INTERSECTING HALF-PLANES
and related problems

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DUALIZING POINTS AND LINES

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In the plane, dualities transform points into lines and lines into points.

One of the most frequently used is:

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\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\
p = (a, b) & \mapsto & D(p) = p* : \; y = ax - b \\
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Properties

1. $D^2 = Id$.
2. D is a bijection between points and non-vertical lines.
3. A point p lies above/on/below a line ℓ if and only if the point ℓ^* lies above/on/below the line p^*.

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   Proof: If $p = (a, b)$ and ℓ is the line of equation $y = mx + n$, the relative position of p wrt ℓ is given by the (in)equation $b \lesseqgtr ma + n$, which is equivalent to $-n \lesseqgtr am - b$, and this expresses the relative position of $\ell^* = (m, -n)$ wrt $p^* : \; y = ax - b$.

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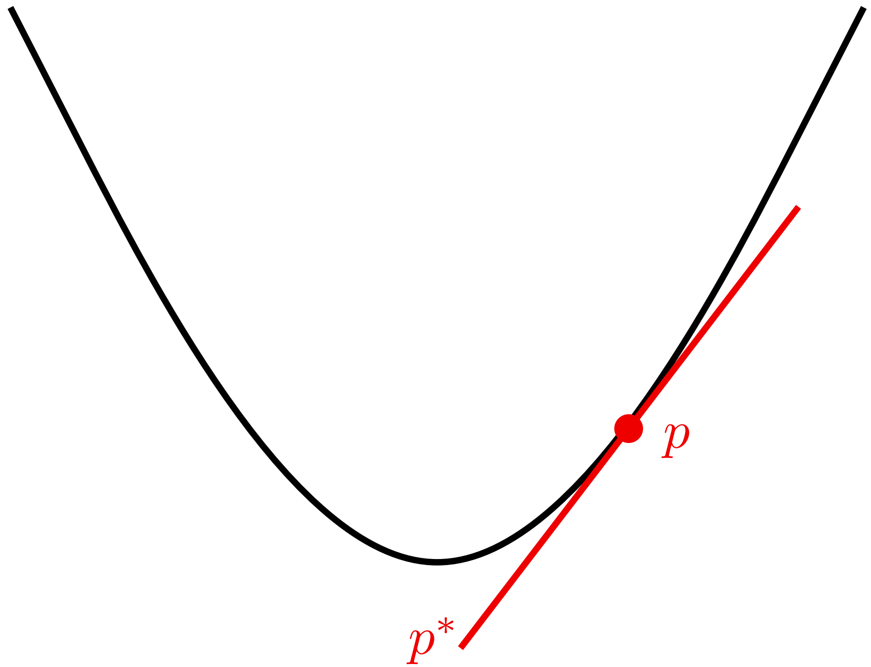
Duality and parabola

Consider the parabola $y = x^2/2$.

- If p is a point on the parabola, then p^* is the line tangent to the parabola at point p.



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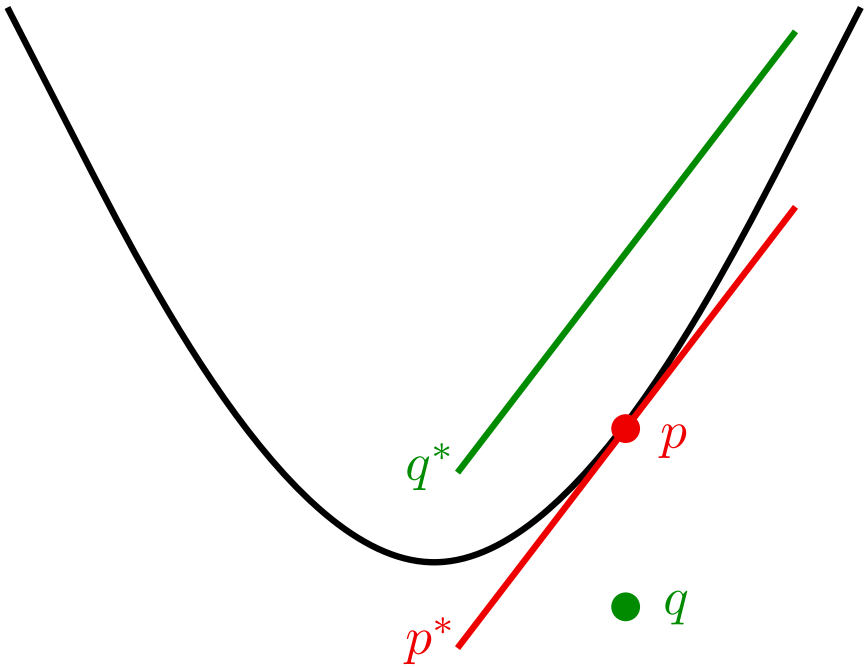

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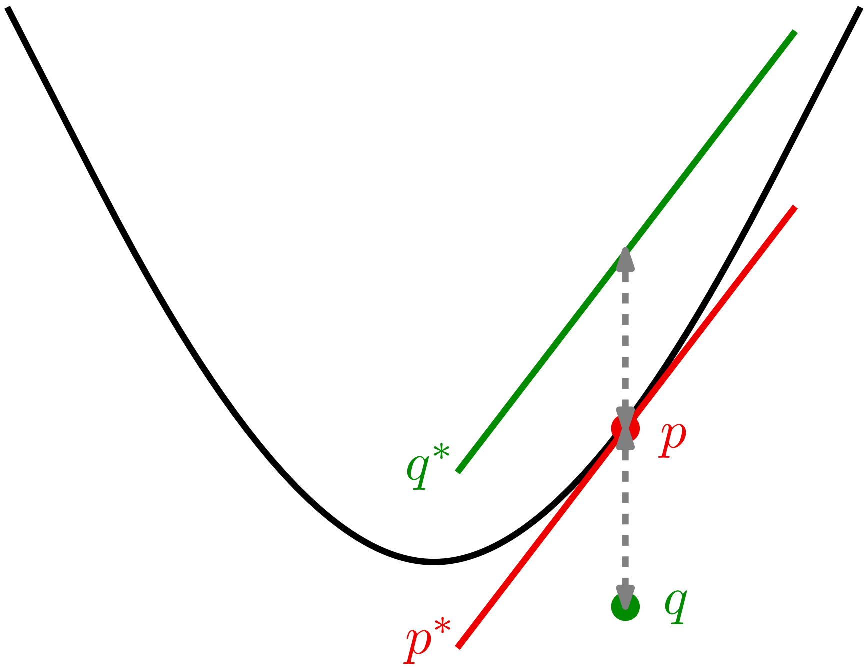

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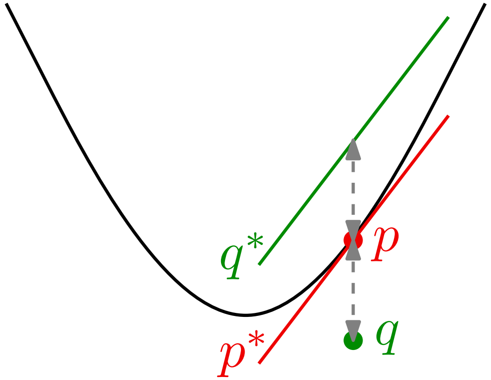

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Proof:
Since $y'(x)=x$, the line tangent to the parabola at a point $p=(a,a^2/2)$ has equation $y-a^2/2=a(x-a)$ i.e., $y=ax-a^2/2$.

For any other point $q=(a,b)$:
$d_{vertical}(p,q)=b-a^2/2$;
$d_{vertical}(p^*,q^*)=-b+a^2/2$.

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https://dccg.upc.edu/teaching/applets/duality.html

Intersection of half-planes and convex hulls

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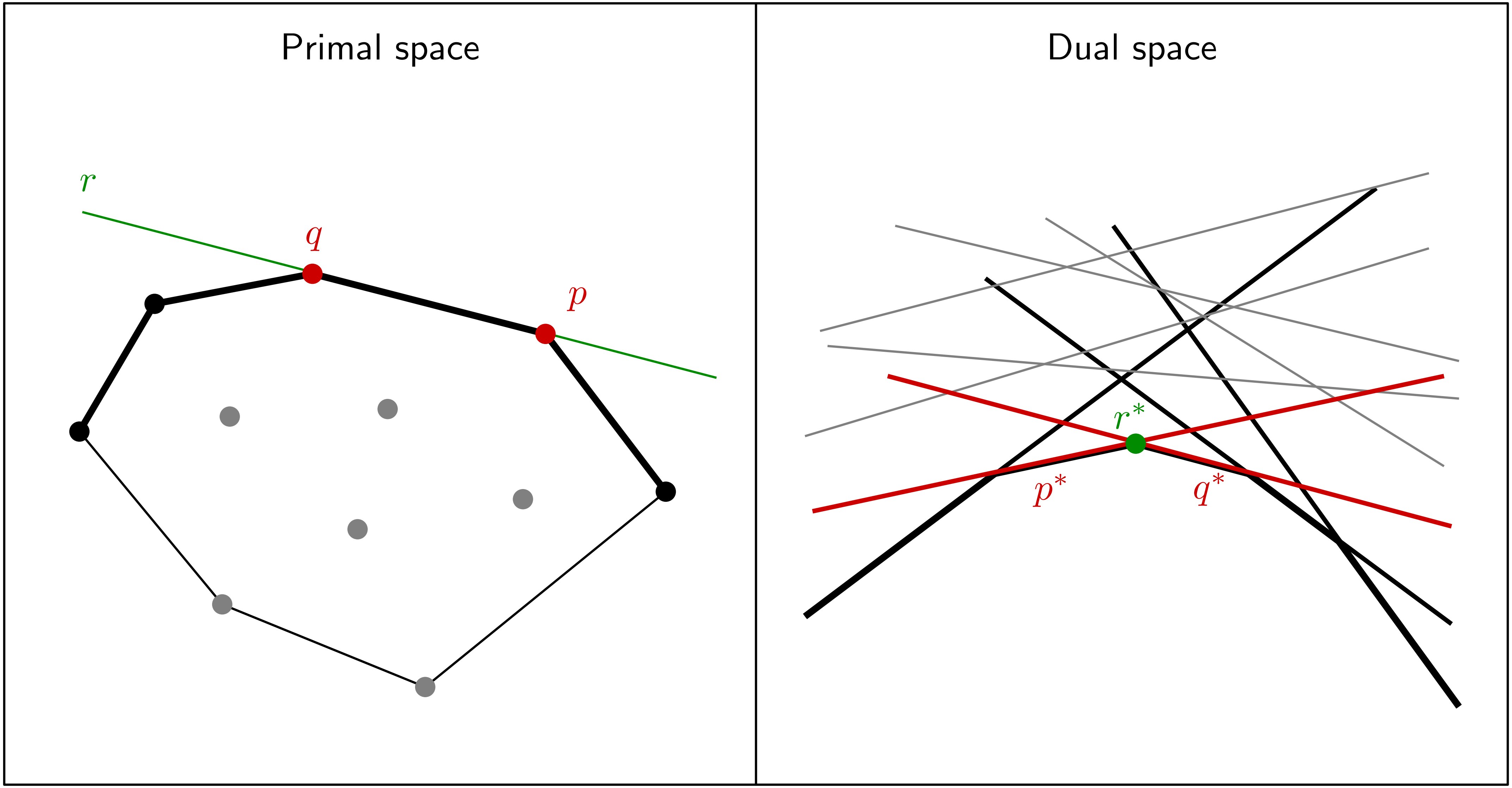

Computing the intersection of half-planes

Input: n half-planes
Output: Sorted list of vertices (and, possibly, half-lines) defining the boundary of their intersection (a convex polygon or an unbounded convex polygonal region).

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Lower bound

Duality reduces in $O(n)$ time the problem of computing the convex hull of n points to the problem of computing the intersection of n half-planes. Therefore, computing the intersection of n half-planes has an $\Omega(n \log n)$ lower bound.

SOLVING LINEAR PROGRAMS

Linear program

Optimize a linear function $ax + by$
subject to n linear restictions $a_i x + b_i y + c_i \leq 0$, where $i = 1, \ldots, n$.

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restricted to:

$$50x + 24y \leq 40 \times 60$$

$$30x + 33y \leq 35 \times 60$$

$$x \geq 75 - 30$$

$$y \geq 95 - 90$$

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Algo 2 (Megiddo, Dyer)

In $O(n)$ time, it is possible to find the vertex of R achieving the optimum, without computing the entire feasible region R.

This is done by a **prune and search strategy**, which at each step eliminates a constant fraction of the restrictions:

- either because they are redundant,
- or because they are not needed to define the solution vertex.

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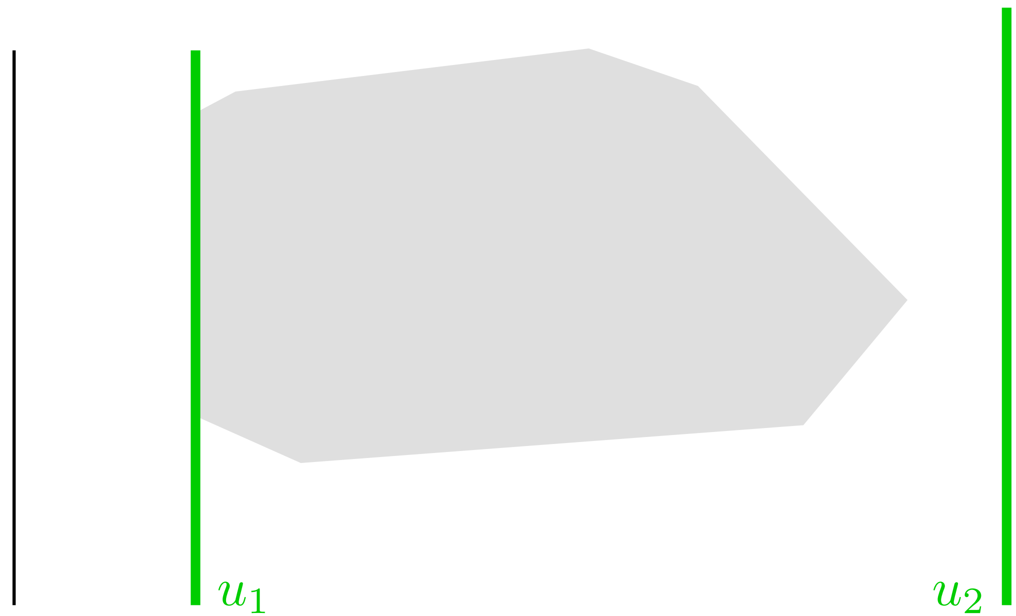

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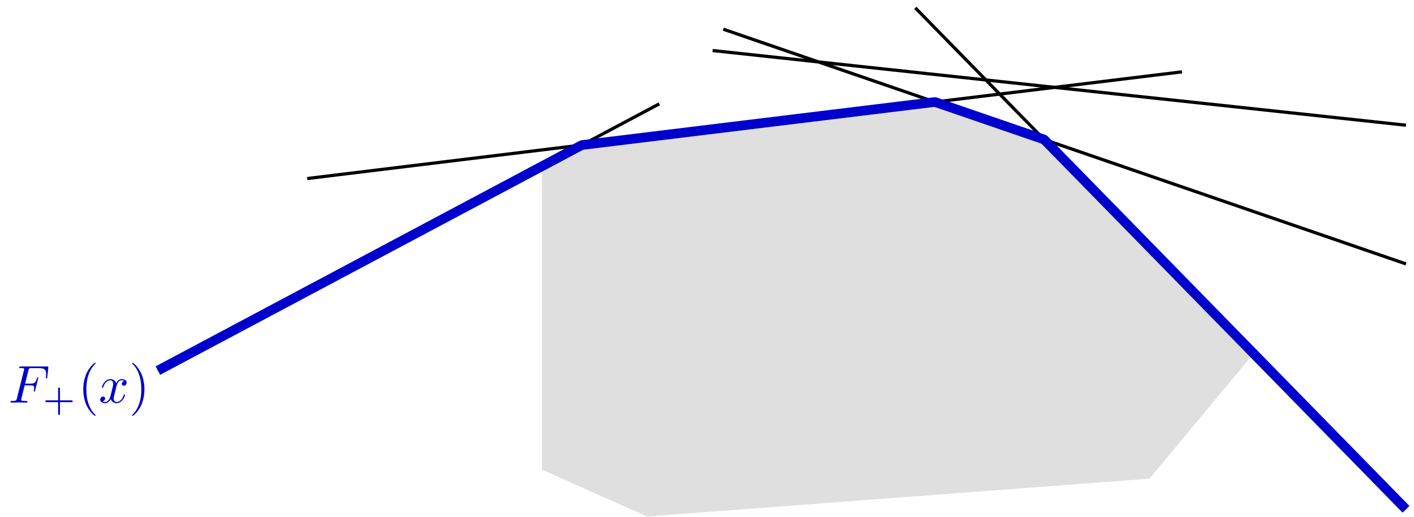

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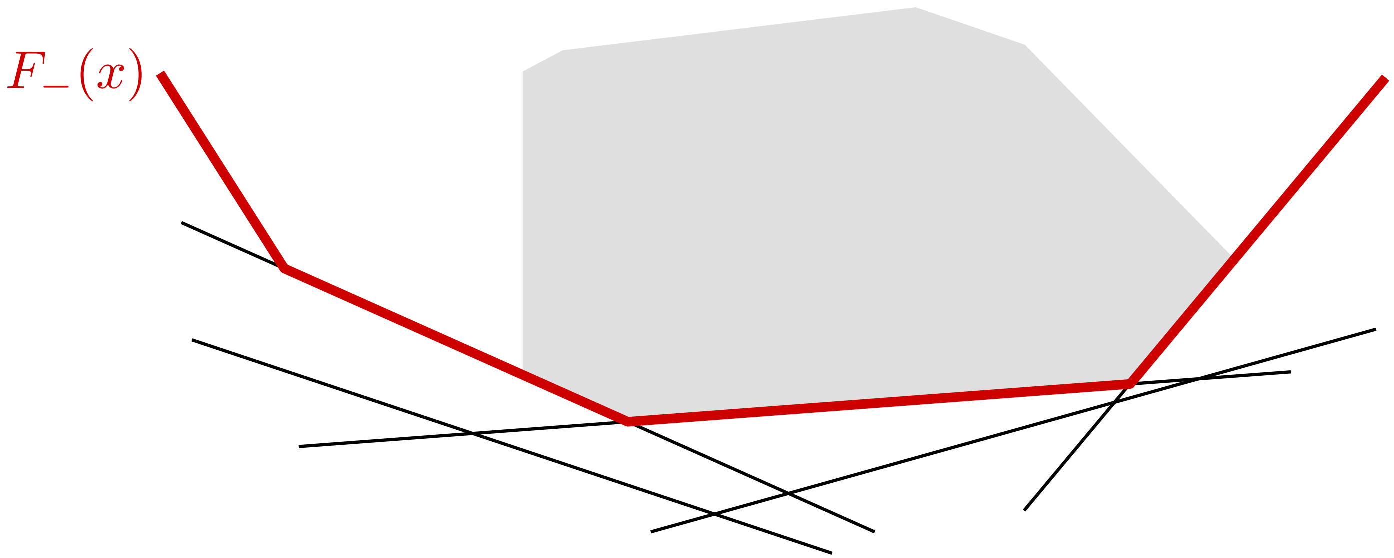

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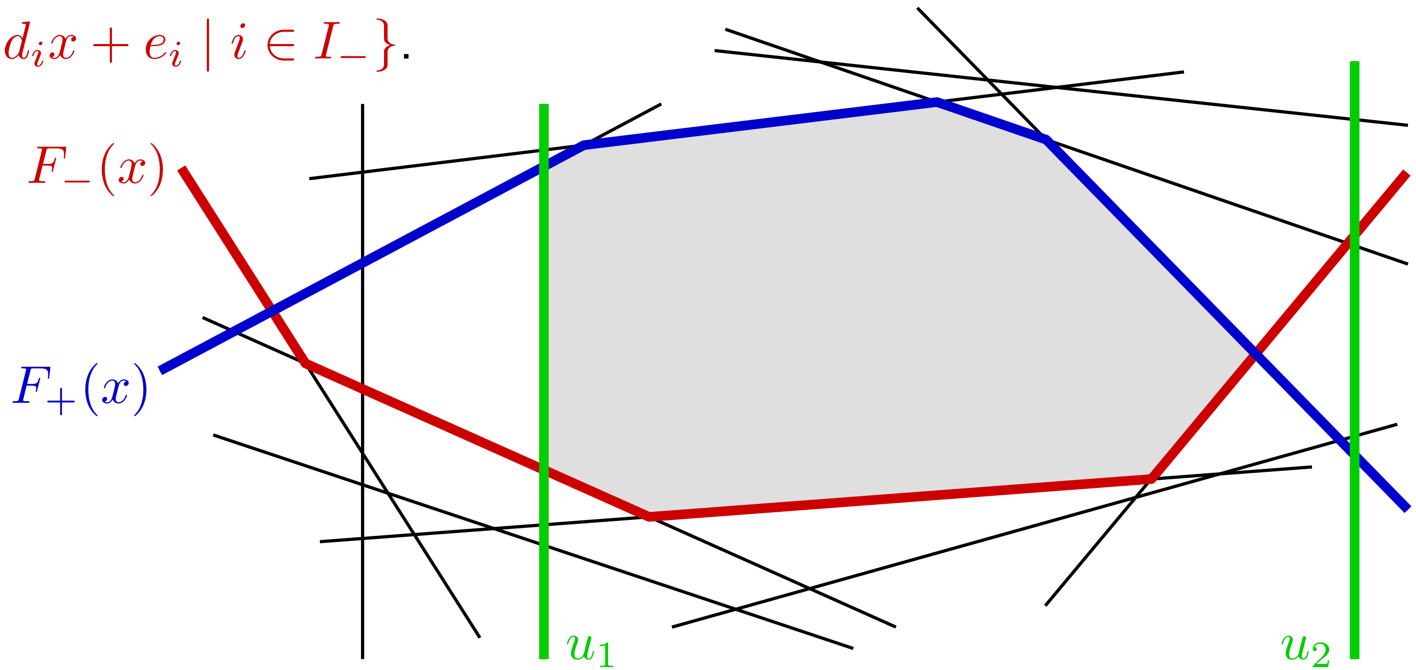

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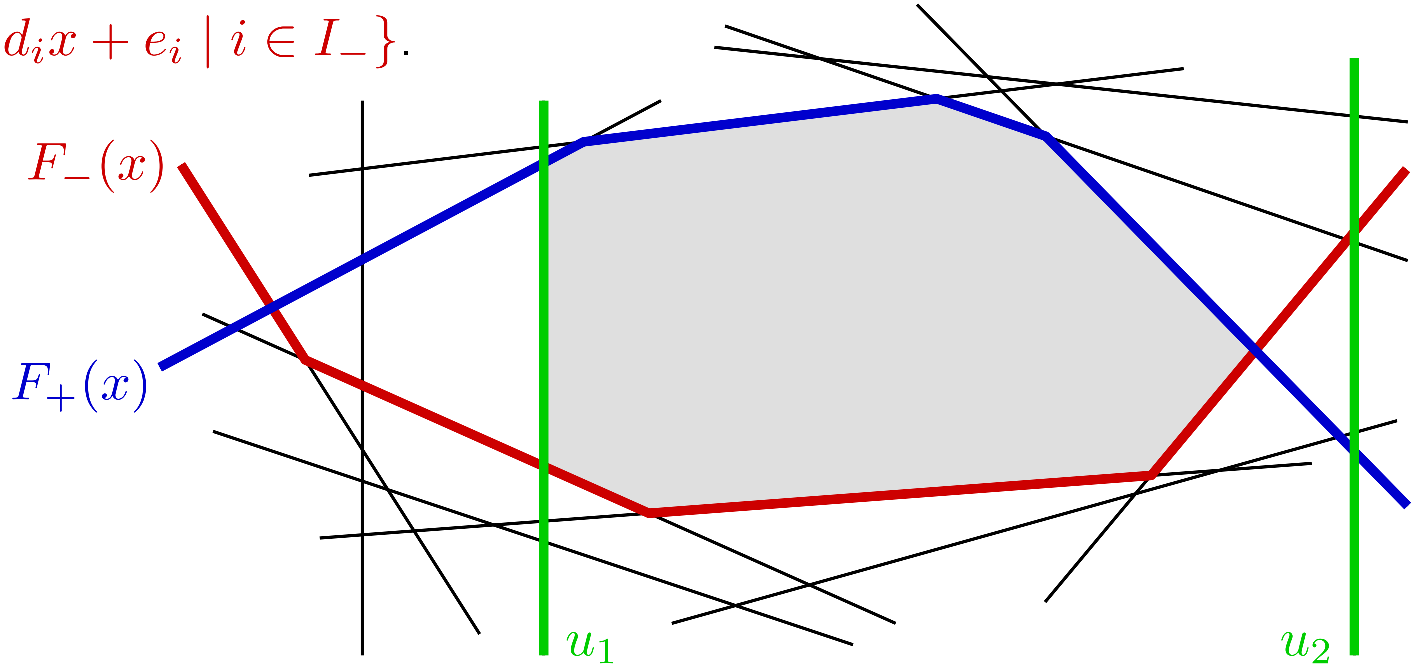

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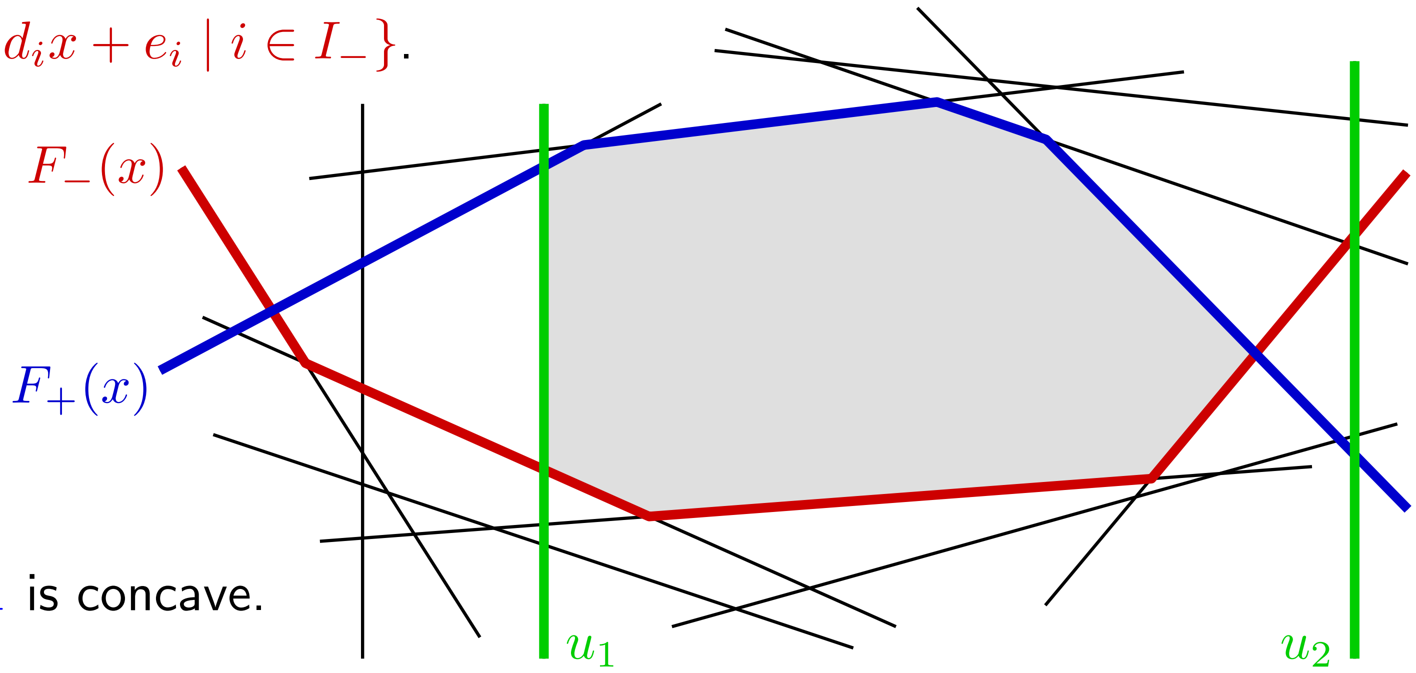

Notice that F_- is convex and F_+ is concave.

The search

SOLVING LINEAR PROGRAMS

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Given $x' \in [u_1, u_2]$, in $O(n)$ time it is possible to get to one of the following conclusions:

1. x' is feasible, and minimizing $F_-(x)$.
2. x' is feasible, and the solution of the problem lies to its right (left).
3. x' is infeasible, and the solution of the problem does not lie to its left (right).
4. x' is infeasible, and the problem has no solution.

