

Describing and implementing basic geometric objects

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Basic geometric objects

- Points: $p = (x, y) \in \mathbb{R}^2$ and $(x, y, z) \in \mathbb{R}^3$
- Lines and line segments: p_1, p_2
- Halflines: p_1, p_2 or p, \vec{v}
- Polygonal lines and polygons: p_1, p_2, \dots, p_n
- Circles and discs: p_1, p_2, p_3 non collinear
- Balls and spheres: p_1, p_2, p_3, p_4 non coplanar

Basic geometric objects

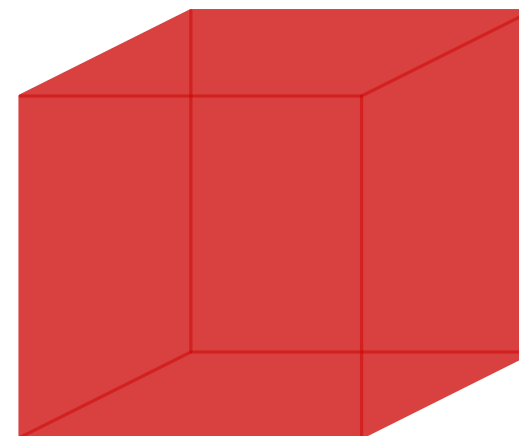
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- Planar graphs such as planar decompositions and polyhedra:
 - For each vertex v :
 - * Its coordinates (x, y, z)
 - * An edge $e(v)$ incident to it
 - For each face f :
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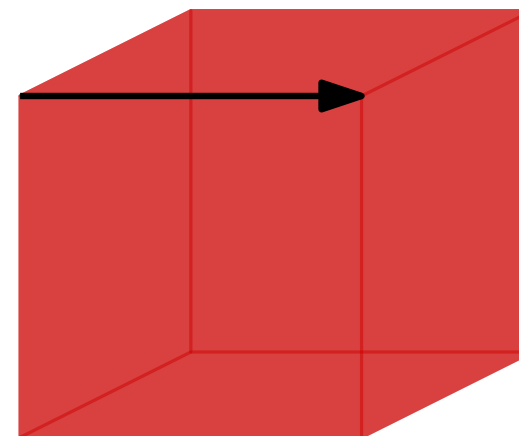
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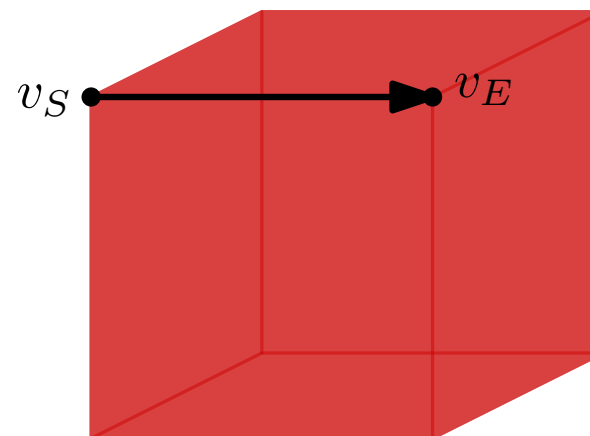
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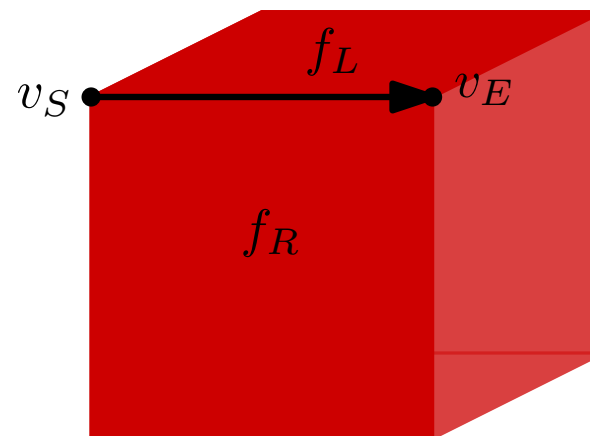
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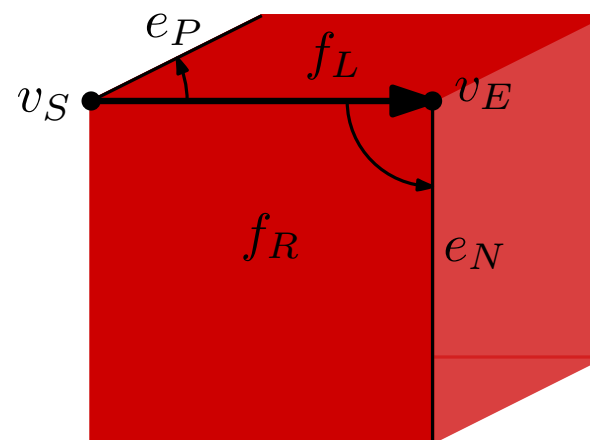
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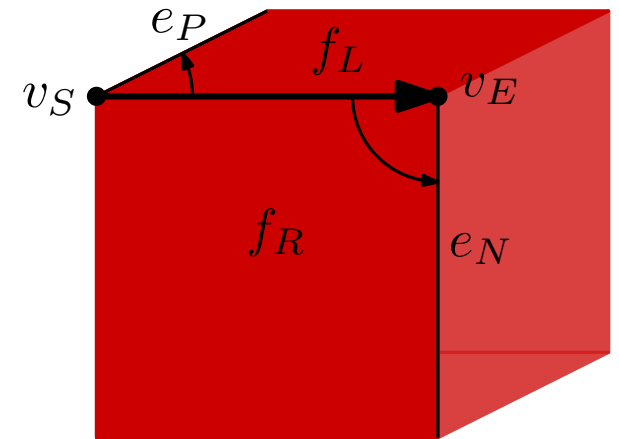


