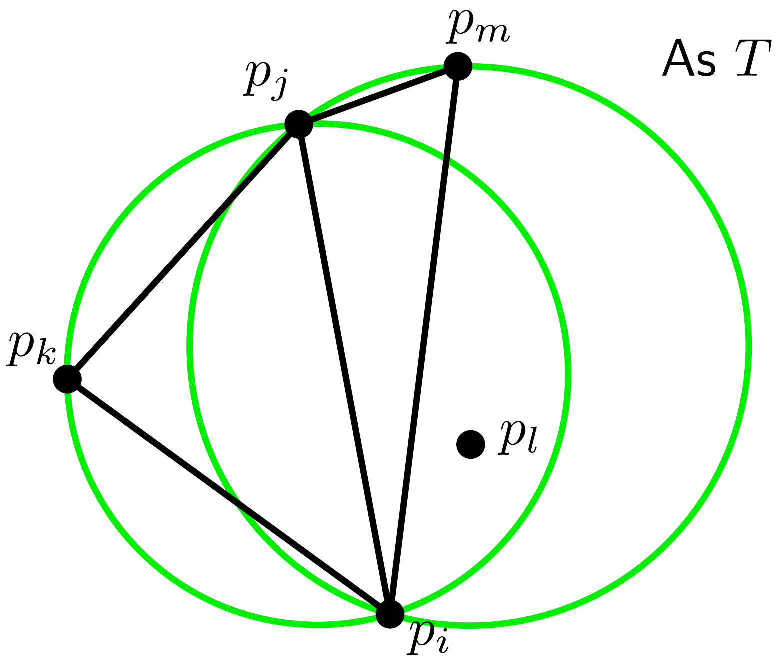

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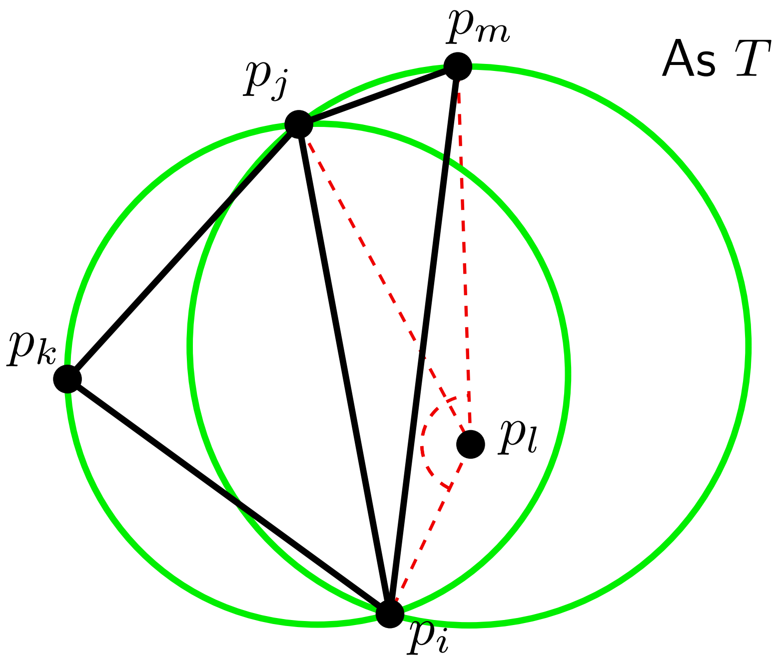

As $T(P)$ is locally Delaunay, $m \neq l$.

Then $p_l \in C_{ijm}$.

Hence, one of the angles $p_i p_l p_m$ or $p_j p_l p_m$ would be greater than $p_i p_l p_j$.

DELAUNAY FLIPS

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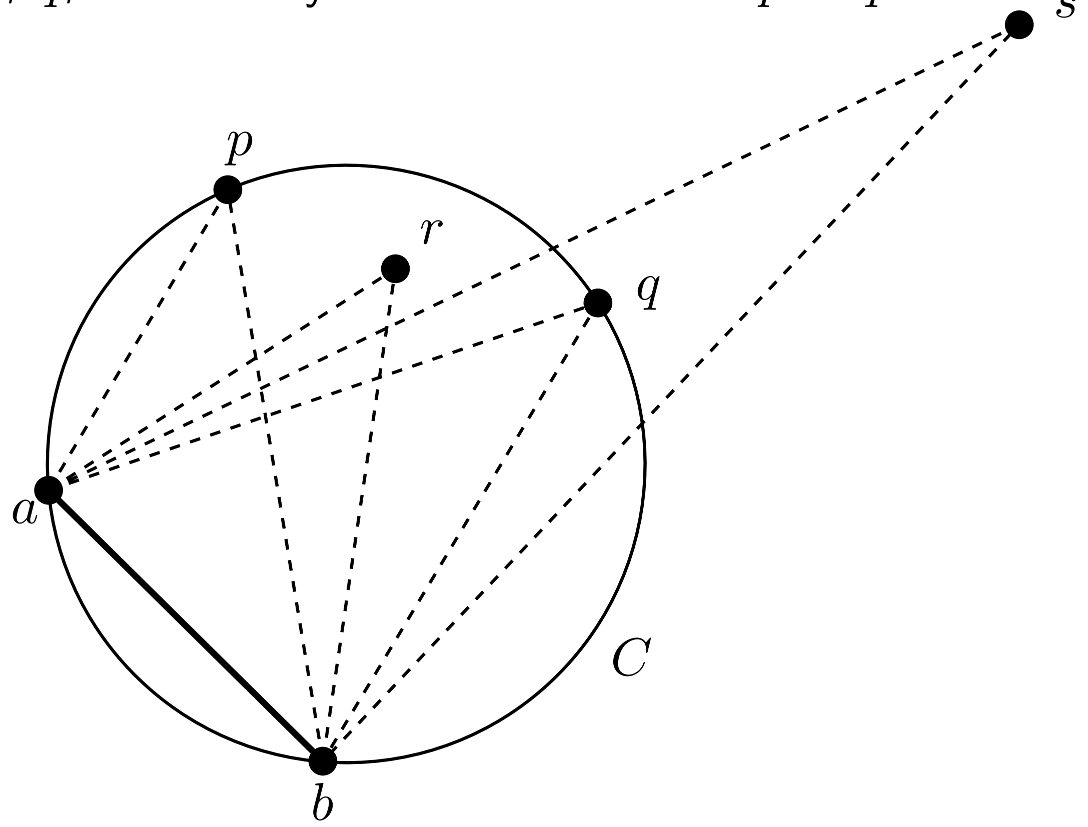

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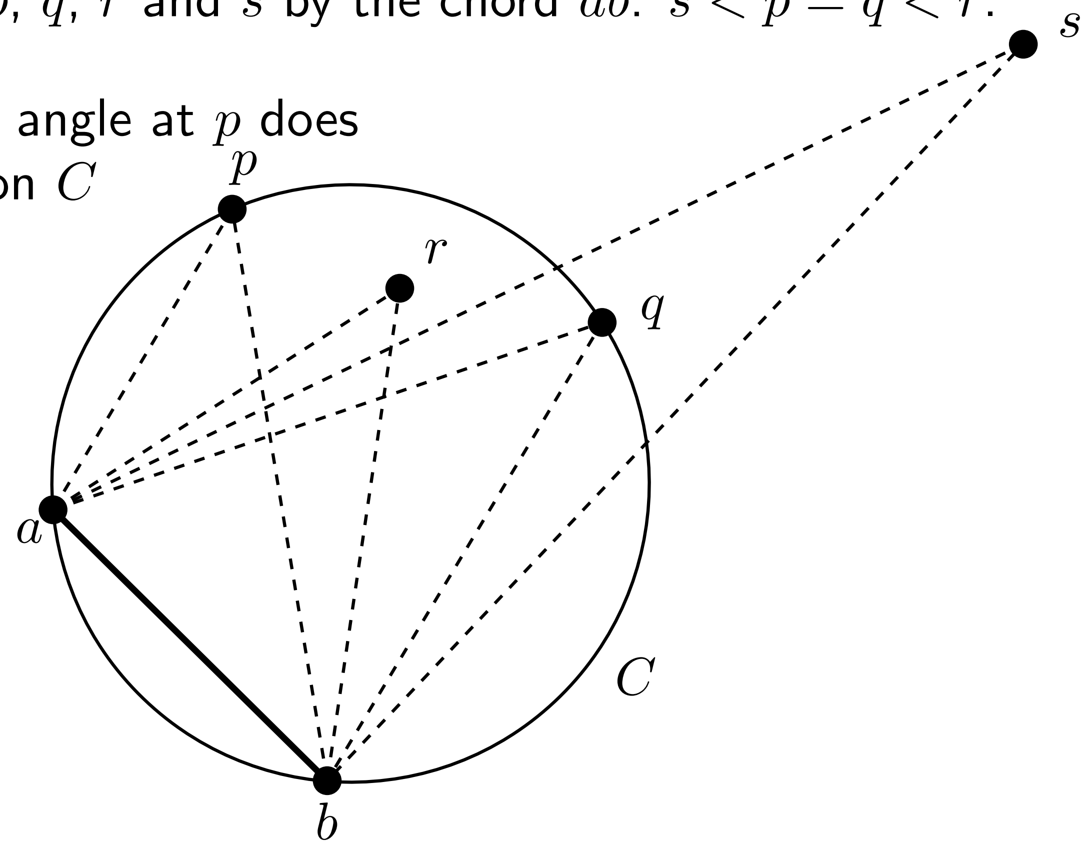

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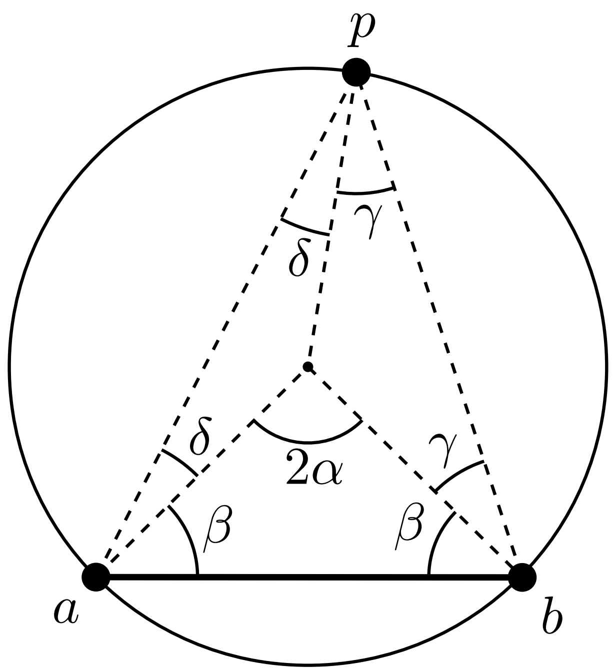

