

INTERSECTING HALF-PLANES and related problems

Vera Sacristán

Computational Geometry
Facultat d'Informtica de Barcelona
Universitat Politcnica de Catalunya

DUALIZING POINTS AND LINES

Duality

DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

Properties

1. $D^2 = Id$.
2. D is a bijection between points and non-vertical lines.
3. A point p lies above/on/below a line ℓ if and only if the point ℓ^* lies above/on/below the line p^* .

DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

Properties

1. $D^2 = Id$.
2. D is a bijection between points and non-vertical lines.
3. A point p lies above/on/below a line ℓ if and only if the point ℓ^* lies above/on/below the line p^* .

Proof: If $p = (a, b)$ and ℓ is the line of equation $y = mx + n$, the relative position of p wrt ℓ is given by the (in)equation $b \lessgtr ma + n$, which is equivalent to $-n \lessgtr am - b$, and this expresses the relative position of $\ell^* = (m, -n)$ wrt $p^* : y = ax - b$.

DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

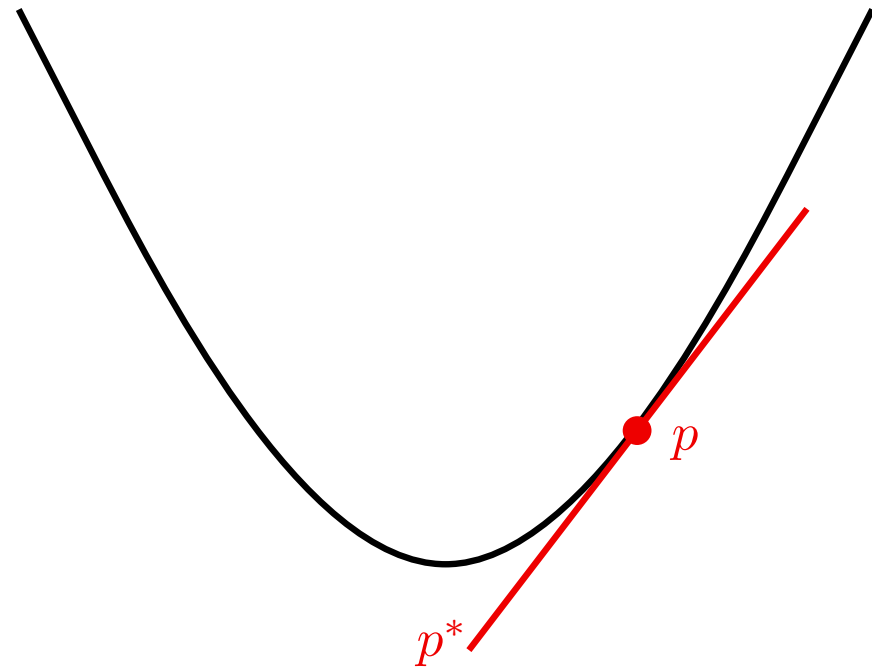
One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

Duality and parabola

Consider the parabola $y = x^2/2$.

- If p is a point on the parabola, then p^* is the line tangent to the parabola at point p .



DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

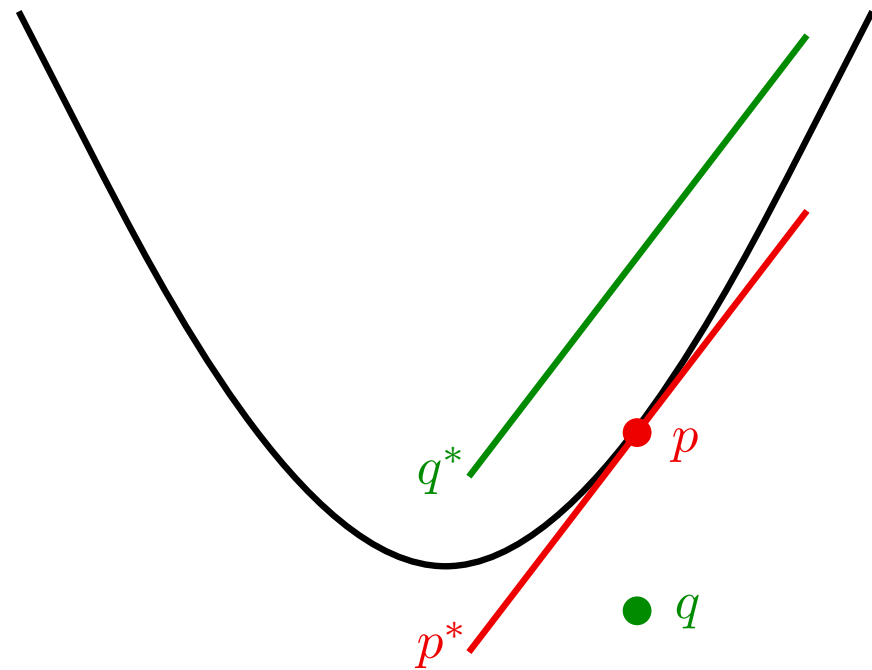
One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

Duality and parabola

Consider the parabola $y = x^2/2$.

- If p is a point on the parabola, then p^* is the line tangent to the parabola at point p .
- Let q be any point in the plane and let p be its vertical projection onto the parabola. Then q^* is the line parallel to p^* whose vertical distance from p^* is the opposite to the vertical distance from q to p .



DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

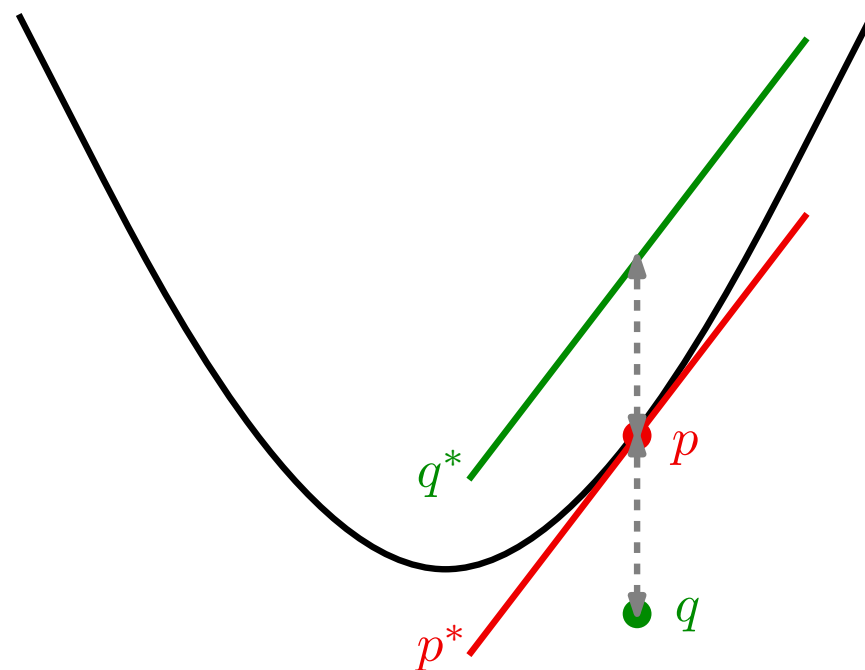
One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

Duality and parabola

Consider the parabola $y = x^2/2$.

- If p is a point on the parabola, then p^* is the line tangent to the parabola at point p .
- Let q be any point in the plane and let p be its vertical projection onto the parabola. Then q^* is the line parallel to p^* whose vertical distance from p^* is the opposite to the vertical distance from q to p .



DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

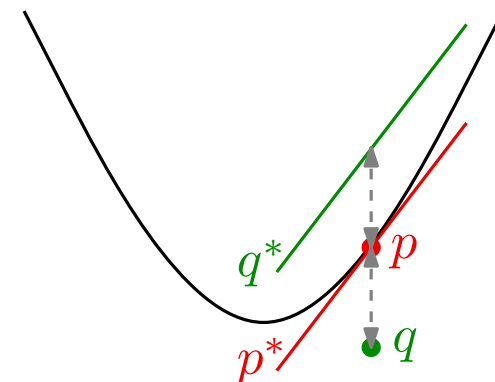
One of the most frequently used is:

$$\begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

Duality and parabola

Consider the parabola $y = x^2/2$.

- If p is a point on the parabola, then p^* is the line tangent to the parabola at point p .
- Let q be any point in the plane and let p be its vertical projection onto the parabola. Then q^* is the line parallel to p^* whose vertical distance from p^* is the opposite to the vertical distance from q to p .



Proof:

Since $y'(x) = x$, the line tangent to the parabola at a point $p = (a, a^2/2)$ has equation $y - a^2/2 = a(x - a)$ i.e., $y = ax - a^2/2$.

For any other point $q = (a, b)$:
 $d_{vertical}(p, q) = b - a^2/2$;
 $d_{vertical}(p^*, q^*) = -b + a^2/2$.

DUALIZING POINTS AND LINES

Duality

In the plane, dualities transform points into lines and lines into points.

One of the most frequently used is:

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ p = (a, b) & \mapsto & D(p) = p^* : y = ax - b \\ \ell : y = mx + n & \mapsto & D(\ell) = \ell^* = (m, -n) \end{array}$$

ENJOY OUR APPLET!

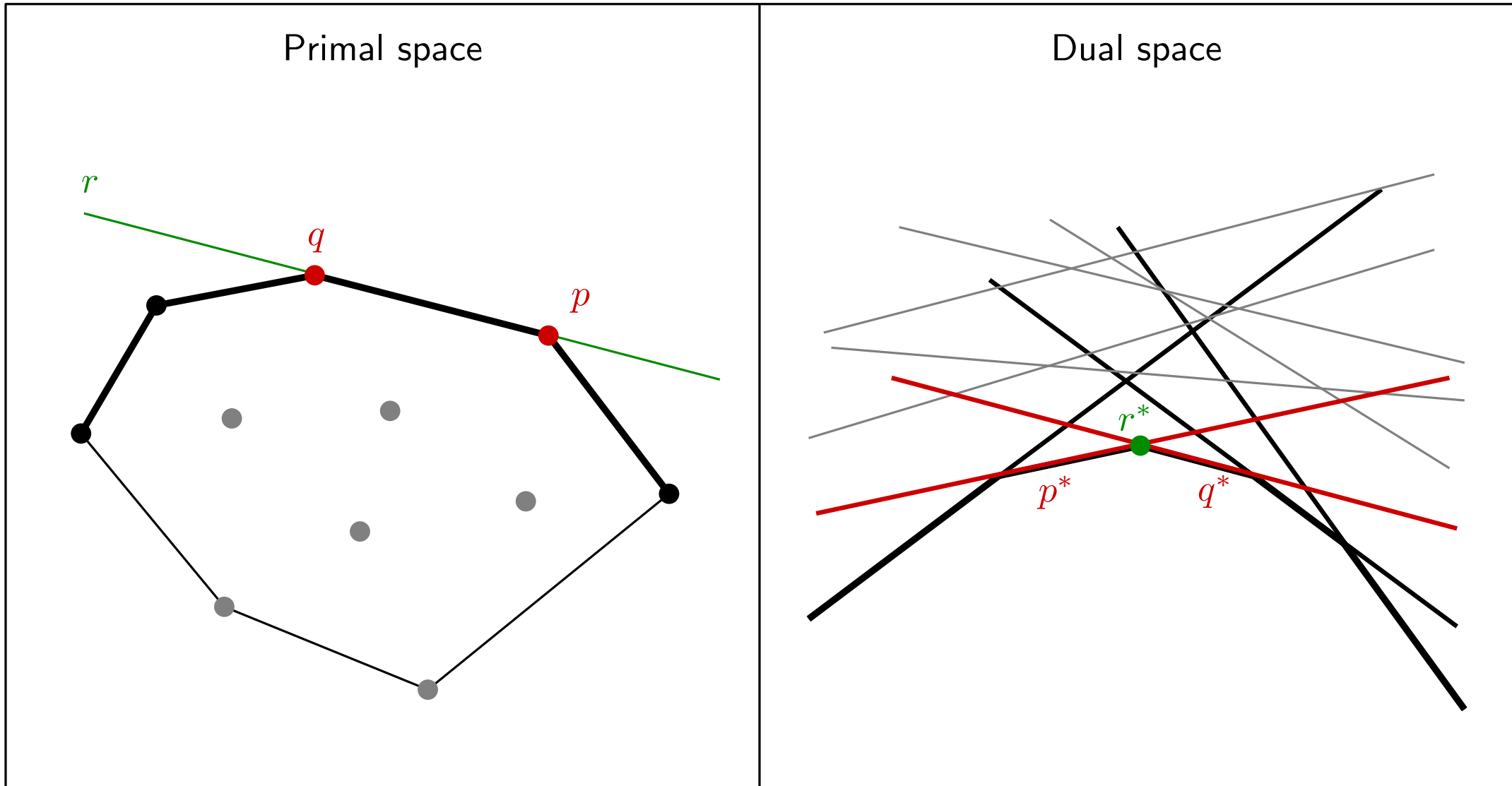
<http://dccg.upc.edu/people/vera/teaching/courses/discrete-and-algorithmic-geometry/the-parabola-and-the-circle-dualities-applet/>

