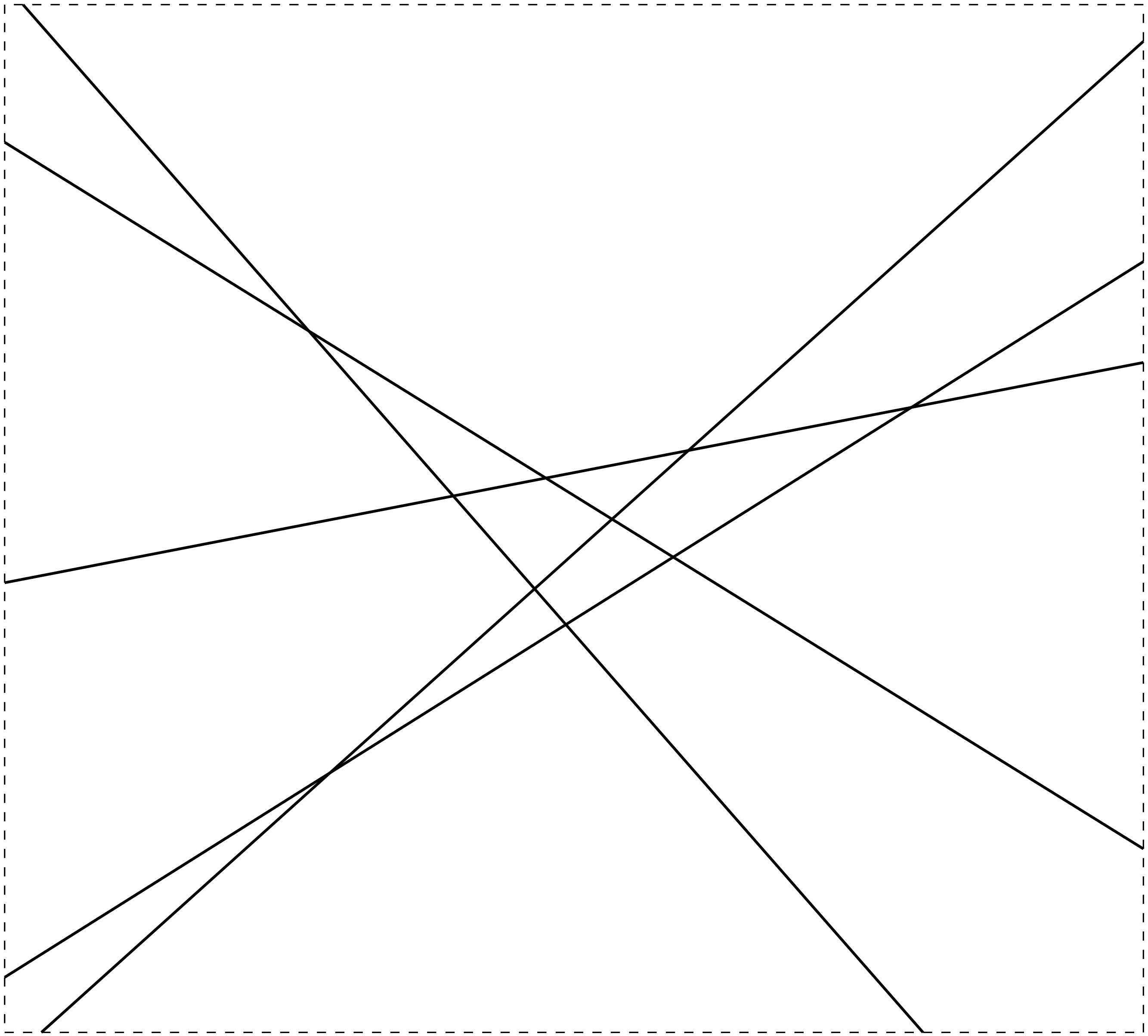
ARRANGEMENTS

Vera Sacristán

Discrete and Algorithmic Geometry
Facultat de Matemàtiques i Estadística
Universitat Politècnica de Catalunya

DEFINITION

Let $L = \{l_1, \ldots, l_n\}$ be a finite set of lines in the plane.



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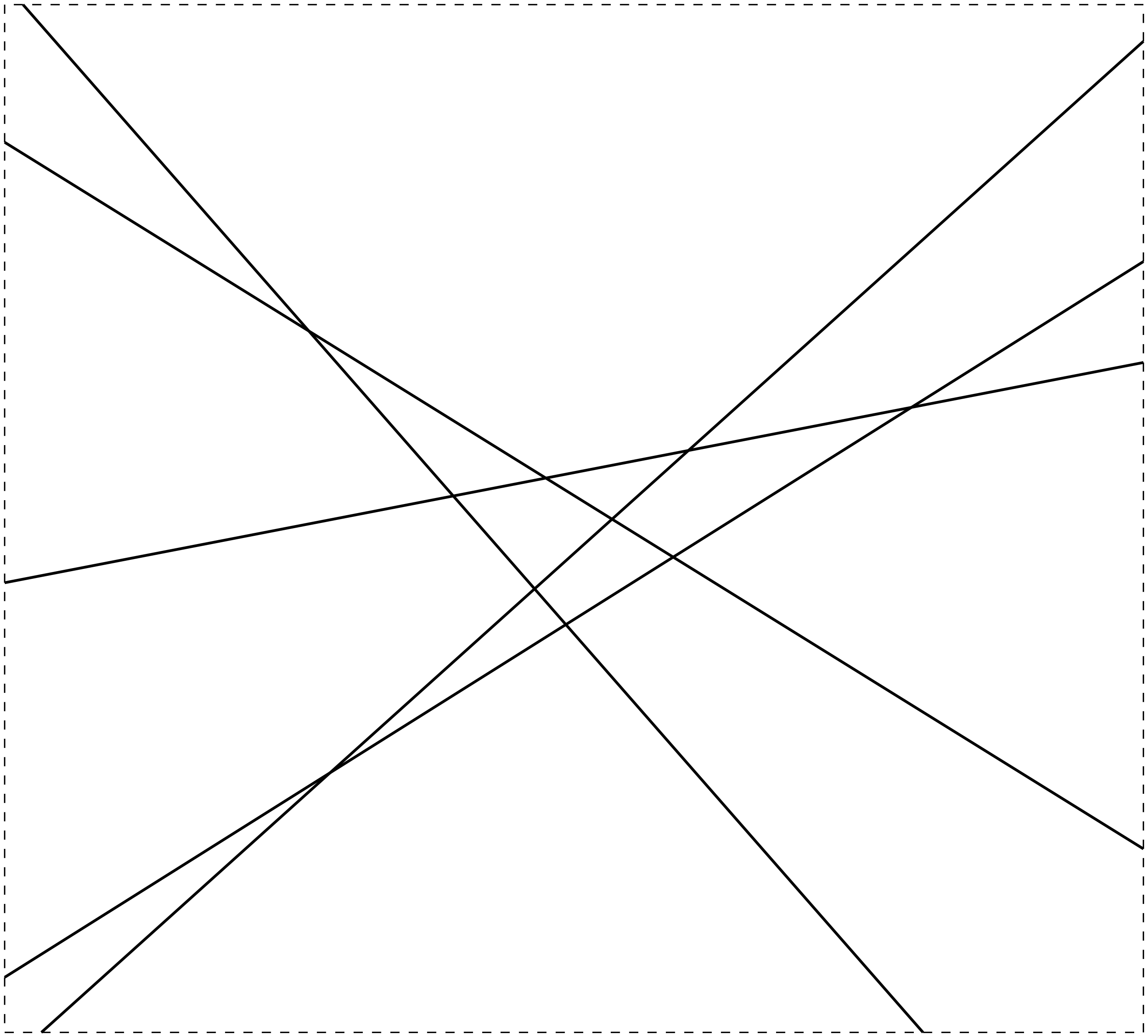
The *arrangement* $\mathcal{A}(L)$ is the decomposition of the plane into *faces*, *edges* and *vertices* produced by L.



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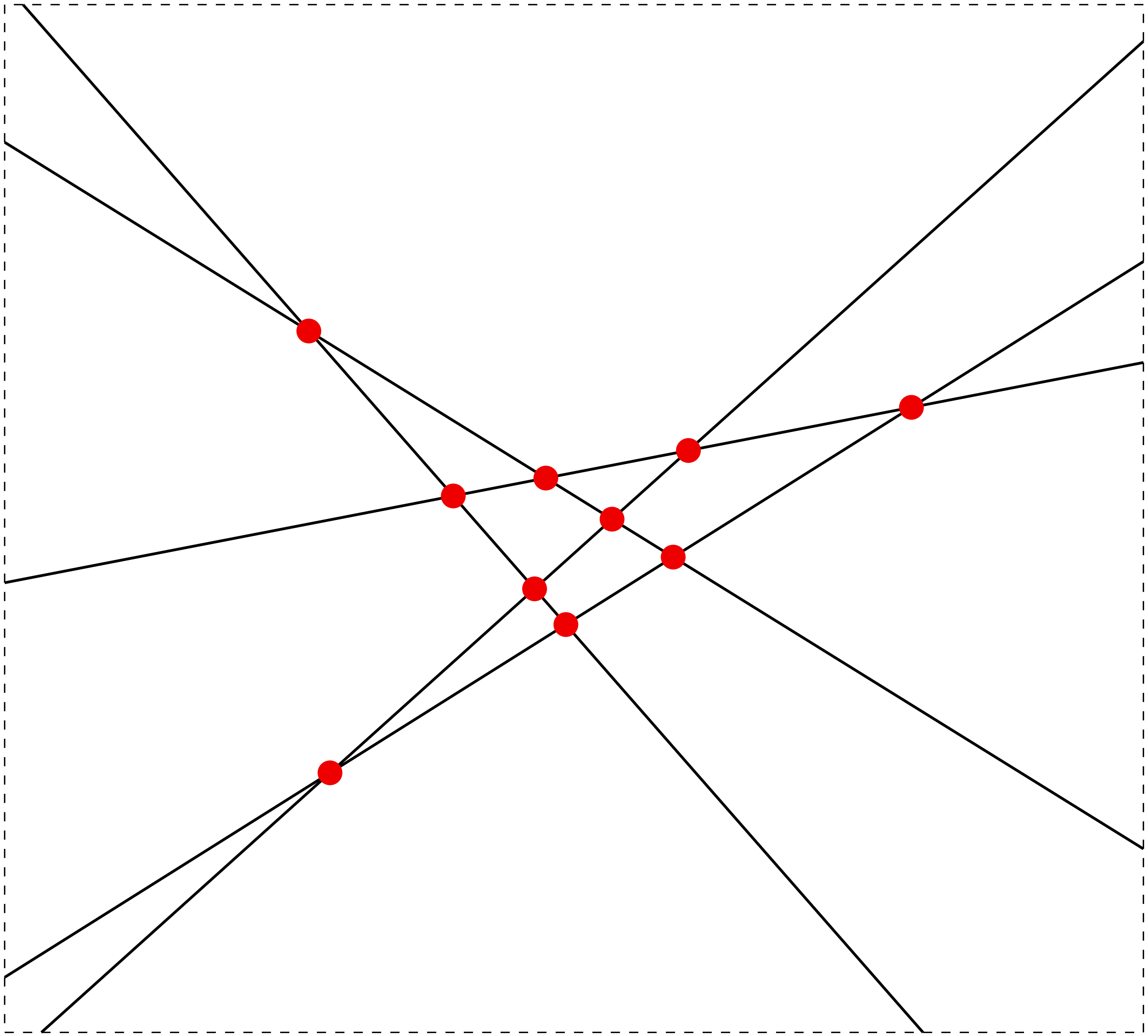

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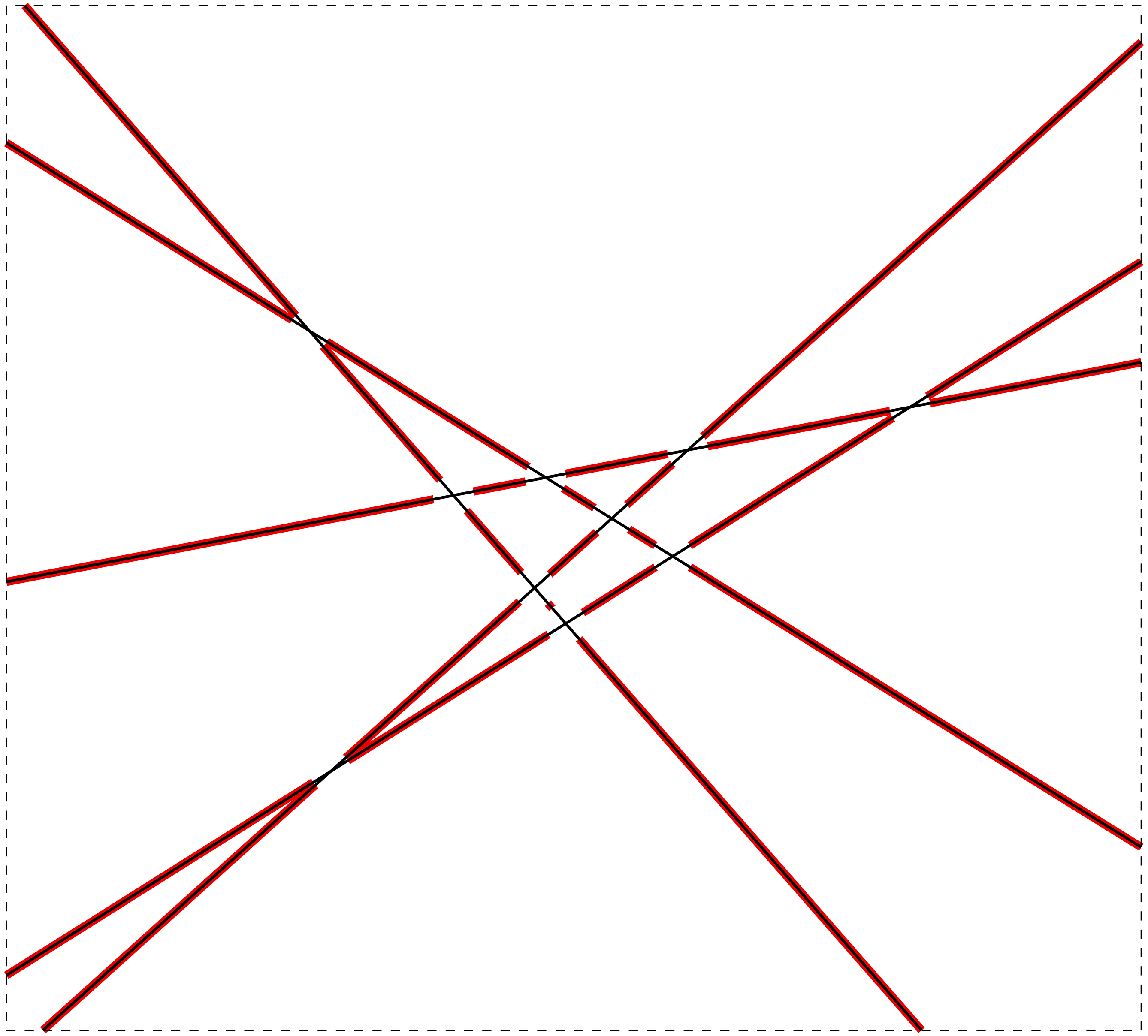

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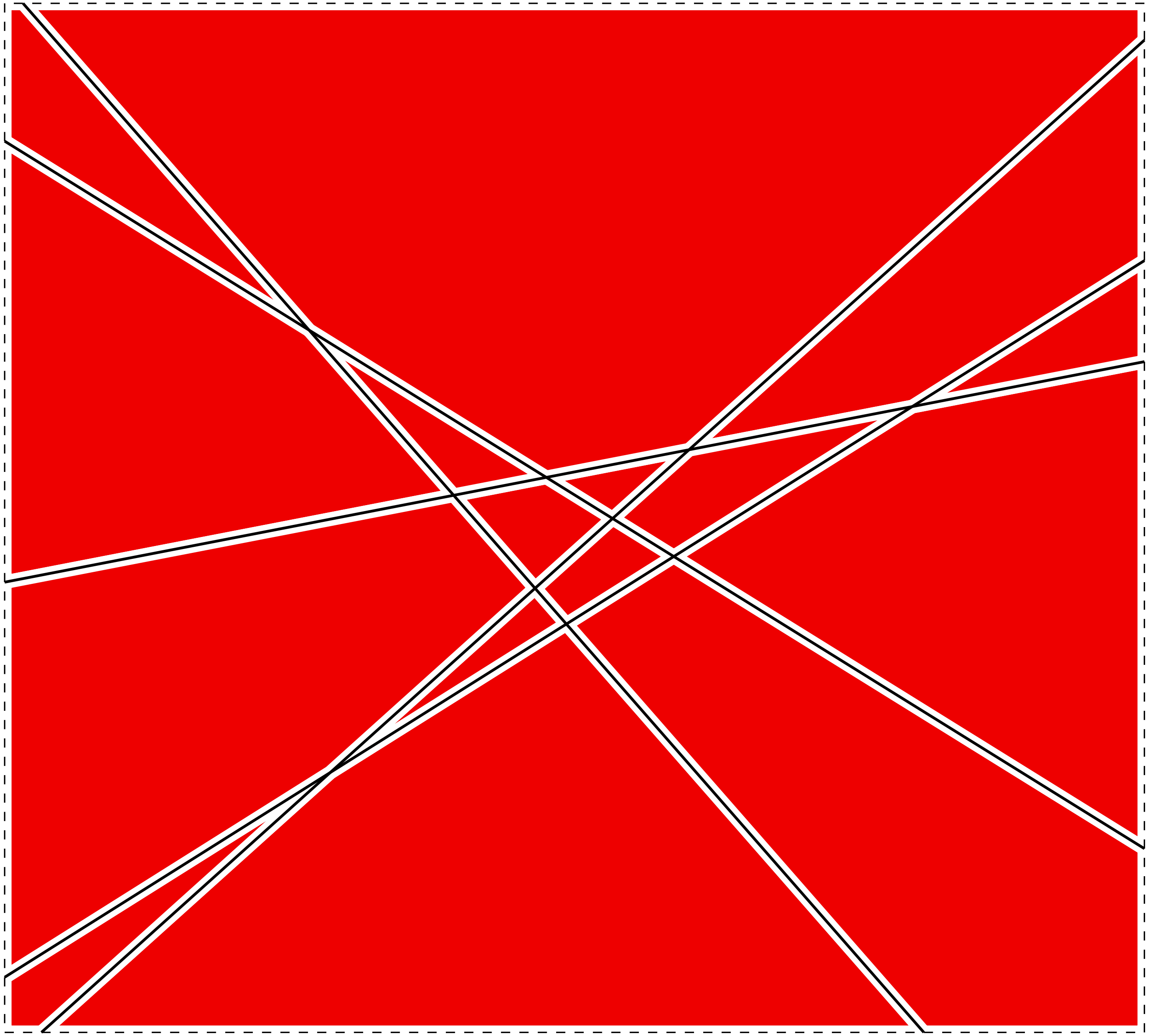

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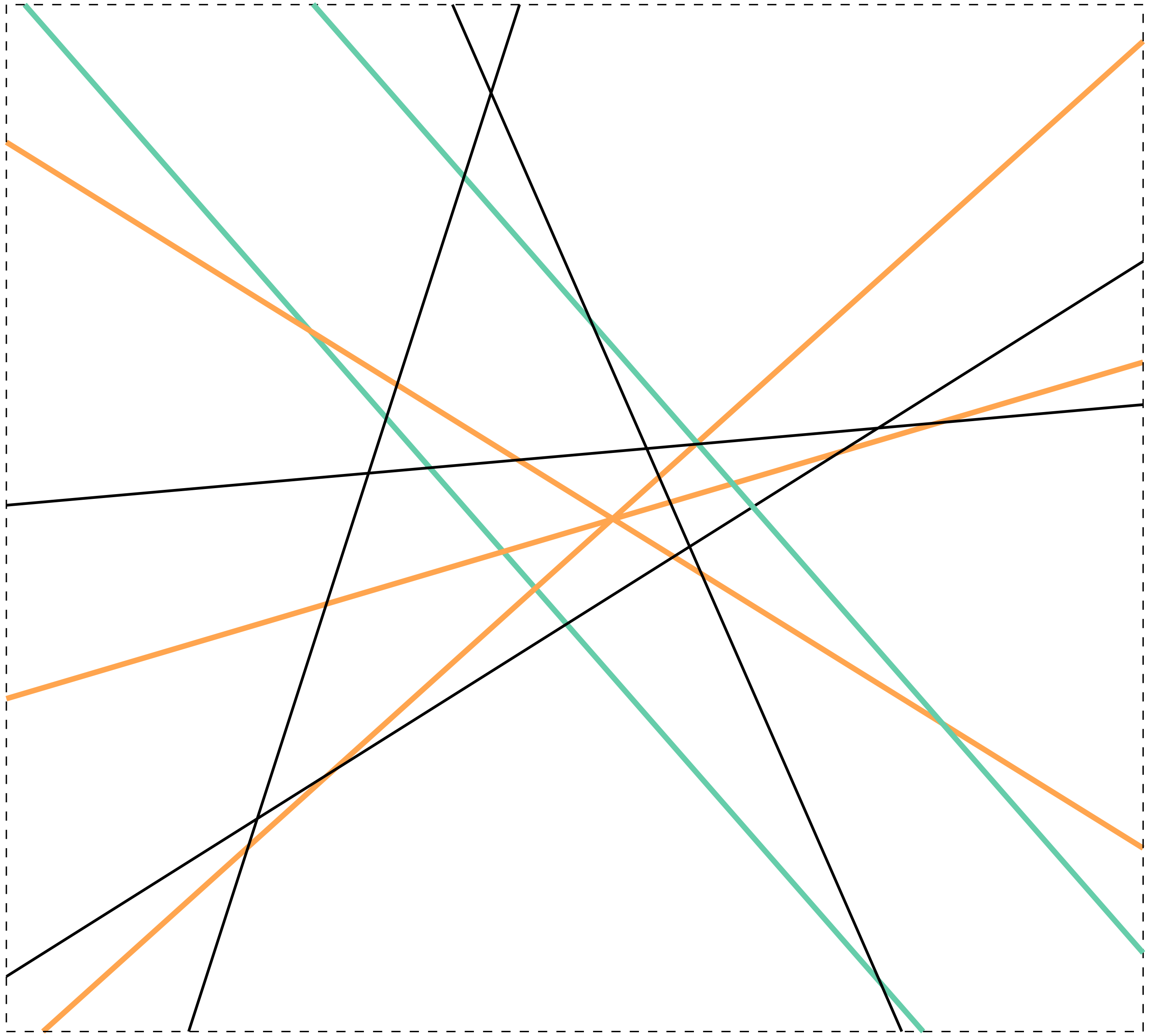
The arrangement $\mathcal{A}(L)$ is said to be *simple* if L does not contain any two parallel lines nor any three lines through the same point.



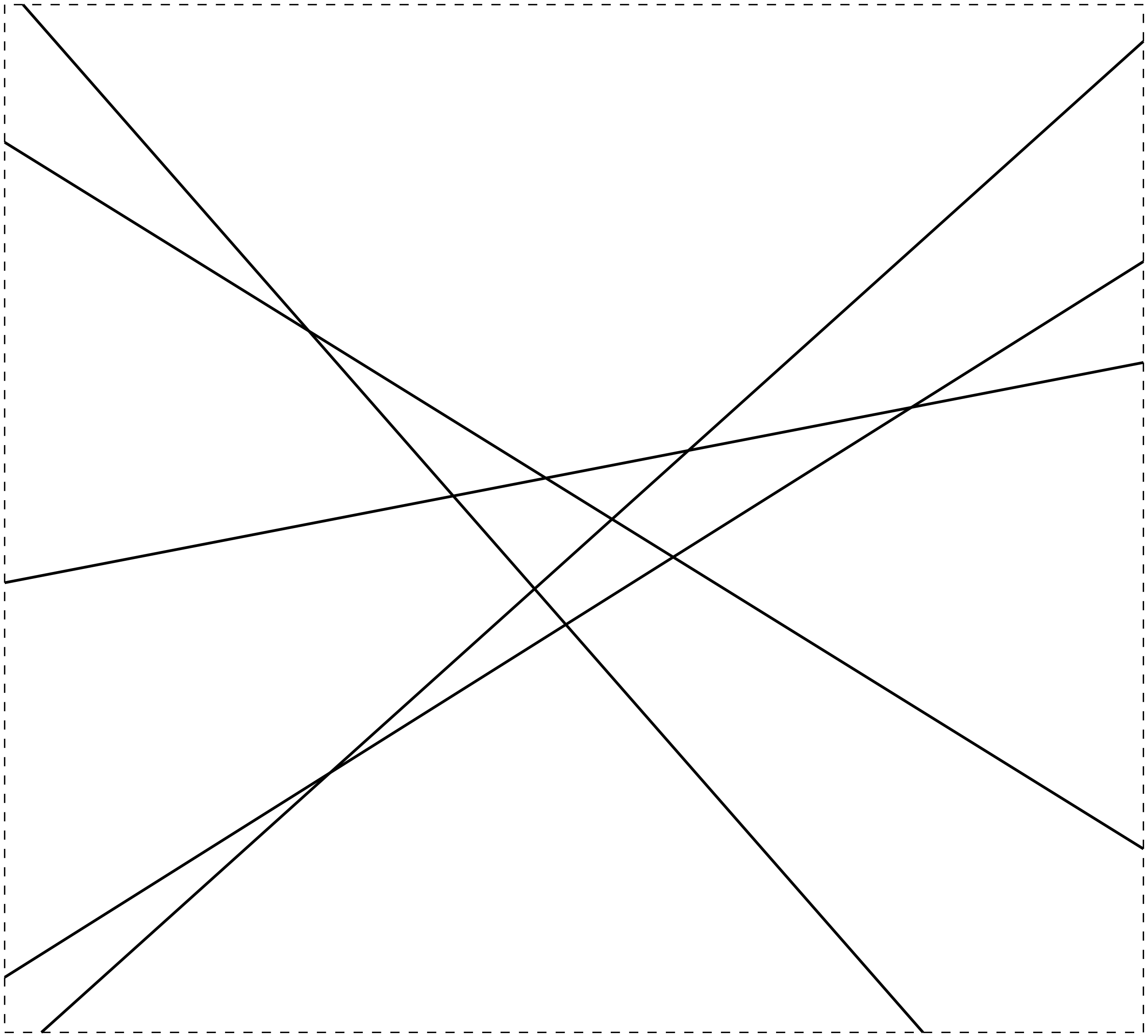
COMPLEXITY

The *(combinatorial) complexity* of an arrangement $\mathcal{A}(L)$ is its number of faces, edges and vertices.

The complexity of $\mathcal{A}(L)$ is $O(n^2)$, where $n = \#L$. Specifically:

- $v \leq \frac{n(n-1)}{2}$
- $e \leq n^2$
- $f \leq \frac{n^2}{2} + \frac{n}{2} + 1$

These bounds are achieved when the arrangement is simple.



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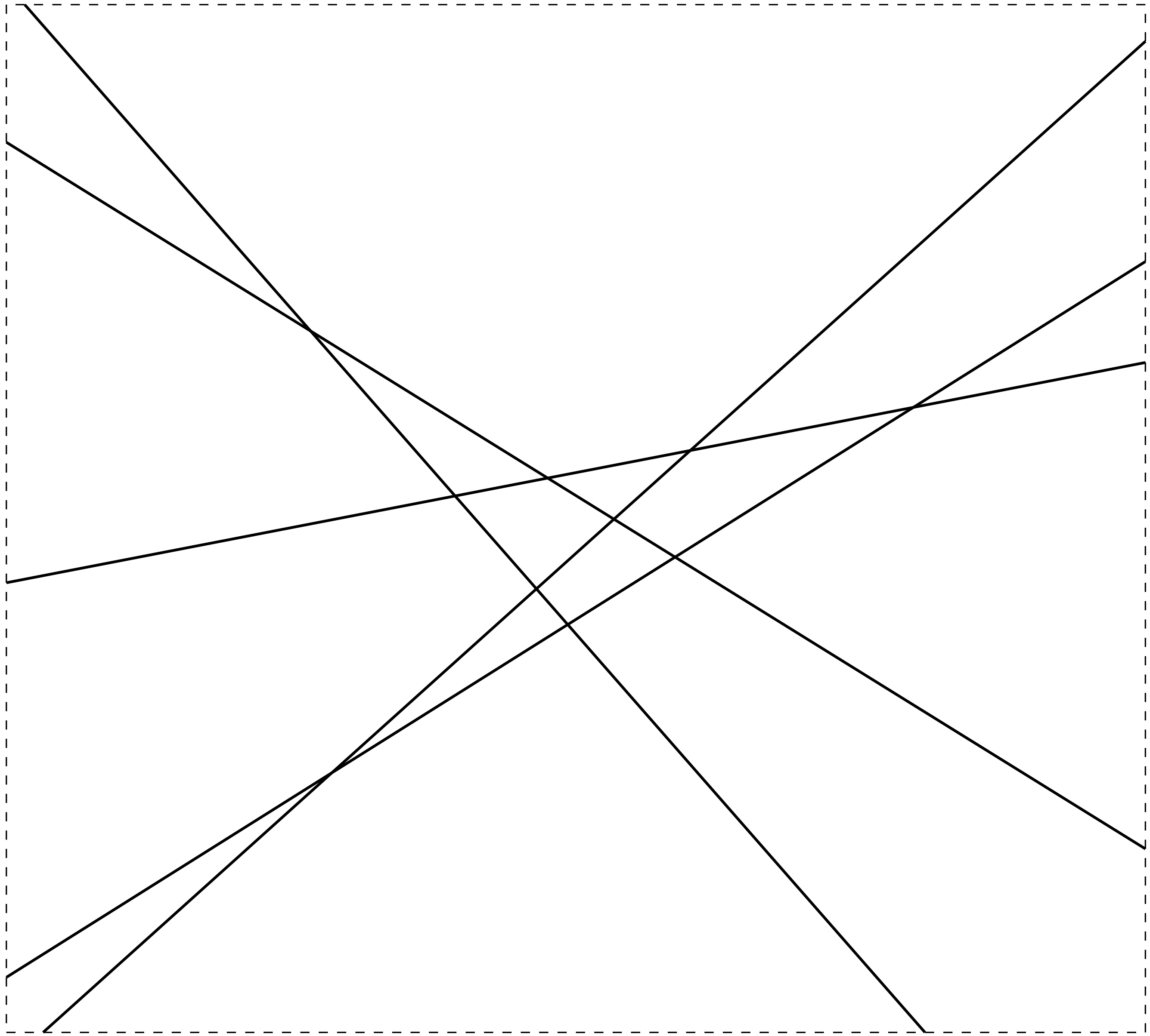

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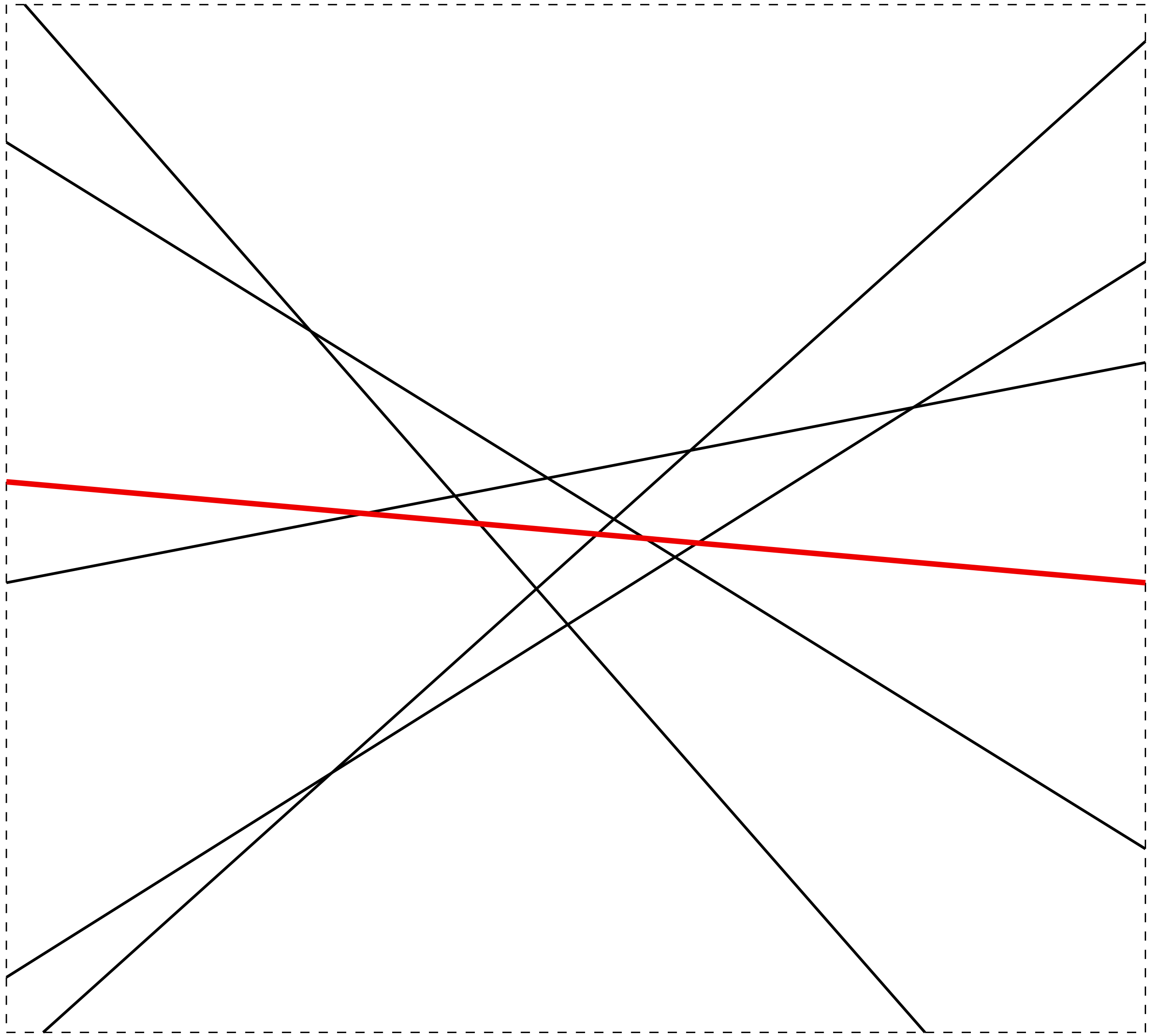