First case:

$$\left.\begin{aligned} 2\delta + 2\gamma + 2\beta &= \pi \\ 2\alpha + 2\beta &= \pi \end{aligned}\right\} \Rightarrow 2\alpha = 2\gamma + 2\delta \Rightarrow \alpha = \gamma + \delta \Rightarrow \widehat{p} = \widehat{q} = \alpha$$

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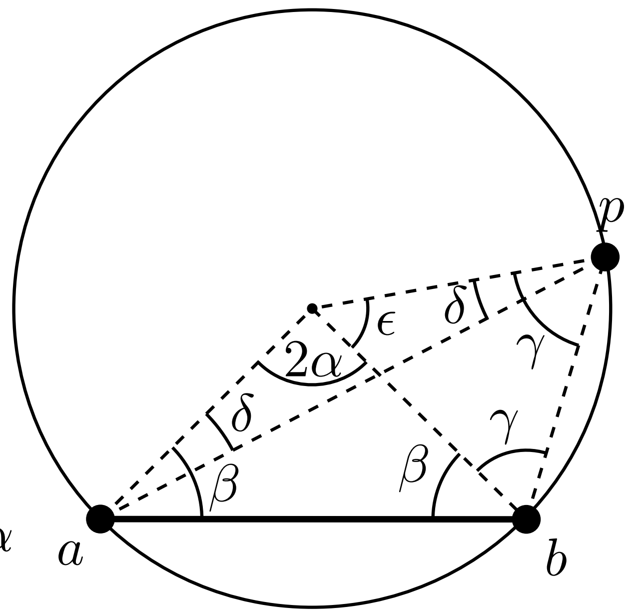

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Second case:

$$\left.\begin{aligned} 2\alpha + \epsilon + 2\delta &= \pi \\ 2\gamma + \epsilon &= \pi \end{aligned}\right\} \Rightarrow 2\alpha + 2\delta - 2\gamma = 0 \Rightarrow \alpha = \gamma - \delta \Rightarrow \widehat{p} = \widehat{q} = \alpha$$

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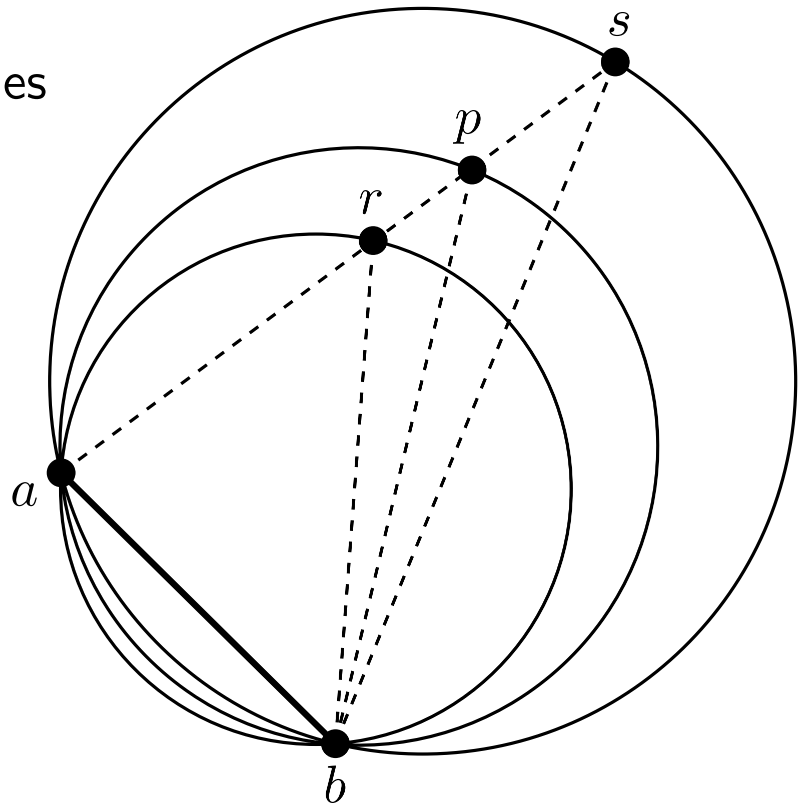

The remaining relations follow immediately.

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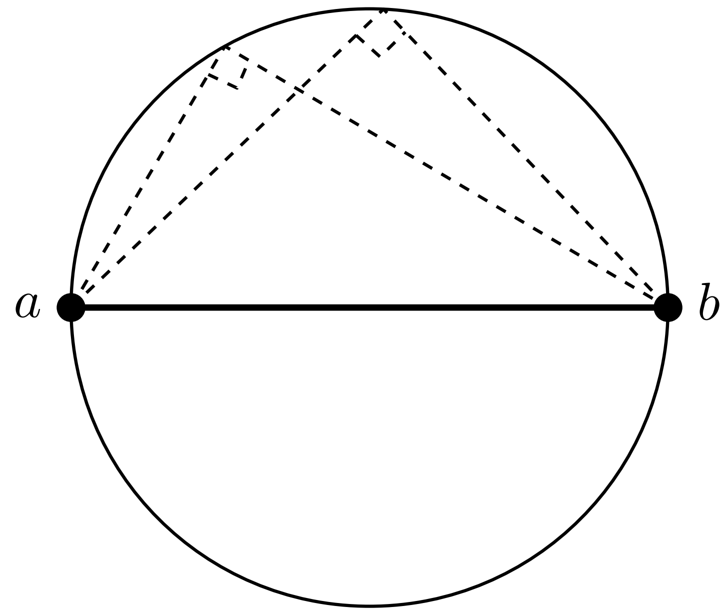
Since in this case $2\alpha = \pi$.



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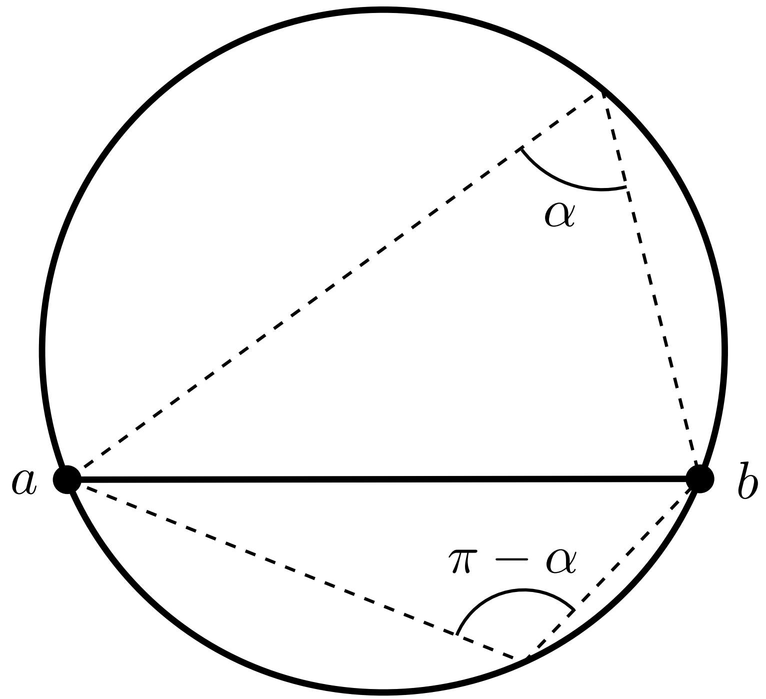

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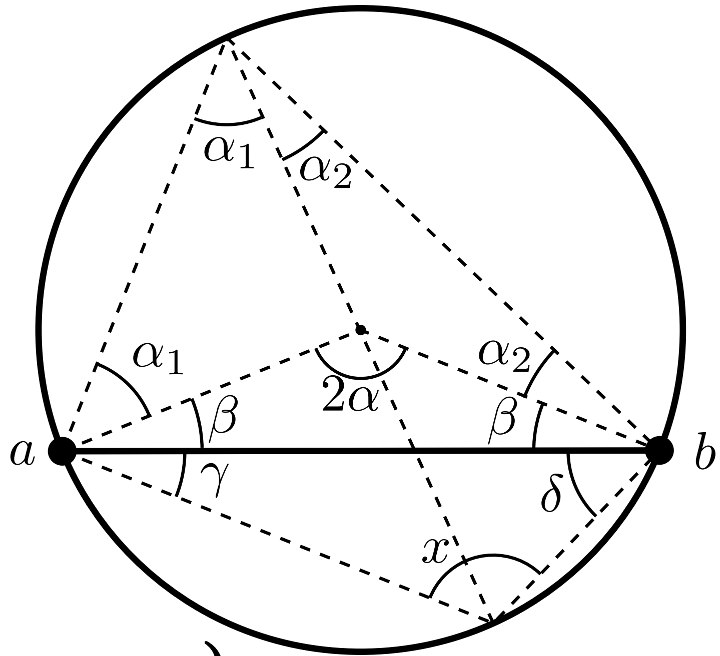

$$\left.\begin{array}{r}\left.\begin{array}{r}\left.\begin{array}{r}\alpha_1+\beta+\gamma=\frac{\pi}{2}\\ \alpha_2+\beta+\delta=\frac{\pi}{2}\end{array}\right\}\Rightarrow \alpha+2\beta+\gamma+\delta=\pi\\ x+\gamma+\delta=\pi\end{array}\right\}\Rightarrow x=\alpha+2\beta\\ 2\alpha+2\beta=\pi\end{array}\right\}\Rightarrow x=\pi-\alpha$$