The search

Given $x' \in [u_1, u_2]$, in $O(n)$ time it is possible to get to one of the following conclusions:

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This is done by analizing the values of F_+, F_- and their slopes $f_+^l, f_+^r, f_-^l, f_-^r$ in x', where the slopes are defined as follows:

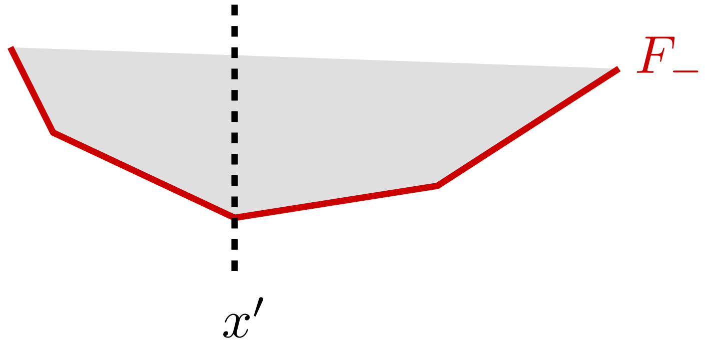
- If $F_-(x')$ (resp. $F_+(x')$) is defined by a unique index $i \in I_-$ (resp. I_+), then $f_-^l(x') = f_-^r(x') = d_i$ (resp. $f_+^l(x') = f_+^r(x') = d_i$).
- If $F_-(x')$ (resp. $F_+(x')$) is defined by two indexes $i, j \in I_-$ (resp. I_+), then $f_-^l(x') = \min(d_i, d_j)$ and $f_-^r(x') = \max(d_i, d_j)$ (resp. $f_+^l(x') = \max(d_i, d_j)$ and $f_+^r(x') = \min(d_i, d_j)$).

The search

Case 1. $F_-(x') \leq F_+(x')$ (x' is feasible)

The search

Case 1. $F_{-}(x') \leq F_{+}(x')$ (x' is feasible)



If $f_{-}^{l}(x') < 0 < f_{-}^{r}(x')$,
then x' is the solution.

SOLVING LINEAR PROGRAMS

The search

Case 1. $F_-(x') \leq F_+(x')$ (x' is feasible)

If $f_-^l(x') < 0 < f_-^r(x')$,
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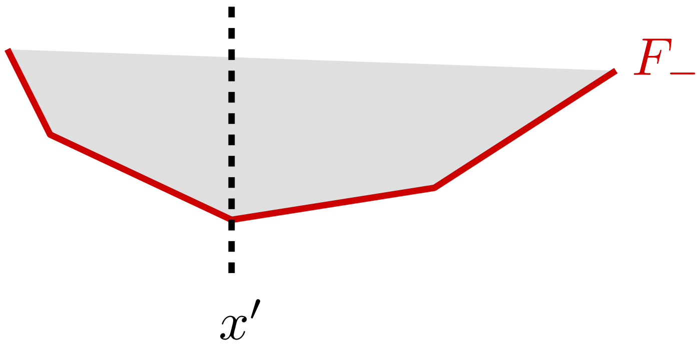
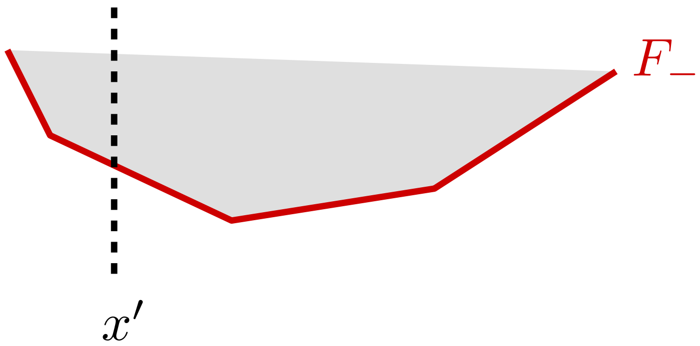


If $f_-^r(x') < 0$,
then search to the right of x'.



SOLVING LINEAR PROGRAMS

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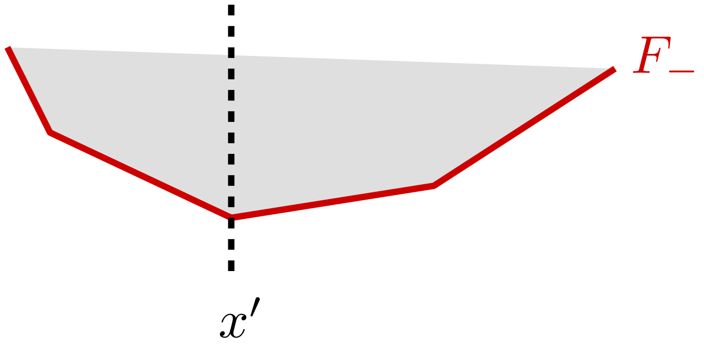

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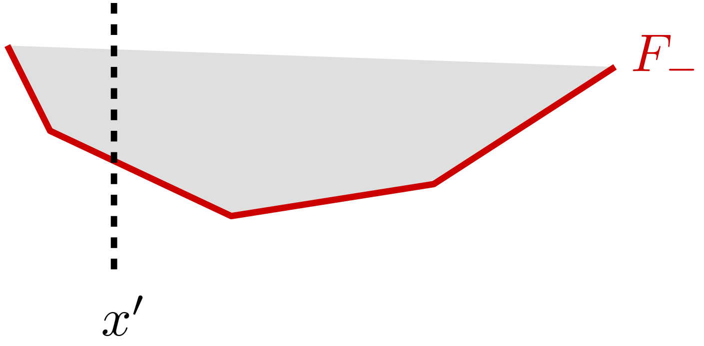
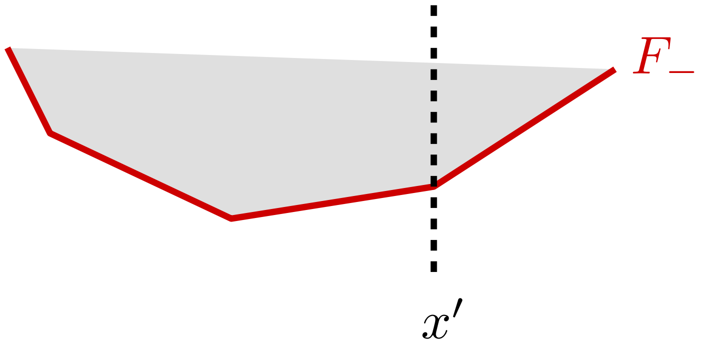


If $f_-^l(x') > 0$,
then search to the left of x'.



The search

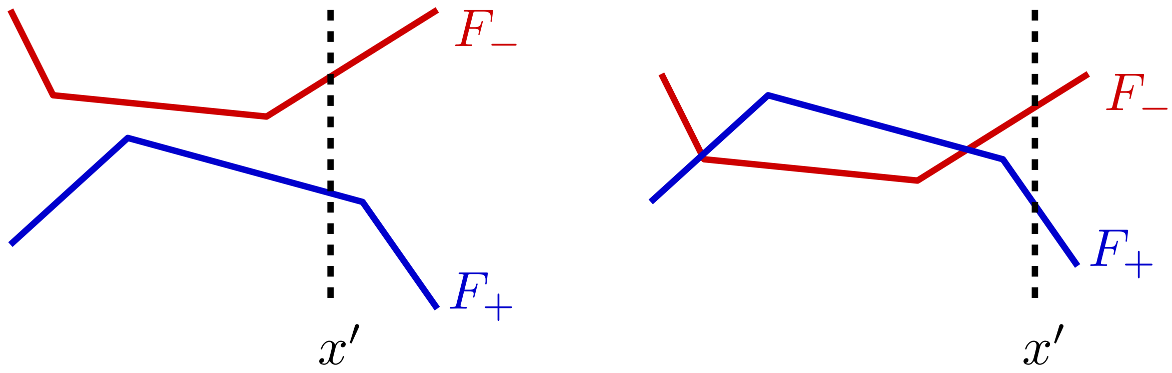
Case 2. $F_-(x') > F_+(x')$ (x' is infeasible)

SOLVING LINEAR PROGRAMS

The search

Case 2. $F_-(x') > F_+(x')$ (x' is infeasible)

If $f^l_-(x') > f^l_+(x')$,
then search to the left of x'.



The search

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If $f_-^l(x') > f_+^l(x')$,
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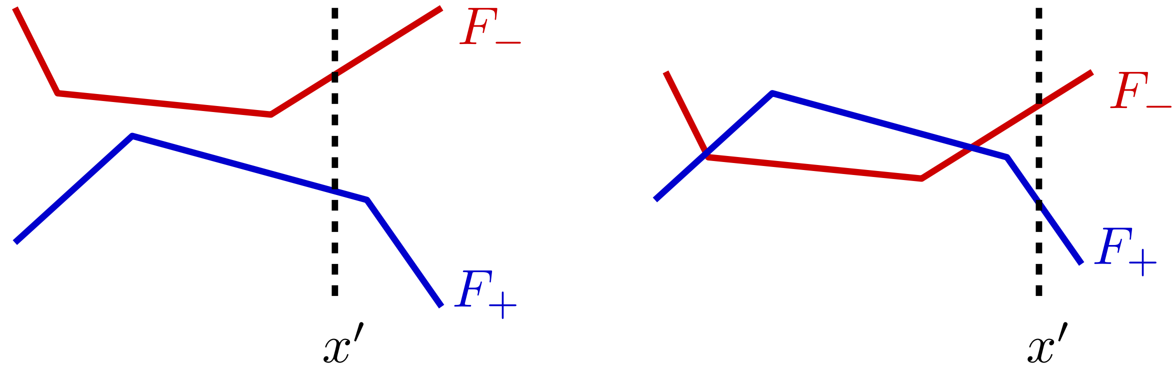

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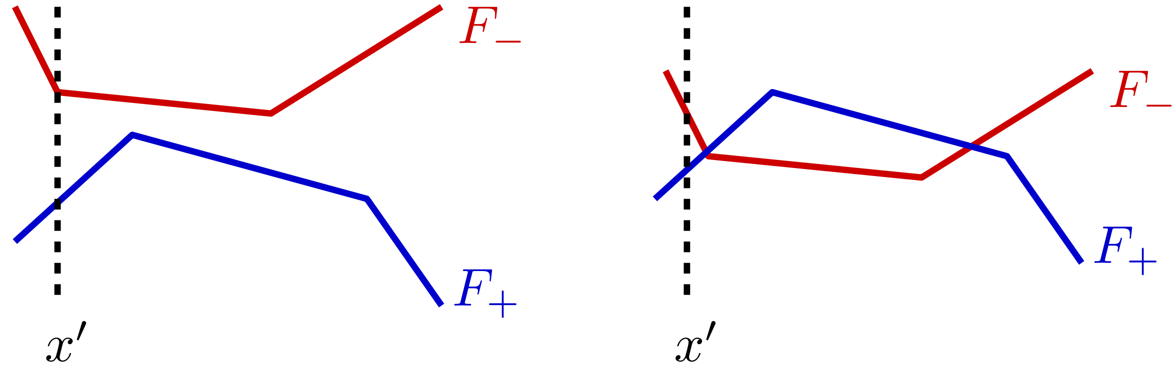

SOLVING LINEAR PROGRAMS

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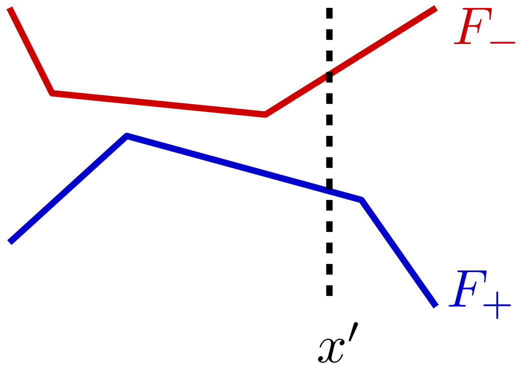

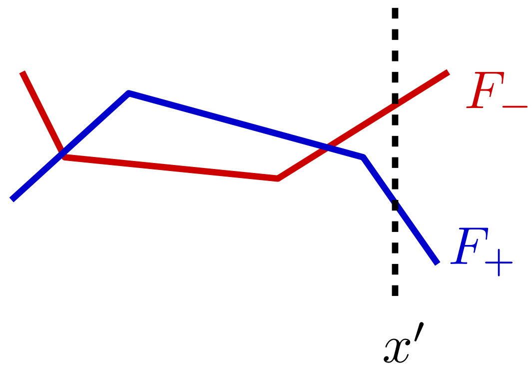

If $f_-^r(x') < f_+^r(x')$,
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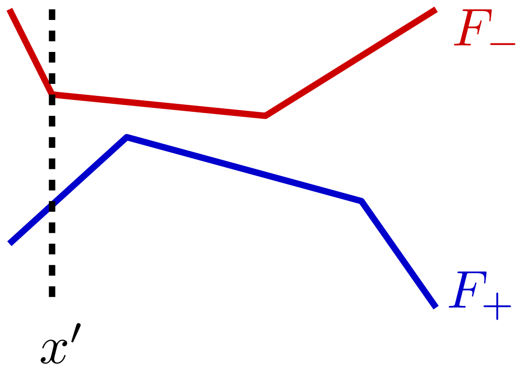
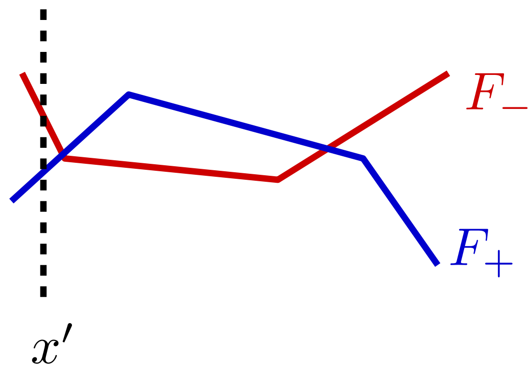
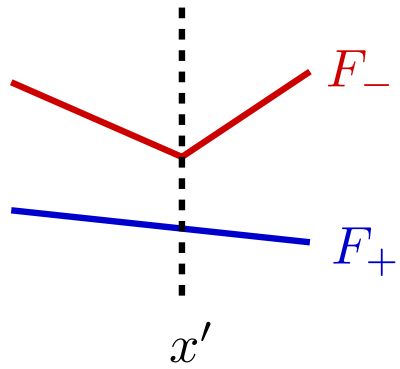
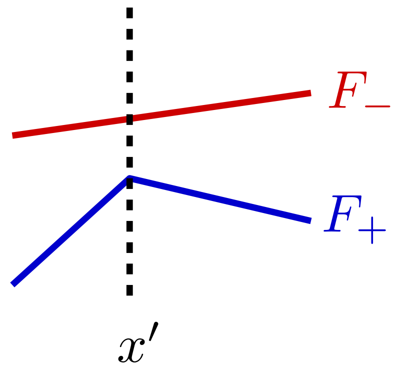




If $f_-^l(x') \leq f_+^l(x')$ and $f_-^r(x') \geq f_+^r(x')$
then there is no solution.

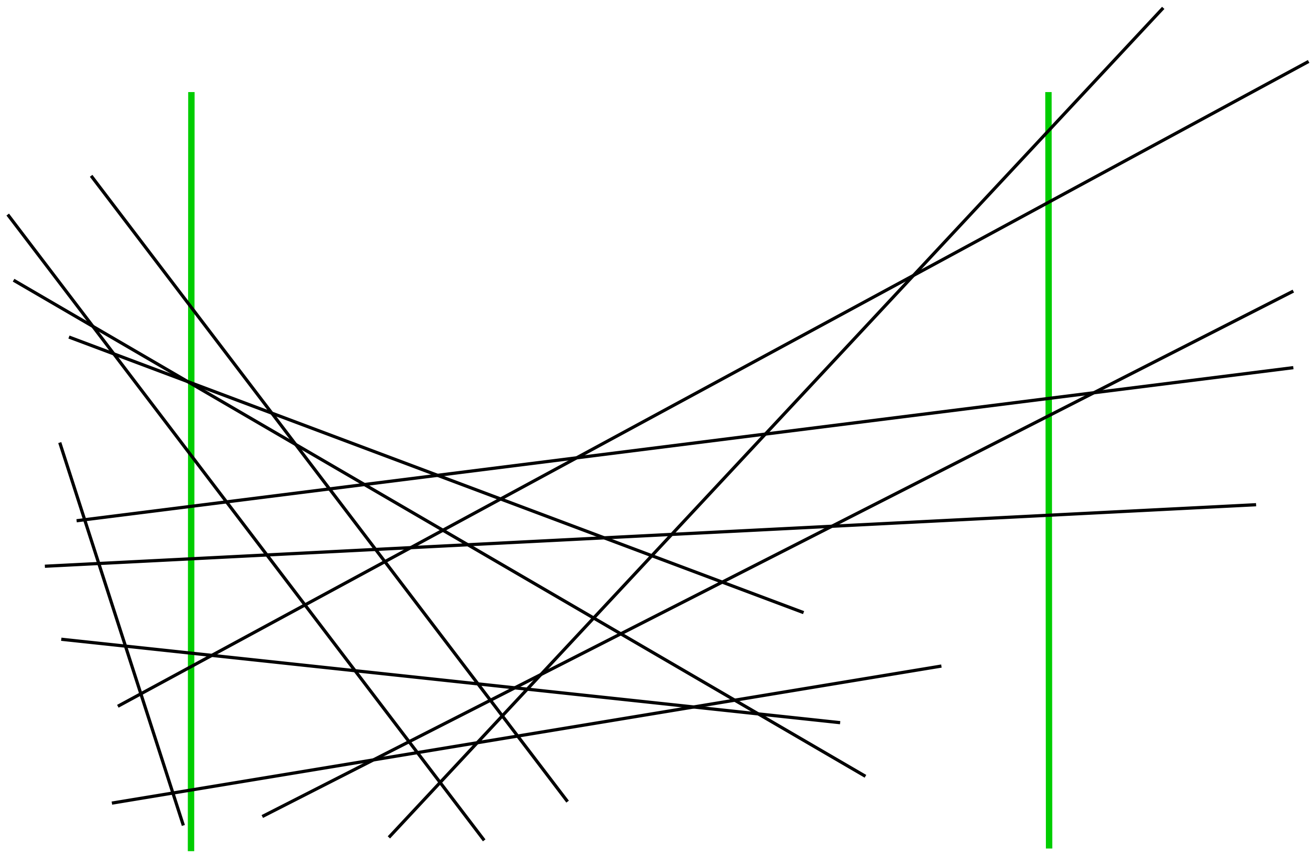




The algorithm

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2, if they exist.

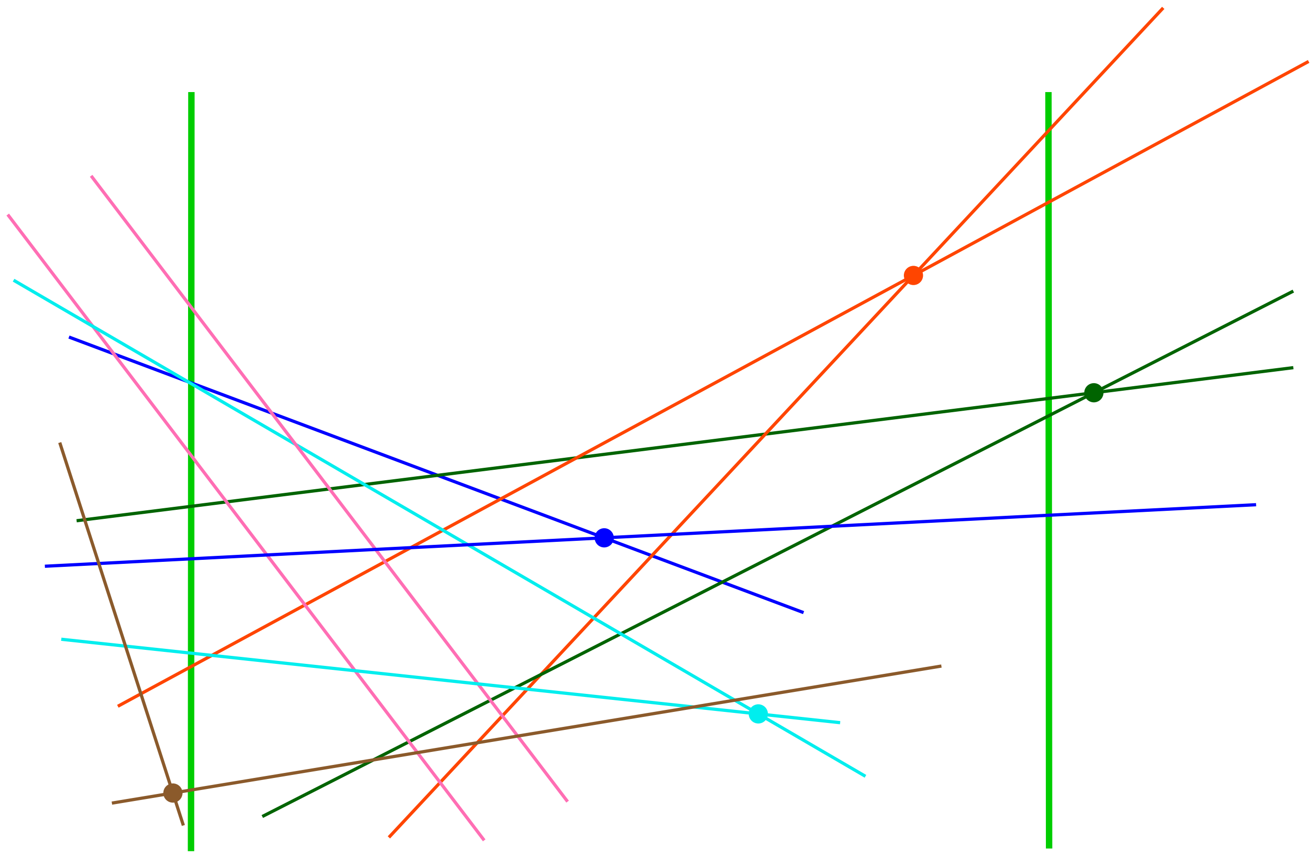


The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2, if they exist.

Advance. Repeat as many times as necessary:

1. Pair up all restrictions corresponding to I_- (possibly but one). Analogously for I_+. Compute the abscissa of the intersection point of each pair of lines, if it exists.



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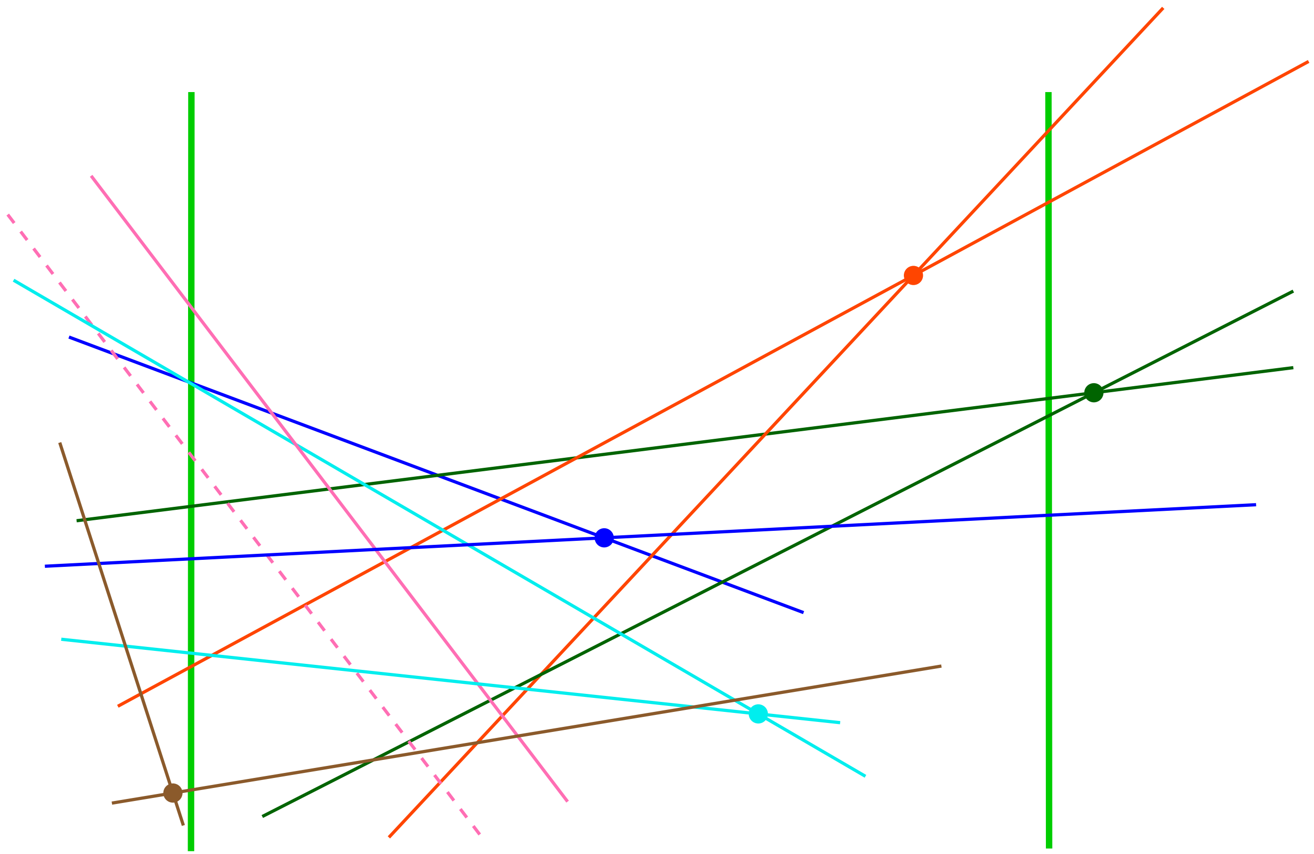
Advance. Repeat as many times as necessary:

2. For all pairs $i, j \in I_-$ (analogously for I_+), do:

    - If $d_i = d_j$, eliminate the line corresponding to $\min(e_i, e_j)$ (resp. $\max(e_i, e_j)$).



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Advance. Repeat as many times as necessary:

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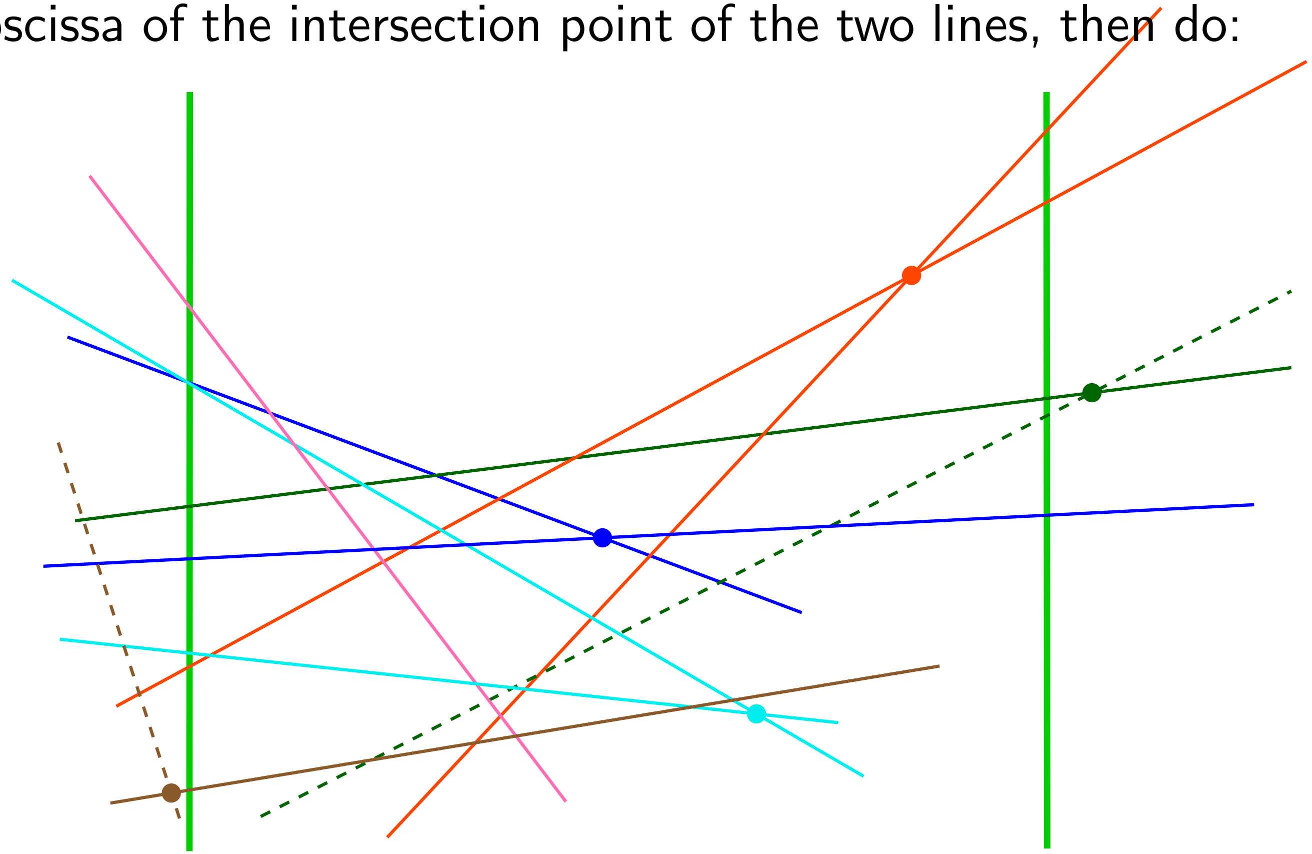
   - If $d_i = d_j$, eliminate the line corresponding to $\min(e_i, e_j)$ (resp. $\max(e_i, e_j)$).
   - If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
     - If $x_{ij} < u_1$, eliminate the line with $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).
     - If $x_{ij} > u_2$, eliminate the line with $\max(d_i, d_j)$ (resp. $\min(d_i, d_j)$).