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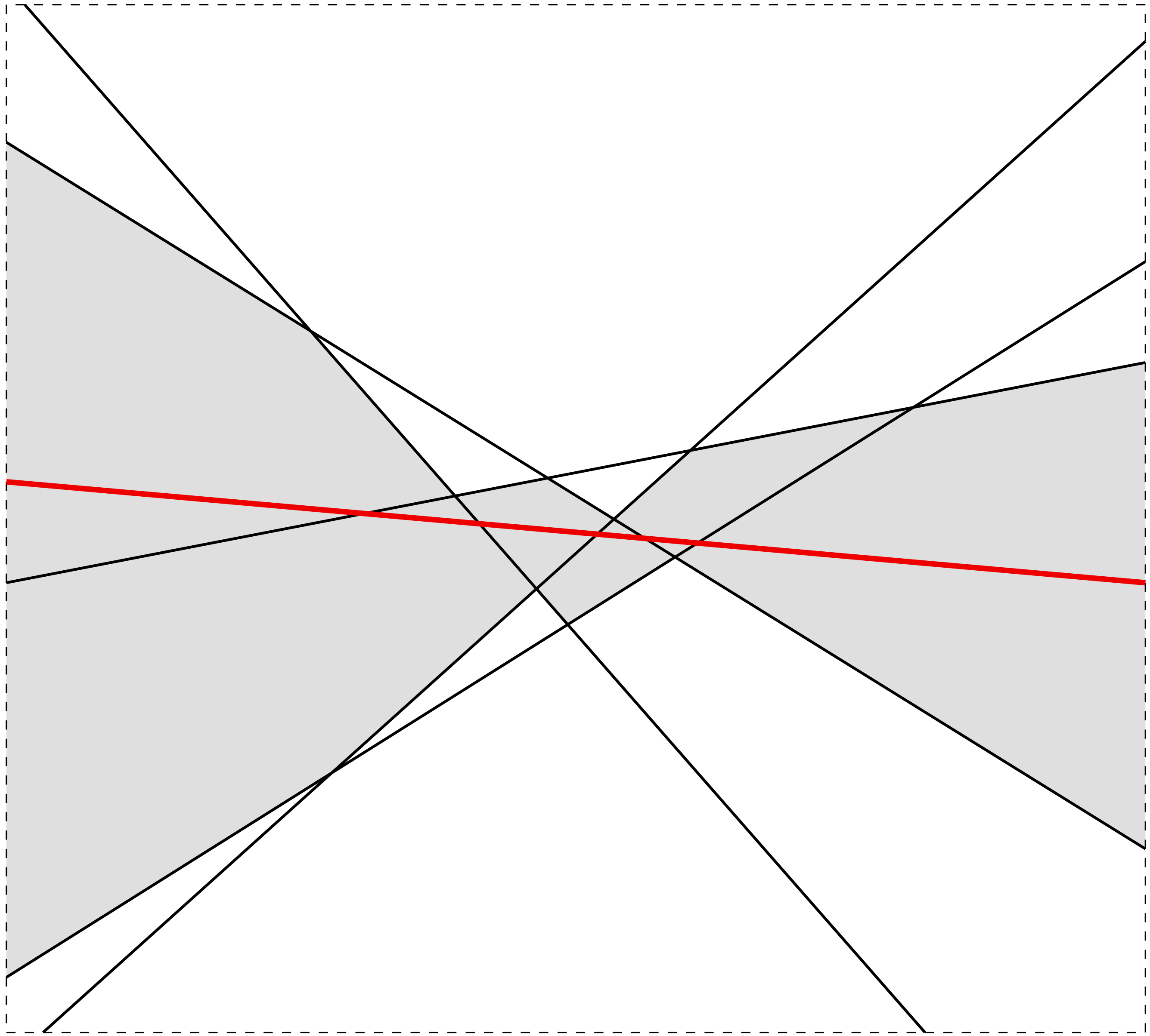

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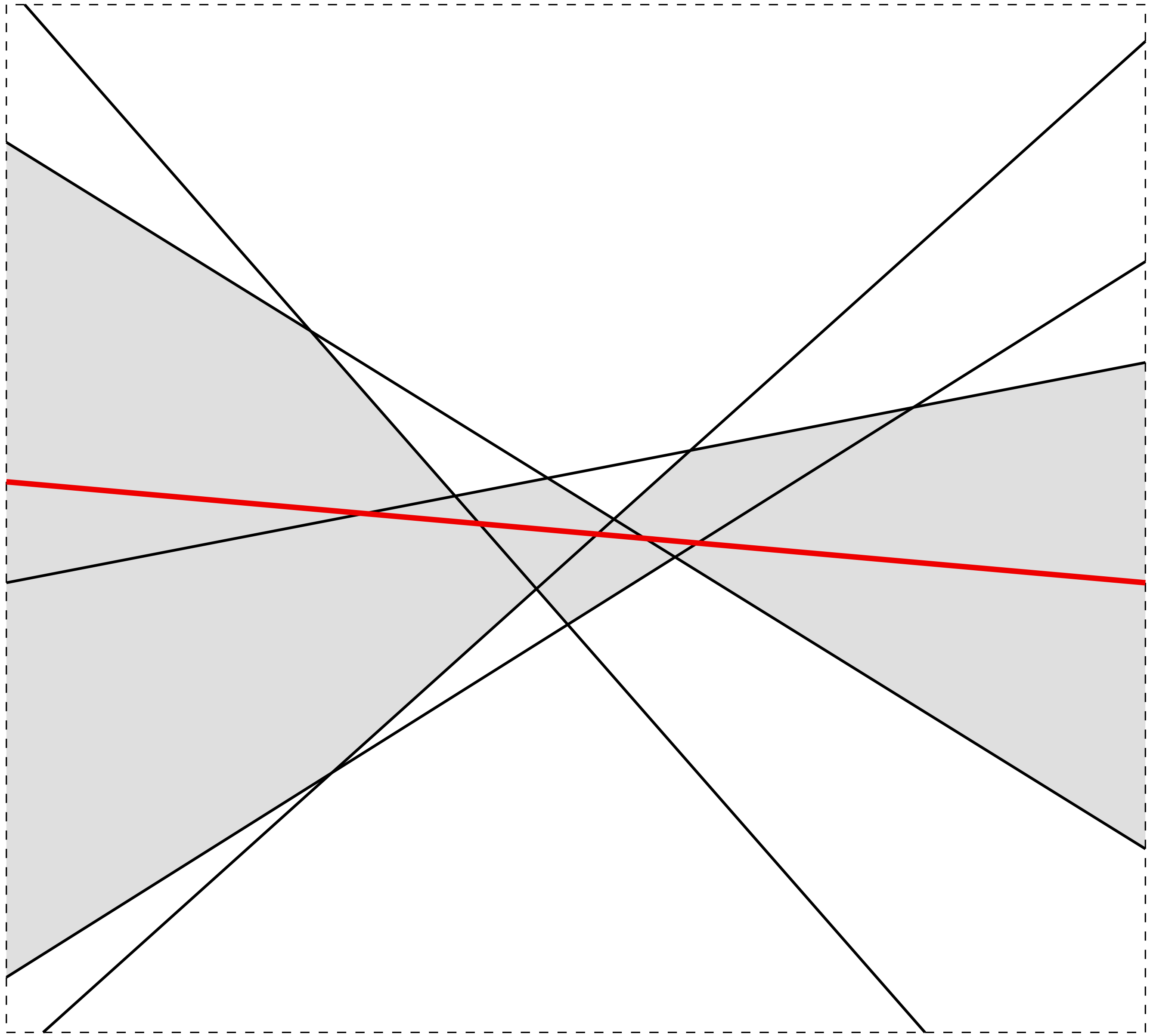
Theorem: The complexity of the zone of a line h in an arrangement of n lines is $O(n)$. More precisely, the total number of edges of the faces of the zone is $\leq 8n$.



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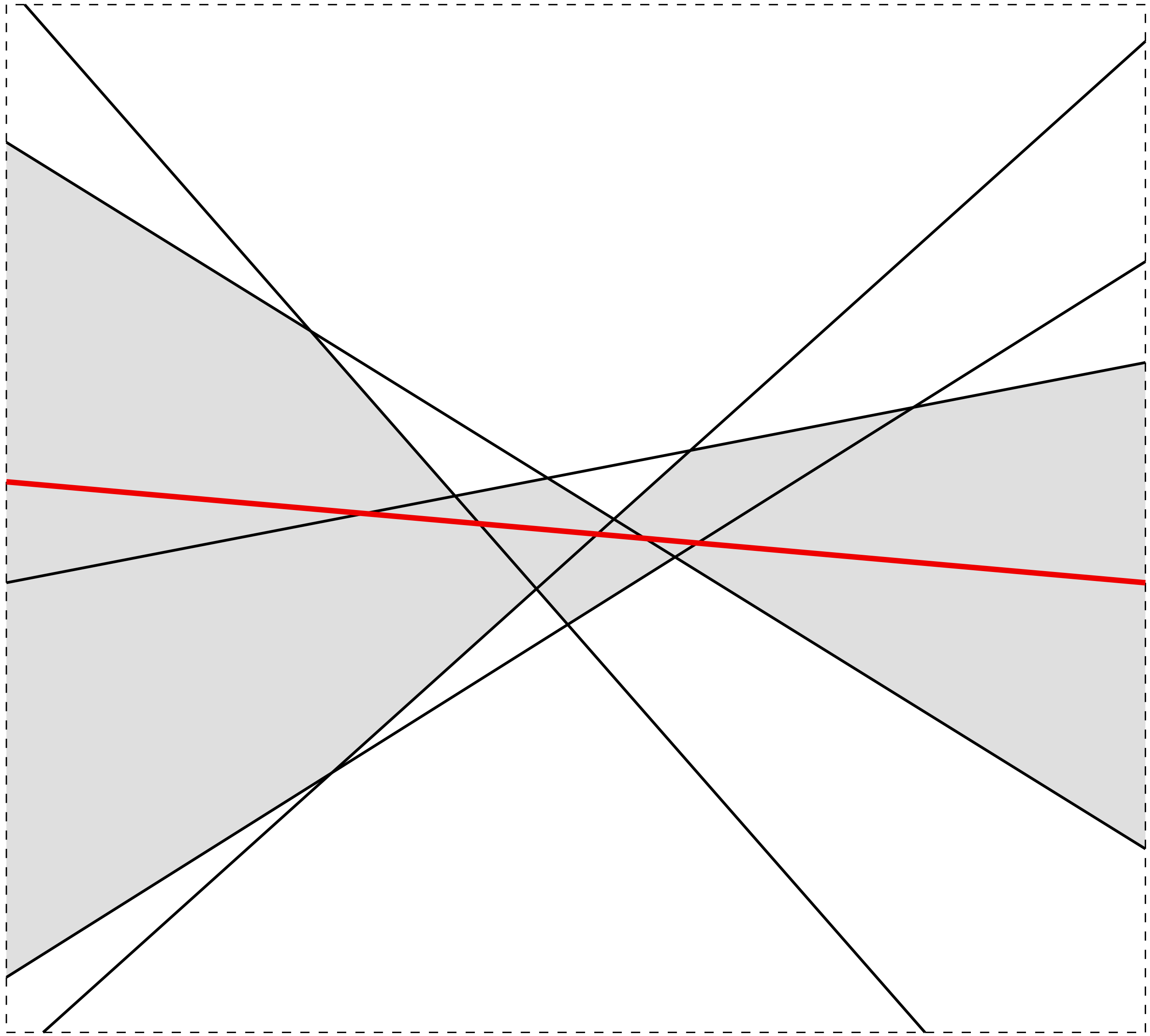

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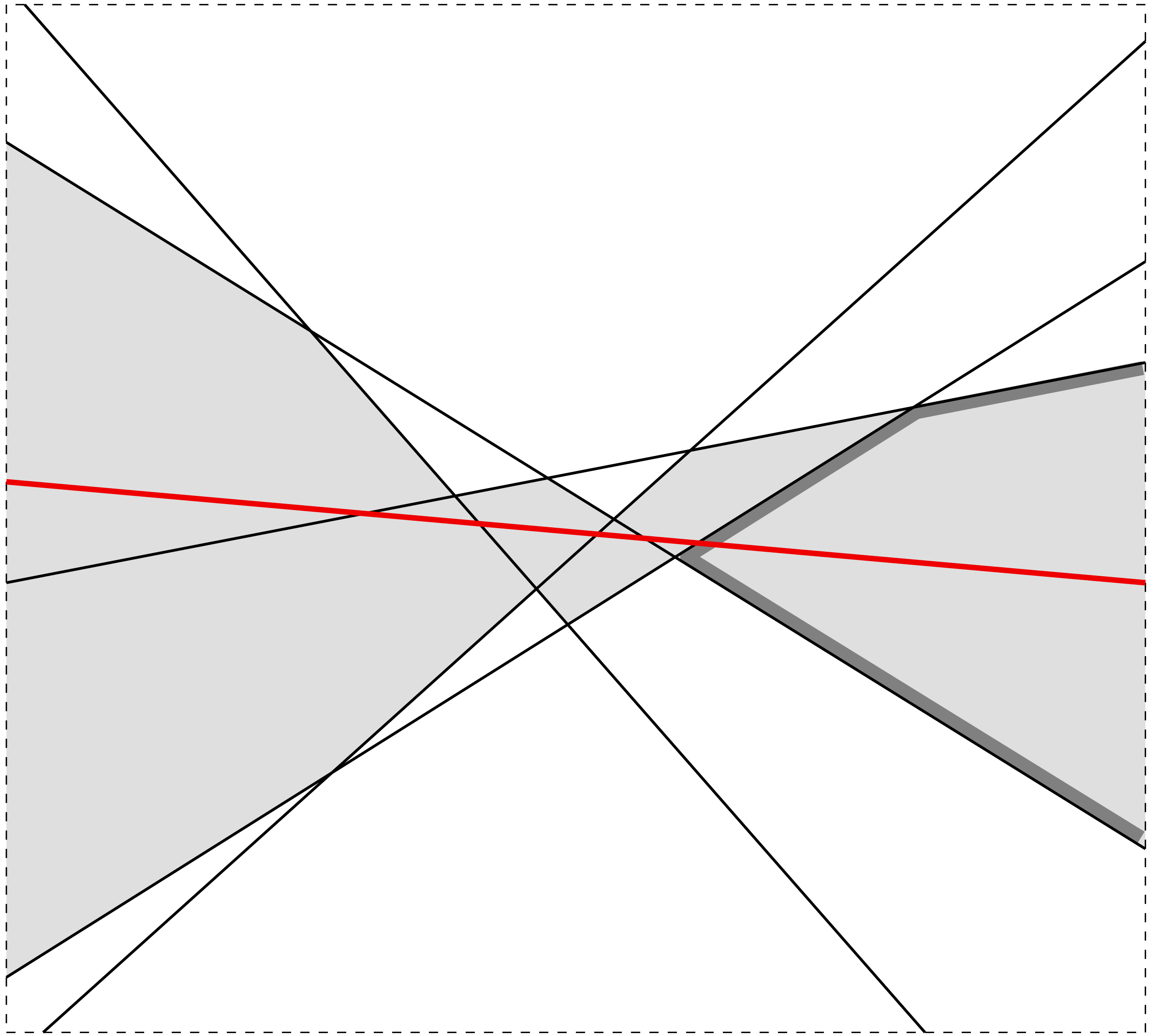

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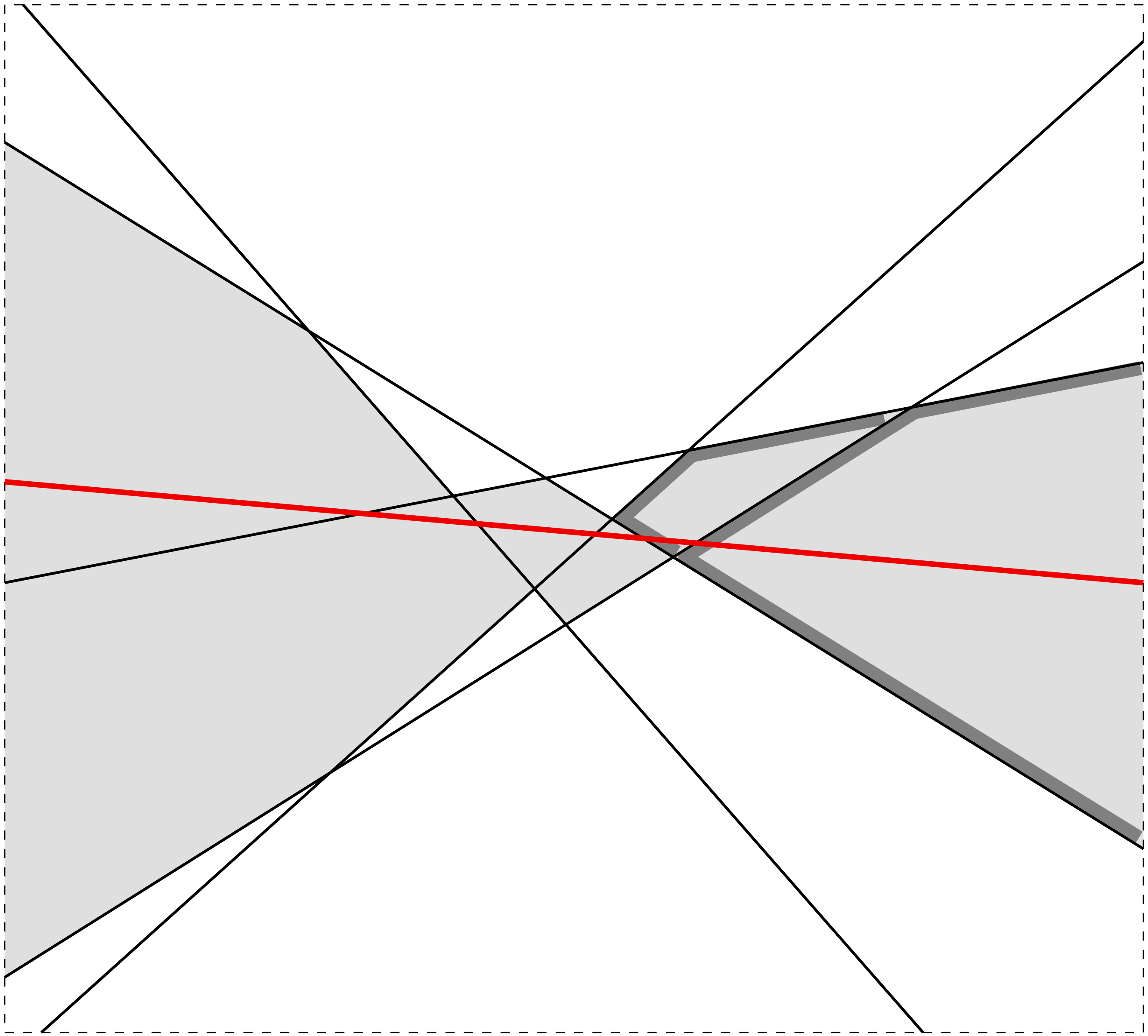

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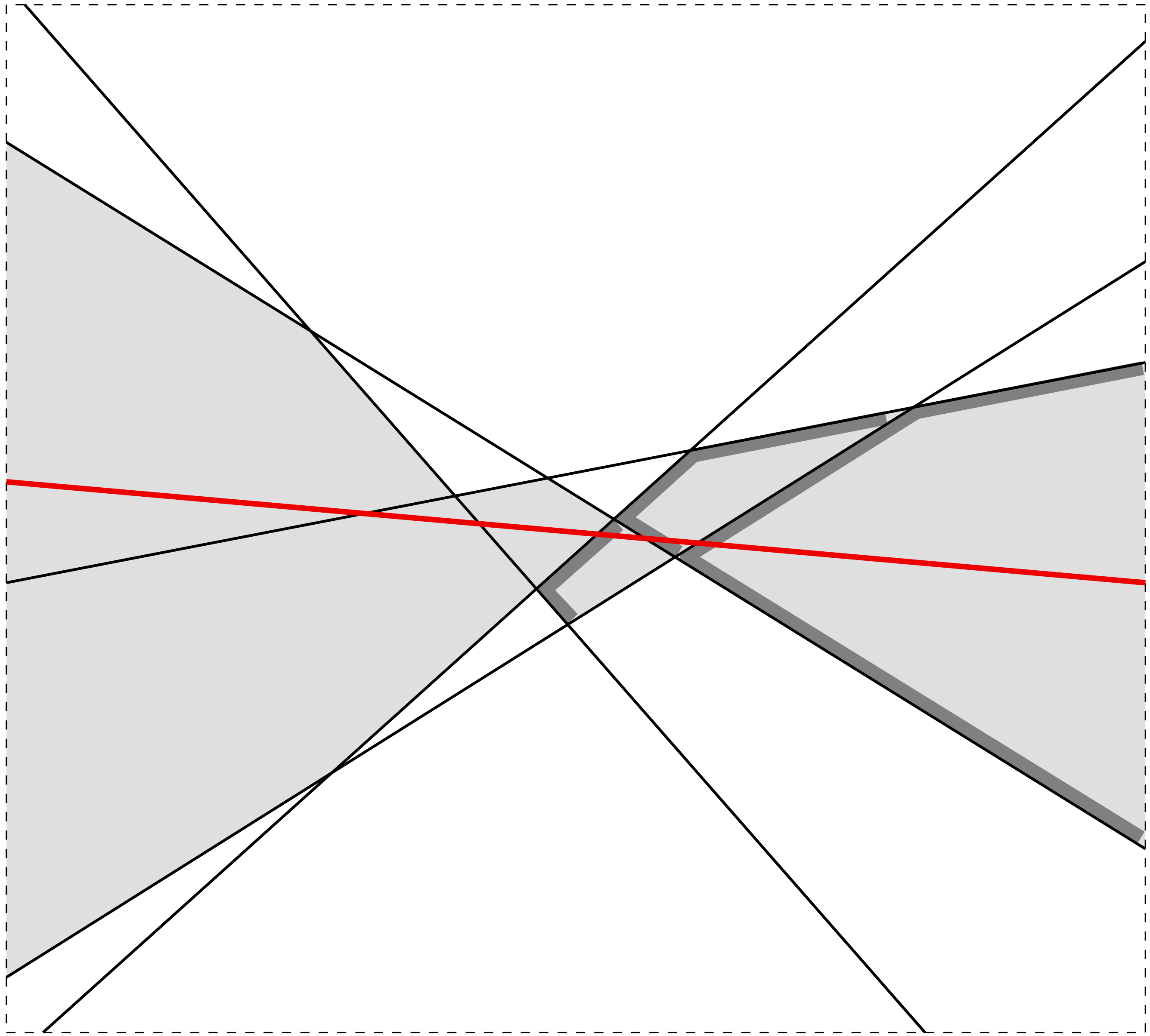

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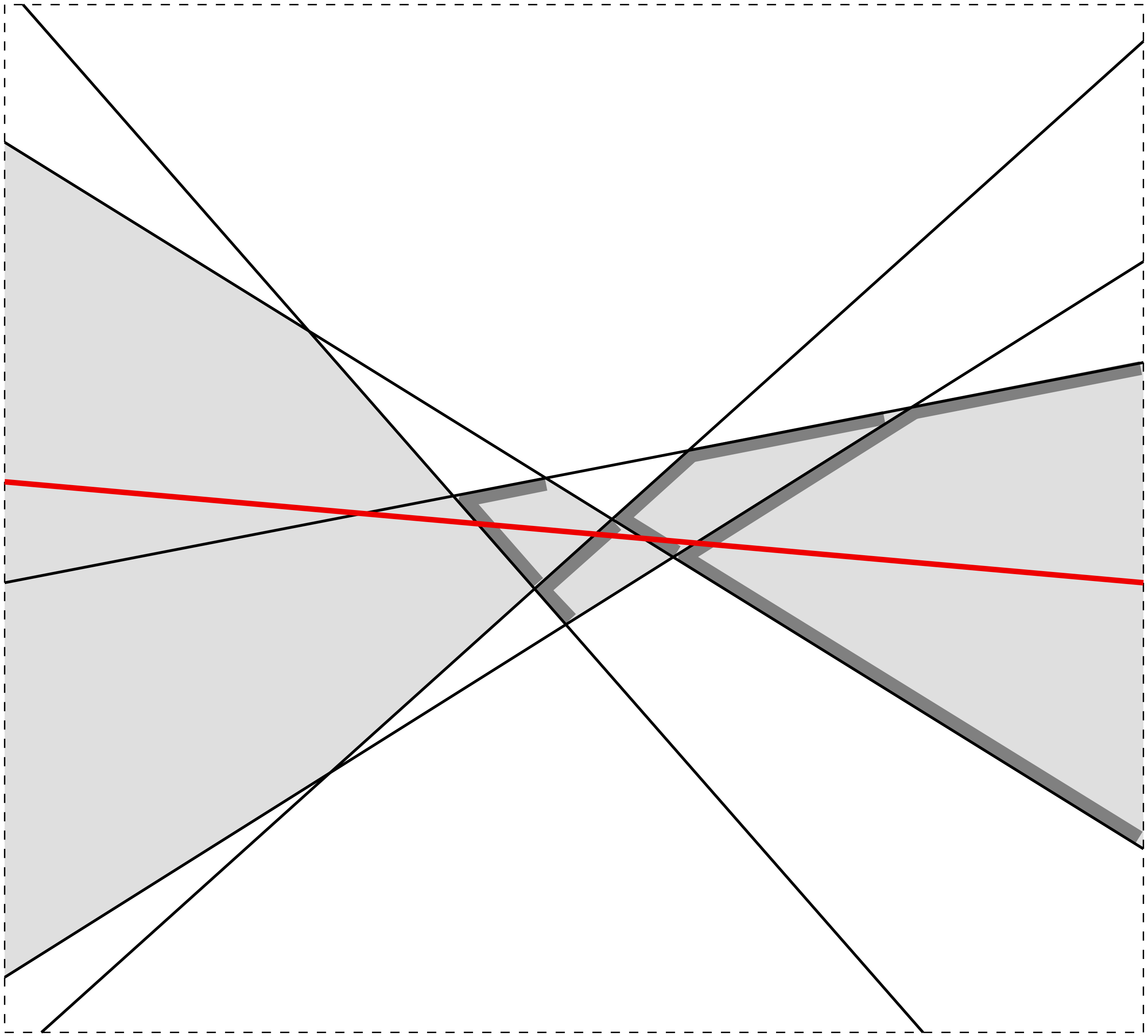

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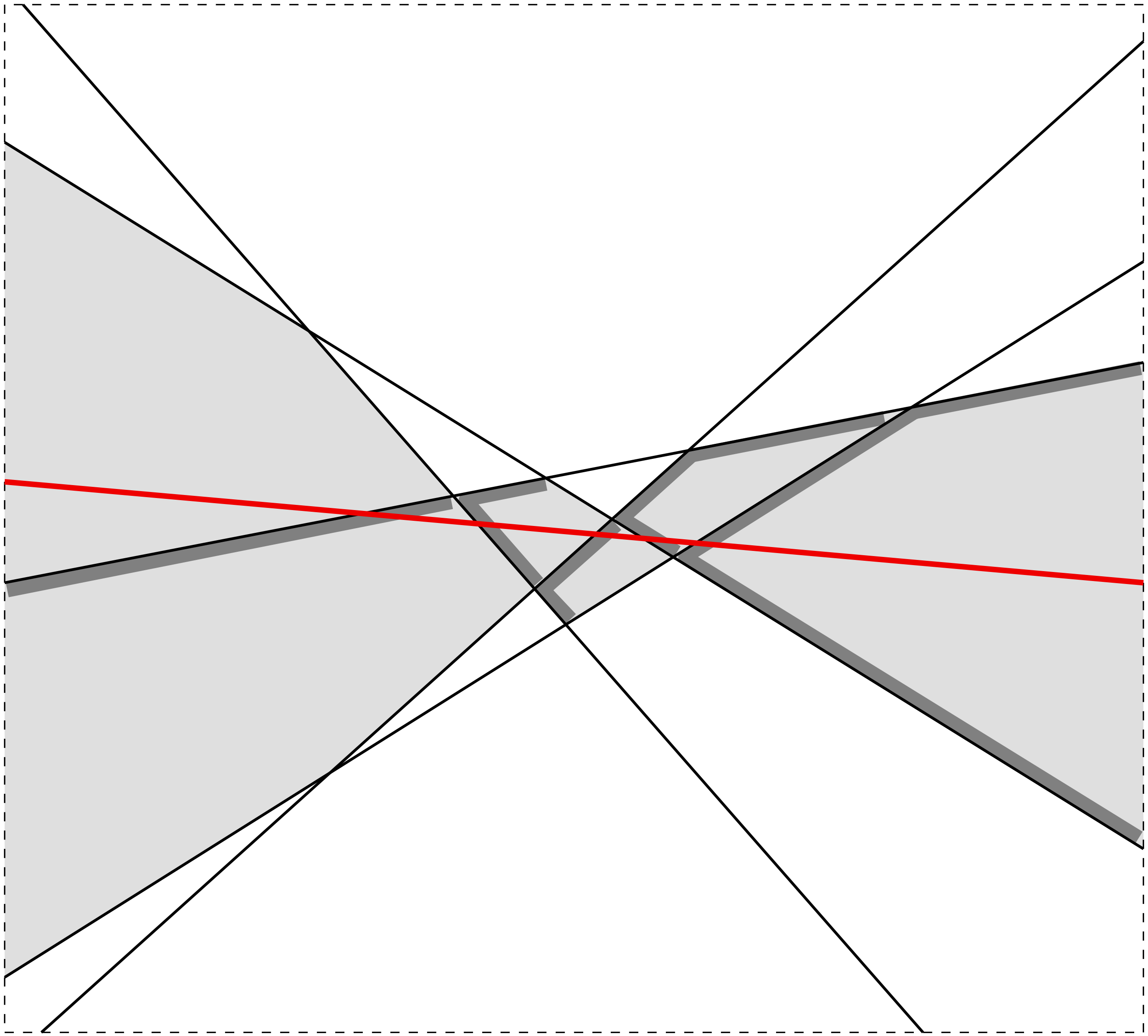

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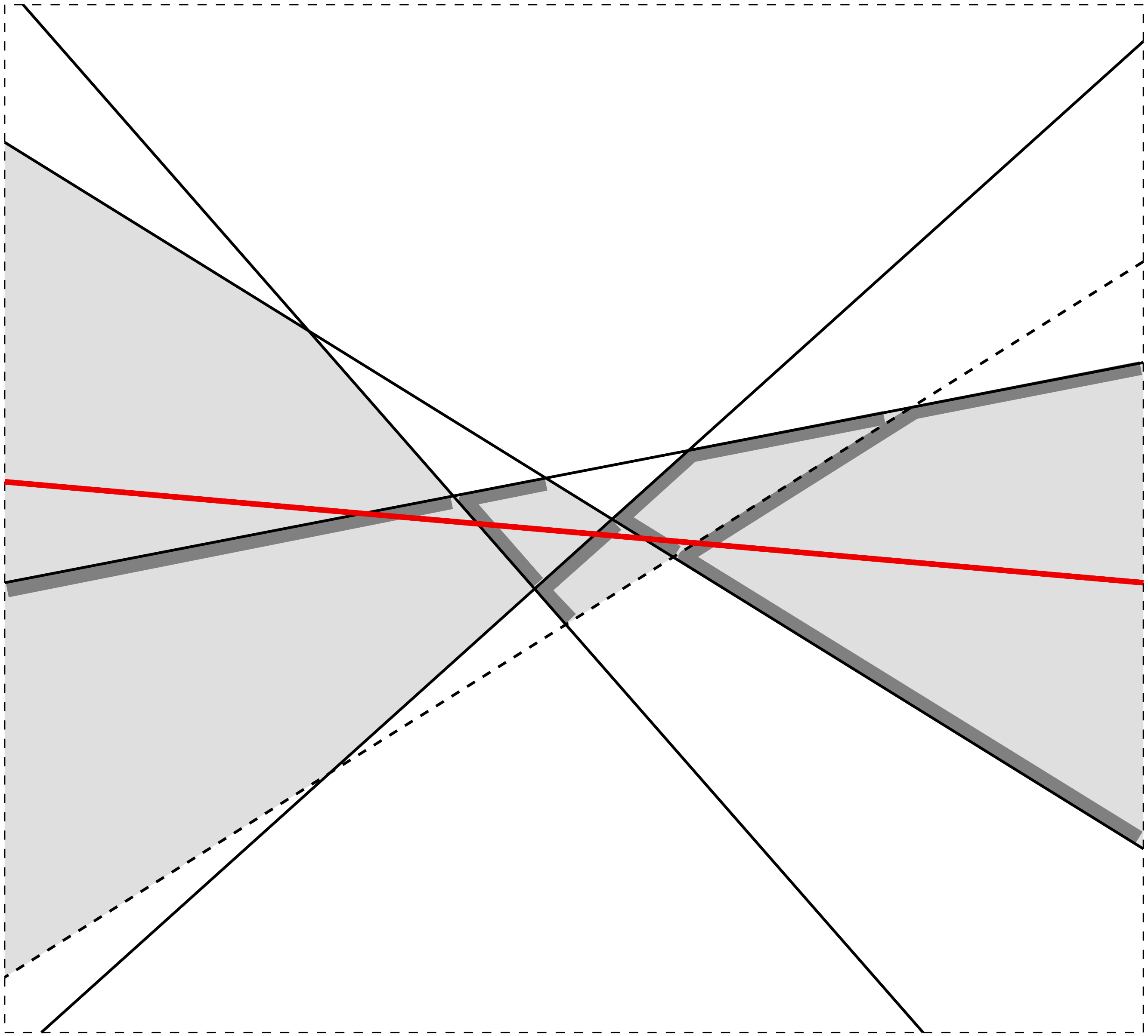

