

INTERSECTING HALF-PLANES and related problems

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DUALIZING POINTS AND LINES

Duality

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In the plane, dualities transform points into lines and lines into points.

One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

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Properties

1. $D^2 = Id$.
2. D is a bijection between points and non-vertical lines.
3. A point p lies above/on/below a line ℓ if and only if the point ℓ^* lies above/on/below the line p^* .

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Proof: If $p = (a, b)$ and ℓ is the line of equation $y = mx + n$, the relative position of p wrt ℓ is given by the (in)equation $b \lessgtr ma + n$, which is equivalent to $-n \lessgtr am - b$, and this expresses the relative position of $\ell^* = (m, -n)$ wrt $p^* : y = ax - b$.

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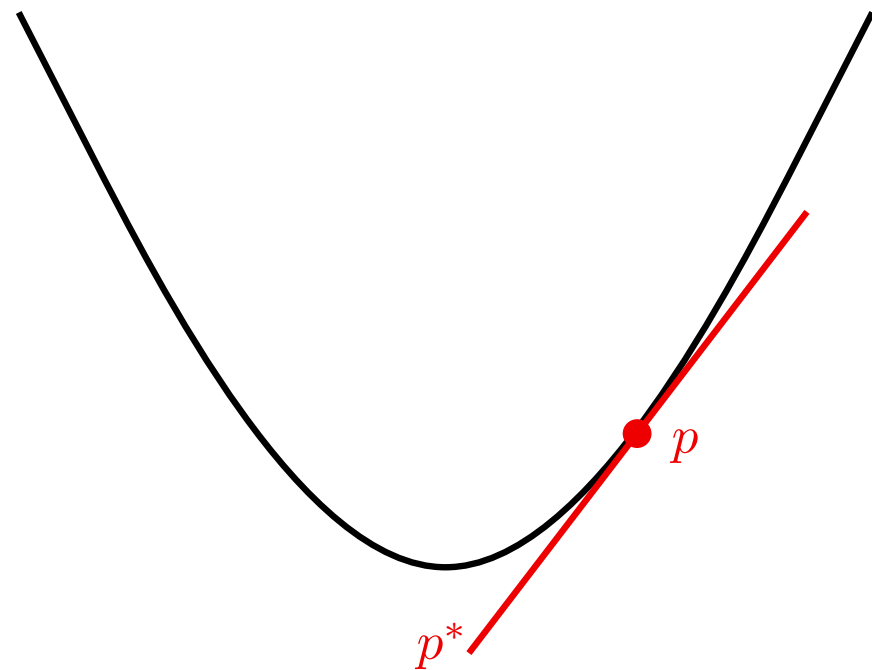
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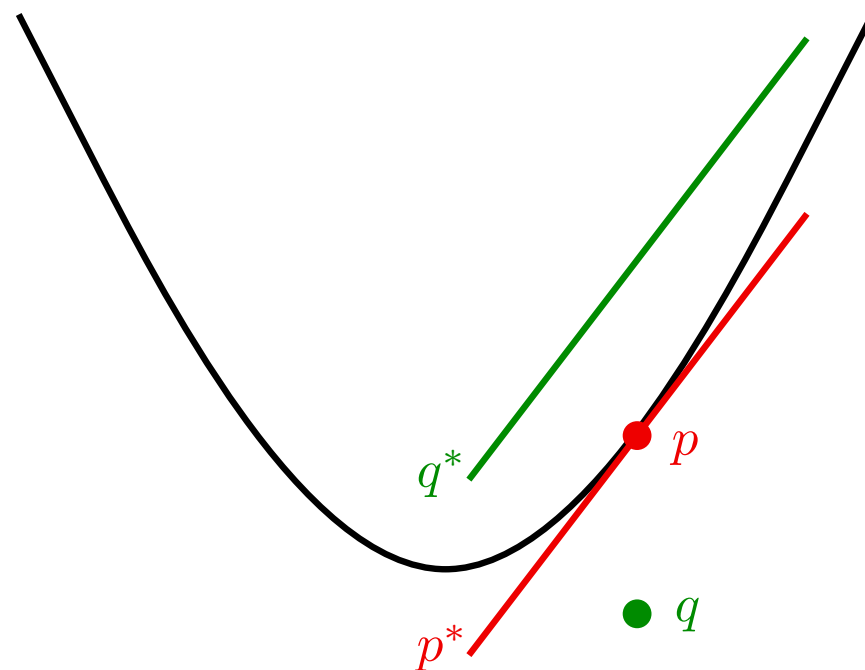
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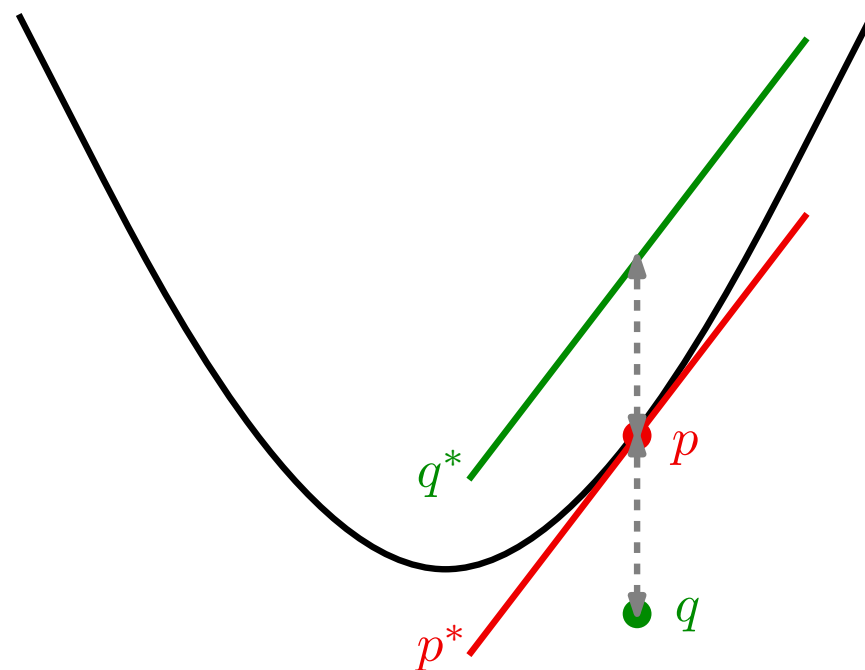
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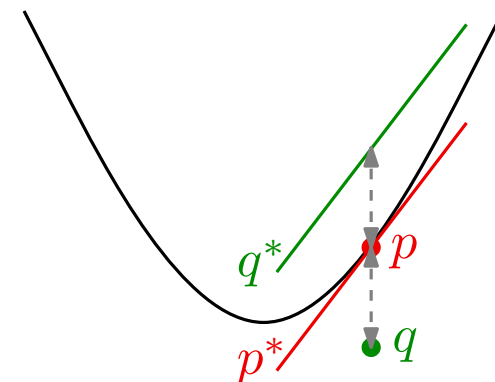
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Proof:

Since $y'(x) = x$, the line tangent to the parabola at a point $p = (a, a^2/2)$ has equation $y - a^2/2 = a(x - a)$ i.e., $y = ax - a^2/2$.

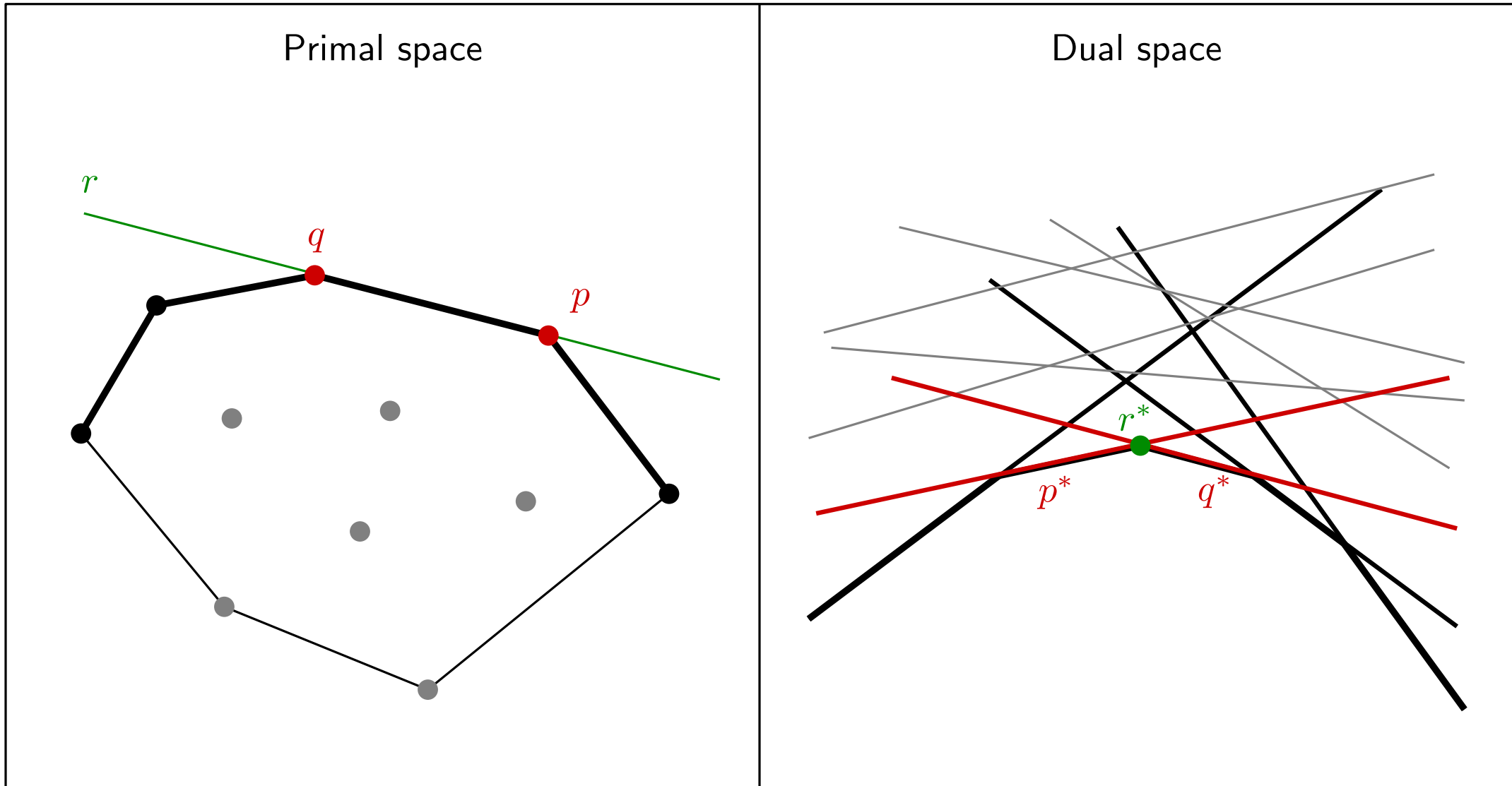
For any other point $q = (a, b)$:
 $d_{vertical}(p, q) = b - a^2/2$;
 $d_{vertical}(p^*, q^*) = -b + a^2/2$.

INTERSECTING HALF-PLANES

Intersection of half-planes and convex hulls

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INTERSECTING HALF-PLANES

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Input: n half-planes

Output: Sorted list of vertices (and, possibly, half-lines) defining the boundary of their intersection (a convex polygon or an unbounded convex polygonal region).

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Lower bound

Duality reduces in $O(n)$ time the problem of computing the convex hull of n points to the problem of computing the intersection of n half-planes. Therefore, computing the intersection of n half-planes has an $\Omega(n \log n)$ lower bound.

SOLVING LINEAR PROGRAMS

Linear program

Optimize a linear function $ax + by$

subject to n linear restrictions $a_i x + b_i y + c_i \leq 0$, where $i = 1, \dots, n$.

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The cost of producing each unit of X is:

- for machine A : 50 minutes,
- for machine B : 30 minutes.

The cost of producing each unit of Y is:

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The week starts with:

- A stock of 30 units of X and 90 of Y ,
- A demand of 75 units of X and 95 of Y .

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$$\text{Maximize } (x + 30 - 75) + (y + 90 - 95)$$

x = units of X to be produced

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Maximize $(x + 30 - 75) + (y + 90 - 95)$

x = units of X to be produced

y = units of Y to be produced

restricted to:

$$50x + 24y \leq 40 \times 60$$

$$30x + 33y \leq 35 \times 60$$

$$x \geq 75 - 30$$

$$y \geq 95 - 90$$

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Compute the feasible region R in $O(n \log n)$ time. Any linear function on R can then be optimized in $O(\log n)$ time by binary search.

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Algo 2 (Megiddo, Dyer)

In $O(n)$ time, it is possible to find the vertex of R achieving the optimum, without computing the entire feasible region R .

This is done by a prune and search strategy, which at each step eliminates a constant fraction of the restrictions:

- either because they are redundant,
- or because they are not needed to define the solution vertex.

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Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.



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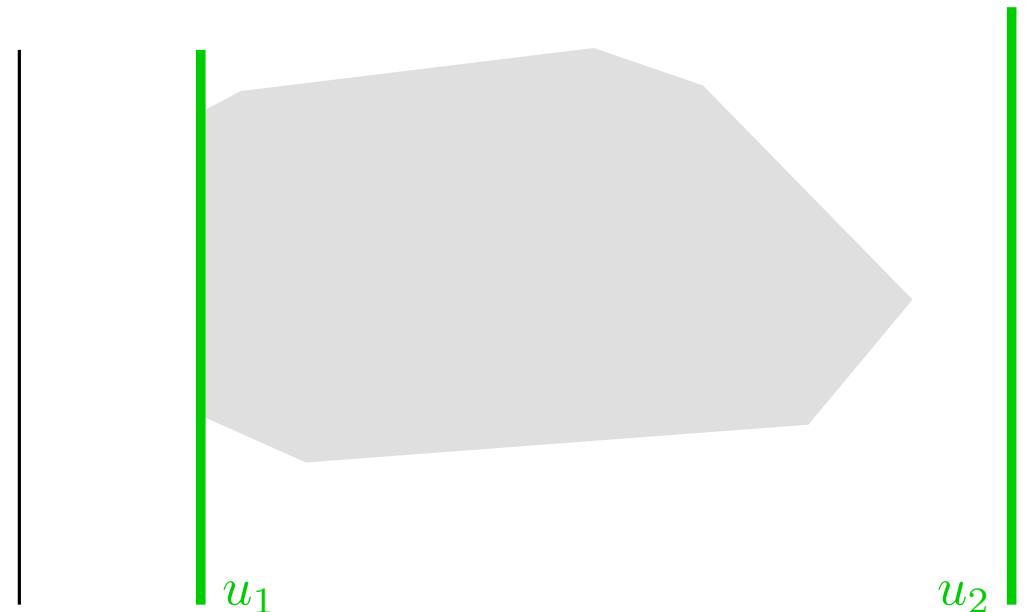
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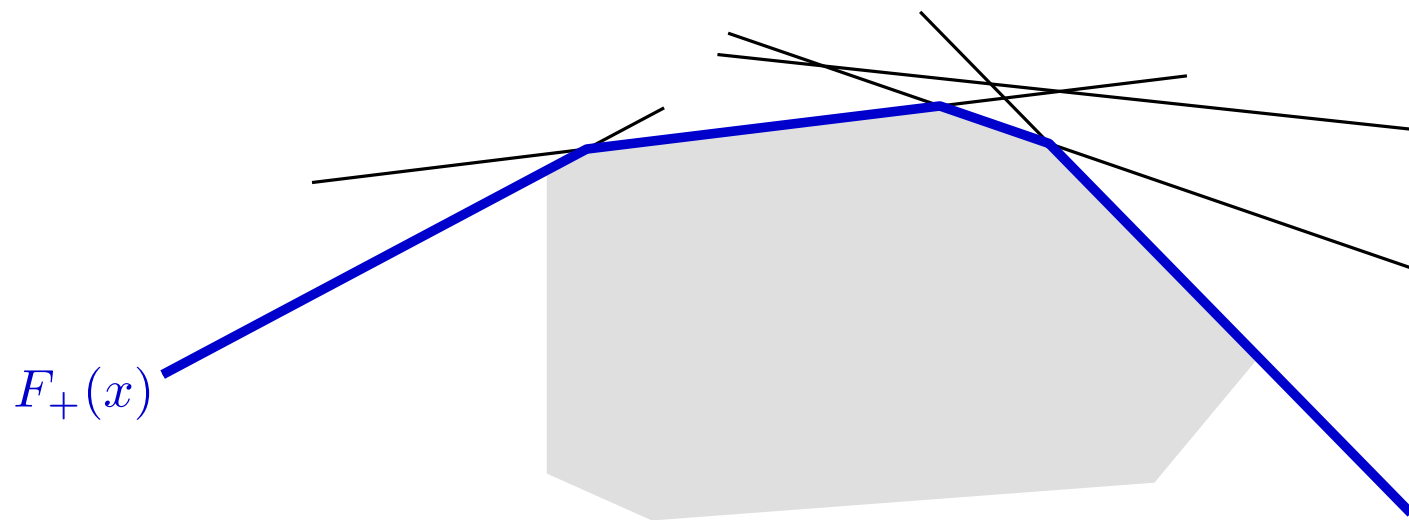
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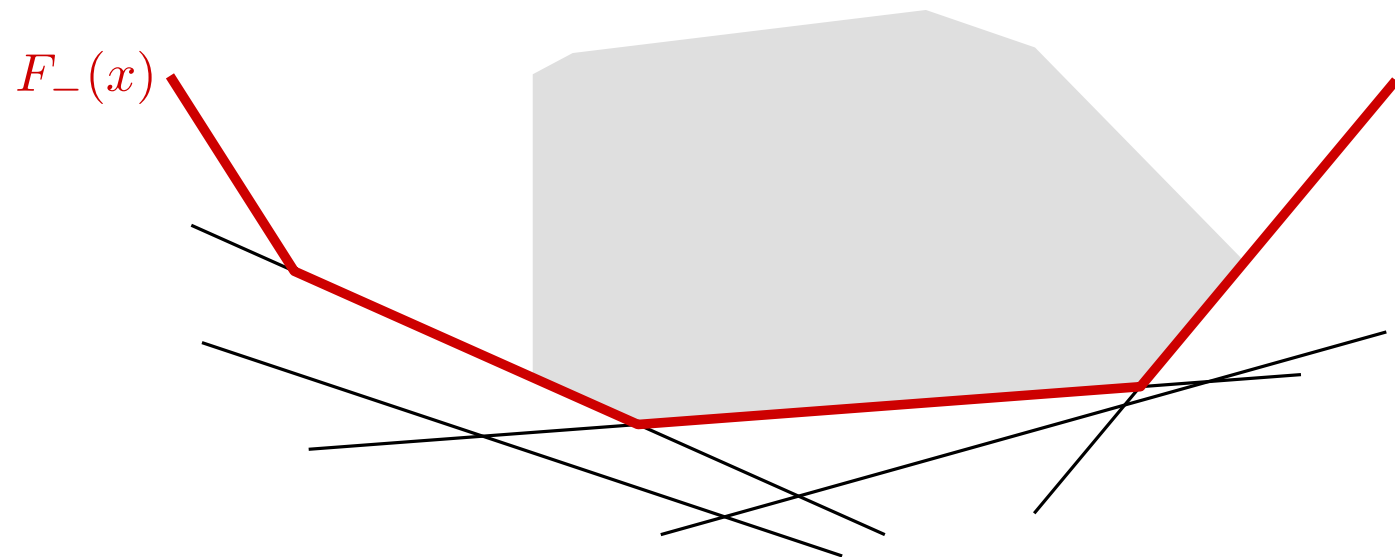
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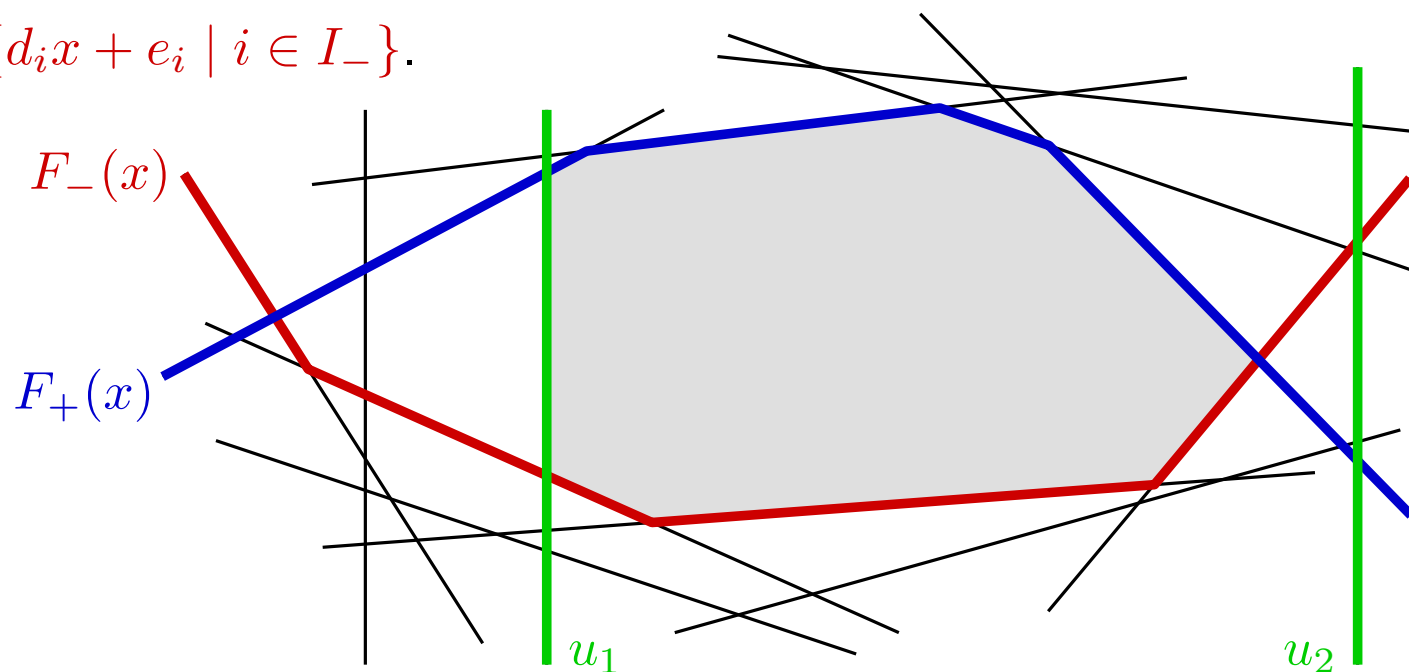
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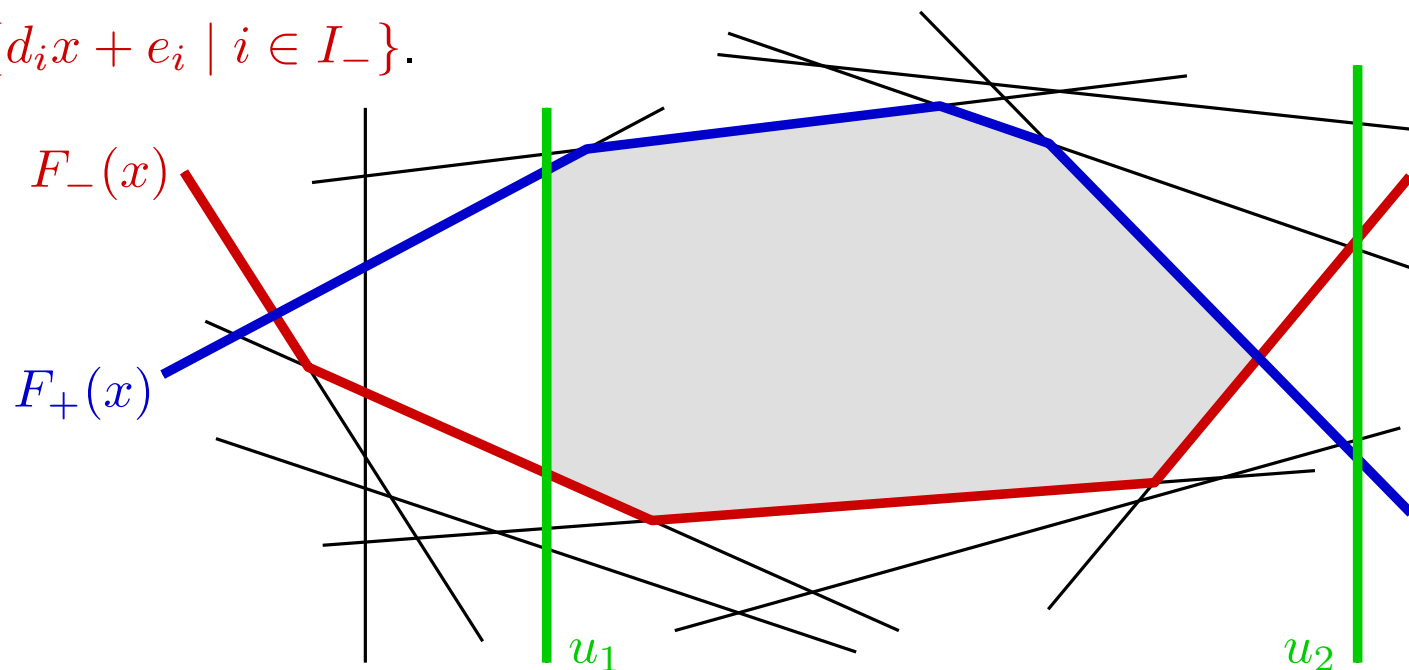
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$$F_-(x) \leq F_+(x)$$



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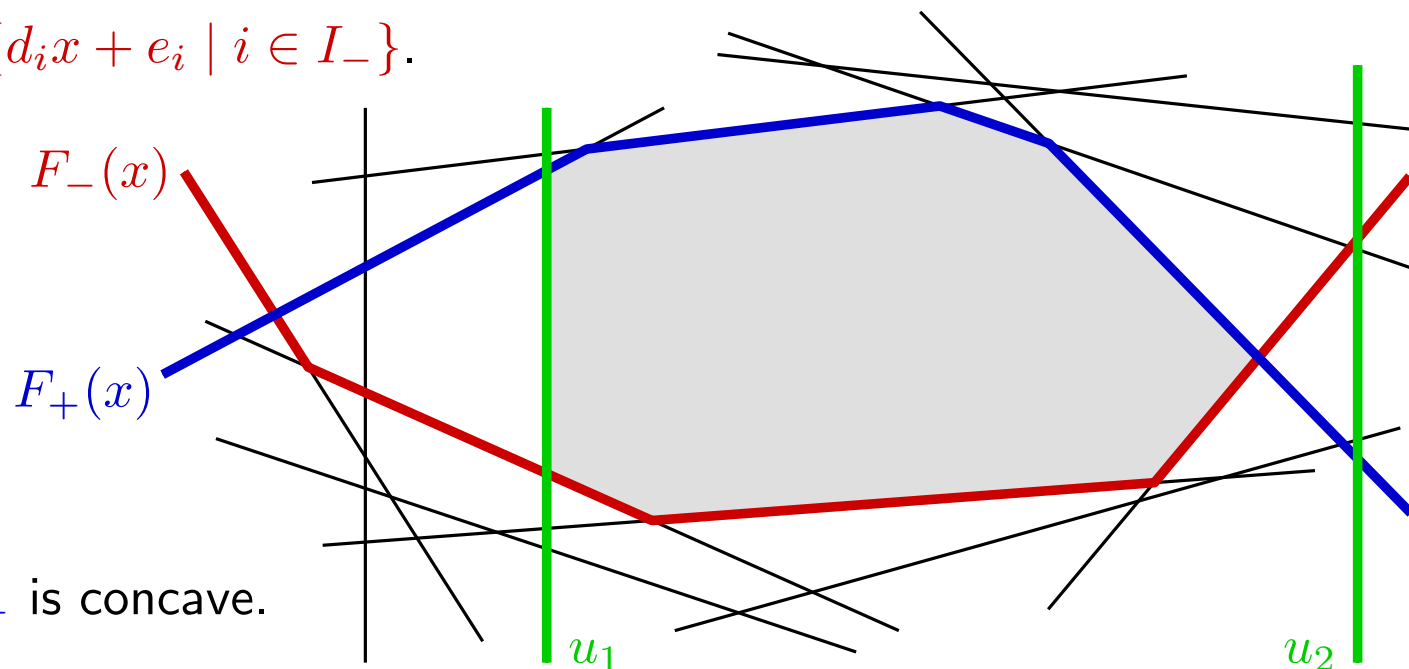
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$$F_-(x) \leq F_+(x)$$

Notice that F_- is convex and F_+ is concave.



SOLVING LINEAR PROGRAMS

The search

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Given $x' \in [u_1, u_2]$, in $O(n)$ time it is possible to get to one of the following conclusions:

1. x' is feasible, and minimizing $F_-(x)$.
2. x' is feasible, and the solution of the problem lies to its right (left).
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4. x' is infeasible, and the problem has no solution.