The algorithm

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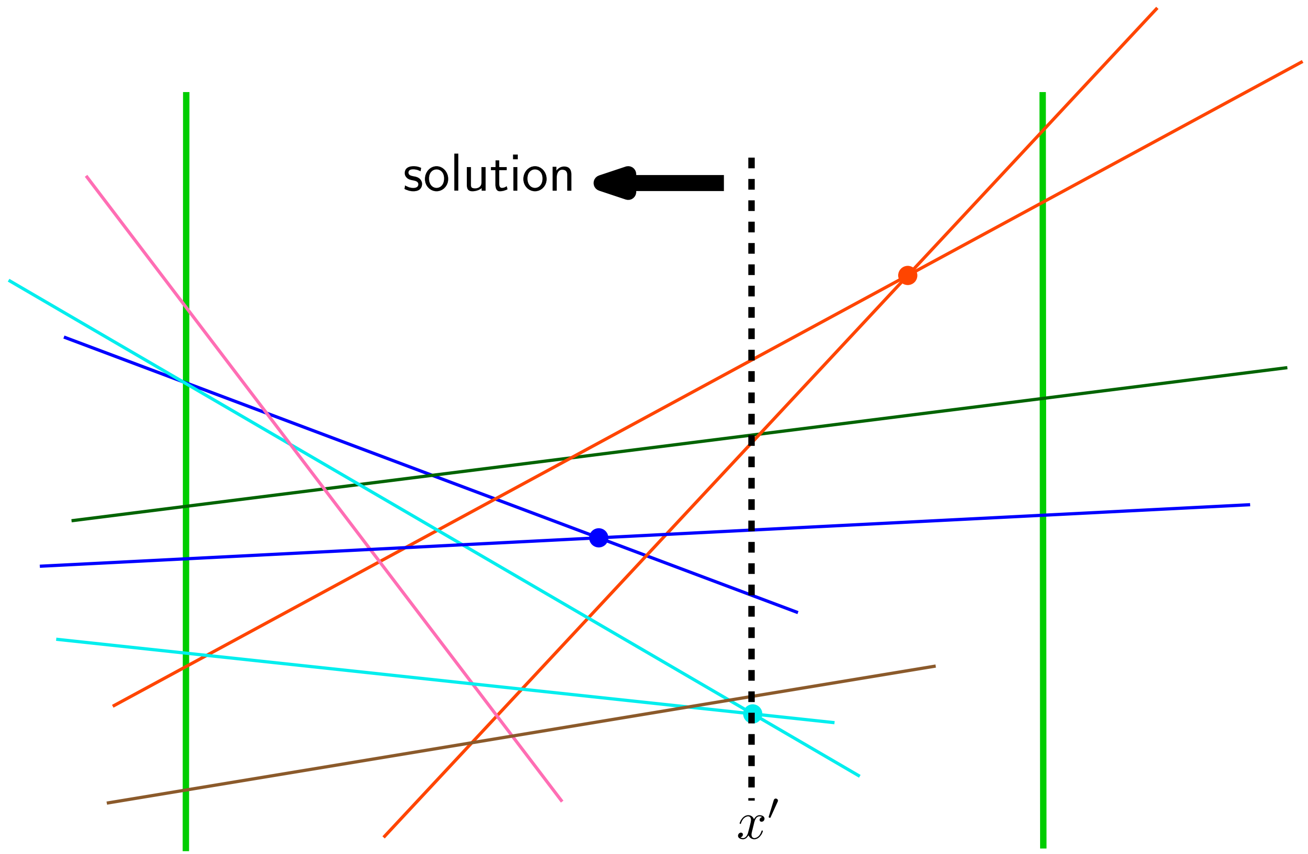
Advance. Repeat as many times as necessary:

3. Compute the median value x' of the surviving x_{ij}.
4. Search: Apply the search procedure to x'. If the answer is x', return x'.



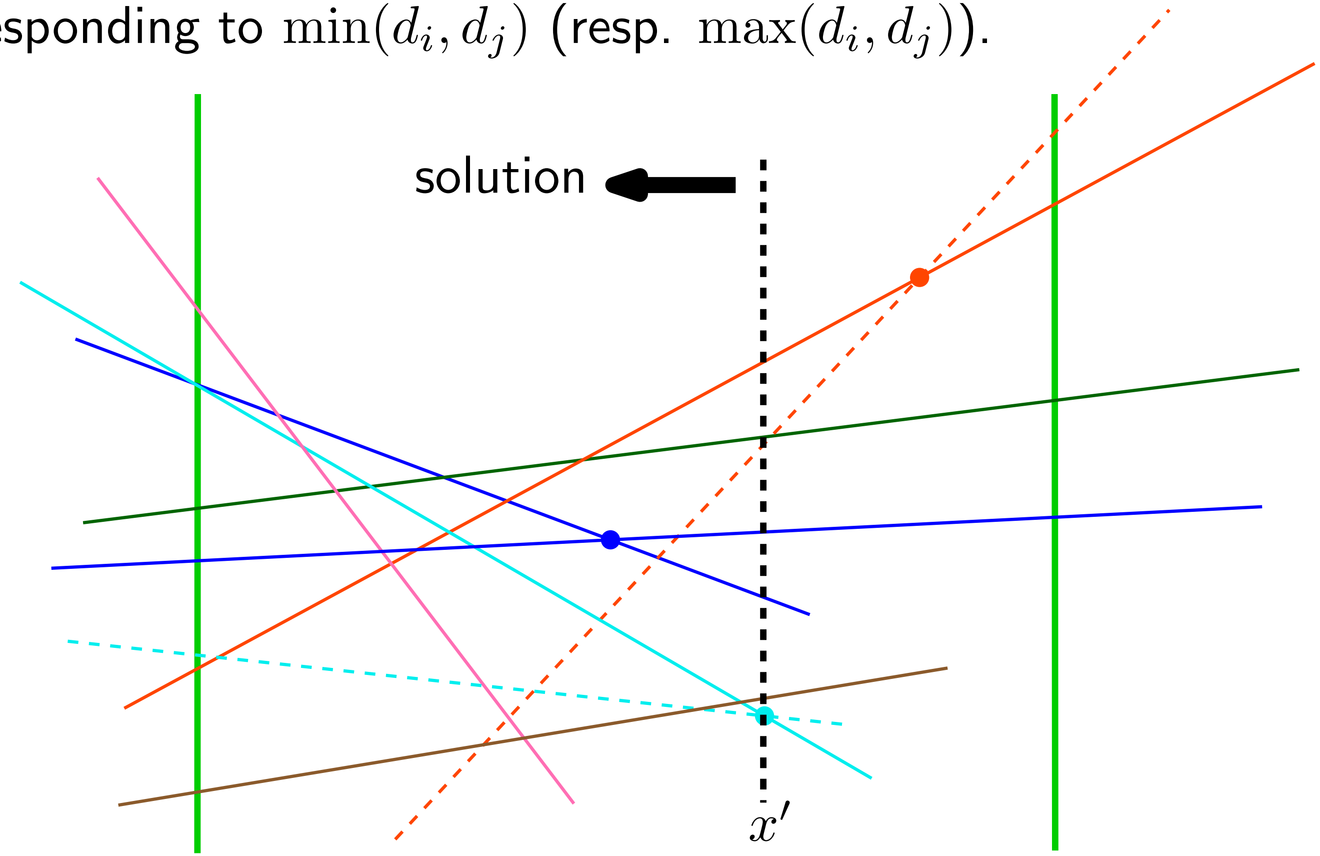
SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2, if they exist.

Advance. Repeat as many times as necessary:

5. Prune:
   If the answer is "search to the left" then, for each $x_{ij} \geq x'$ eliminate the line corresponding to $\max(d_i, d_j)$ (resp. $\min(d_i, d_j)$). If the answer is "search to the right" then, for each $x_{ij} < x'$ eliminate the line corresponding to $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).



SOLVING LINEAR PROGRAMS

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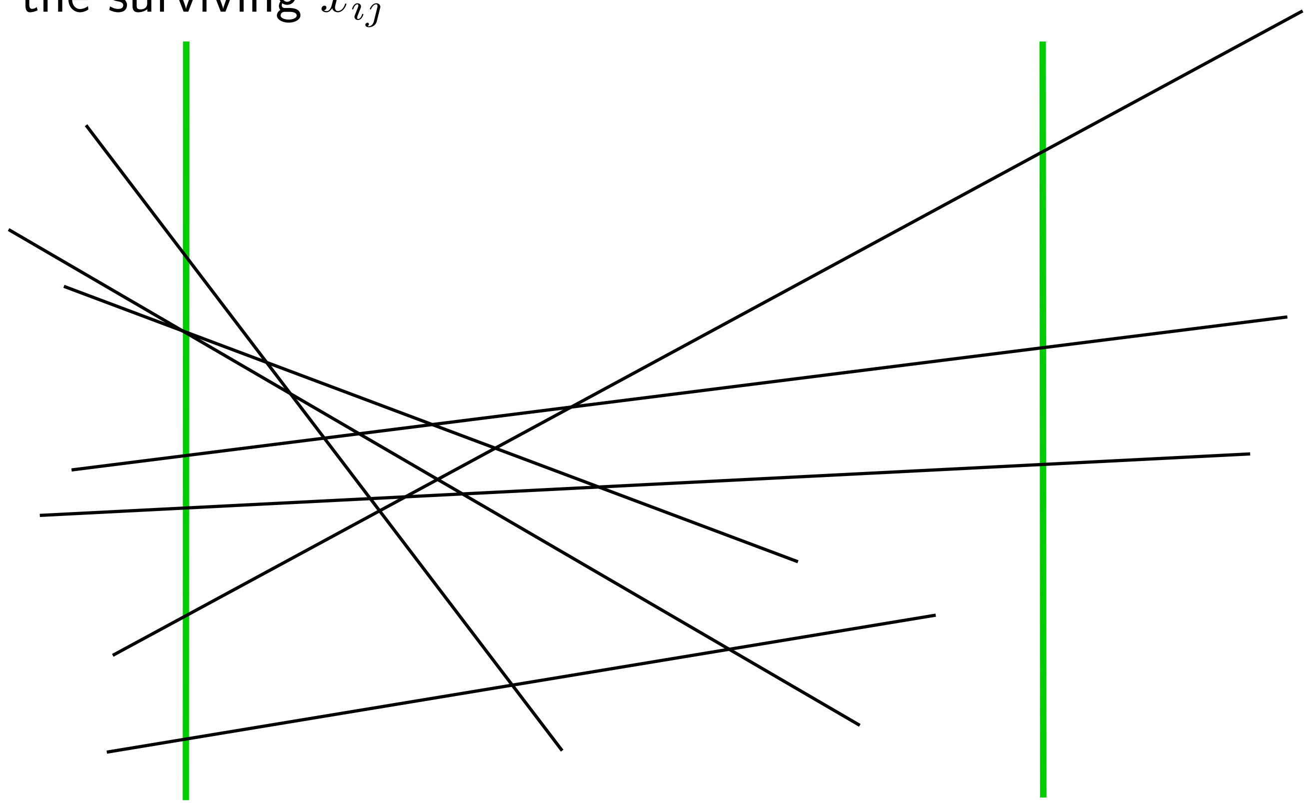
Advance. Repeat as many times as necessary:

1. Pair lines up
2. Prune parallel and external lines
3. Compute the median value x' of the surviving x_{ij}
4. Search
5. Prune



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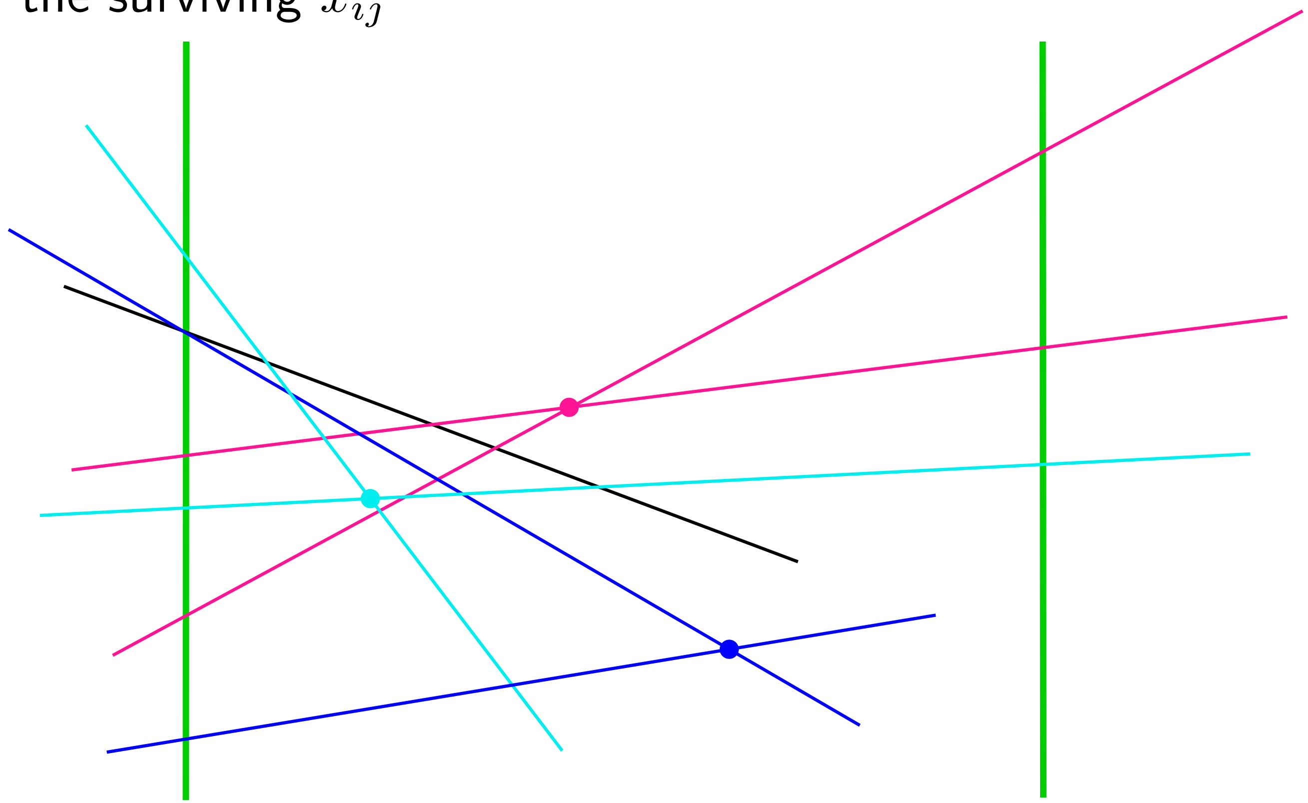

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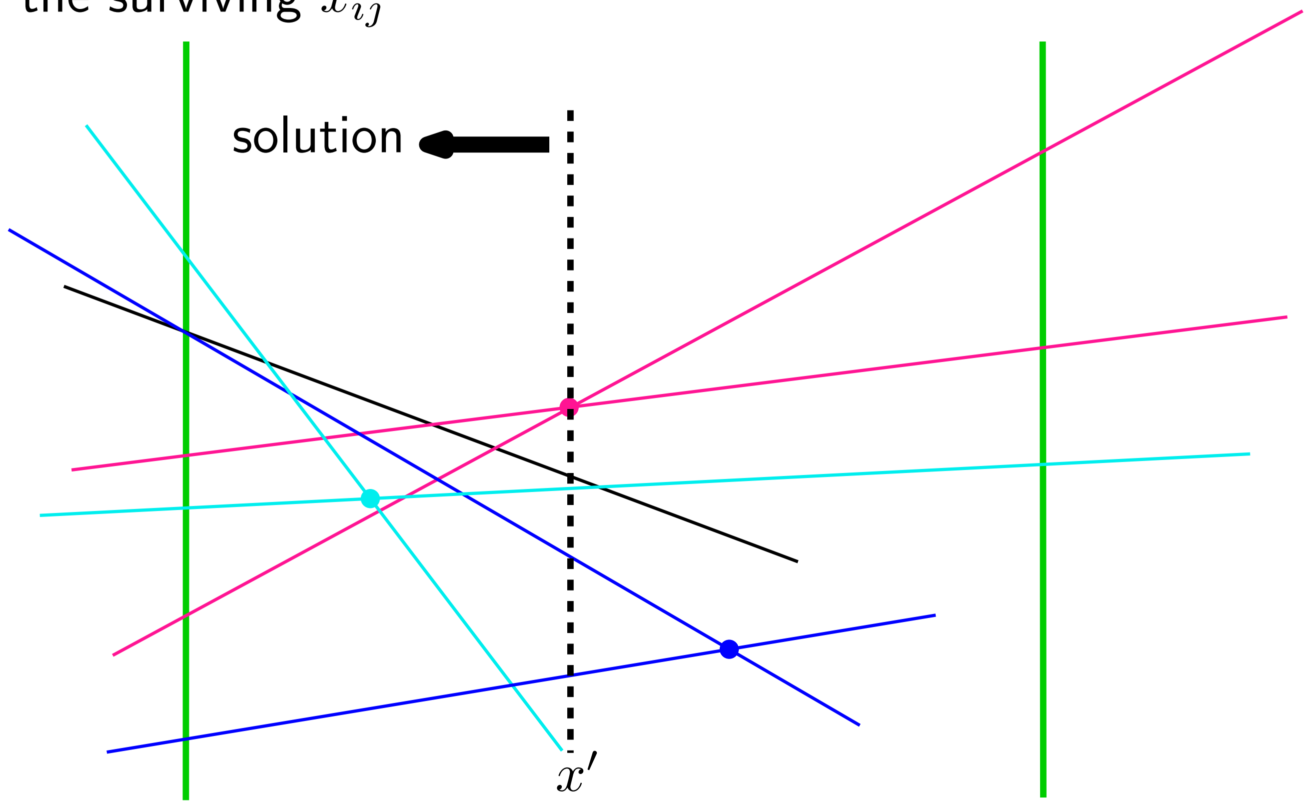

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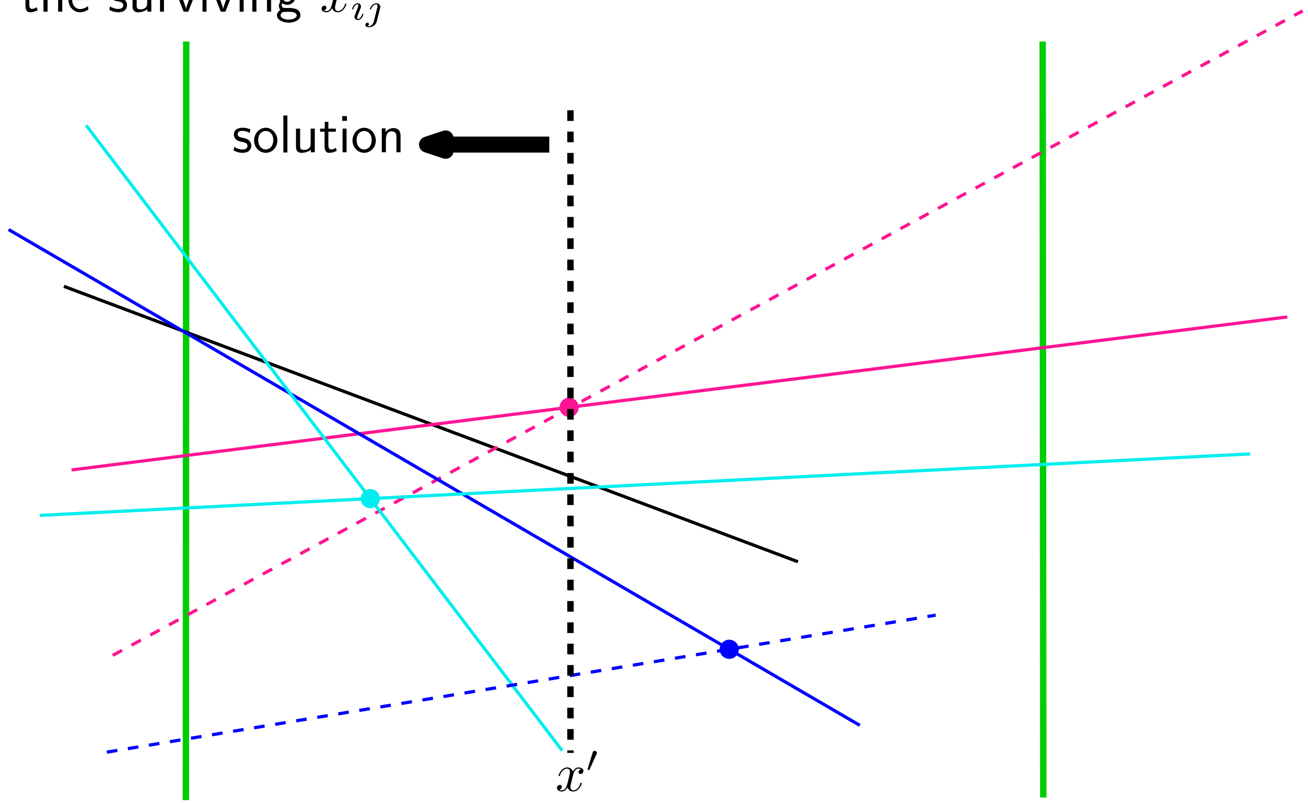

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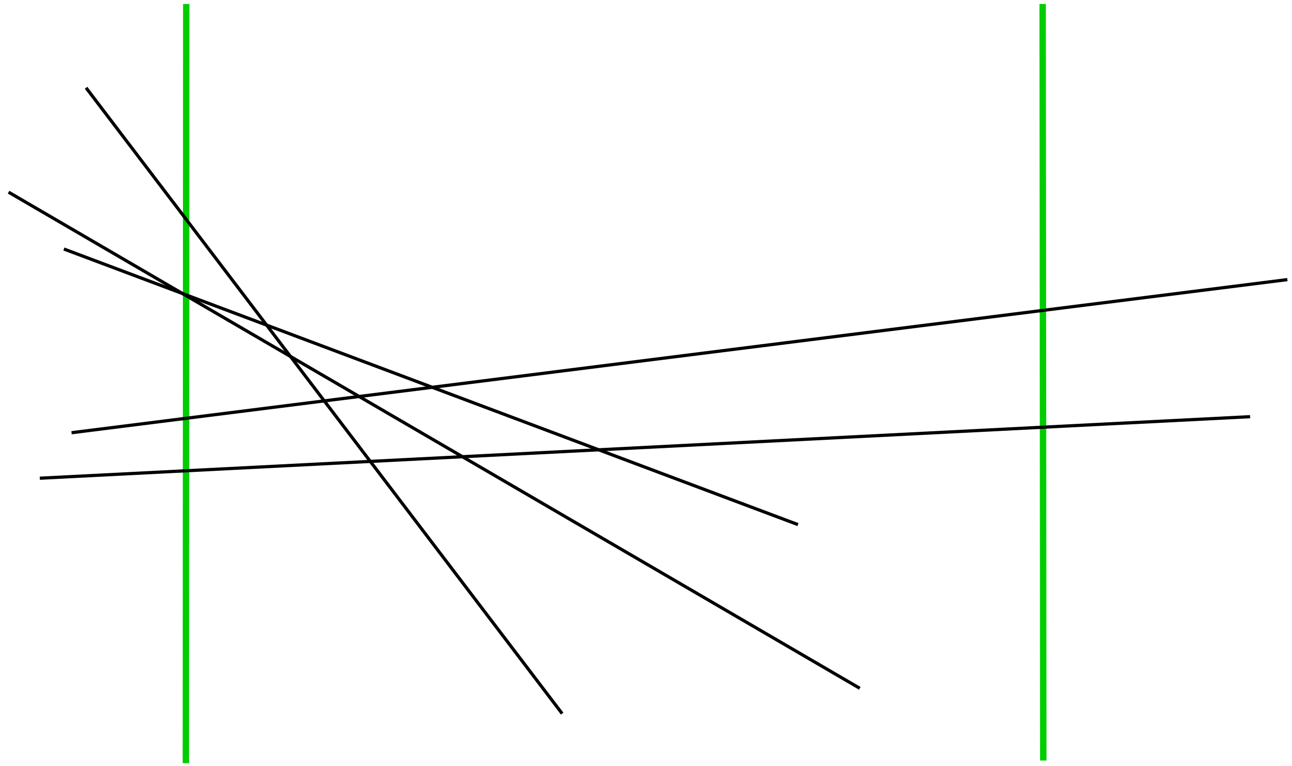

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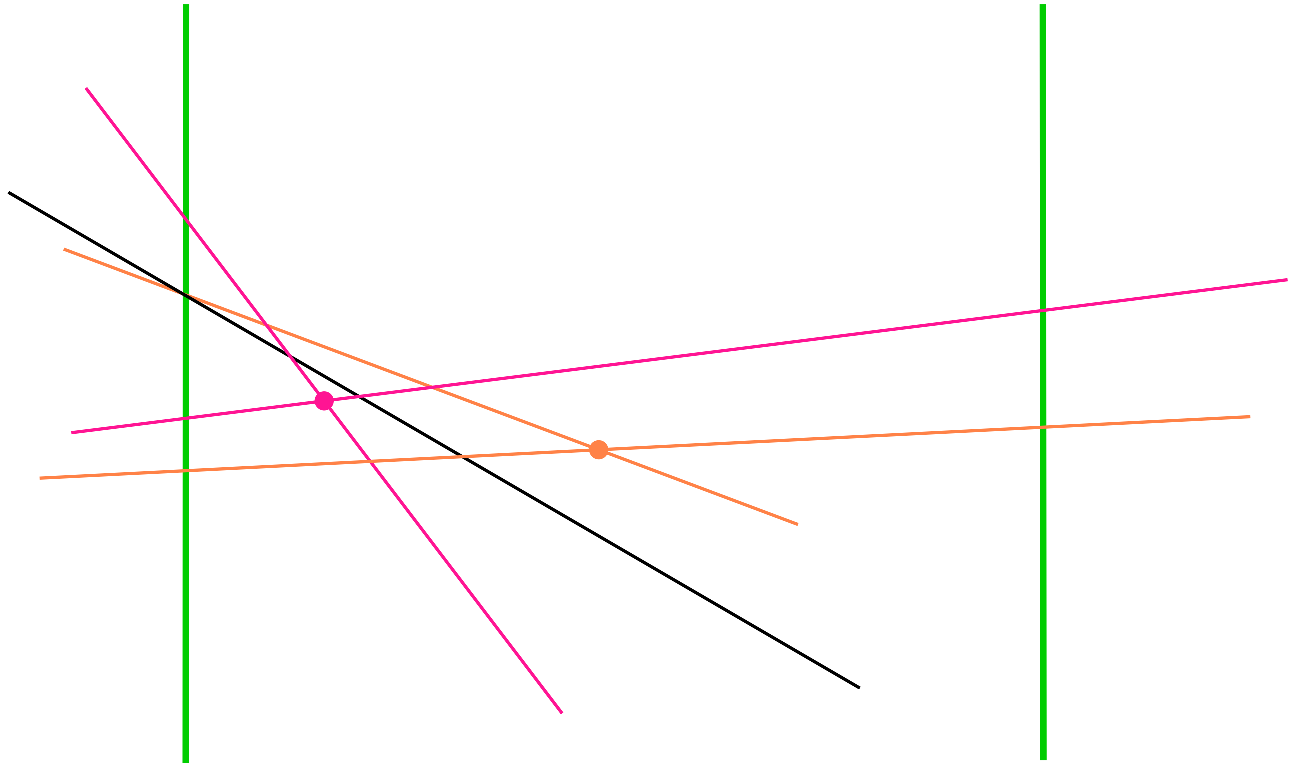

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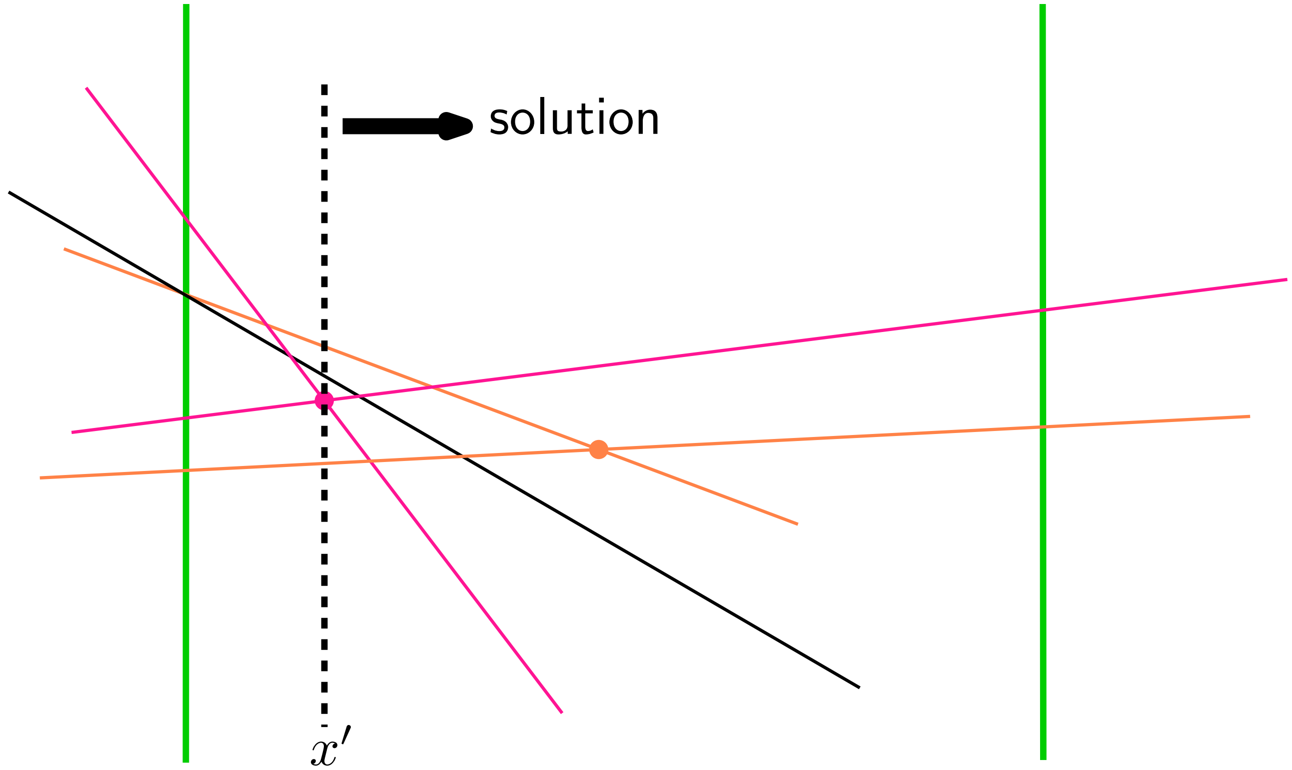