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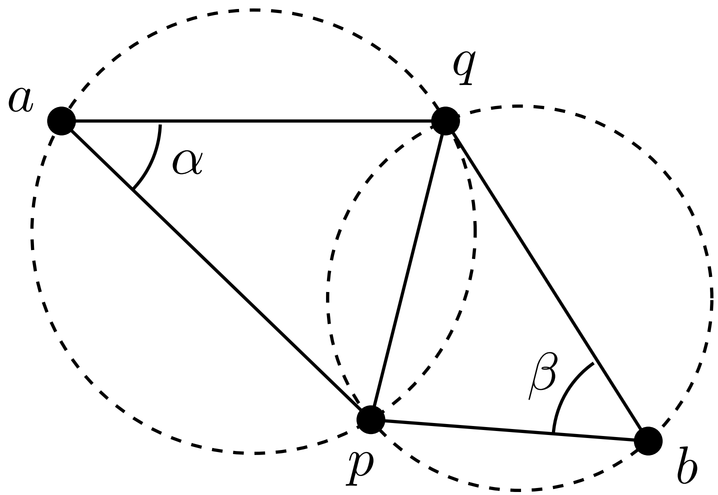
Lemma 4. Let $\overline{pq}$ be the common edge of the triangles pqa and pqb, forming a convex quatrilateral. Then:

$$a \in ext(C_{pqb}) \iff b \in ext(C_{pqa})$$



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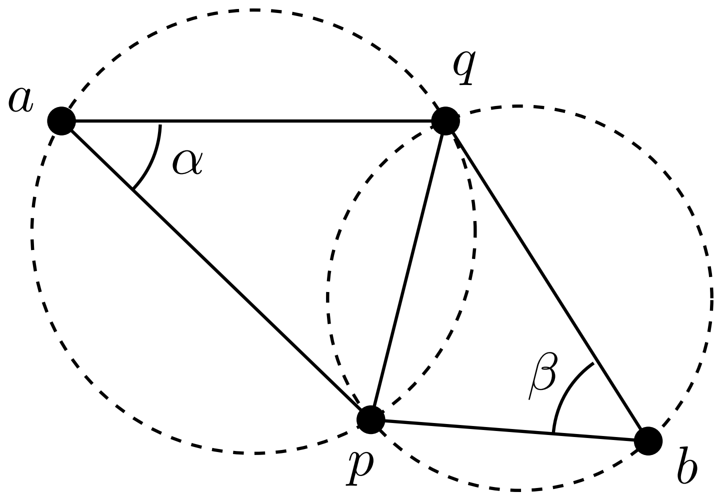

$$a \in ext(C_{pqb}) \iff \alpha < \pi - \beta \iff \beta < \pi - \alpha \iff b \in ext(C_{pqa})$$

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Lemma 5. Consider a convex quadrilateral with diagonals $\overline{ab}$ and $\overline{pq}$. Then:

$$\overline{ab} \text{ is not locally Delaunay} \Longleftrightarrow \overline{pq} \text{ is locally Delaunay}$$

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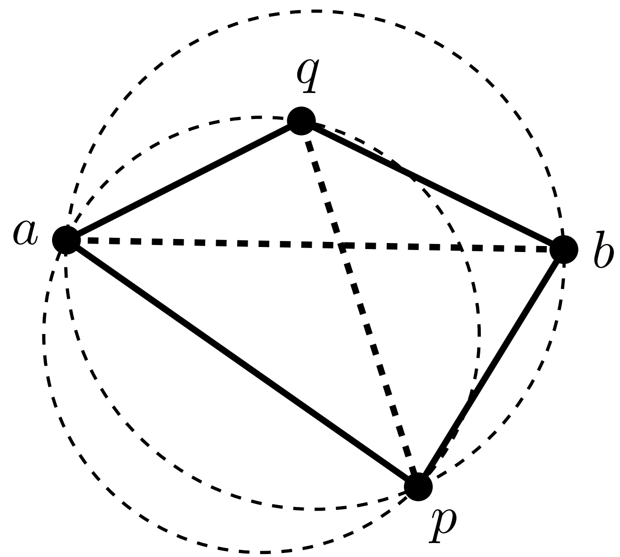
$\overline{ab}$ is not locally Delaunay

$\iff q \in int(C_{abp})$

$\iff \widehat{aqp} > \widehat{abp}$

$\iff b \in ext(C_{apq})$

$\iff \overline{pq}$ is locally Delaunay



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Observation. Let P be a set of points $p_i = (x_i, y_i, 0)$ in the plane, and let P^* be the set of their vertical projections $p^* = (x_i, y_i, x_i^2 + y_i^2)$ onto the unit paraboloid. Producing a Delaunay flip in a triangulation of P corresponds to "sticking" a tetrahedron from below to the corresponding polyhedrization of P^*.

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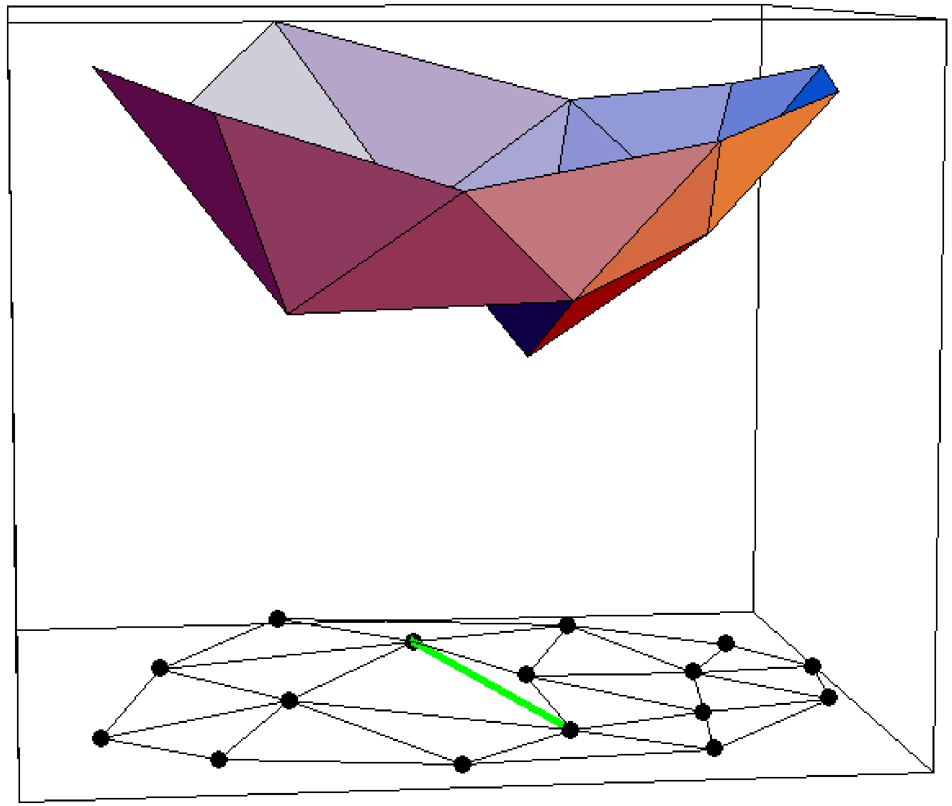

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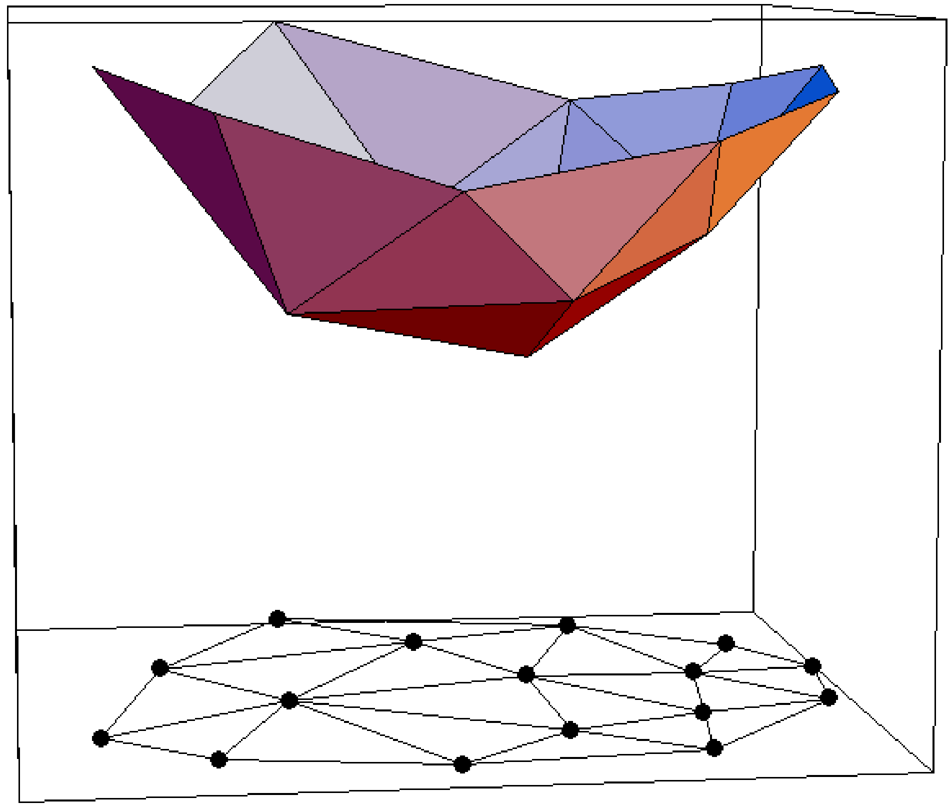

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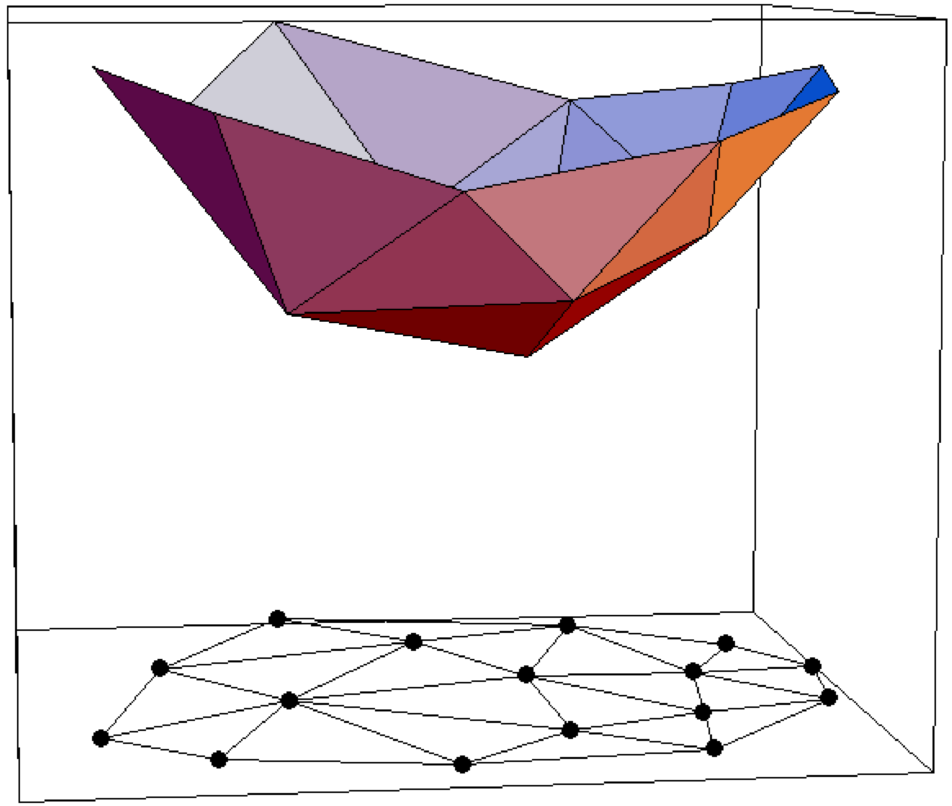
Once flipped, the quatrilateral is locally Delaunay: the fourth point lies in the exterior of the circumcircle of the triangle.