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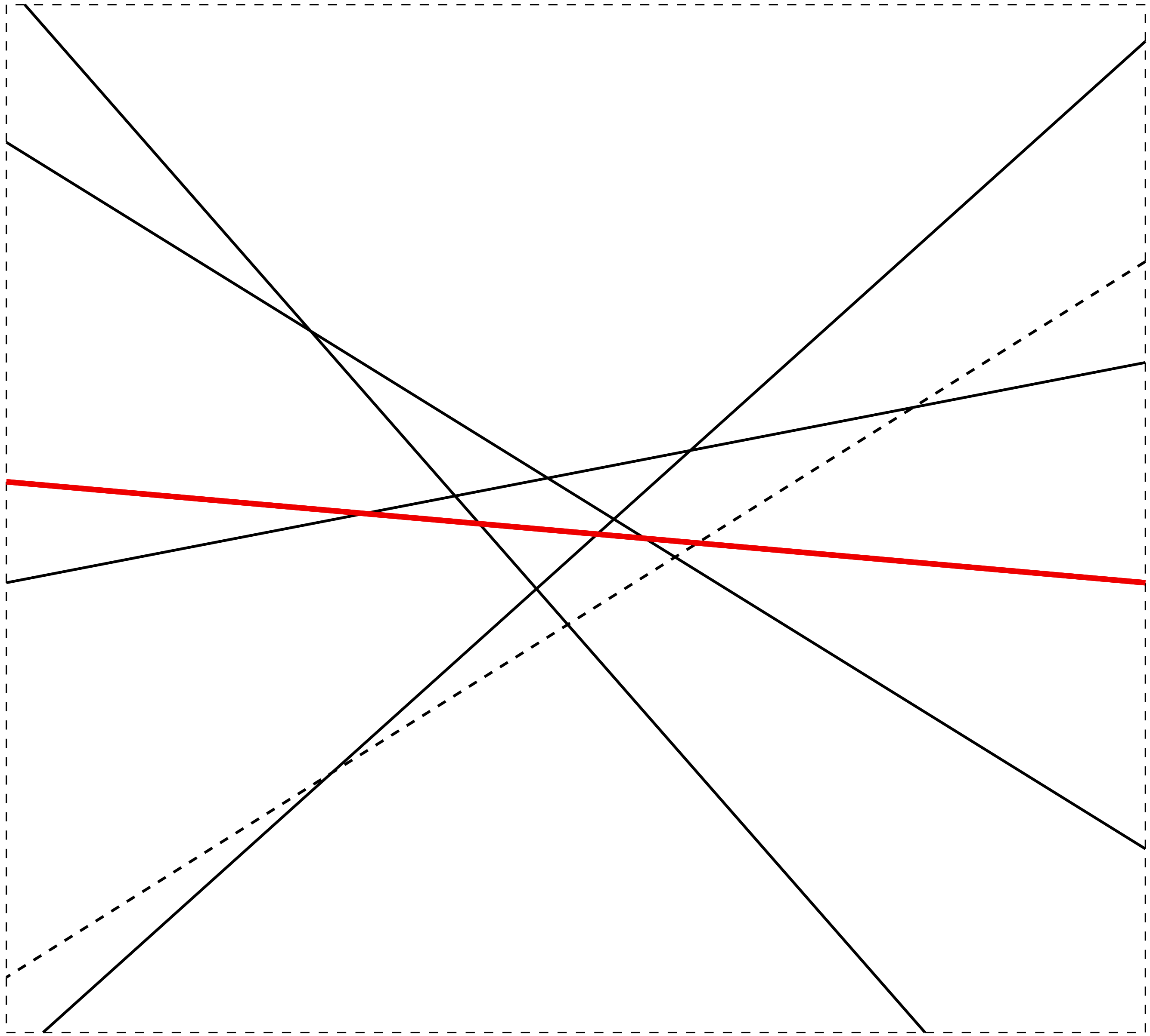

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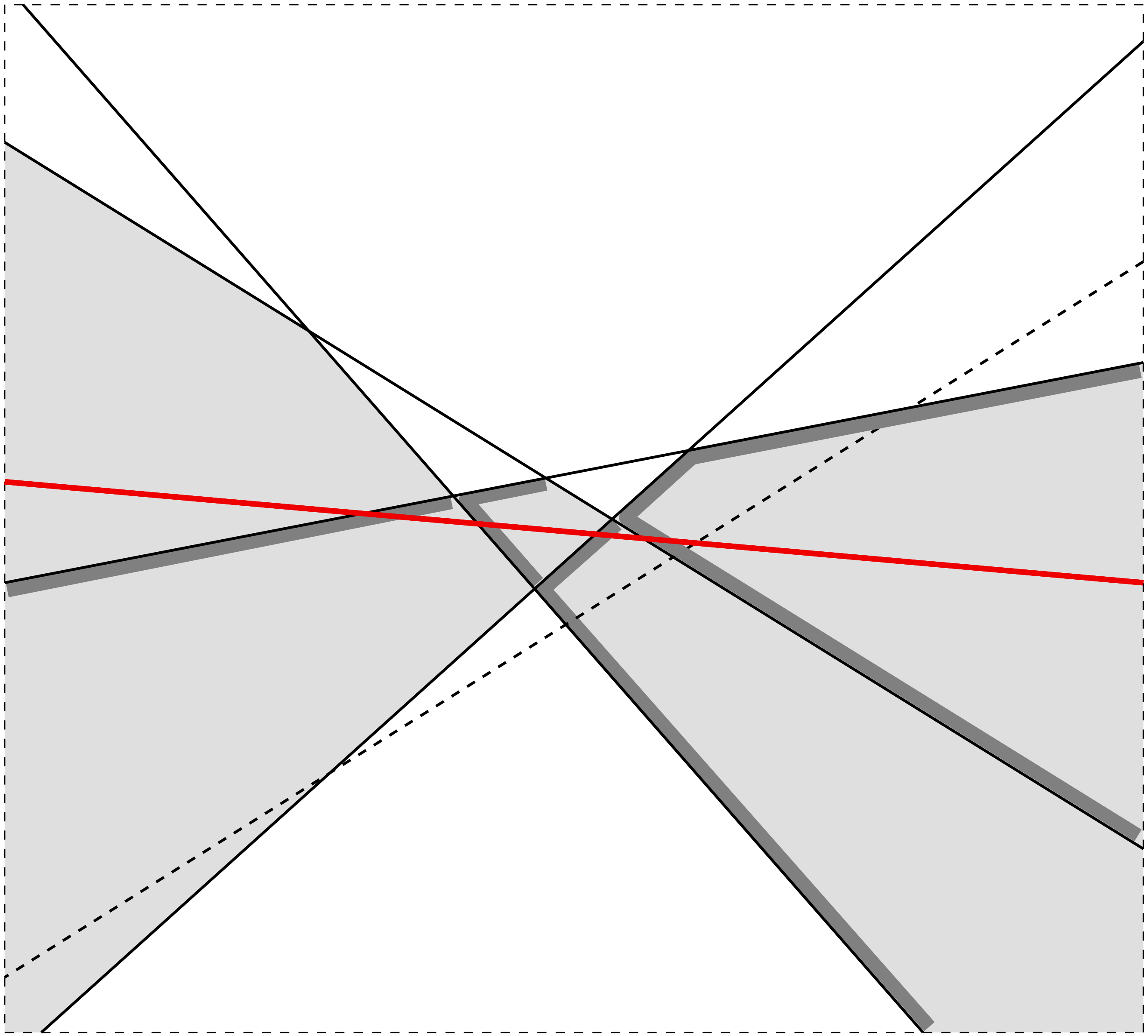

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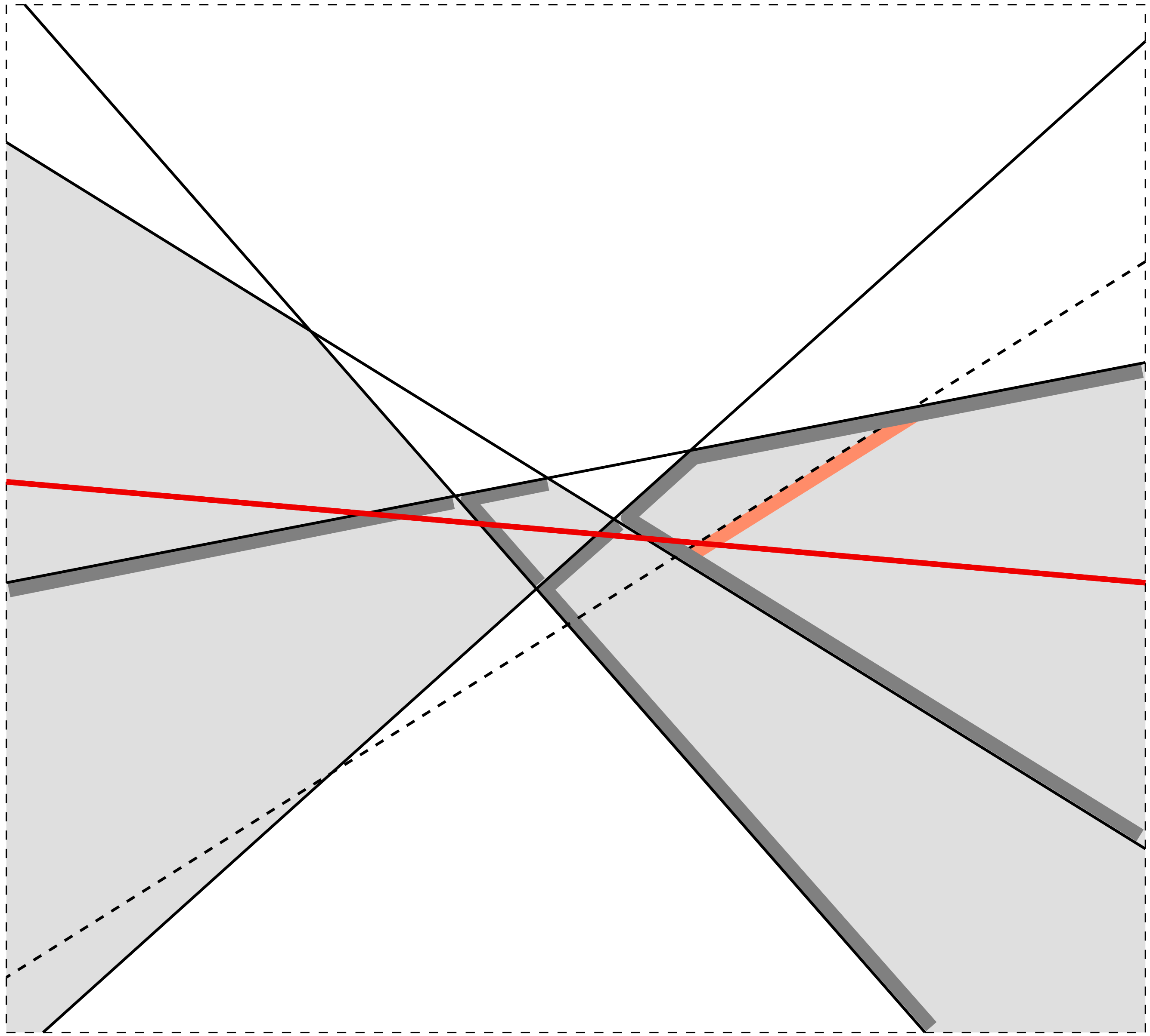

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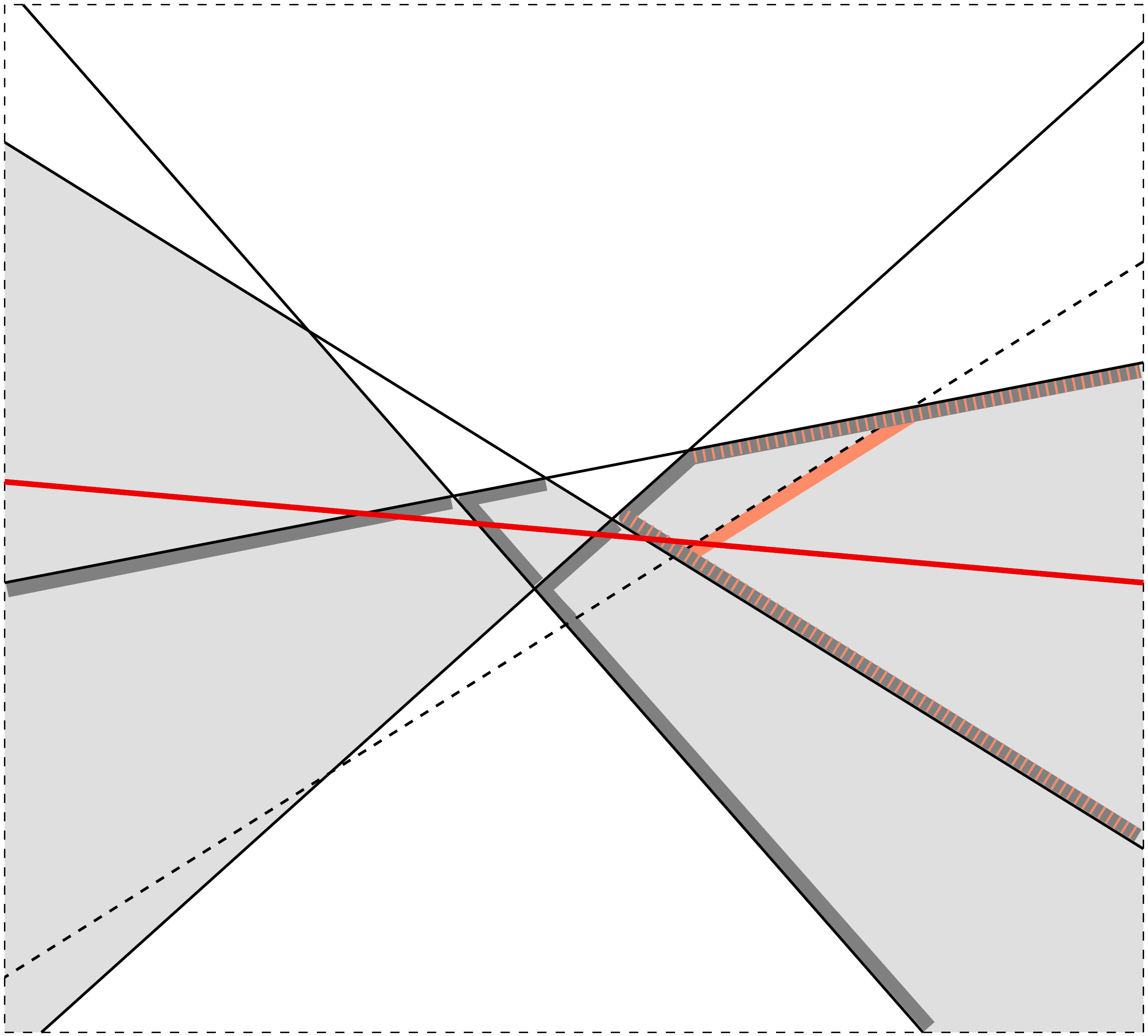

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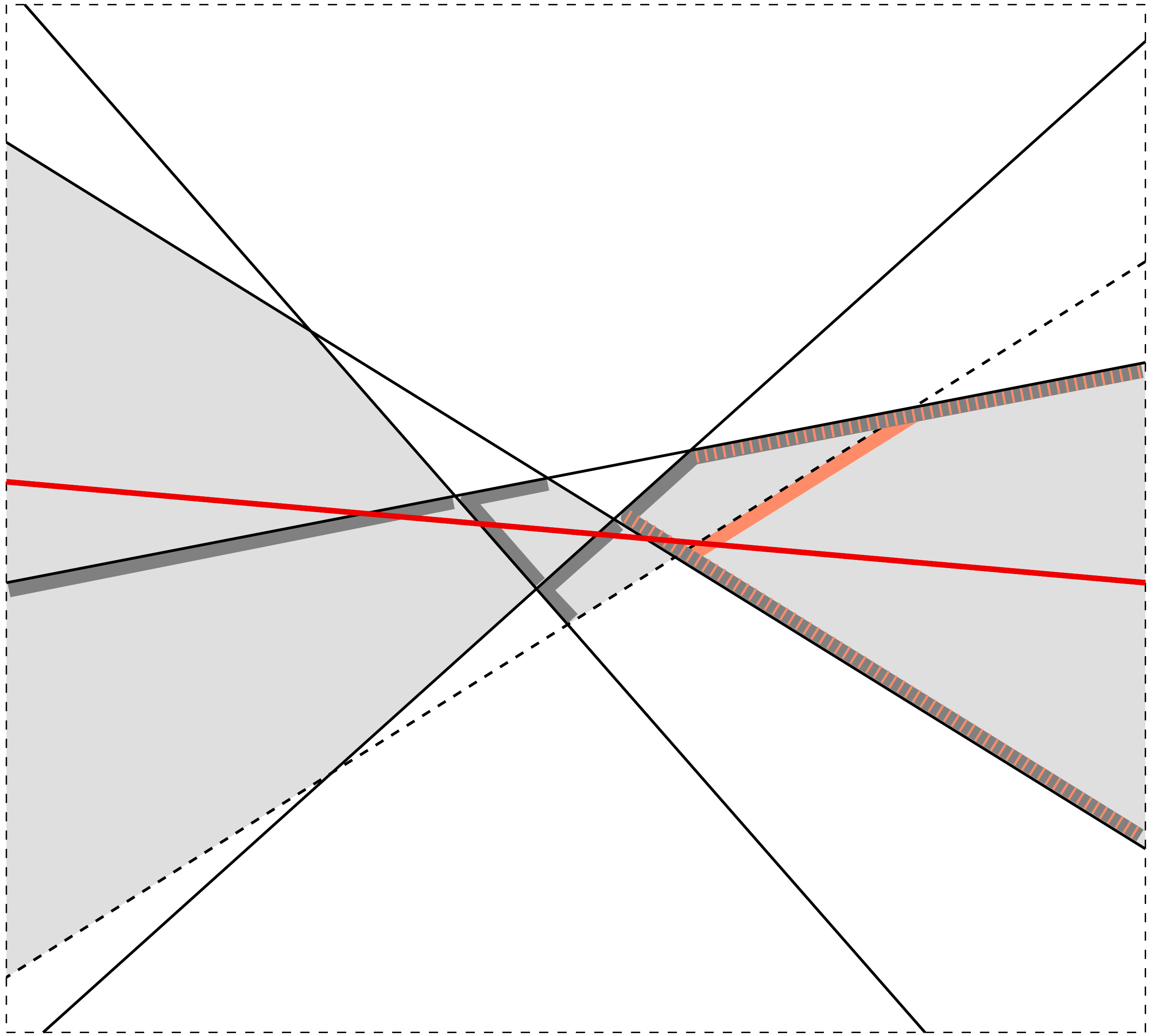

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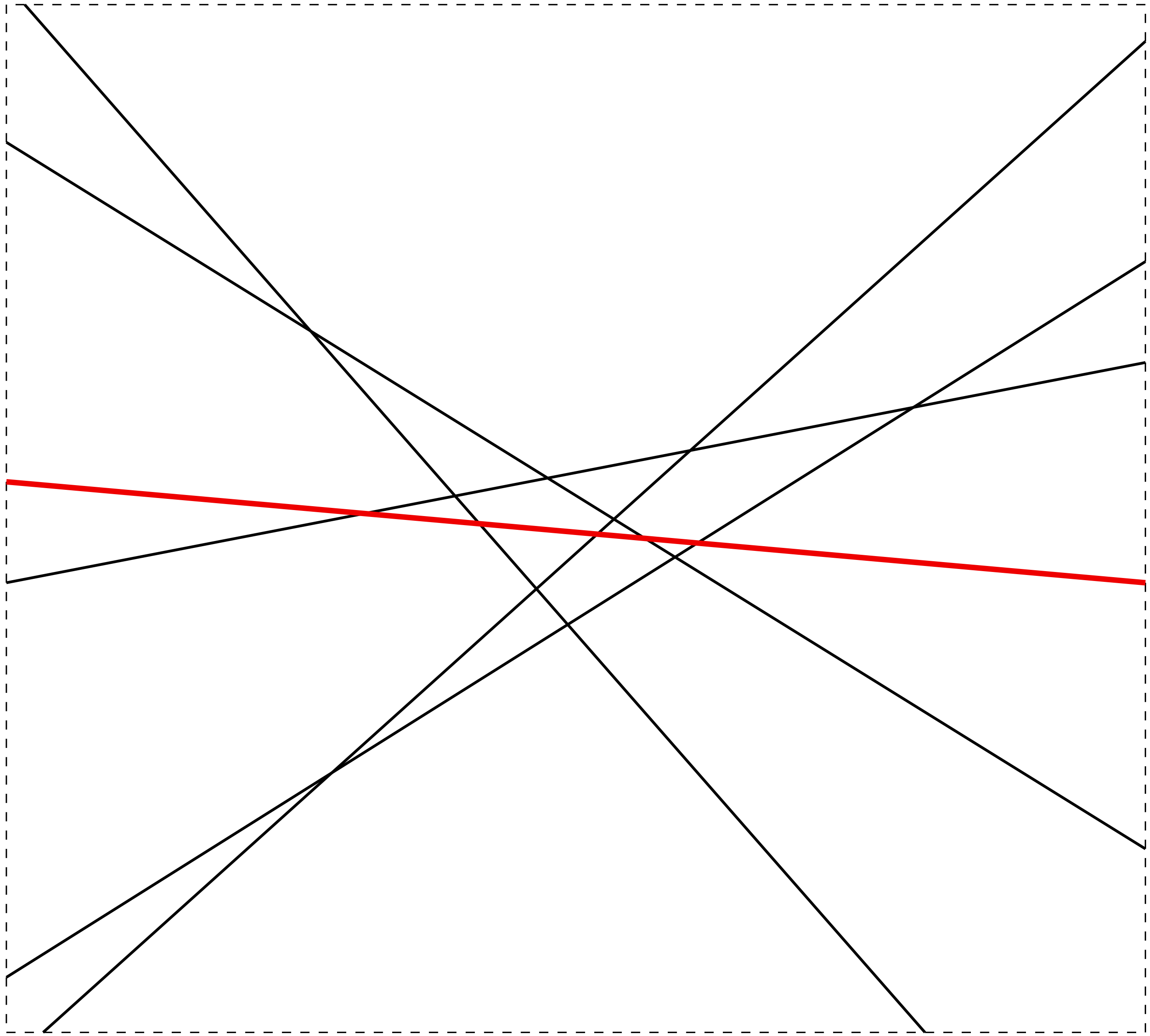

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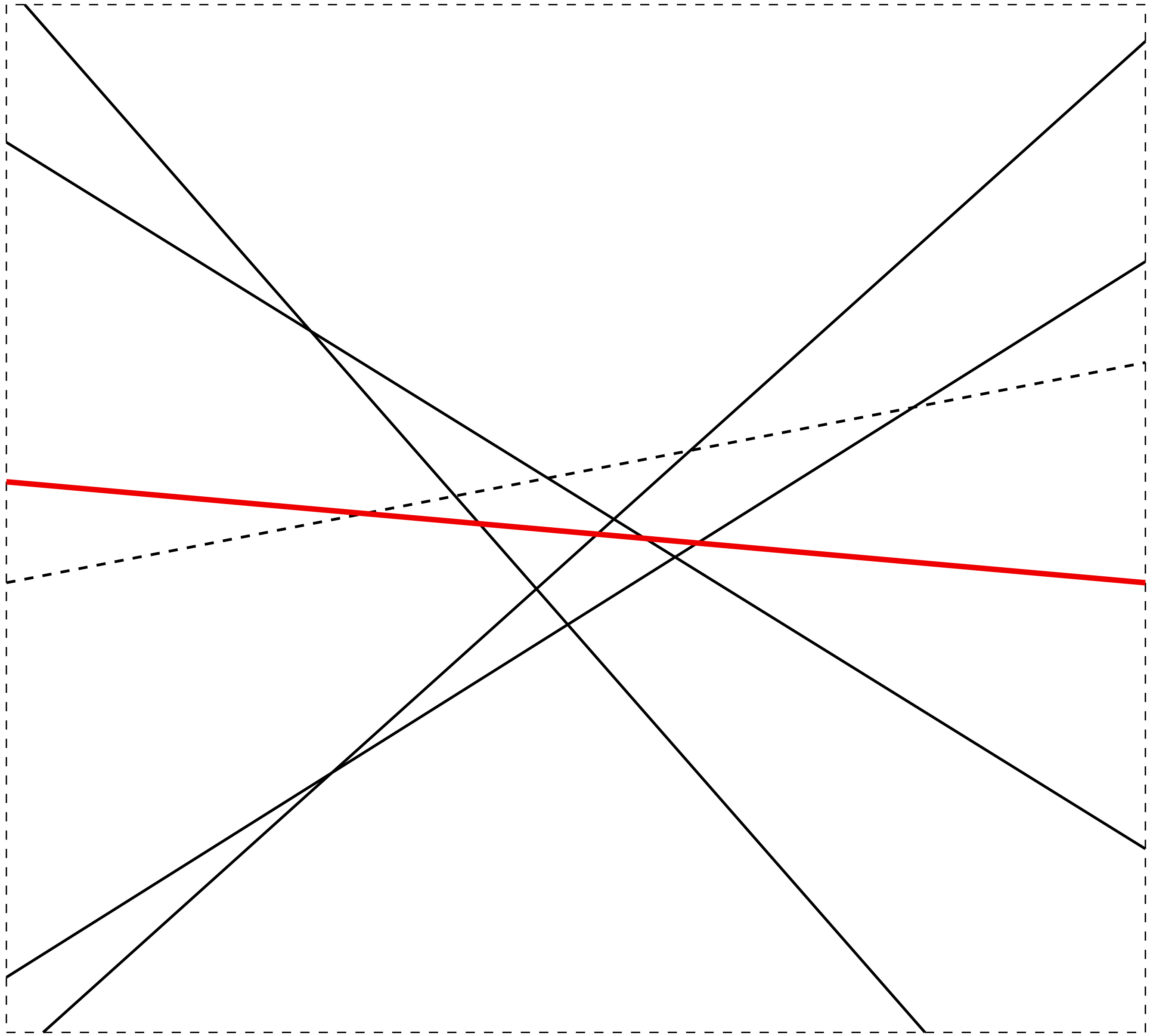

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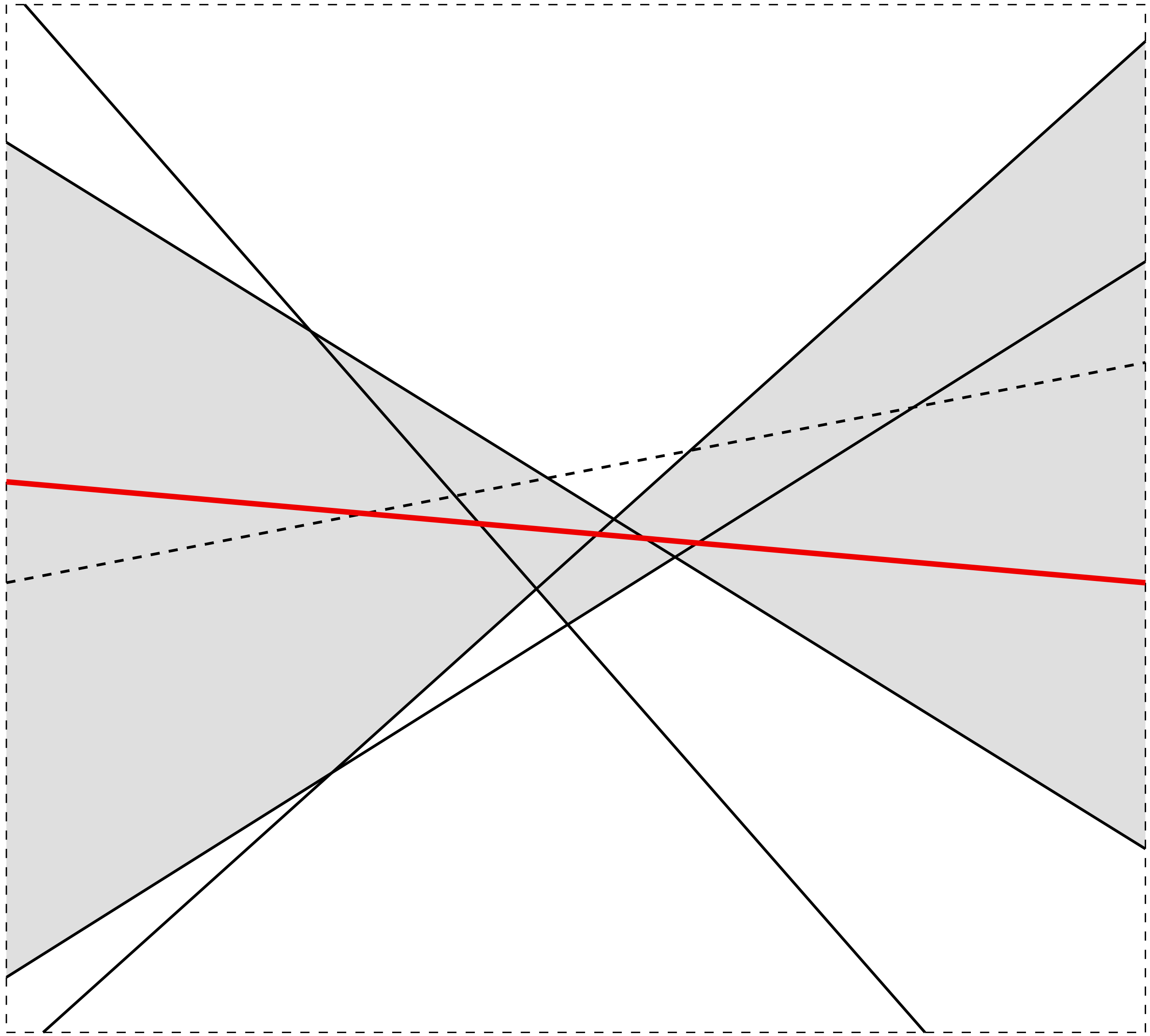

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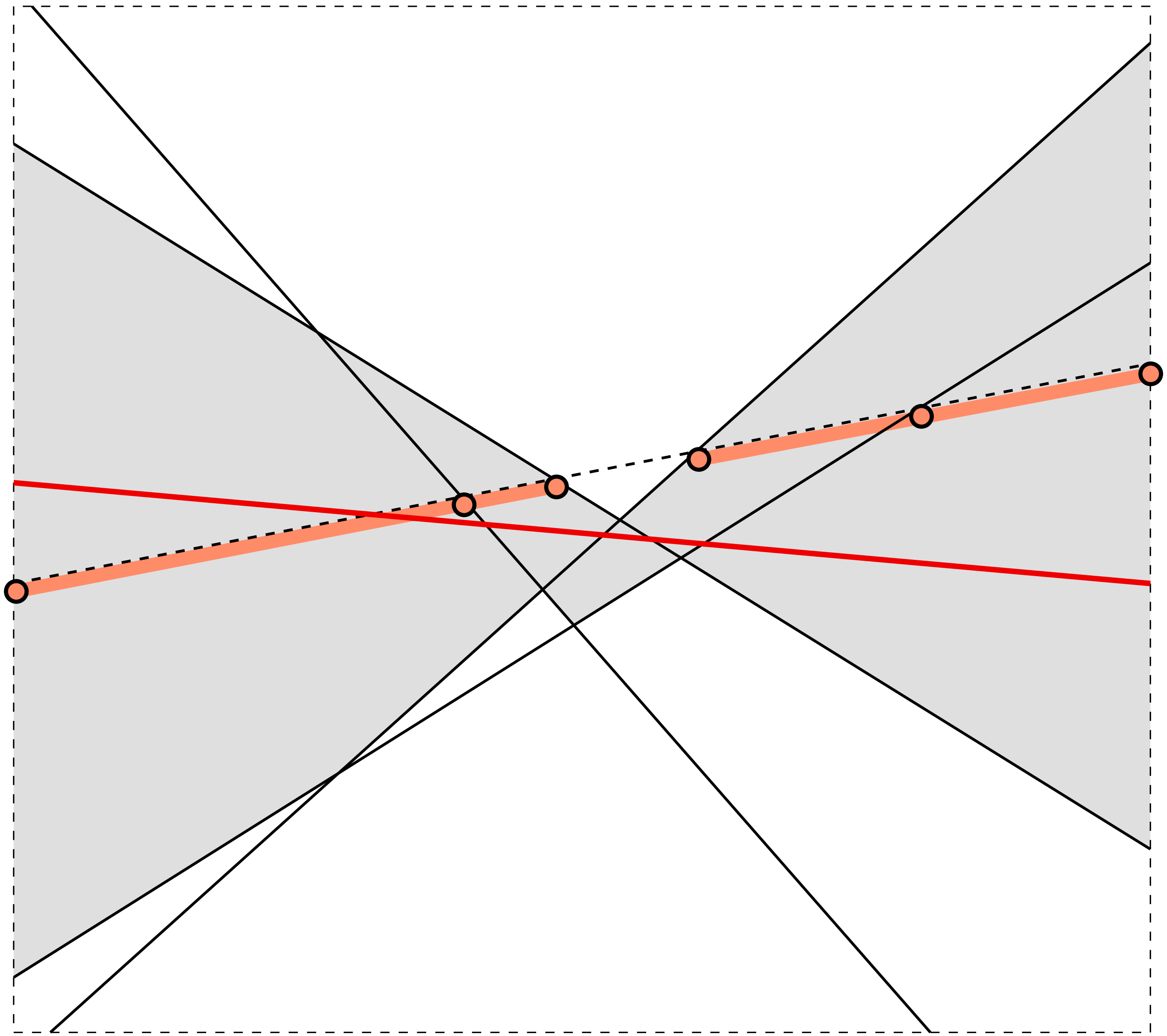

