

# Computational Geometry - Problems

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1. Propose an algorithm to compute the area of a simple polygon.
2. Let  $S$  be the set of non-ordered  $n$  segments that are the edges of a convex polygon  $P$ . Describe an algorithm to compute in  $O(n \log n)$  time the list of vertices of  $P$  from  $S$ .
3. Let  $S$  be a set of  $n$  pairwise disjoint line segments in the plane. One segment in  $S$  can be separated in the  $(1, 0)$  direction if it is possible to translate it horizontally towards  $+\infty$  without it colliding with any of the remaining segments.
  - (a) Prove that there always exists at least one segment in  $S$  that can be separated in the  $(1, 0)$  direction.
  - (b) Obtain an algorithm that gives an ordering of the segments of  $S$  that can be used to separate them, one by one.
4. Let  $S$  be a set of  $n$  pairwise disjoint line segments,  $s_1, \dots, s_n$ , and let  $C$  be the set of their endpoints.
  - (a) Prove that, for each point  $q$  lying in the exterior of the convex hull of  $C$ , there exists at least one line segment in  $S$  that can be entirely seen from  $q$ .
  - (b) Obtain an efficient algorithm to find it.
  - (c) Give an example where  $q$  is interior to the convex hull of  $C$  and it cannot see any line segment of  $S$  entirely.
5. Let  $S$  be a set of  $n$  pairwise disjoint line segments in the plane. Let  $p$  be a point not lying in any of the segments of  $S$ . We want to decide which of the segments of  $S$  can be (weakly) seen from  $p$ . Give an algorithm to solve this problem in  $O(n \log n)$  time.
6. Given  $n$  line segments in the plane (without assuming them to be disjoint) and two points  $p$  and  $q$ , find an algorithm to decide whether there exists any point  $x$  in the plane that can simultaneously see  $p$  and  $q$ , i.e., such that the line segments  $\overline{xp}$  and  $\overline{xq}$  do not intersect any of the given  $n$  segments.
7. Find an algorithm to compute the area of the union of  $n$  isothetic rectangles. If it suits you, you can suppose that their edges are not aligned.
8. Let  $R$  be a set of  $n$  rectangles,  $R_1, \dots, R_n$ , all them having their base on the  $x$ -axis. Each rectangle  $R_i$  is defined by its lower left vertex,  $(x_i, 0)$ , its width,  $a_i$ , and its height,  $h_i$ . The *profile* of  $R$  is the upper envelope of the rectangles and the  $x$ -axis.

- (a) It is quite obvious that the profile of  $R$  has at most  $2n$  vertical segments (the verticals edges of the rectangles). Prove that it has at most  $2n + 1$  horizontal segments.
- (b) Consider the profiles corresponding to two sets of rectangles,  $R'$  and  $R''$ , consisting of a total number of  $n'$ , respectively  $n''$  horizontal and vertical segments. Prove that the number of intersections of the two profiles is  $O(n' + n'')$ . Note: you may suppose that there is no overlapping among the segments of the two profiles.
- (c) Give an algorithm to compute the profile of a set  $R$  of  $n$  rectangles.
9. Propose an algorithm for decomposing an isothetic polygon (a polygon with only horizontal and vertical edges) into rectangles, using segments aligned with the edges of the polygon. Your algorithm must produce the minimum number of pieces, and the output must give a complete description of the partition.
10. Given a set  $P$  with  $n$  points in the plane, propose an algorithm to find the position of the isothetic unit square containing the maximum number of points from  $P$ .
11. Given a simple polygon  $P$  and a line  $\ell$  which does not intersect  $P$ , propose an algorithm for computing the point of  $P$  closest to  $\ell$ . Can improve the previous algorithm for convex polygons? If so, how?
12. Propose an algorithm to compute the the external and the internal common tangent lines (more properly speaking, supporting lines) for disjoint convex polygons. Does this problem make sense when the polygons are simple?
13. Propose an algorithm that, given a point  $p$  external to a convex polygon  $P$ , finds the point of  $P$  closest to  $p$ . What if instead of finding the closest point we look for the farthest? What if we restrict the search to the vertices of  $P$ ?
14. Propose an algorithm that, given two disjoint convex polygons,  $P$  and  $Q$ , finds the closest points  $p \in P$  and  $q \in Q$ .
15. Given a convex polygon  $P$  and a point  $q$  external to  $P$ , find an algorithm to compute the line through  $q$  maximizing the distance to  $P$ .
16. Characterize the orthogonal star-shaped polygons and their kernel. Infer an algorithm to compute the kernel of such a polygon in optimal time. Note: An *orthogonal polygon* is a polygon whose angles are all right angles.
17. Given  $n$  points in the plane,  $p_1, \dots, p_n$ , find an algorithm to decide, for any new point  $q$ , whether  $q$  belongs to the exterior, the interior or the boundary of the convex hull of  $p_1, \dots, p_n$ . Is it possible to solve this problem in  $o(n \log n)$  time?
18. Given  $n$  vertical segments,  $s_1, \dots, s_n$ , we want to *i*) decide whether they have any common stabbing line and *ii*) in the affirmative, find one.
- (a) Find the dual formulation of this problem.
- (b) Give an algorithm to solve it.
19. The following duality, often called *polarity*, is a transformation among points in the plane, different from the origin, and lines in the plane that do not go through the origin:

$$(a, b) \longleftrightarrow \{(x, y) \in \mathbb{R}^2 \mid ax + by = 1\}.$$

- (a) Prove that the polar transform of a point  $p = (a, b)$  is the line perpendicular to the vector  $\vec{v} = (a, b)$  passing through the point  $q = \frac{p}{\|p\|^2}$ .

- (b) Obtain as a consequence that the polar transform of a point  $p$  on the unit circle,  $C$ , is the line tangent to  $C$  in  $p$ .
  - (c) If  $p$  is at distance  $d$  from the origin, prove that its polar transform is at distance  $\frac{1}{d}$  from the origin.
  - (d) What is the result of dualizing a regular polygon centered at the origin?
20. Let  $P$  be an  $x$ -monotone polygonal line with  $n$  vertices, all with positive  $y$ -coordinate, and let  $a$  and  $b$  respectively be the minimal and maximal  $x$ -coordinates of the points in  $P$ . We call *terrain* the portion of the plane enclosed by the  $x$ -axis, the vertical lines  $x = a$  and  $x = b$ , and the polygonal line  $P$ . We say that  $P$  is the profile of the terrain.
- (a) Characterize the set  $S$  of all the points lying in the strip  $a \leq x \leq b$  and above  $P$  which can see all of  $P$ . Give an algorithm to find the point of  $S$  of minimum  $y$ -coordinate.
  - (b) Give an algorithm to find the point of the profile allowing to construct the shortest tower guarding the entire profile  $P$ .
21. Find an  $O(n)$  algorithm to triangulate a star-shaped  $n$ -gon, assuming that a point in its kernel is known.
22. Find an algorithm to decide, given a simple polygon  $P$ , whether there exists a direction for which  $P$  is monotone.
23. The usual definition of convexity reads as follows: A set  $S$  is convex iff for all points  $p, q \in S$  the line segment  $pq$  lies in  $S$ .
- (a) Prove that the following is an alternative definition: A set  $S$  is convex iff for all lines  $\ell$ , the intersection  $S \cap \ell$  is a connected set.
  - (b) Obtain as a corollary that a set  $S$  is convex iff it is monotone in all directions.
24. Given  $n$  points in the plane, we want to construct a simple polygon having them as vertices.
- (a) Prove that this problem has complexity  $\Omega(n \log n)$ .
  - (b) Propose an algorithm to solve it.
  - (c) Prove that any set of  $n$  2D points in general position admits an  $x$ -monotone polygonization, i.e., that there always exists a simple and  $x$ -monotone polygon having the given points as vertices.
  - (d) Give an algorithm to construct it.
25. Find an algorithm to decompose an orthogonal pyramid into convex quadrilaterals, by insertion of internal diagonals. Is it possible to do so in  $O(n)$  time? Note: An orthogonal pyramid is an orthogonal  $y$ -monotone polygon having one horizontal edge whose length coincides with the sum of the lengths of all the remaining horizontal edges.
26. Which value or values can the sum of all the internal angles of a simple  $n$ -gon take and why? The same question, now for the sum of the external angles.
27. About the number of triangulations of a polygon.
- (a) Which are the polygons that have the least number of triangulations?
  - (b) Can a simple  $n$ -gon have a unique triangulation?
  - (c) Which are the polygons that have the most triangulations?
  - (d) Compute the number  $T_n$  of different triangulations of a convex  $n$ -gon, following these steps:

- i. By counting the number of triangulations that can be built from the initial insertion of one diagonal of the polygon, prove that

$$T_n = \frac{n(T_3T_{n-1} + T_4T_{n-2} + T_5T_{n-3} + \cdots + T_{n-3}T_5 + T_{n-2}T_4 + T_{n-1}T_3)}{2(n-3)}.$$

- ii. By counting the number of triangulations that can be built from the initial insertion of a triangle whose base is on the edge  $p_1p_2$  of the polygon, prove:

$$T_{n+1} = T_n + T_3T_{n-1} + T_4T_{n-2} + T_5T_{n-3} + \cdots + T_{n-3}T_5 + T_{n-2}T_4 + T_{n-1}T_3 + T_n.$$

- iii. Infer the recursive formula  $T_{n+1} = \frac{4n-6}{n}T_n$ .

- iv. Finally, prove the expression  $T_{n+2} = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$ . These numbers are known as Catalan numbers, in honor to the Belgian mathematician Charles Catalan (1814-1894).

28. A *visibility-independent set* in a polygon  $P$  is a finite set of points on  $P$ ,  $IS(P)$ , such that the visibility polygons of its elements are pairwise disjoint, i.e., for all  $p, q \in IS(P)$ , it holds that  $Vis(p, P) \cap Vis(q, P) = \emptyset$ . Prove that no single point guard in  $P$  is able to see more than one point of  $IS(P)$ . Conclude that  $g(P) \geq is(P)$ , where  $g(P)$  is the number of point guards in a minimum-cardinality set of point guards of  $P$  and  $is(P)$  is the number of points in a maximum-cardinality visibility-independent set of  $P$ . Is the bound tight?
29. Propose an algorithm to compute the convex hull of a monotone polygon in optimal time.
30. How many connected components can the intersection of two  $x$ -monotone polygons have? Give an algorithm to efficiently compute them by exploiting the monotonicity of the polygons.
31. Prove that if two convex polygons are disjoint there always exists a separating line. Use this fact to design an algorithm to decide, in  $O(s \log s + t \log t)$  time, whether there exists a line separating two point sets  $S$  and  $T$  in the plane with cardinality  $s$ , respectively  $t$ . Less easy: this problem can be solved in  $O(s + t)$  time.
32. A polygon  $P$  is *weakly visible* from an edge  $a$  if each (internal or boundary) point  $p \in P$  can see at least one point  $m$  of  $a$ . Give a linear time algorithm for triangulating polygons weakly visible from a given edge.
33. A vertical line leaves to its left  $n$  blue points and to its right  $n$  red points. If the union of all red and blue points does not contain any three aligned points, prove that it is possible to match the blue and the red points pairwise, such that the resulting red-blue segments do not intersect, and give an algorithm to construct such a matching.
34. Prove that any simple polygon with holes can be triangulated. What can you say about the number of triangles of its triangulations?
35. Let  $P$  be a convex  $n$ -gon and  $Q$  be a simple  $m$ -gon. Suppose that we are given a triangulation of the interior of  $Q$ , as well as of each of its pockets. Prove that it is possible to compute the intersection of  $P$  and  $Q$  in  $O(n + m)$  time. Suggestion: start enclosing  $Q$  in a triangle  $T$  big enough to enclose also  $P$ , and triangulate  $T \setminus Q$ . Note: if  $P$  is a simple polygon and  $a$  is an edge of the convex hull of  $P$  but it is not an edge of  $P$ , then  $a$  is the *lid* of a *pocket* of  $P$ , limited by  $a$  and the edge chain of  $P$  connecting the two endpoints of  $a$ .
36. Prove that the *flip distance* between two triangulations of a given  $n$ -gon can be as big as  $\Omega(n^2)$ .