In the paraboloid, this means that the fourth point lies above the triangular face of the polyhedrization.



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Corollary. Given any triangulation of P, performing locally Delaunay flips is a procedure converging to $Del(P)$.

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3. Incremental algorithm

Compute an enclosing triangle for $\{p_1, \ldots, p_n\}$

Compute $Del(p_1, \ldots, p_{i+1})$ from $Del(p_1, \ldots, p_i)$

INCREMENTAL ALGORITHM

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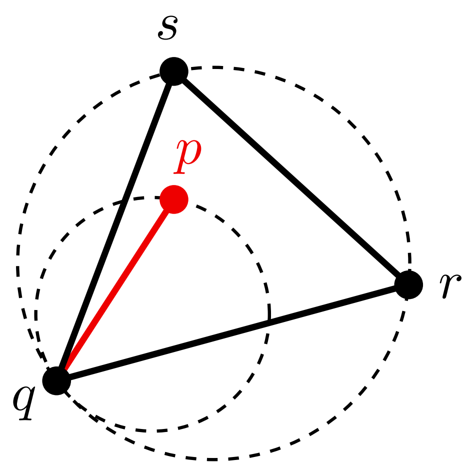
As C_{qrs} is empty, there exist empty circles C_{pq}, such as the circle through p and q tangent to C_{qrs} in q. Similarly for r and s.



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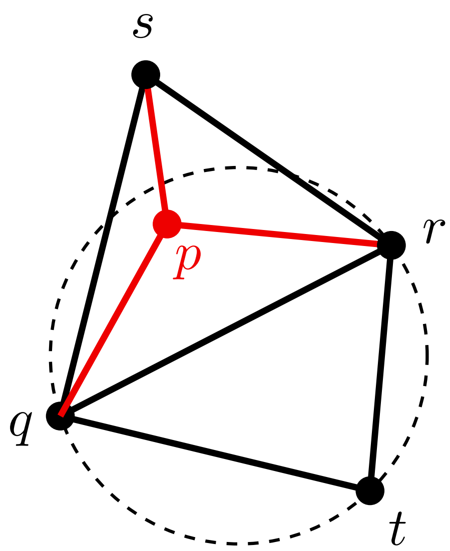
Since p may lie in the interior of C_{qrt}.



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> This holds because the property is local: it affects only quadrilaterals formed by two triangles sharing an edge.

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Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still edges of triangles incident to p that are non-locally Delaunay, flip them.

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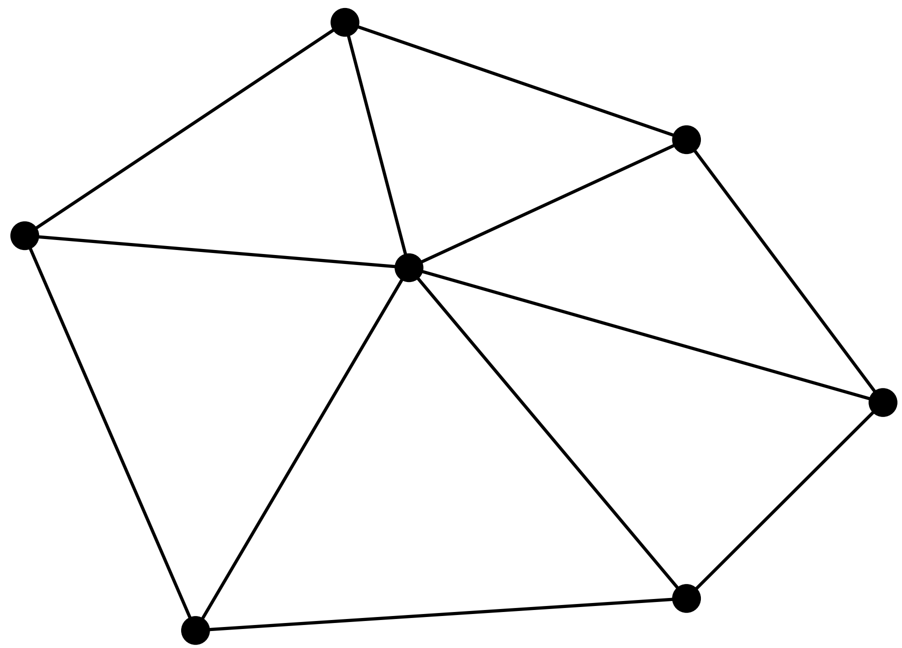

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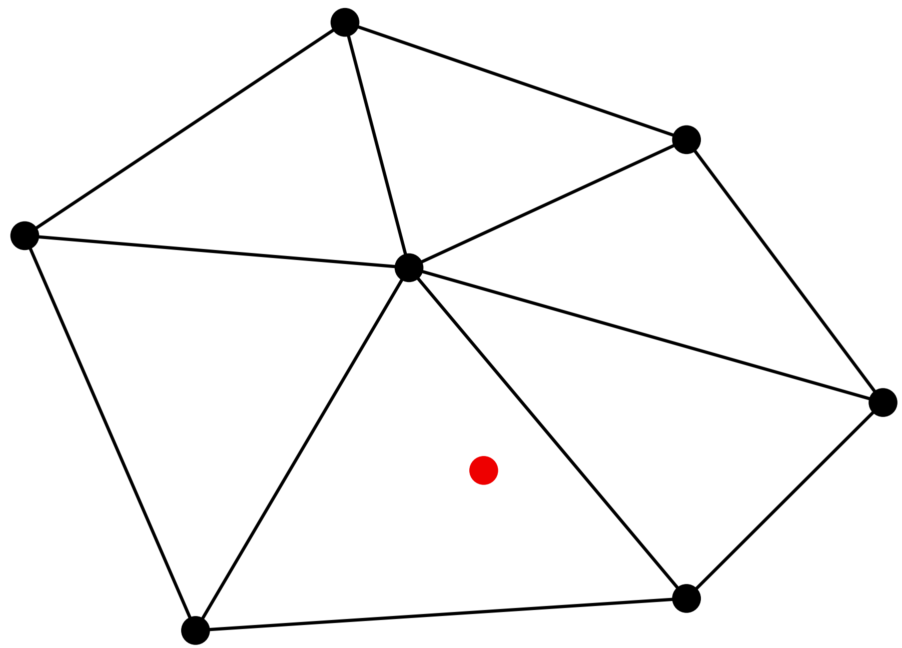

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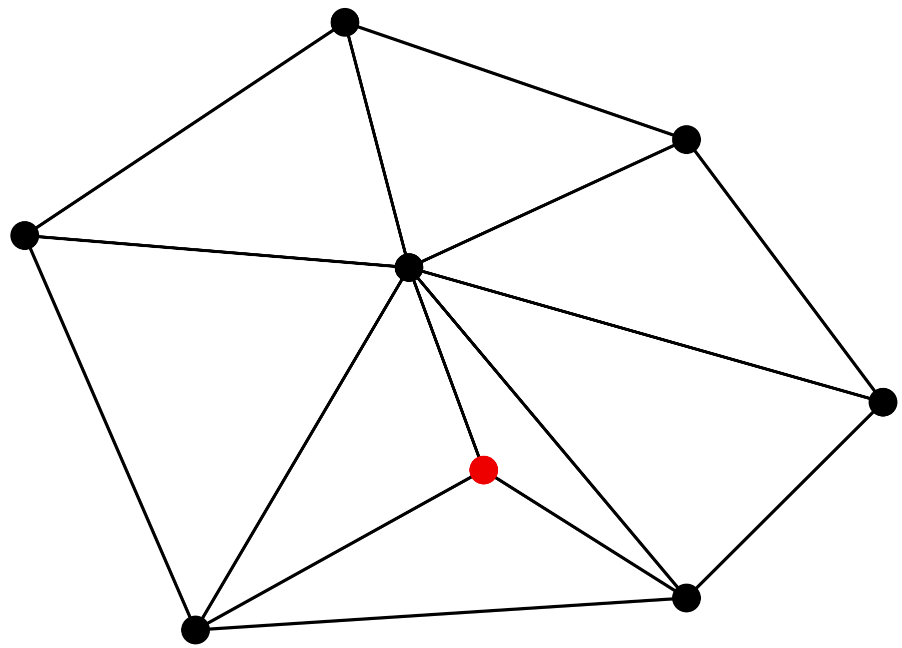