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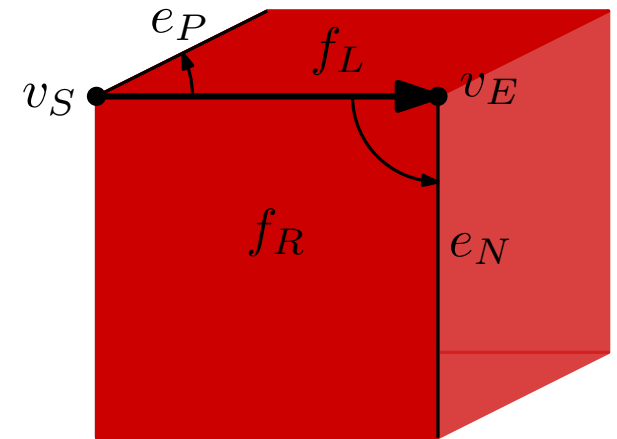


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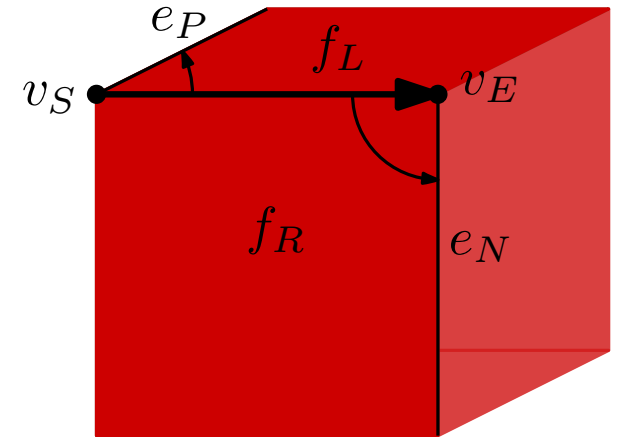
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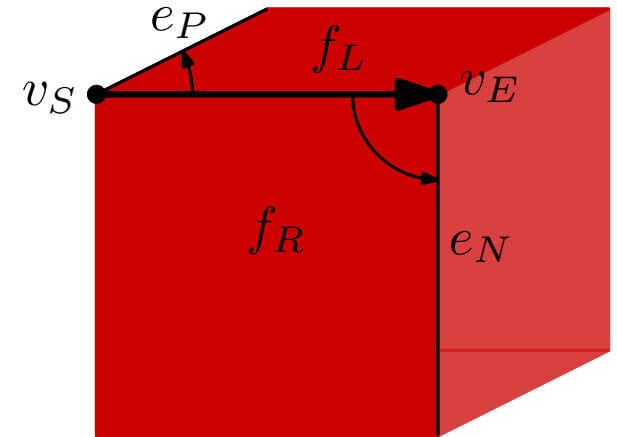
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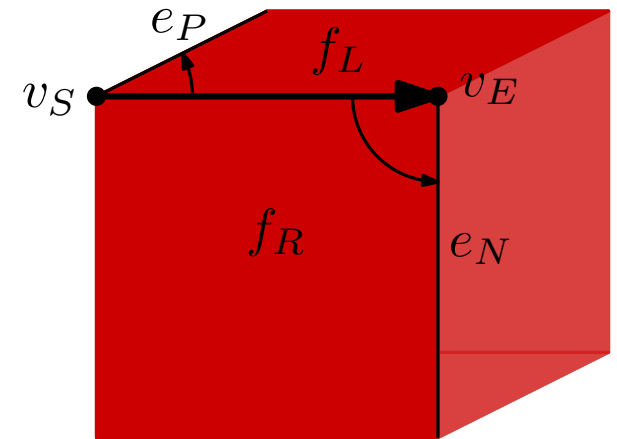


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Proof: Since the graph is planar, $V + F = E + 2$.