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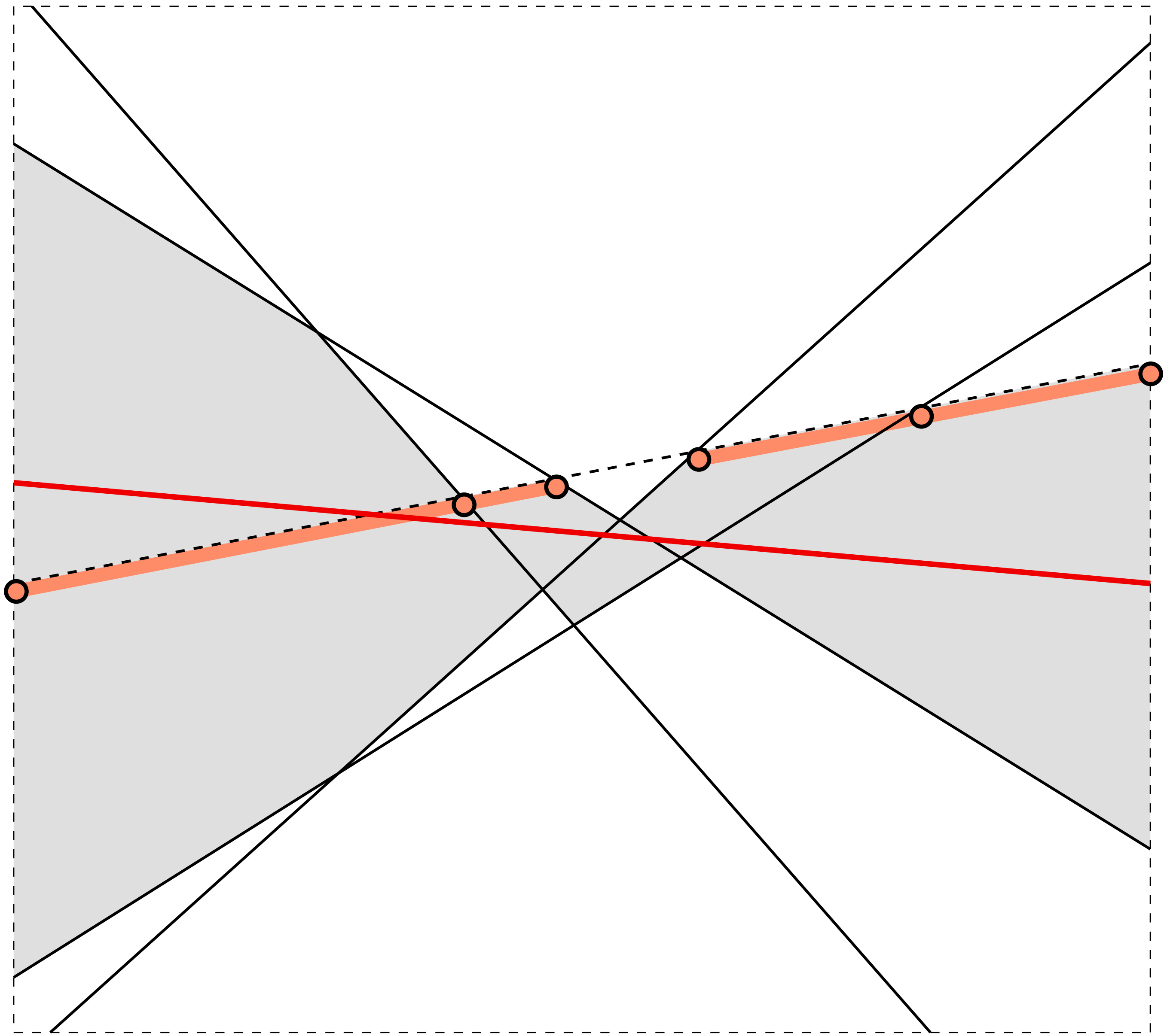

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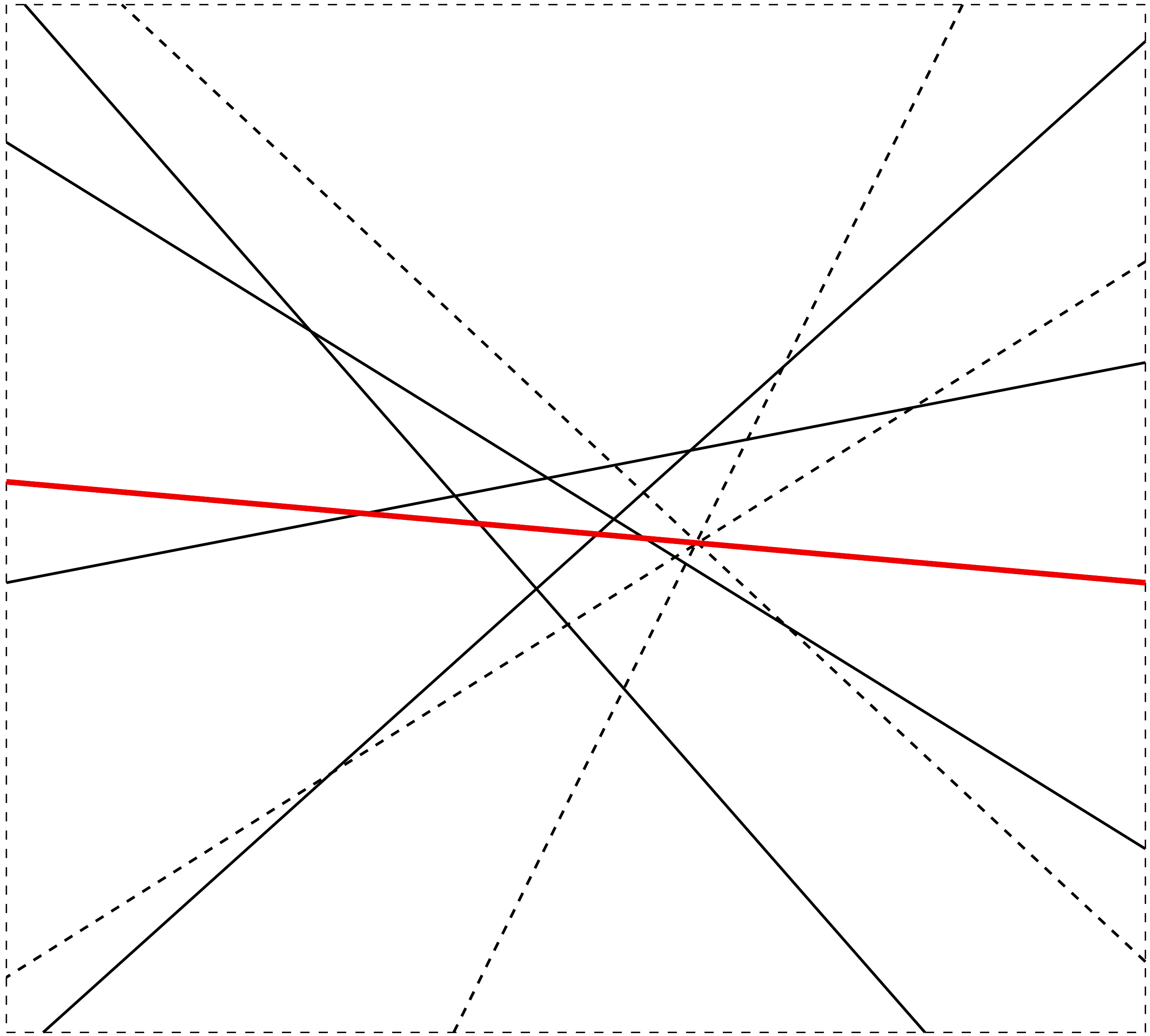

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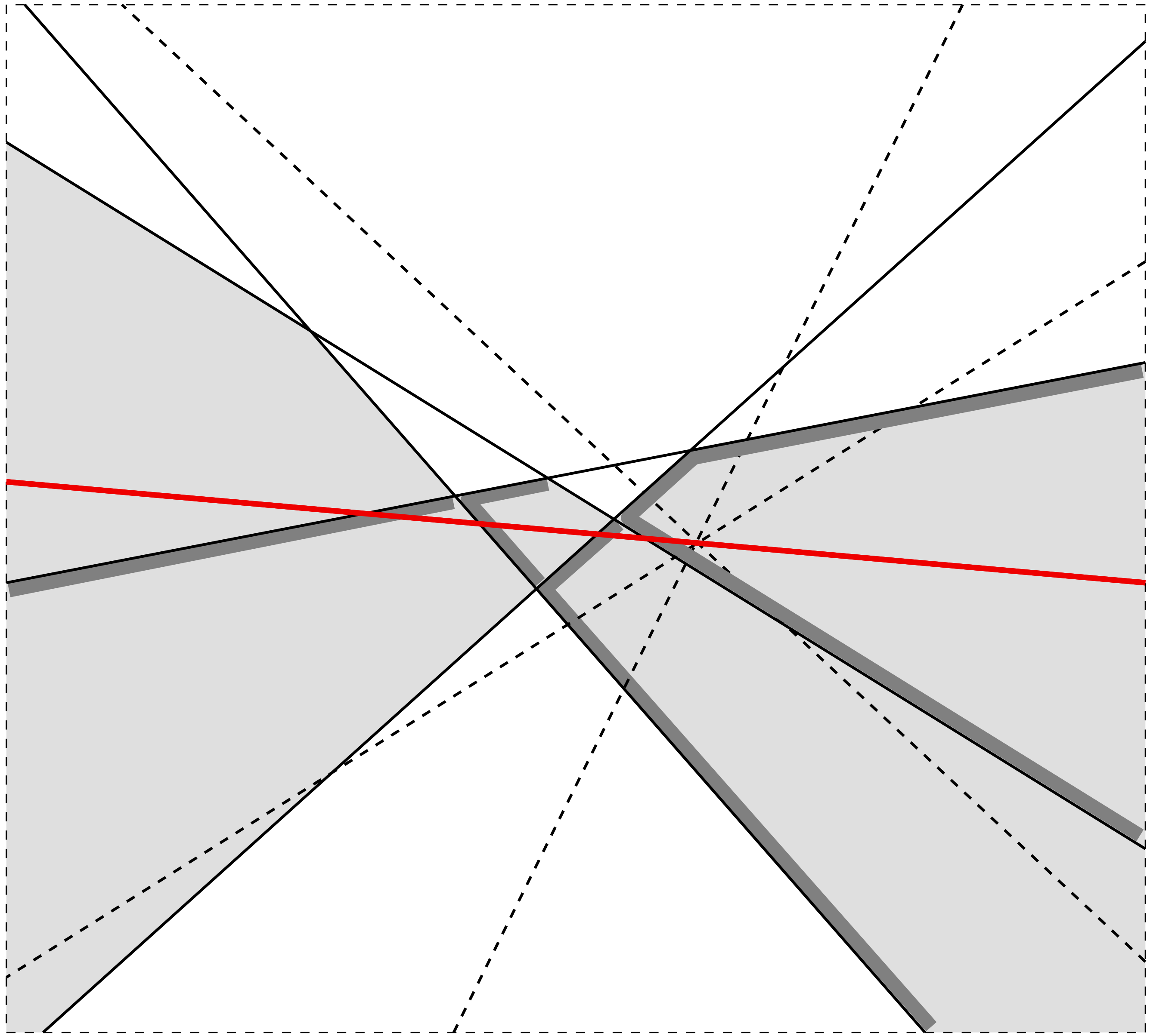

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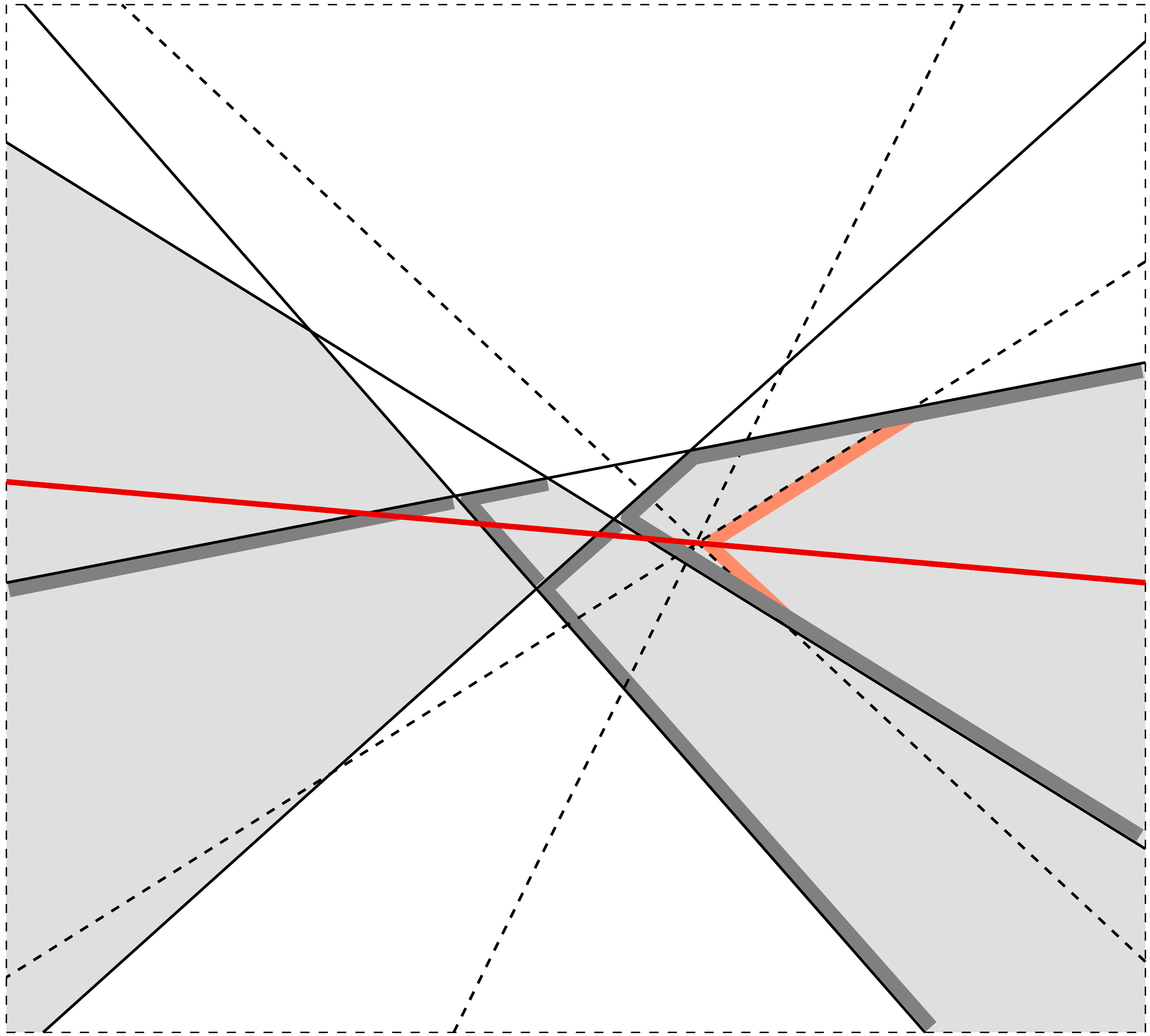

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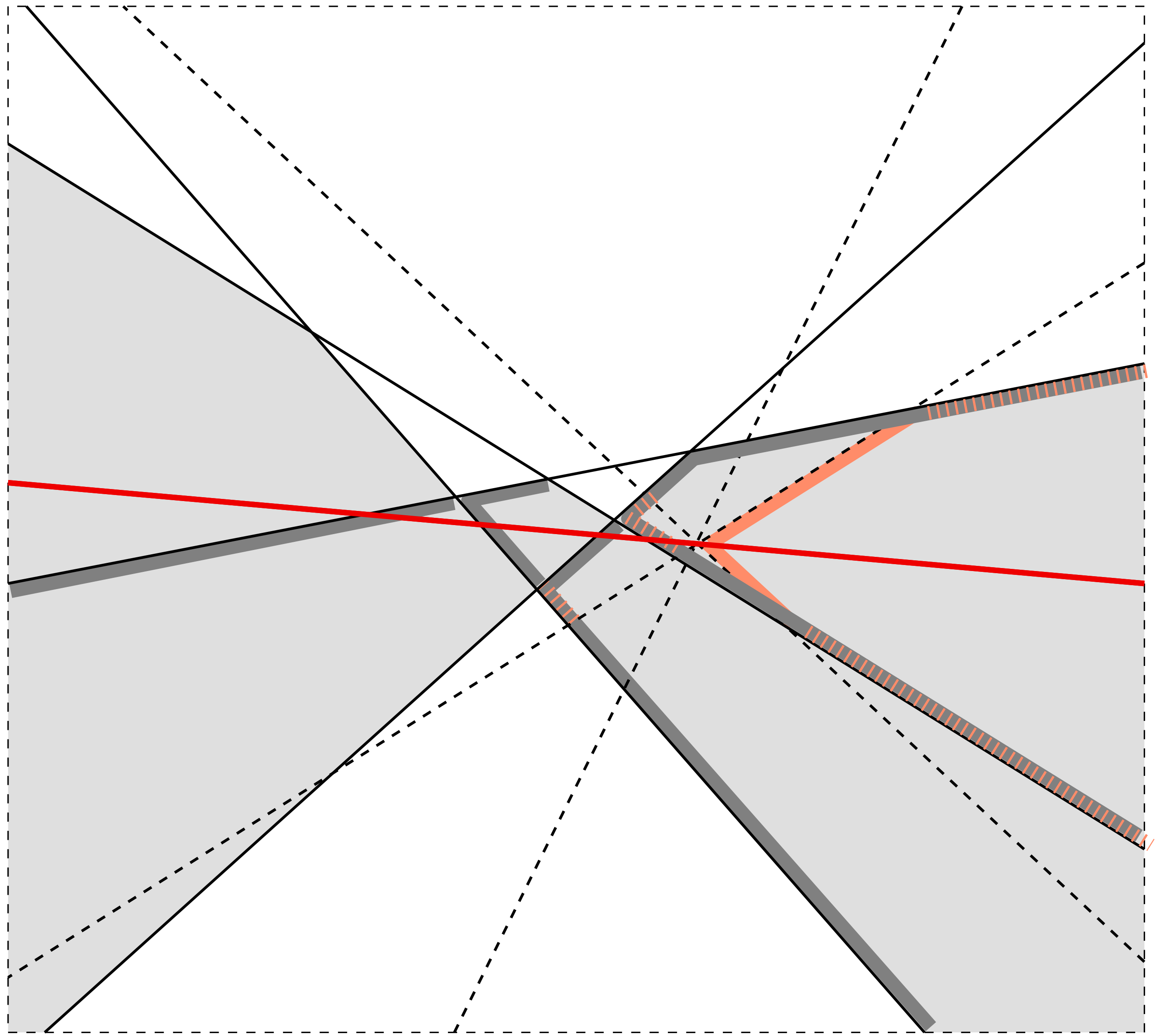

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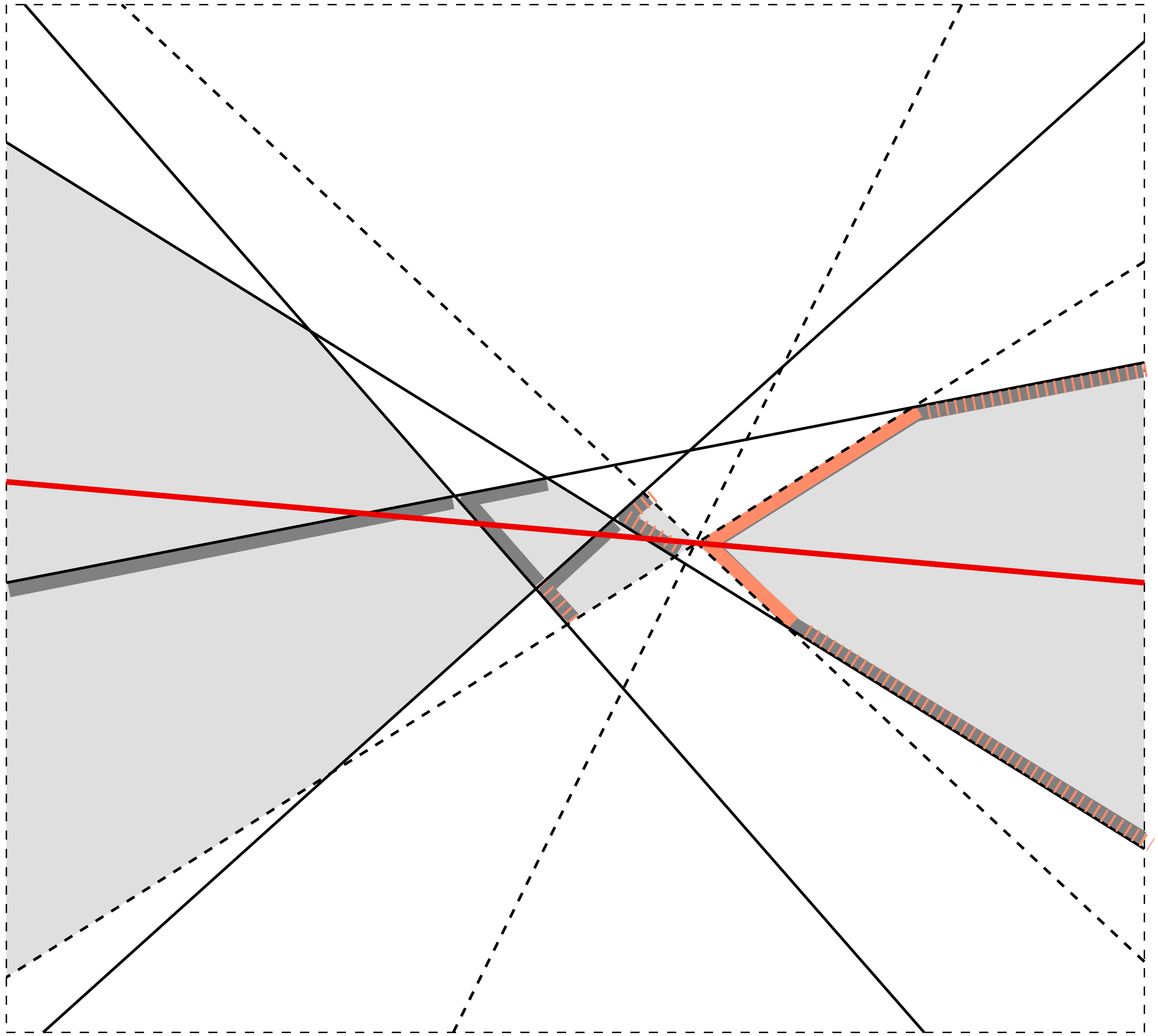

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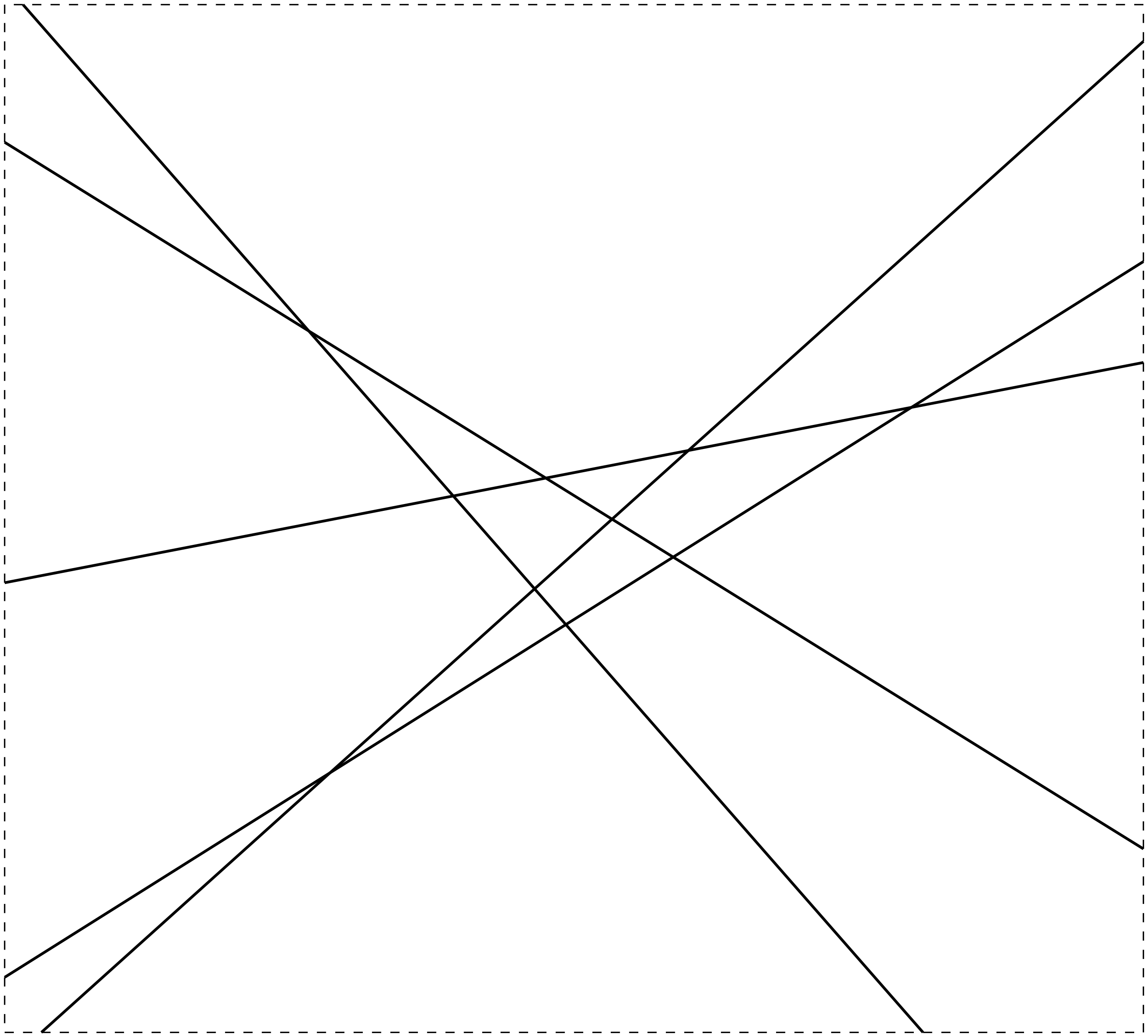
- The number of lines above p is $\leq k-1$.
- The number of lines below p is $\leq n-k$.



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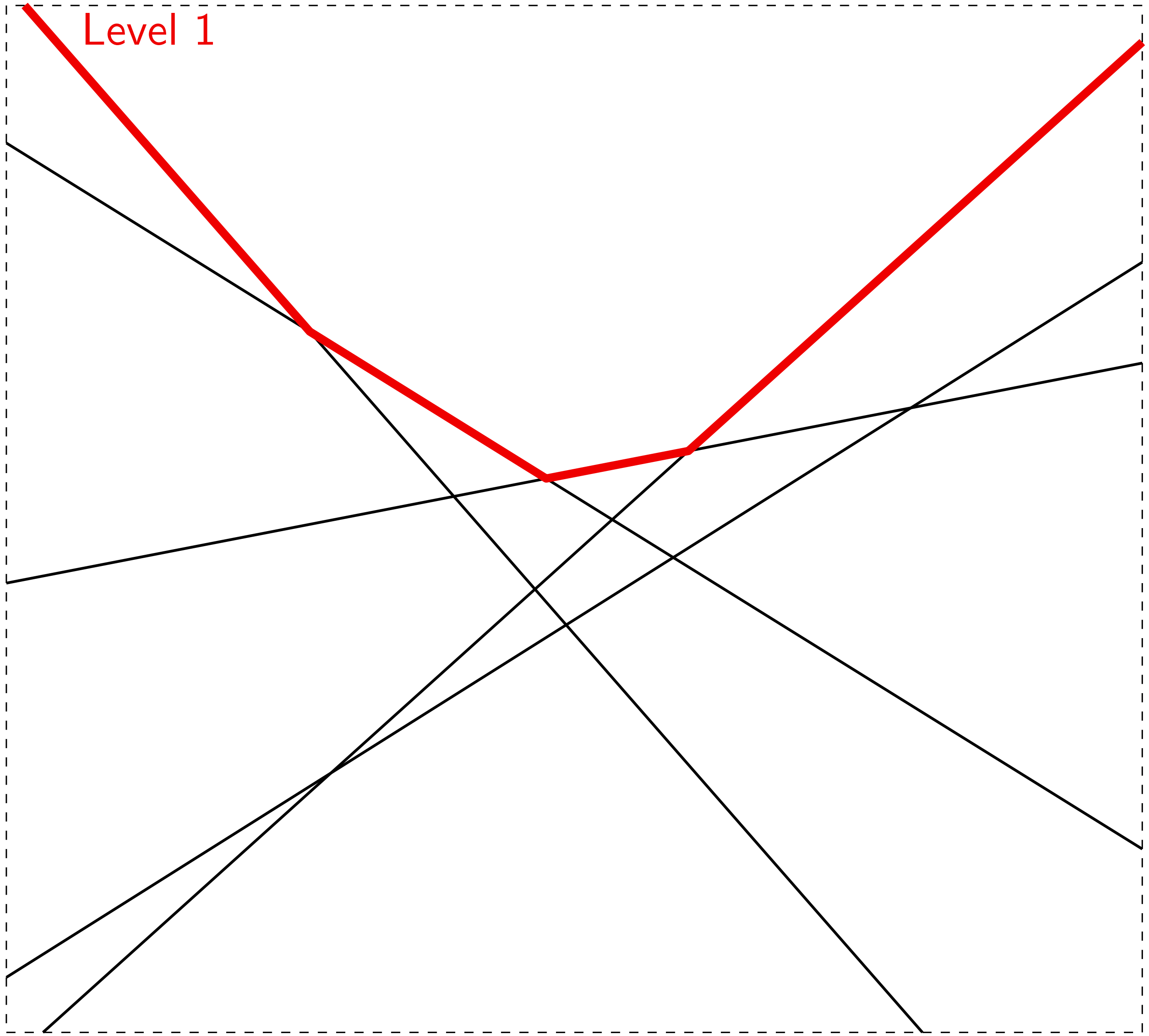

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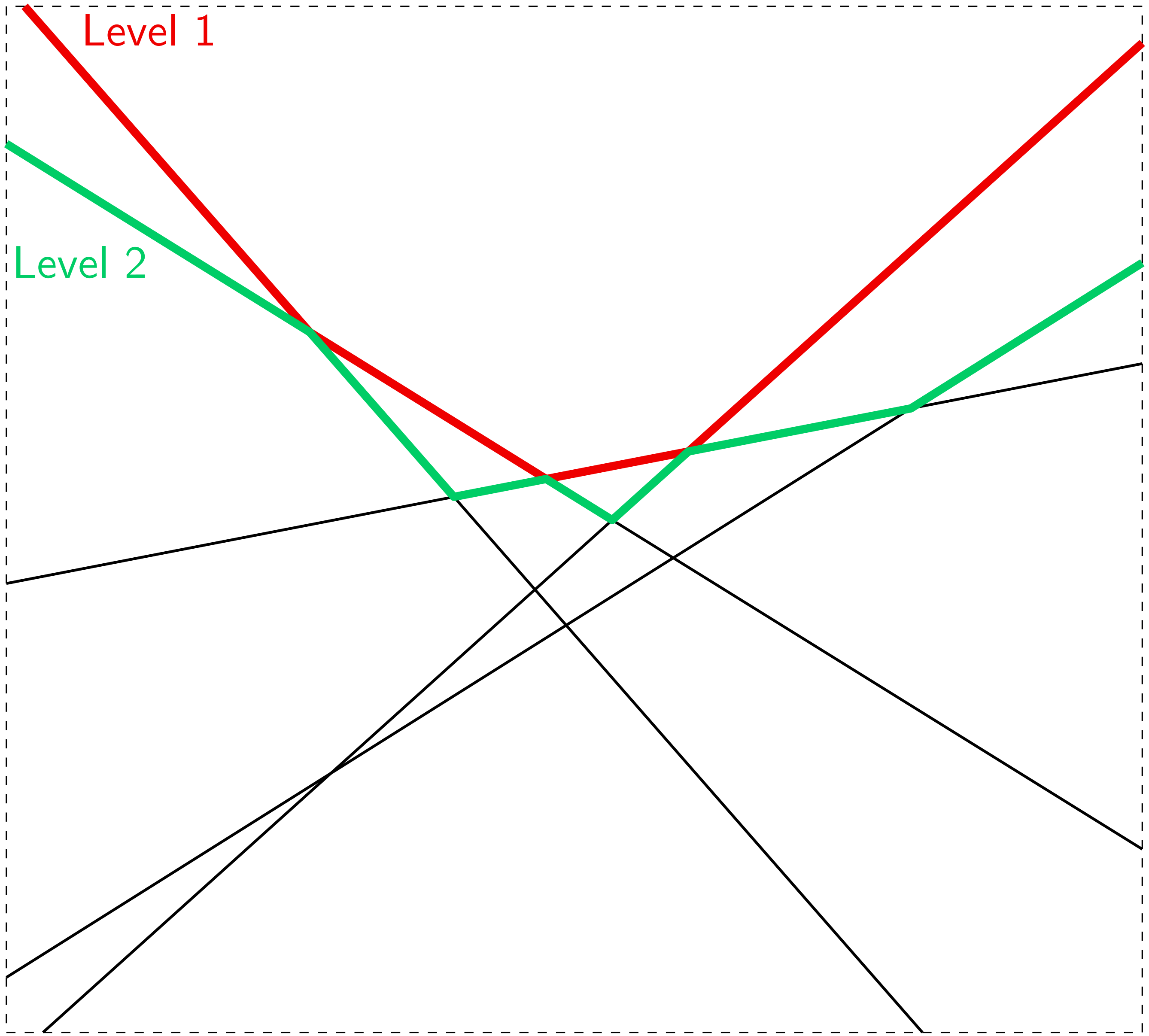

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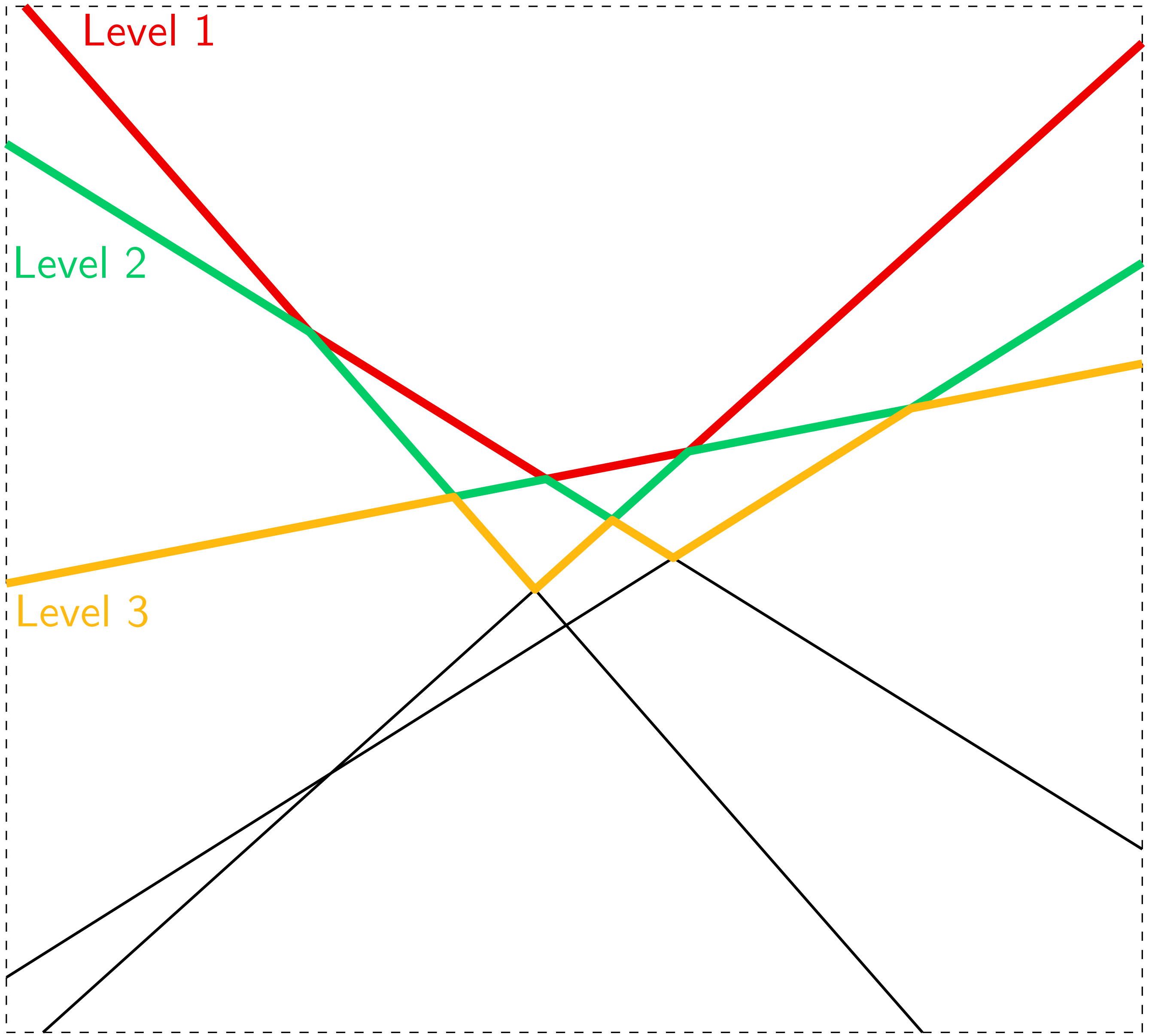