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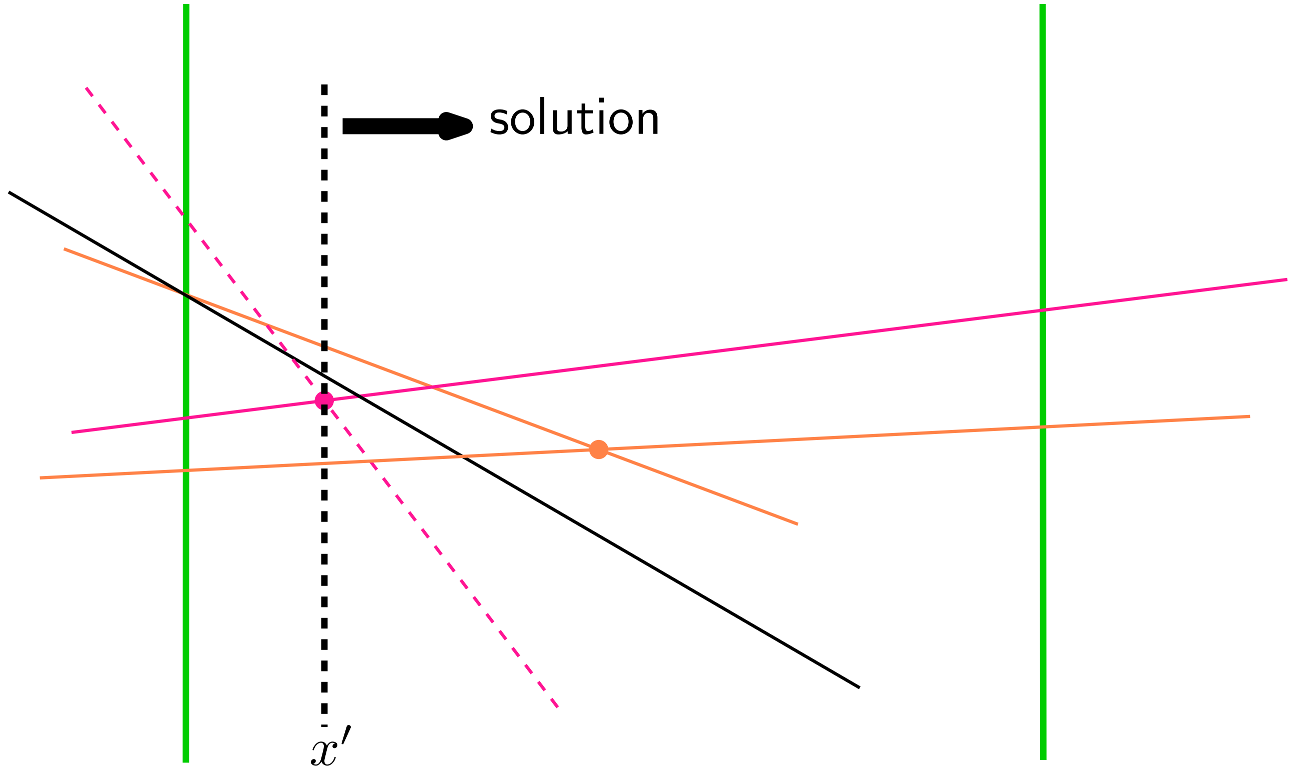

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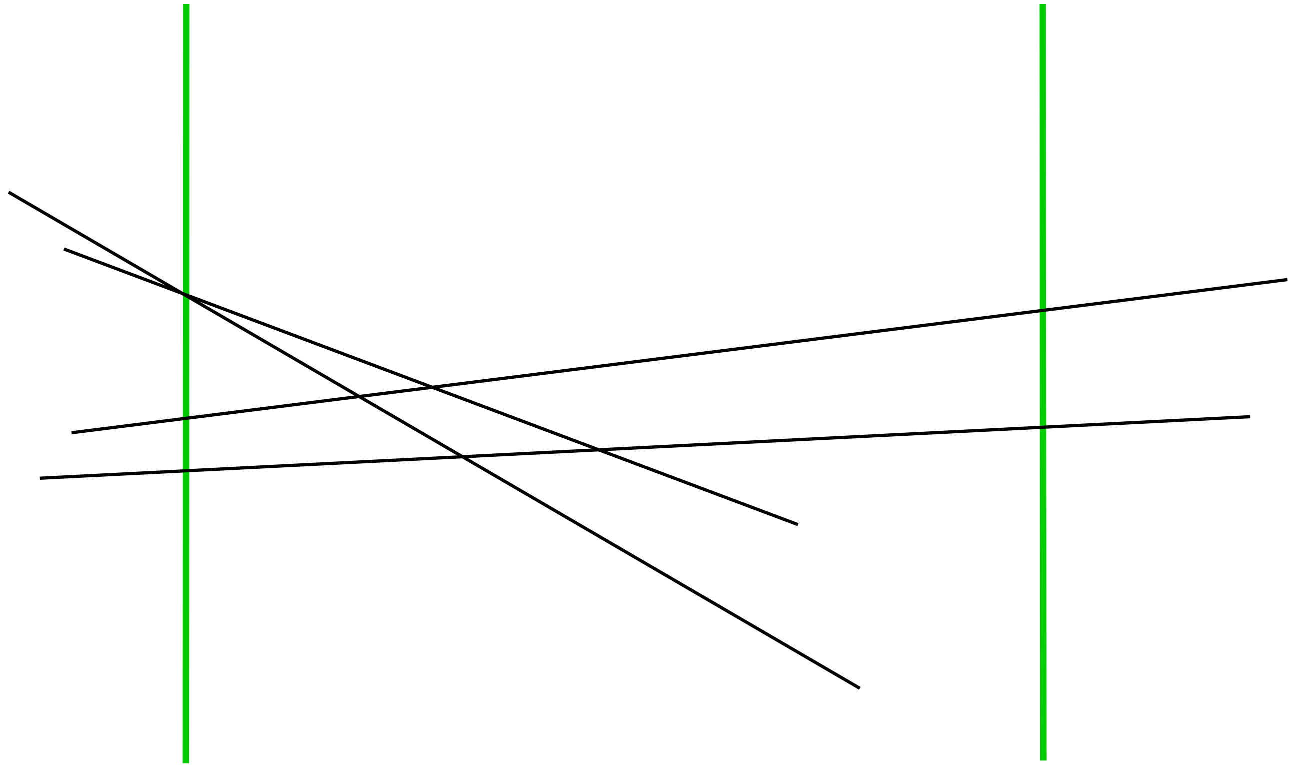

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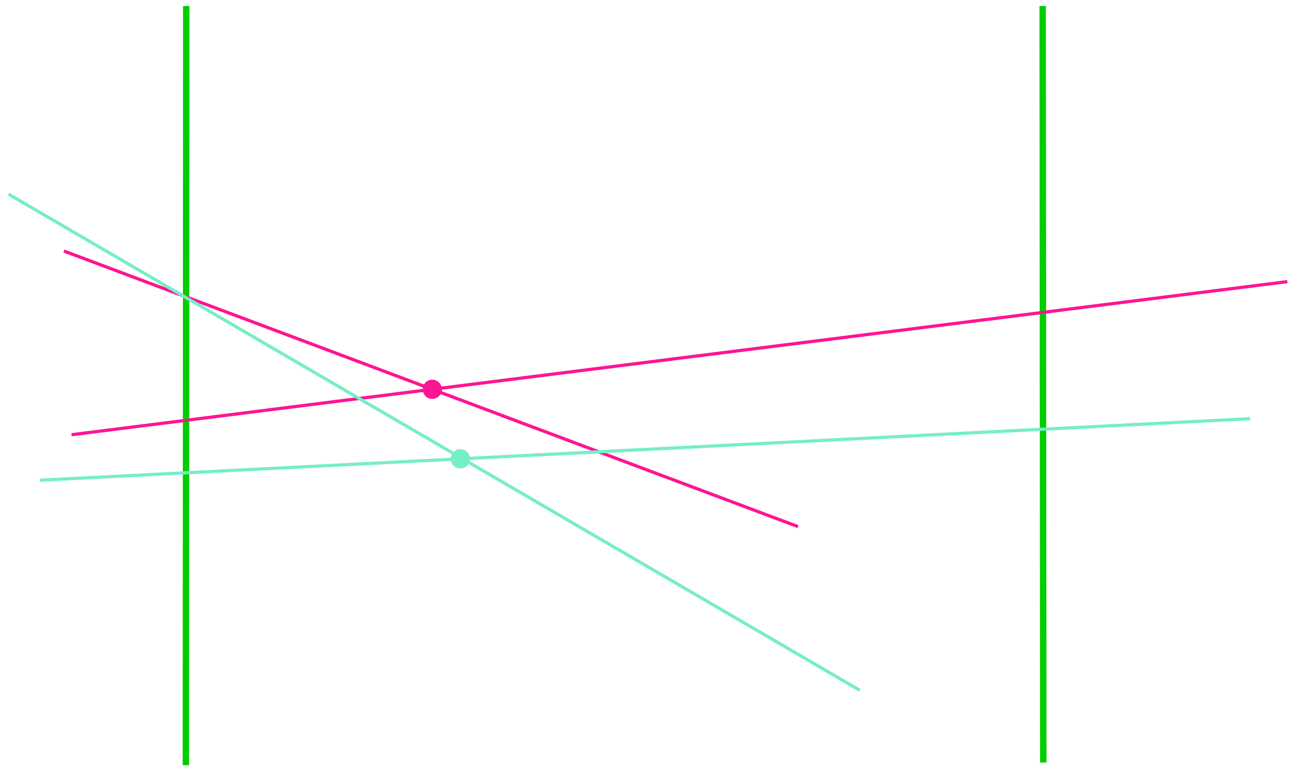

SOLVING LINEAR PROGRAMS

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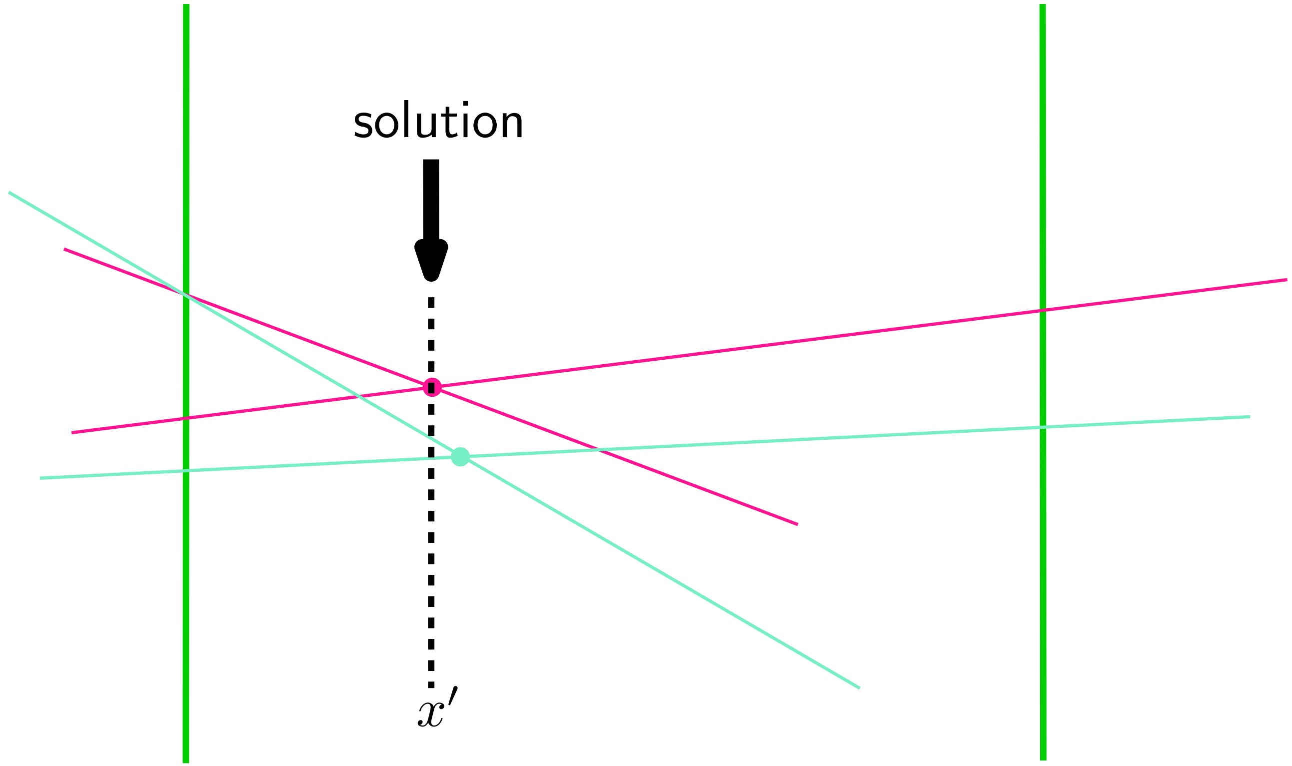

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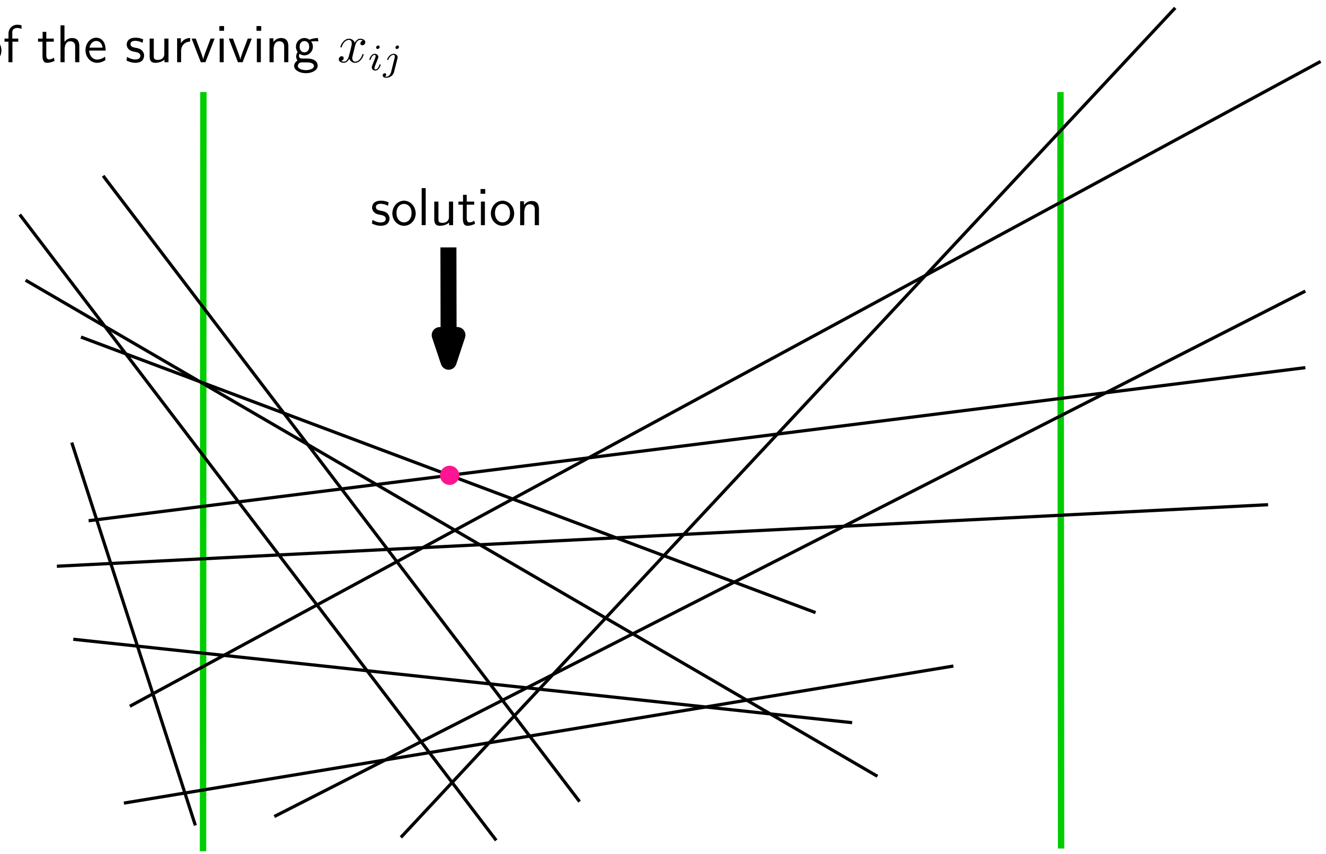

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Advance. Repeat as many times as necessary:

1. Pair up restrictions and compute the abscissa of the intersection points. $O(n)$
2. For all pairs, do: $O(n)$
   - If $d_i = d_j$, eliminate one of the two.
   - If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
     - If $x_{ij} < u_1$, eliminate one of the two.
     - If $x_{ij} > u_2$, eliminate one of the two.
3. Compute the median value x' of the surviving x_{ij}. $O(n)$ (see the appropriate reference)
4. Search: $O(n)$
5. Prune: $O(n)$

SOLVING LINEAR PROGRAMS

The algorithm

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How many restrictions are pruned at each step?

SOLVING LINEAR PROGRAMS

The algorithm

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5. Prune: $O(n)$

How many restrictions are pruned at each step? $O(n/4)$

The algorithm

Exact counting:

- The initial number of vertical restrictions is k, where $k \geq 0$.
- The vertical restrictions prune step eliminates all of them but at most 2.
- The initial number of pairs of non vertical lines (i.e., points x_{ij}) is $\frac{m}{2}$, $\frac{m-1}{2}$ or $\frac{m}{2} - 1$, depending on the parities of $|I_-|$ and $|I_+|$, where $m + k = n$.
- The prune step eliminates roughly half of them.

Alltogether, the eliminated restrictions are, at least:

$$k + \left\lfloor \frac{\frac{m}{2} - 1}{2} \right\rfloor = \left\lfloor \frac{m}{4} + k - \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{4} + \frac{3}{4}k - \frac{1}{2} \right\rfloor \geq \frac{n}{4}.$$

SOLVING LINEAR PROGRAMS

The algorithm

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Running time

Hence, the running time of the algorithm is

$$cn + c\tfrac{3}{4}n + c\left(\tfrac{3}{4}\right)^2 n + c\left(\tfrac{3}{4}\right)^3 n + \cdots = cn \sum_{k=o}^{\lceil \log_{3/4} n \rceil} \left(\tfrac{3}{4}\right)^k < cn \frac{1}{1 - \frac{3}{4}} = 4cn \in O(n).$$

MINIMUM SPANNING (or ENCLOSING) CIRCLE

MIN-MAX FACILITY LOCATION

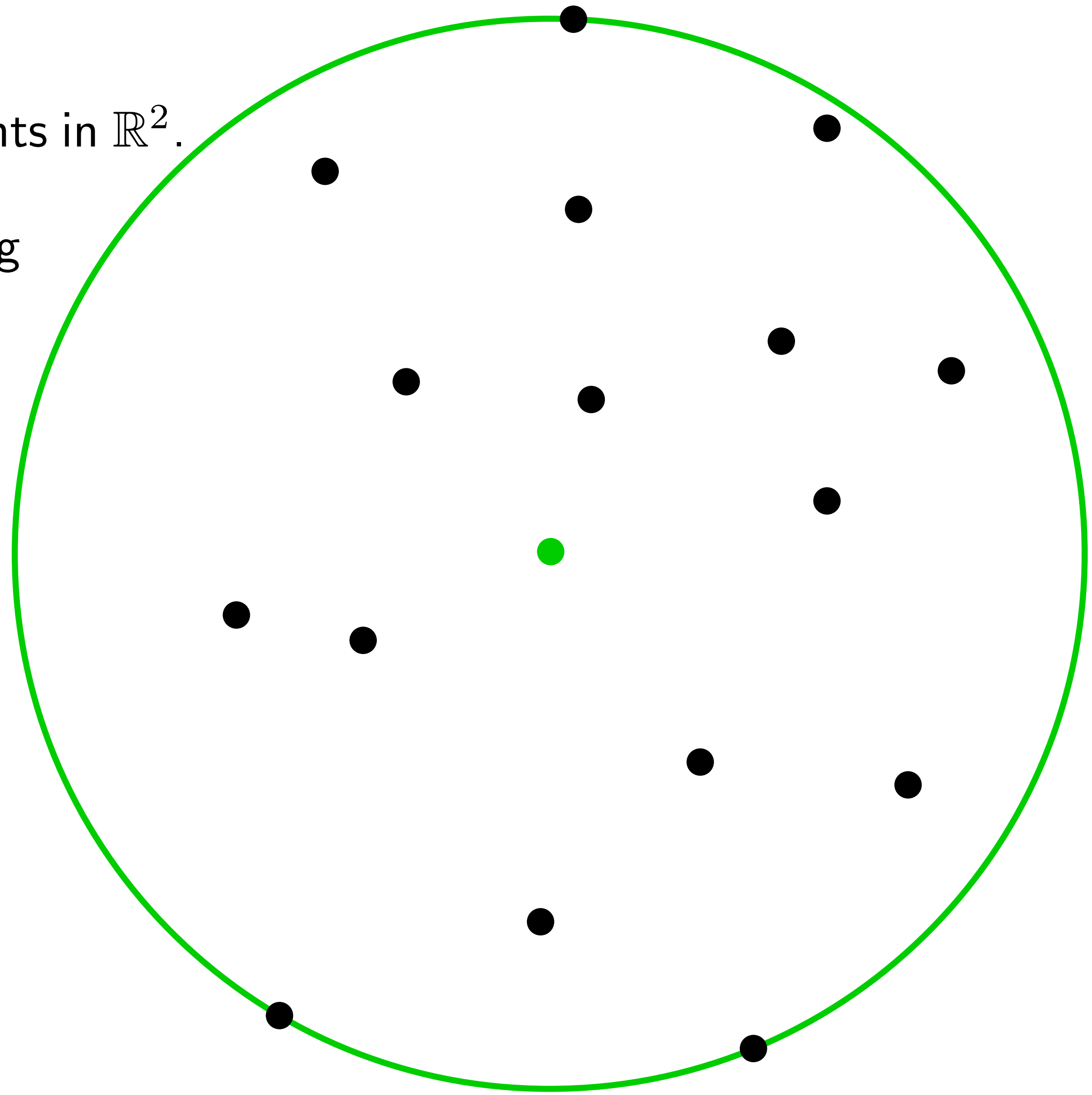
Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{R}^2$.

Find the point x on the plane achieving

$$\min_{x \in \mathbb{R}^2} \max_{p_i \in P} d(x, p_i).$$

Geometrically

Find the center of the circle of minimum radius enclosing P.



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

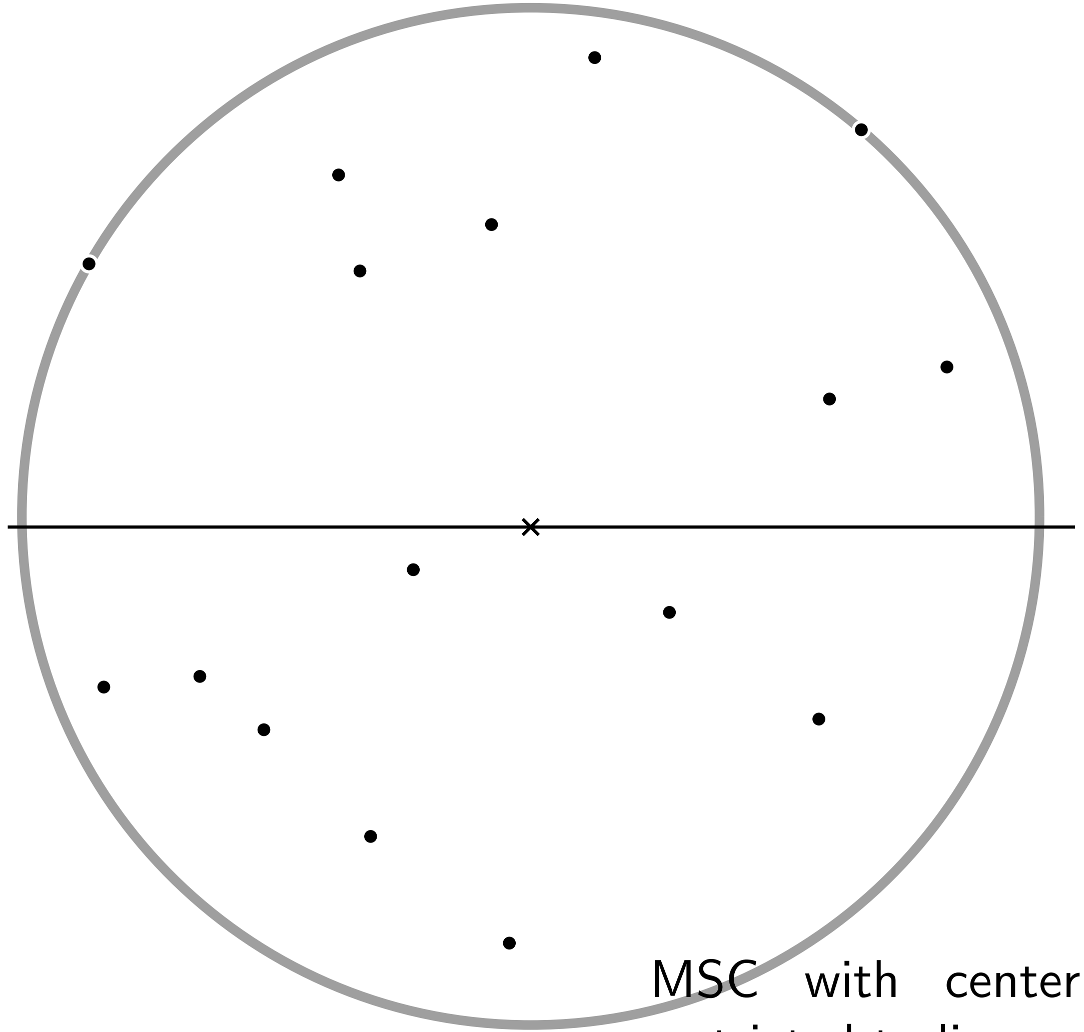
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1\ldots n} (x - a_i)^2 + b_i^2$.

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \dots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.