Since the faces are n -gons (with $n \geq 3$), $2E \geq 3F$.

Therefore:

$$2V + 2F = 2E + 4 \geq 3F + 4 \implies F \leq 2V - 4.$$

$$3E + 6 = 3V + 3F \leq 3V + 2E \implies E \leq 3V - 6.$$

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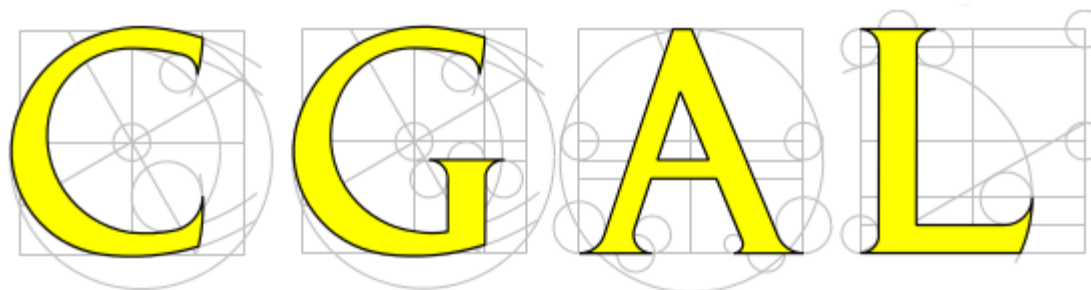
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Recommended libraries

The GNU multiple-precision arithmetic library: <https://gmplib.org/>

The GNU multiple-precision floating-point library: <http://www.mpfr.org/>

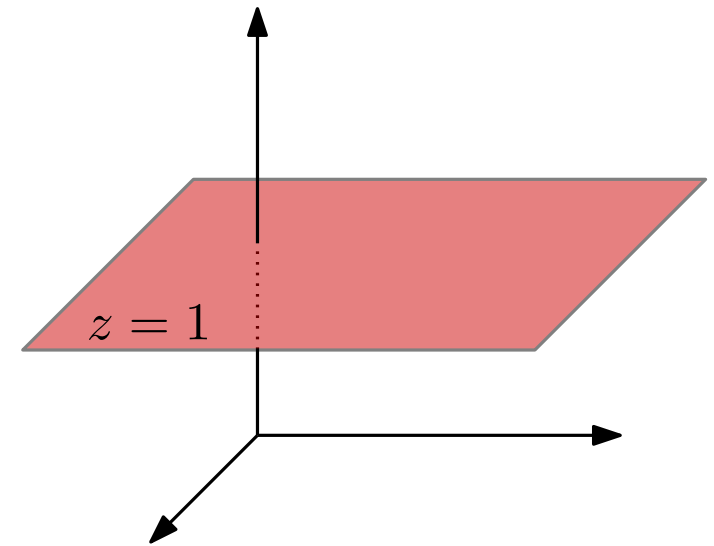
The Computational Geometry Algorithms Library: <http://www.cgal.org/>