INTERSECTING HALF-PLANES

Intersection of half-planes and convex hulls

INTERSECTING HALF-PLANES

Intersection of half-planes and convex hulls



INTERSECTING HALF-PLANES

Computing the intersection of half-planes

Input: n half-planes

Output: Sorted list of vertices (and, possibly, half-lines) defining the boundary of their intersection (a convex polygon or an unbounded convex polygonal region).

INTERSECTING HALF-PLANES

Computing the intersection of half-planes

Input: n half-planes

Output: Sorted list of vertices (and, possibly, half-lines) defining the boundary of their intersection (a convex polygon or an unbounded convex polygonal region).

Algorithm 1: by duality

Solves the problem in $O(n \log n)$ time by dualizing and computing two convex hulls.

INTERSECTING HALF-PLANES

Computing the intersection of half-planes

Input: n half-planes

Output: Sorted list of vertices (and, possibly, half-lines) defining the boundary of their intersection (a convex polygon or an unbounded convex polygonal region).

Algorithm 1: by duality

Solves the problem in $O(n \log n)$ time by dualizing and computing two convex hulls.

Algorithm 2: incremental

Solves the problem in $O(n \log n)$ time, by incrementally intersecting a convex polygon (or unbounded convex polygonal region) with a halfplane.

INTERSECTING HALF-PLANES

Computing the intersection of half-planes

Input: n half-planes

Output: Sorted list of vertices (and, possibly, half-lines) defining the boundary of their intersection (a convex polygon or an unbounded convex polygonal region).

Algorithm 1: by duality

Solves the problem in $O(n \log n)$ time by dualizing and computing two convex hulls.

Algorithm 2: incremental

Solves the problem in $O(n \log n)$ time, by incrementally intersecting a convex polygon (or unbounded convex polygonal region) with a halfplane.

Algorithm 3: by divide and conquer

Solves the problem in $O(n \log n)$ time: the merging step consists in computing the intersection of two convex polygons (or unbounded convex polygonal regions) in $O(n)$ time.

INTERSECTING HALF-PLANES

Computing the intersection of half-planes

Input: n half-planes

Output: Sorted list of vertices (and, possibly, half-lines) defining the boundary of their intersection (a convex polygon or an unbounded convex polygonal region).

Algorithm 1: by duality

Solves the problem in $O(n \log n)$ time by dualizing and computing two convex hulls.