37. The *Gabriel graph* of a point set  $P$  of the plane,  $GG(P)$ , is defined as follows: its vertices are the points of  $P$ ; the segment  $\overline{pq}$  connecting two points of  $P$  is an edge of the graph if and only if the disc that has  $\overline{pq}$  as diameter does not enclose any other point of  $P$ .
- Prove that  $GG(P) \subseteq Del(P)$ .
  - Prove that two points  $p$  and  $q$  are adjacent in  $GG(P)$  if and only if the Delaunay edge  $\overline{pq}$  intersects its dual edge in  $Vor(P)$ .
  - Propose an algorithm to compute  $GG(P)$ .
38. Two points  $p$  and  $q$  of a set  $P$  in the plane are *relative neighbors* if there is no other point from  $P$  simultaneously closer to  $p$  and  $q$ . In other words, there is no other point  $x \in P$  such that  $d(x, p) < d(p, q)$  and  $d(x, q) < d(p, q)$ . The *relative neighborhood graph*  $RNG(P)$  of  $P$  is defined as follows: its vertices are the points of  $P$ ; a segment  $\overline{pq}$  connecting two points of  $P$  is an edge of the graph if, and only if,  $p$  and  $q$  are relative neighbors.
- Given two points,  $p$  and  $q$ , the *lens*  $lens(p, q)$  is the intersection of the discs of radius  $d(p, q)$  centered at  $p$  and  $q$ . Prove that  $p$  and  $q$  are relative neighbors if, and only if,  $lens(p, q)$  does not contain any other point of  $P$ .
  - Prove that  $RNG(P) \subseteq Del(P)$ .
  - Propose an algorithm to compute  $RNG(P)$ .
39. Let  $P = \{p_1, \dots, p_n\}$  be a set of  $n$  points in the plane. We define  $m(p_i, p_j)$  as the minimum number obtained when counting the number of points from  $P$  contained in any closed disk containing  $p_i$  and  $p_j$ ; and  $m(P) = \max_{i,j=1\dots n} m(p_i, p_j)$ .
- Find the value of  $m(P)$  for all configurations of  $P$  with  $n = 3$  and  $n = 4$ .
  - If there exists a disk  $D$  containing  $p_i, p_j$  and some other  $k$  points from  $P$ , prove that there then also exists a disk  $D' \subseteq D$  having  $p_i$  and  $p_j$  on its boundary and such that it contains at most  $k$  other points from  $P$ .
  - Use the previous result to obtain an algorithm to compute  $m(p_i, p_j)$  for any given pair of points  $p_i$  and  $p_j$ .
  - Give an algorithm for computing  $m(P)$ .
40. Find an algorithm to compute the disk of minimum radius enclosing a given convex polygon and centered in a point of its boundary.
41. Design an algorithm that, given two sets  $A$  and  $B$  with respectively  $n$  and  $m$  points, computes (if it exists) a disk containing all the points in  $A$  and none in  $B$ . What is the complexity of your algorithm?
42. The distance  $d_\infty$  between two points in the plane is defined as  $d_\infty((a, b), (c, d)) = \max(|c - a|, |d - b|)$ .
- Disk centered on  $p$  with radius  $r$ : if  $r > 0$ , what is the geometric locus of the points at distance  $r$  from a given point  $p$ ?
  - Bisector of  $p$  and  $q$ : what is the geometric locus of the points that are at the same distance from two given points,  $p$  and  $q$ ? Hint: it may be convenient to distinguish some cases, depending on the relative position of  $p$  and  $q$ .
  - What can you say about the Voronoi region of a point?
    - Is it convex?
    - Is it star-shaped?