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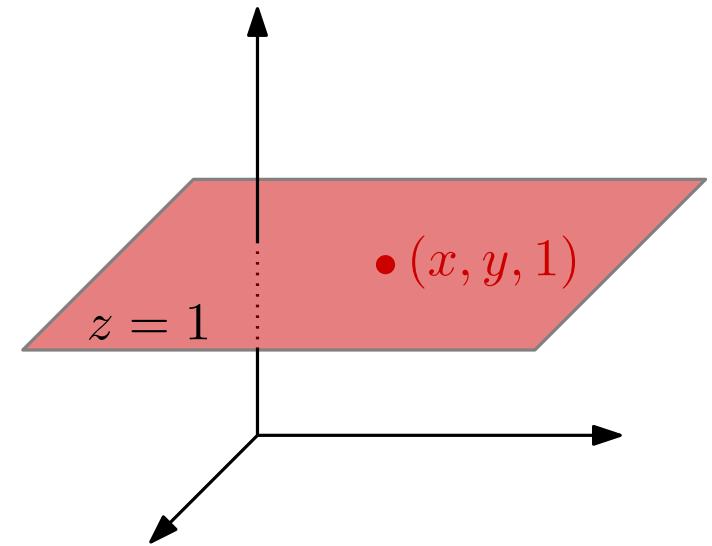
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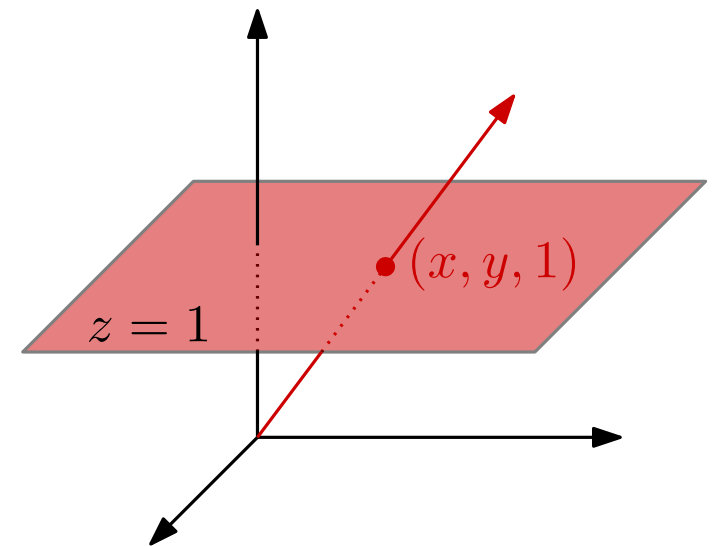
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Then identify point $(x, y, 1)$ with all the points in the ray connecting it to the origin:

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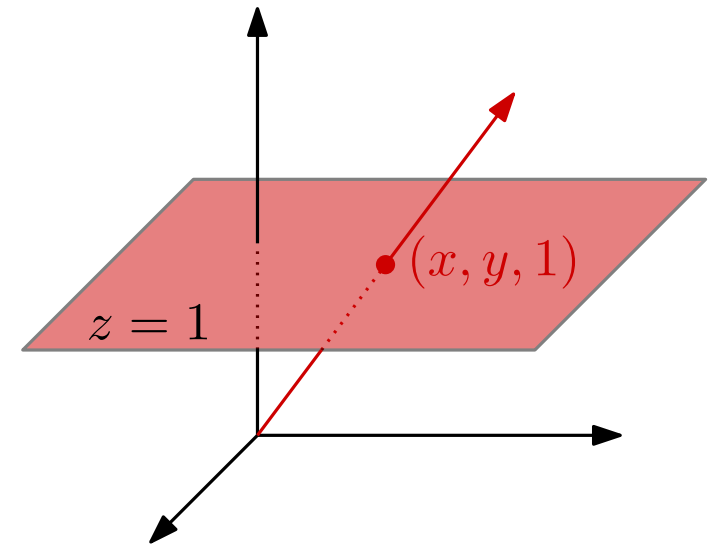
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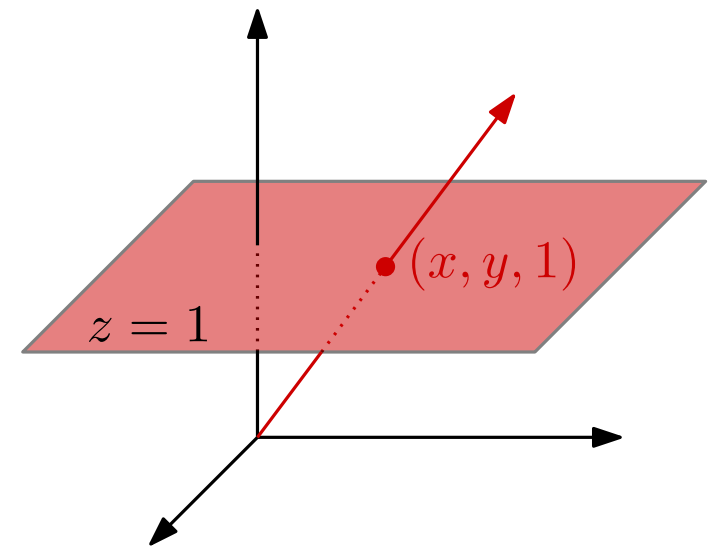
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Welcome to **projective geometry**!

What about vectors?

- Vectors have coordinates $(x : y : 0)$ because they are parallel to the plane $z = 1$.
- They correspond to horizontal rays, i.e., to points at infinity.



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Intersecting two lines

$$\left. \begin{array}{l} p \text{ lies in } \ell_1 \implies p \perp \ell_1 \\ p \text{ lies in } \ell_2 \implies p \perp \ell_2 \end{array} \right\} \implies p = \ell_1 \times \ell_2$$

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Parallel lines

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Identical lines (or points)

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FURTHER READING

Read about CGAL!

- At this course web page (topic 13)
- At CGAL web page