This is done by analyzing the values of F_+ , F_- and their slopes f_+^l , f_+^r , f_-^l , f_-^r in x' , where the slopes are defined as follows:

- If $F_-(x')$ (resp. $F_+(x')$) is defined by a unique index $i \in I_-$ (resp. I_+), then $f_-^l(x') = f_-^r(x') = d_i$ (resp. $f_+^l(x') = f_+^r(x') = d_i$).
- If $F_-(x')$ (resp. $F_+(x')$) is defined by two indexes $i, j \in I_-$ (resp. I_+), then $f_-^l(x') = \min(d_i, d_j)$ and $f_-^r(x') = \max(d_i, d_j)$ (resp. $f_+^l(x') = \max(d_i, d_j)$ and $f_+^r(x') = \min(d_i, d_j)$).

SOLVING LINEAR PROGRAMS

The search

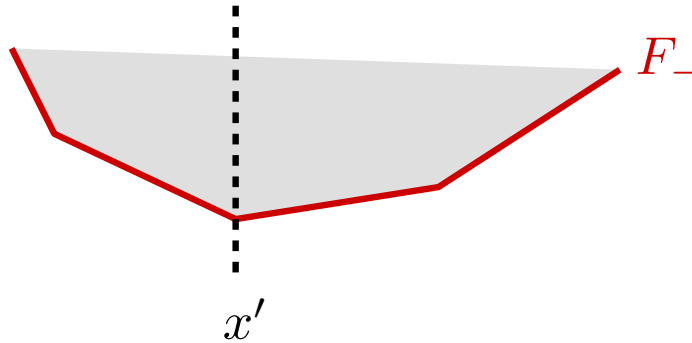
Case 1. $F_-(x') \leq F_+(x')$ (x' is feasible)

SOLVING LINEAR PROGRAMS

The search

Case 1. $F_-(x') \leq F_+(x')$ (x' is feasible)

If $f_-^l(x') < 0 < f_-^r(x')$,
then x' is the solution.

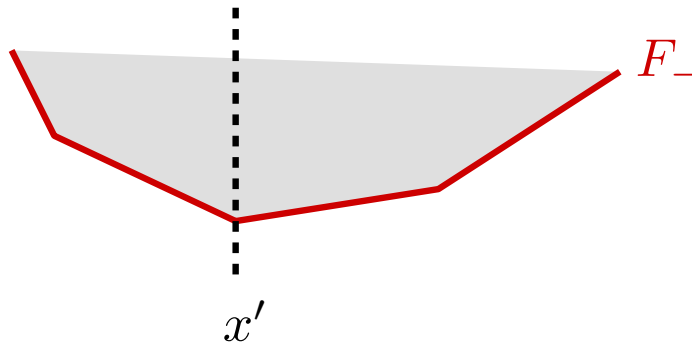


SOLVING LINEAR PROGRAMS

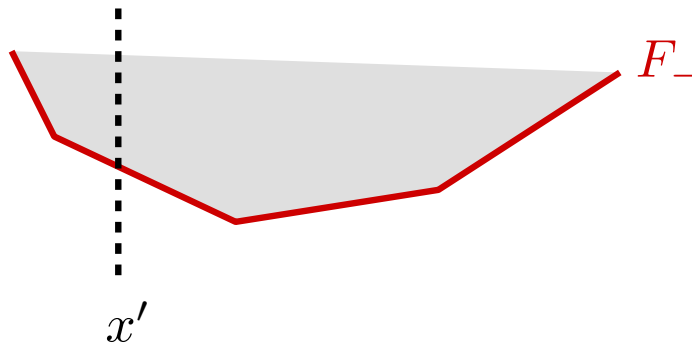
The search

Case 1. $F_-(x') \leq F_+(x')$ (x' is feasible)

If $f_-^l(x') < 0 < f_-^r(x')$,
then x' is the solution.



If $f_-^r(x') < 0$,
then search to the right of x' .

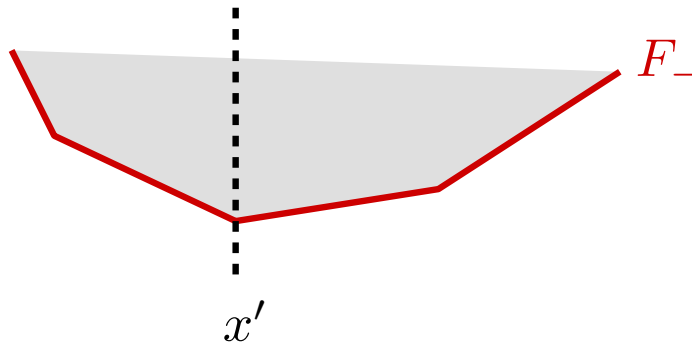


SOLVING LINEAR PROGRAMS

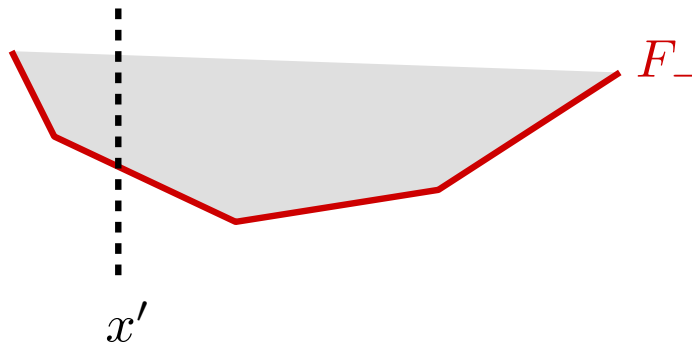
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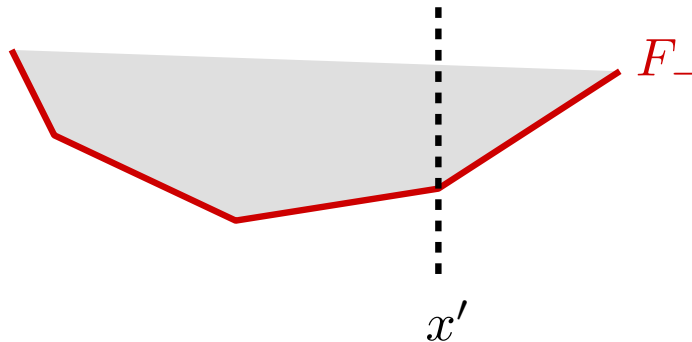
If $f_-^l(x') < 0 < f_-^r(x')$,
then x' is the solution.



If $f_-^r(x') < 0$,
then search to the right of x' .



If $f_-^l(x') > 0$,
then search to the left of x' .



SOLVING LINEAR PROGRAMS

The search

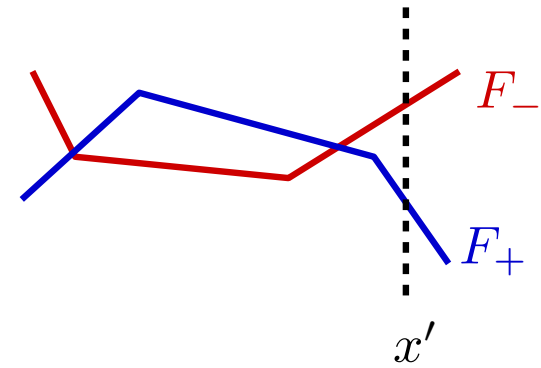
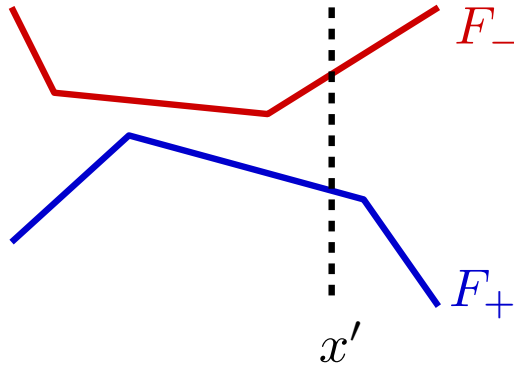
Case 2. $F_-(x') > F_+(x')$ (x' is infeasible)

SOLVING LINEAR PROGRAMS

The search

Case 2. $F_-(x') > F_+(x')$ (x' is infeasible)

If $f_-^l(x') > f_+^l(x')$,
then search to the left of x' .

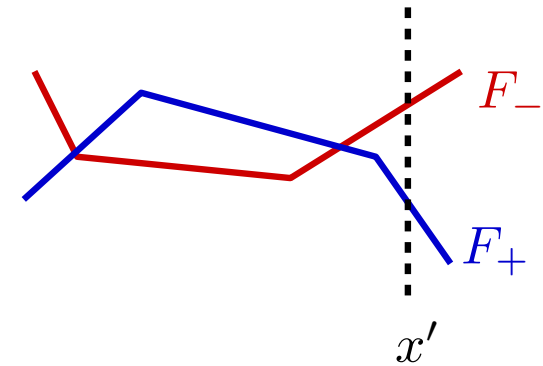
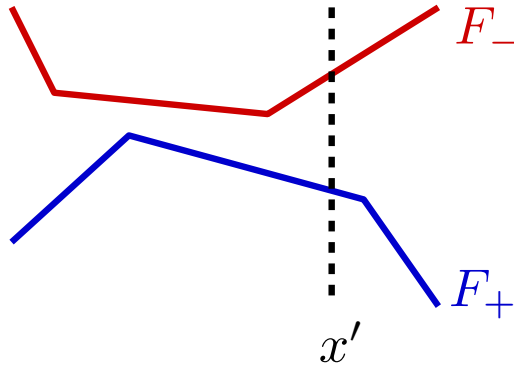


SOLVING LINEAR PROGRAMS

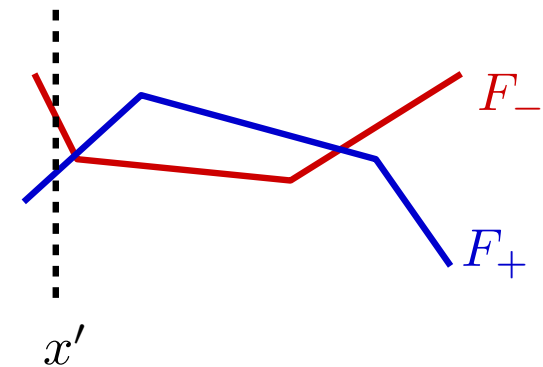
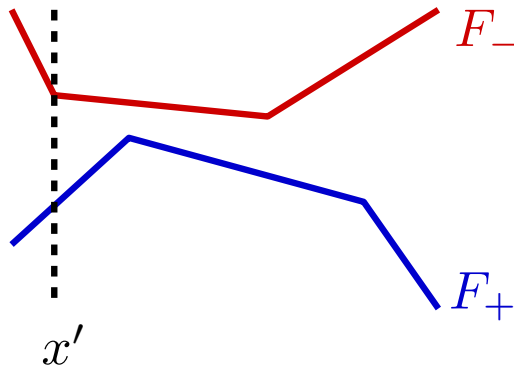
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If $f_-^l(x') > f_+^l(x')$,
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If $f_-^r(x') < f_+^r(x')$,
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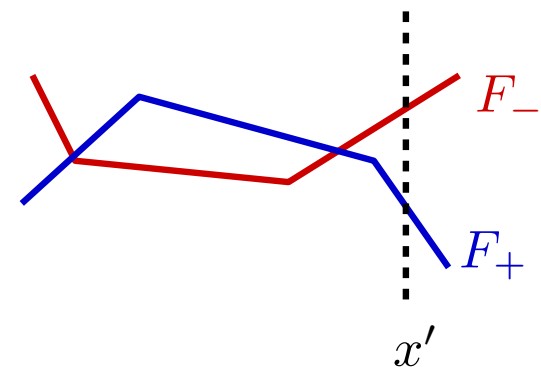
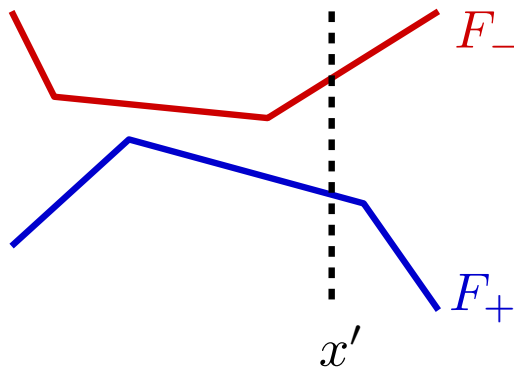


SOLVING LINEAR PROGRAMS

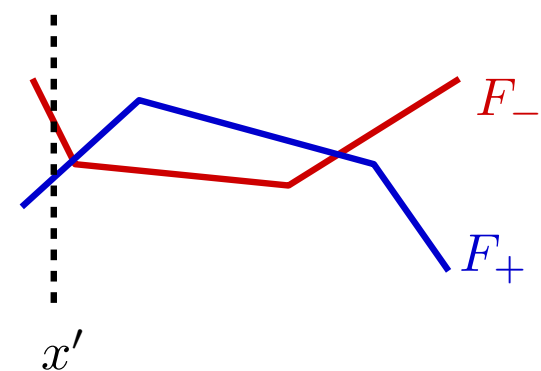
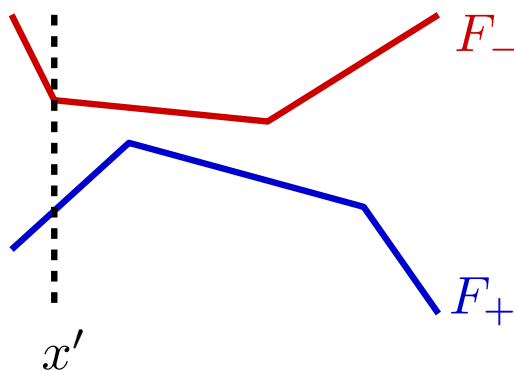
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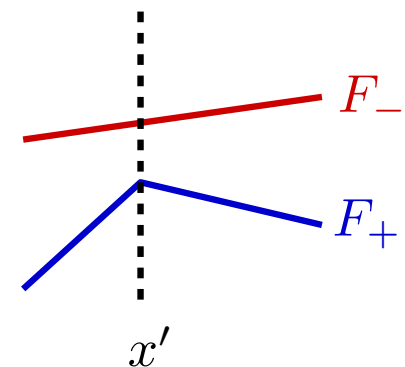
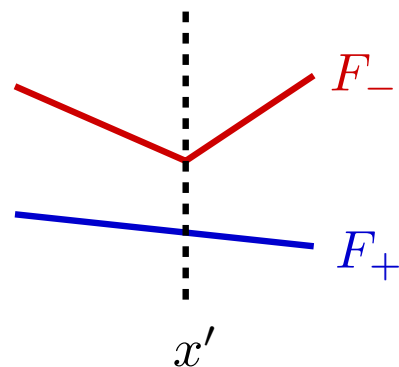
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If $f_-^r(x') < f_+^r(x')$,
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If $f_-^l(x') \leq f_+^l(x')$ and $f_-^r(x') \geq f_+^r(x')$
then there is no solution.



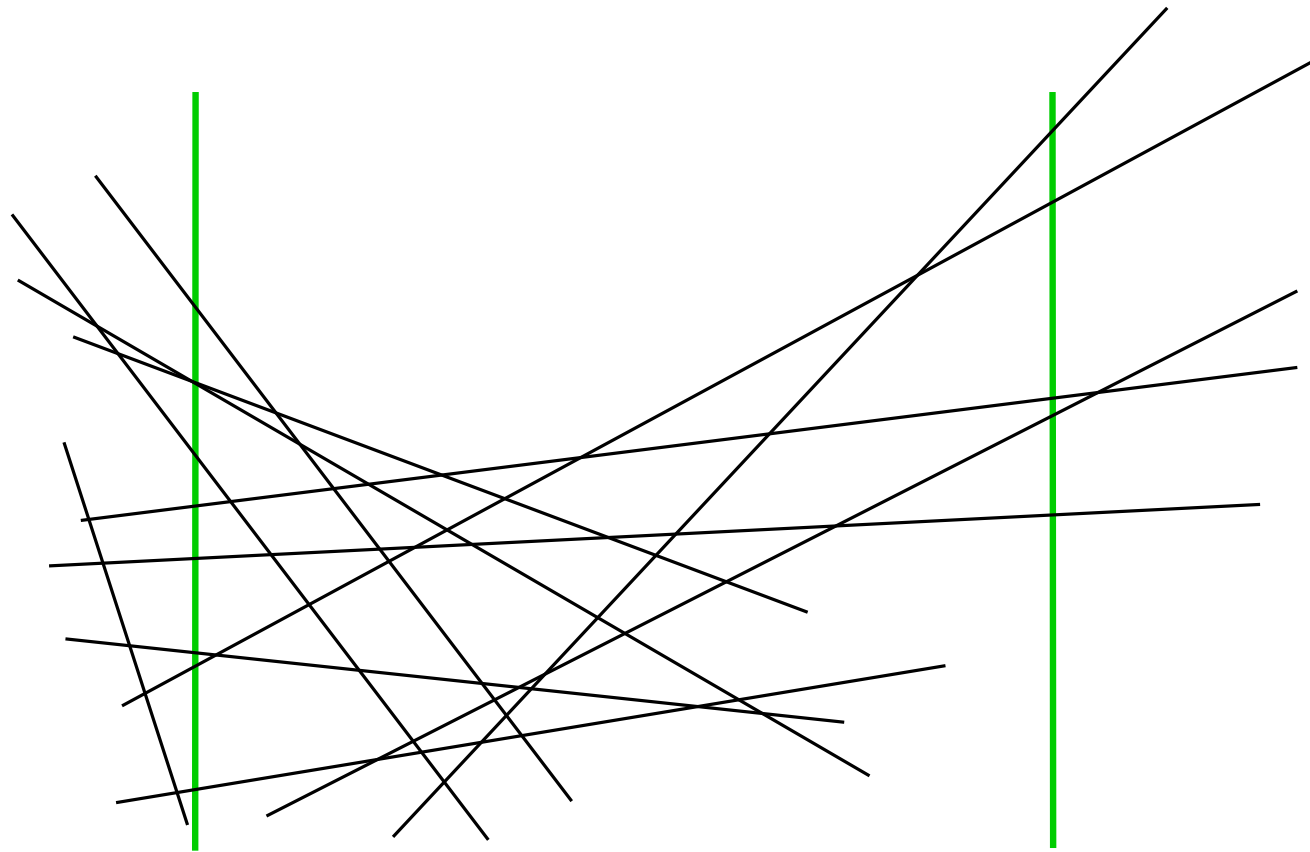
SOLVING LINEAR PROGRAMS

The algorithm

SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist.



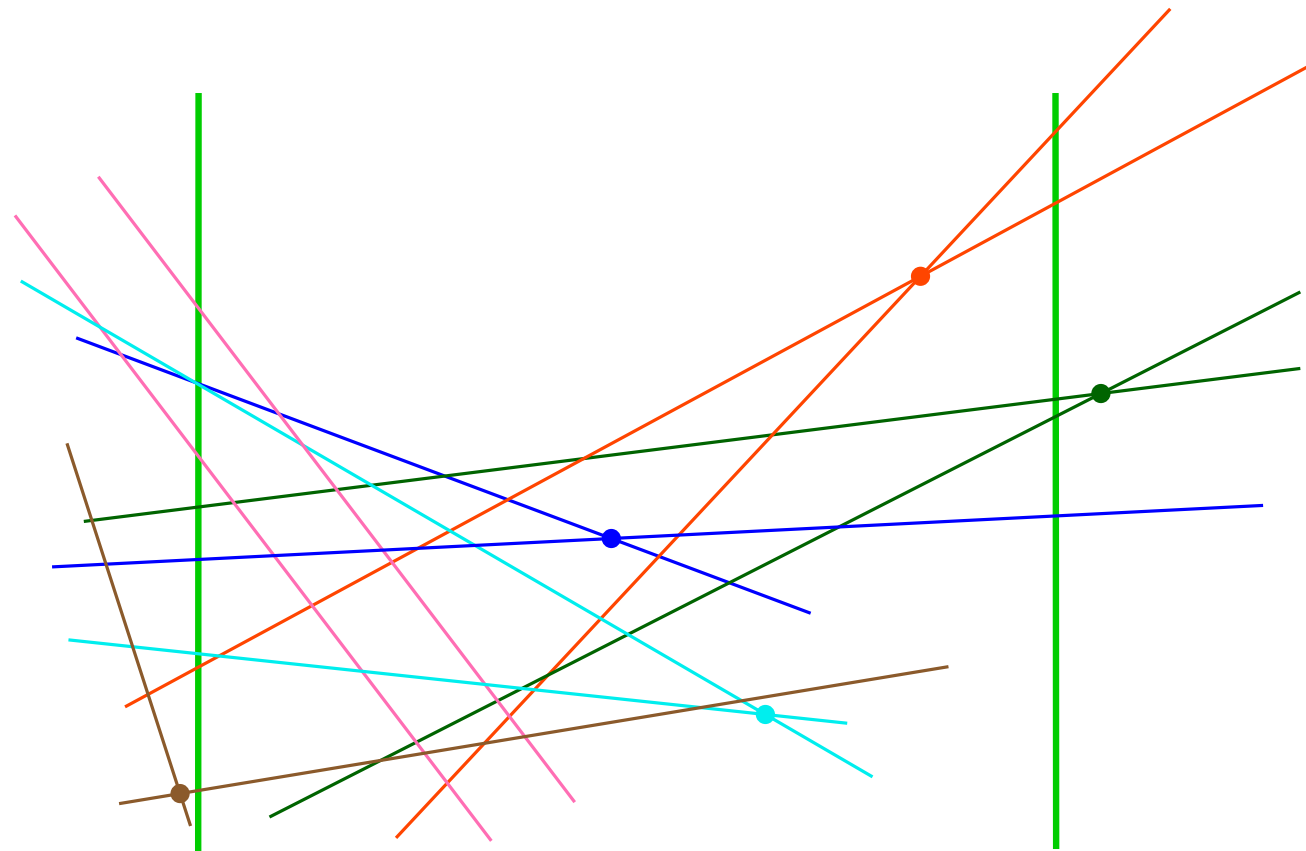
SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist.

Advance. Repeat as many times as necessary:

1. Pair up all restrictions corresponding to I_- (possibly but one). Analogously for I_+ . Compute the abscissa of the intersection point of each pair of lines, if it exists.



SOLVING LINEAR PROGRAMS

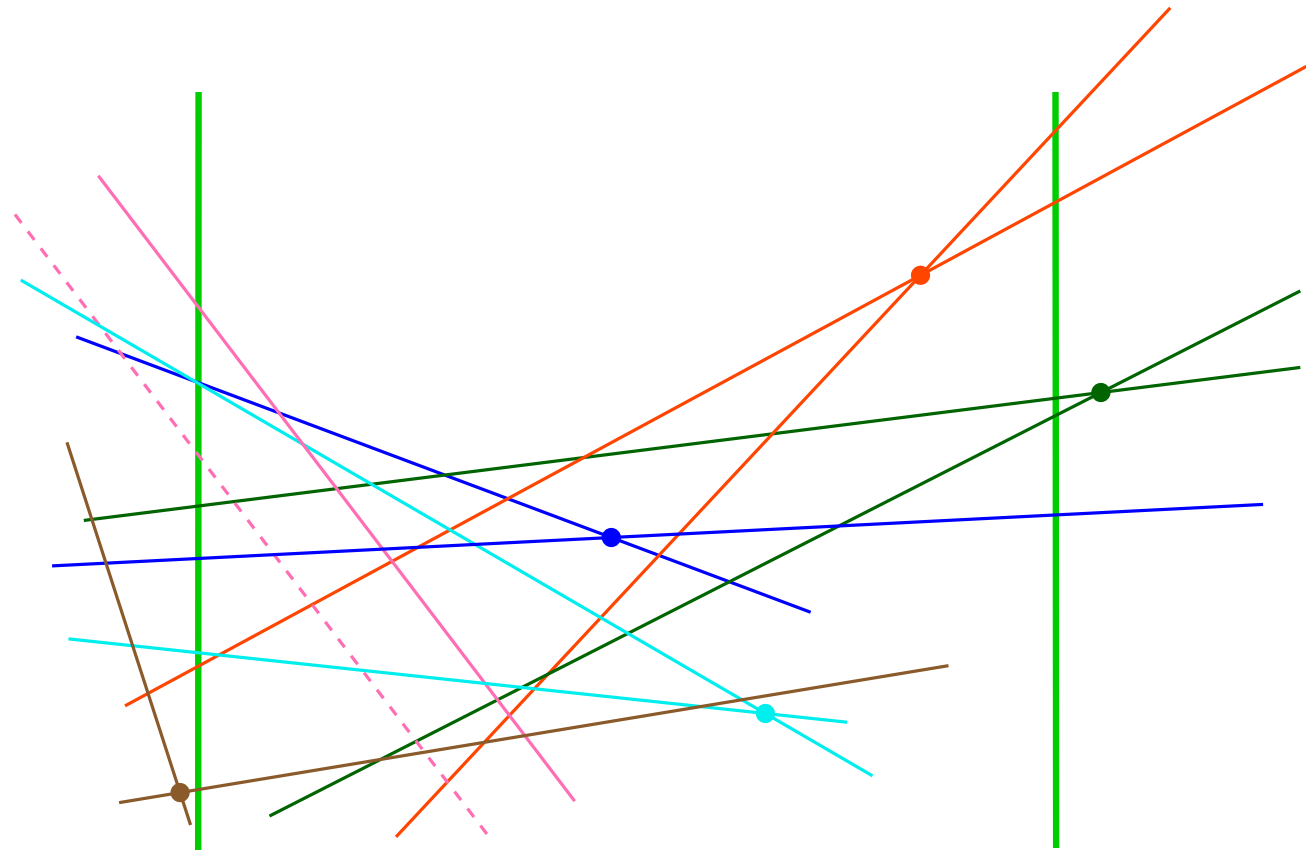
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Advance. Repeat as many times as necessary:

2. For all pairs $i, j \in I_-$ (analogously for I_+), do:

- If $d_i = d_j$, eliminate the line corresponding to $\min(e_i, e_j)$ (resp. $\max(e_i, e_j)$).



SOLVING LINEAR PROGRAMS

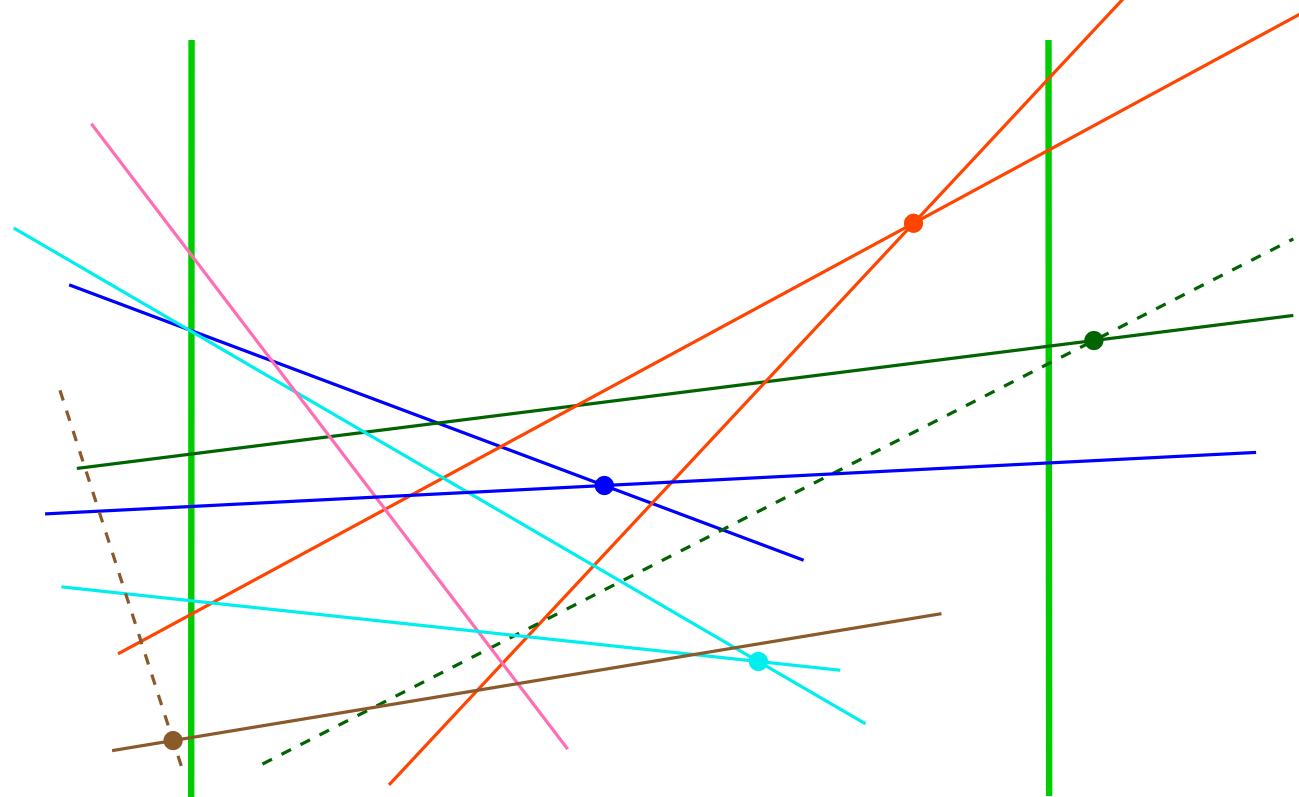
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- If $d_i = d_j$, eliminate the line corresponding to $\min(e_i, e_j)$ (resp. $\max(e_i, e_j)$).
- If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
 - If $x_{ij} < u_1$, eliminate the line with $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).
 - If $x_{ij} > u_2$, eliminate the line with $\max(d_i, d_j)$ (resp. $\min(d_i, d_j)$).