MSC with center restricted to line

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
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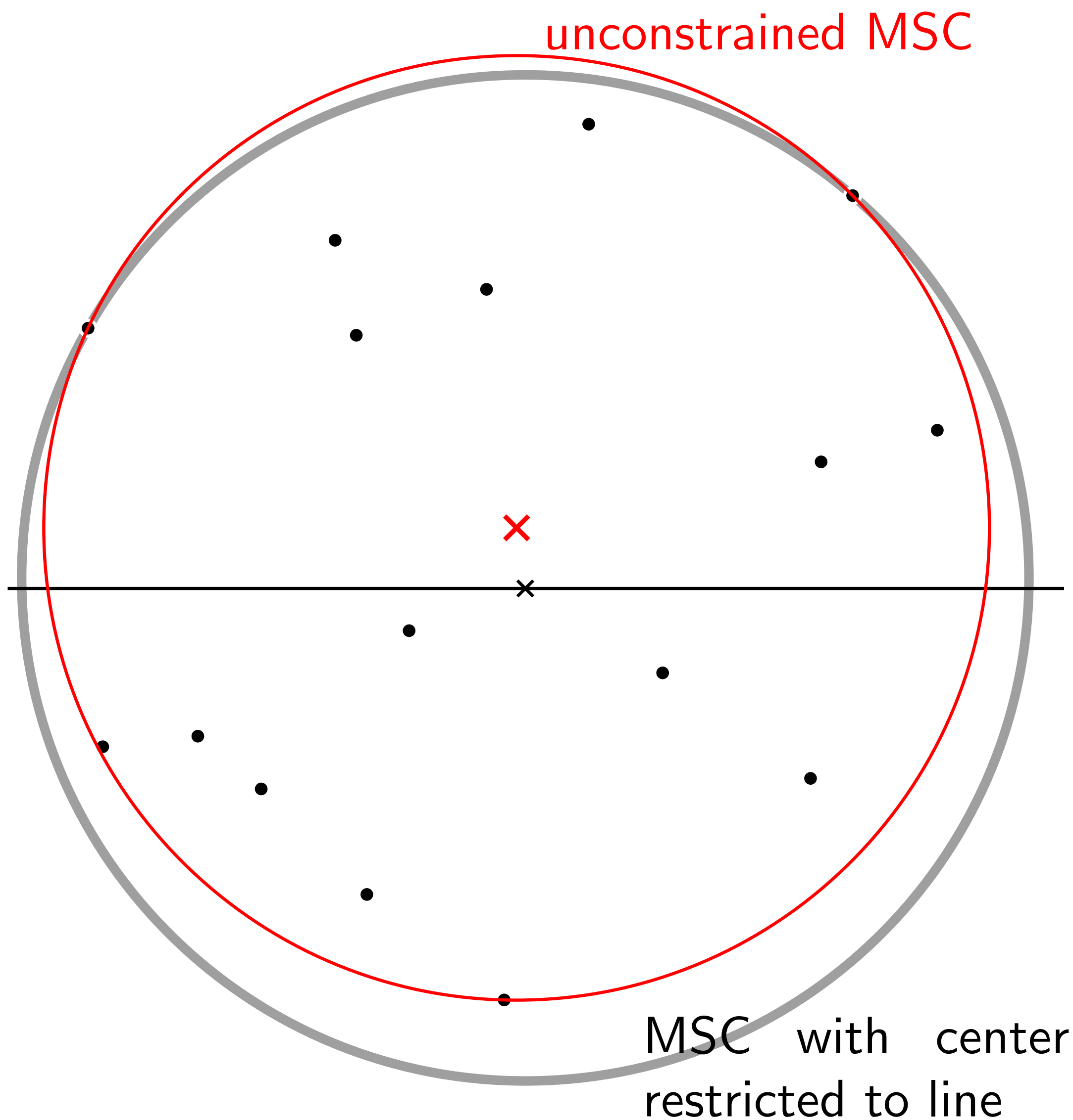

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1\ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j, compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
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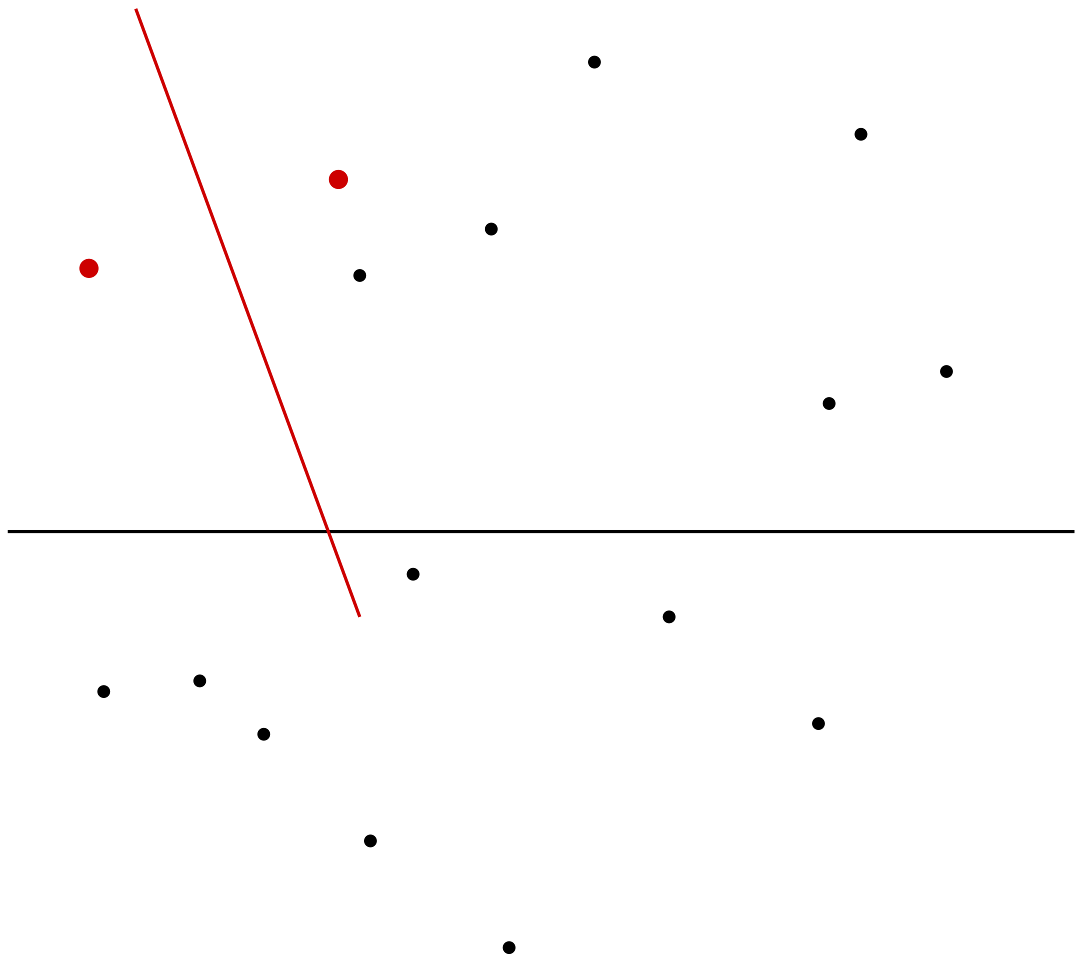

COMPUTING THE MINIMUM SPANNING CIRCLE

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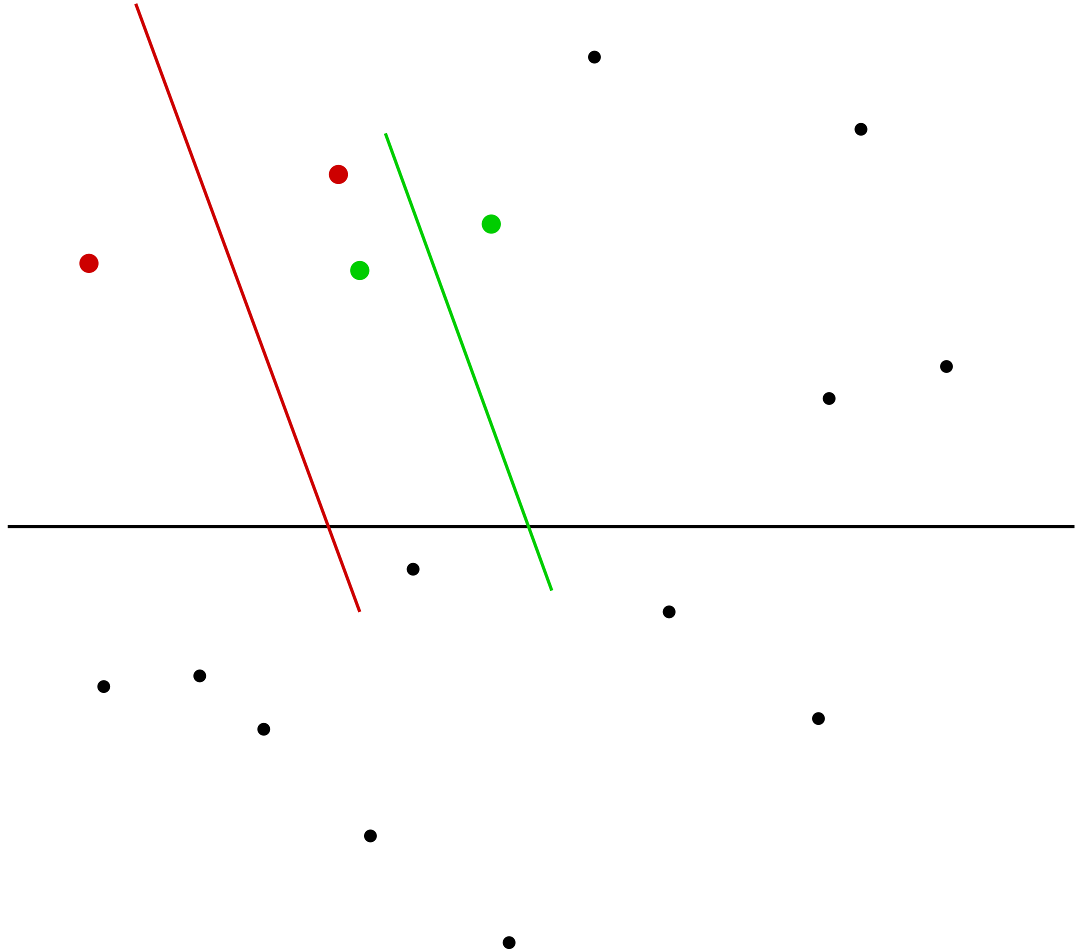

COMPUTING THE MINIMUM SPANNING CIRCLE

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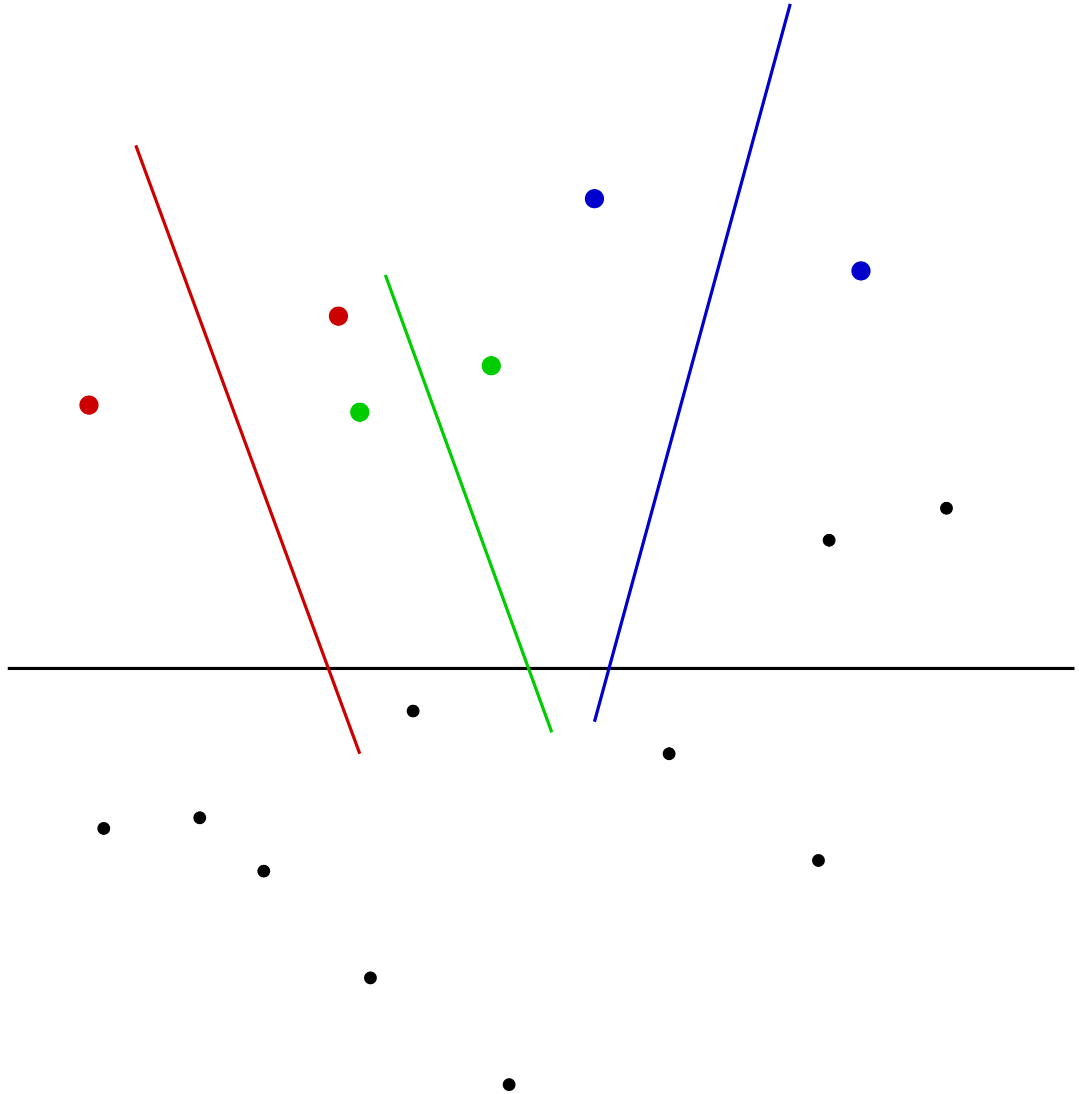

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \dots, n$.
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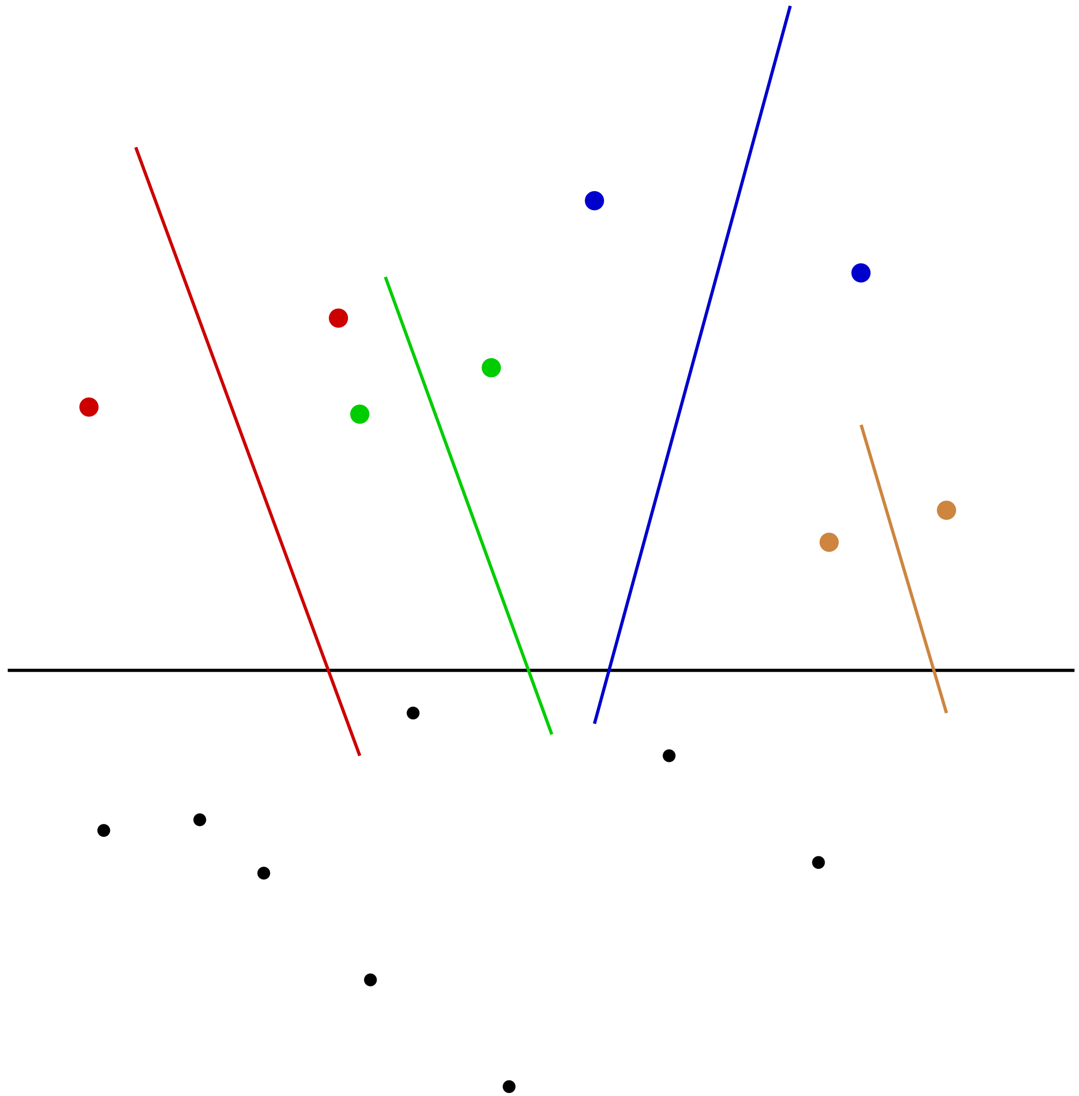

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1\ldots n} (x - a_i)^2 + b_i^2$.

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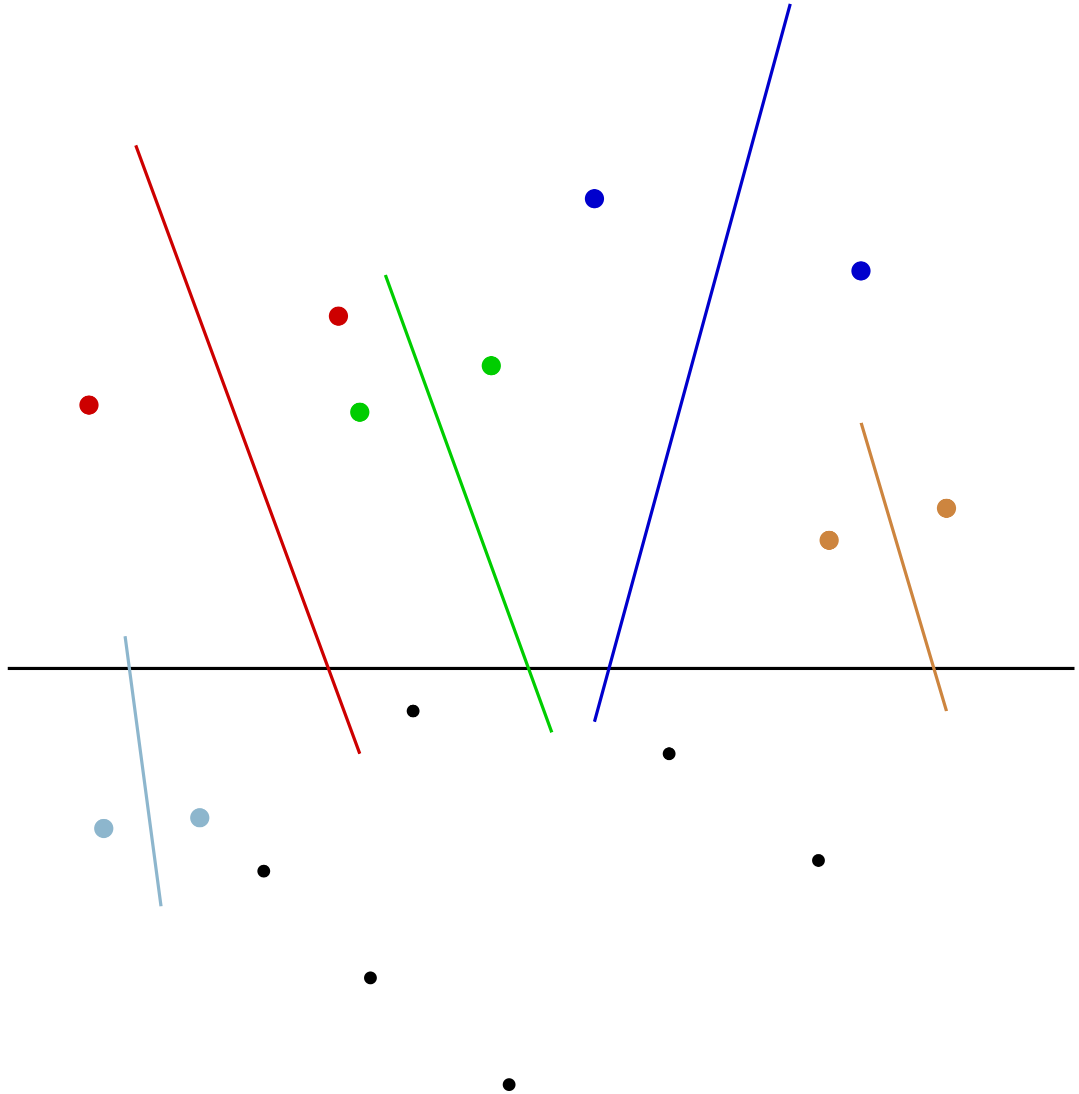

COMPUTING THE MINIMUM SPANNING CIRCLE

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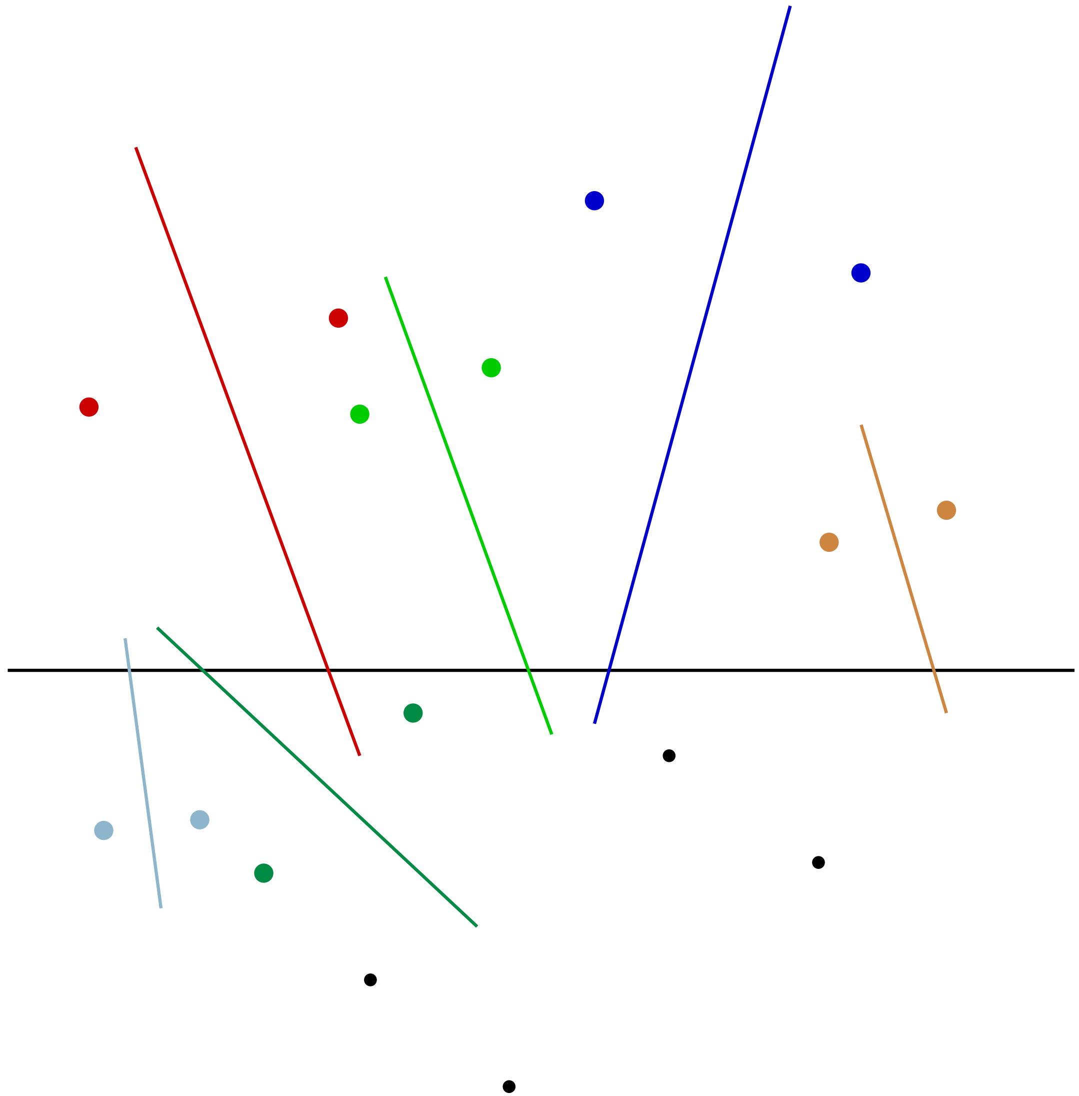

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
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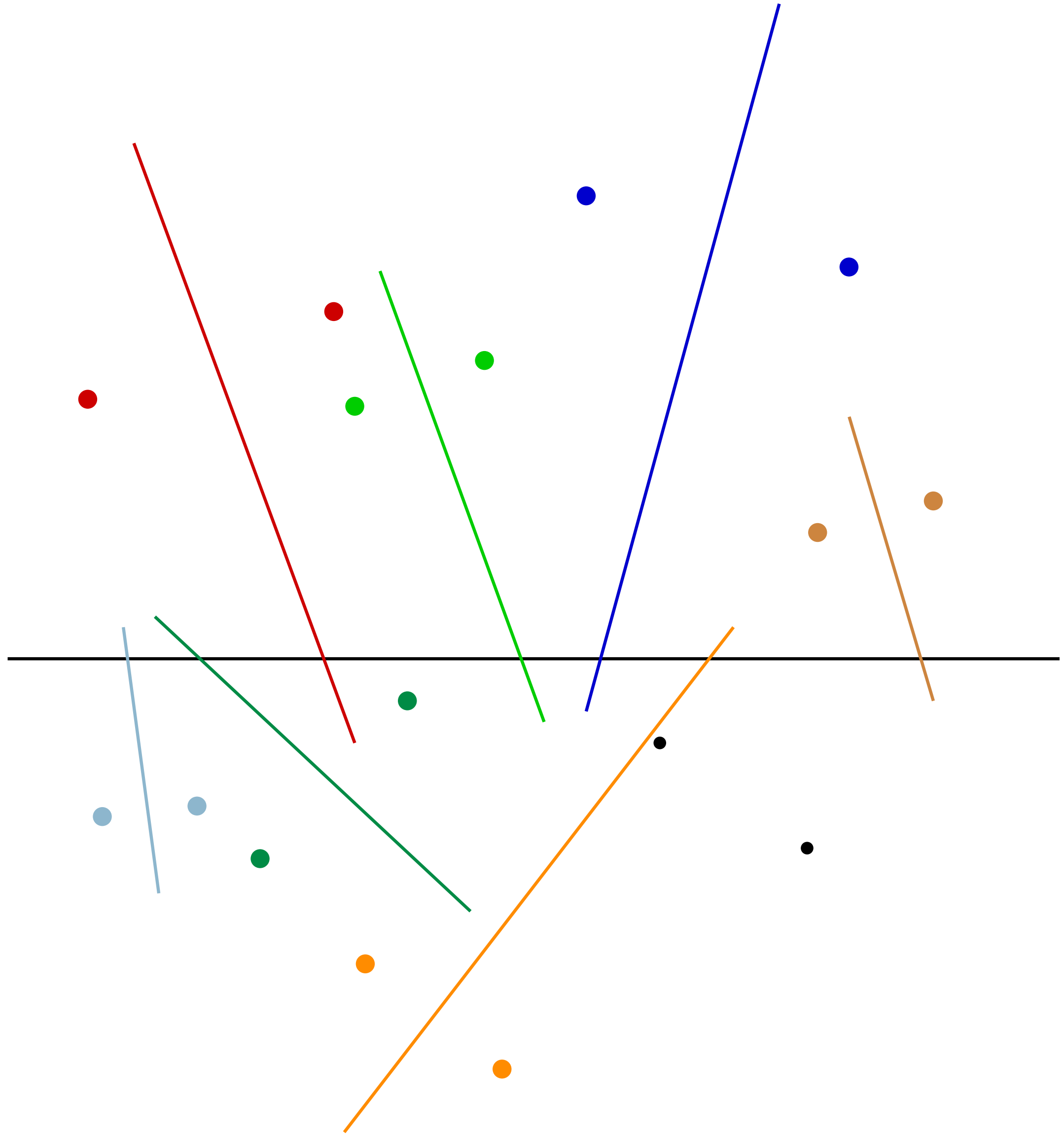

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1 \ldots n} (x - a_i)^2 + b_i^2$.