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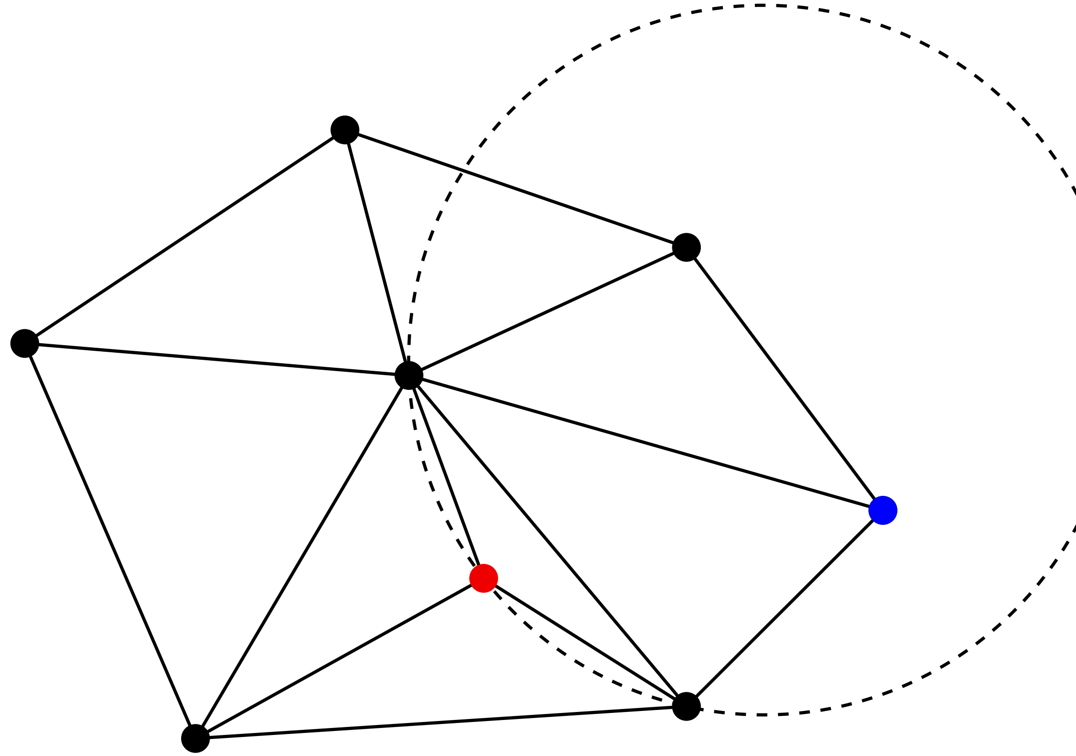

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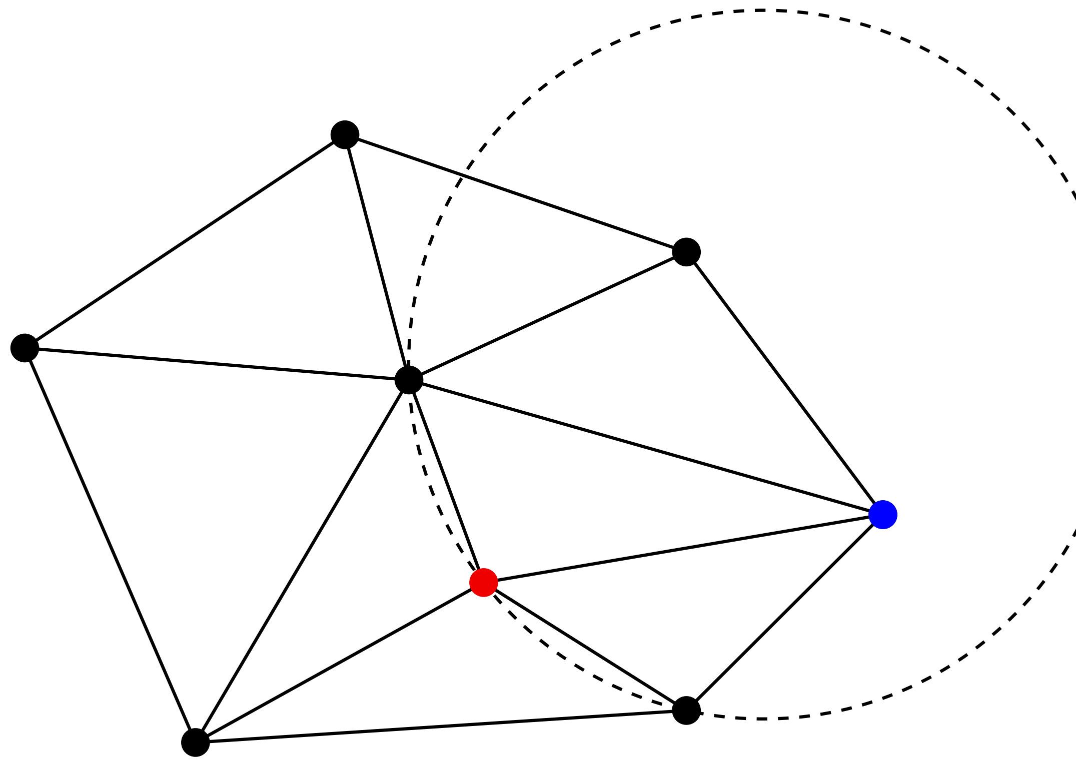

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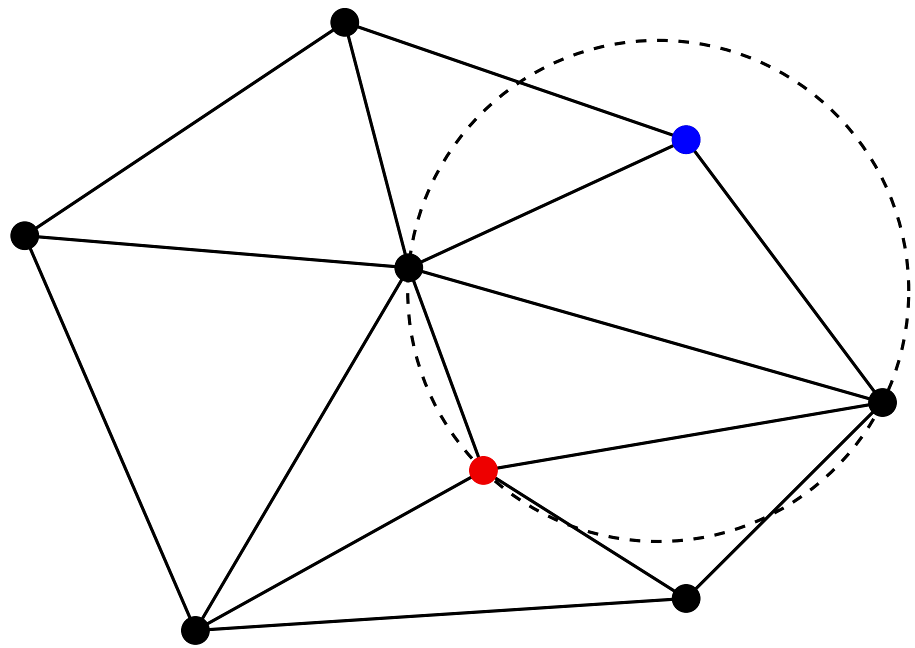

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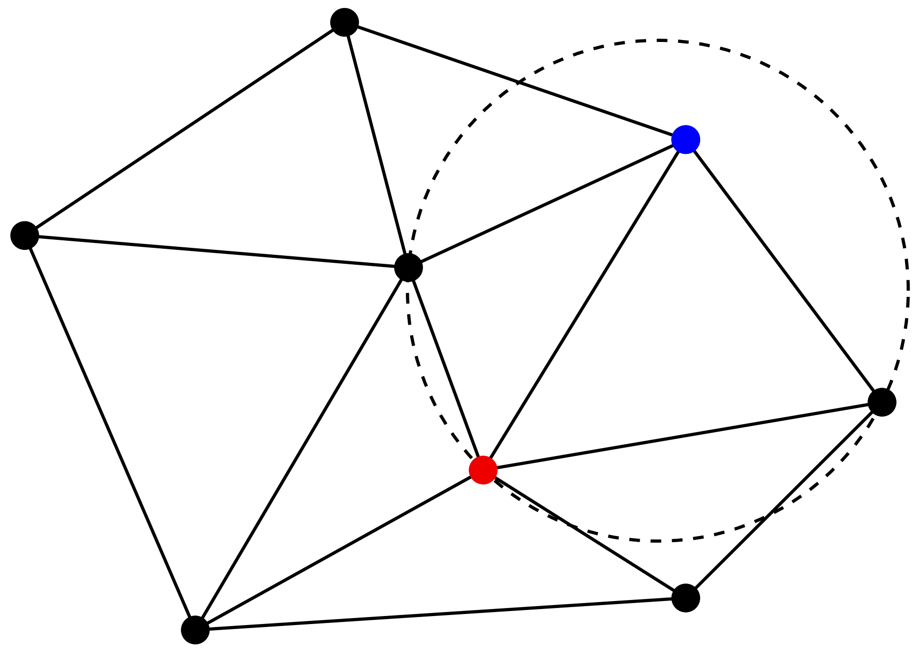

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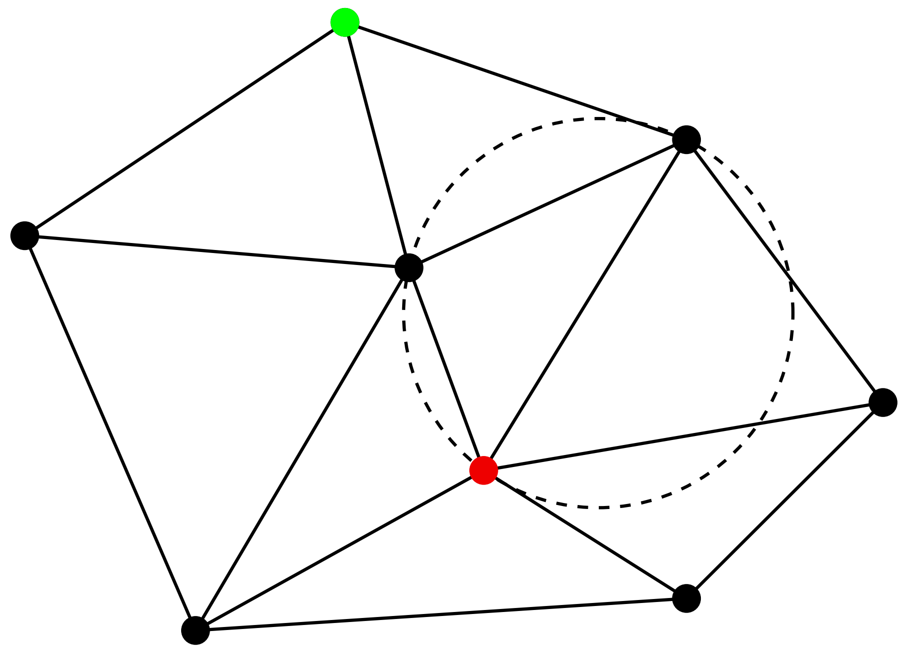