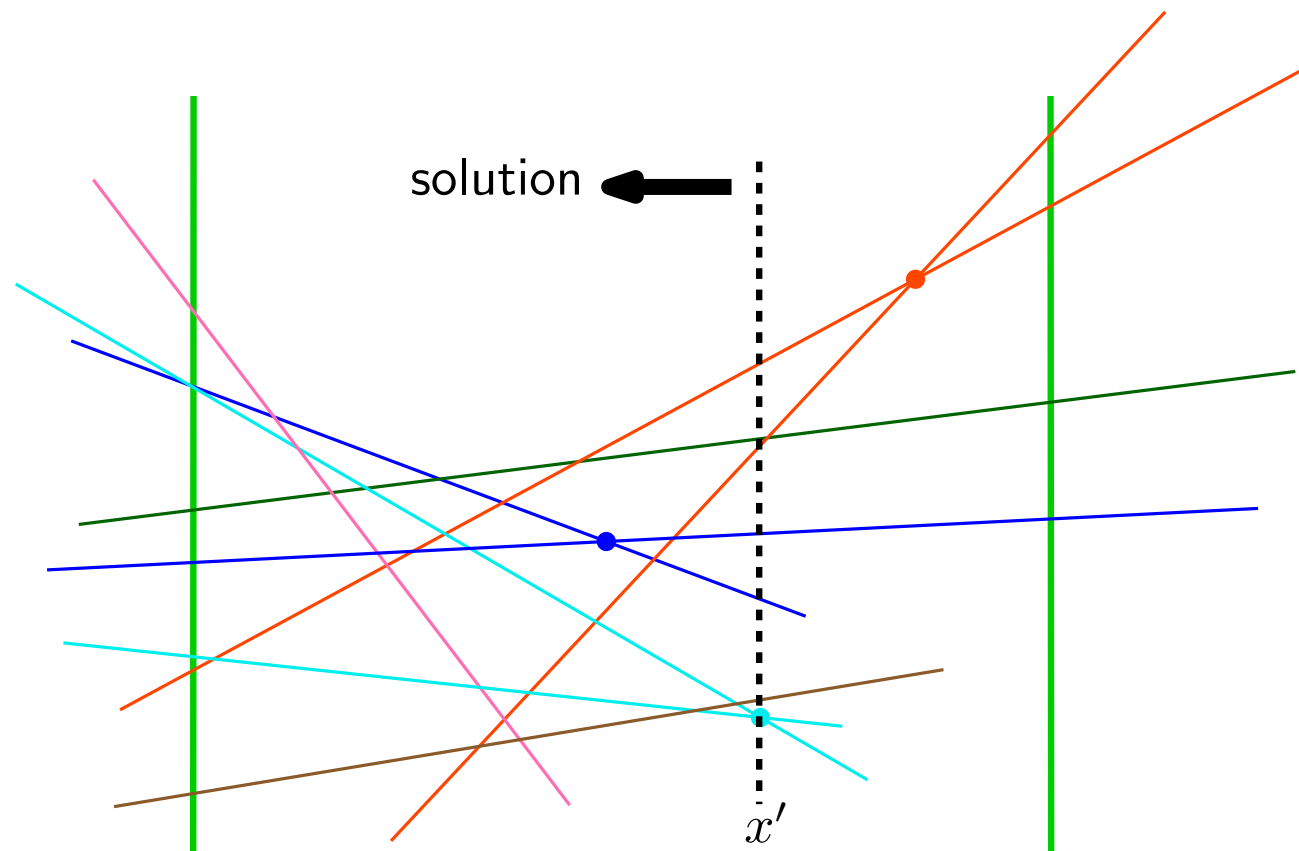
SOLVING LINEAR PROGRAMS

The algorithm

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Advance. Repeat as many times as necessary:

3. Compute the median value x' of the surviving x_{ij} .
4. Search: Apply the search procedure to x' . If the answer is x' , return x' .



SOLVING LINEAR PROGRAMS

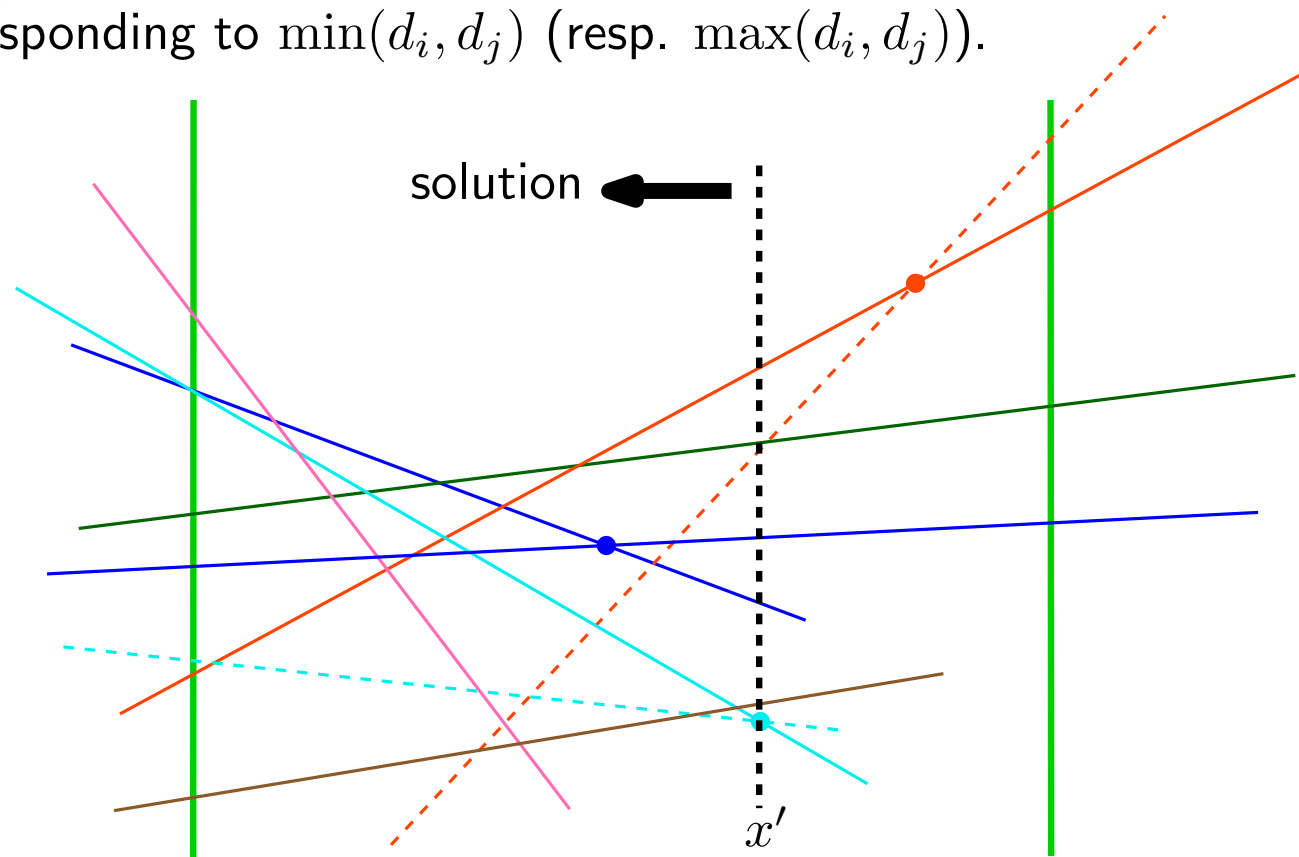
The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist.

Advance. Repeat as many times as necessary:

5. Prune:

If the answer is “search to the left” then, for each $x_{ij} \geq x'$ eliminate the line corresponding to $\max(d_i, d_j)$ (resp. $\min(d_i, d_j)$). If the answer is “search to the right” then, for each $x_{ij} < x'$ eliminate the line corresponding to $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).



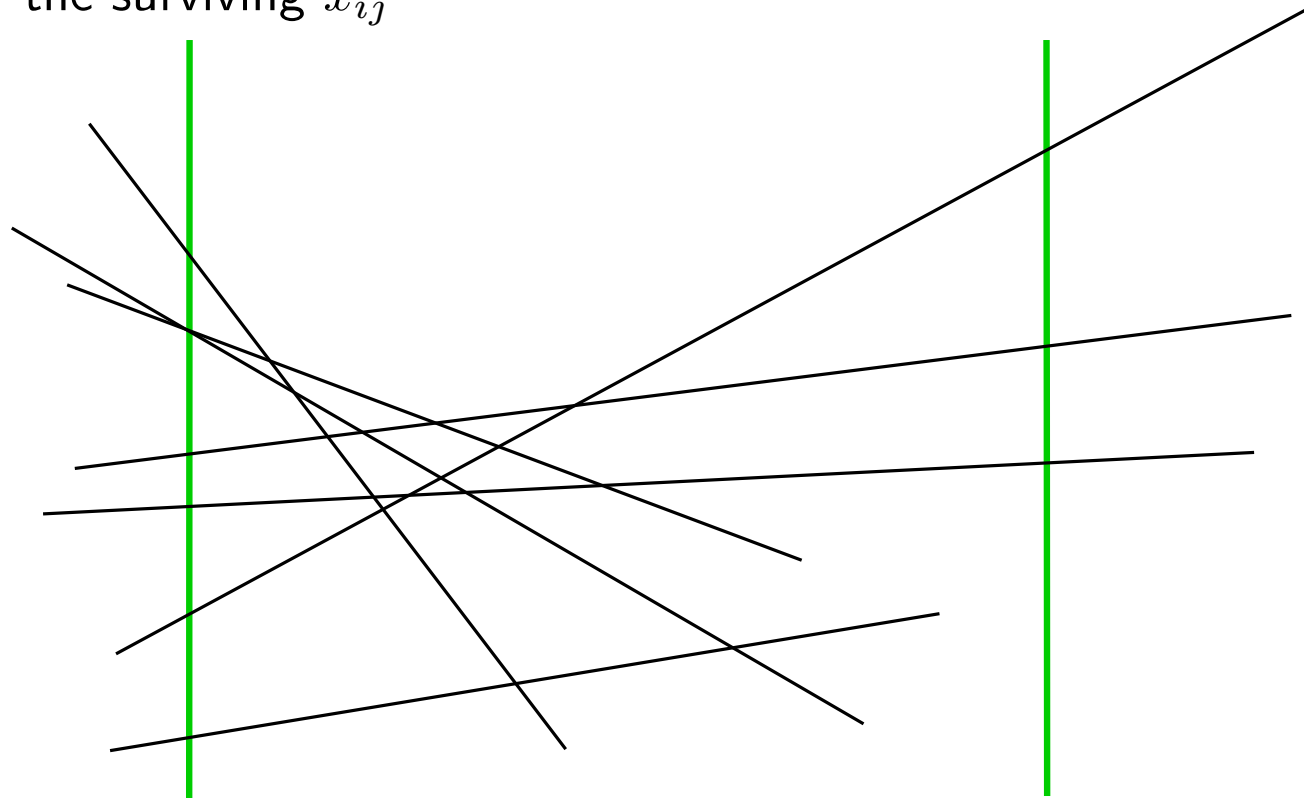
SOLVING LINEAR PROGRAMS

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Advance. Repeat as many times as necessary:

1. Pair lines up
2. Prune parallel and external lines
3. Compute the median value x' of the surviving x_{ij}
4. Search
5. Prune



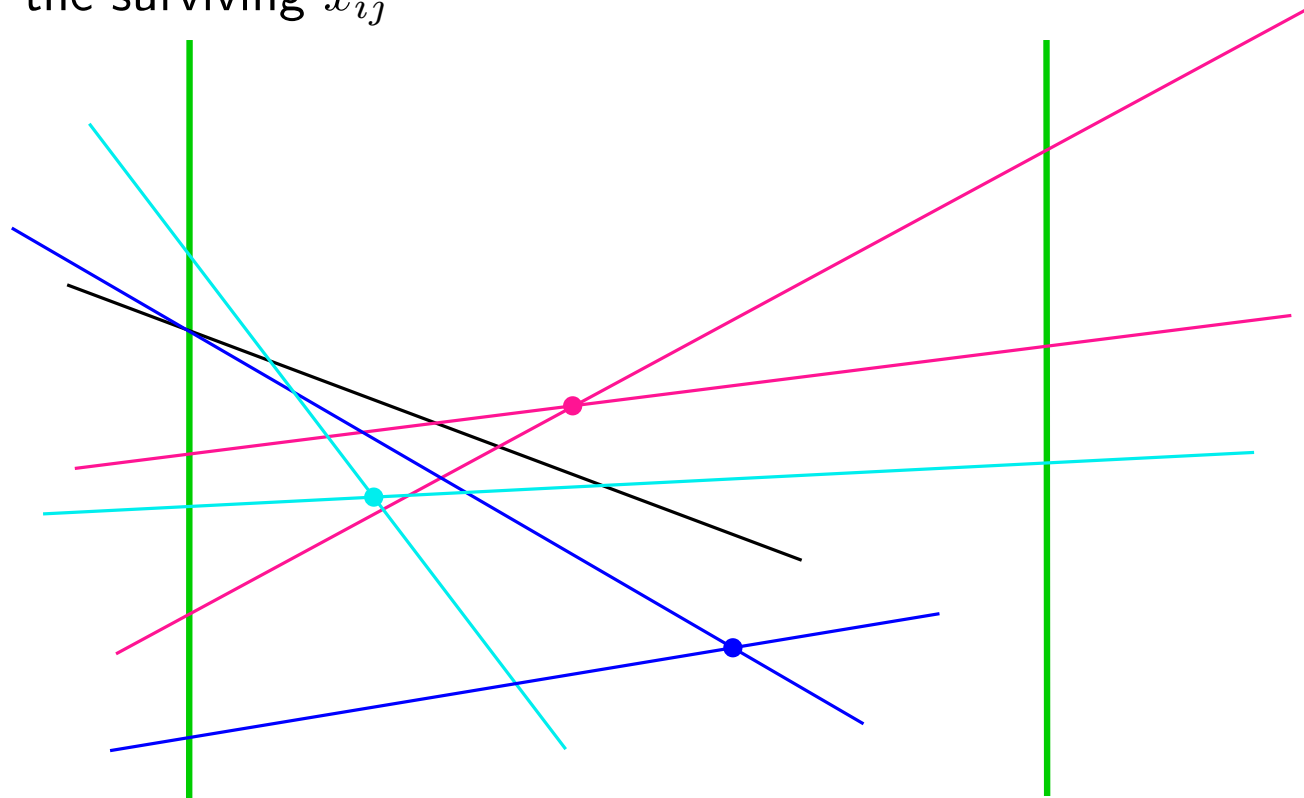
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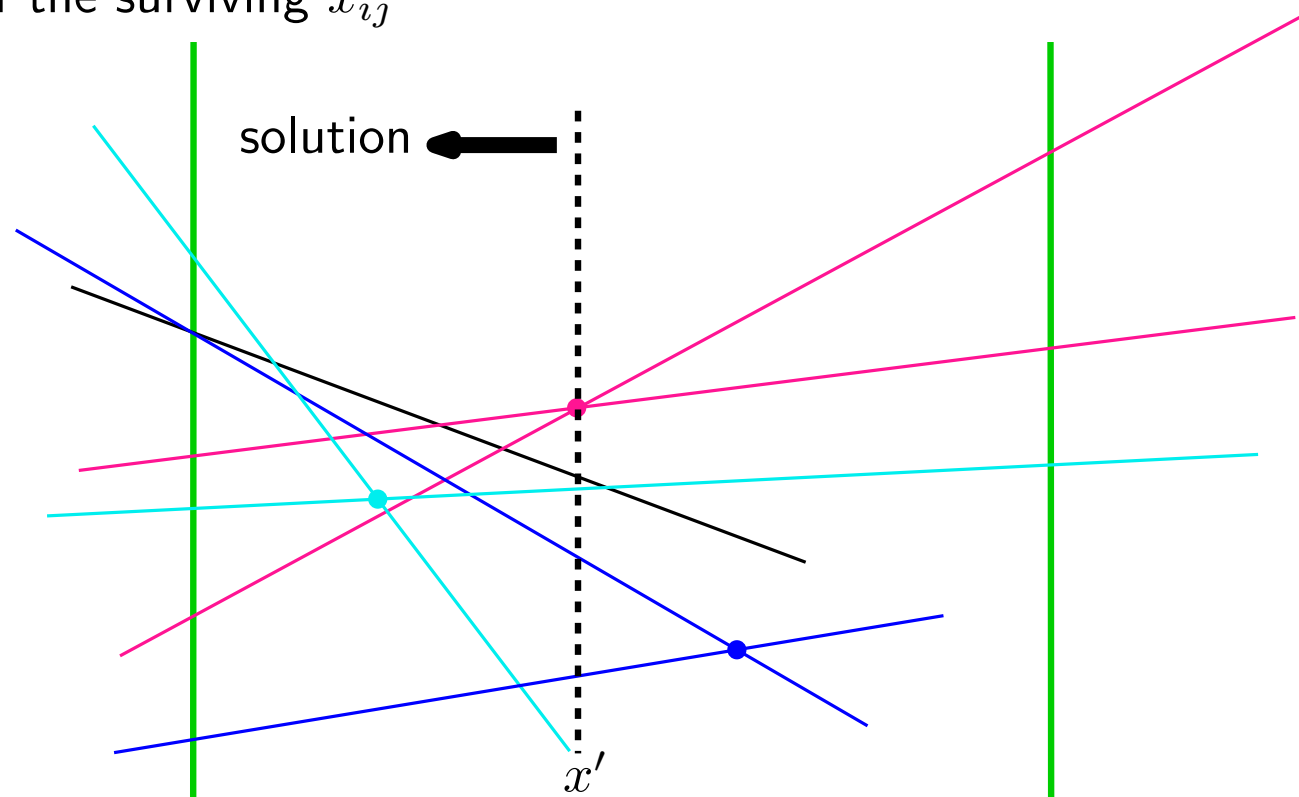
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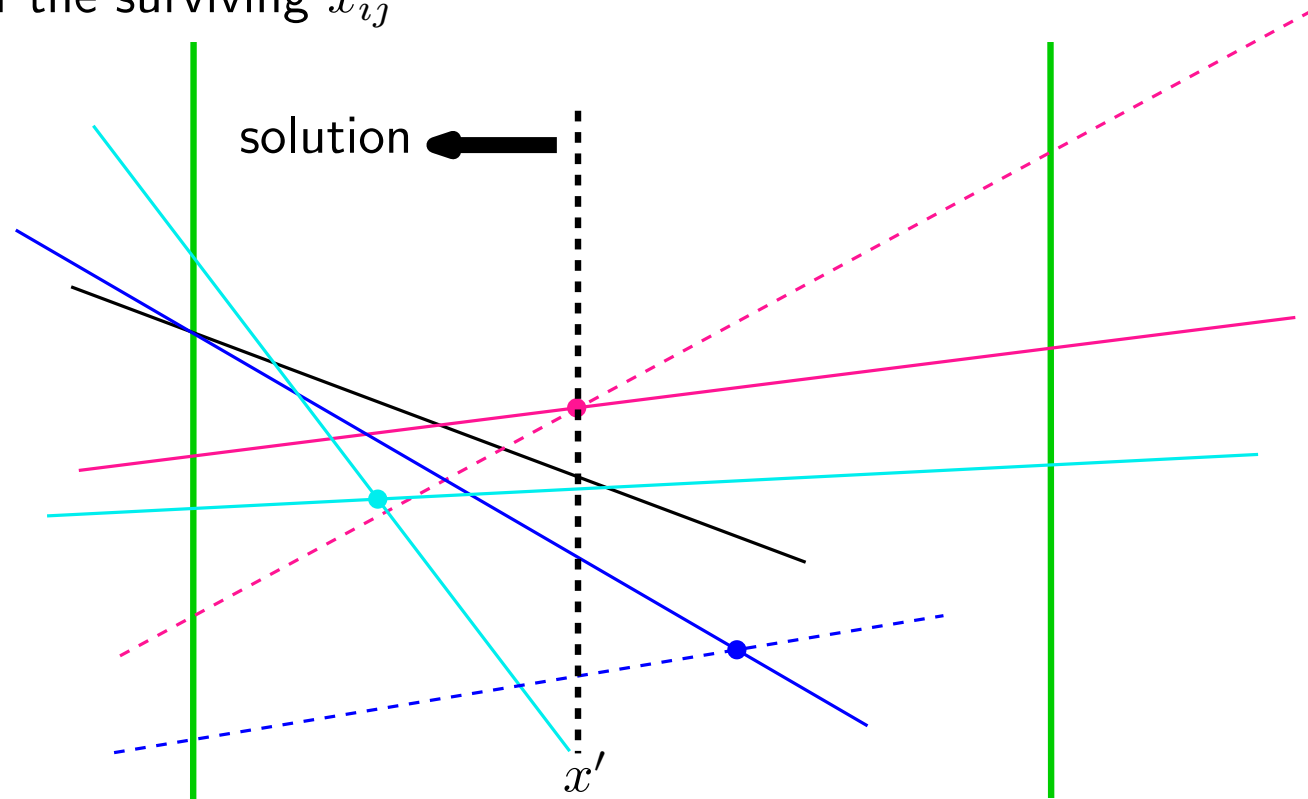
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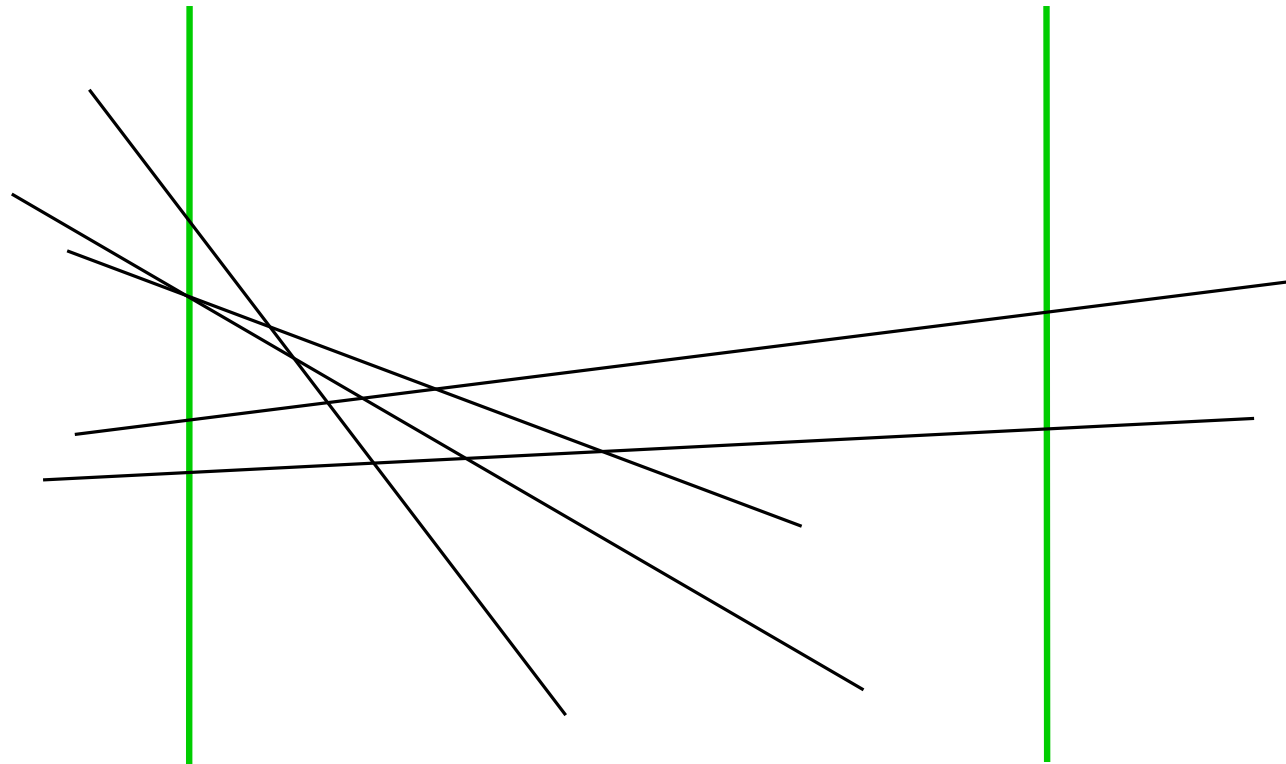
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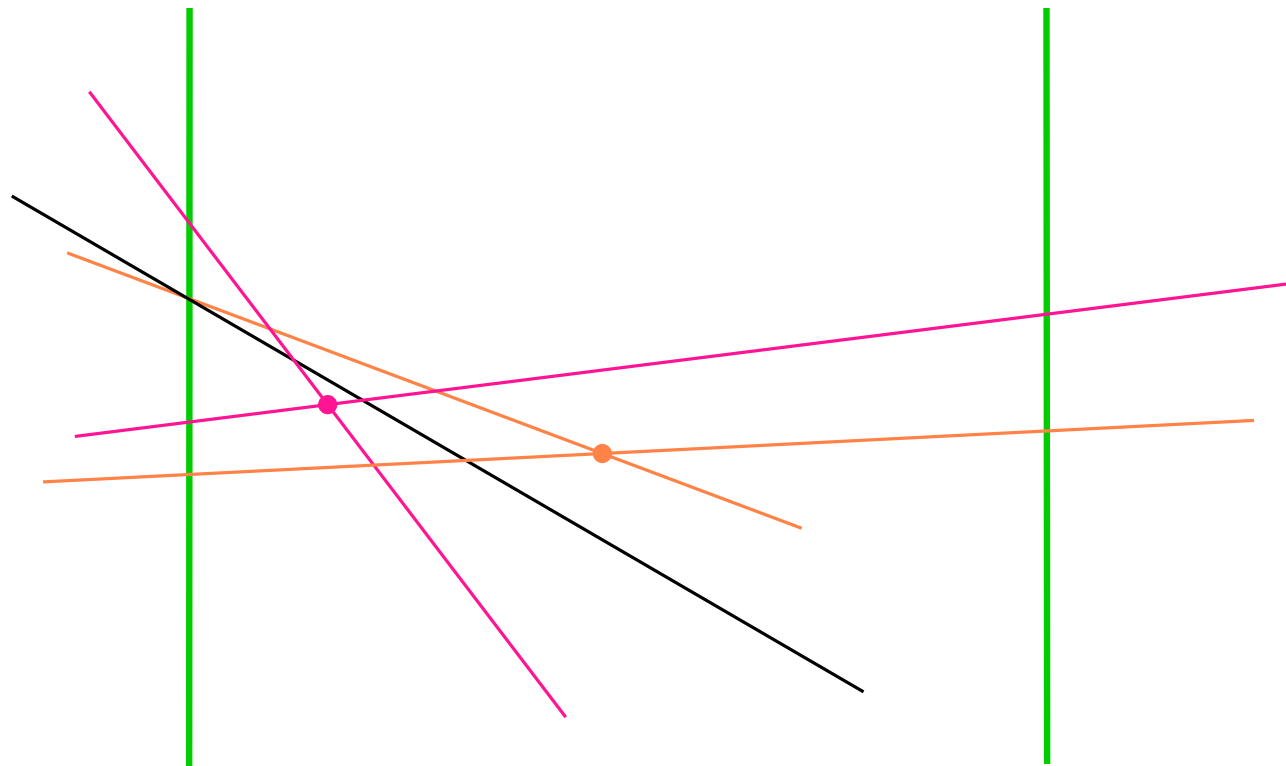
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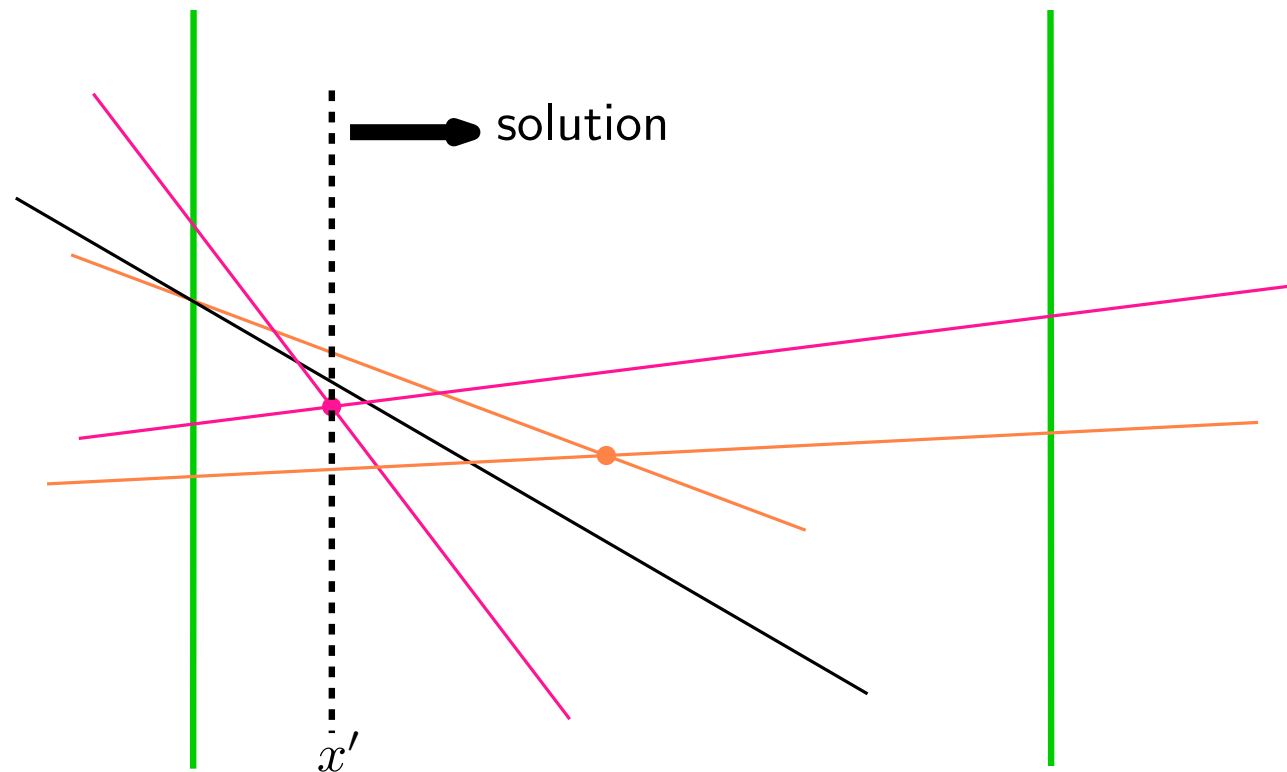
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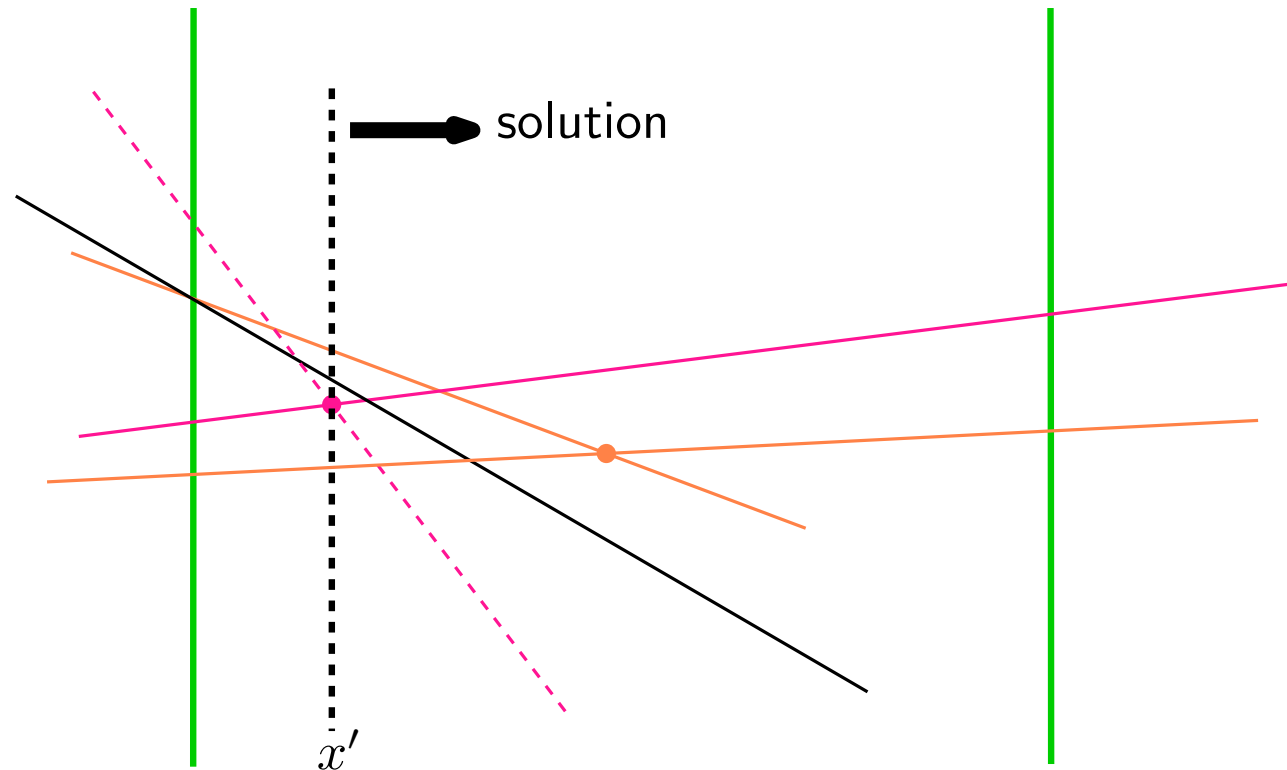
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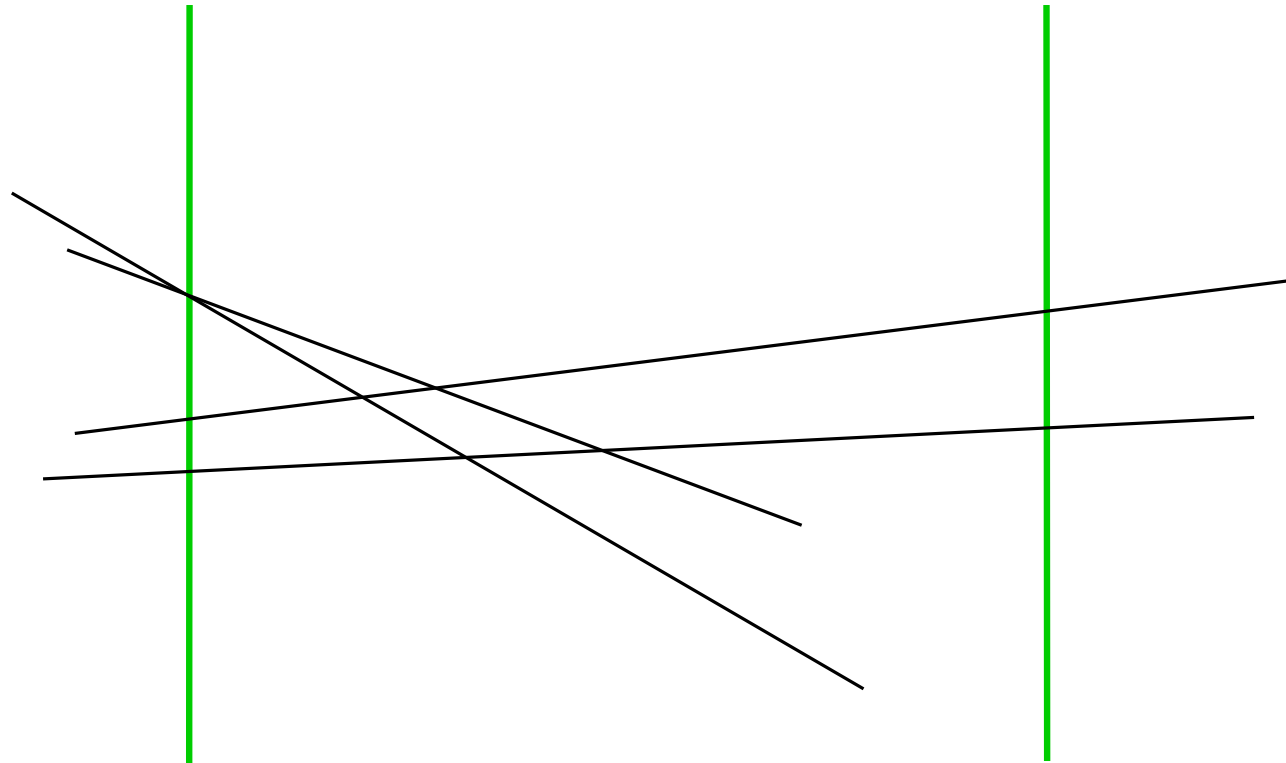
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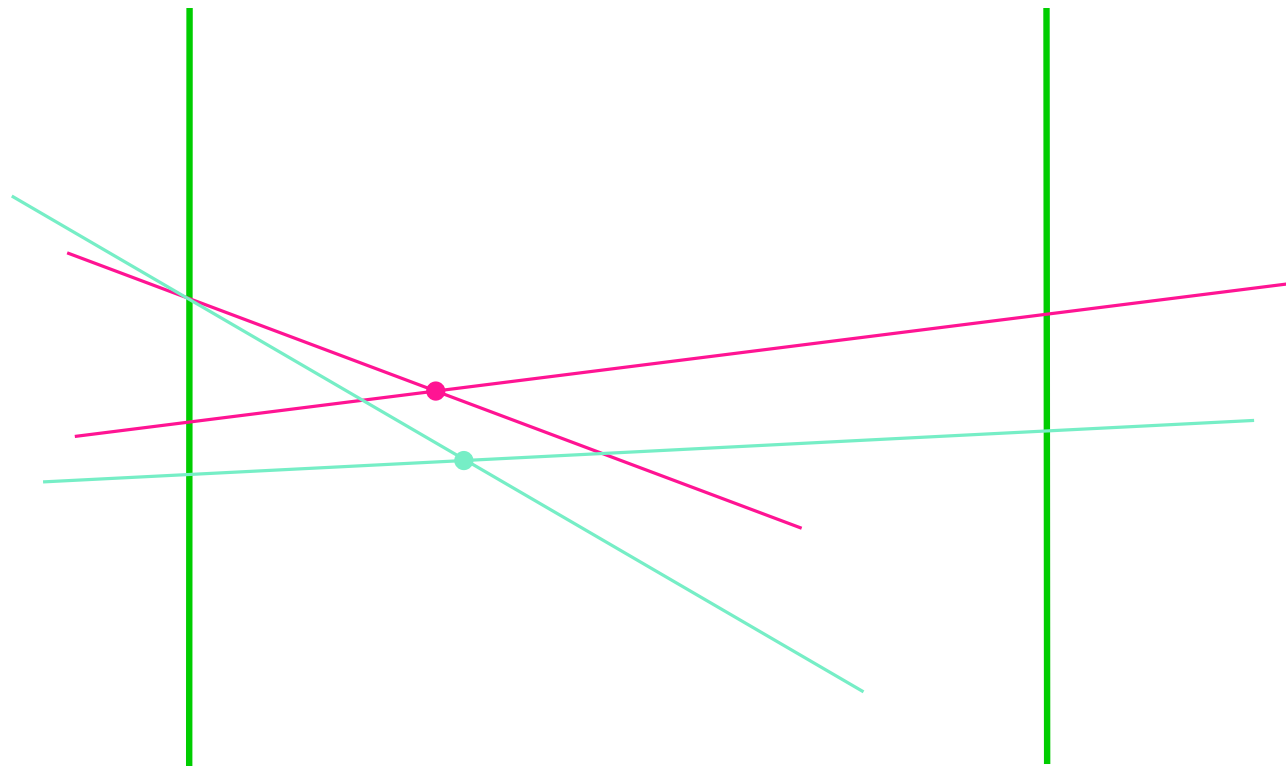
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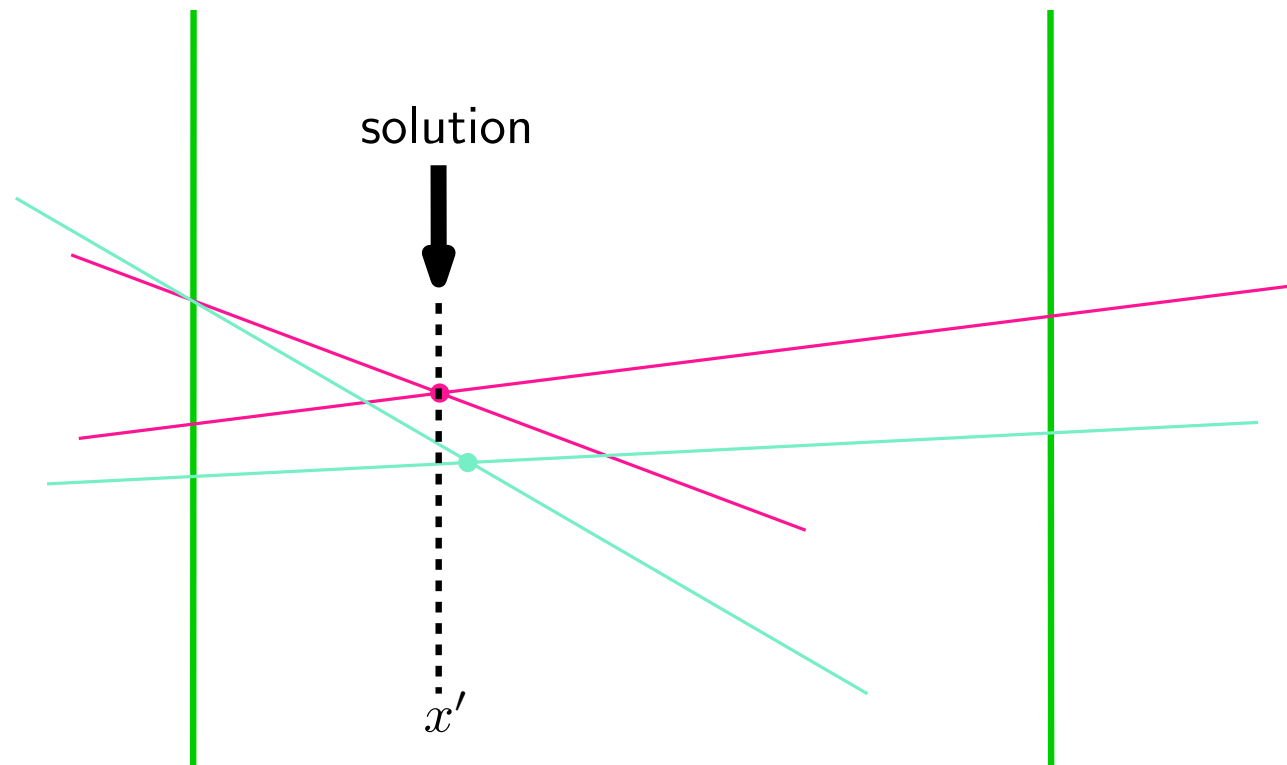
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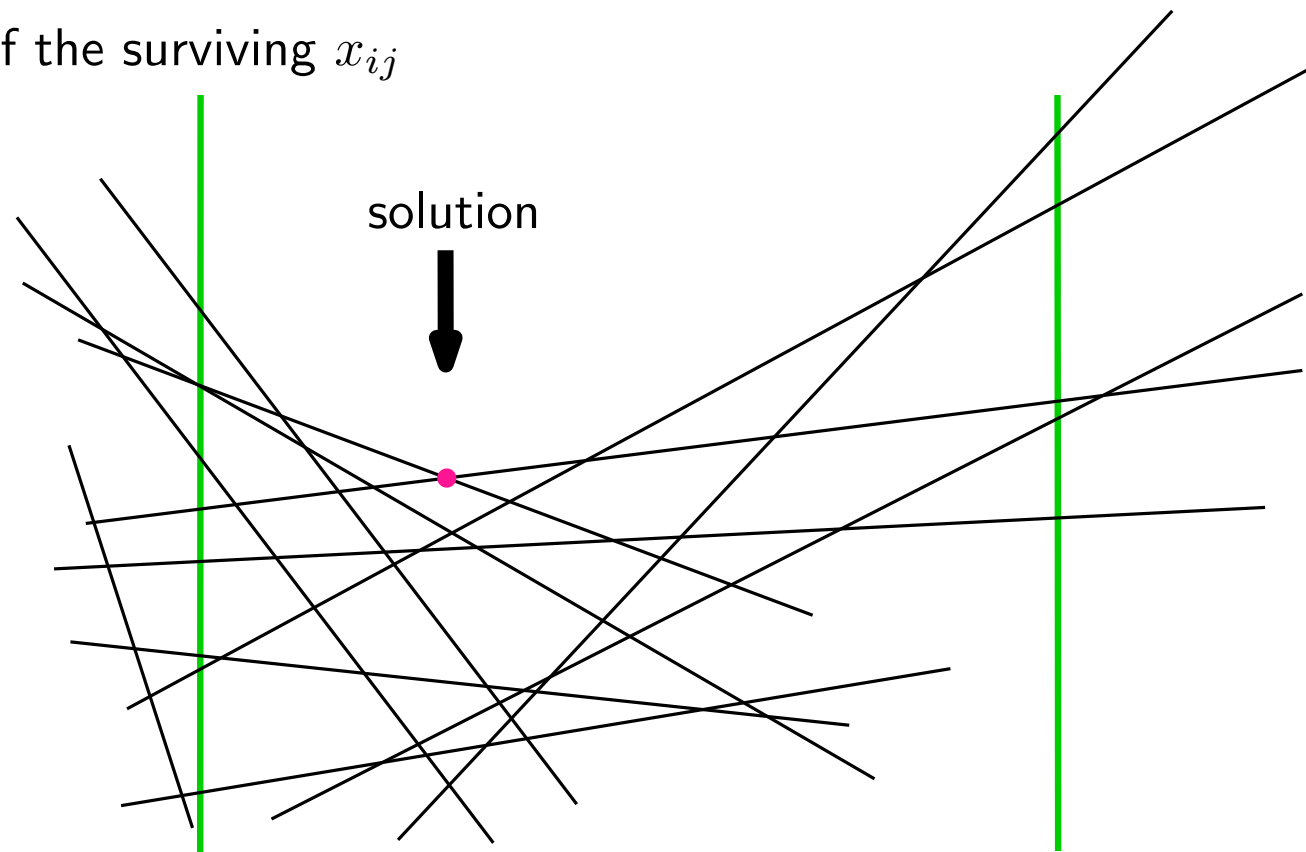
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SOLVING LINEAR PROGRAMS