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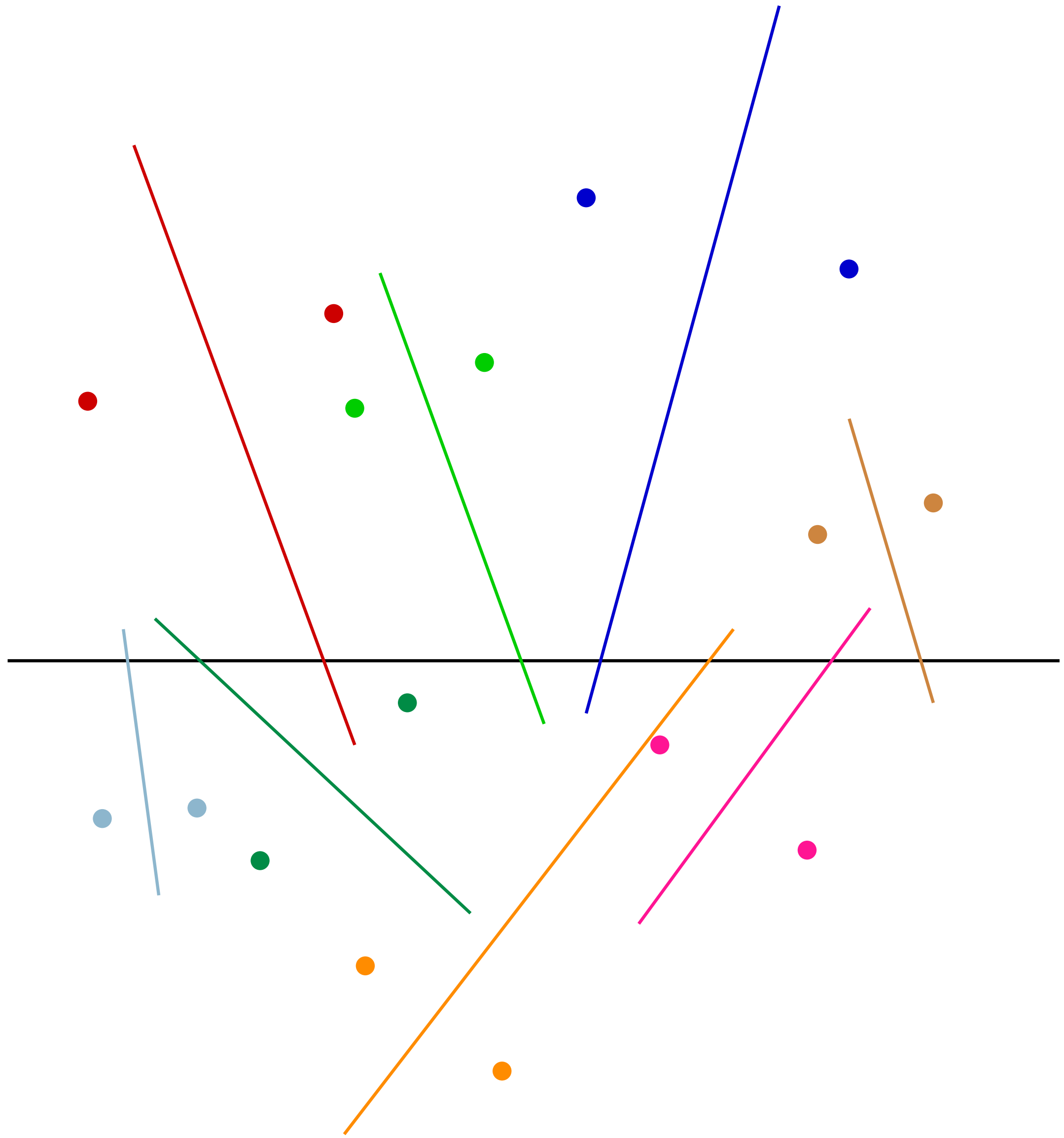


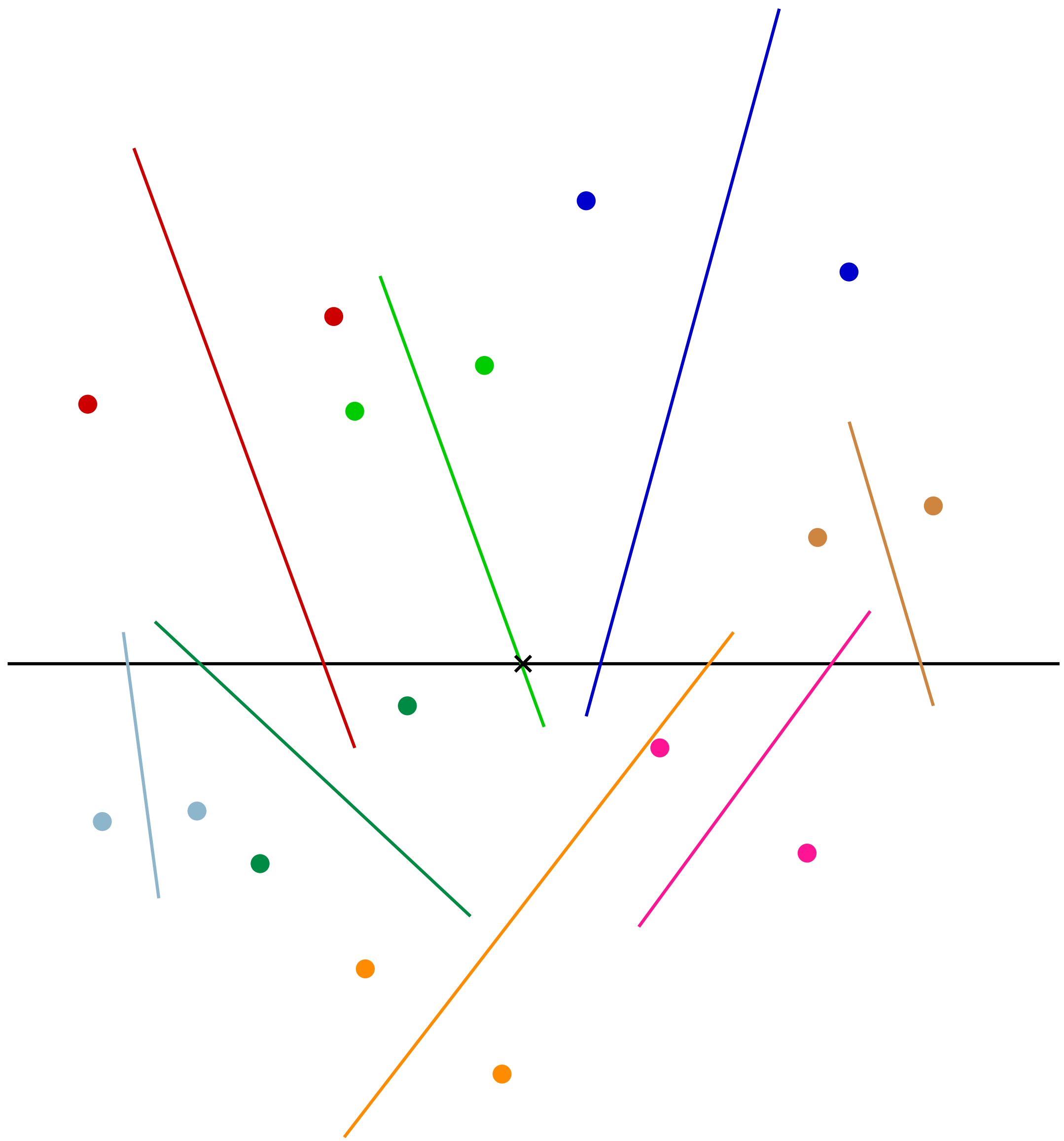
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1\ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j, compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.

2. Compute x_m, the median value of the x_{ij}.

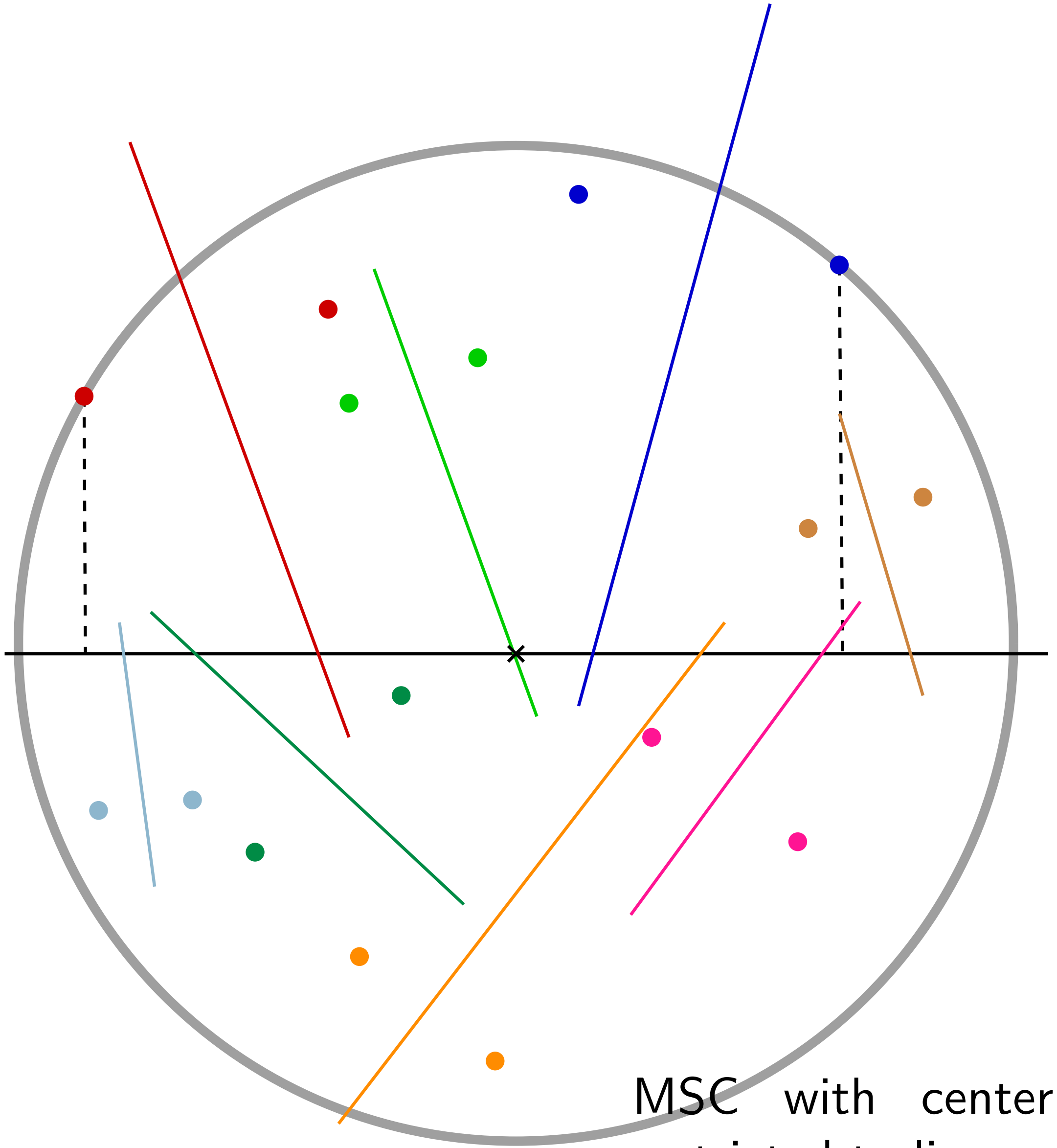


COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1\ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j, compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m, the median value of the x_{ij}.
3. Search: Compute C_m, the MSC centered at x_m. If the points p_i lying in the boundary of C_m project ortogonally onto $y = 0$ to different sides of x_m, then x_m is the solution. If they all project onto the same side then search on that side.



MSC with center restricted to line

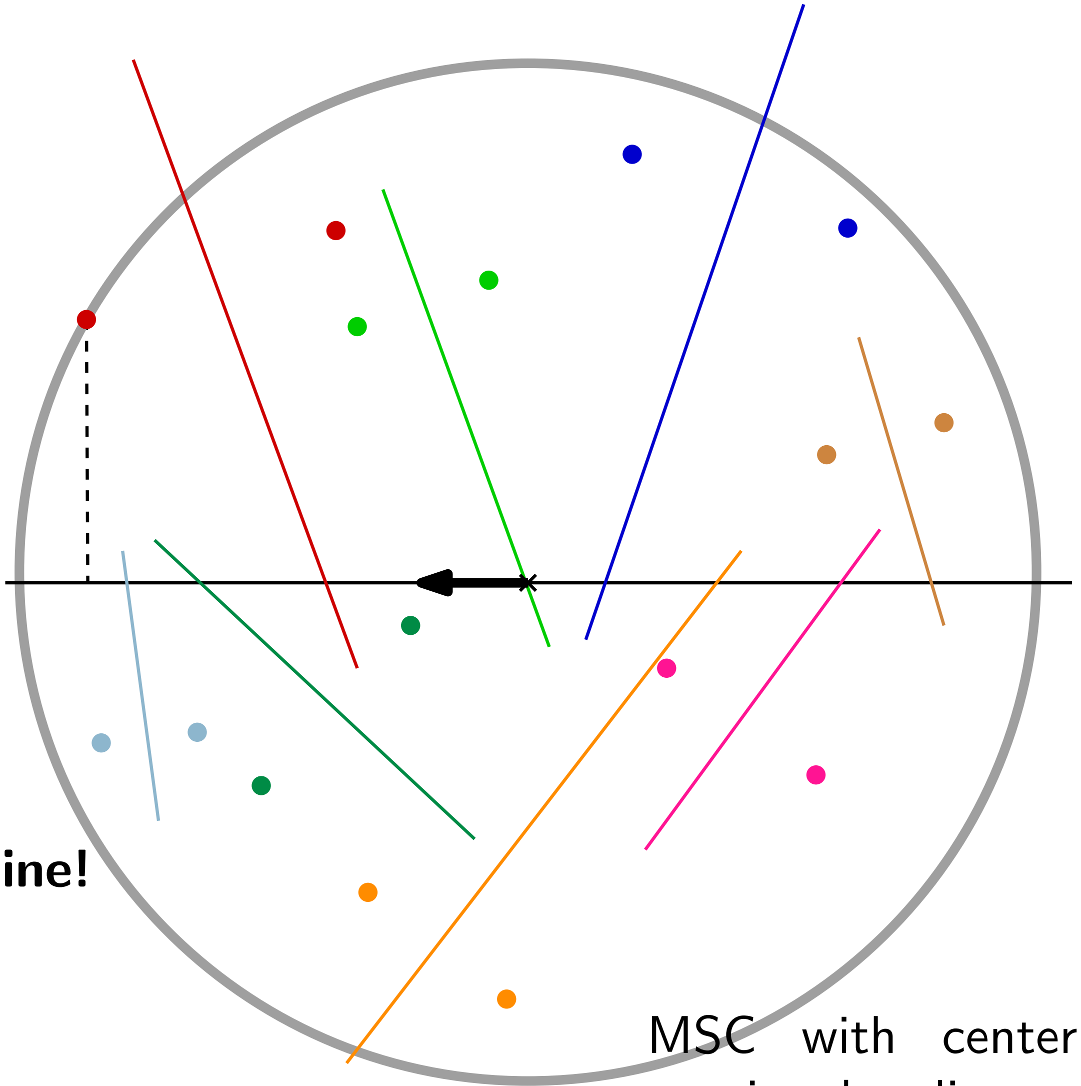
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min\limits_{x \in \mathbb{R}} \max\limits_{i=1 \ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j, compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m, the median value of the x_{ij}.
3. Search: Compute C_m, the MSC centered at x_m. If the points p_i lying in the boundary of C_m project ortogonally onto $y = 0$ to different sides of x_m, then x_m is the solution. If they all project onto the same side then search on that side.

Notice that the radius is unimodal along the line!



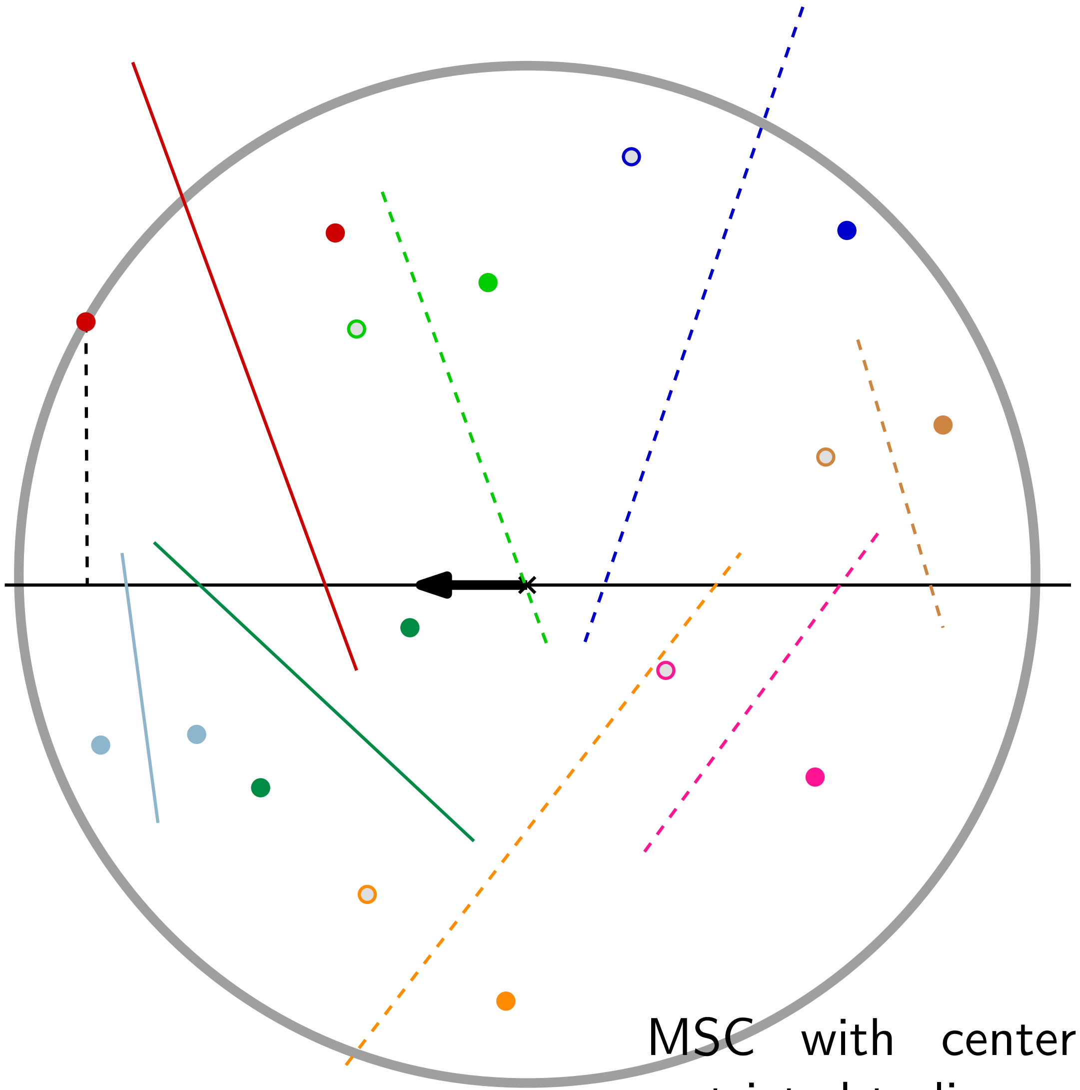
MSC with center restricted to line

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1\ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j, compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m, the median value of the x_{ij}.
3. Search: Compute C_m, the MSC centered at x_m. If the points p_i lying in the boundary of C_m project ortogonally onto $y = 0$ to different sides of x_m, then x_m is the solution. If they all project onto the same side then search on that side.
4. Prune: For all x_{ij} located in the side opposite to the solution, eliminate the point p_i or p_j which is closest to x_m.



MSC with center restricted to line

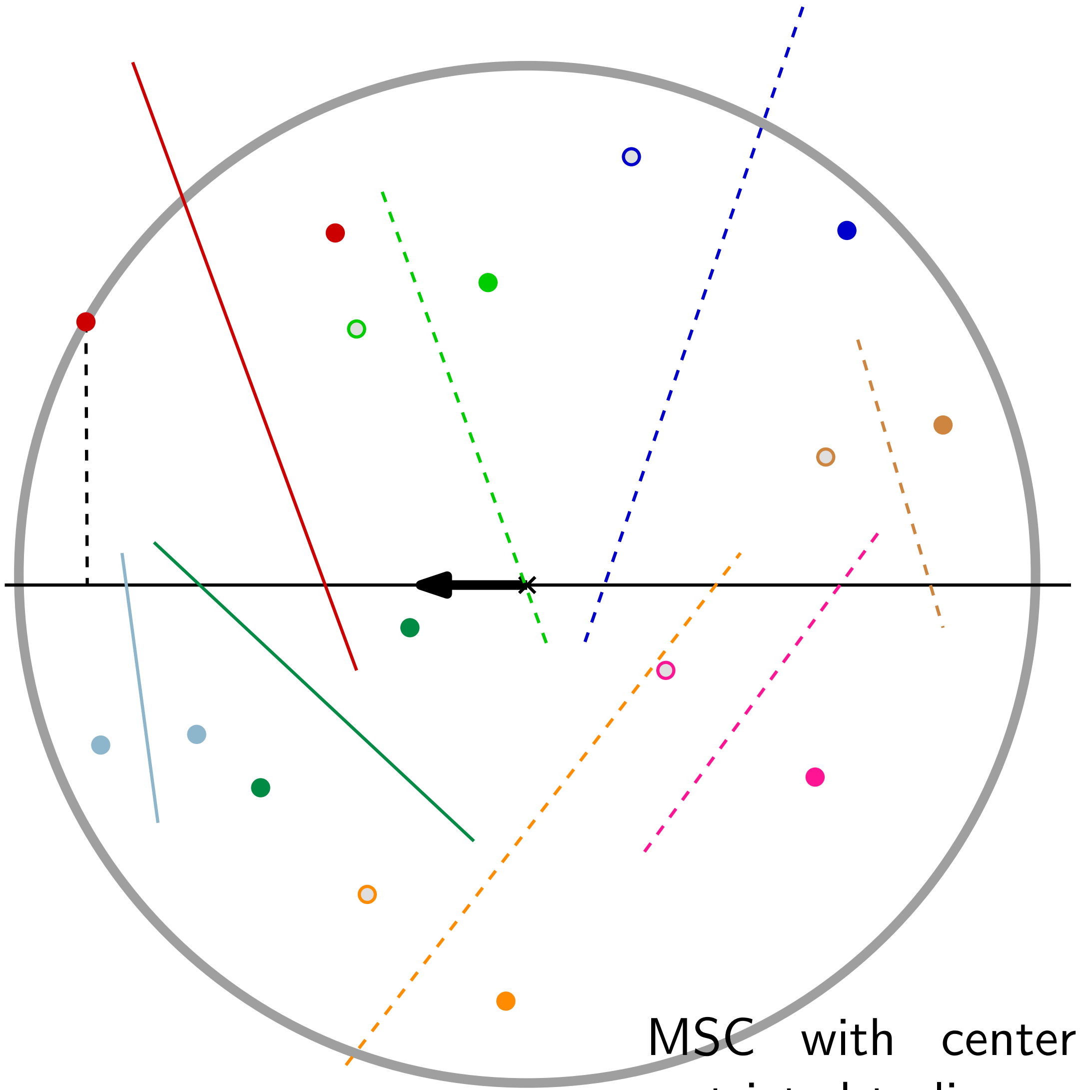
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line $y = 0$

Input: a set of n points (a_i, b_i), $i = 1, \ldots, n$.
Output: $\min_{x \in \mathbb{R}} \max_{i=1\ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points, compute the perpendicular bisector of each pair, and find its intersection with the line $y = 0$. $O(n)$
2. Compute a median value. $O(n)$
3. Search: $O(n)$
4. Prune: In $O(n)$ time, at least $1/4$ of the input is pruned.

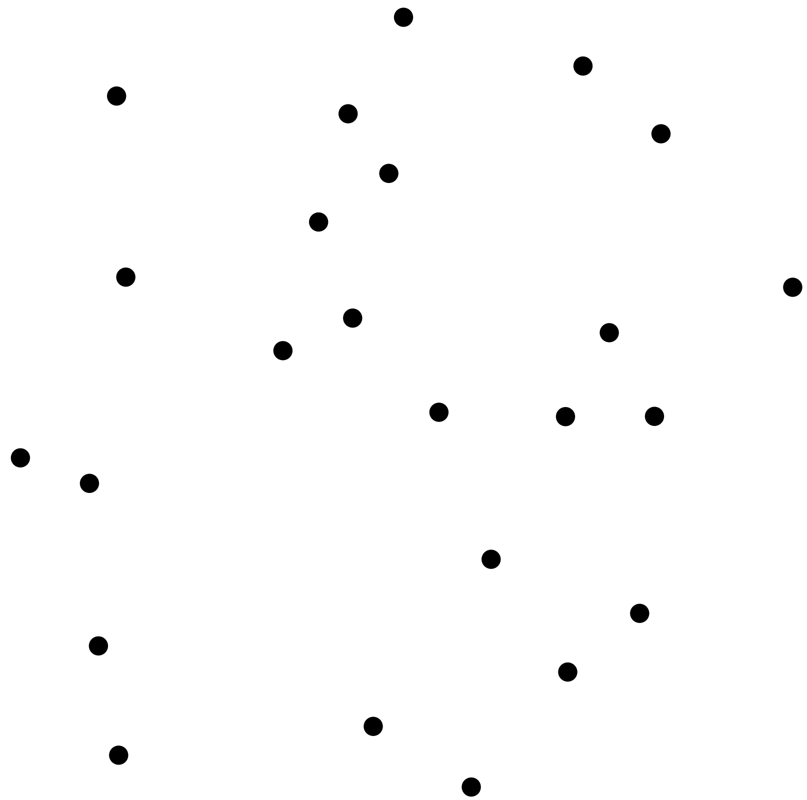
The problem is solved in $O(n)$ time.



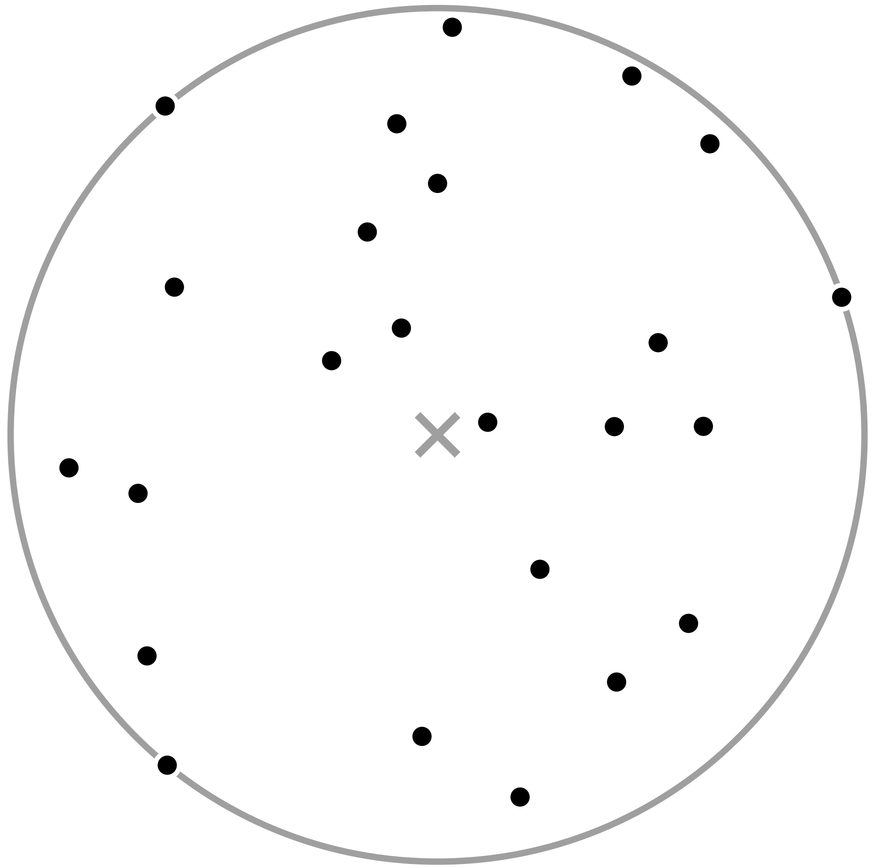
MSC with center restricted to line

MSC in the plane

MSC in the plane

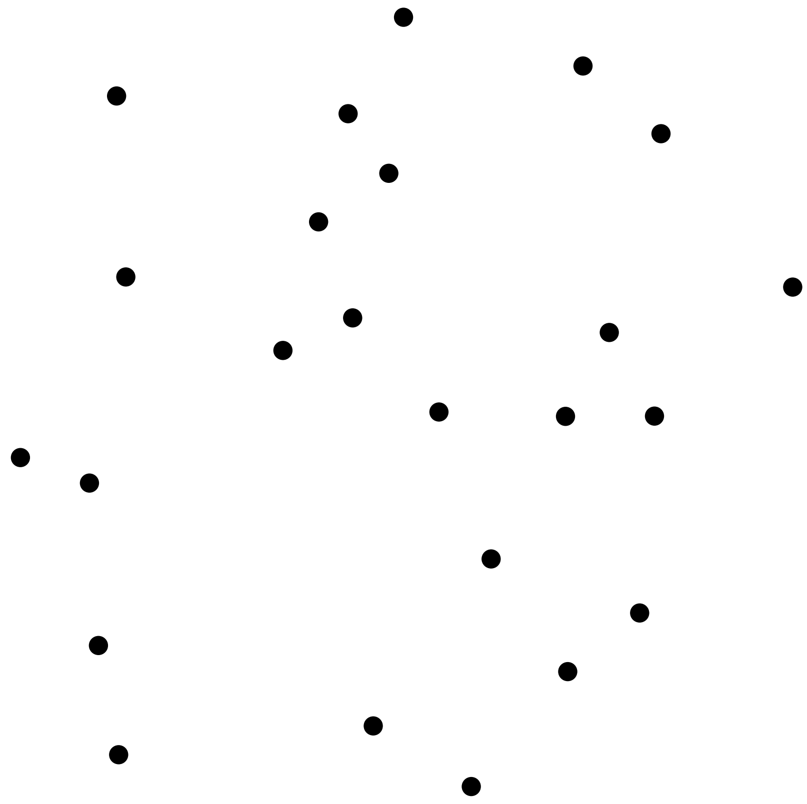


MSC in the plane



MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair.