Algorithm 2: incremental

Solves the problem in $O(n \log n)$ time, by incrementally intersecting a convex polygon (or unbounded convex polygonal region) with a halfplane.

Algorithm 3: by divide and conquer

Solves the problem in $O(n \log n)$ time: the merging step consists in computing the intersection of two convex polygons (or unbounded convex polygonal regions) in $O(n)$ time.

Lower bound

Duality reduces in $O(n)$ time the problem of computing the convex hull of n points to the problem of computing the intersection of n half-planes. Therefore, computing the intersection of n half-planes has an $\Omega(n \log n)$ lower bound.

SOLVING LINEAR PROGRAMS

Linear program

Optimize a linear function $ax + by$

subject to n linear restrictions $a_i x + b_i y + c_i \leq 0$, where $i = 1, \dots, n$.

SOLVING LINEAR PROGRAMS

Example

SOLVING LINEAR PROGRAMS

Example

A manufacturer produces two products, X and Y , with two machines, A and B .

SOLVING LINEAR PROGRAMS

Example

A manufacturer produces two products, X and Y , with two machines, A and B .

The cost of producing each unit of X is:

- for machine A : 50 minutes,
- for machine B : 30 minutes.

The cost of producing each unit of Y is:

- for machine A : 24 minutes,
- for machine B : 33 minutes.

SOLVING LINEAR PROGRAMS

Example

A manufacturer produces two products, X and Y , with two machines, A and B .

The cost of producing each unit of X is:

- for machine A : 50 minutes,
- for machine B : 30 minutes.

The cost of producing each unit of Y is:

- for machine A : 24 minutes,
- for machine B : 33 minutes.

Working plans for a particular week are:

- 40 hours of work on machine A ,
- 35 hours of work on machine B .

SOLVING LINEAR PROGRAMS

Example

A manufacturer produces two products, X and Y , with two machines, A and B .

The cost of producing each unit of X is:

- for machine A : 50 minutes,
- for machine B : 30 minutes.

The cost of producing each unit of Y is:

- for machine A : 24 minutes,
- for machine B : 33 minutes.

Working plans for a particular week are:

- 40 hours of work on machine A ,
- 35 hours of work on machine B .

The week starts with:

- A stock of 30 units of X and 90 of Y ,
- A demand of 75 units of X and 95 of Y .

SOLVING LINEAR PROGRAMS

Example

A manufacturer produces two products, X and Y , with two machines, A and B .

The cost of producing each unit of X is:

- for machine A : 50 minutes,
- for machine B : 30 minutes.

The cost of producing each unit of Y is:

- for machine A : 24 minutes,
- for machine B : 33 minutes.

Working plans for a particular week are:

- 40 hours of work on machine A ,
- 35 hours of work on machine B .

The week starts with:

- A stock of 30 units of X and 90 of Y ,
- A demand of 75 units of X and 95 of Y .

How to plan the production, in order to end the week with the maximum stock?

SOLVING LINEAR PROGRAMS

Example

A manufacturer produces two products, X and Y , with two machines, A and B .

The cost of producing each unit of X is:

- for machine A : 50 minutes,
- for machine B : 30 minutes.

The cost of producing each unit of Y is:

- for machine A : 24 minutes,
- for machine B : 33 minutes.

Working plans for a particular week are:

- 40 hours of work on machine A ,
- 35 hours of work on machine B .

The week starts with:

- A stock of 30 units of X and 90 of Y ,
- A demand of 75 units of X and 95 of Y .

How to plan the production, in order to end the week with the maximum stock?

$$\text{Maximize } (x + 30 - 75) + (y + 90 - 95)$$

x = units of X to be produced

y = units of Y to be produced

SOLVING LINEAR PROGRAMS

Example

A manufacturer produces two products, X and Y , with two machines, A and B .

The cost of producing each unit of X is:

- for machine A : 50 minutes,
- for machine B : 30 minutes.