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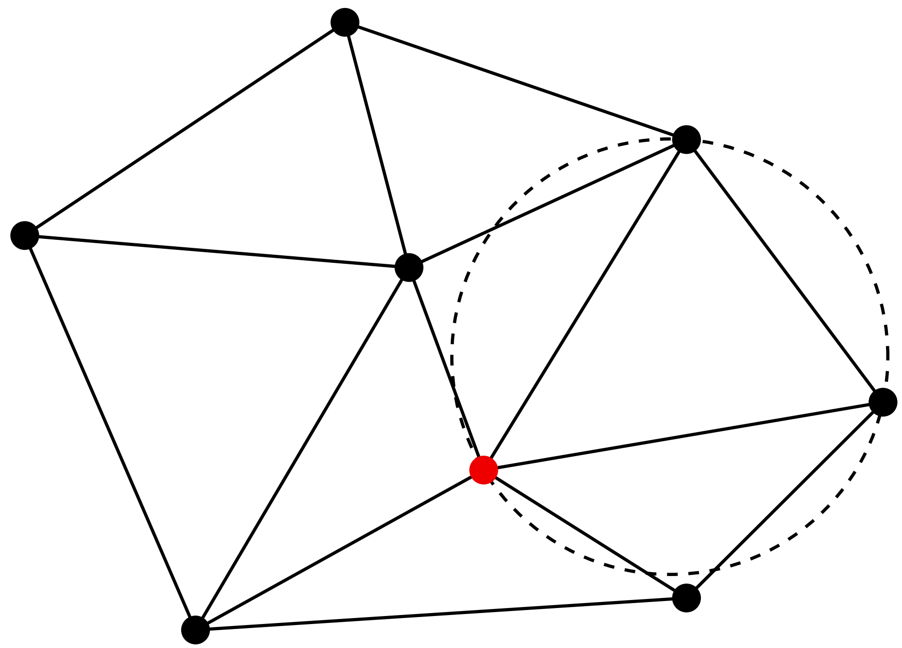

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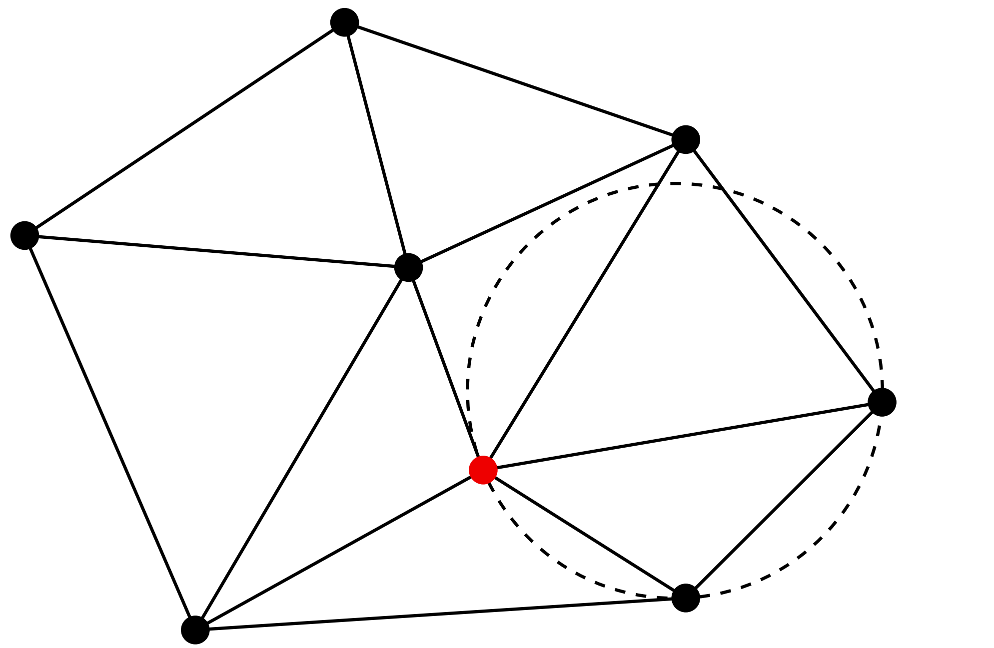

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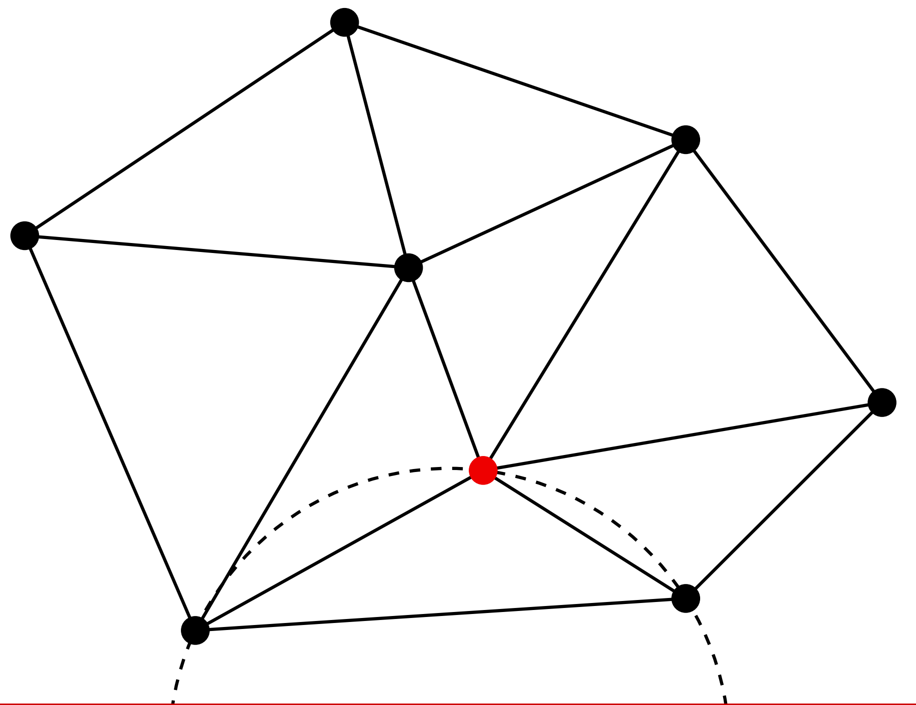

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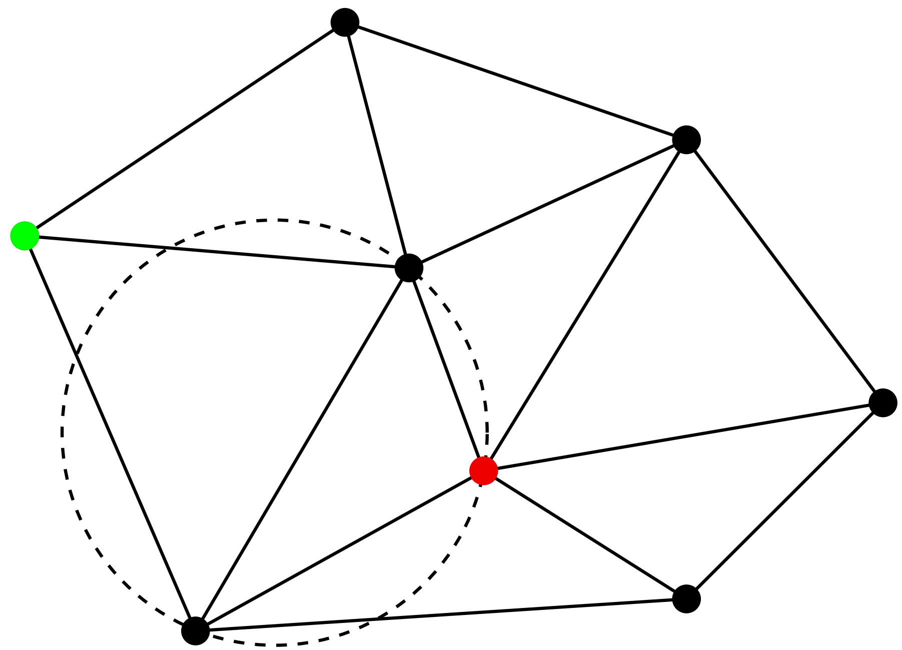

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Worst case: $O(n^2)$ time. However, as average degree is smaller than 6, the **expected** running time, if points are added in random order, is $O(n \log n)$.

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The proof of this statement requires a last lemma.

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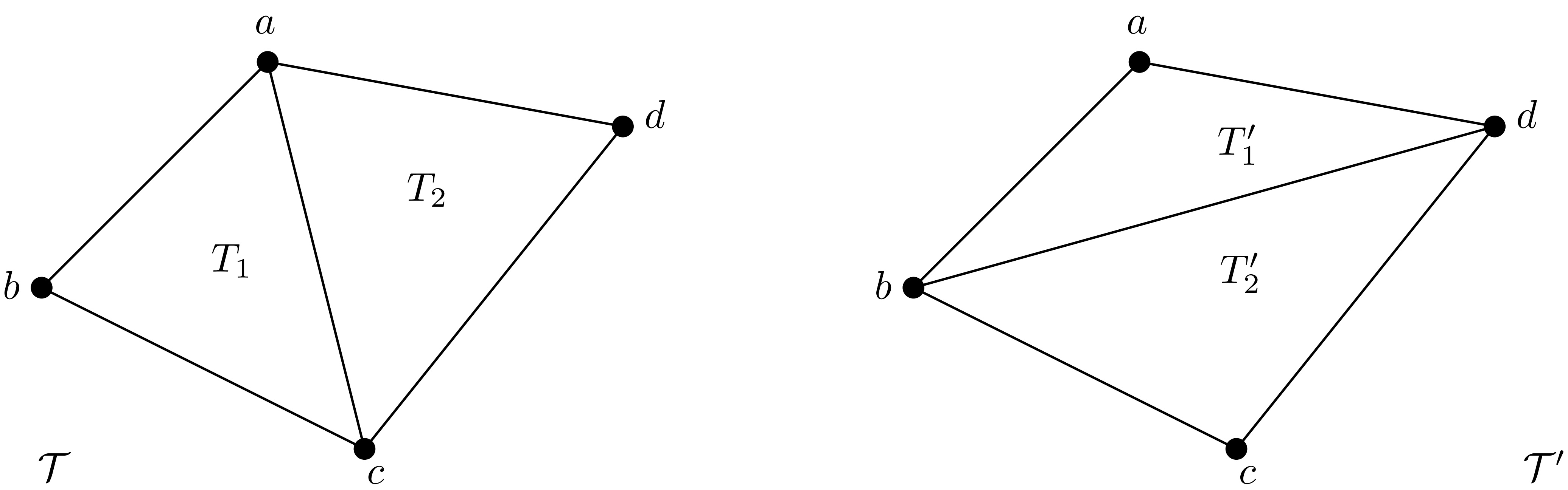
Lemma 7. Let a, b, c and d be four points forming a convex quadrilateral, in counterclockwise order. Let $\mathcal{T}$ and $\mathcal{T}'$ be the two possible triangulations of the quadrilateral: $\mathcal{T}$ uses the diagonal $\overline{ac}$ and $\mathcal{T}'$ uses $\overline{bd}$. Let ϵ and ϵ' respectively be the minimum angles of $\mathcal{T}$ and $\mathcal{T}'$. Then:

$$\epsilon > \epsilon' \iff d \in ext(C_{abc})$$

$$\epsilon = \epsilon' \iff d \in \partial(C_{abc})$$

$$\epsilon < \epsilon' \iff d \in int(C_{abc})$$



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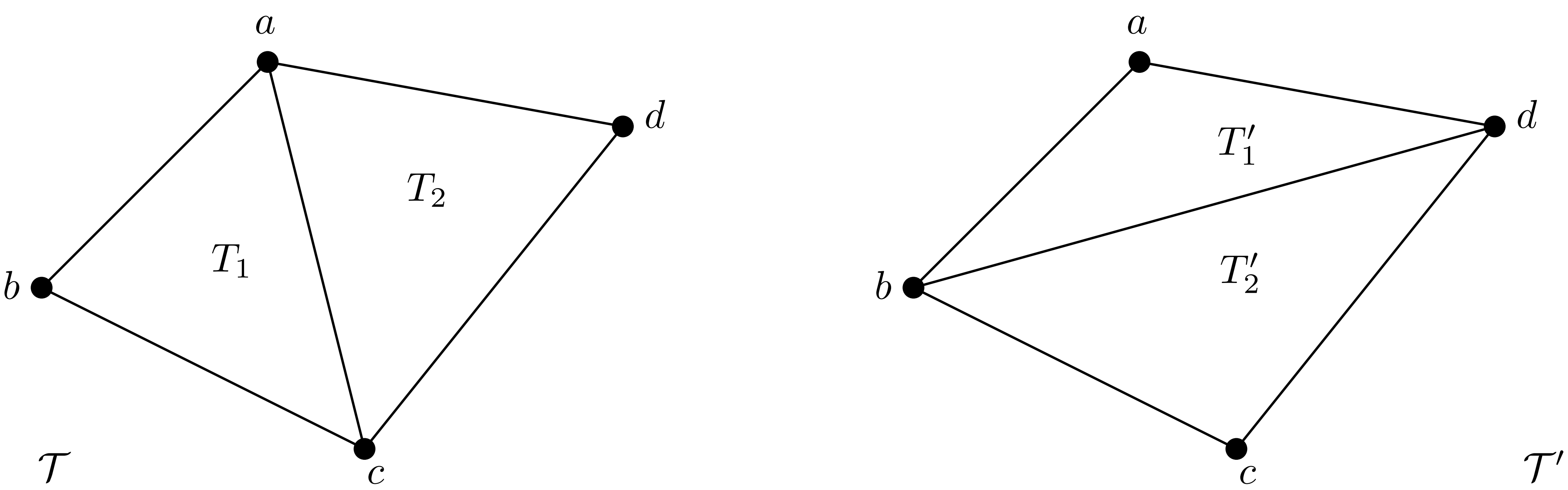

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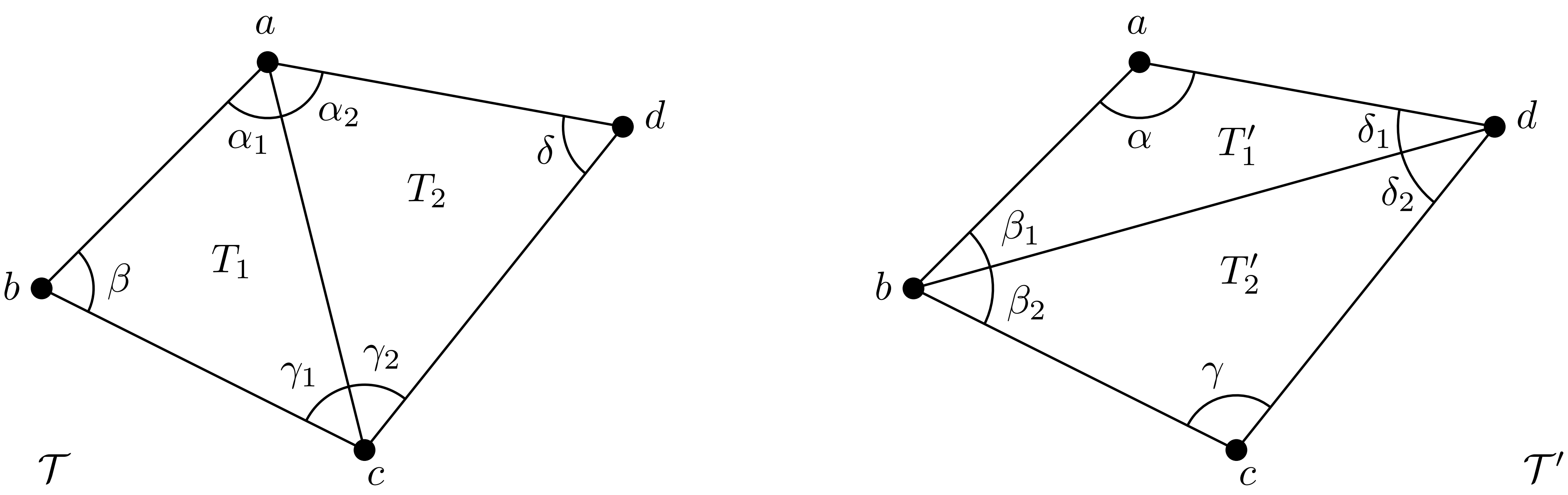

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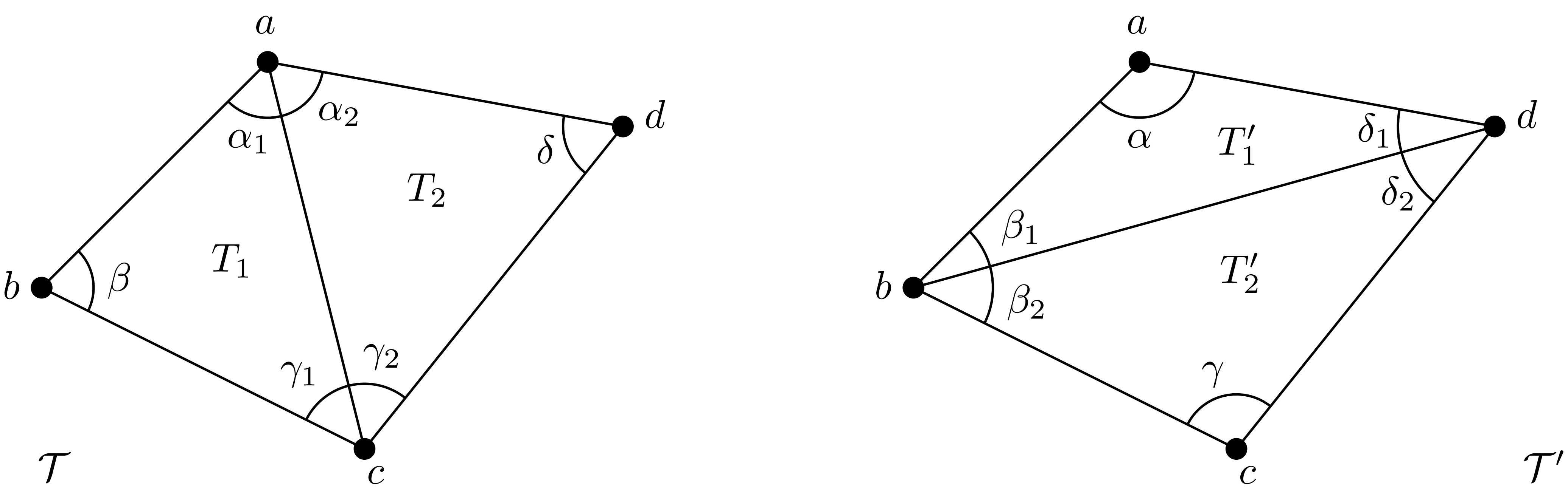

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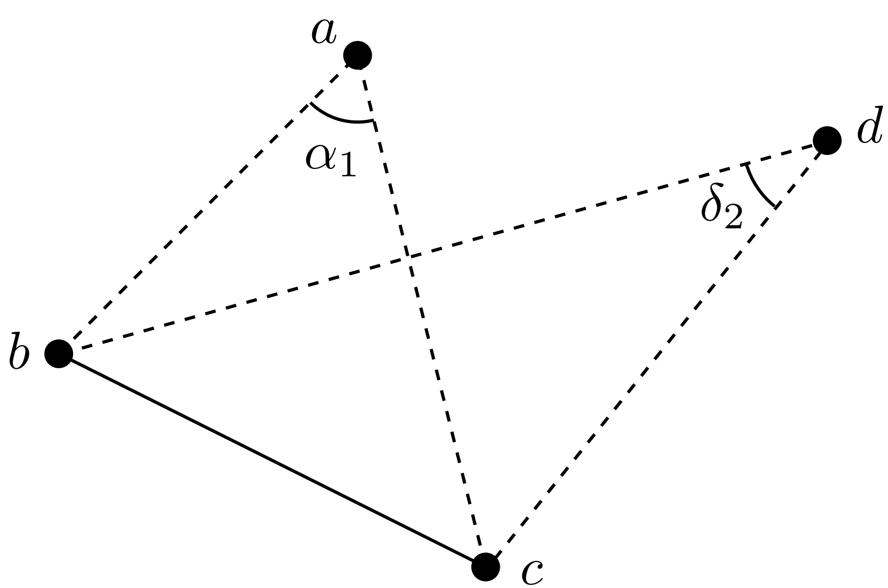