The algorithm

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Advance. Repeat as many times as necessary:

1. Pair up restrictions and compute the abscissa of the intersection points. $O(n)$
2. For all pairs, do: $O(n)$
 - If $d_i = d_j$, eliminate one of the two.
 - If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
 - If $x_{ij} < u_1$, eliminate one of the two.
 - If $x_{ij} > u_2$, eliminate one of the two.
3. Compute the median value x' of the surviving x_{ij} . $O(n)$ (see the appropriate reference)
4. Search: $O(n)$
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SOLVING LINEAR PROGRAMS

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4. Search: $O(n)$
5. Prune: $O(n)$

How many restrictions are pruned at each step?

SOLVING LINEAR PROGRAMS

The algorithm

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How many restrictions are pruned at each step? $O(n/4)$

SOLVING LINEAR PROGRAMS

The algorithm

Exact counting:

- The initial number of vertical restrictions is k , where $k \geq 0$.
- The vertical restrictions prune step eliminates all of them but at most 2.
- The initial number of pairs of non vertical lines (i.e., points x_{ij}) is $\frac{m}{2}$, $\frac{m-1}{2}$ or $\frac{m}{2} - 1$, depending on the parities of $|I_-|$ and $|I_+|$, where $m + k = n$.
- The prune step eliminates roughly half of them.

Alltogether, the eliminated restrictions are, at least:

$$k + \left\lfloor \frac{\frac{m}{2} - 1}{2} \right\rfloor = \left\lfloor \frac{m}{4} + k - \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{4} + \frac{3}{4}k - \frac{1}{2} \right\rfloor > \frac{n}{4} - 1.$$

SOLVING LINEAR PROGRAMS

The algorithm

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Running time

Hence, the running time of the algorithm is

$$cn + c\frac{3}{4}n + c\left(\frac{3}{4}\right)^2 n + c\left(\frac{3}{4}\right)^3 n + \dots = cn \sum_{k=0}^{\lceil \log_{3/4} n \rceil} \left(\frac{3}{4}\right)^k < cn \frac{1}{1 - \frac{3}{4}} = 4cn \in O(n).$$

COMPUTING THE MINIMUM SPANNING CIRCLE

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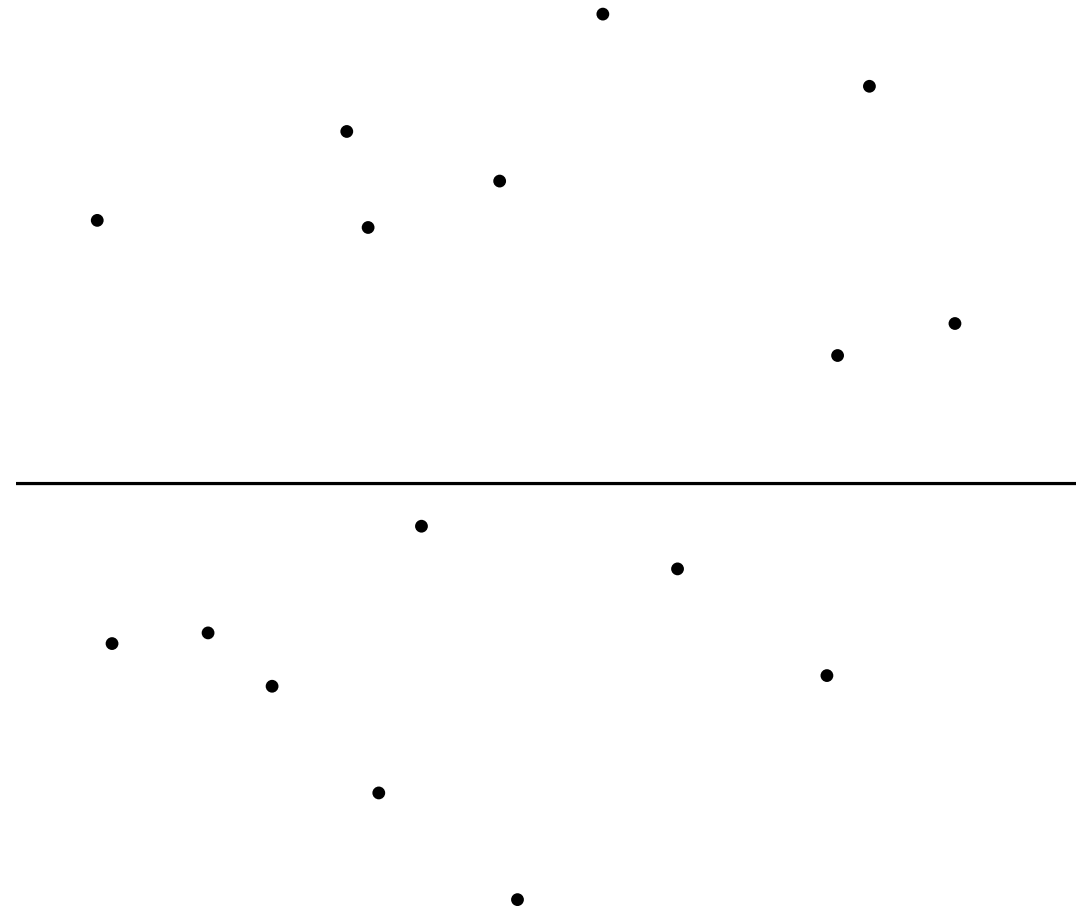
MSC restricted to a line

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

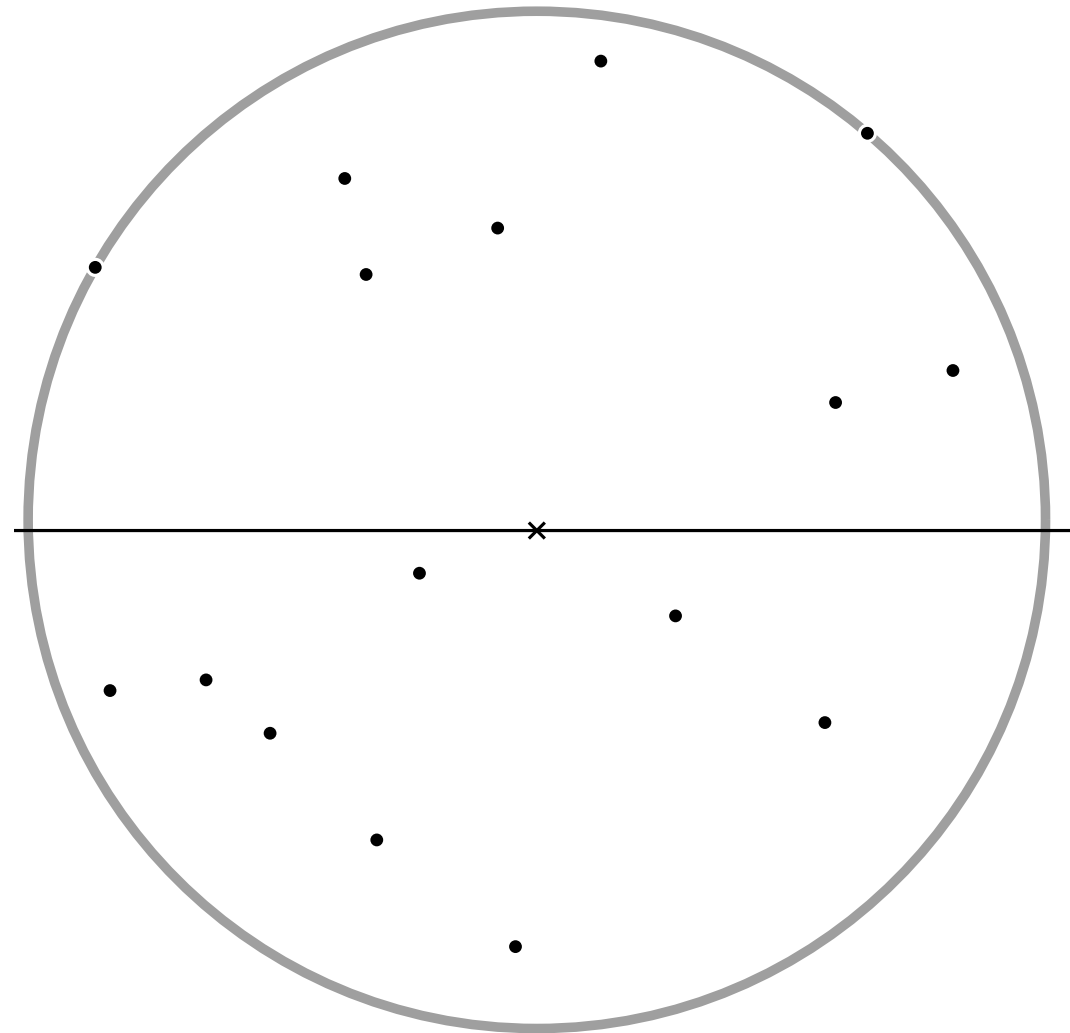


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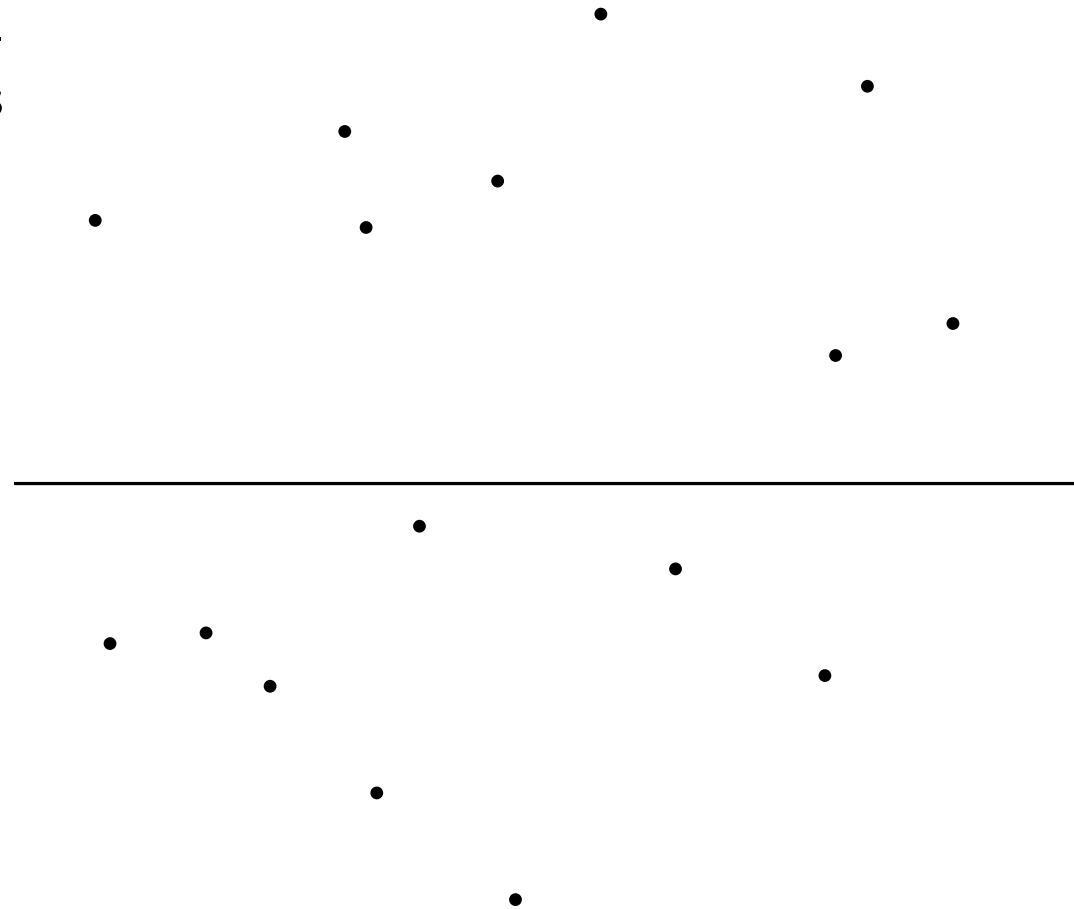
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1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



