Computational Geometry - GEI

Euler's formula for planar graphs

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Theorem 1 *Let $G = (V, E)$ be a connected plane graph, and let v, e, and f respectively denote its number of vertices, edges and faces. We have:*

$$v + f = e + 2.$$

Proof: We proceed by induction on the number of edges.

Base case: If $e = 0$, the connectivity of the graph guarantees that $v = 1$, and $f = 1$. Therefore, $v + f = 1 + 1 = 2 = 0 + 2 = e + 2$, and the formula holds.

Inductive step: If G is a tree, then $v = e + 1$ and $f = 1$. As a result, $v + f = (e + 1) + 1 = e + 2$. If G is not a tree, since it is connected it must contain a cycle. Let a be an edge in such a cycle. The Graph $G\ \{a\}$ has v vertices, $e - 1$ edges and $f - 1$ faces. By inductive hypothesis, $v + (f - 1) = (e - 1) + 2$. Therefore, $v + f = e + 2$. $\square$