MSC in the plane

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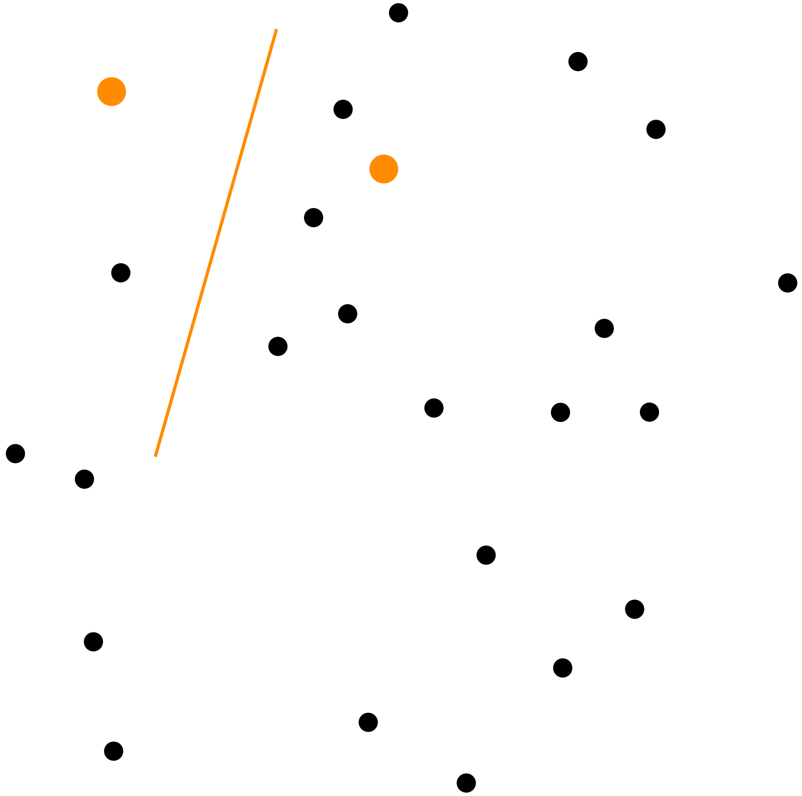

MSC in the plane

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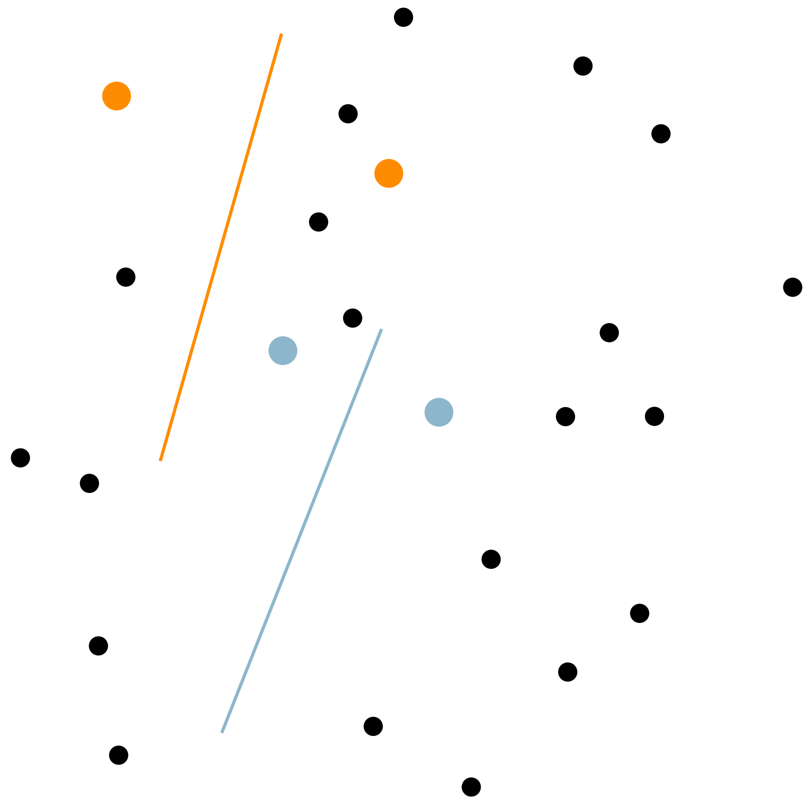

MSC in the plane

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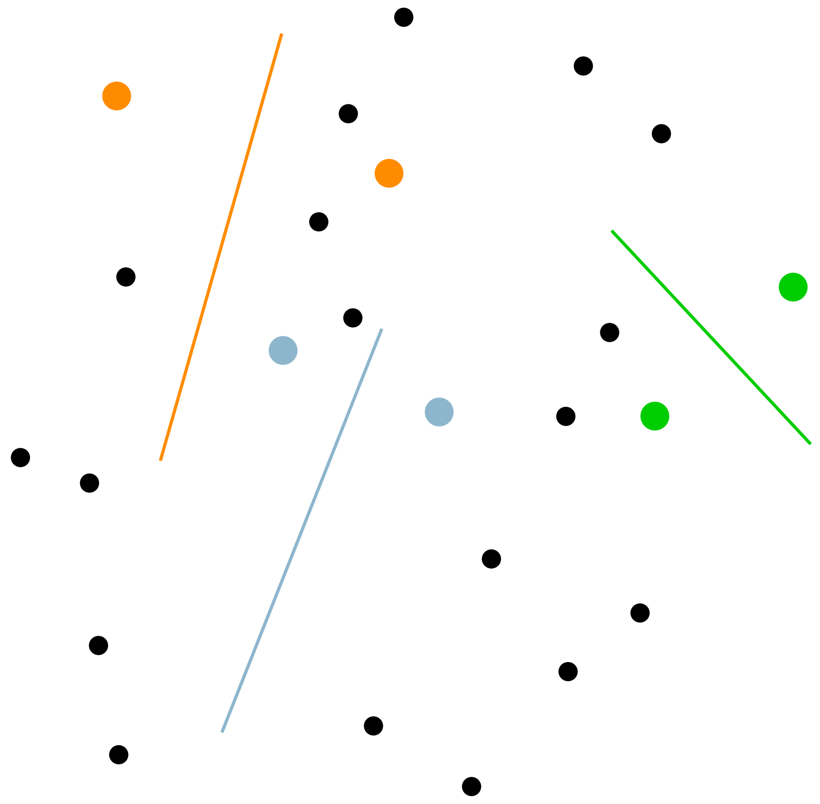

MSC in the plane

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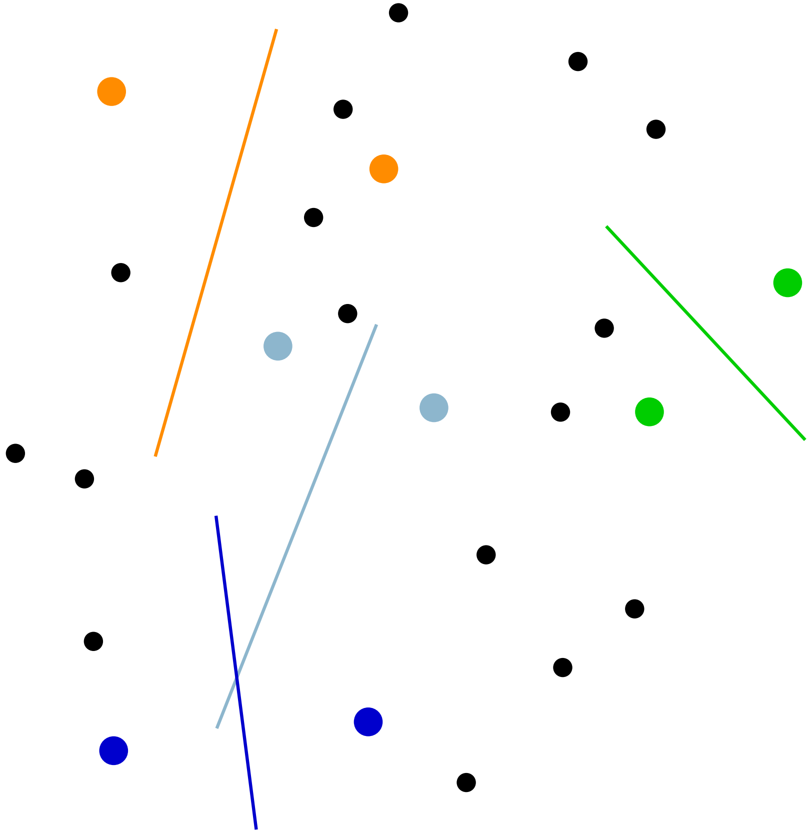

MSC in the plane

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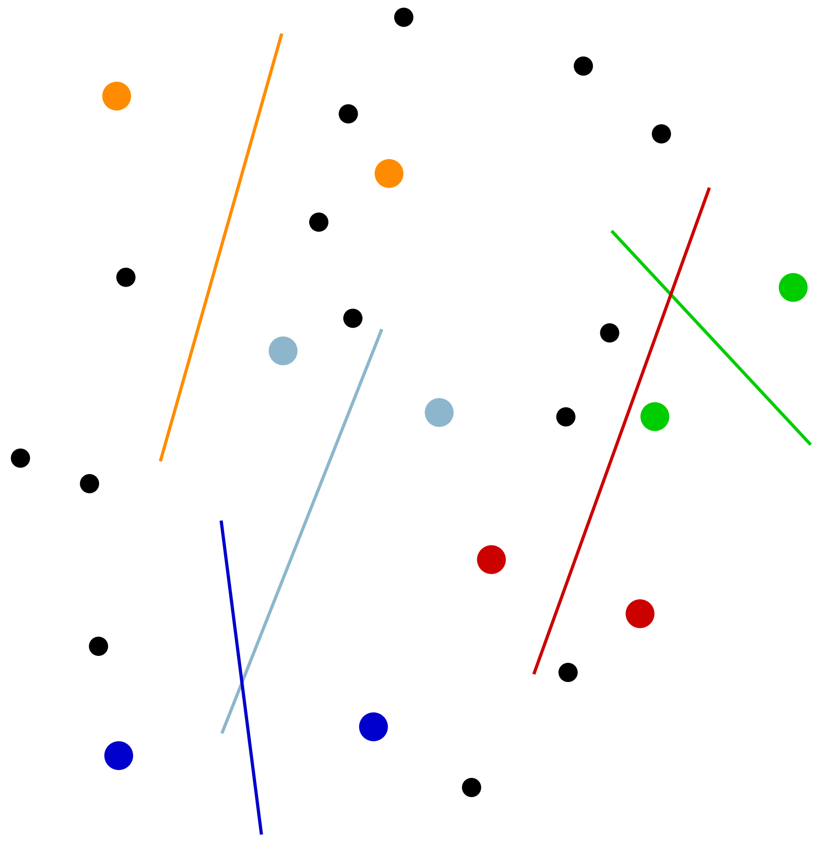

MSC in the plane

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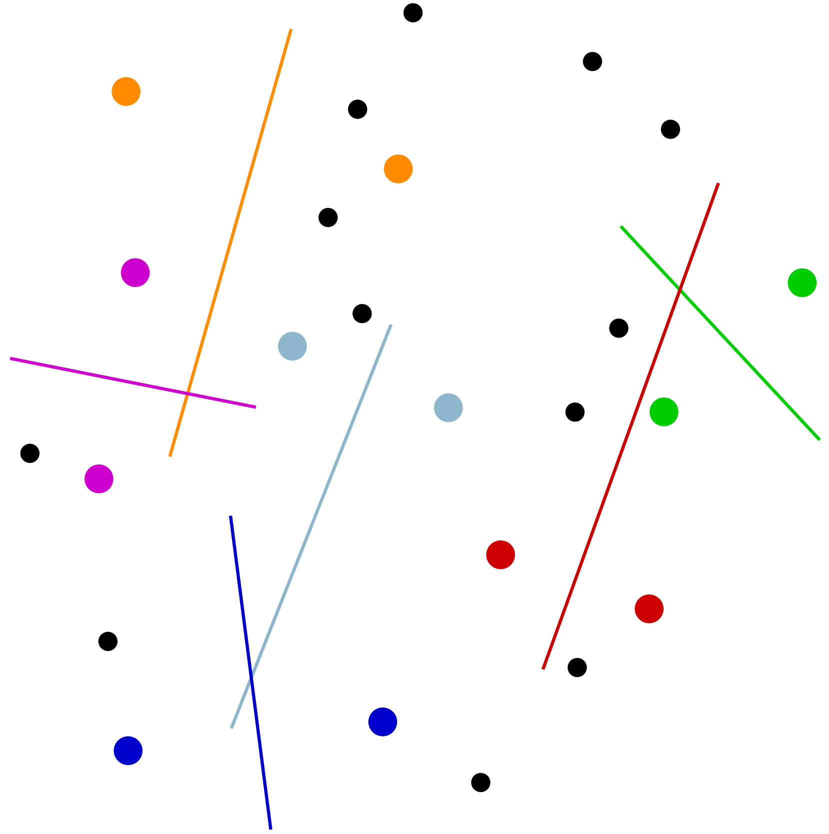
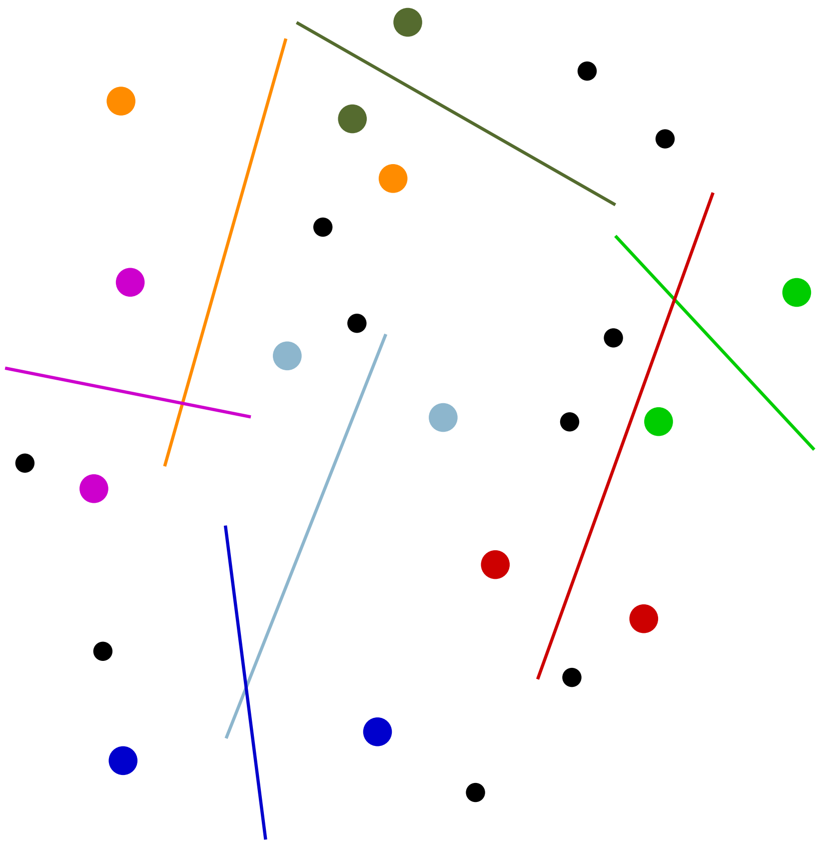


COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair.



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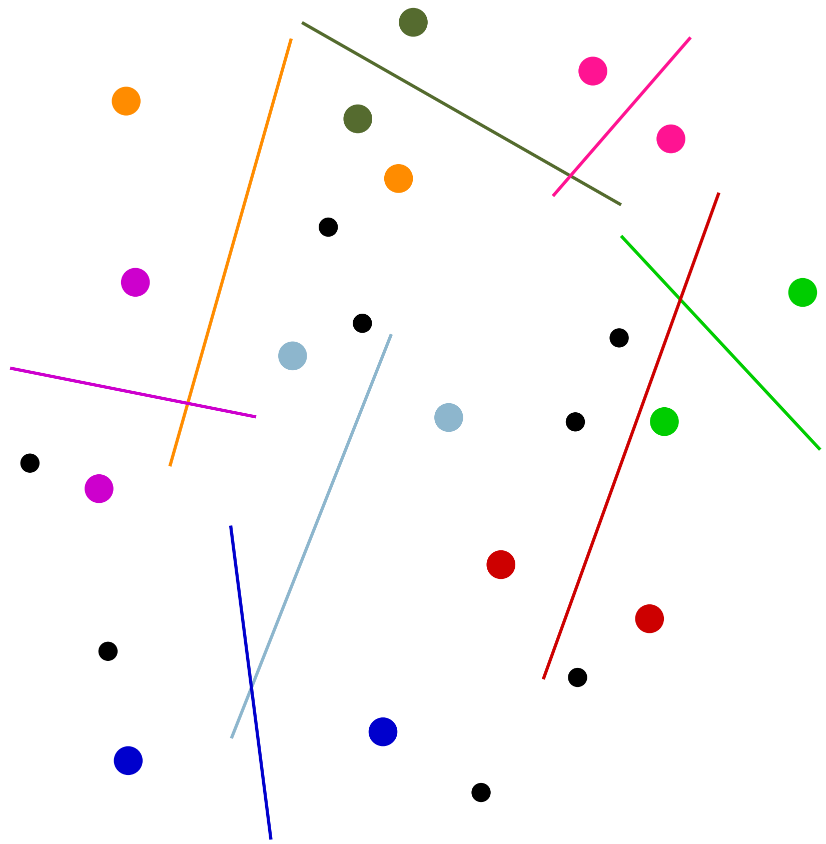

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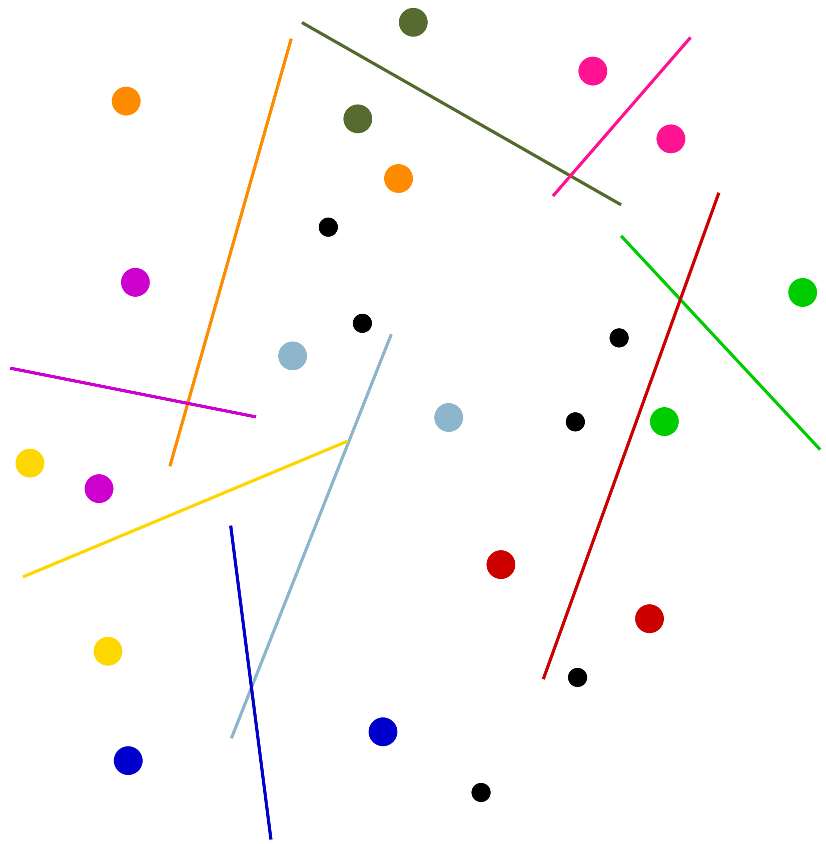

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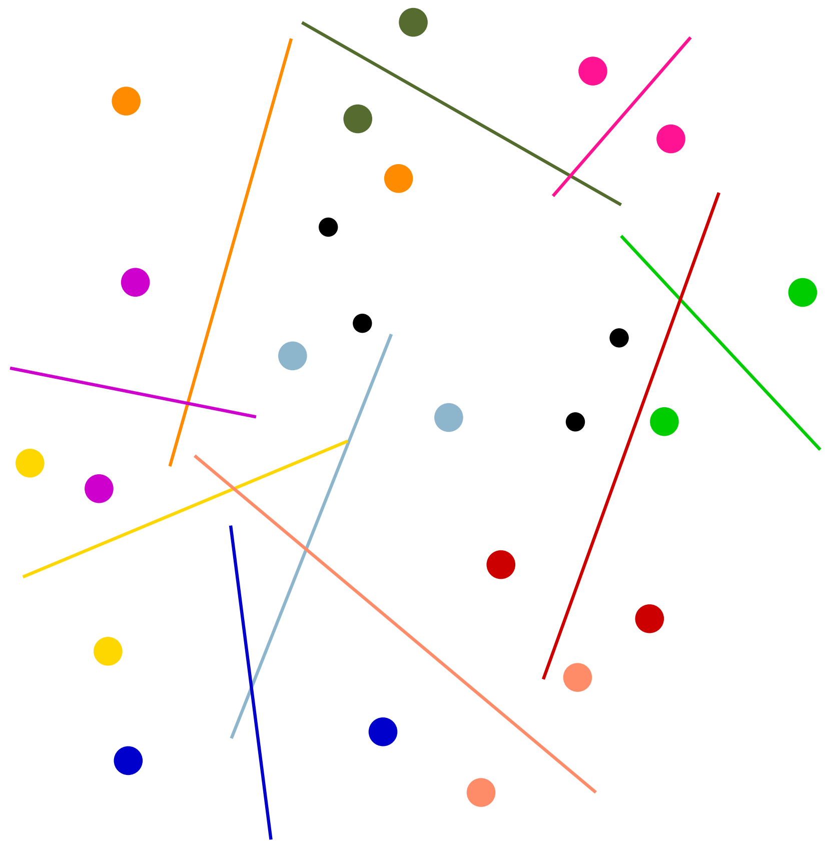

MSC in the plane

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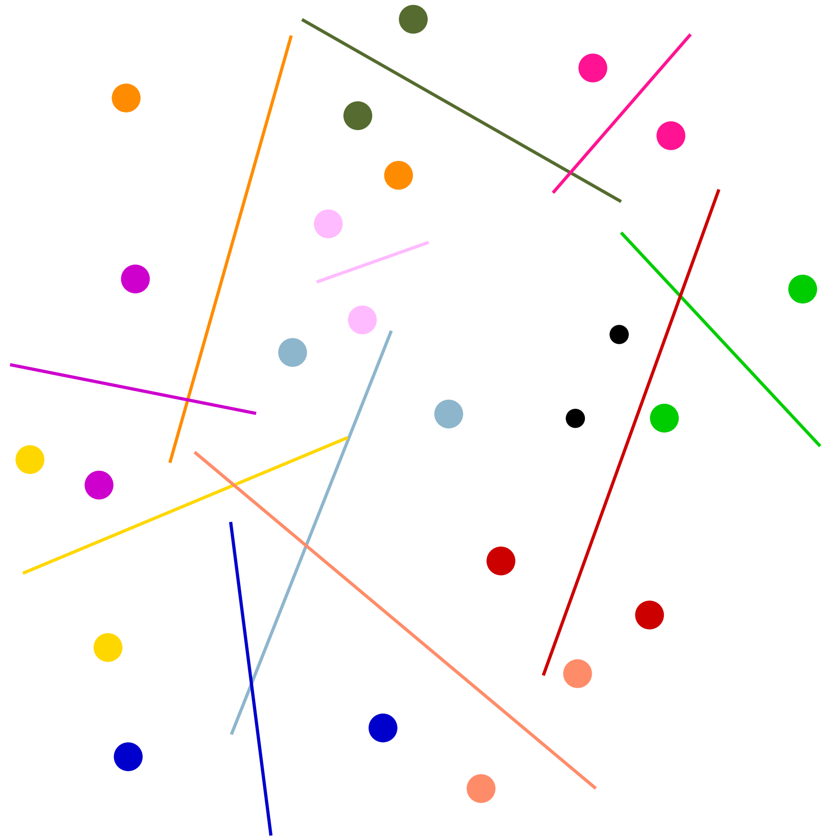

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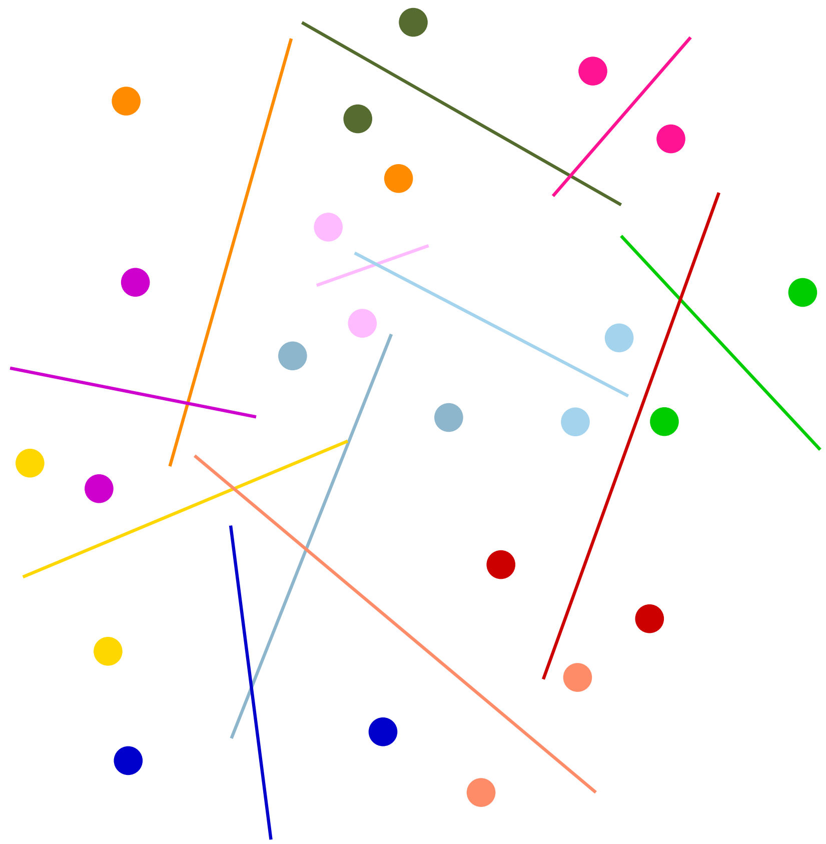


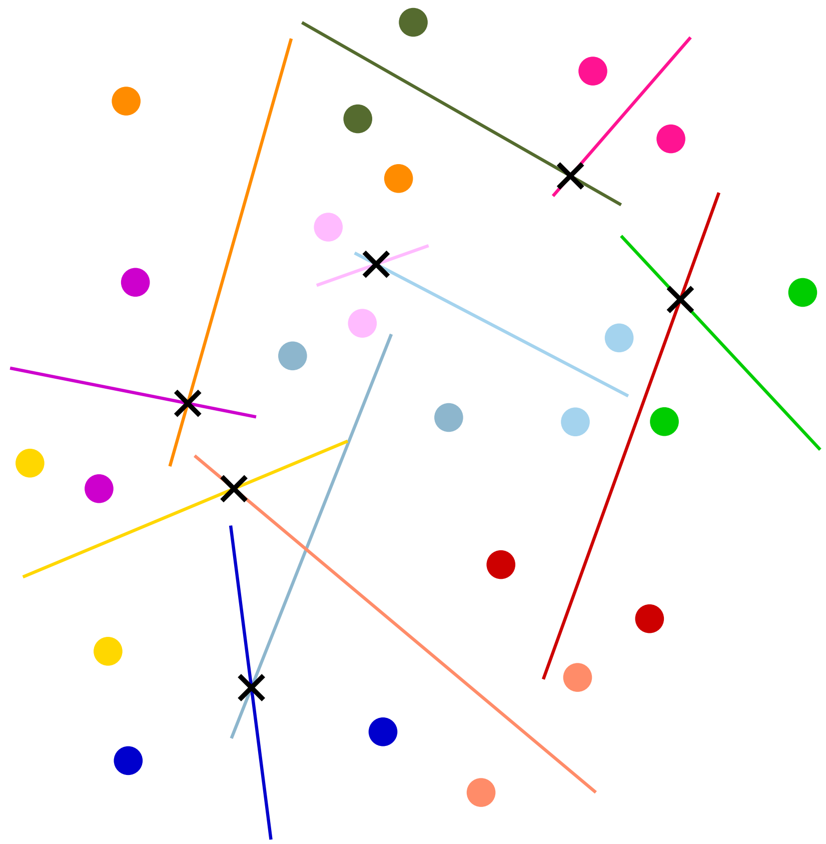
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair.