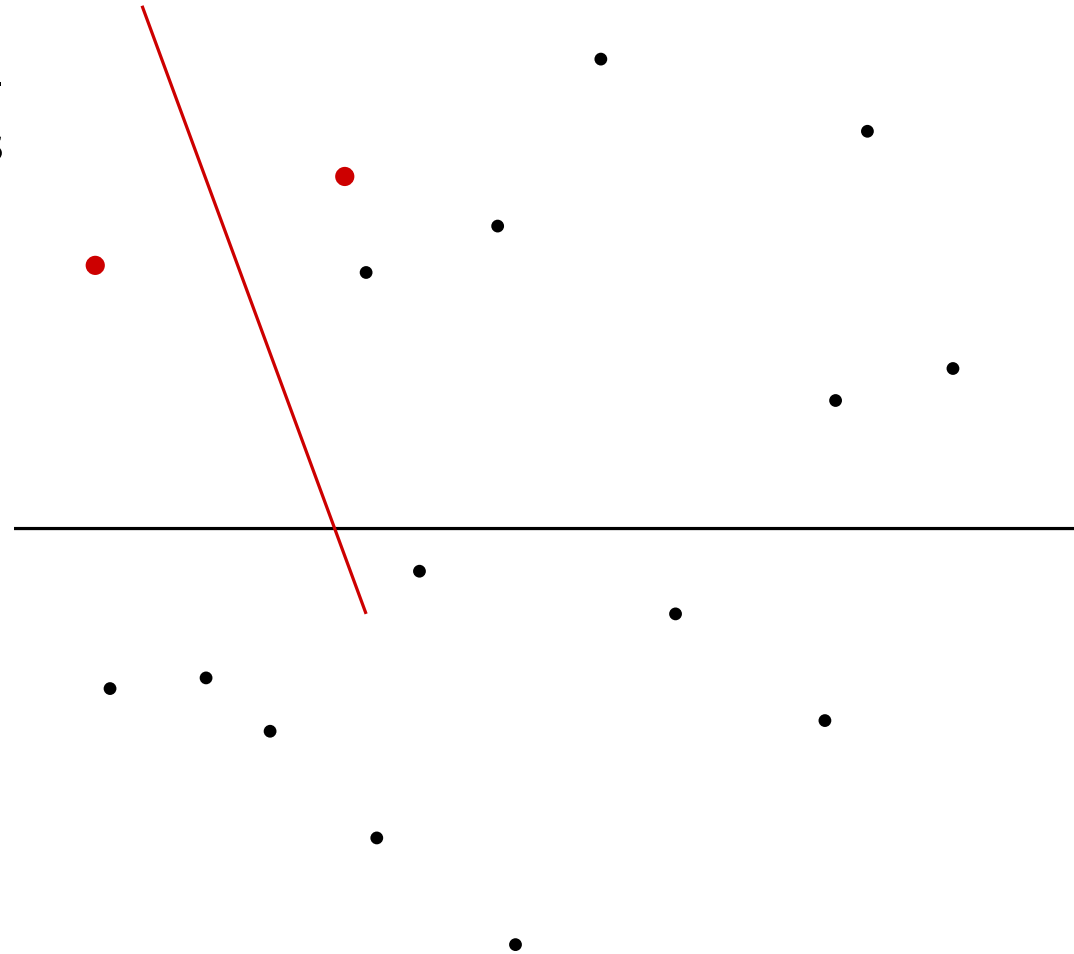
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Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



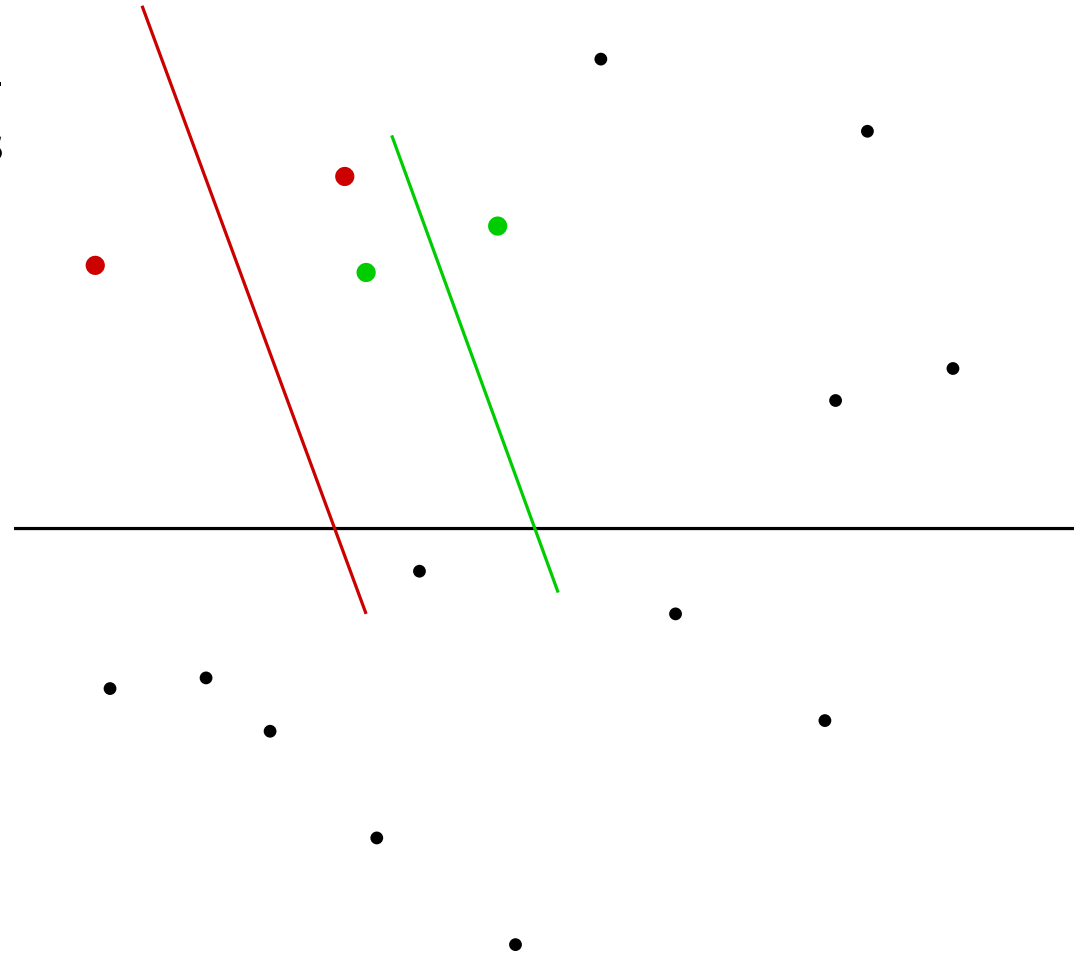
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

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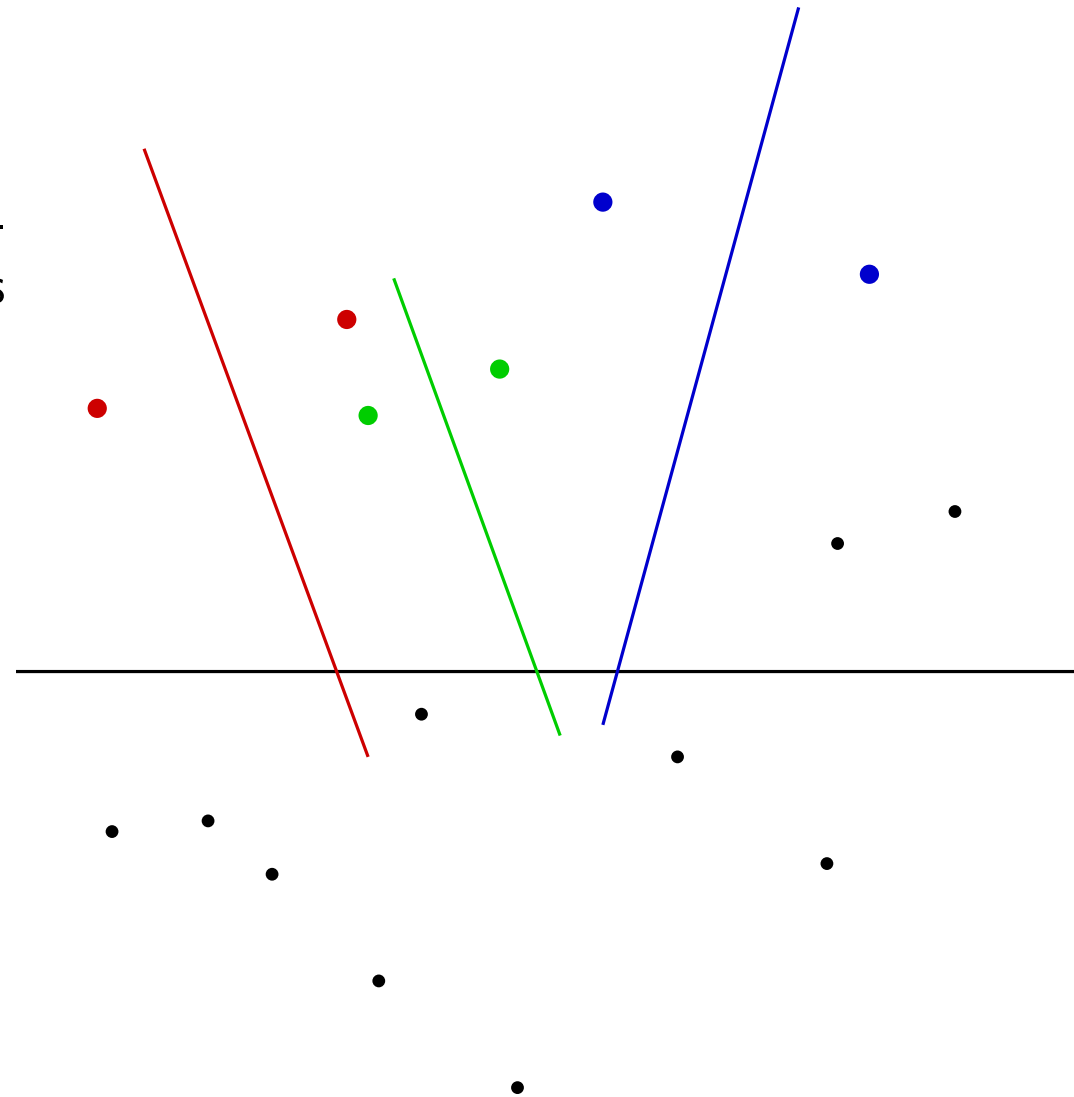
COMPUTING THE MINIMUM SPANNING CIRCLE

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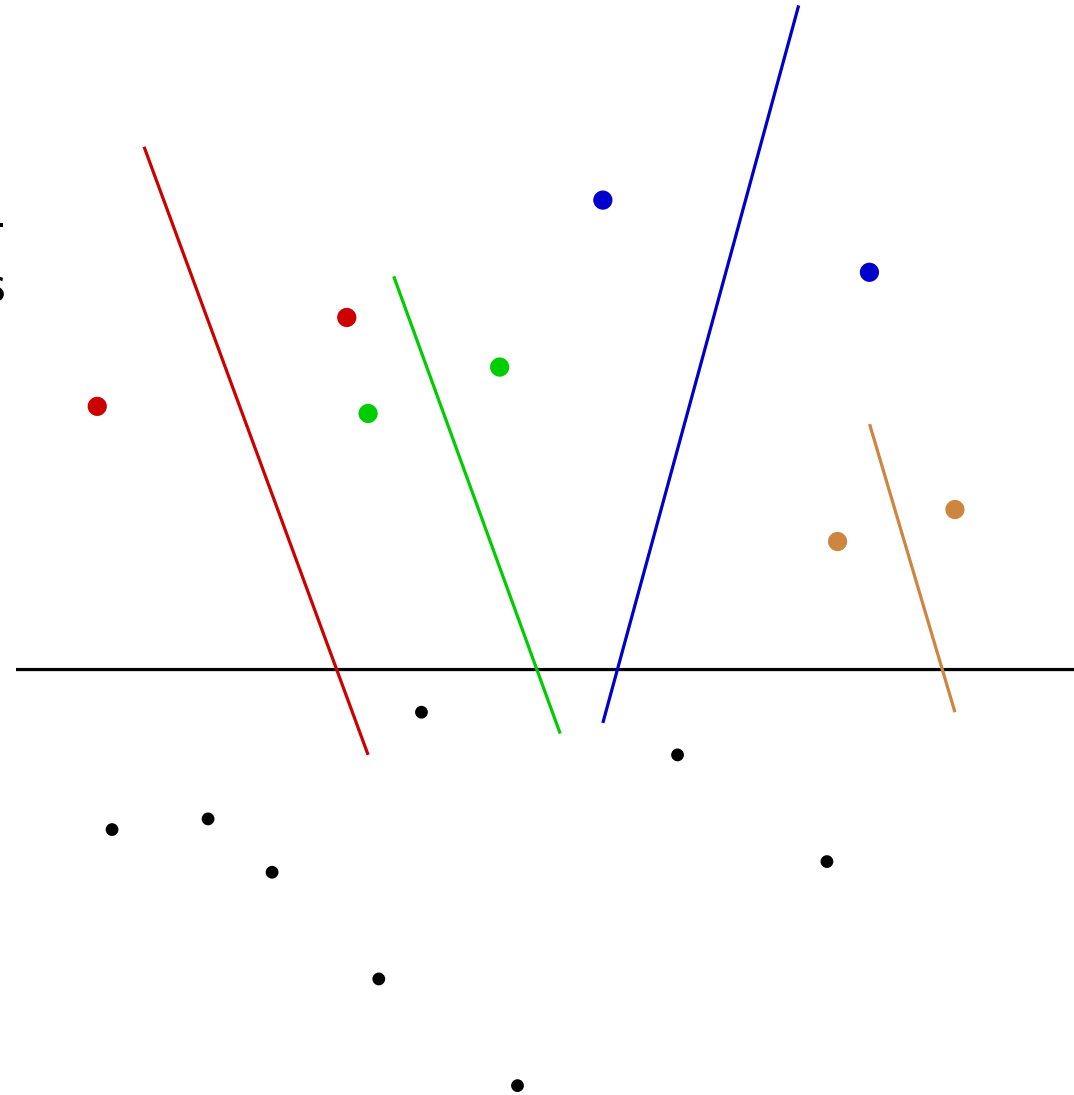
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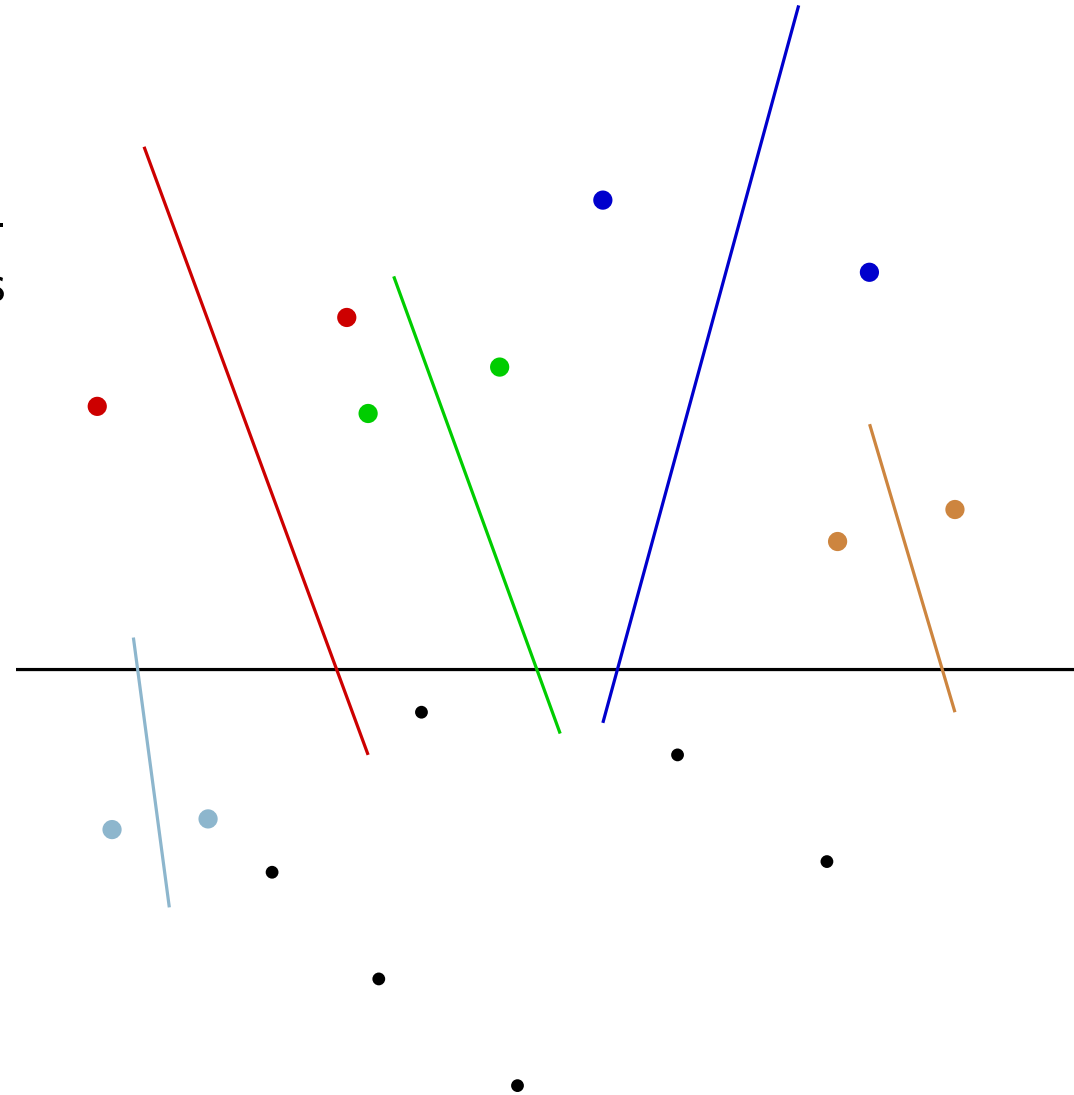
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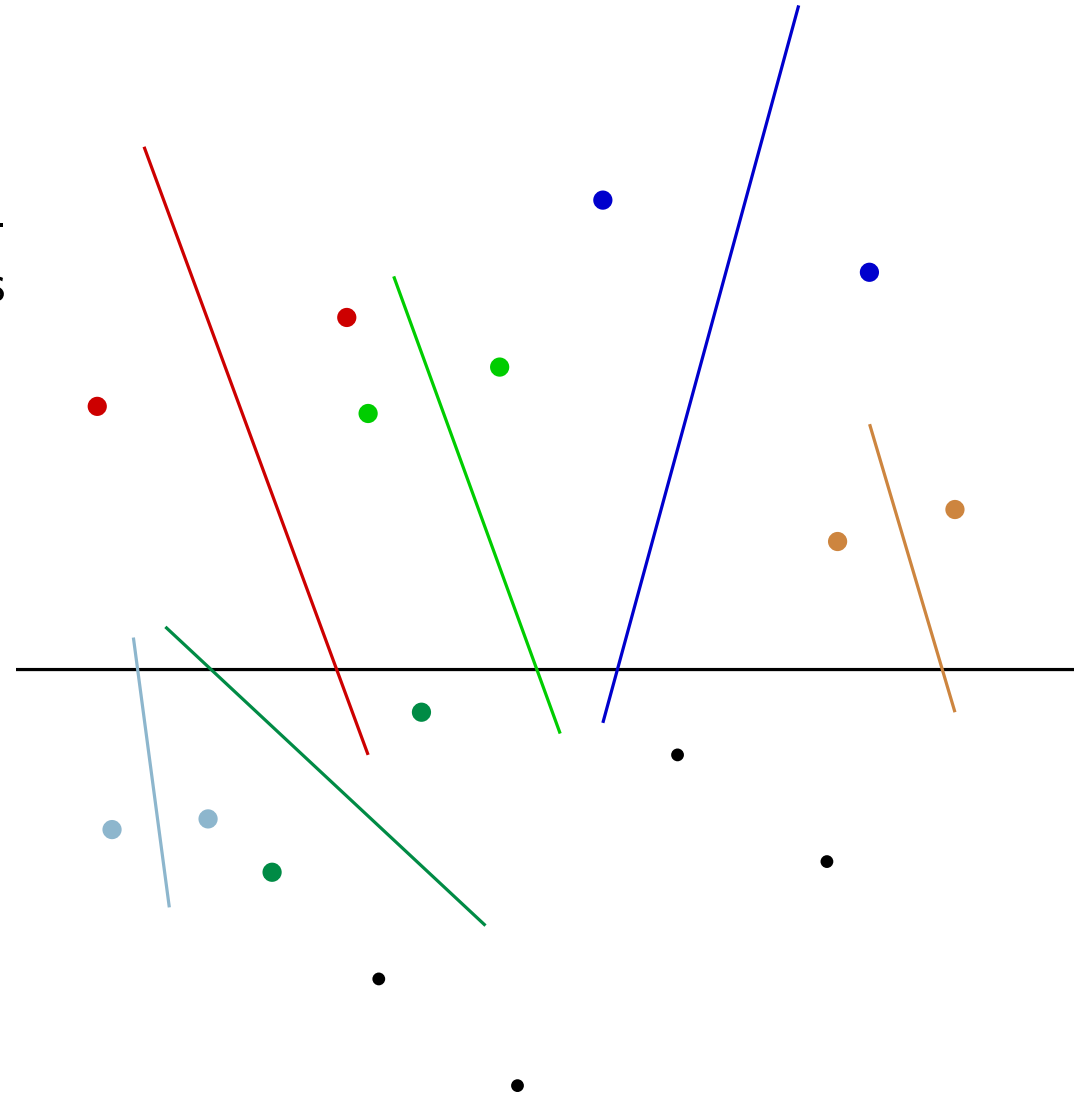
COMPUTING THE MINIMUM SPANNING CIRCLE

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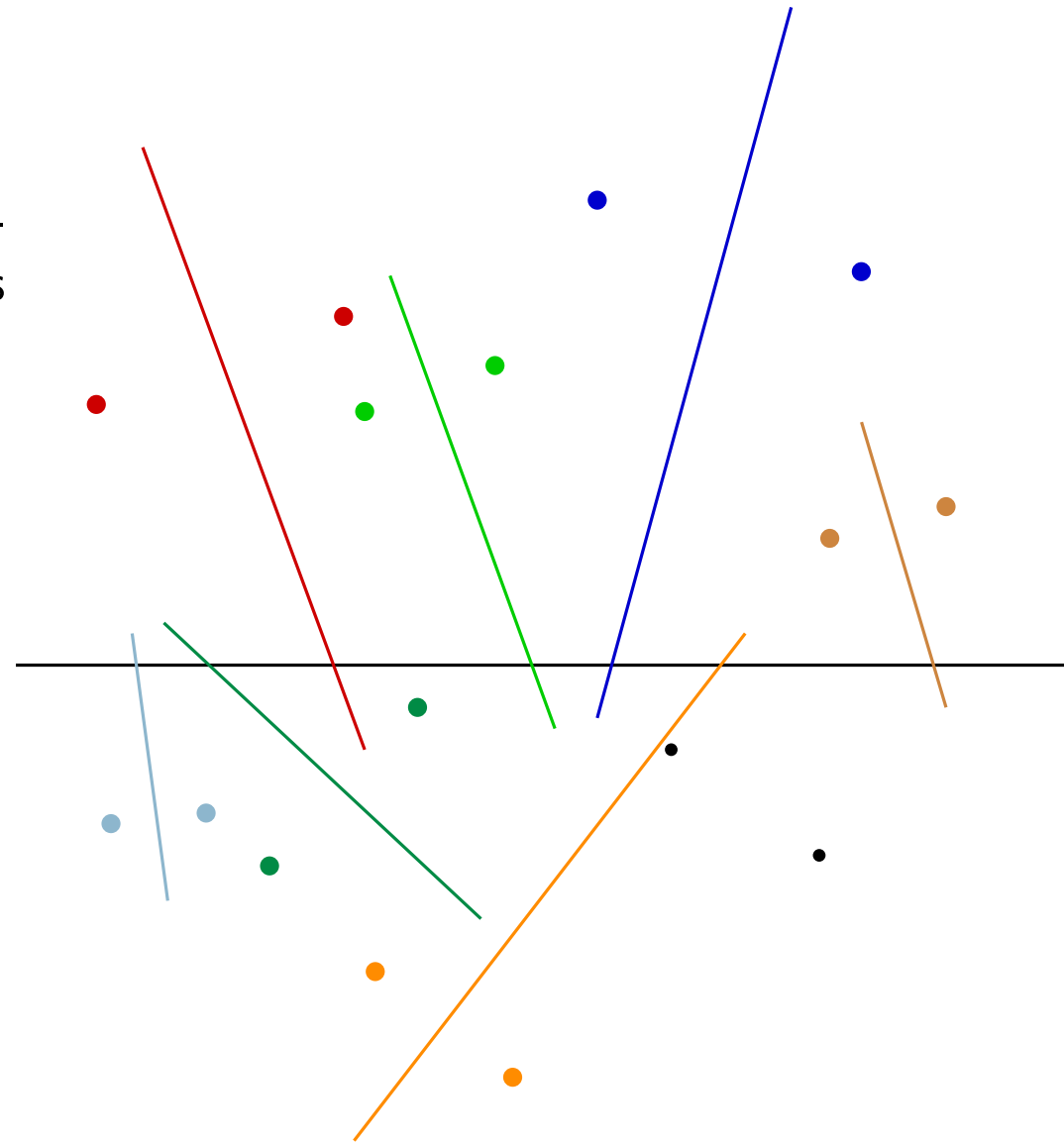
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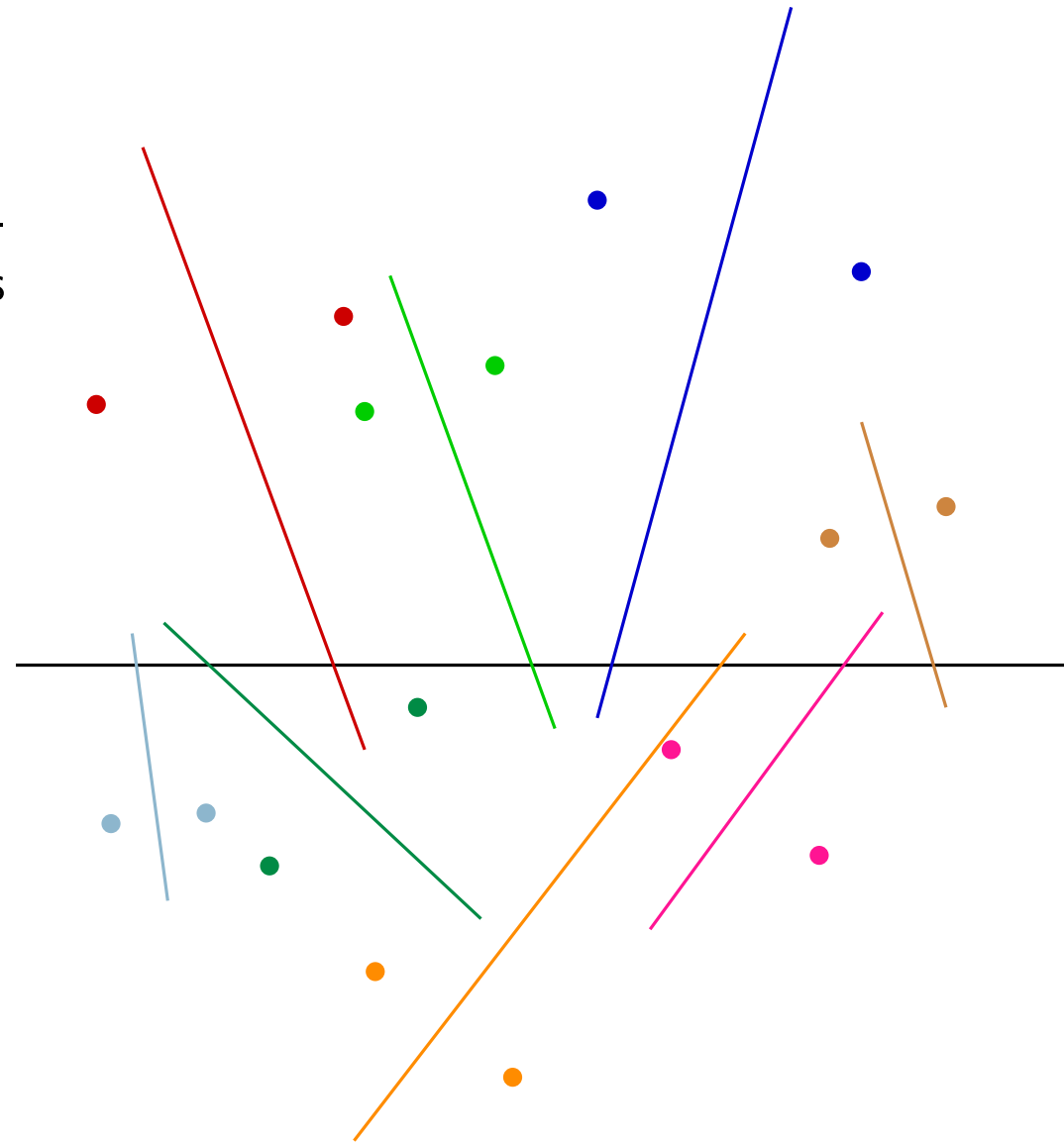
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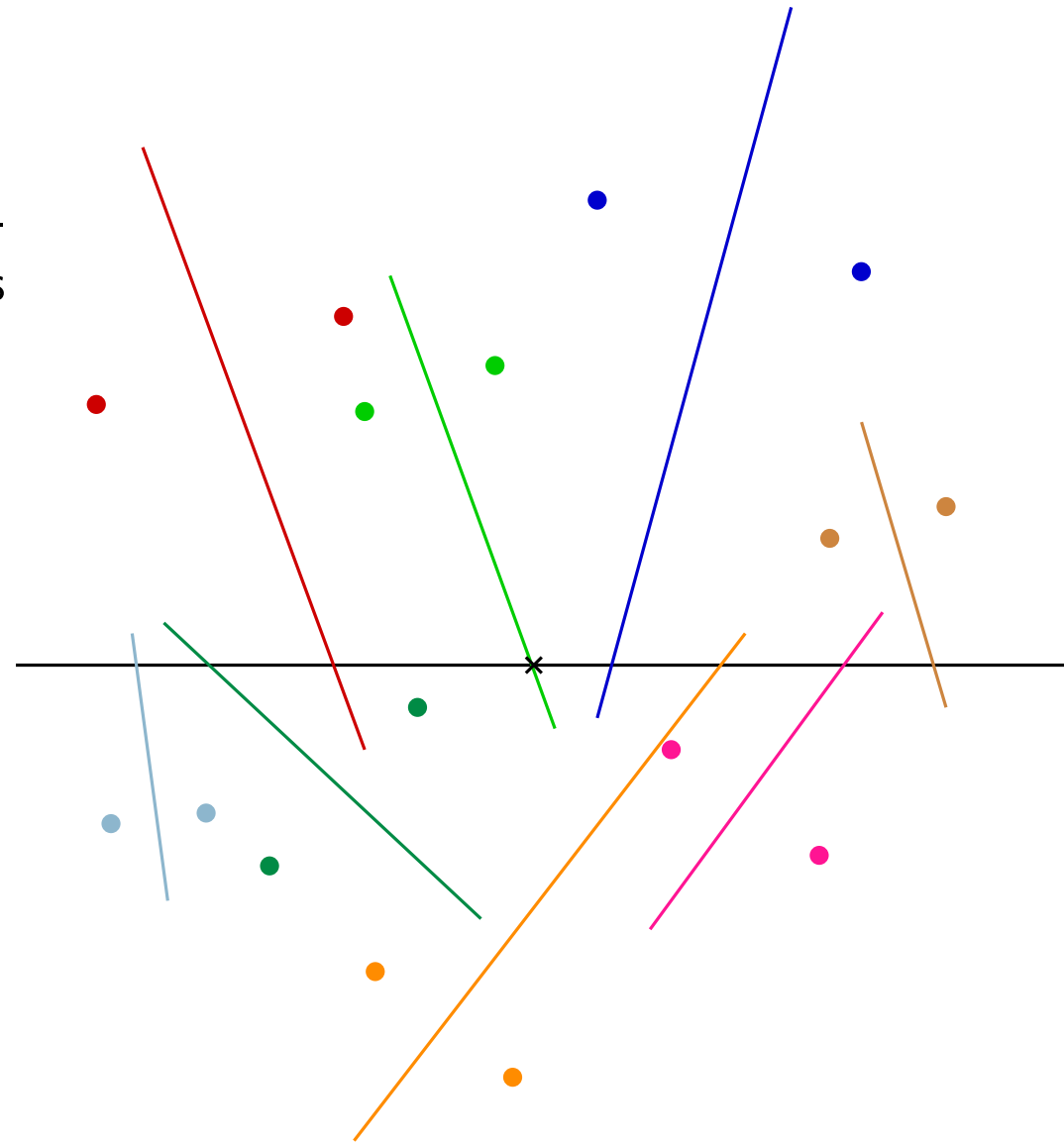
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1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m , the median value of the x_{ij} .