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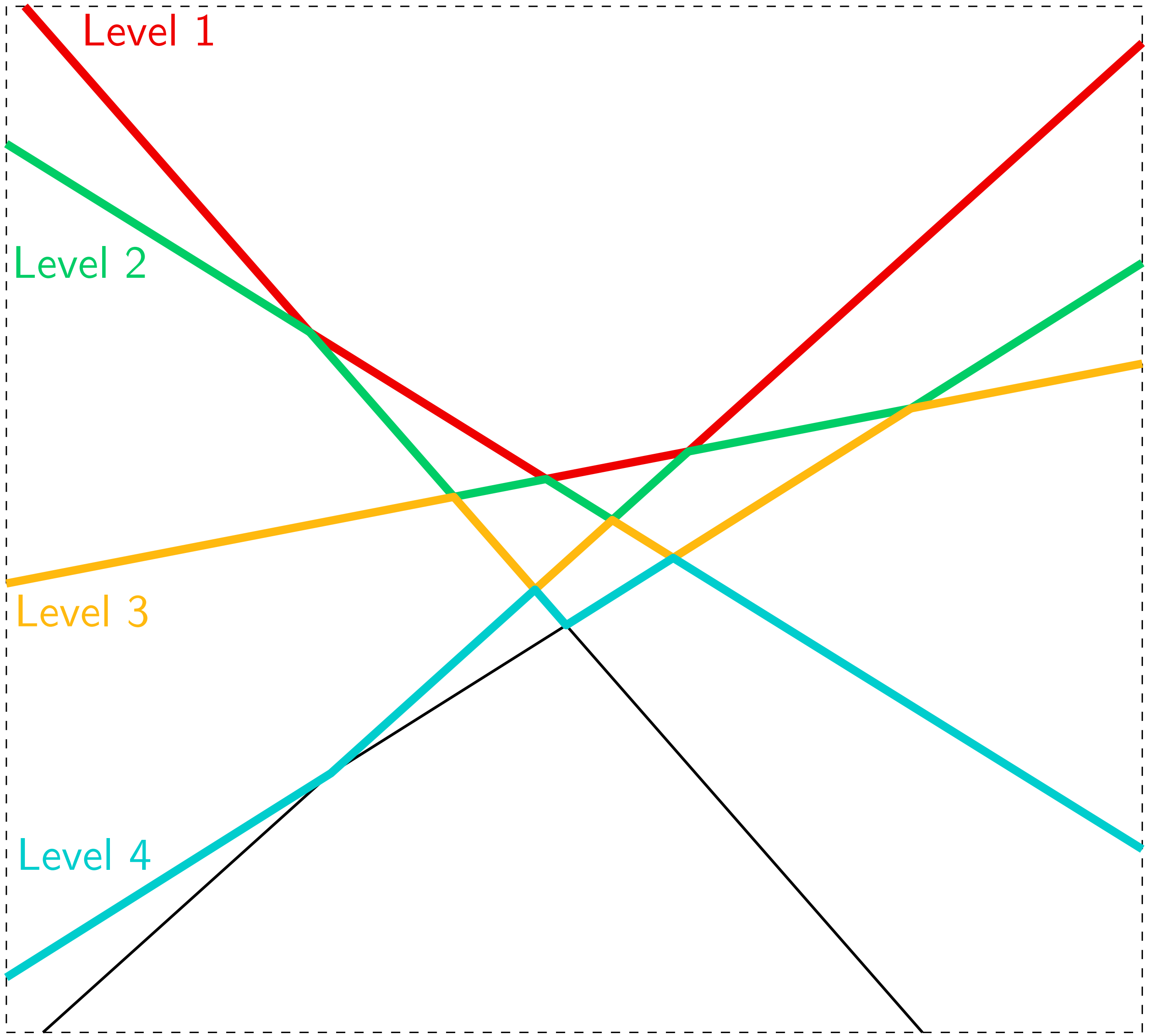

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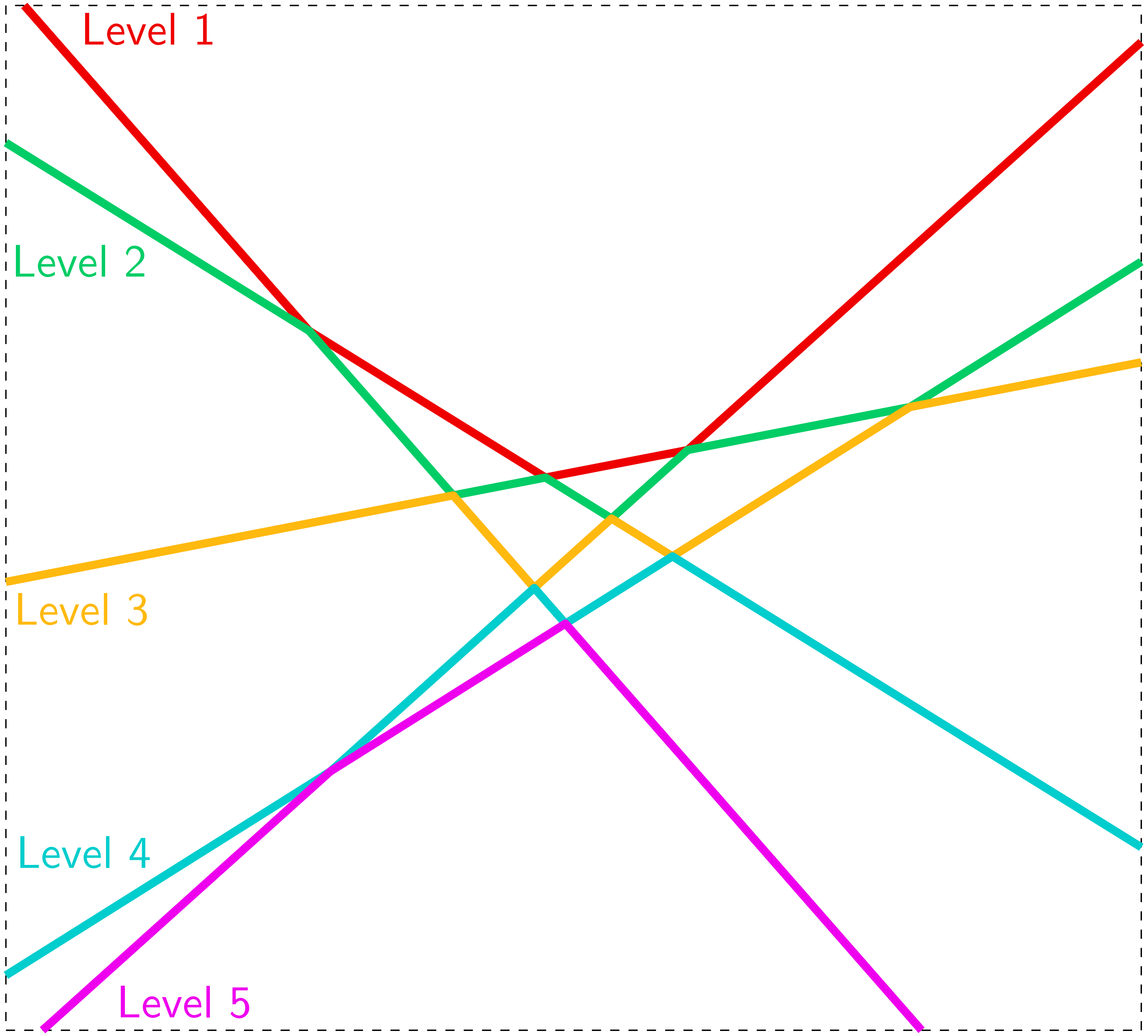
- The number of lines above p is $\leq k-1$.
- The number of lines below p is $\leq n-k$.



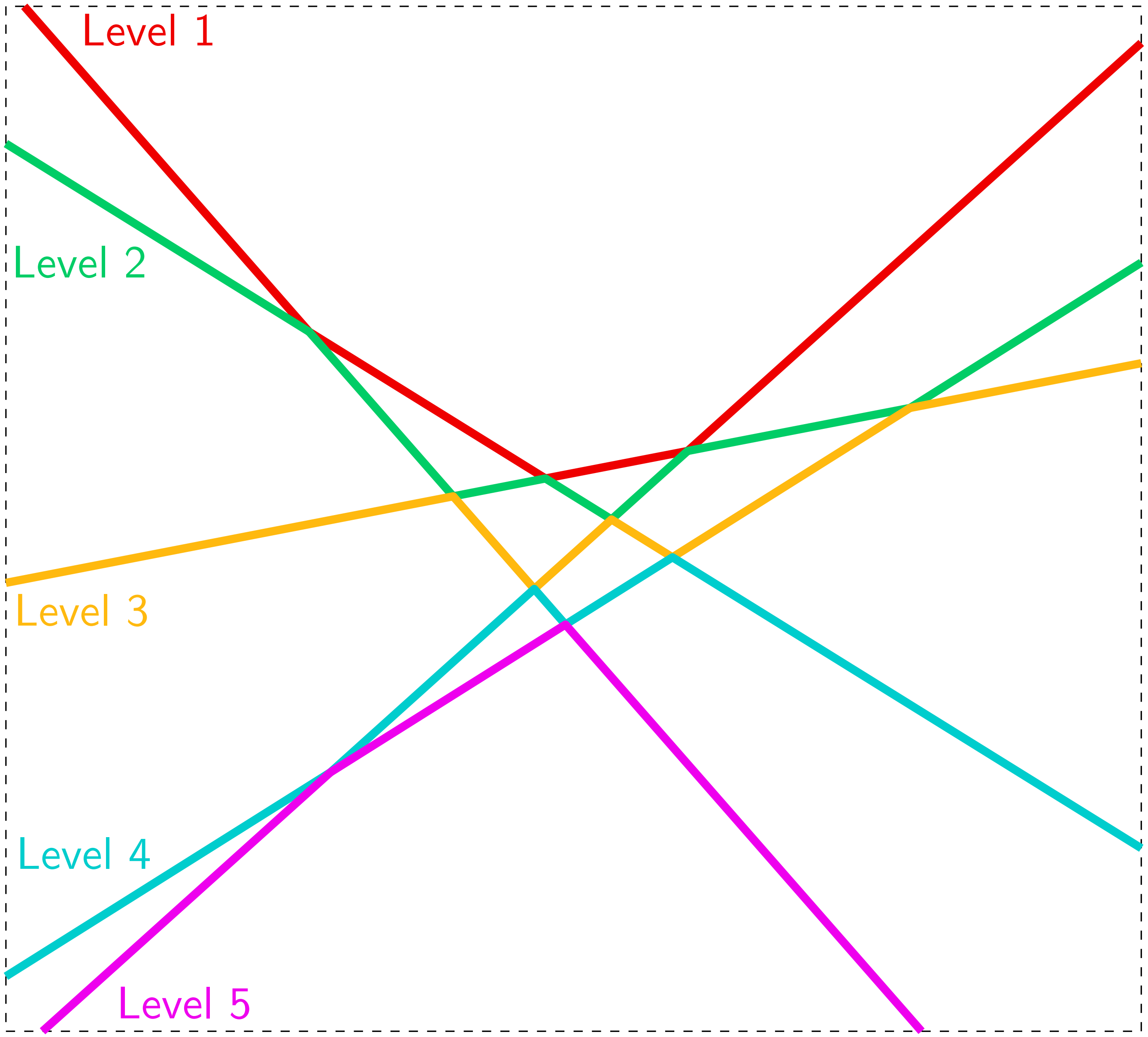
LEVELS

Properties

1. A point p belongs to one (more than one) level if and only if it belongs to one (more than one) of the lines.

2. Any vertical line intersects each level in a single point.

3. If a vertical line intersects two levels, $i \leq j$ in two points p_i and p_j, then p_i is above p_j. Consequently, the levels are totally ordered.



ARRANGEMENTS AND VORONOI DIAGRAMS

All previous definitions and properties extend to arrangements of hyperplanes in $\mathbb{R}^d$ (simplicity, combinatorics, levels,...), but the complexity of the arrangement depends on the dimension.

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Proposition: Let $P = \{p_1, \ldots, p_n\}$ be a finite set of points $p_i = (a_i, b_i, 0)$. Let $H = \{h_1, \ldots, h_n\}$ be the set of planes which are tangent to the paraboloid $z = x^2 + y^2$ at the points $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$. The 1-skeleton of the kth-order Voronoi diagram of P is the orthogonal projection onto the plane $z = 0$ of the intersection of the levels k and $k+1$ of $\mathcal{A}(H)$.

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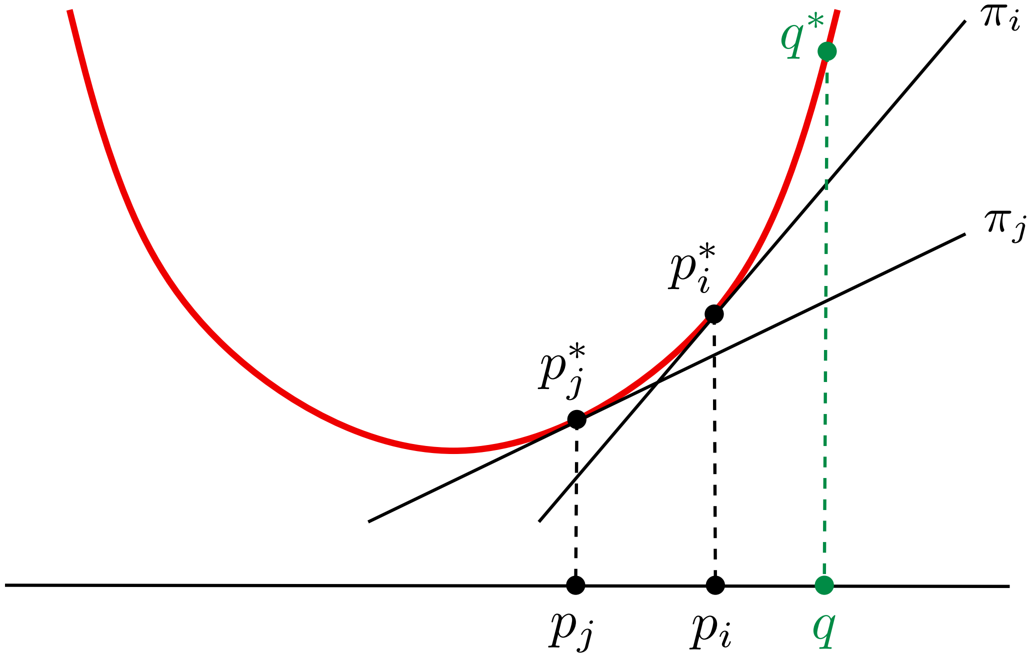

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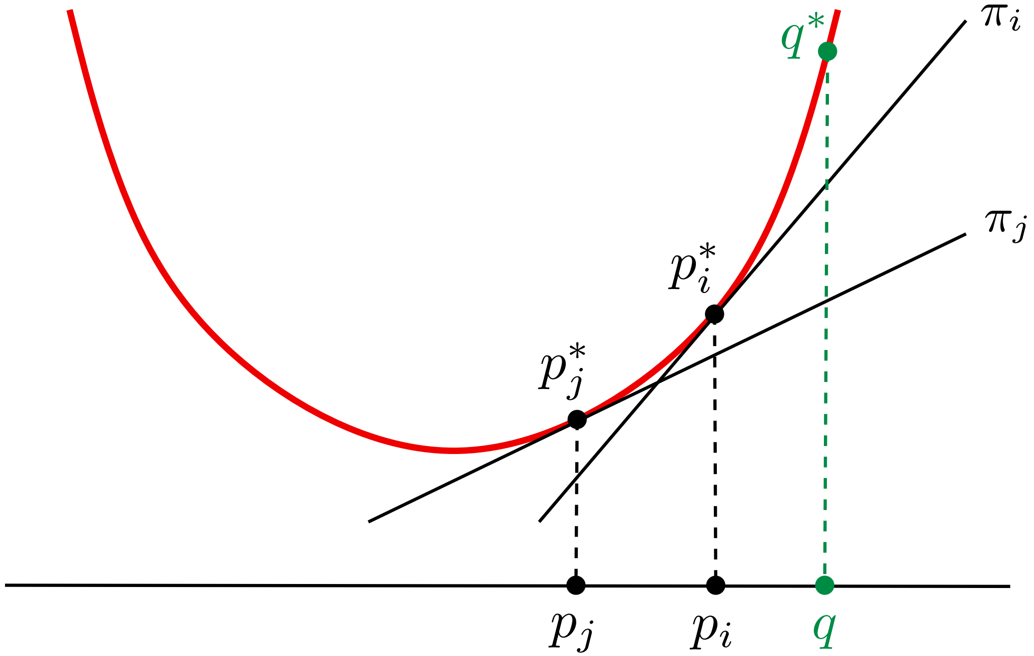
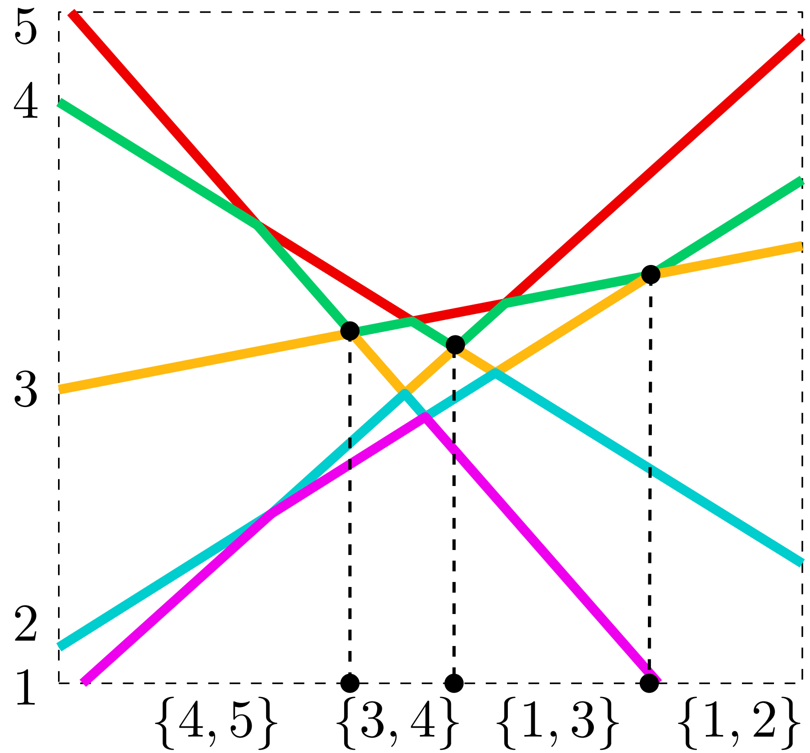
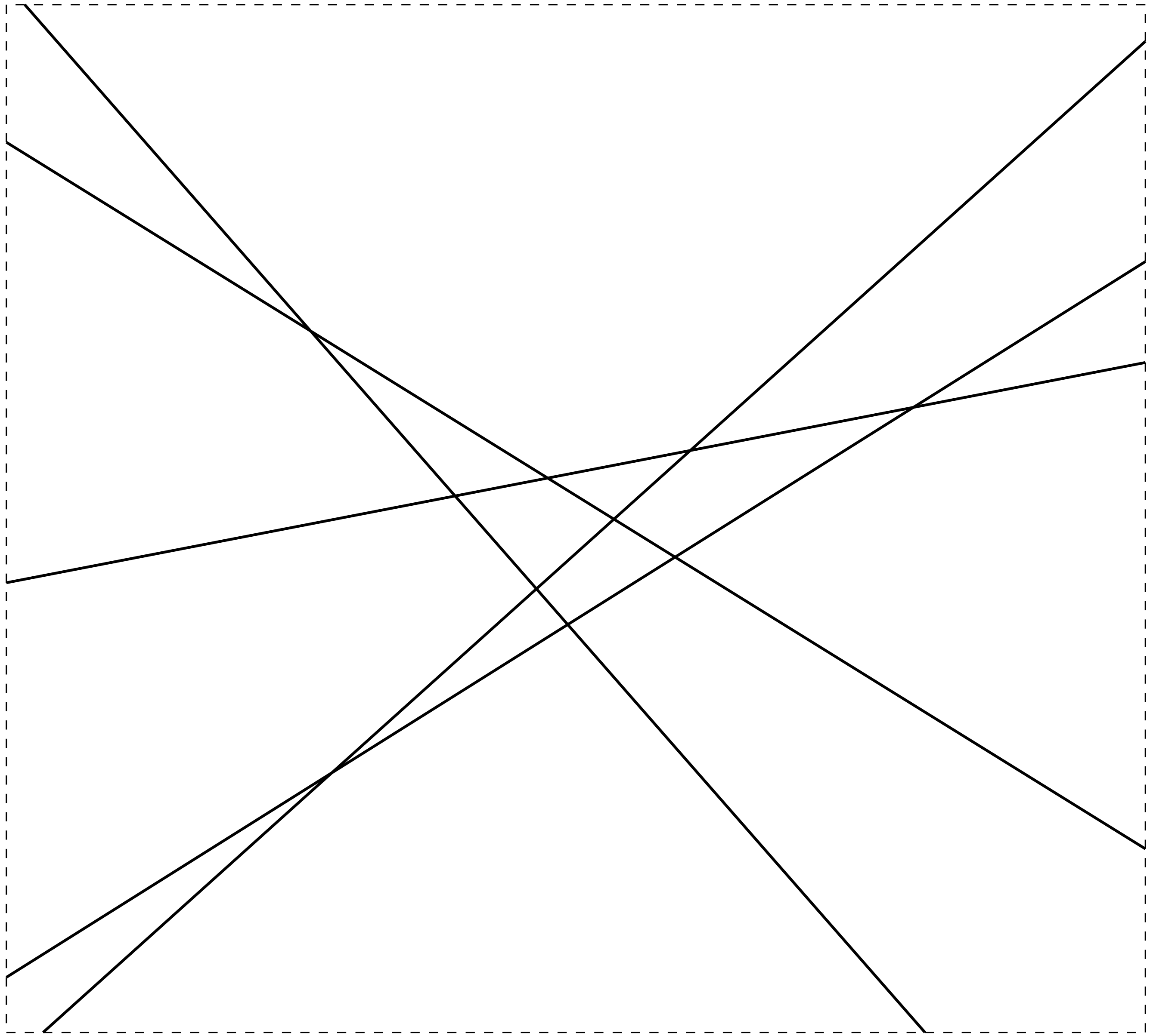




ARRANGEMENTS

STORAGE

As an arragements of lines is a decomposition of the plane, the structure usually used to store it is a DCEL.



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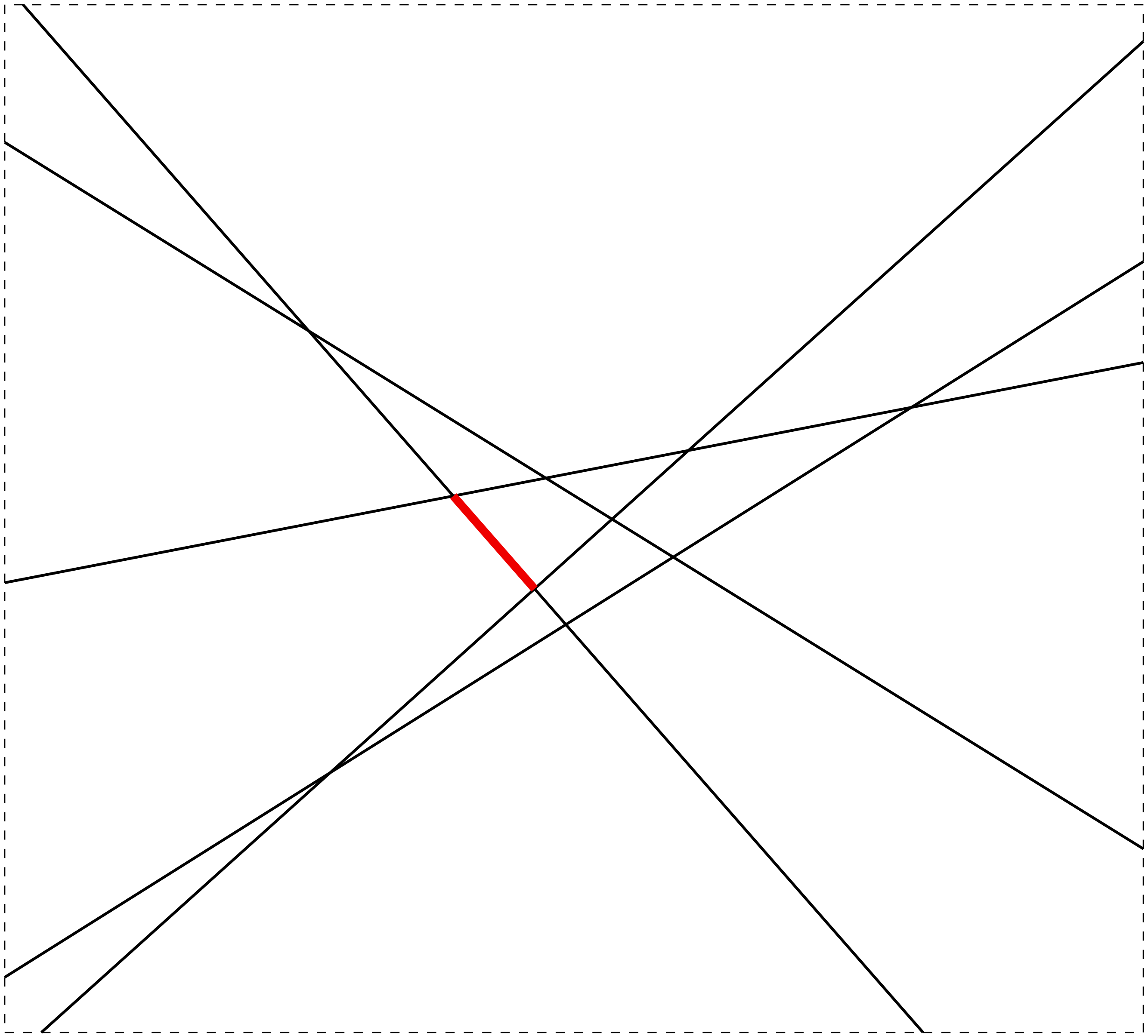
e



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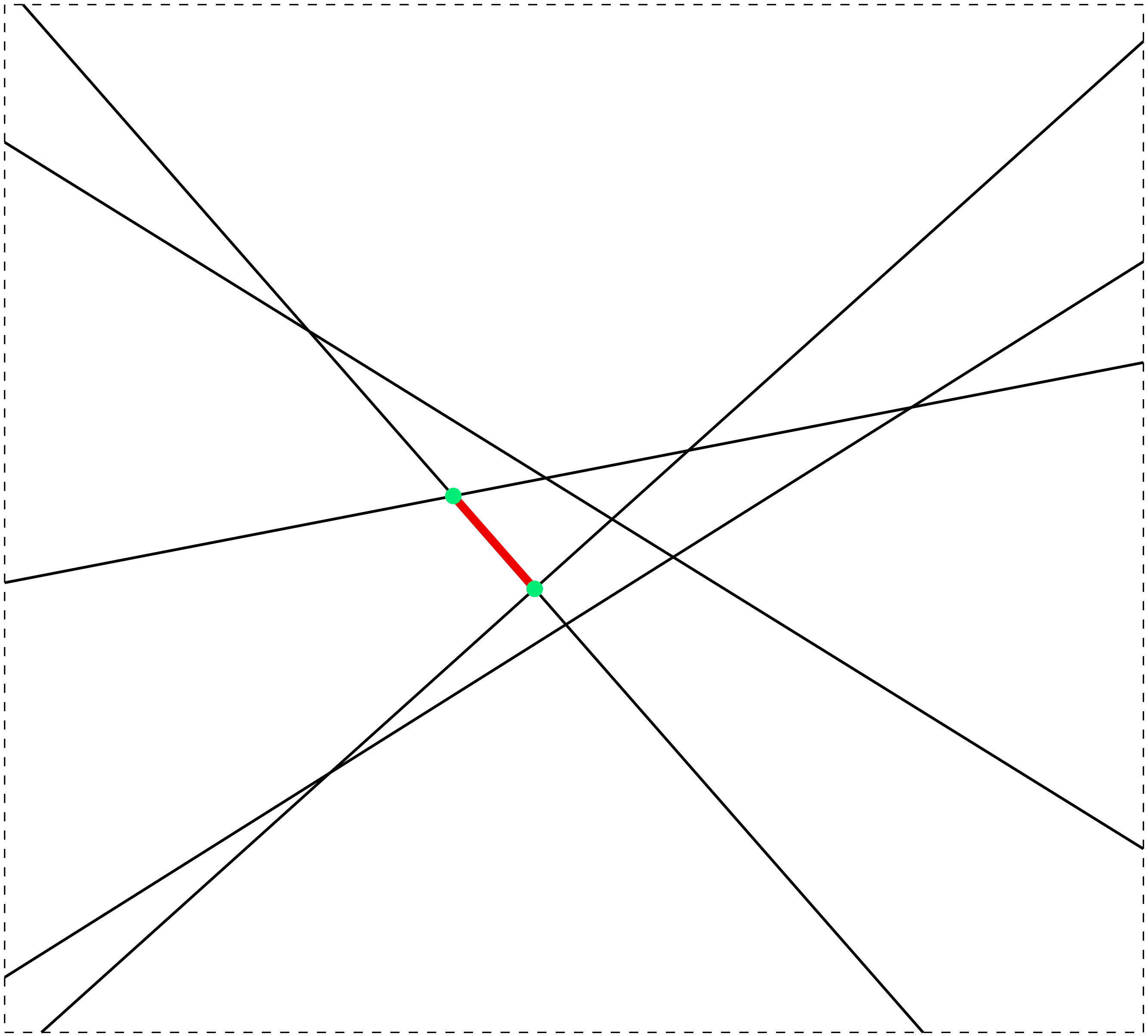
$e \rightarrow v_B \, v_E$



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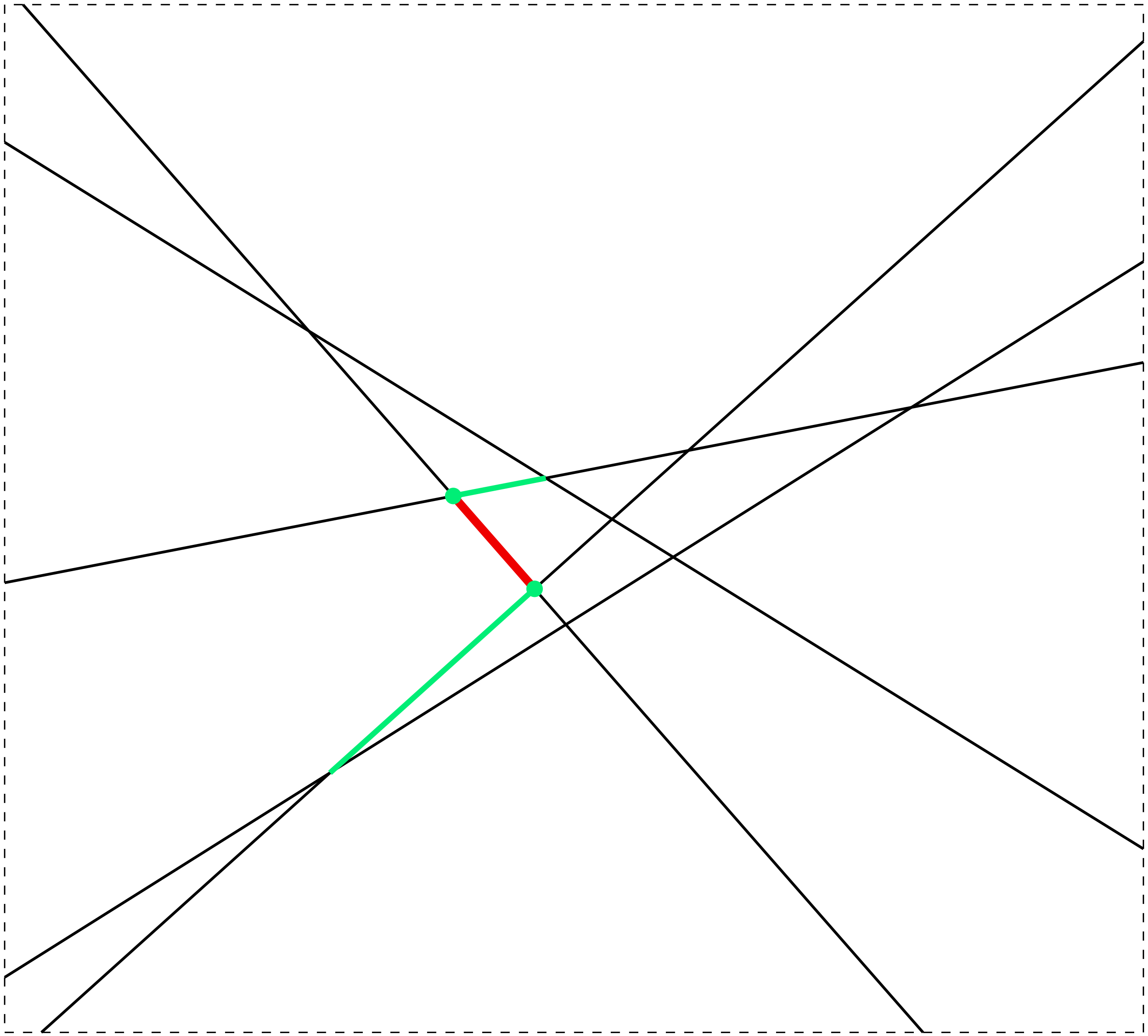
$e \to v_B\, v_E\;\; e_P\, e_N$