2. Compute the median slope of the non-vertical bisectors, and take it as horizontal.

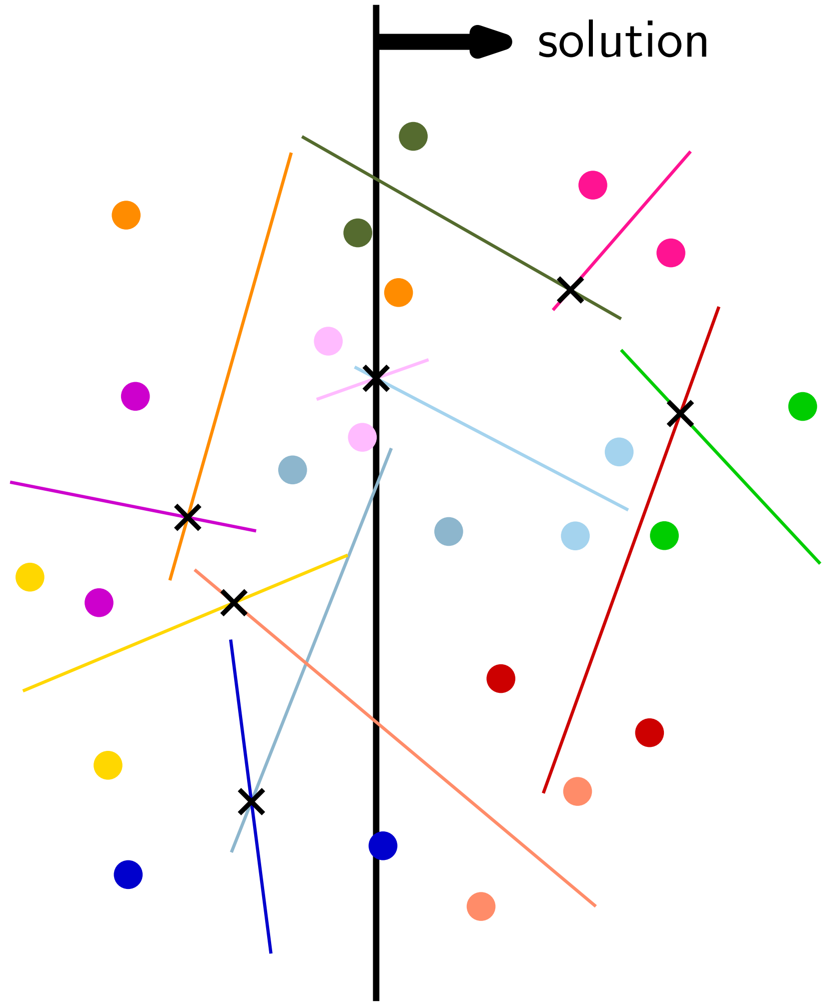
3. Pair up the non-vertical non-horizontal bisectors, always pairing up one positive slope bisector with one negative slope bisector. Compute the intersection point of each pair.



MSC in the plane

4. First search

- Compute the median value x_m of the x-coordinate of the intersection points (and vertical bisectors).
- Solve the MSC problem restricted to the line $x = x_m$.
- Use the solution to decide whether the unrestricted solution has been found or it lies to the left/right of the vertical line $x = x_m$. (More on how to do this later.)



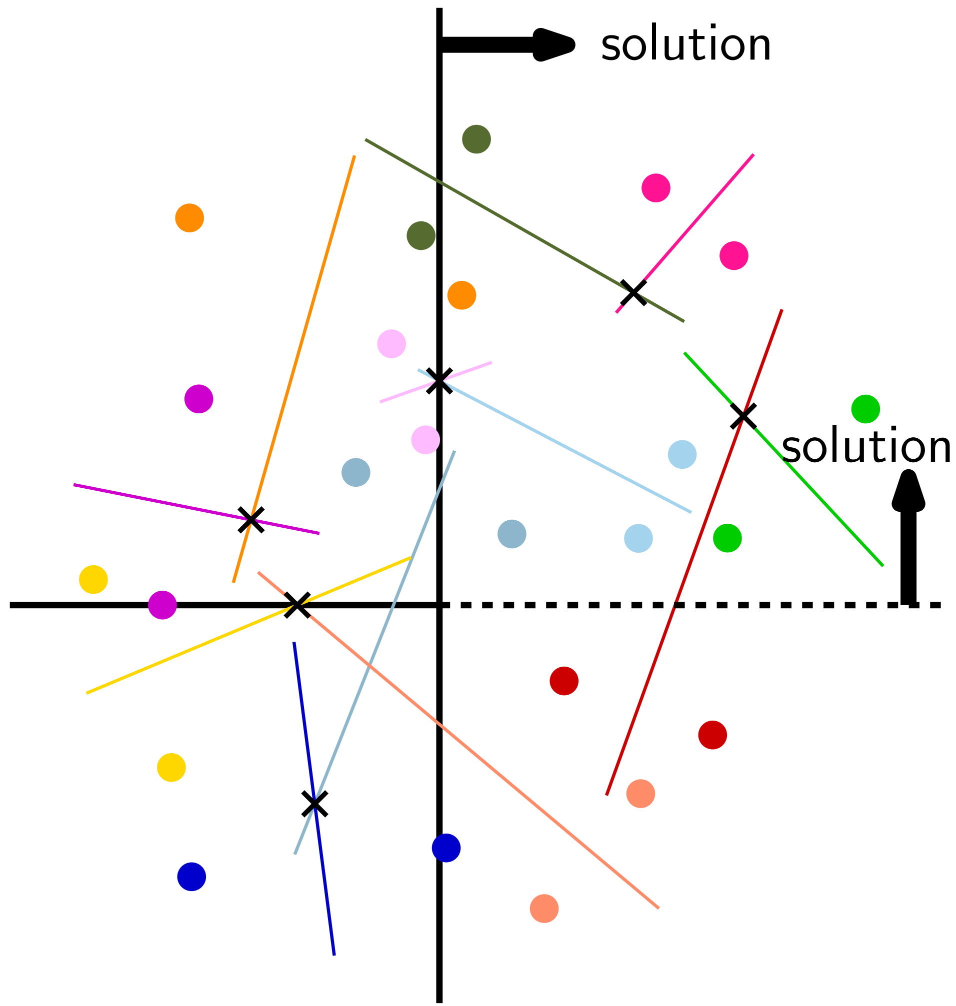
MSC in the plane

4. First search

5. Second search

   If the solution has not been found,

   - Compute the median value y_m of the y-coordinate of the intersection points (and horizontal bisectors) lying in the opposite halfplane.
   - Solve the MSC problem restricted to the line $y = y_m$.
   - Use the solution to decide whether the unrestricted solution has been found or it lies above/below the horizontal line $y = y_m$.



MSC in the plane

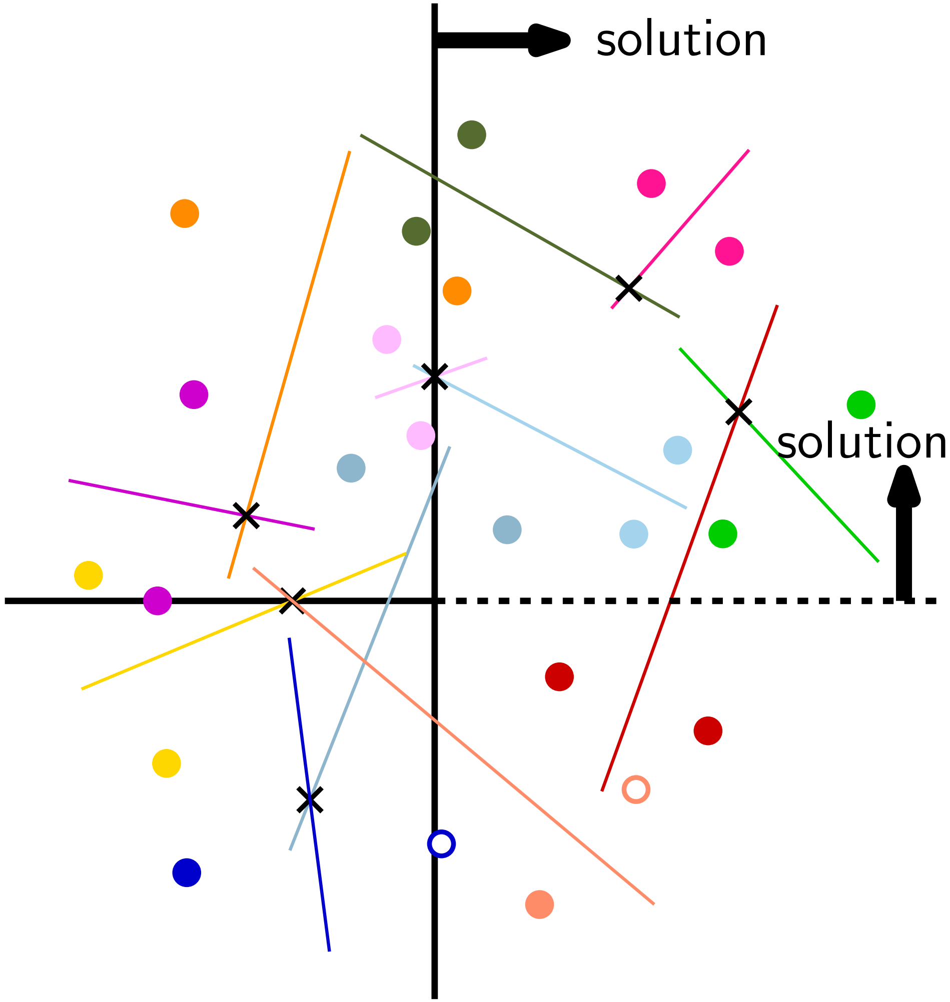
4. First search

5. Second search

6. Prune

   If the solution has not been found, then

   - Each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated.
   - Analogously, for each horizontal/vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated.



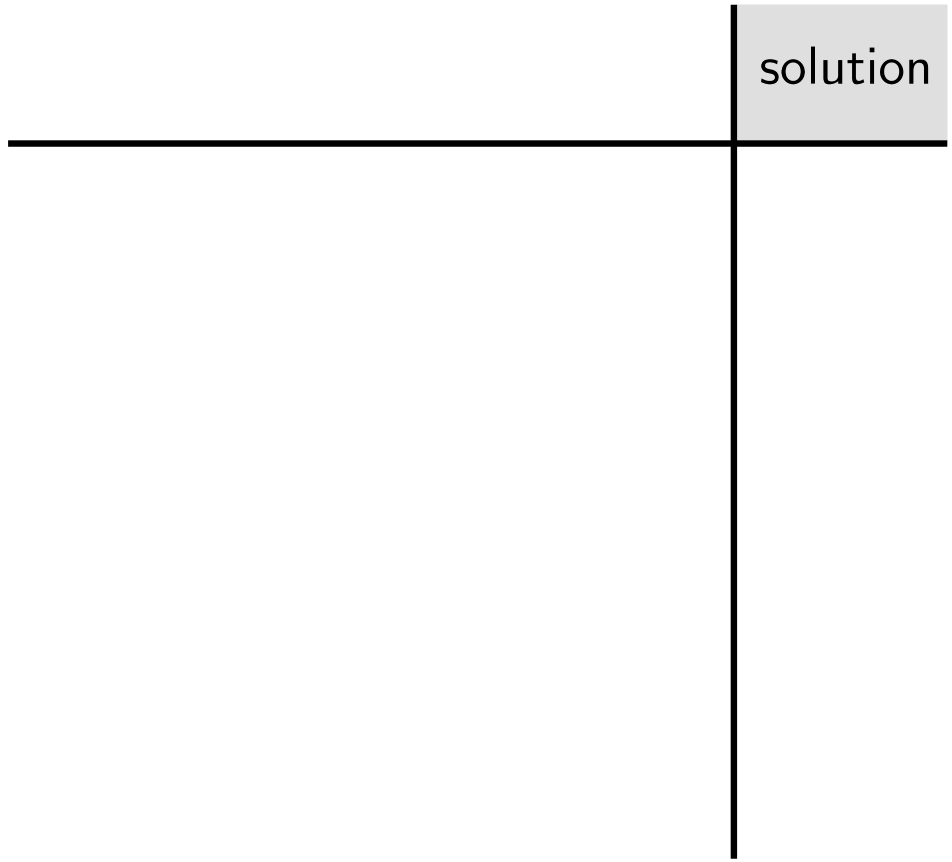
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

4. First search
5. Second search
6. Prune

   If the solution has not been found, then

   - Each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated.
   - Analogously, for each horizontal/vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated.



MSC in the plane

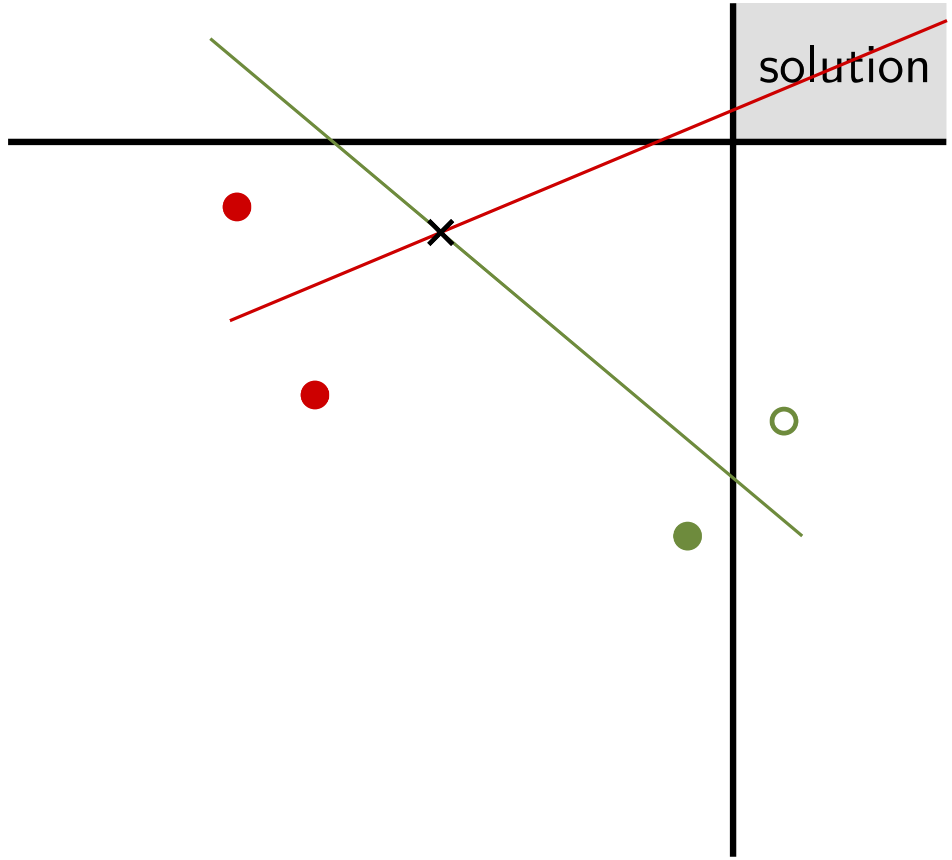
4. First search

5. Second search

6. Prune

   If the solution has not been found, then

   - Each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated.
   - Analogously, for each horizontal/vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated.

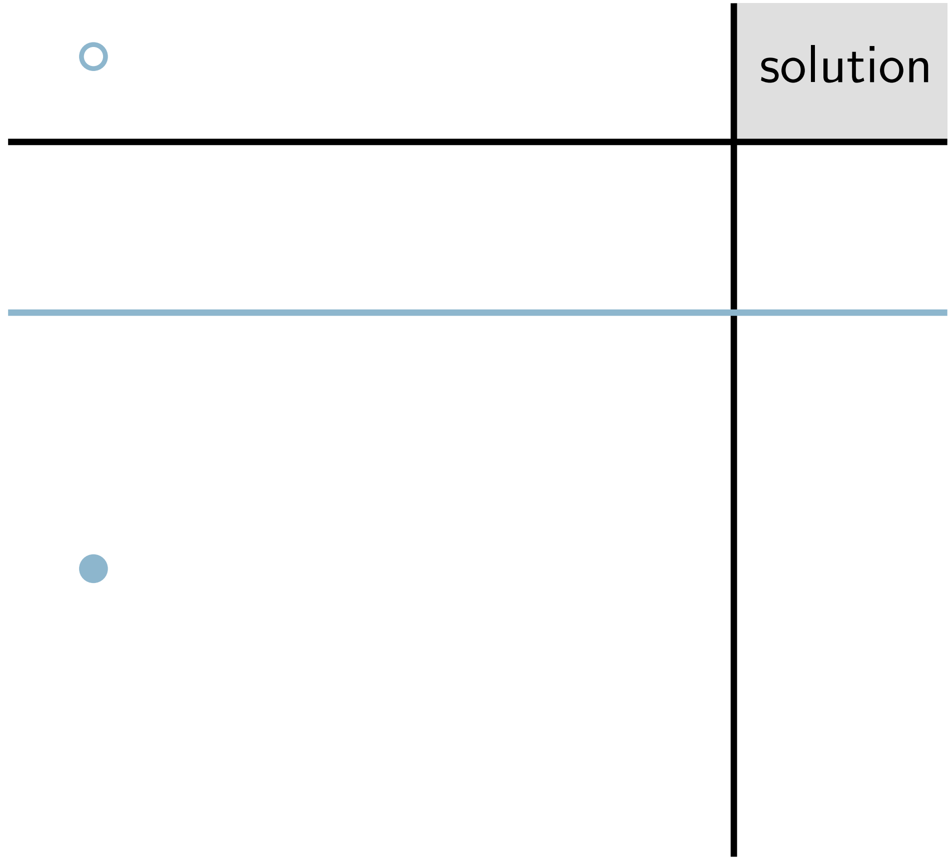


MSC in the plane

4. First search
5. Second search
6. Prune

   If the solution has not been found, then

   - Each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated.
   - Analogously, for each horizontal/vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated.



MSC in the plane

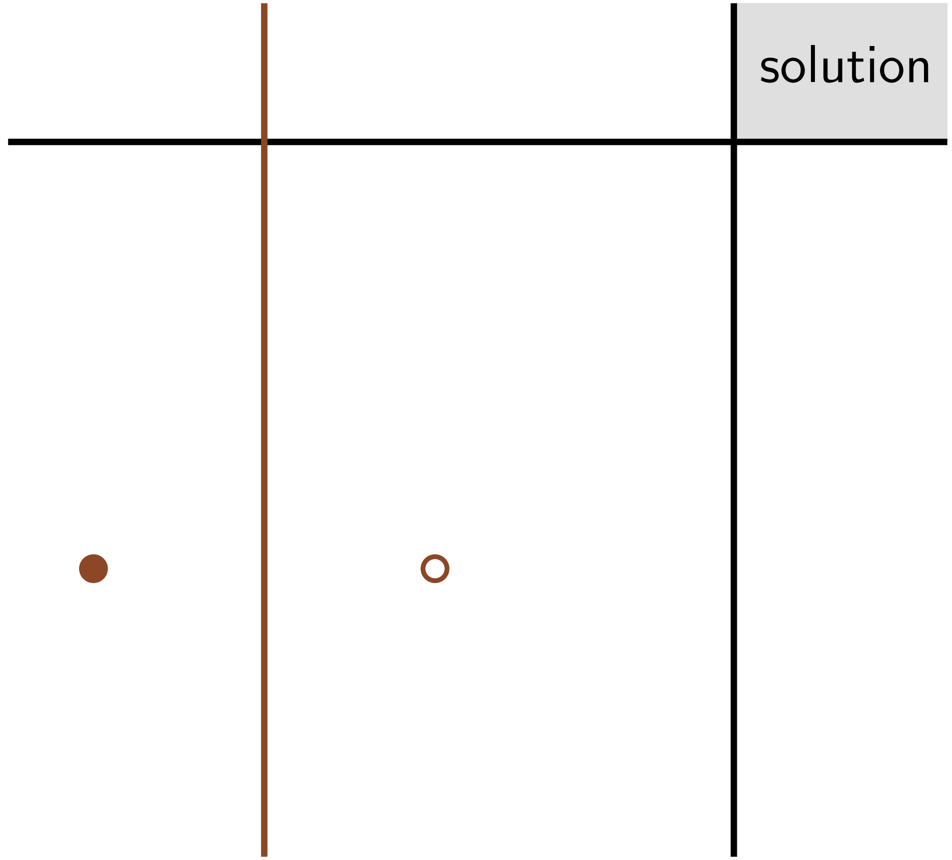
4. First search

5. Second search

6. Prune

   If the solution has not been found, then

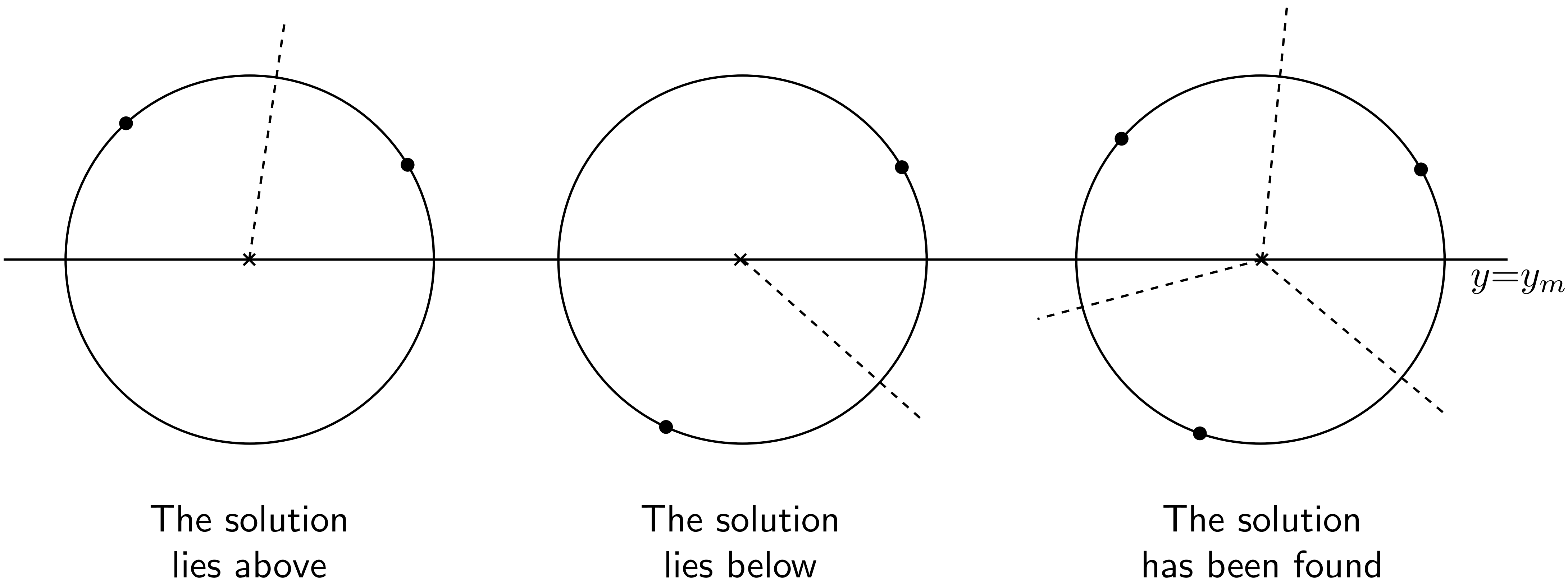
   - Each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated.
   - Analogously, for each horizontal/vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated.



MSC in the plane

Search step

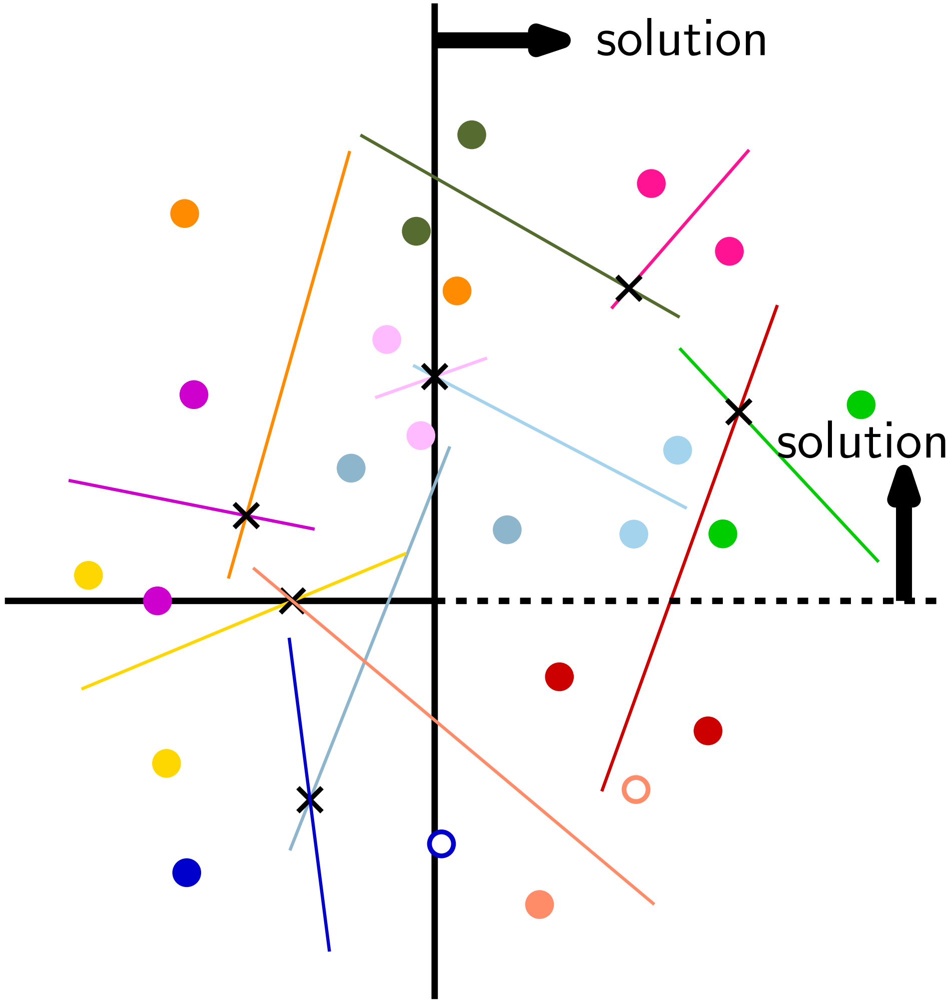
Use the solution to decide whether the unrestricted solution has been found or it lies above/below the horizontal line $y = y_m$



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

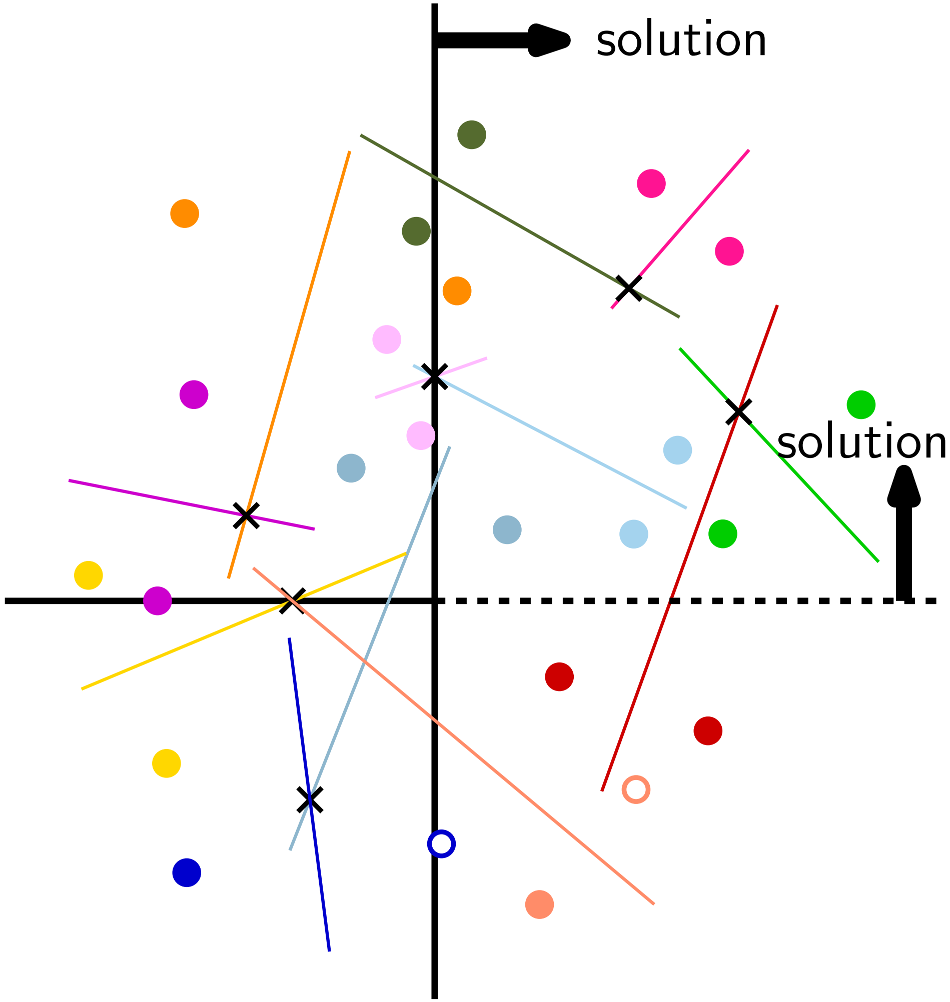
1. Pair up the points, and compute the perpendicular bisector of each pair. $O(n)$
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal. $O(n)$
3. Pair up the non-vertical non-horizontal bisectors, and compute the intersection point of each pair. $O(n)$
4. Fisrt search: $O(n)$
5. Second search: $O(n)$
6. Prune: In $O(n)$ time, at least $1/16$ of the input is pruned.



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair. $O(n)$
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal. $O(n)$
3. Pair up the non-vertical non-horizontal bisectors, and compute the intersection point of each pair. $O(n)$
4. Fisrt search: $O(n)$
5. Second search: $O(n)$
6. Prune: In $O(n)$ time, at least $1/16$ of the input is pruned.



Conclusion

Given n points in the plane, the min-max facility location problem can be solved in linear time.

TO LEARN MORE

- F. P. Preparata and M. I. Shamos, **Computational Geometry: an Introduction** (revised ed.). Springer-Verlag, 1993.
- N. Megiddo, Linear-time algorithms for linear programming in R^3 and related problems, *SIAM Journal on Computing*, Vol. 12, N. 4, pp. 759-776, 1983.