The cost of producing each unit of Y is:

- for machine A : 24 minutes,
- for machine B : 33 minutes.

Working plans for a particular week are:

- 40 hours of work on machine A ,
- 35 hours of work on machine B .

The week starts with:

- A stock of 30 units of X and 90 of Y ,
- A demand of 75 units of X and 95 of Y .

How to plan the production, in order to end the week with the maximum stock?

Maximize $(x + 30 - 75) + (y + 90 - 95)$

x = units of X to be produced

y = units of Y to be produced

restricted to:

$$50x + 24y \leq 40 \times 60$$

$$30x + 33y \leq 35 \times 60$$

$$x \geq 75 - 30$$

$$y \geq 95 - 90$$

SOLVING LINEAR PROGRAMS

Linear program

Optimize a linear function $ax + by$

subject to n linear restrictions $a_i x + b_i y + c_i \leq 0$, where $i = 1, \dots, n$.

SOLVING LINEAR PROGRAMS

Linear program

Optimize a linear function $ax + by$

subject to n linear restrictions $a_i x + b_i y + c_i \leq 0$, where $i = 1, \dots, n$.

or, equivalently (apply the linear transform $Y = \pm(ax + by)$, $X = x$):

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

SOLVING LINEAR PROGRAMS

Linear program

Optimize a linear function $ax + by$

subject to n linear restrictions $a_i x + b_i y + c_i \leq 0$, where $i = 1, \dots, n$.

or, equivalently (apply the linear transform $Y = \pm(ax + by)$, $X = x$):

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Algorithm 1

Compute the feasible region R in $O(n \log n)$ time. Any linear function on R can then be optimized in $O(\log n)$ time by binary search.

SOLVING LINEAR PROGRAMS

Linear program

Optimize a linear function $ax + by$

subject to n linear restrictions $a_i x + b_i y + c_i \leq 0$, where $i = 1, \dots, n$.

or, equivalently (apply the linear transform $Y = \pm(ax + by)$, $X = x$):

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Algorithm 1

Compute the feasible region R in $O(n \log n)$ time. Any linear function on R can then be optimized in $O(\log n)$ time by binary search.

Algo 2 (Megiddo, Dyer)

In $O(n)$ time, it is possible to find the vertex of R achieving the optimum, without computing the entire feasible region R .

This is done by a prune and search strategy, which at each step eliminates a constant fraction of the restrictions:

- either because they are redundant,
- or because they are not needed to define the solution vertex.

SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.



SOLVING LINEAR PROGRAMS

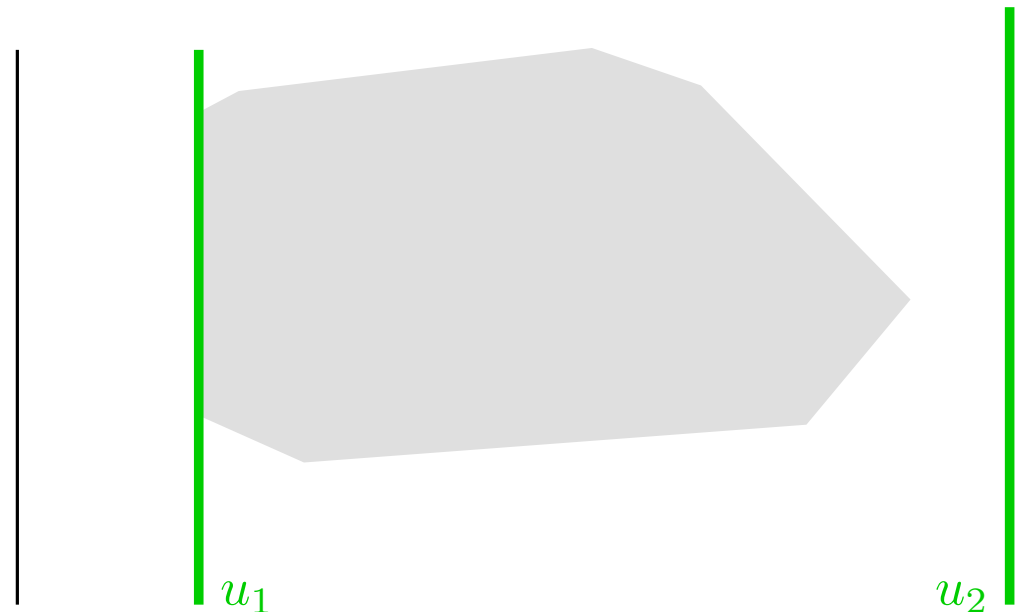
Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.