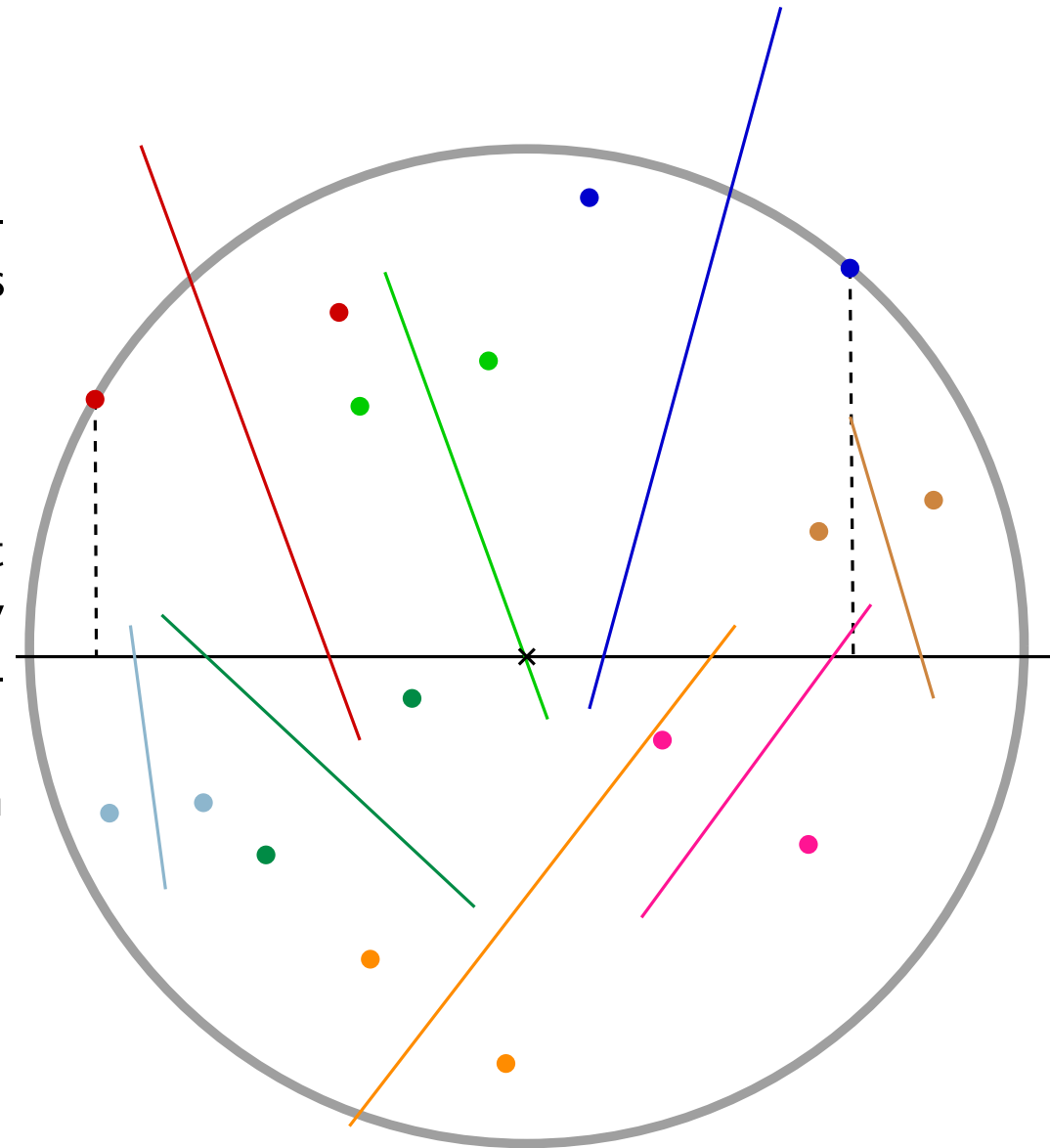
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Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m , the median value of the x_{ij} .
3. Search: Compute C_m , the MSC centered at x_m . If the points p_i lying in the boundary of C_m project orthogonally onto $y = 0$ to different sides of x_m , then x_m is the solution. If they all project onto the same side then search on that side.



COMPUTING THE MINIMUM SPANNING CIRCLE

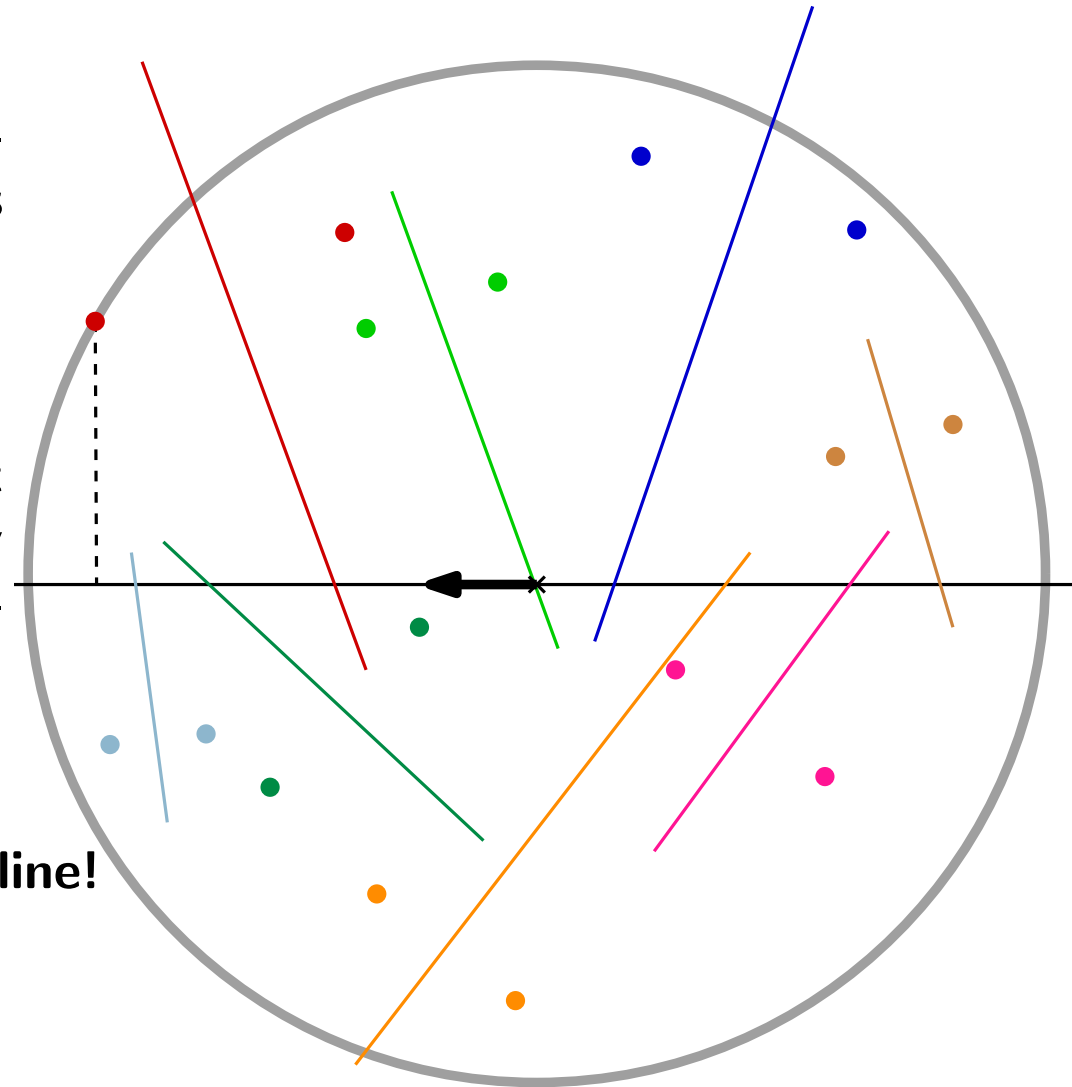
MSC restricted to a line

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Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

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Notice that the radius is unimodal along the line!



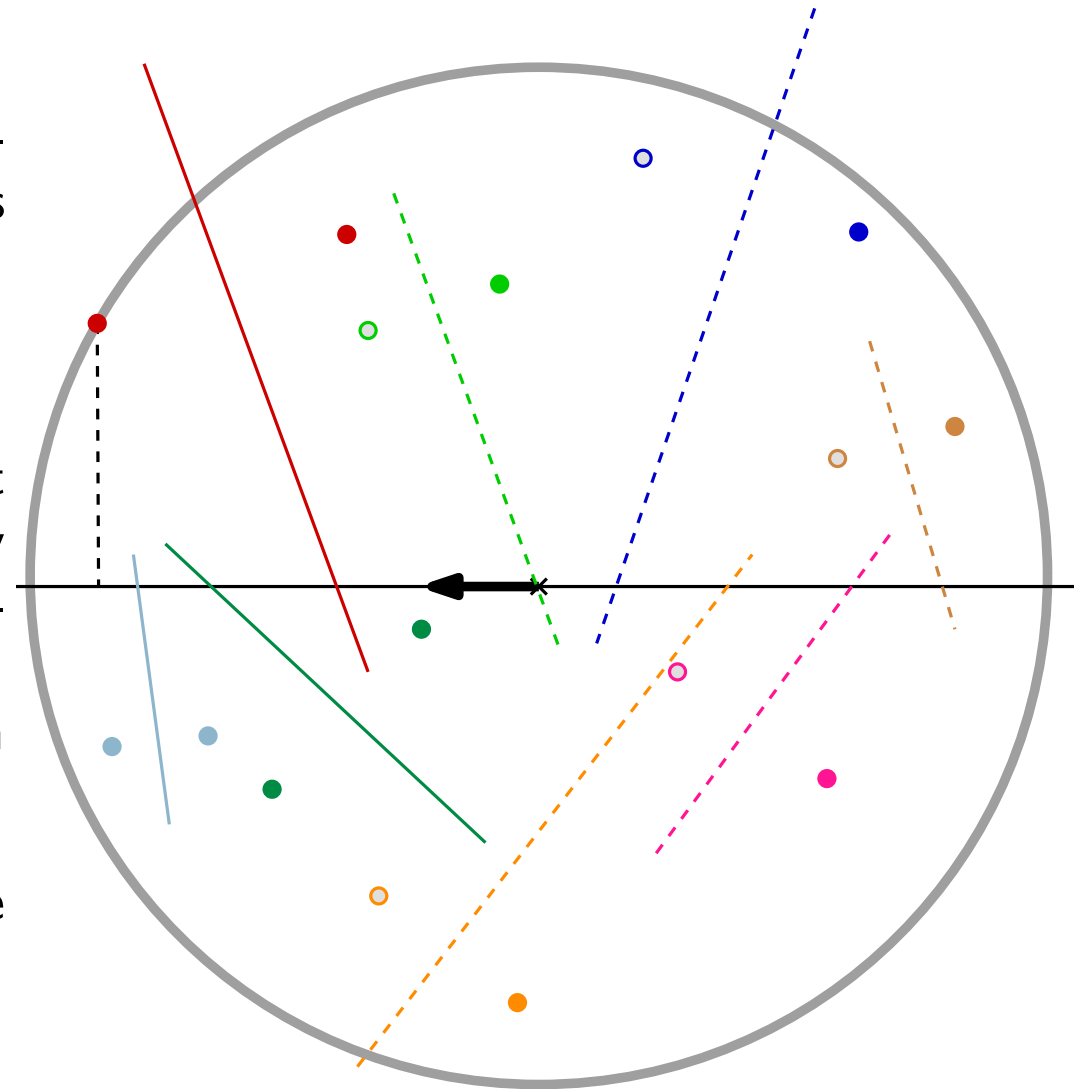
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4. Prune: For all x_{ij} located in the side opposite to the solution, eliminate the point p_i or p_j which is closest to x_m .



COMPUTING THE MINIMUM SPANNING CIRCLE

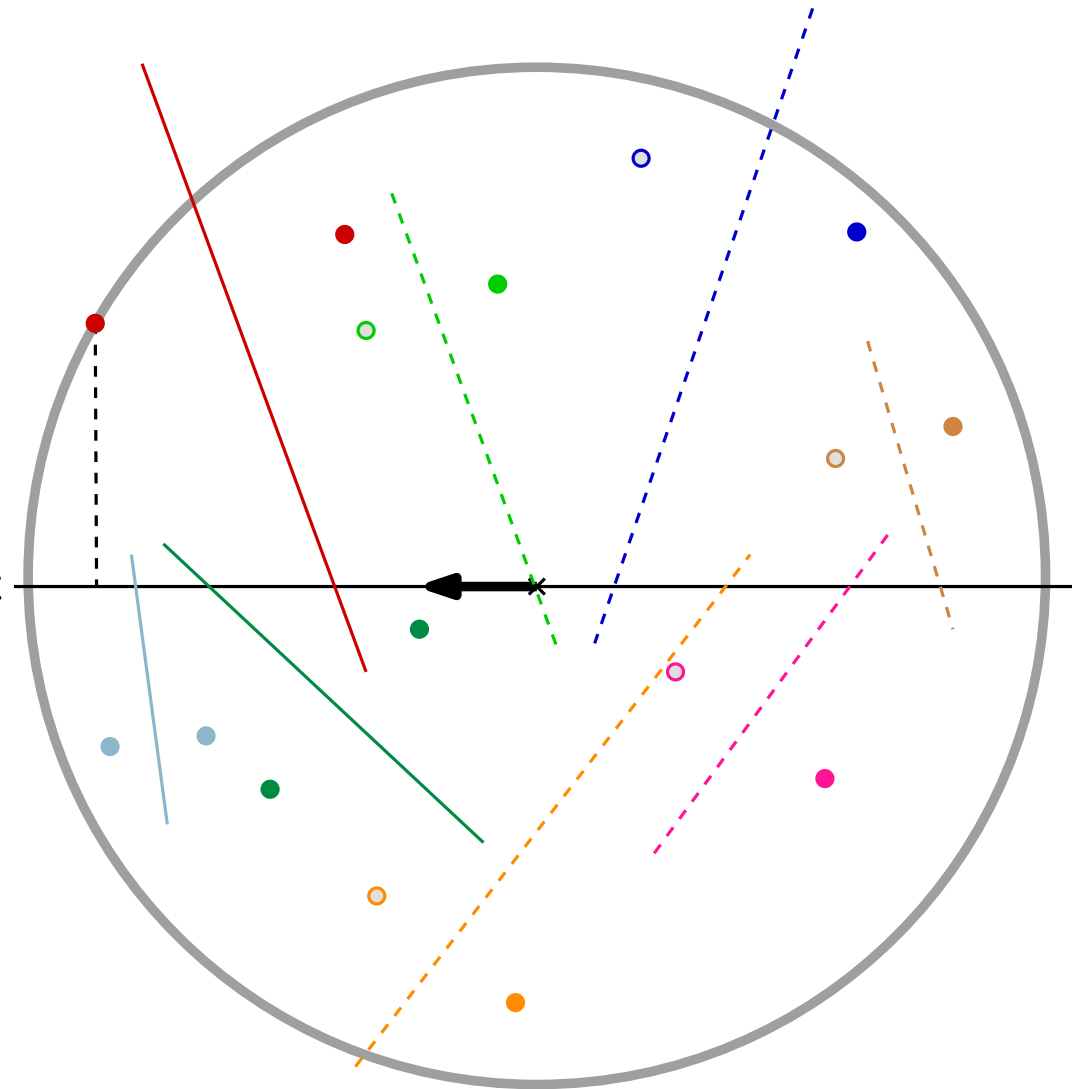
MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points, compute the perpendicular bisector of each pair, and find its intersection with the line $y = 0$. $O(n)$
2. Compute a median value. $O(n)$
3. Search: $O(n)$
4. Prune: In $O(n)$ time, at least $1/4$ of the input is pruned.

The problem is solved in $O(n)$ time.

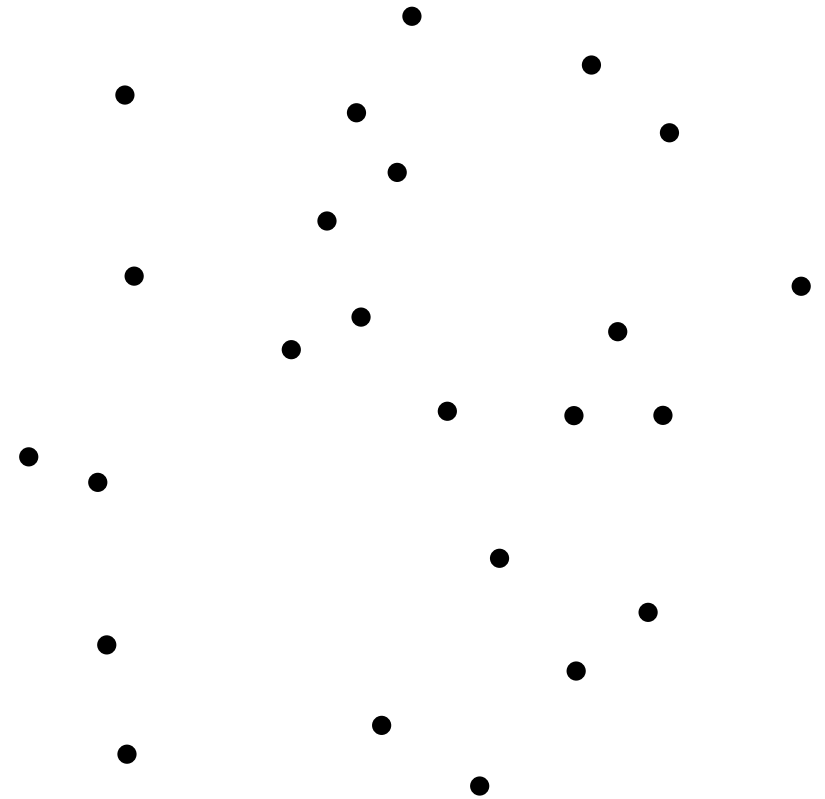


COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

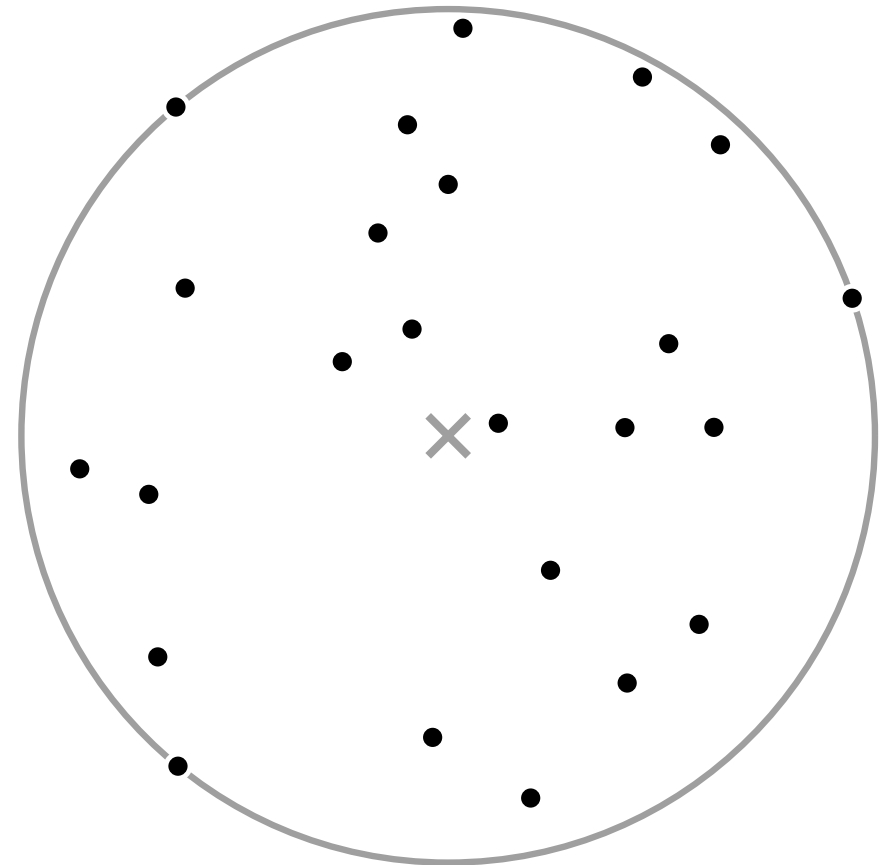
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane



COMPUTING THE MINIMUM SPANNING CIRCLE

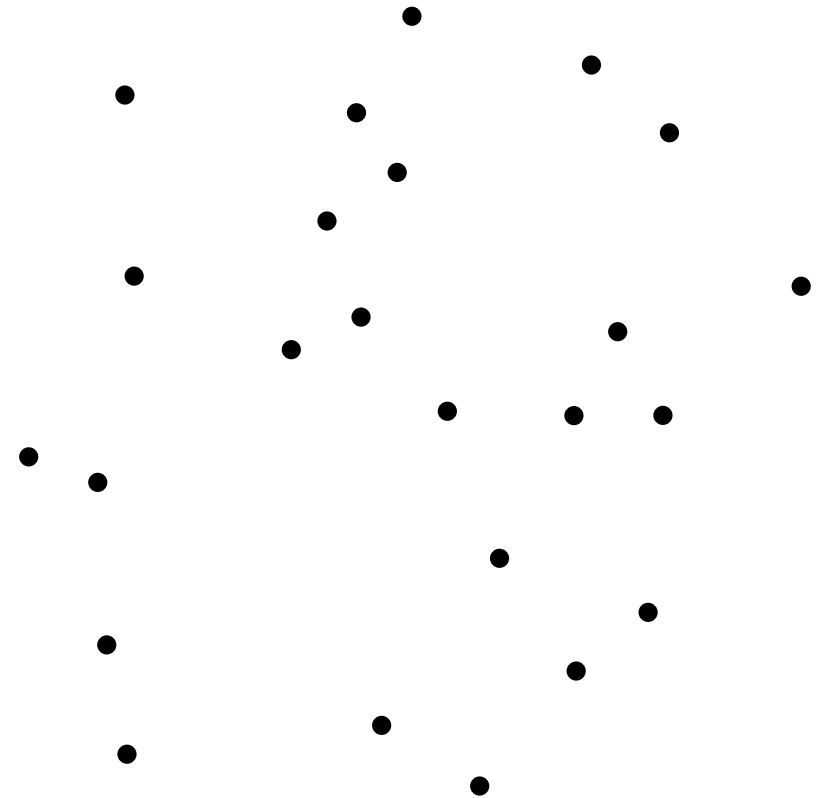
MSC in the plane



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

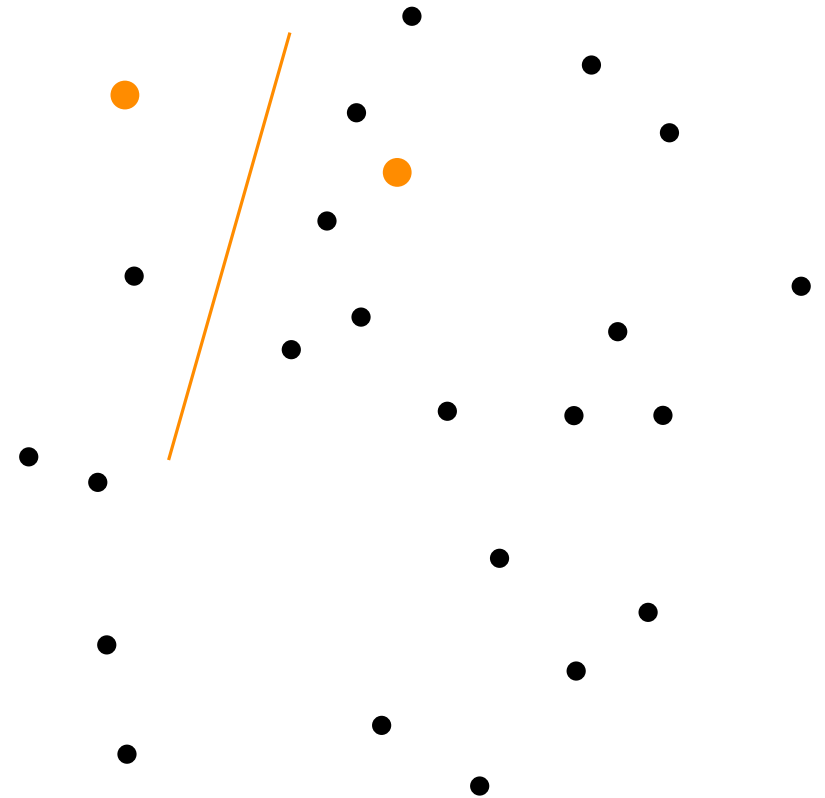
1. Pair up the points, and compute the perpendicular bisector of each pair.