ARRANGEMENTS

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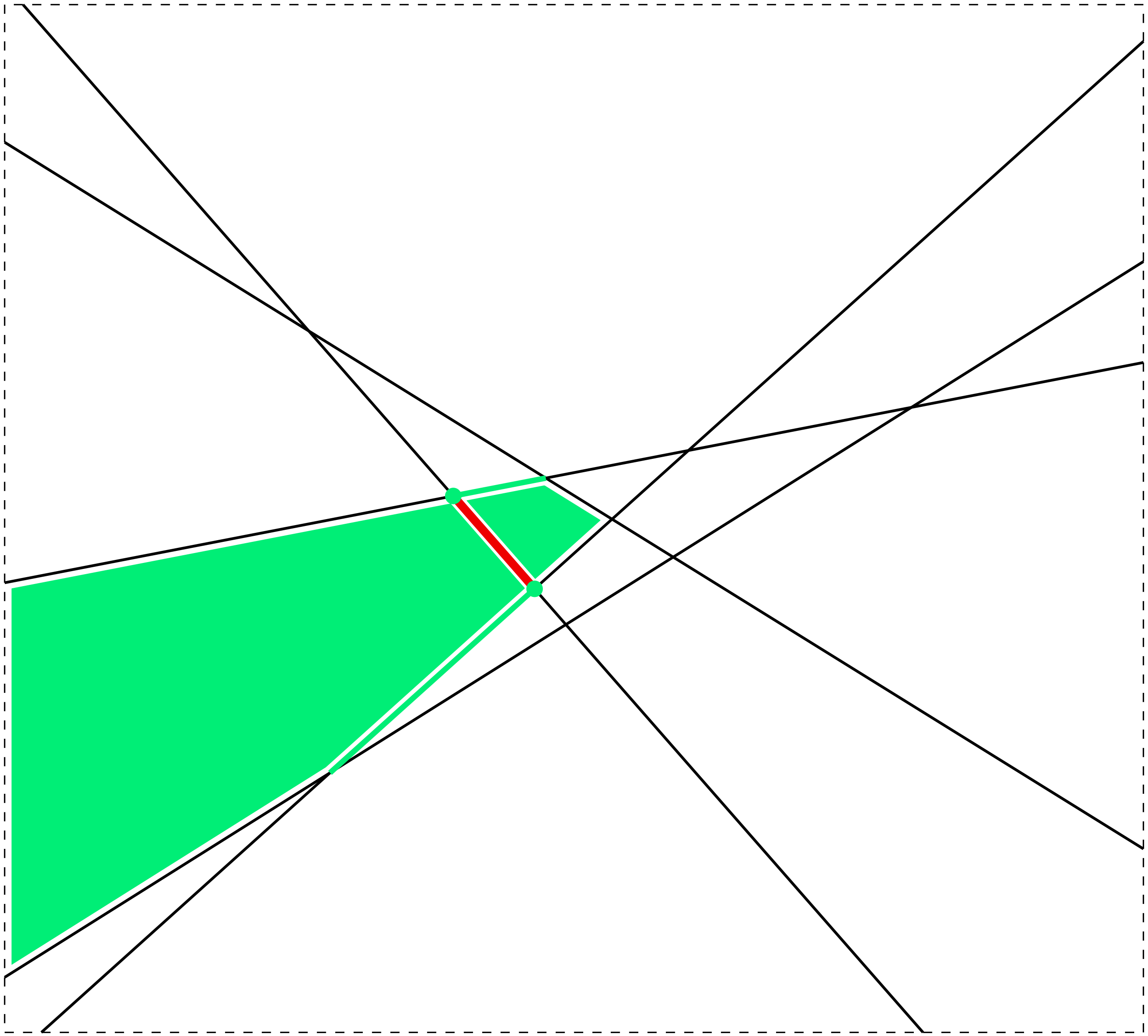
$e \rightarrow v_B\, v_E\;\; e_P\; e_N\;\; f_L\;\; f_R$



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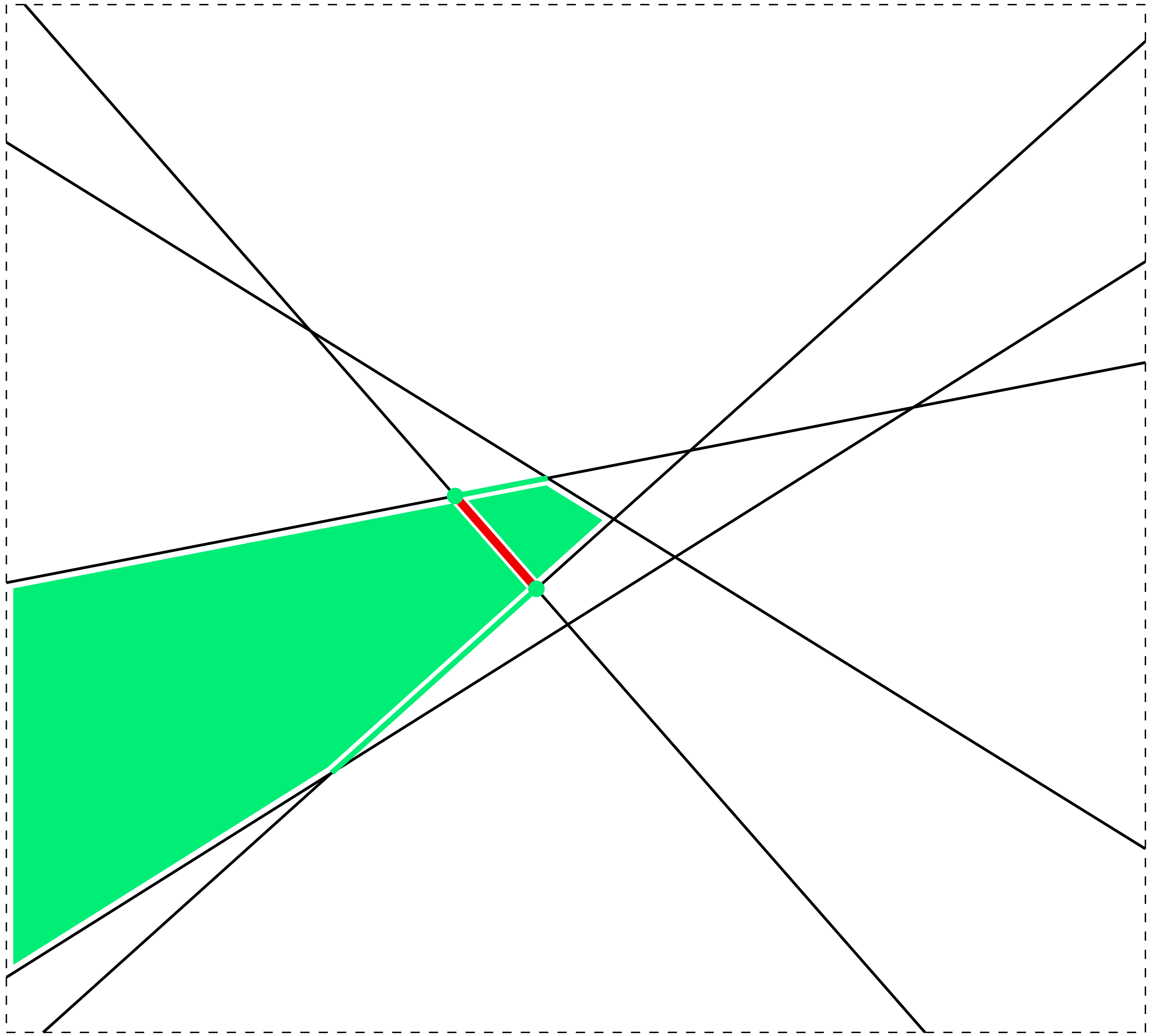


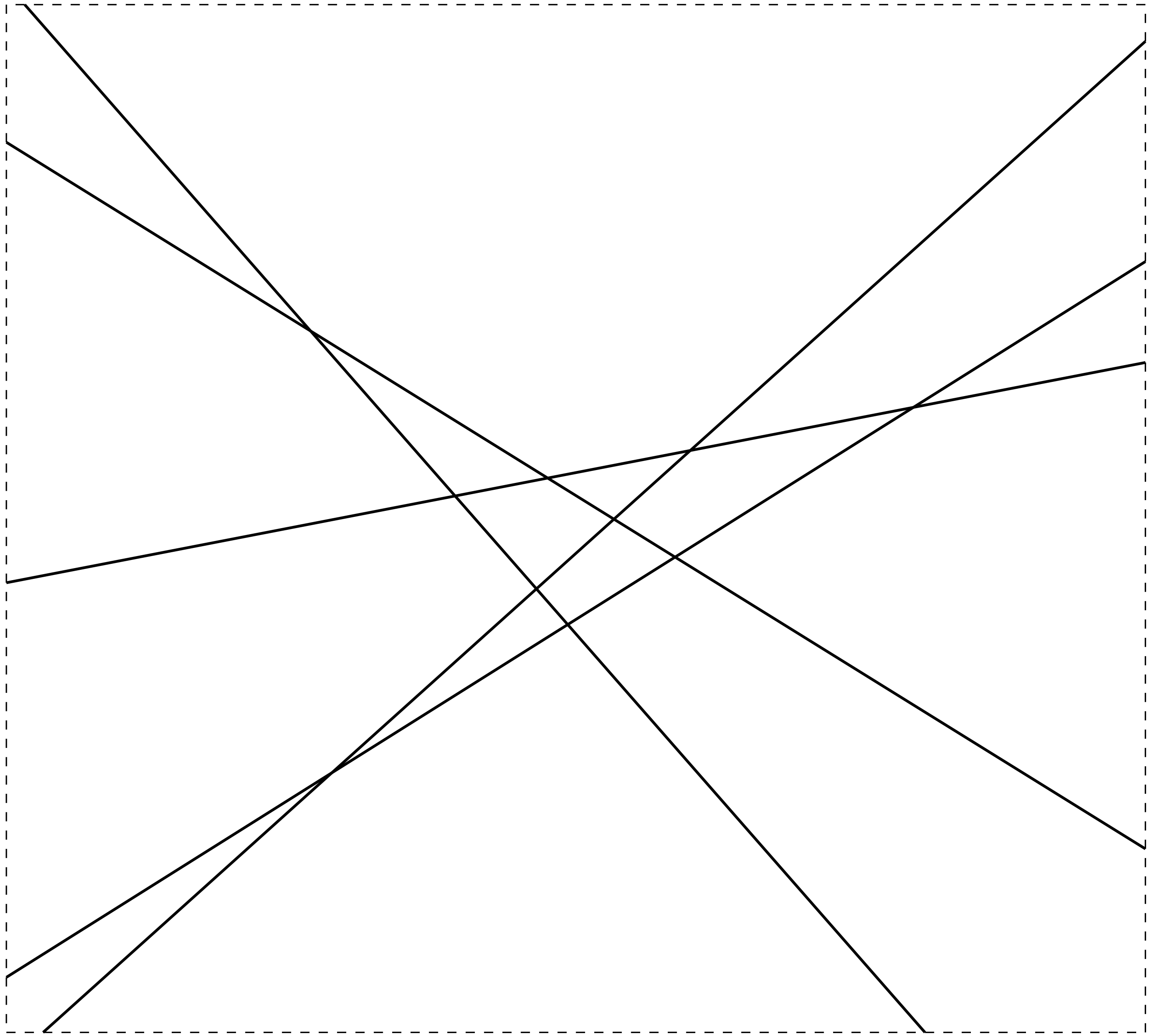
$e \rightarrow v_B\, v_E\;\; e_P\; e_N\;\; f_L\;\; f_R$

$v_i \rightarrow x_i, y_i, e$

$f_j \rightarrow e$

(Recall that halflines require a specific treatment)

COMPUTATION



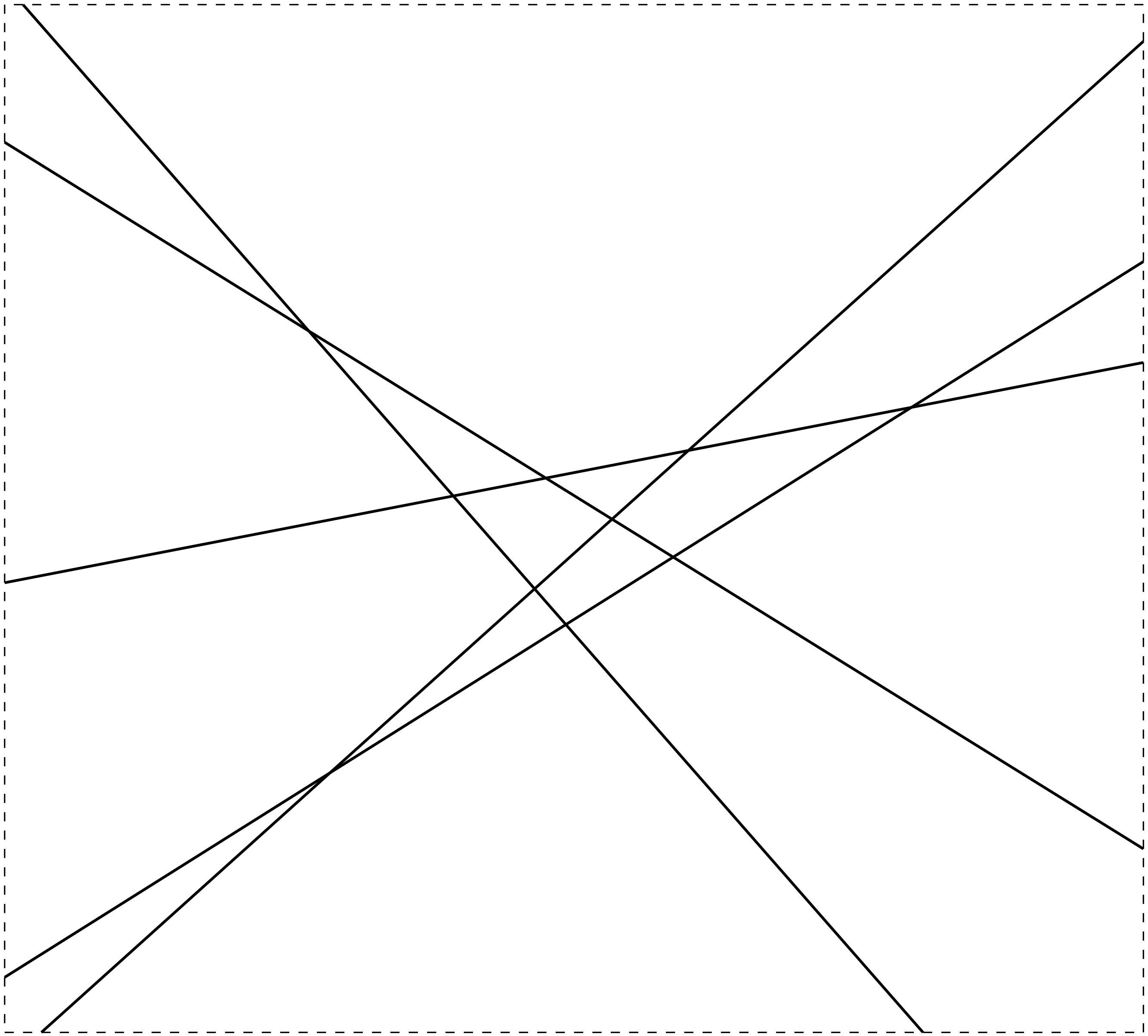
COMPUTATION

Input

A set of n lines $L = \{l_1, \ldots, l_n\}$

Output

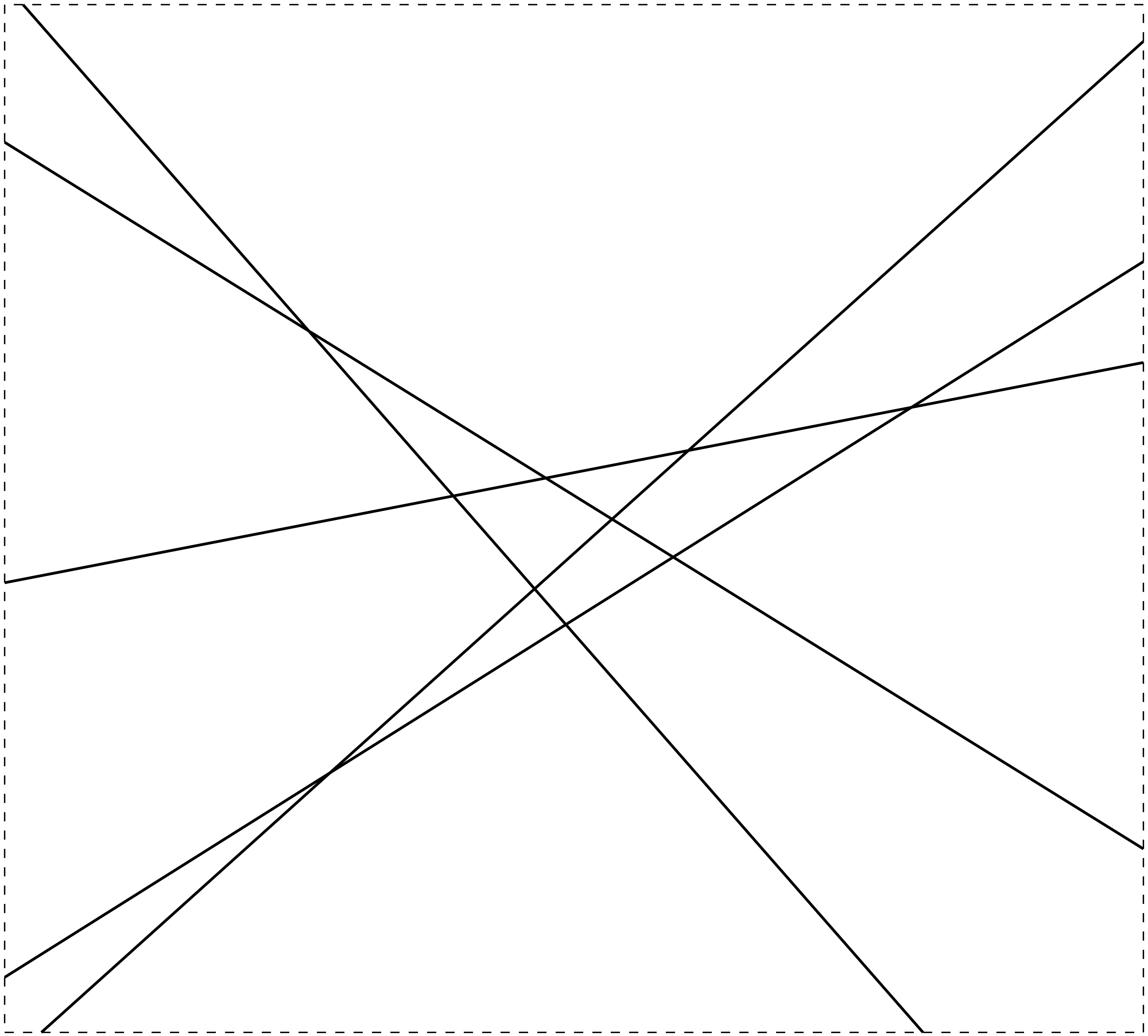
The arrangement $\mathcal{A}(L)$ i.e.,
the DCEL of the arrangement.



ARRANGEMENTS

COMPUTATION

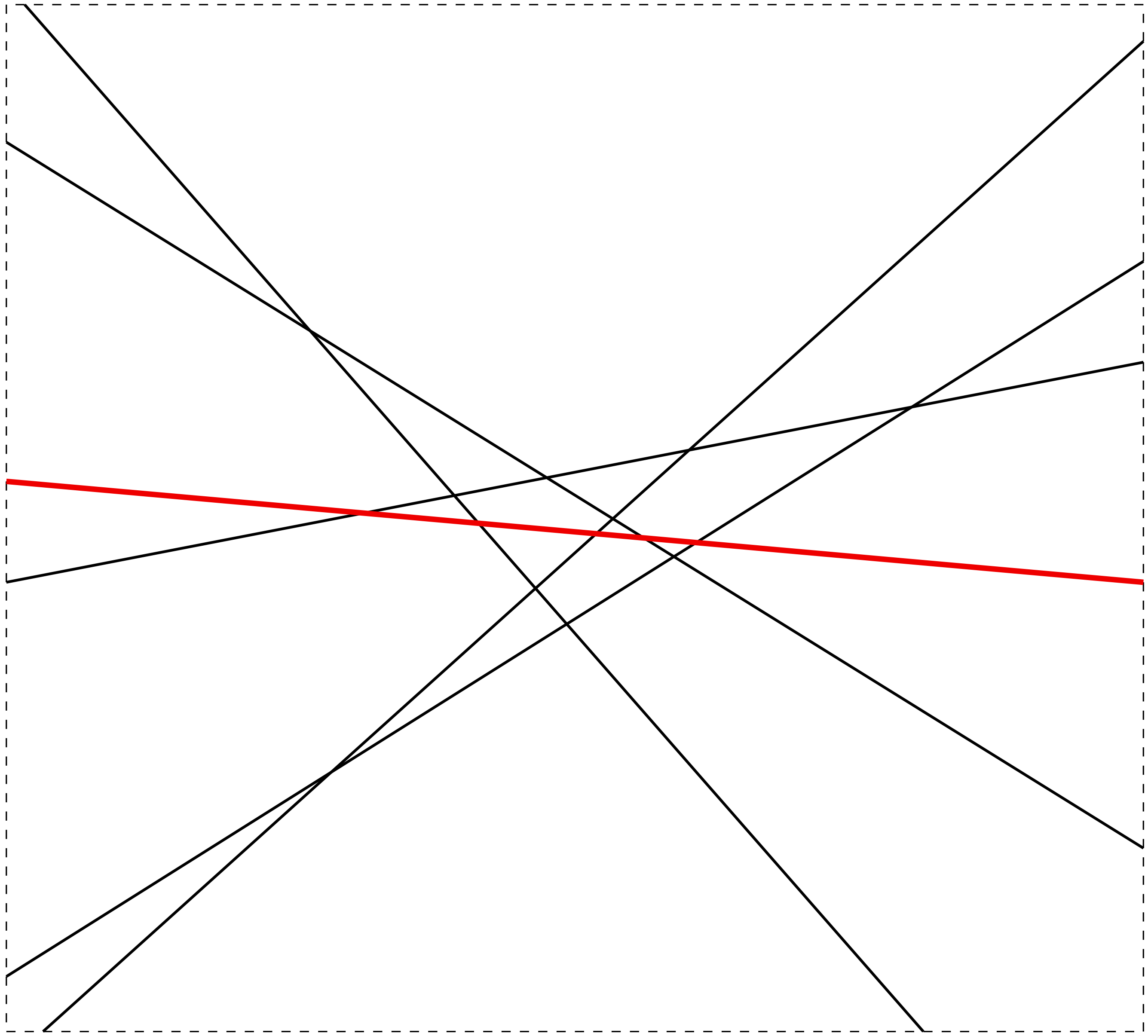
Incremental algorithm



COMPUTATION

Incremental algorithm

For each new line l:



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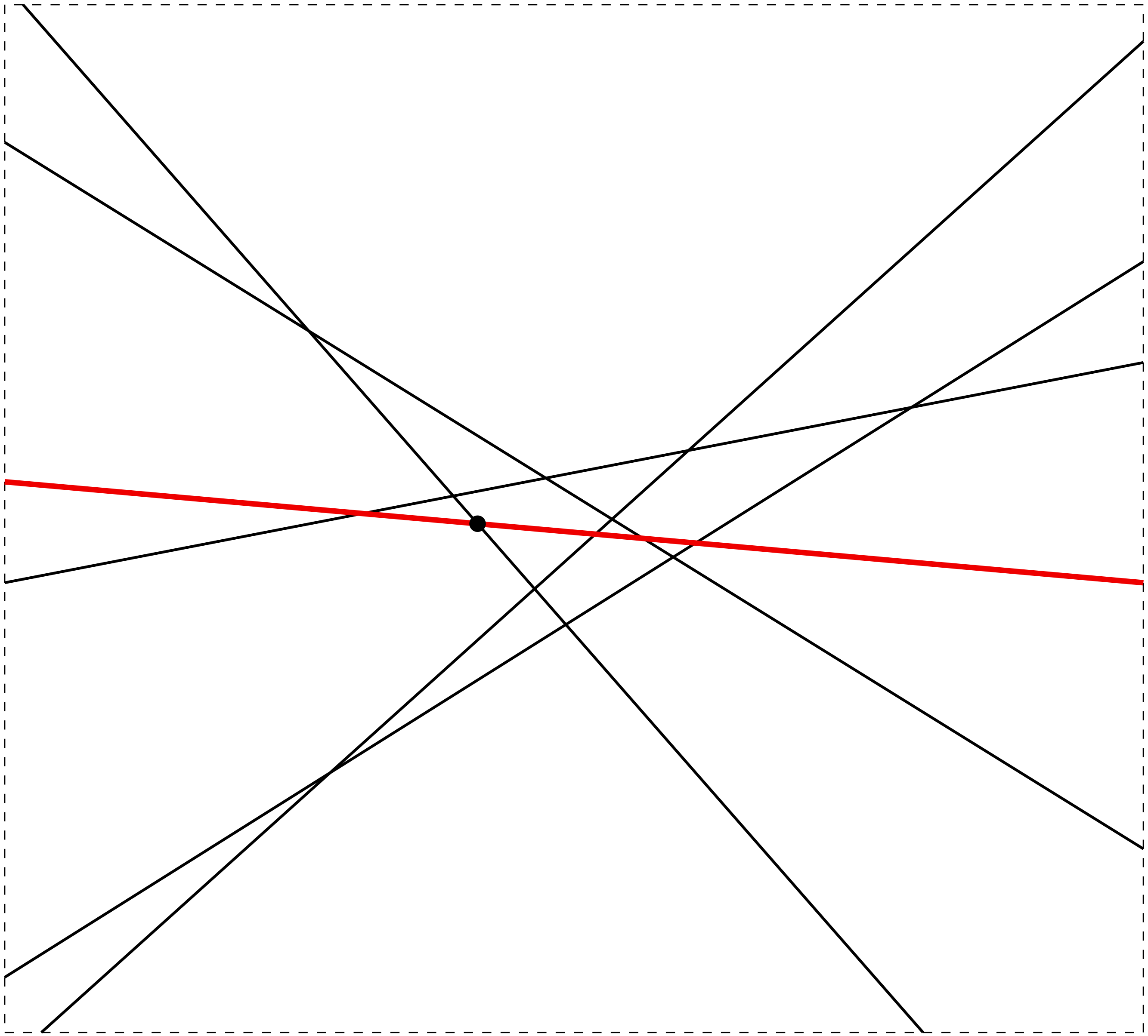
Incremental algorithm

For each new line l:

1. Intersect l with any line l_i pre existent in the arrangement.



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For each new line l:

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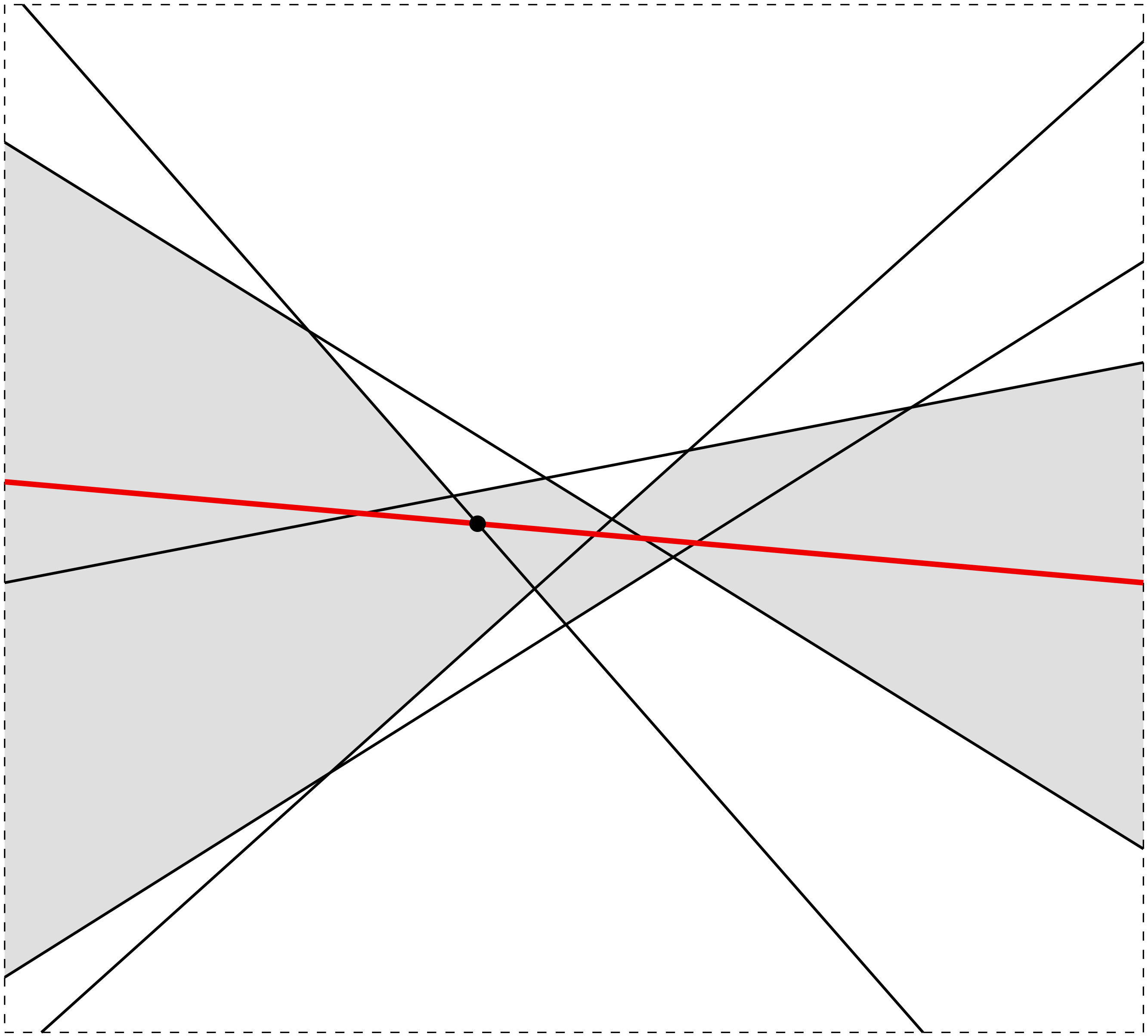
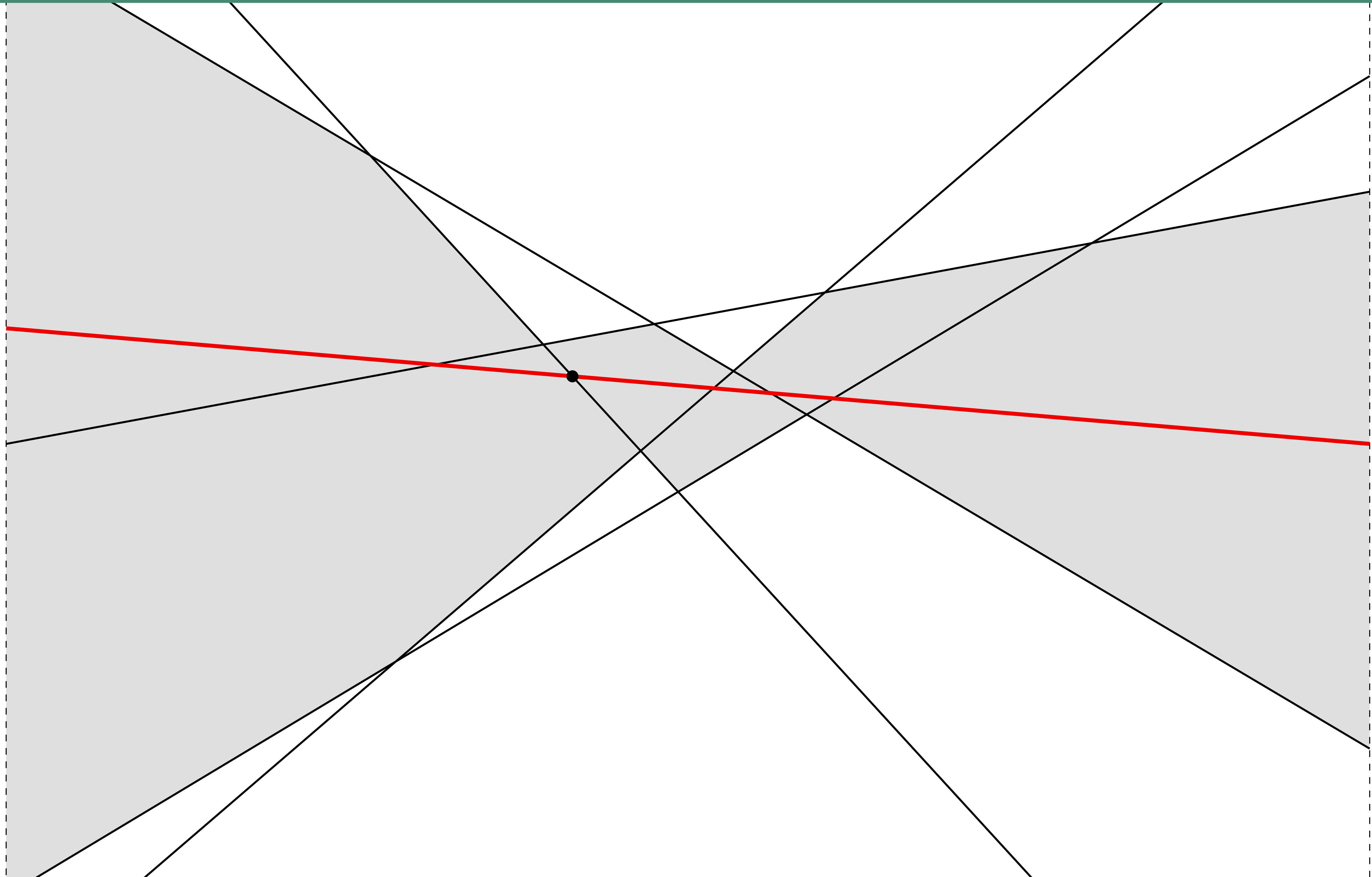
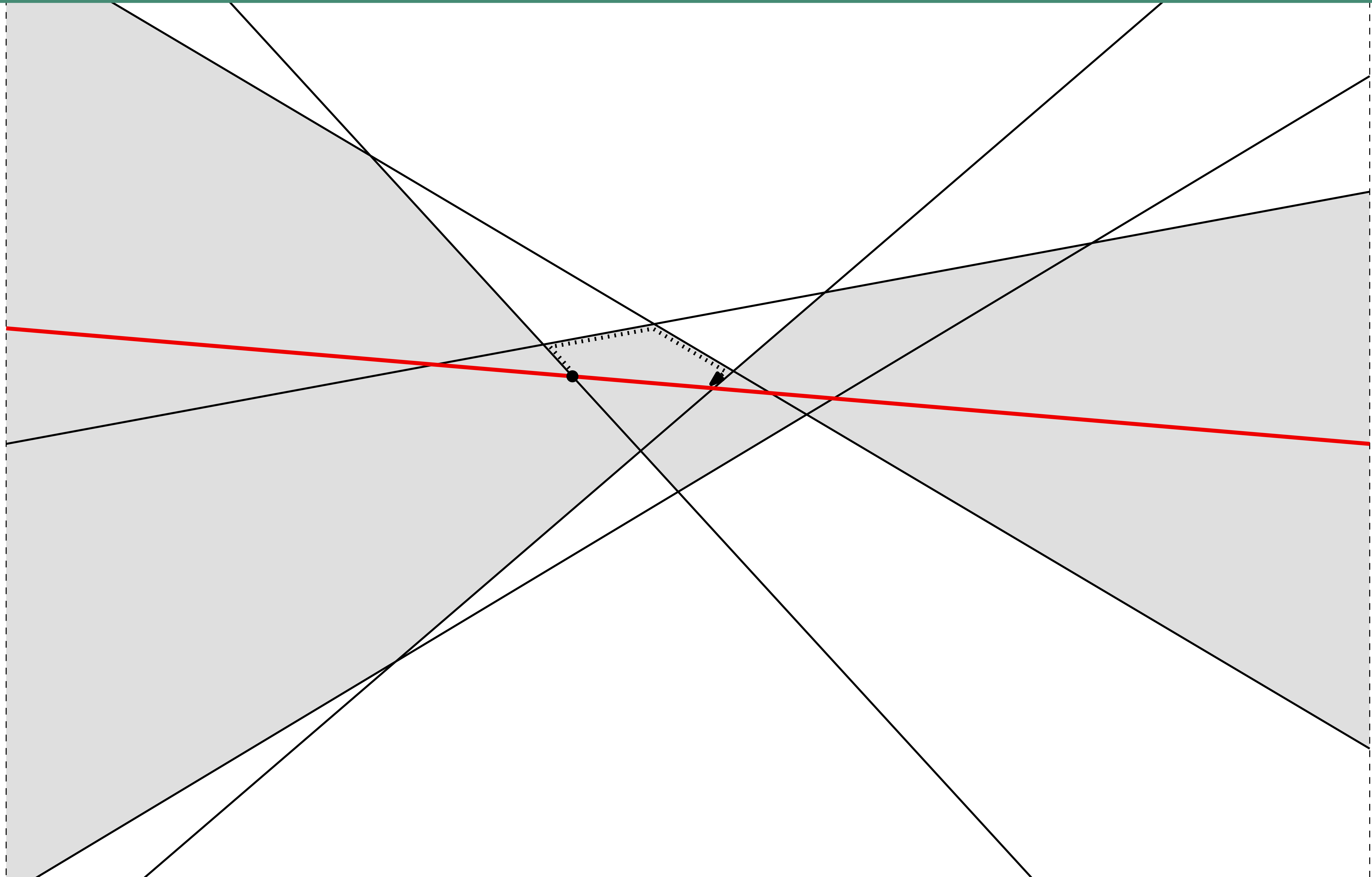
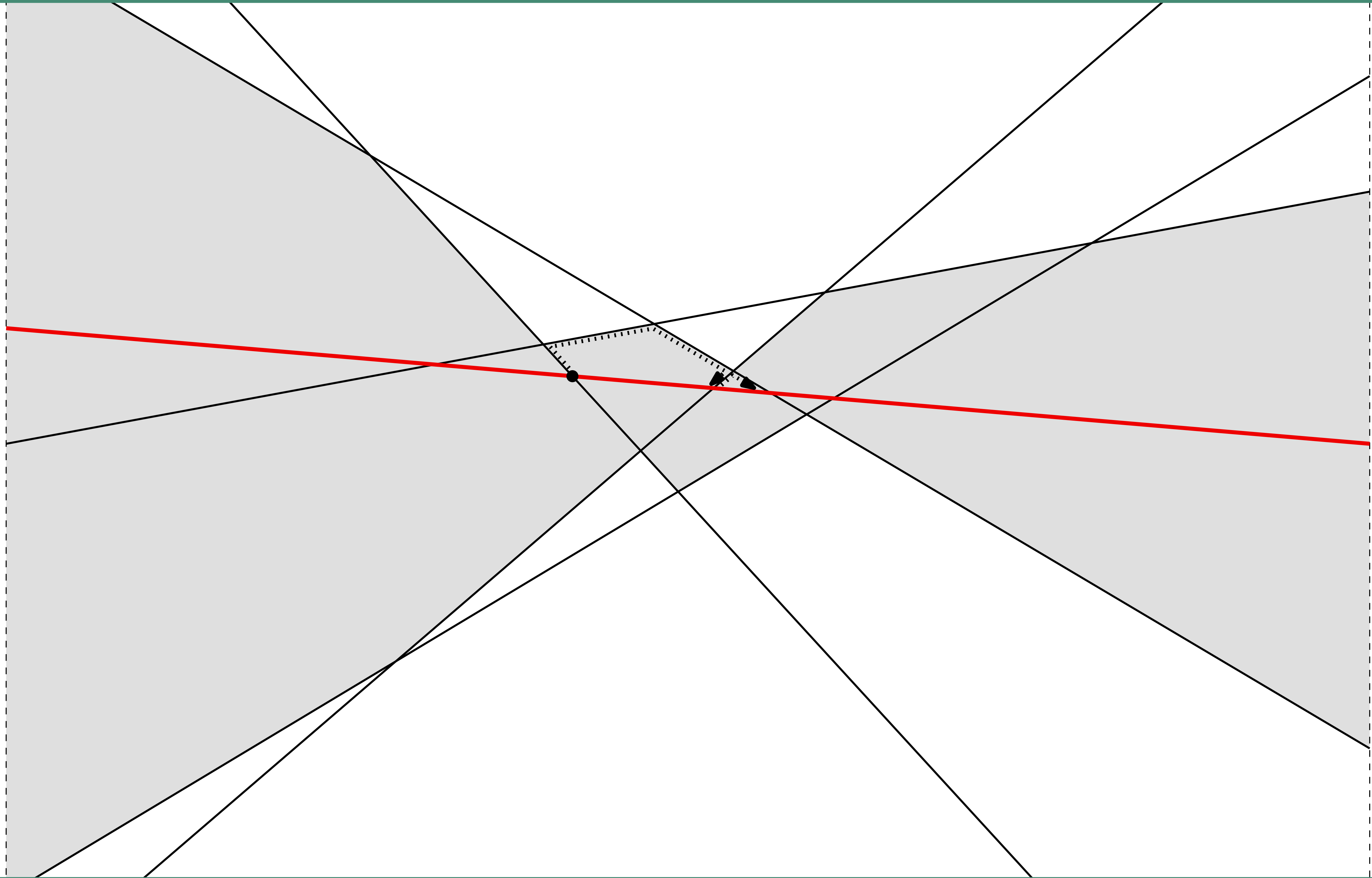
2. Traverse the zone of l...

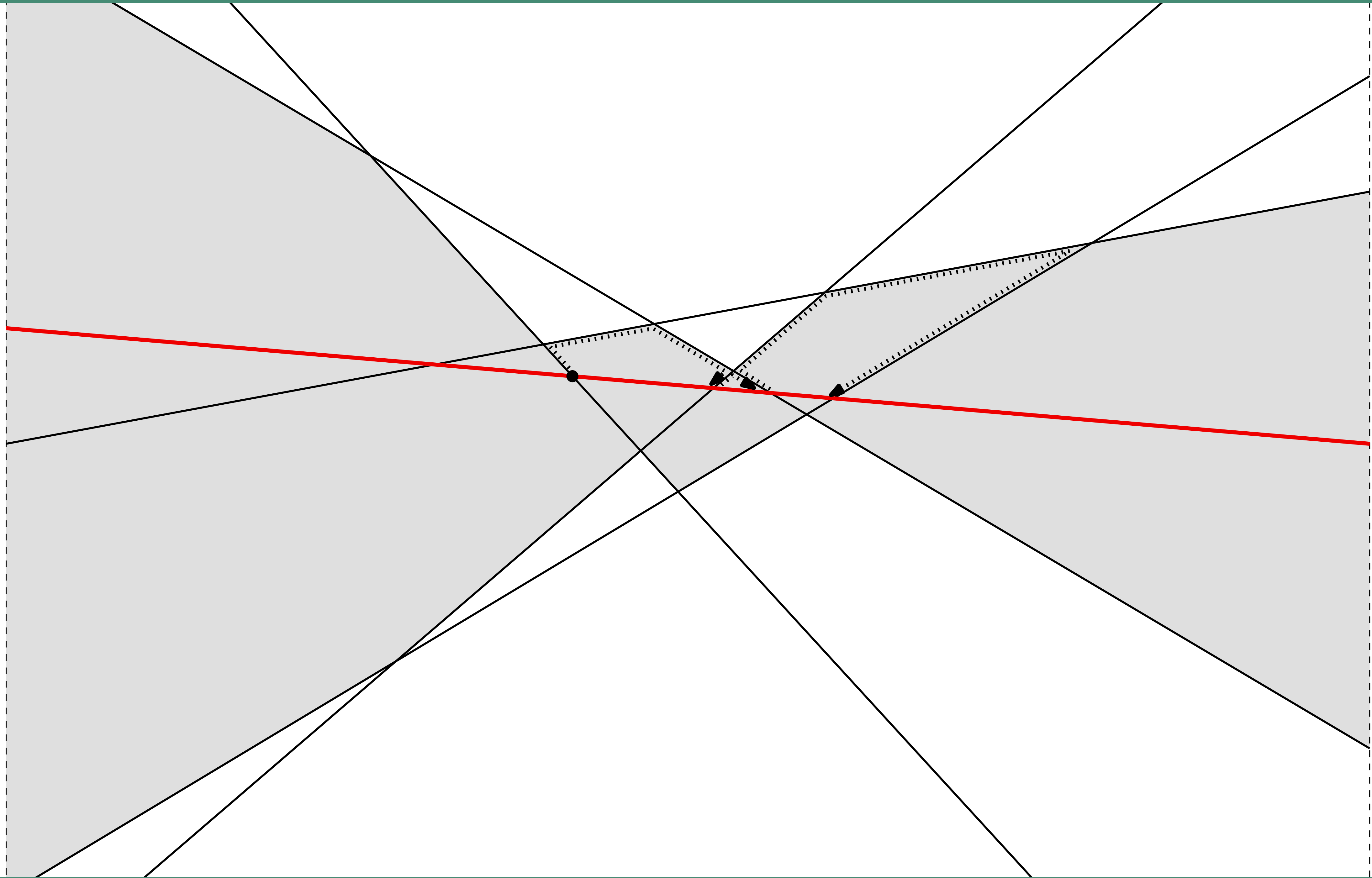


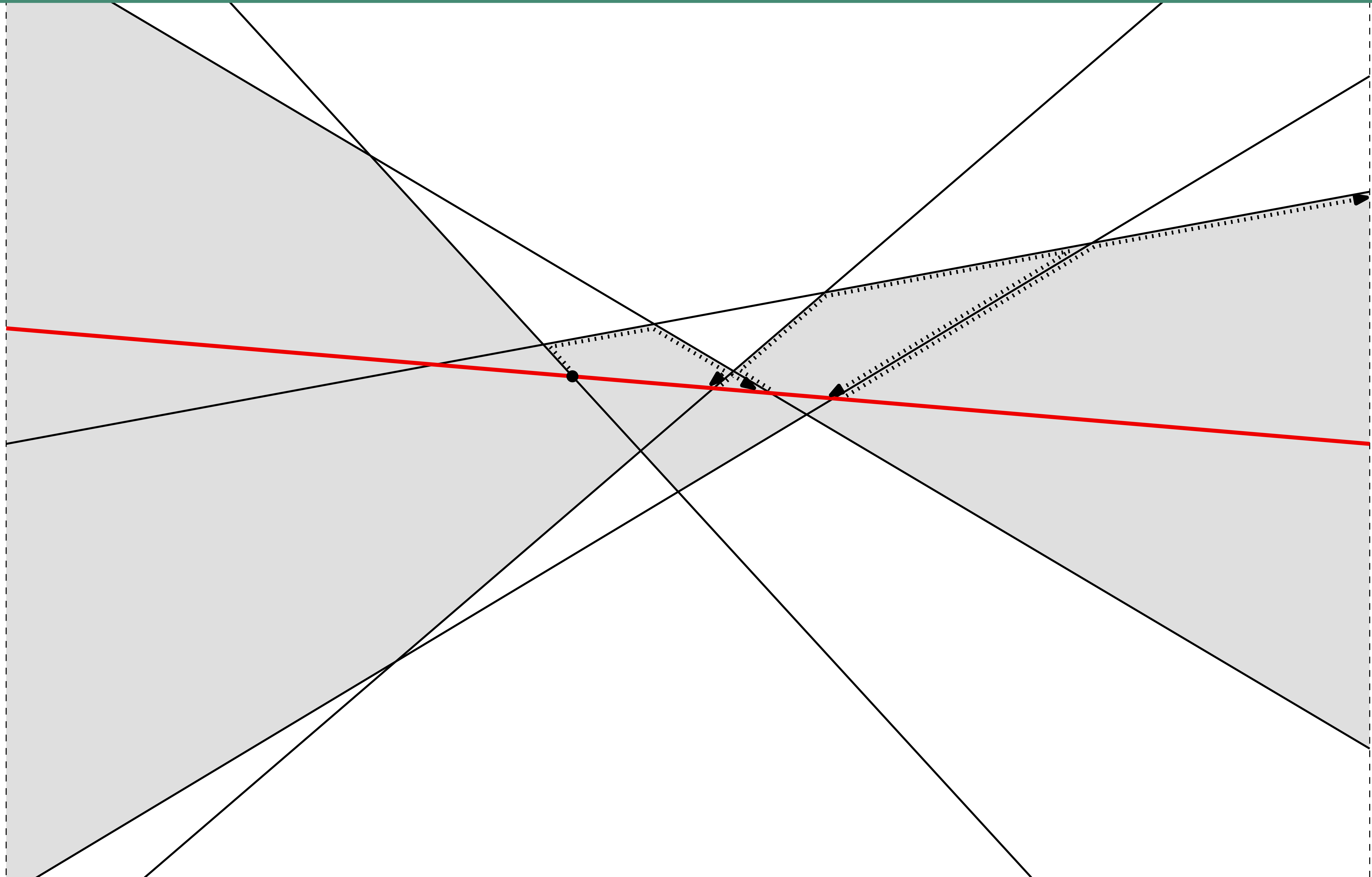




ARRANGEMENTS







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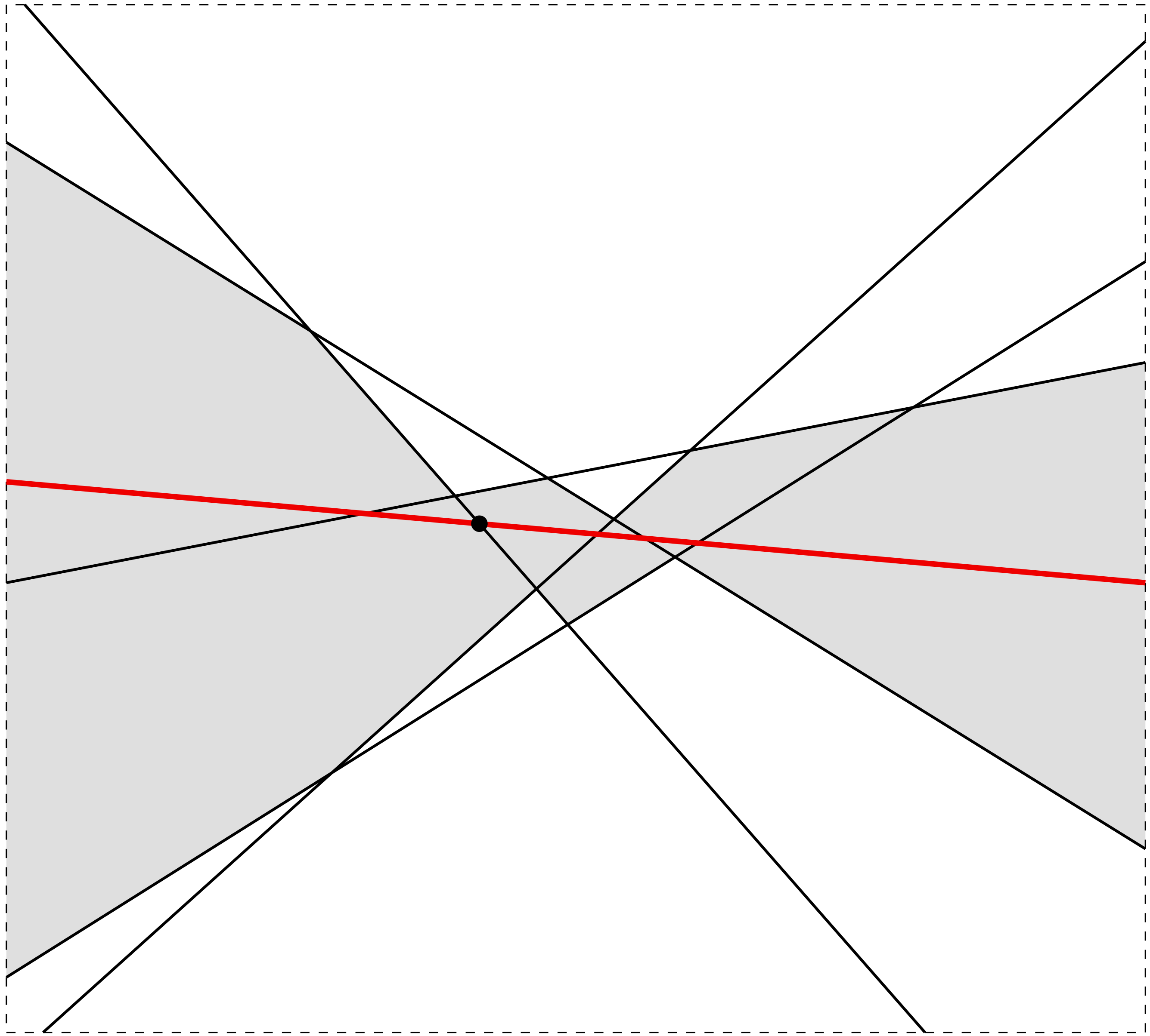

ARRANGEMENTS

COMPUTATION

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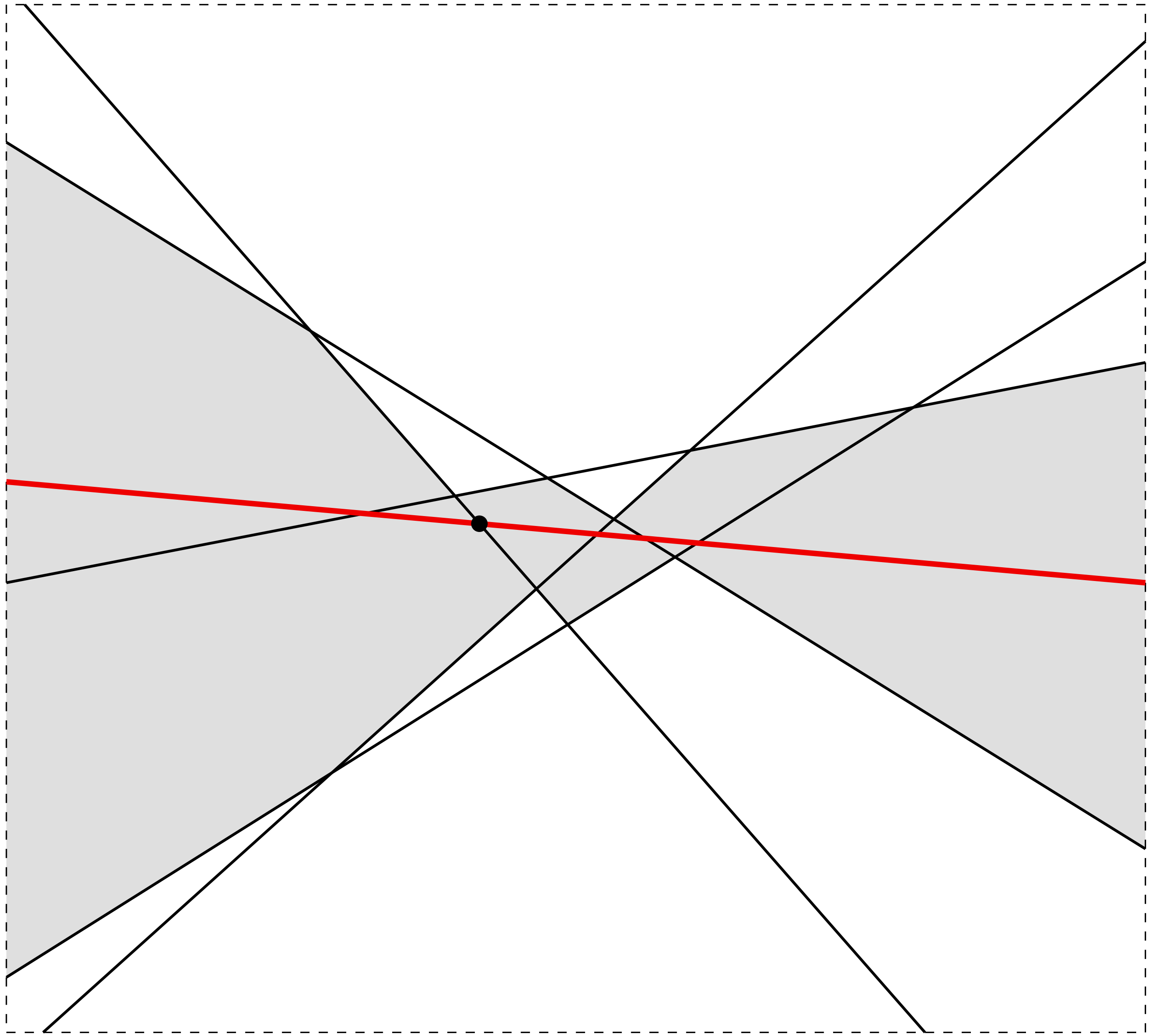
For each new line l:

1. Intersect l with any line l_i pre existent in the arrangement.
2. Traverse the zone of l...
   ...updating the DCEL.



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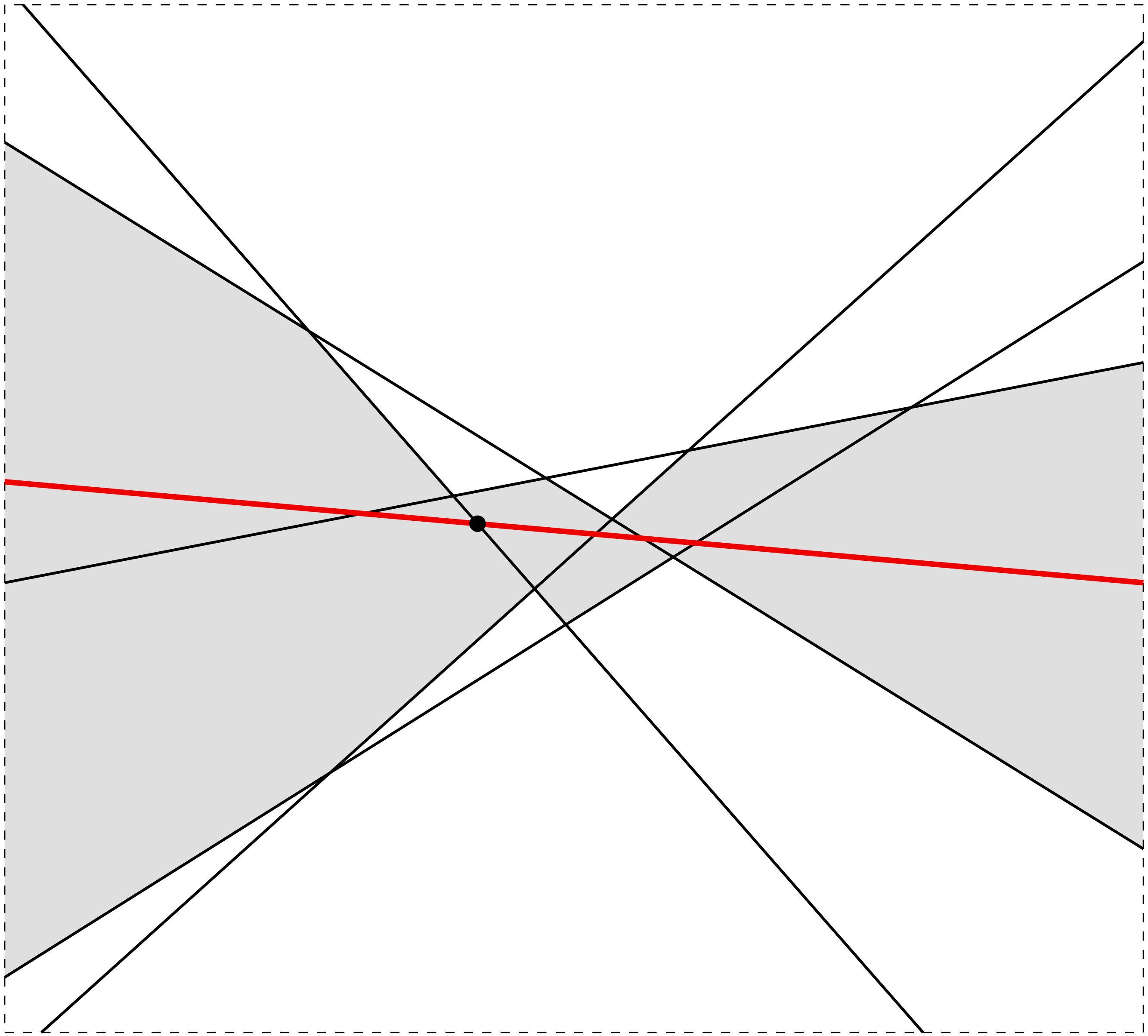
Updating the DCEL:



COMPUTATION

Incremental algorithm

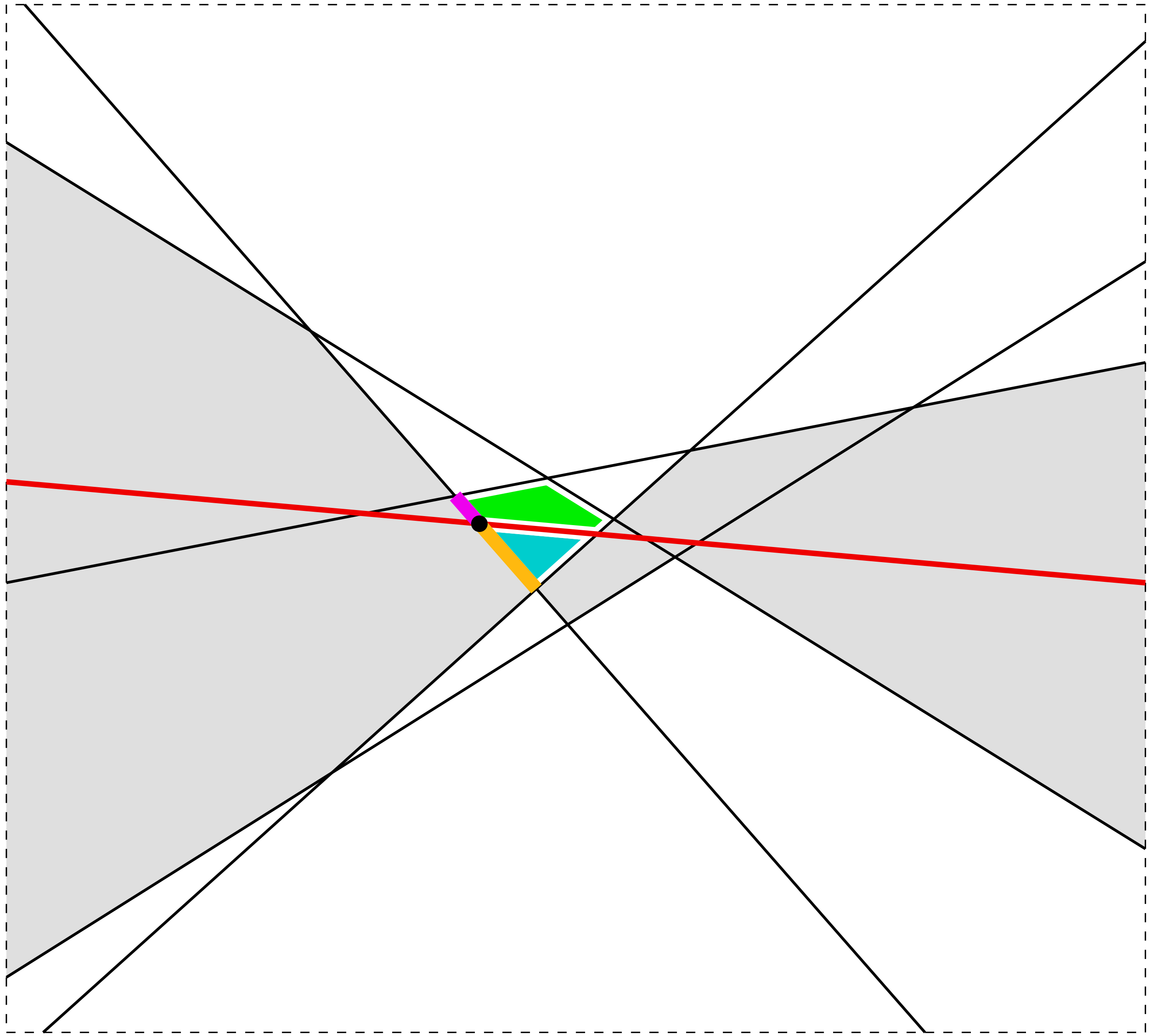
For each new line l:

1. Intersect l with any line l_i pre existent in the arrangement.
2. Traverse the zone of l... ...updating the DCEL.

Updating the DCEL:

At each intersection point v, the line l

- partitions an old edge e into two new edges, e_1 and e_2;
- a new edge b_1 ends and a new edge b_2 starts;
- partitions an old face f into two new faces, f_1 and f_2.



COMPUTATION

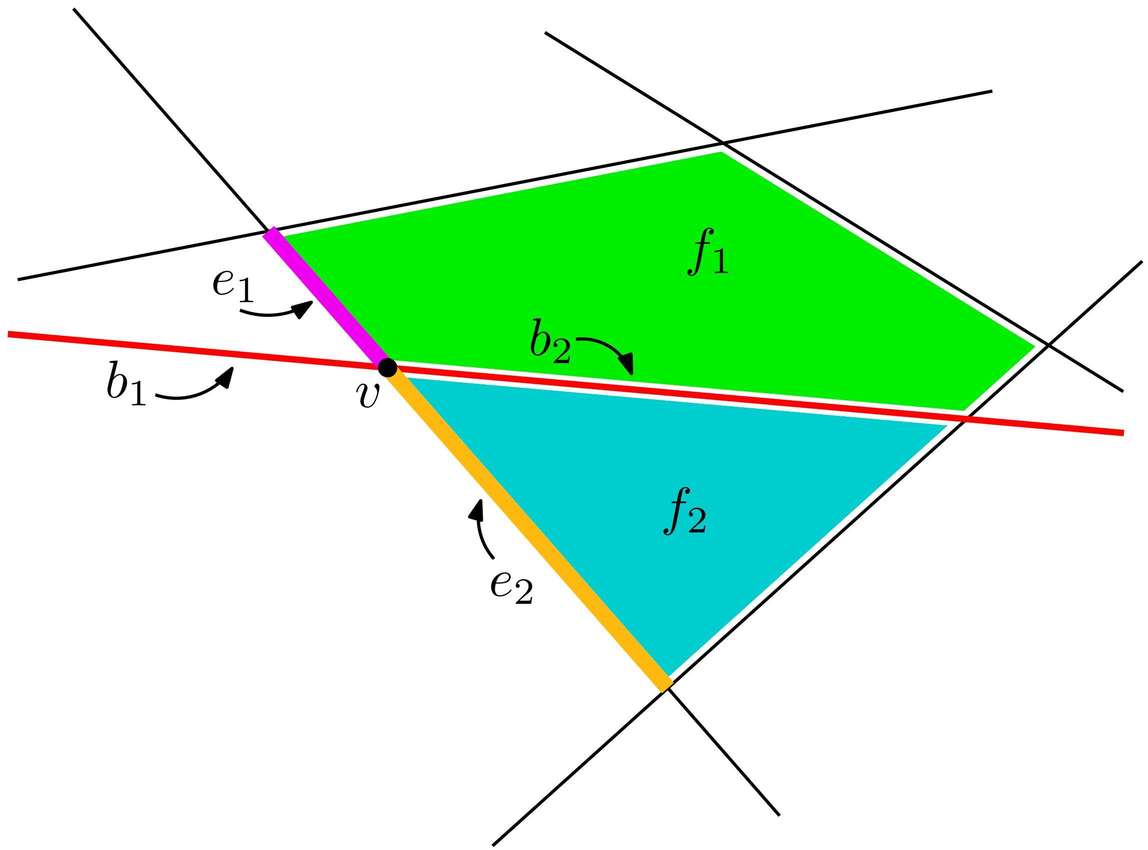
Incremental algorithm

At each intersection point v,

- Add v to the table.
- Delete f and add f_1 and f_2.
- Add v_E and e_N for b_1.
- Add b_2 with all its pointers, excluding v_E and e_N.
- Add e_1 and e_2 and delete e, right after modifying the pointers of $v_B(e)$, $v_E(e)$, $f_L(e)$ i $f_R(e)$.