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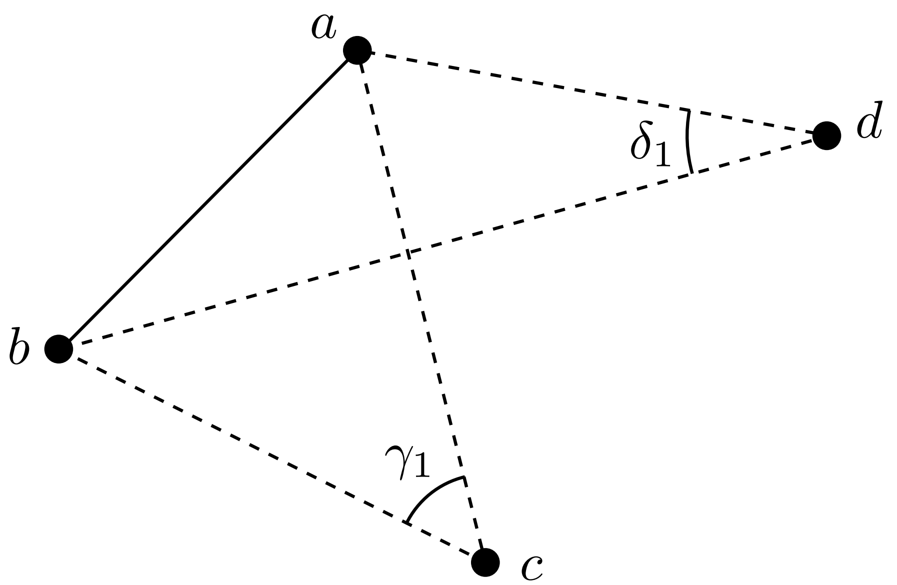

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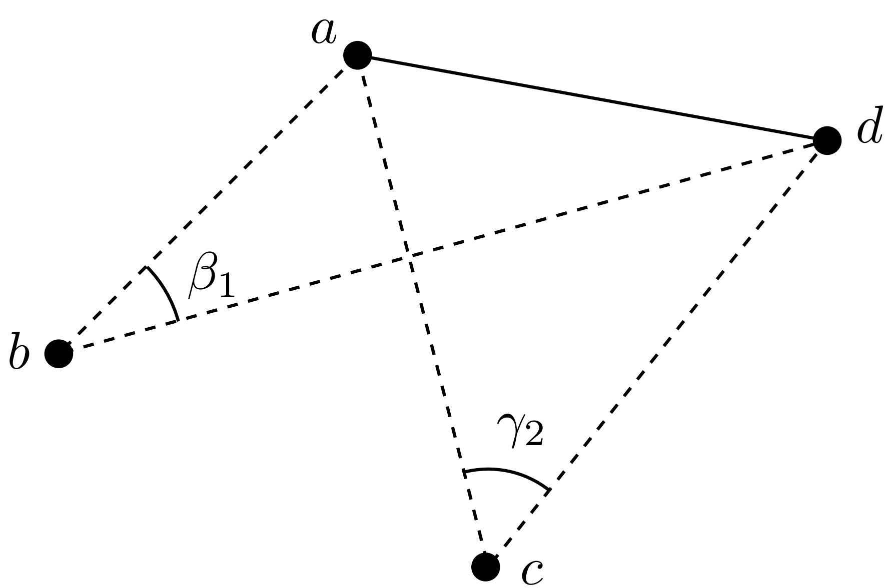

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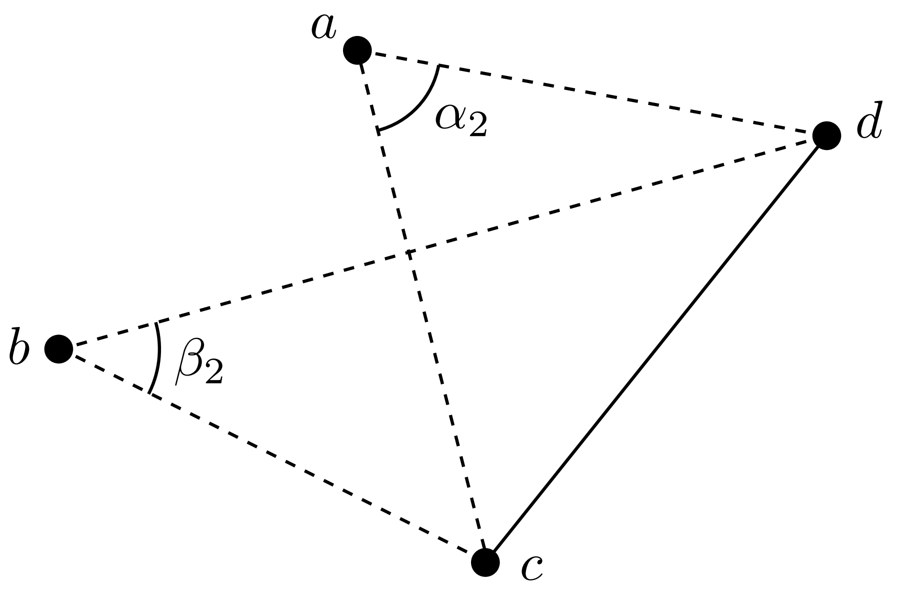

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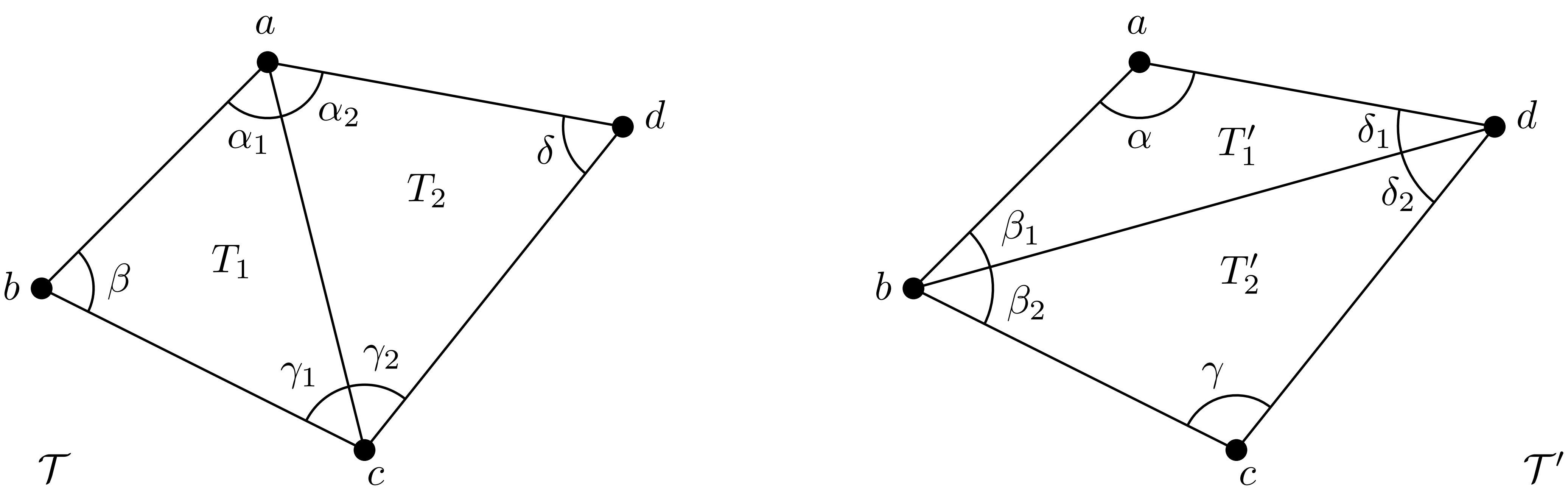

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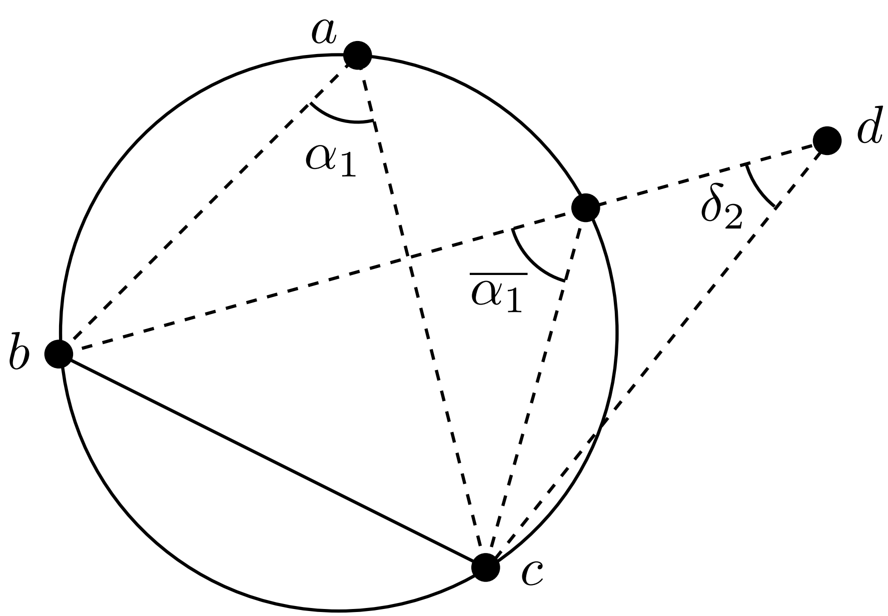

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If P contains four or more concyclic points, $Del(P)$ contains a polygon inscribed in a circle which can be triangulated in several ways. Nevertheless, Lemma 1 (on the geometrical locus of all the points from which a segment is seen under a given angle) guarantees that every triangulation of a polygon inscribed in a circle has the same fineness, since each edge of the polygon belongs to a triangle, and every possible triangle gives rise to the same angle.

DELAUNAY TRIANGULATION AND INTERPOLATION

In the context of terrain modeling, the Delaunay triangulation has another nice property: it minimizes the *roughness* of the terrain (i.e., the integral of the square of the L_2-norm of the terrain's gradient).

It is important to notice that this property is independent from the data, in other words, it is independent from the values of the z-coordinates of the input points.

DELAUNAY TRIANGULATION

TWO BOOKS WITH MUCH MORE INFORMATION

A. Okabe, B. Boots, K. Sugihara, S. N. Chiu
Spatial Tessellations
2nd ed., J. Wiley & Sons, 2000.

F. Aurenhammer, R. Klein, D.-T. Lee
Voronoi Diagrams and Delaunay Triangulations
World Scientific, 2013.