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

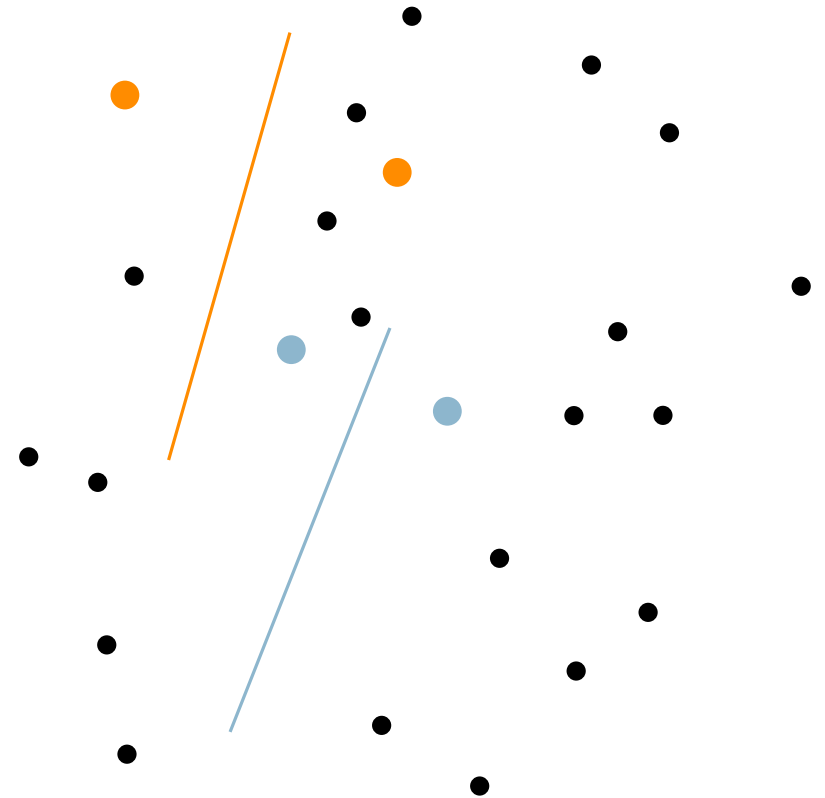
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COMPUTING THE MINIMUM SPANNING CIRCLE

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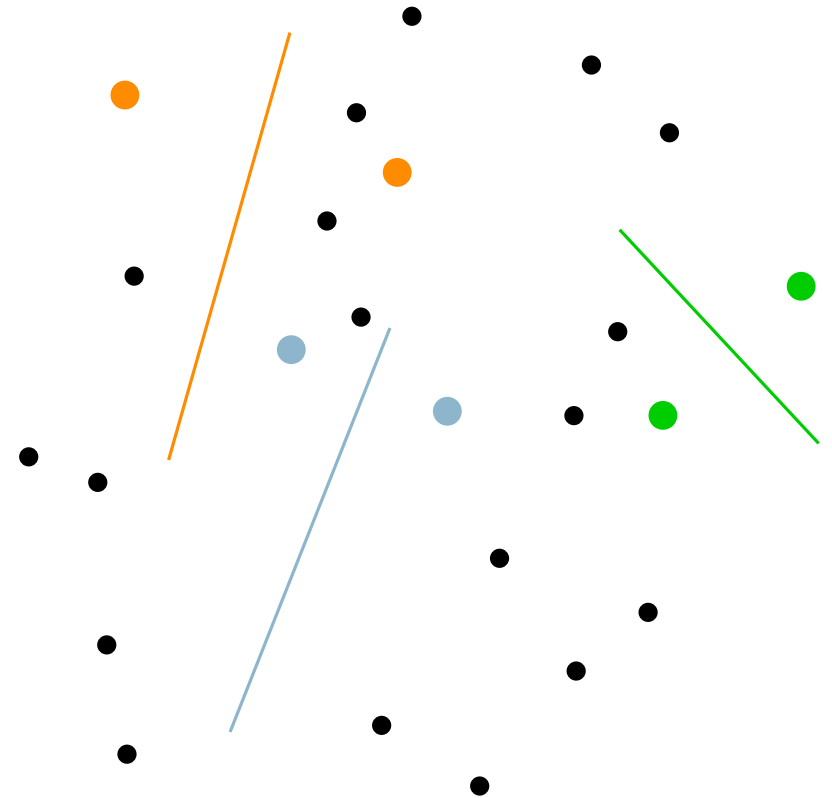
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MSC in the plane

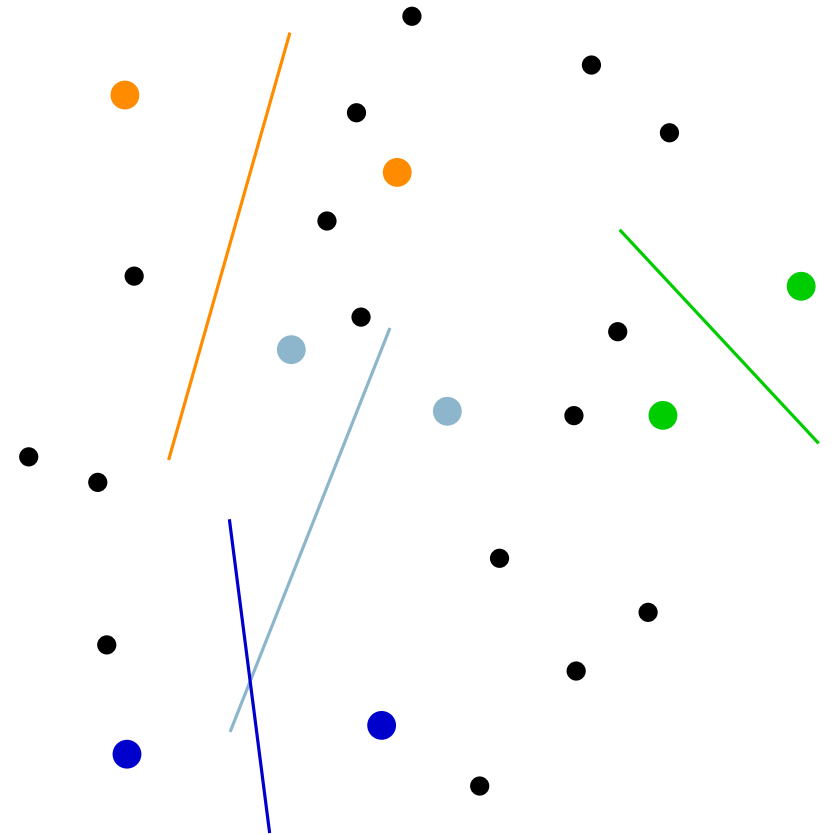
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COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

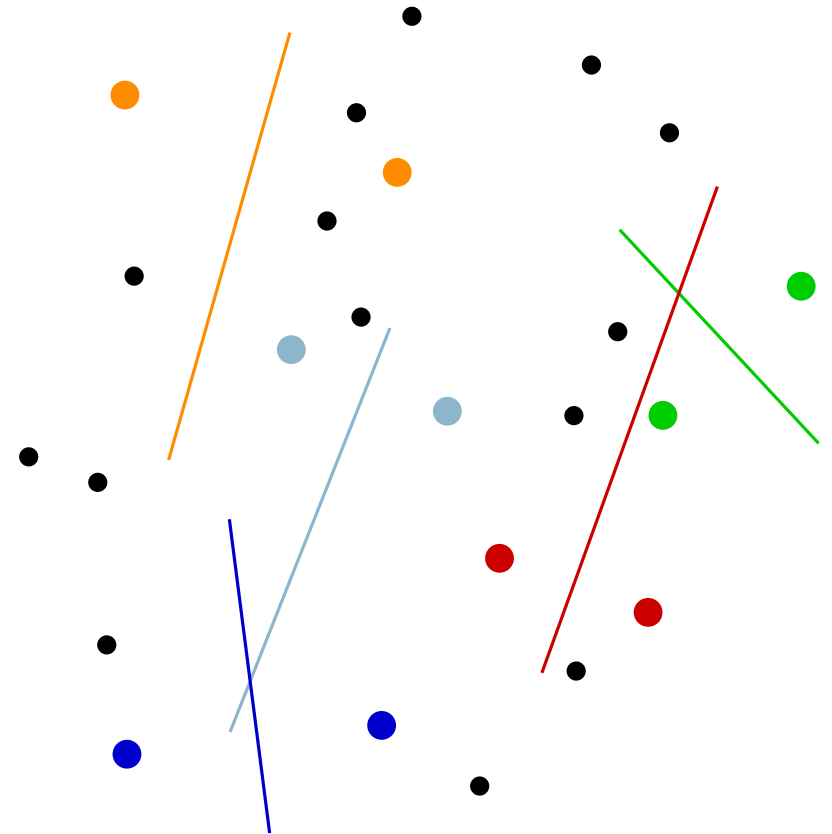
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COMPUTING THE MINIMUM SPANNING CIRCLE

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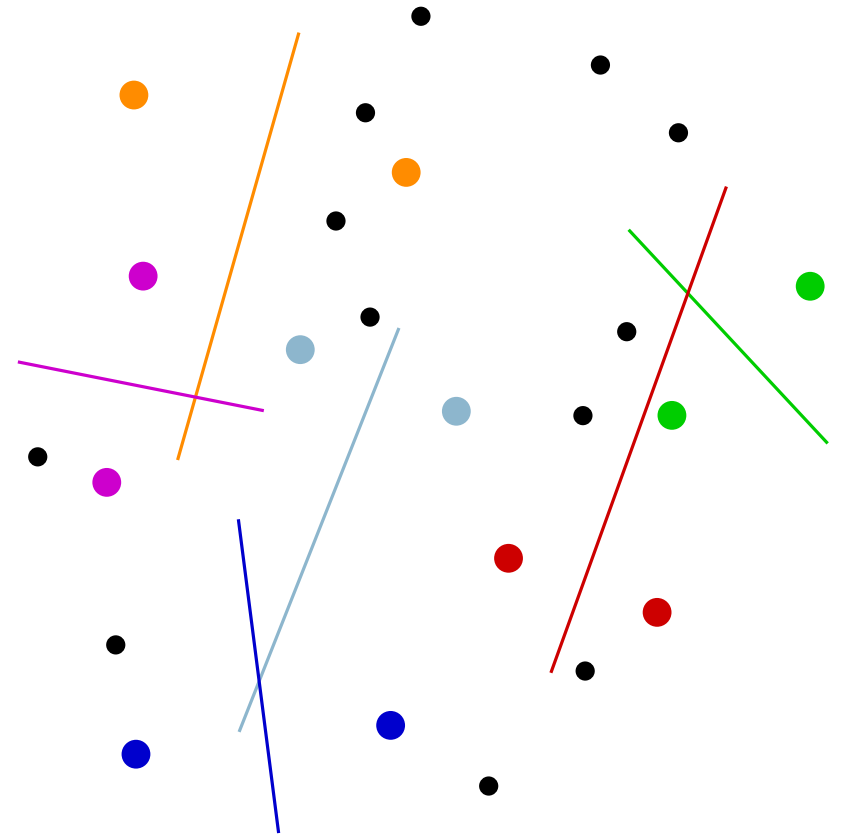
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COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

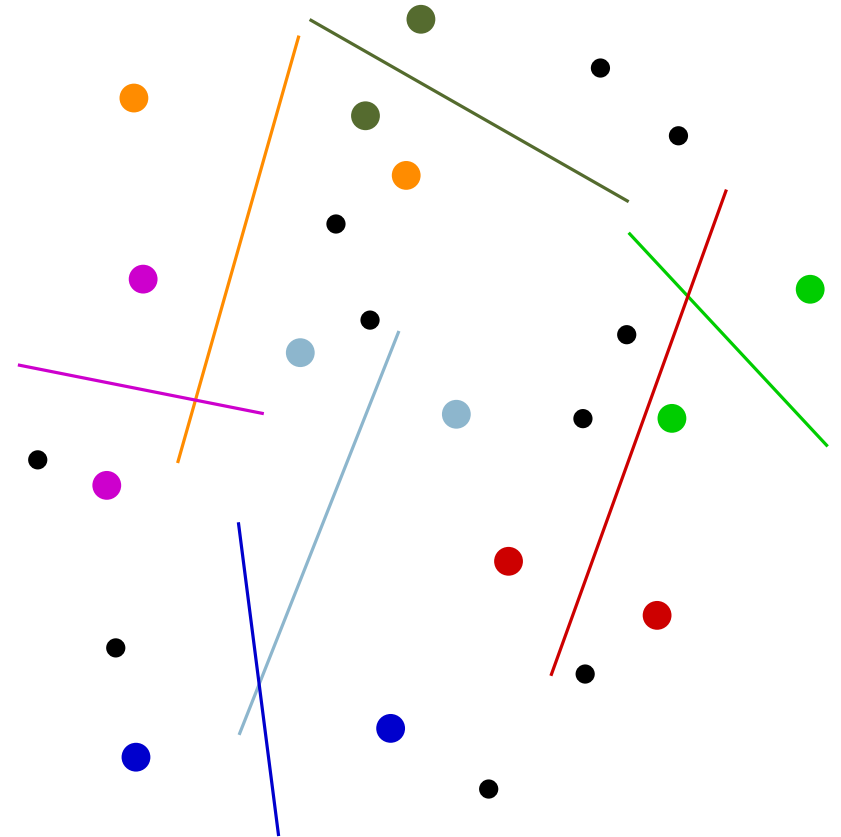
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COMPUTING THE MINIMUM SPANNING CIRCLE

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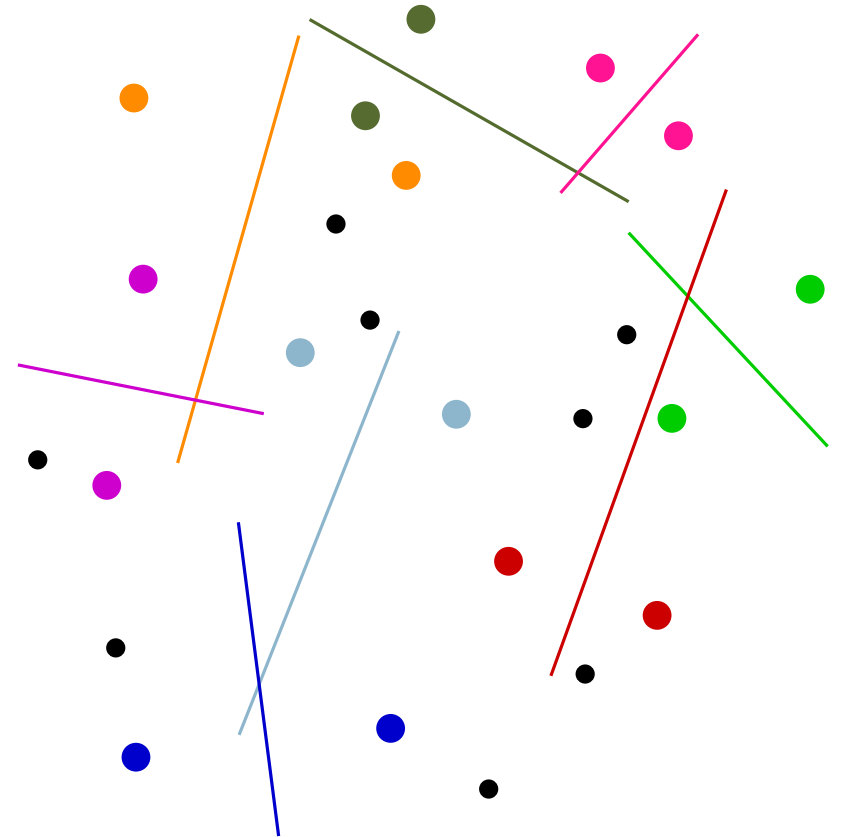
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COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

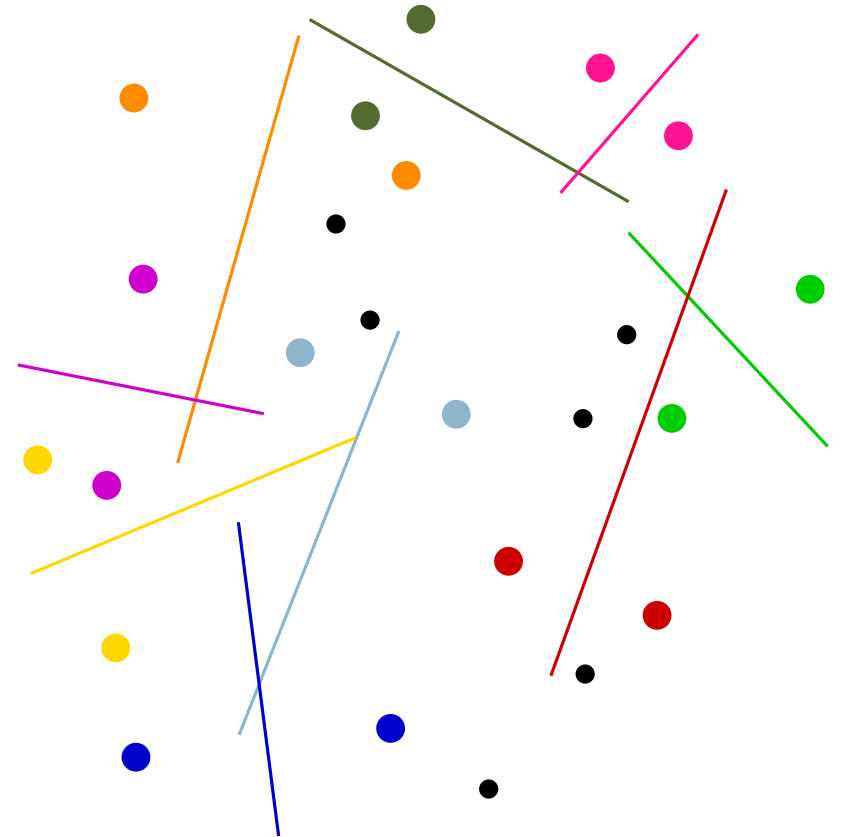
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COMPUTING THE MINIMUM SPANNING CIRCLE

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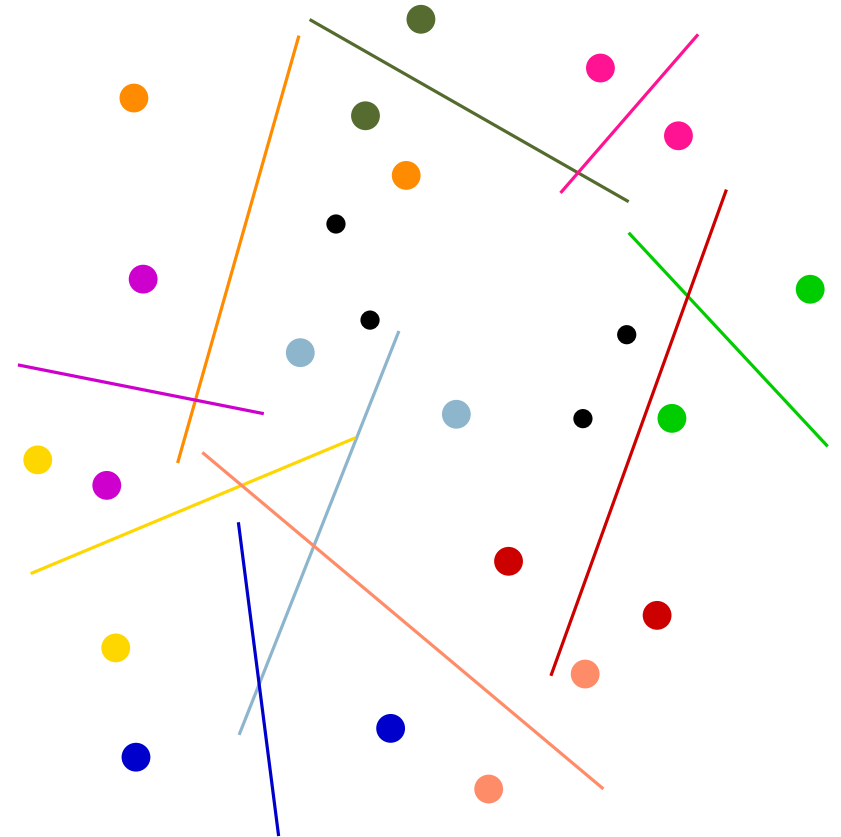
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COMPUTING THE MINIMUM SPANNING CIRCLE

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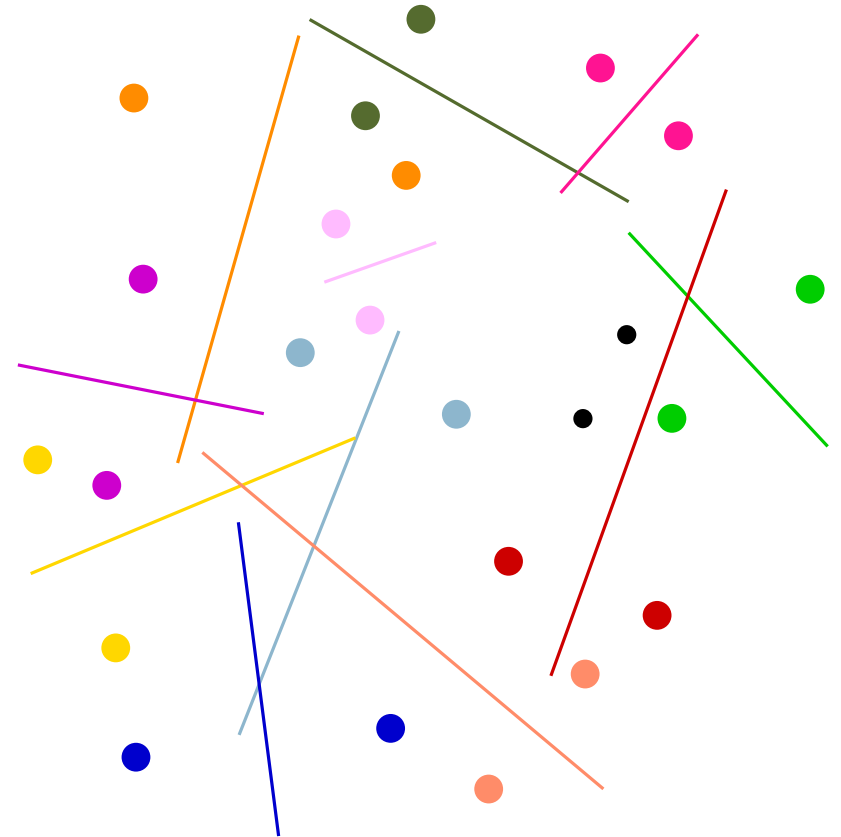
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COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

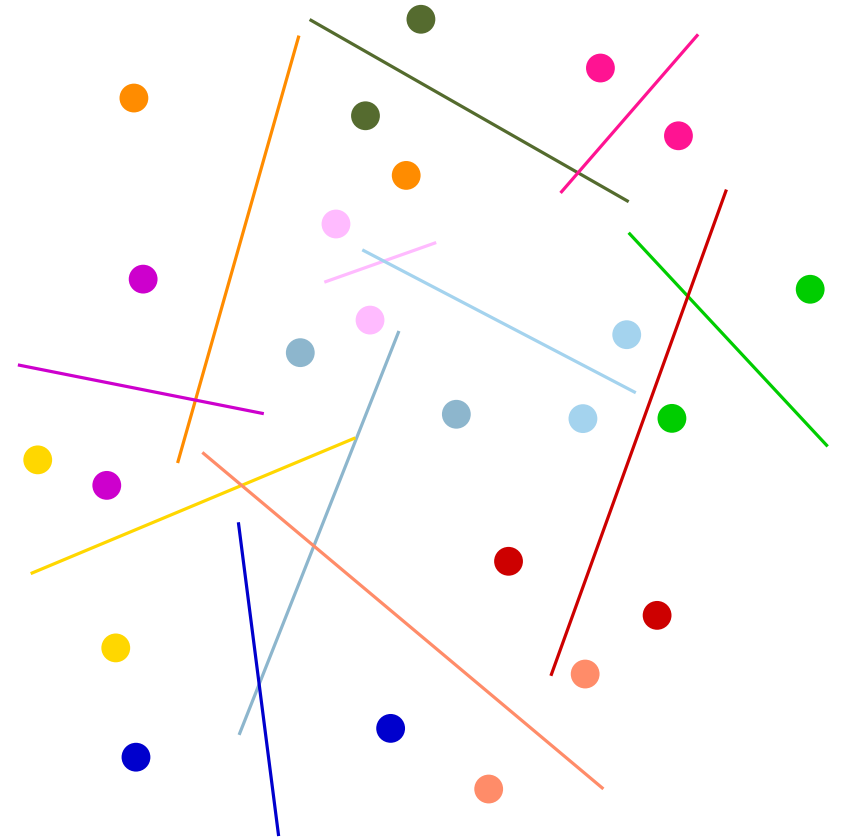
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COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

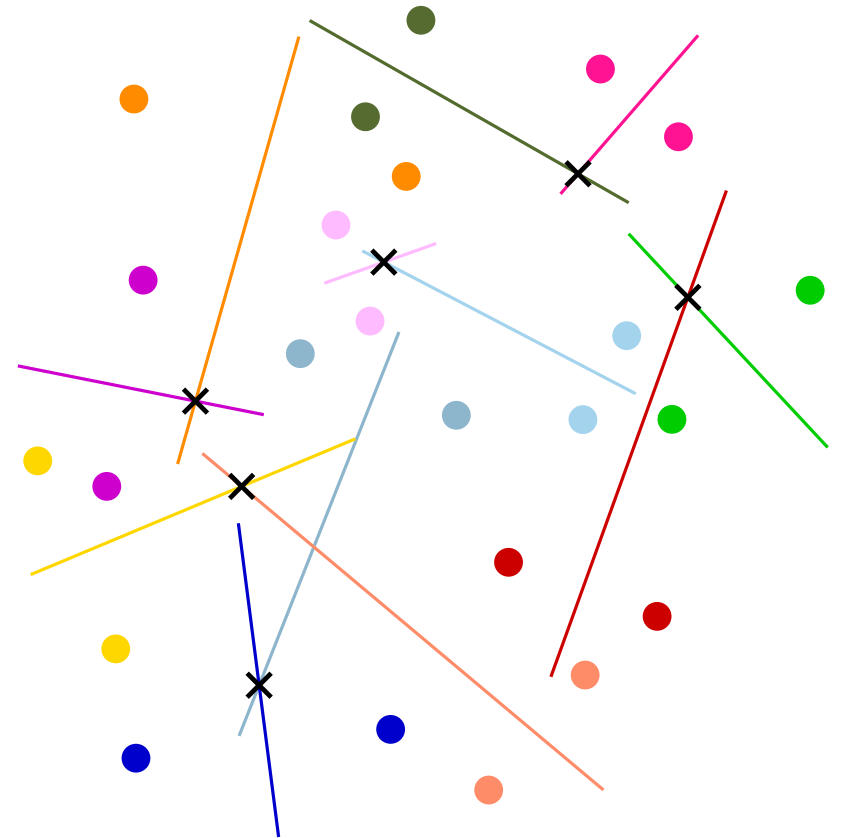
1. Pair up the points, and compute the perpendicular bisector of each pair.



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair.
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal.
3. Pair up the non-vertical non-horizontal bisectors, always pairing up one positive slope bisector with one negative slope bisector. Compute the intersection point of each pair.

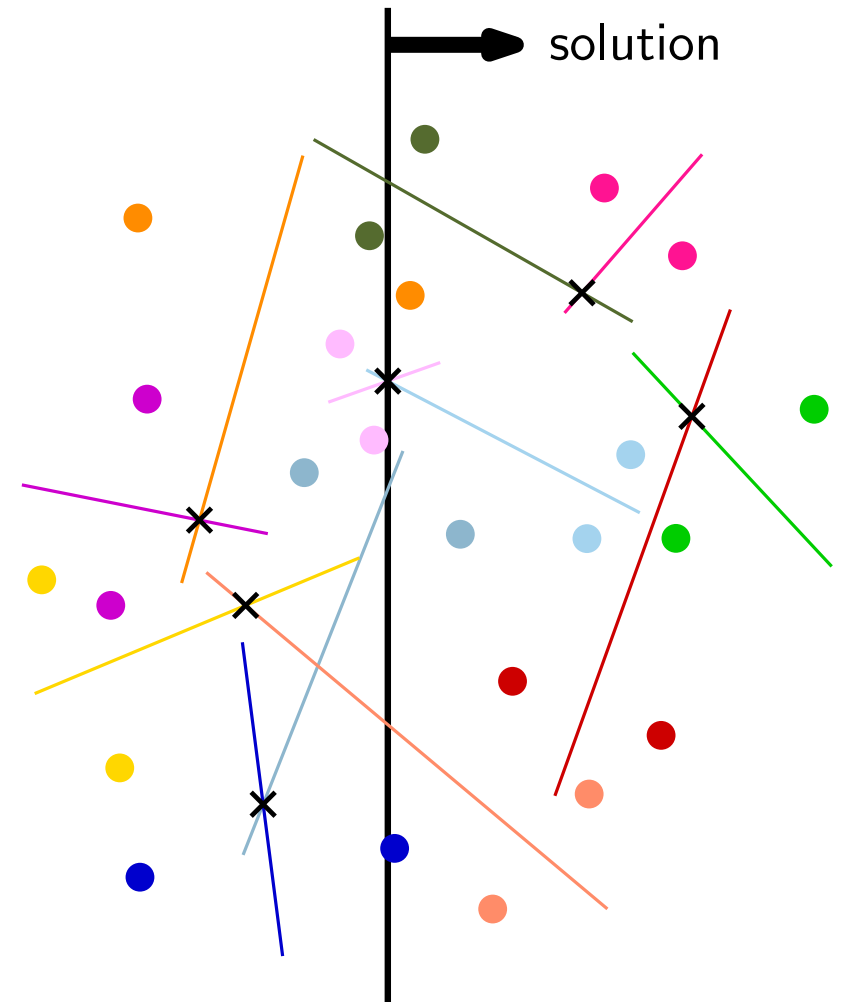


COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

4. First search

- Compute the median value x_m of the x -coordinate of the intersection points (and vertical bisectors).
- Solve the MSC problem restricted to the line $x = x_m$.
- Use the solution to decide whether the unrestricted solution has been found or it lies to the left/right of the vertical line $x = x_m$.