Between two intersection points,

- Modify the pointer of each visited edge to the newly created face.



COMPUTATION

Incremental algorithm

Running time:

For each line (there are n lines):

1. Compute the intersection point in $O(1)$ time and locate, in $O(n)$ time, which edge of the current arrangement contains it.

2. Traverse the zone, which has complexity $O(n)$, updating the DCEL in $O(1)$ time at

   - each intersection point and
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Total running time: $O(n^2)$

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The description of the algorithm we have shown assumes that the arrangement is simple.

An easy modification allows adapting it to the general case.

EXTENSIONS

The arrangement $\mathcal{A}(L)$ of a finite set of *lines* in the *plane* is the decomposition of the plane into faces, edges and vertices produced by L.

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curves

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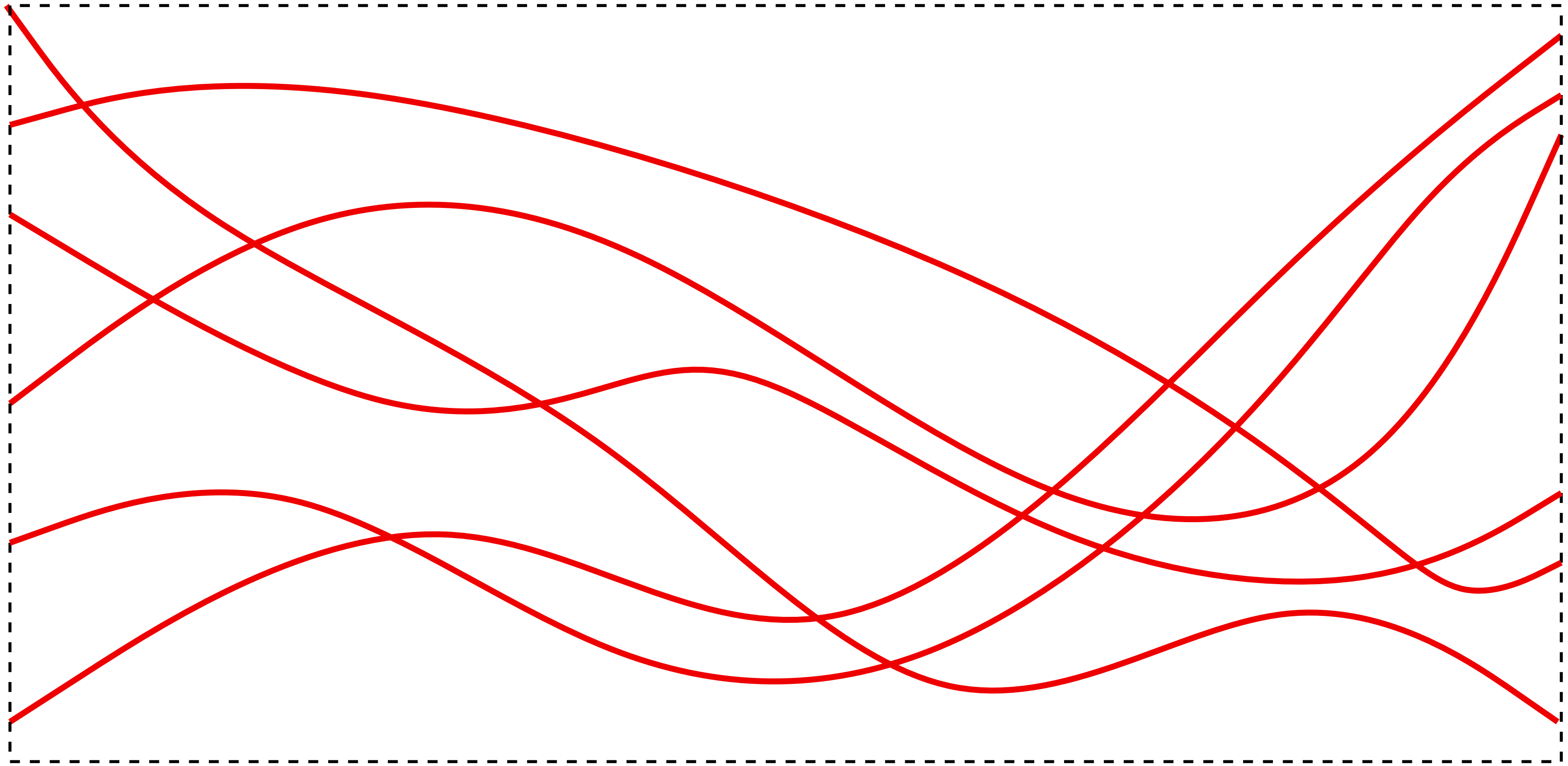

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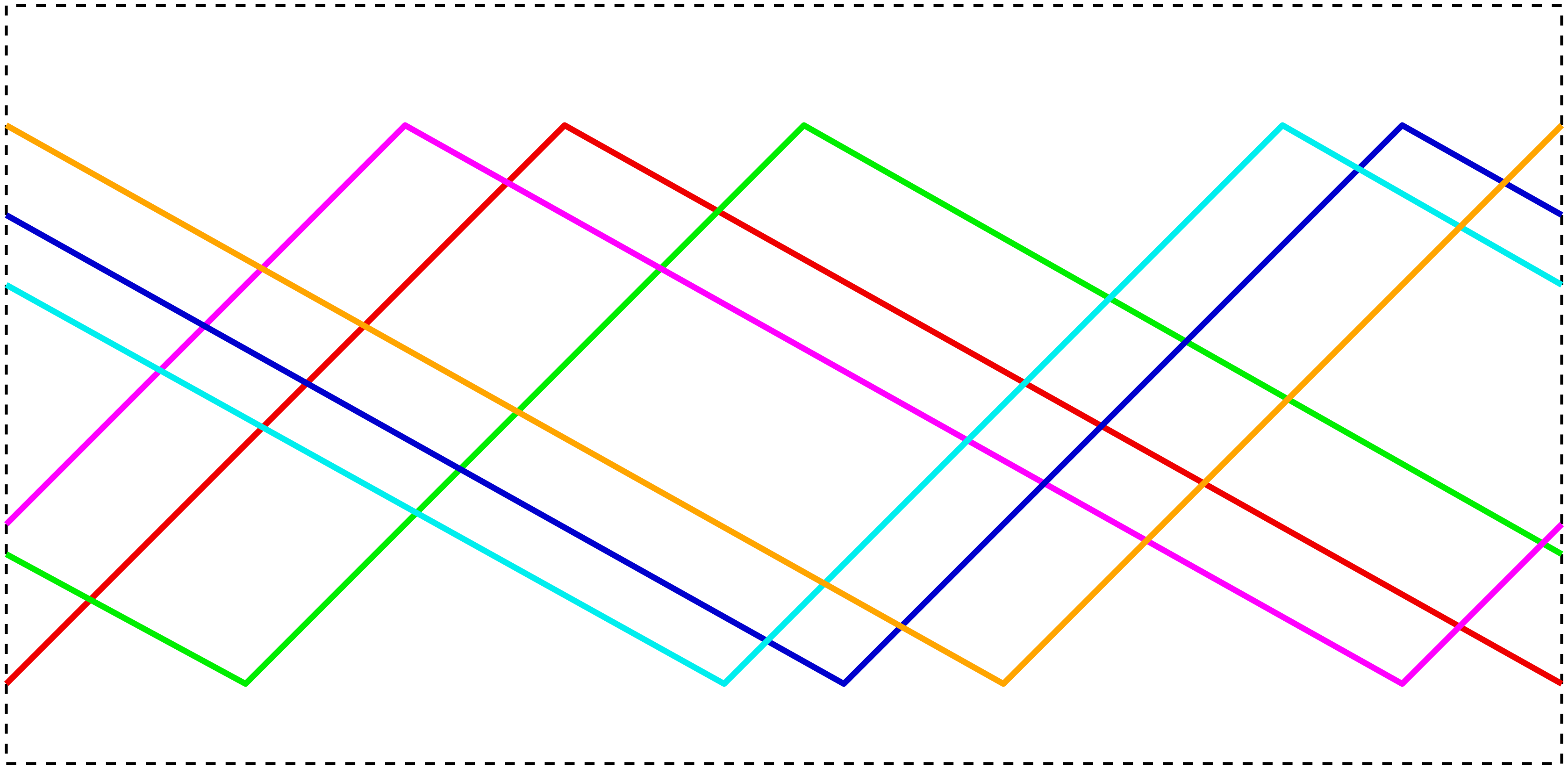

ARRANGEMENTS

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space

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space

hyperplanes

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space

hyperplanes

faces of dimension $1, 2, \ldots, d-1$

ARRANGEMENTS

EXAMPLE: Arrangements of line segments

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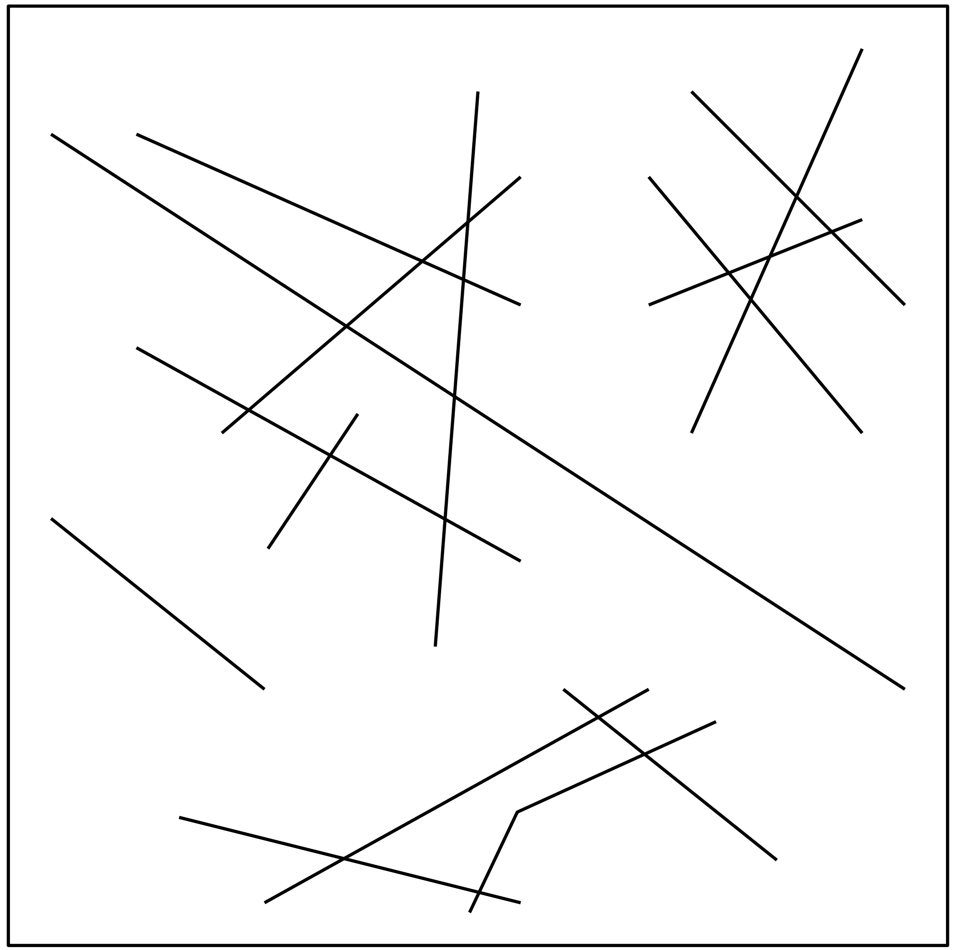

ARRANGEMENTS

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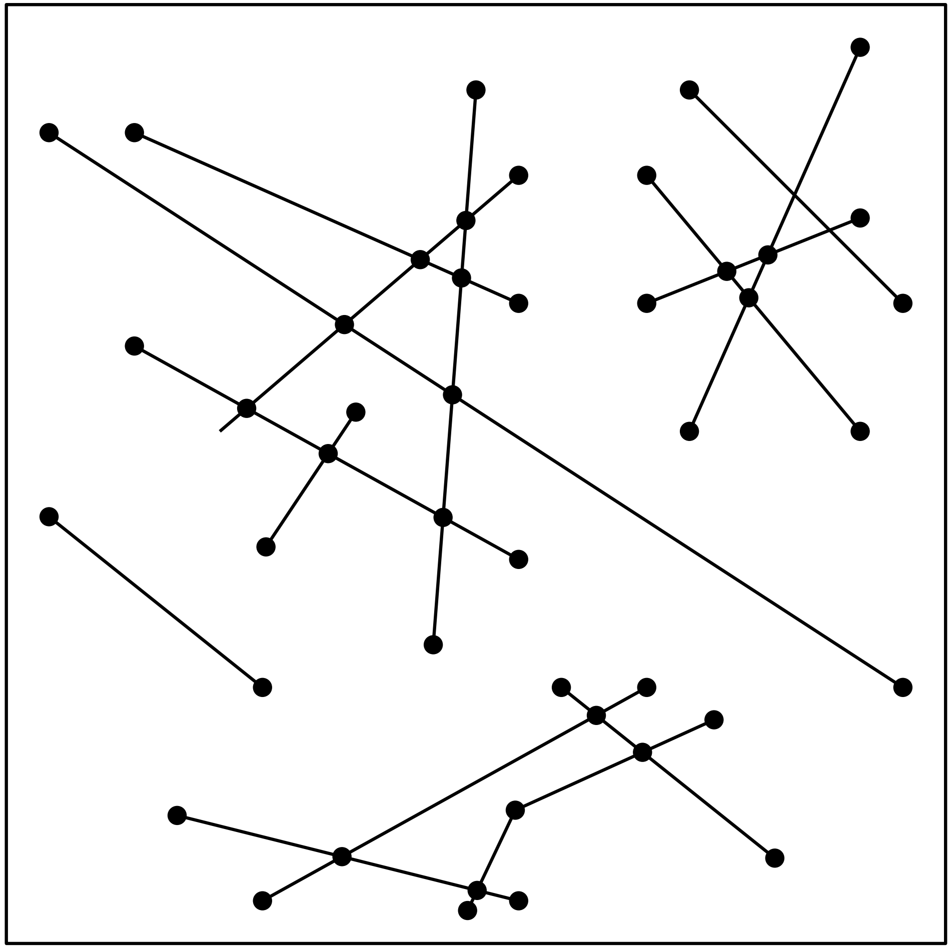

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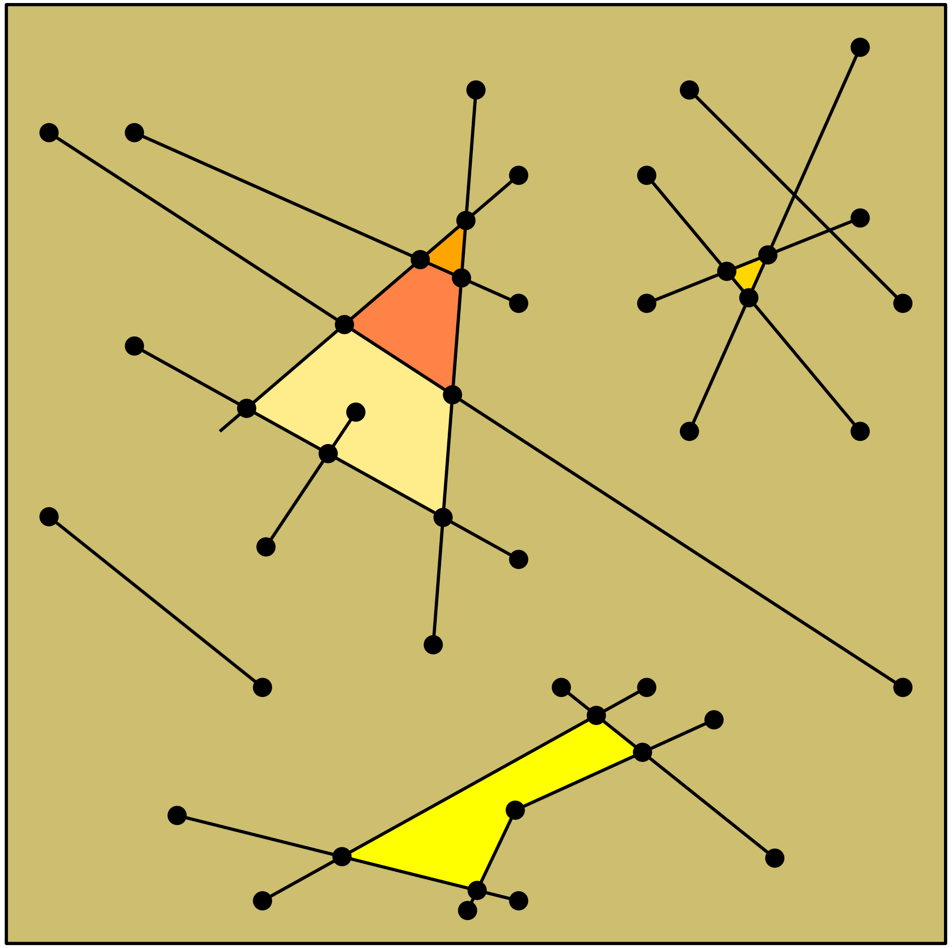
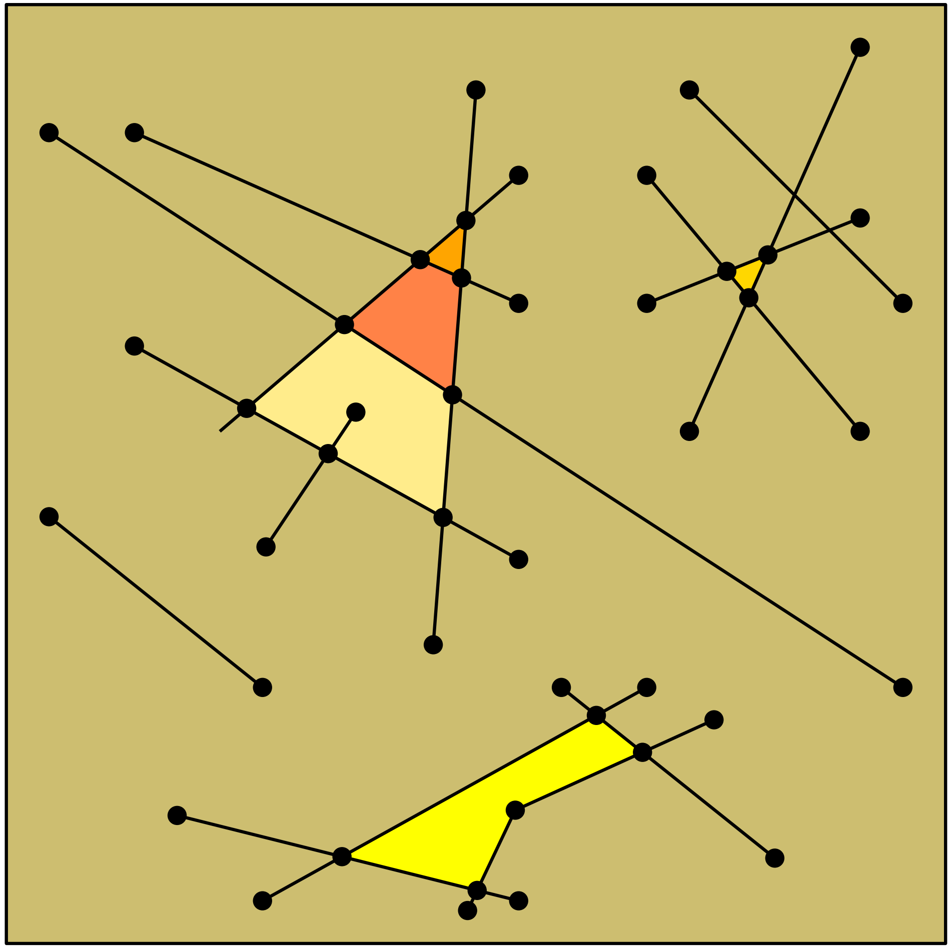


ARRANGEMENTS

EXAMPLE: Arrangements of line segments

In a *simple* arrangement of n line segments

- Faces may be non-convex
- We need to distinguish both "sides" of a segment
- One parameter (n) not enough to describe it. E.g., number of edges can go from n to n^2
  - Take $k = \#$ intersections and $c = \#$ connected components
  - ... then $v = 2n + k$ and $e = n + 2k$
  - ... and $f = k - n + c + 1$.



Lower envelope of line segments

Portion of the arrangement visible from $y = -\infty$

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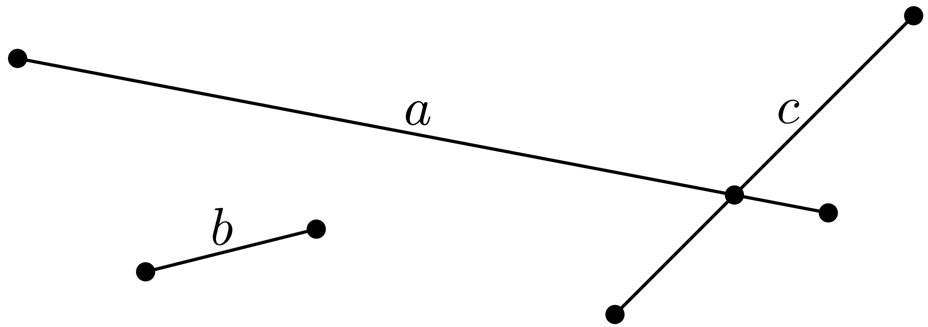

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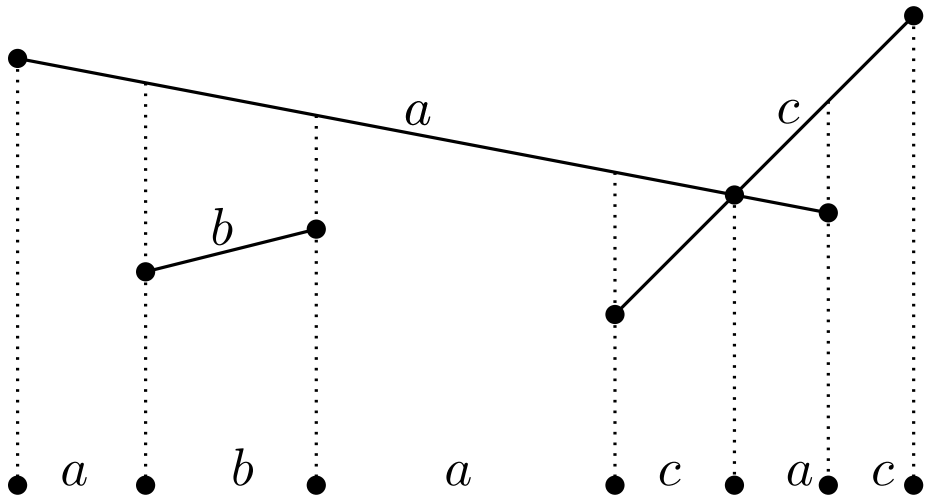
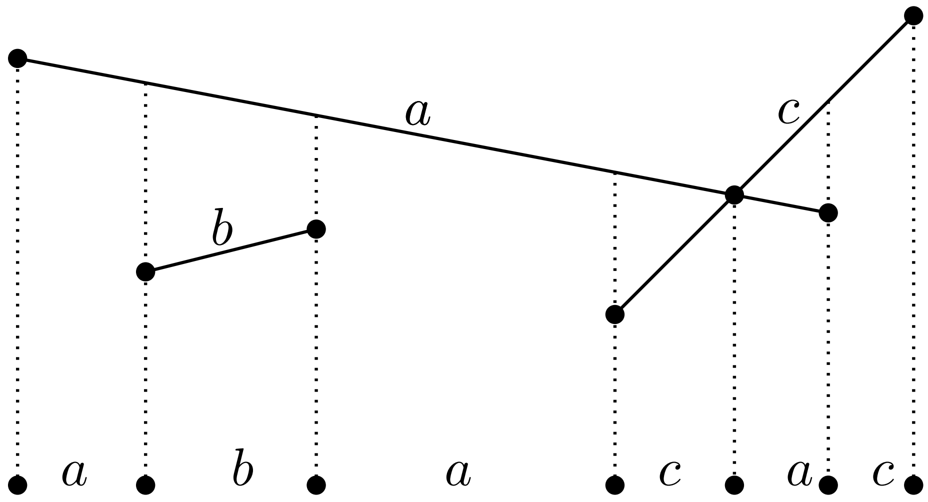


Lower envelope of line segments

Portion of the arrangement visible from $y = -\infty$

What is the complexity of the lower envelope of n segments?

- "At least" $O(n)$
- At most $O(n^2)$ (why?)



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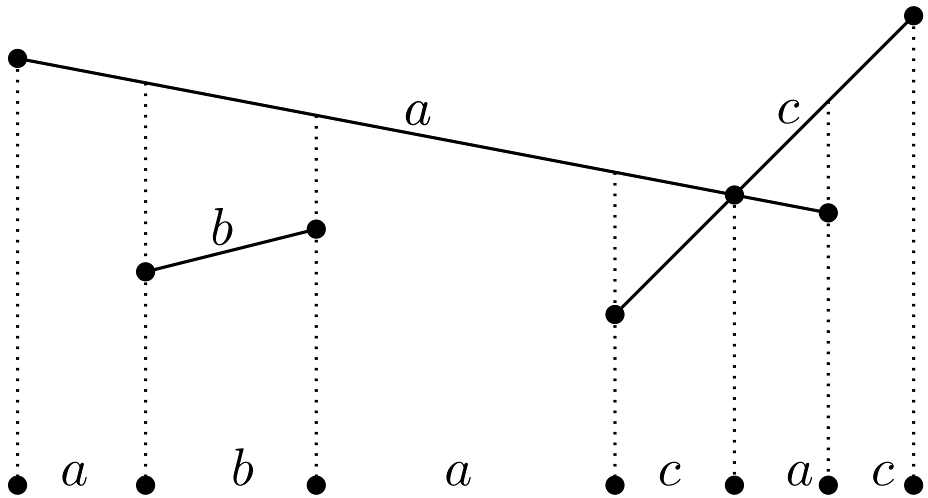

Theorem The complexity of the lower envelope of n line segments is $\Theta(n\alpha(n))$

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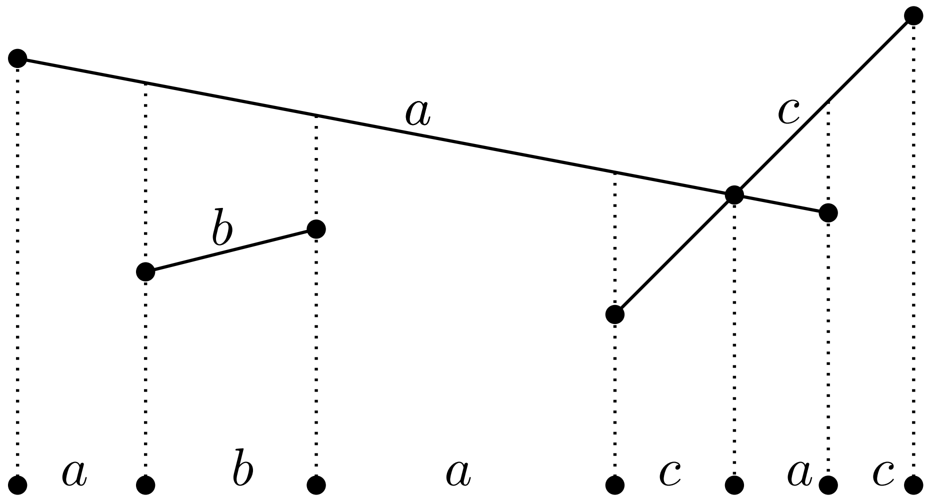


Theorem The complexity of the lower envelope of n line segments is $\Theta(n\alpha(n))$

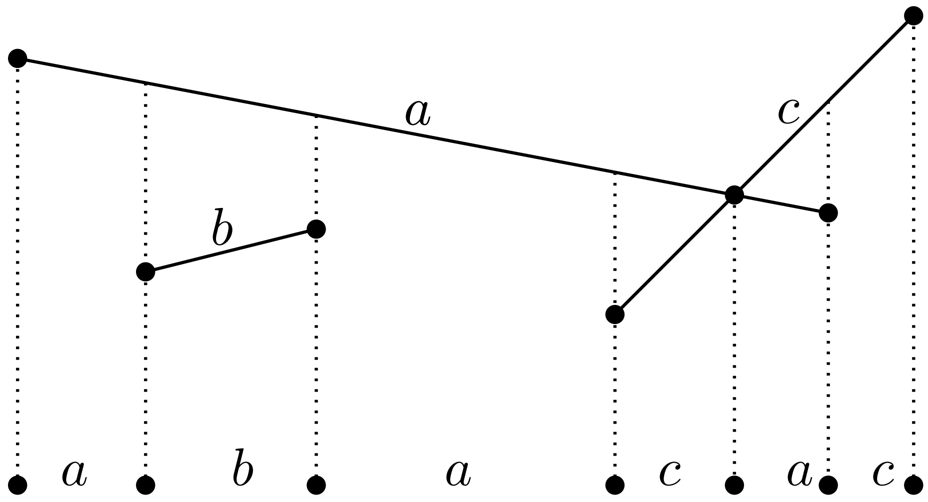
- $\alpha(n)$: inverse of Ackermann function (in practice, a constant ≤ 4)
- Same complexity bound for a single face
- Proof based on *Davenport-Schinzel sequences*

Lower envelope of line segments

Davenport-Schinzel sequence

Given an alphabet $\sum = \{a, b, c, \ldots\}$ of n symbols, a sequence ω is an (n, s)-*Davenport-Schinzel sequence* iff:

- For every $a \in \sum$, aa is not a substring of ω
- For every pair a,b ($a \neq b$), the sequence of $s+2$ alternating a's and b's is not a subsequence of ω



Example: aba is forbidden for $s = 1$, $abab$ for $s = 2$, and $ababa$ for $s = 3$.

We want to bound $\lambda_s(n)$, the maximum length of any (n, s)-sequence. **Why?**

Lower envelope of line segments

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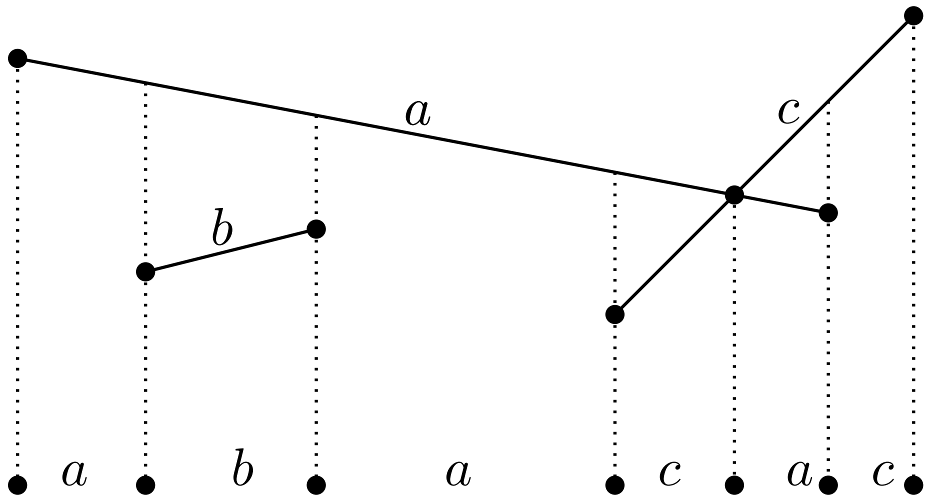

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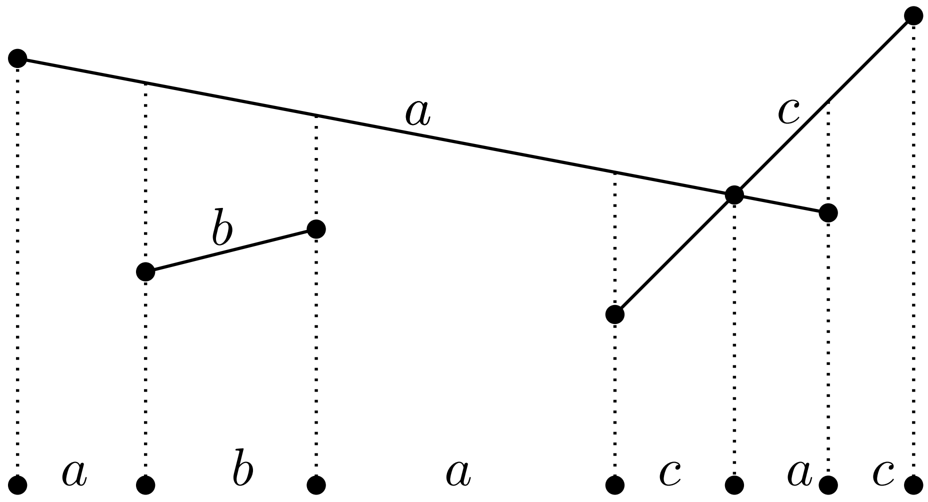

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First bounds: $\lambda_1(n) = n$, $\lambda_2(n) = 2n - 1$, and $\lambda_3(n) = \Theta(n\alpha(n))$

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APPLICATIONS

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- Arrangements and Voronoi diagrams
- Arrangements of lines as point sets duals
  (with much more structure)
  - k-sets
  - ham-sandwich cuts
- Elimination of hidden surfaces
  (for n disjoint polygons in space, solved in $O(n^2)$ time by topological sweep)
- Aspect graphs
- Motion planning for polygonal robots (arrangement of line segments)

ARRANGEMENTS

APPLICATIONS

- Arrangements and Voronoi diagrams
- Arrangements of lines as point sets duals (with much more structure)
  - k-sets
  - ham-sandwich cuts
- Elimination of hidden surfaces (for n disjoint polygons in space, solved in $O(n^2)$ time by topological sweep)
- Aspect graphs
- Motion planning for polygonal robots (arrangement of line segments)

SPECIFIC REFERENCES

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