- iii. Is it connected?
  - (d) Characterize the points whose Voronoi region is unbounded.
43. Let  $V_f(P)$  be the farthest-point Voronoi diagram of a 2-dimensional point set  $P = \{p_1, \dots, p_n\}$ .
- (a) Prove that the Voronoi region of a point,  $V_f(p_i)$ , exists (i.e., is not empty) iff  $p_i$  is a vertex of the convex hull of  $P$ .
  - (b) Prove that all the non empty regions  $V_f(p_i)$  in  $V_f(P)$  are unbounded.
  - (c) Prove that  $V_f(p_i)$  is always a convex polygonal region.
  - (d) What can you say about the 1-skeleton (i.e., the edge set) of  $V_f(P)$ ?
44. Given a finite point set  $S$ , consider two new points  $x, y \notin S$ . Let  $Reg(x)$  be the Voronoi region of  $x$  in  $Vor(S \cup \{x\})$ . Let  $Reg(y)$  be the Voronoi region of  $y$  in  $Vor(S \cup \{y\})$ . Prove that the boundaries of  $Reg(x)$  and  $Reg(y)$  can intersect at most in 2 points.
45. Consider a 2-dimensional robot like Roomba, which can be described as a disk of a given radius  $r$ . We want to plan the trajectory of the robot from a source position  $s$  to a target position  $t$  in the presence of obstacles that are described as points,  $Obs = \{p_1, \dots, p_m\}$ . Is this always possible? Propose an algorithm to decide whether it is or not. When the robot can move from  $s$  to  $t$ , your algorithm should compute a possible path. What is the complexity of your algorithm? Can you tell how long is your path, compared to the shortest one?
46. Given a planar decomposition  $S$  of complexity  $n$ , consider a point set  $P$  of cardinality  $m$ . How can we compute the region of  $S$  where each point of  $P$  is located?
47. Given a set  $R$  with  $n$  lines of the plane, find an algorithm to compute the smallest isothetic rectangle containing all the intersection points of the lines in  $R$ .
48. Let  $P$  be a convex  $n$ -gon, and  $t$  a real number greater than the value of the area of  $P$ . For each point  $q$  external to  $P$ , let  $P(q)$  be the polygon resulting from computing the convex hull of  $P \cup \{q\}$ .
- (a) Prove that the points  $q$  such that the area of  $P(q)$  equals  $t$  form the boundary of a convex polygon  $Q(t)$  enclosing  $P$ .
  - (b) Prove that the number of edges of  $Q(t)$  is between  $n$  and  $2n$ , and that, in fact, it is  $2n$ , except for a finite number of values of  $t$ .
  - (c) Give an algorithm to compute  $Q(t)$ .
49. Let  $S$  be a set of  $n$  points in the plane. Give an algorithm to find a line through the maximum number of points from  $S$ .
50. Let  $P$  be a simple polygon. Propose an algorithm to find the shortest path within  $P$  connecting two (internal or boundary) points  $s$  and  $t$  of  $P$ . In order to do so, it is convenient to figure out and prove which is the structure of such shortest path in the first place. Suggestion for the algorithm: start triangulating the polygon.
51. Let  $P$  be a convex polygon, and consider two points  $a$  and  $b$  external to  $P$ . We will say that  $a$  conquers  $b$  (or that  $b$  is conquered by  $a$ ) when  $b$  belongs to the convex hull of  $P \cup \{a\}$ . Give an algorithm, as efficient as possible, receiving as input a convex polygon  $P$  with  $n$  vertices and a set  $C$  with  $k$  points, external to  $P$ , and producing as output the points of  $C$  that are not conquered by any other point in  $C$ .