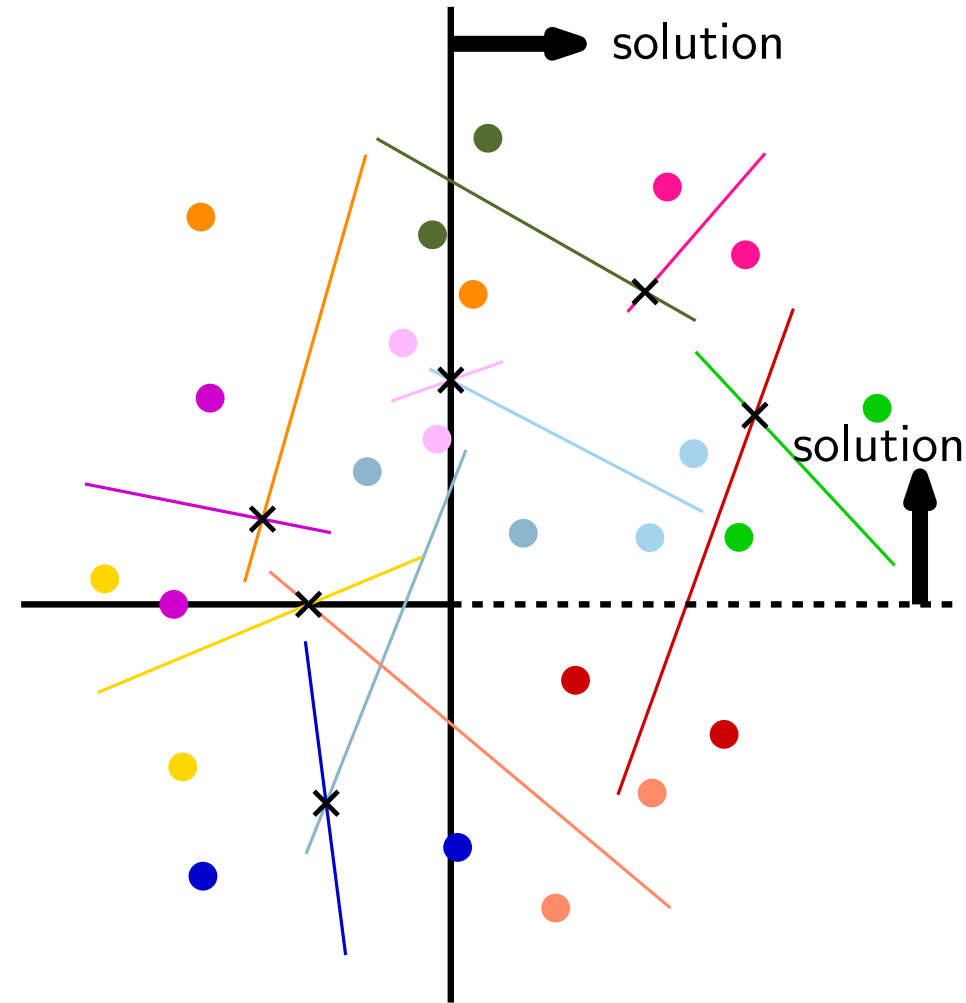
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

4. First search
5. Second search

If the solution has not been found,

- Compute the median value y_m of the y -coordinate of the intersection points (and horizontal bisectors) lying in the opposite halfplane.
- Solve the MSC problem restricted to the line $y = y_m$.
- Use the solution to decide whether the unrestricted solution has been found or it lies above/below the horizontal line $y = y_m$.



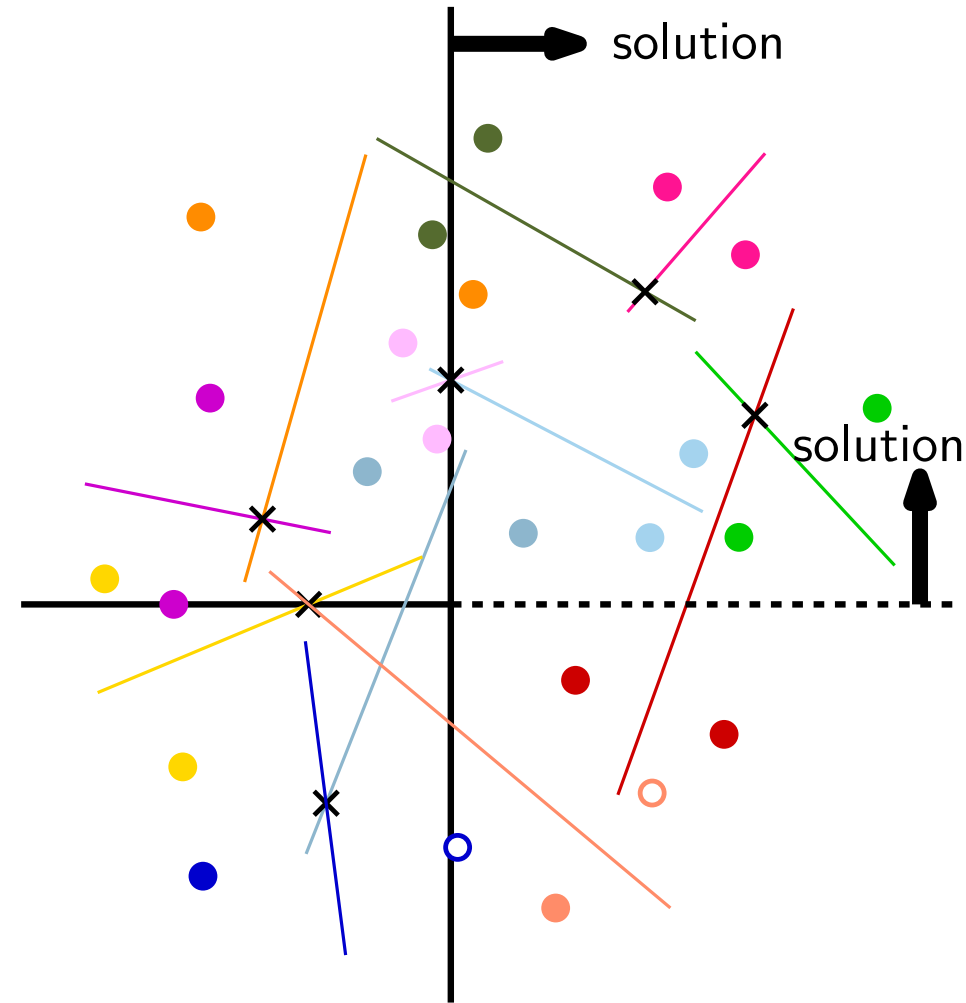
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

4. First search
5. Second search
6. Prune

If the solution has not been found, then

- Each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated.
- Analogously, for each horizontal/vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated.



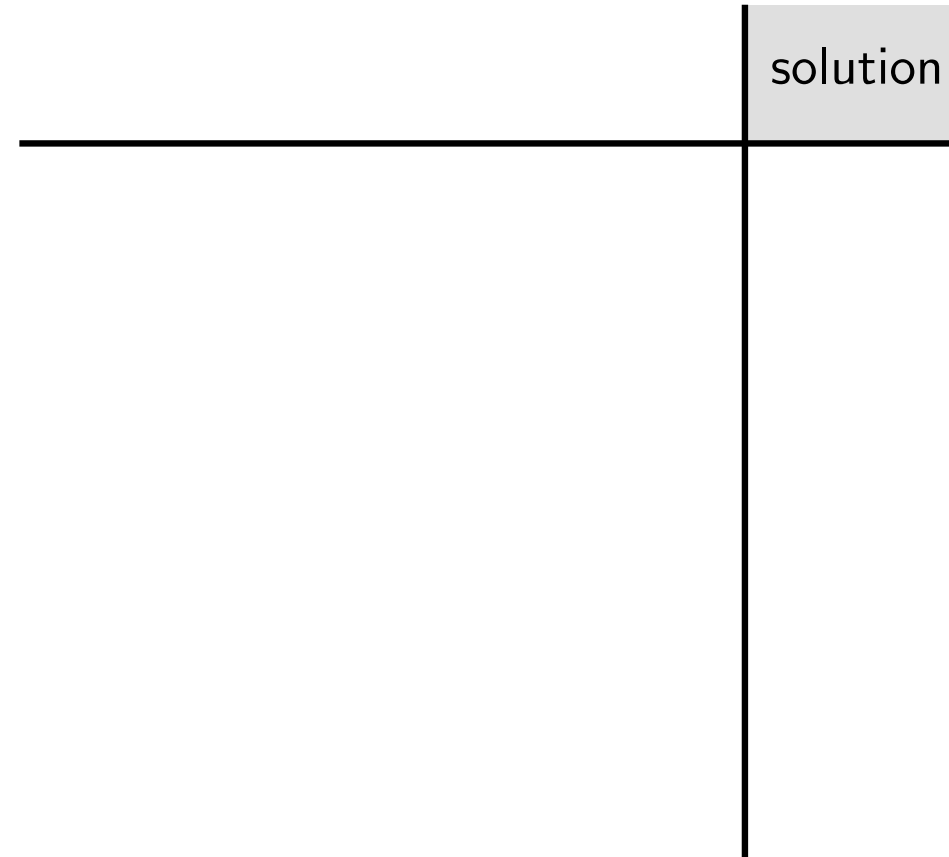
COMPUTING THE MINIMUM SPANNING CIRCLE

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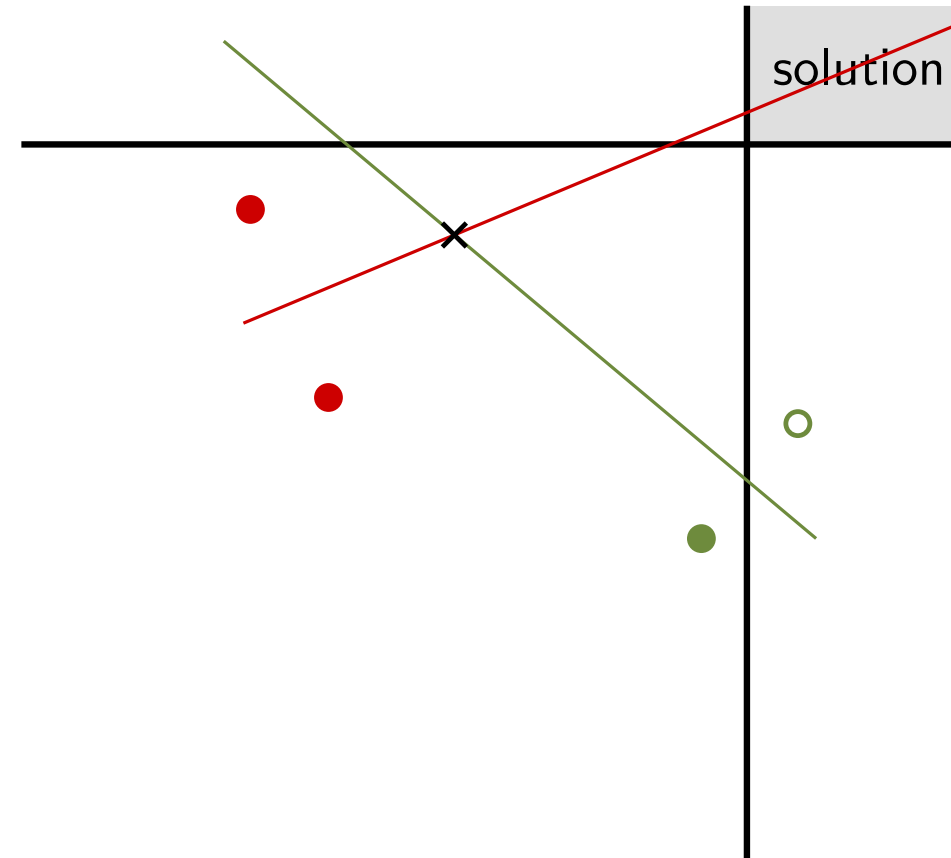
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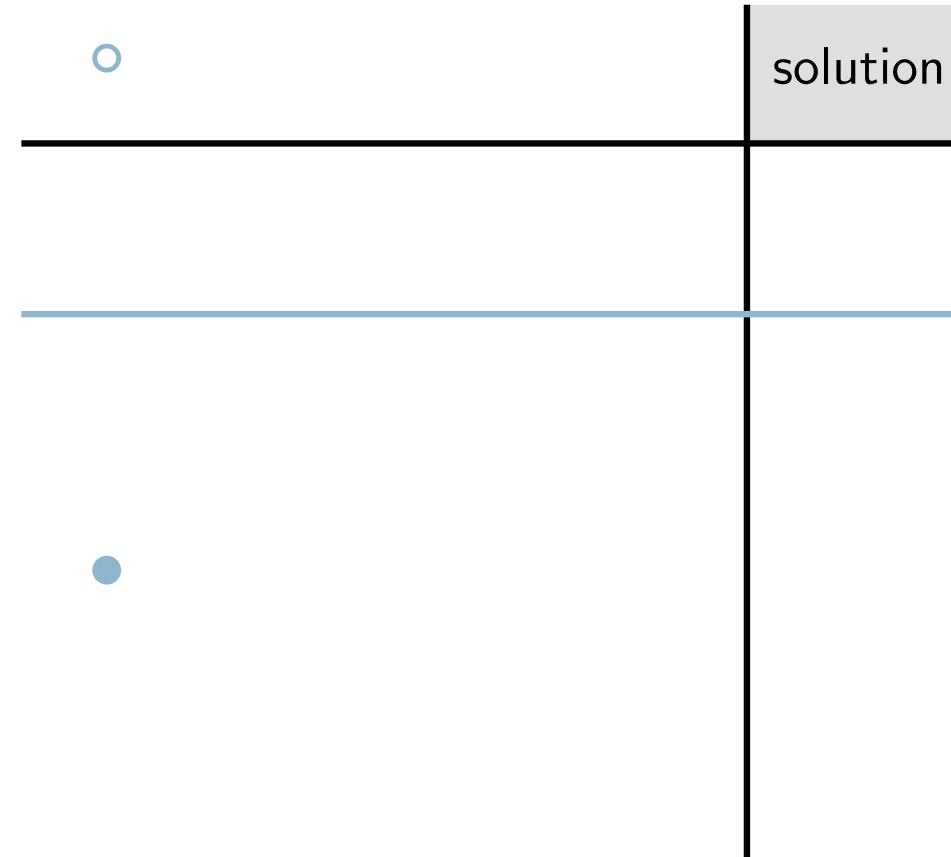
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MSC in the plane

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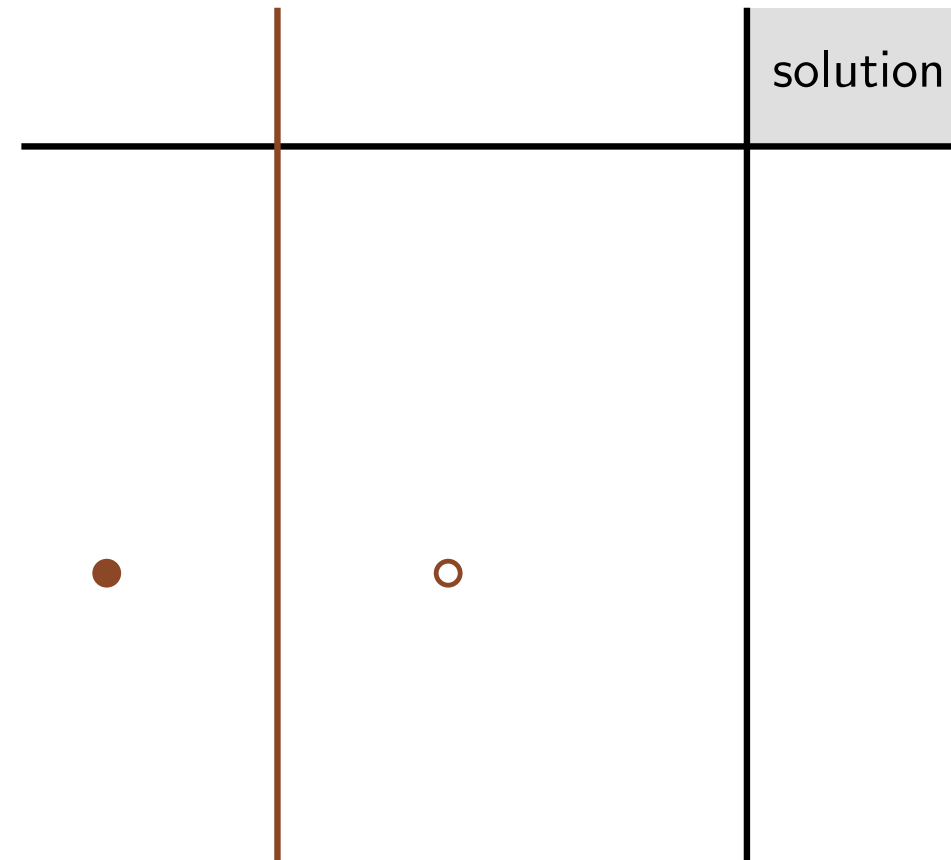
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

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If the solution has not been found, then

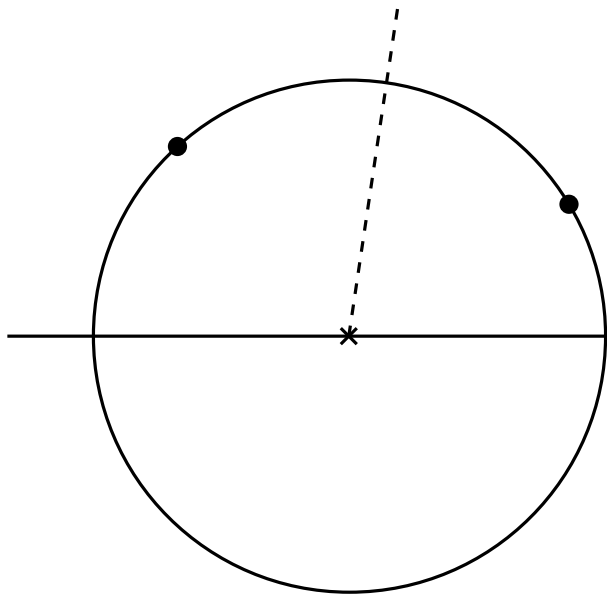
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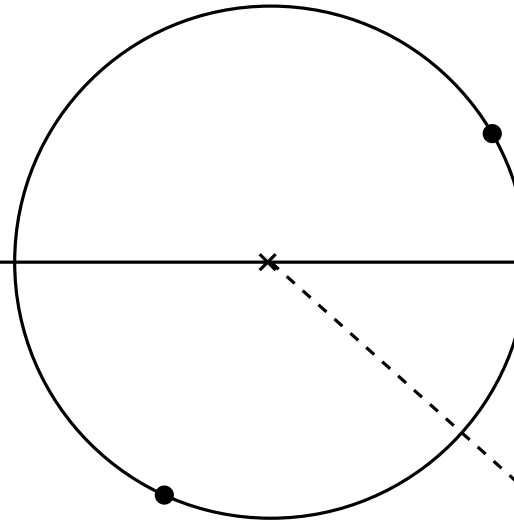
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

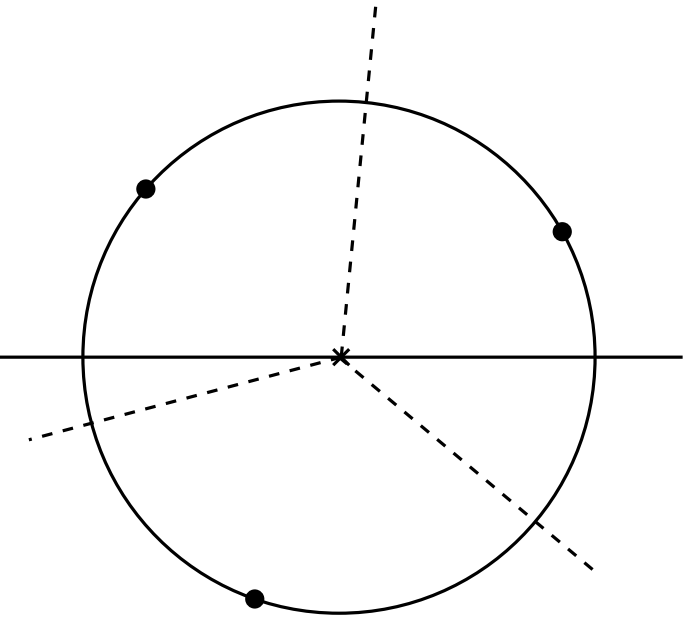
Search step



The solution
lies above



The solution
lies below

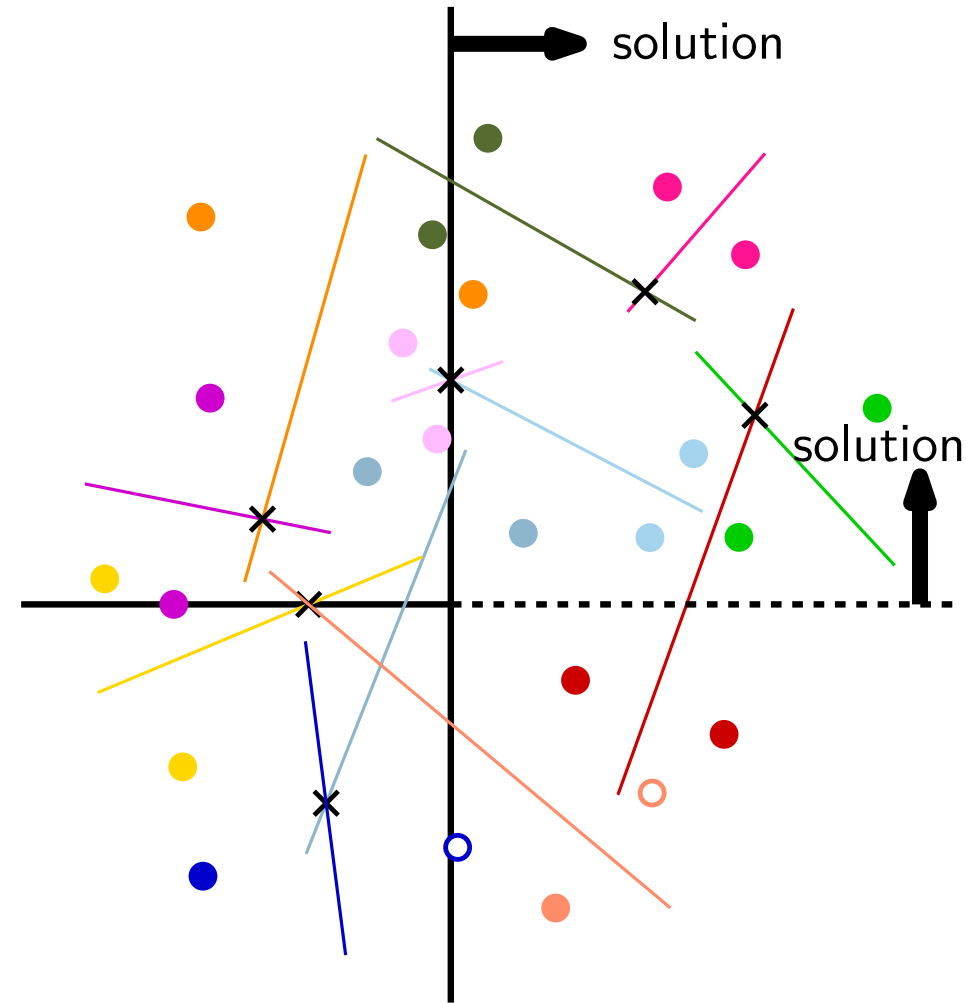


The solution
has been found

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair. $O(n)$
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal. $O(n)$
3. Pair up the non-vertical non-horizontal bisectors, and compute the intersection point of each pair. $O(n)$
4. First search: $O(n)$
5. Second search: $O(n)$
6. Prune: In $O(n)$ time, at least $1/16$ of the input is pruned.



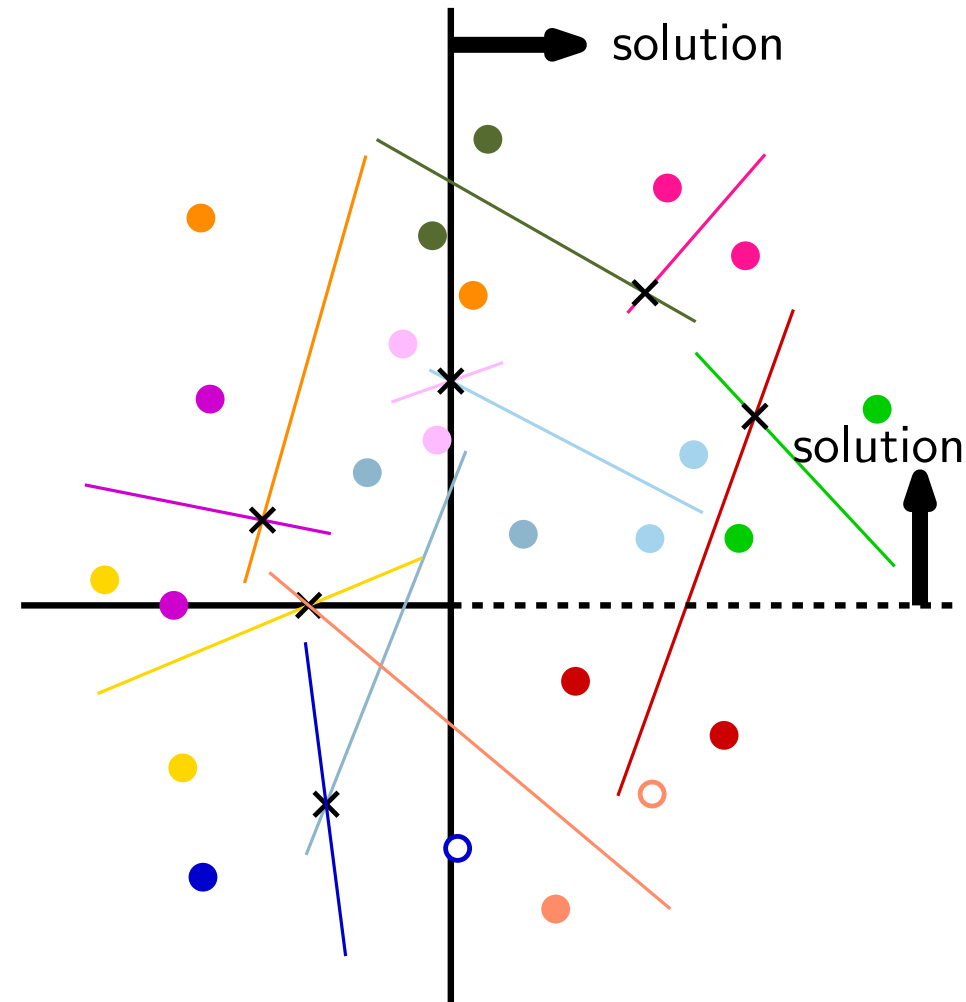
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2. Compute the median slope of the non-vertical bisectors, and take it as horizontal. $O(n)$
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4. First search: $O(n)$
5. Second search: $O(n)$
6. Prune: In $O(n)$ time, at least $1/16$ of the input is pruned.

Conclusion

Given n points in the plane, the min-max facility location problem can be solved in linear time.



TO LEARN MORE

- F. P. Preparata and M. I. Shamos, **Computational Geometry: an Introduction** (revised ed.). Springer-Verlag, 1993.
- N. Megiddo, Linear-time algorithms for linear programming in R^3 and related problems, *SIAM Journal on Computing*, Vol. 12, N. 4, pp. 759-776, 1983.
- M. E. Dyer, On a multidimensional search technique and its application to the euclidean one-centre problem, *SIAM Journal on Computing*, Vol. 15, N. 3, pp. 725-738, 1986.
- F. Hurtado, V. Sacristán, G. Toussaint, Some Constrained Minimax and Maximin Location Problems, *Studies in Locational Analysis*, Vol. 15, pp. 17-35, 2000.
- F. Gómez, F. Hurtado, S. Ramaswami, V. Sacristán, G. Toussaint, Implicit Convex Polygons, *Journal of Mathematical Modelling and Algorithms*, Vol. 1, N. 1, pp. 57-85, 2002.