$I_+ = \{i \mid b_i > 0\}$, corresponds to restrictions of the form $y \leq d_i x + e_i$.

Let's define $F_+(x) = \min\{d_i x + e_i \mid i \in I_+\}$.



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

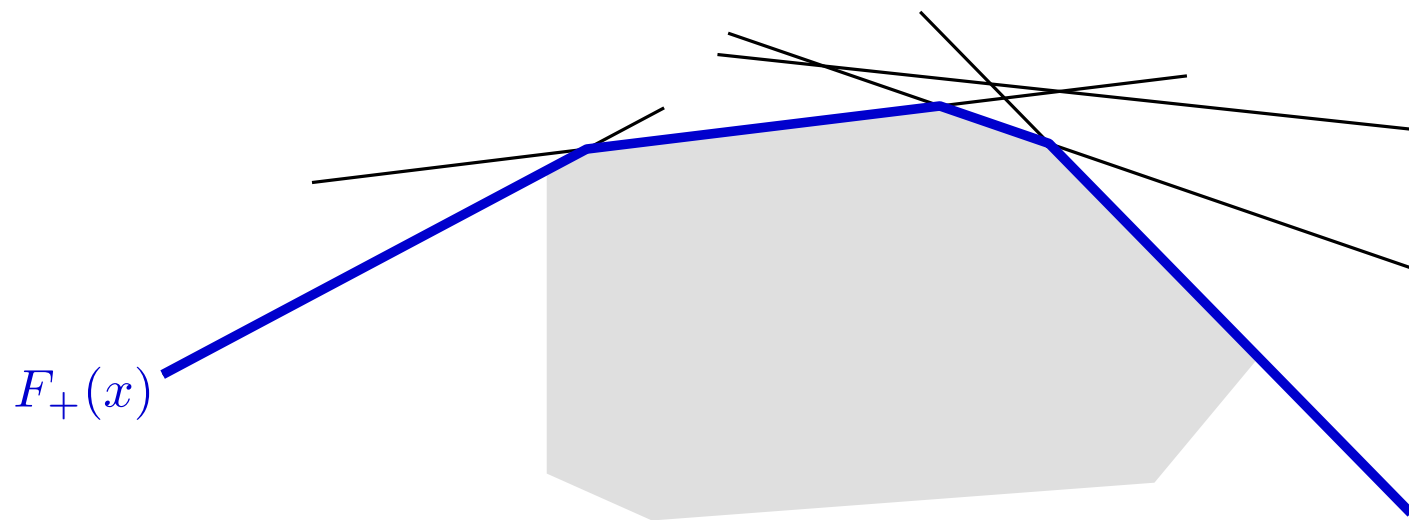
Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.

$I_+ = \{i \mid b_i > 0\}$, corresponds to restrictions of the form $y \leq d_i x + e_i$.

Let's define $F_+(x) = \min\{d_i x + e_i \mid i \in I_+\}$.



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.

$I_+ = \{i \mid b_i > 0\}$, corresponds to restrictions of the form $y \leq d_i x + e_i$.

Let's define $F_+(x) = \min\{d_i x + e_i \mid i \in I_+\}$.

$I_- = \{i \mid b_i < 0\}$, corresponds to restrictions of the form $y \geq d_i x + e_i$.

Let's define $F_-(x) = \max\{d_i x + e_i \mid i \in I_-\}$.



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

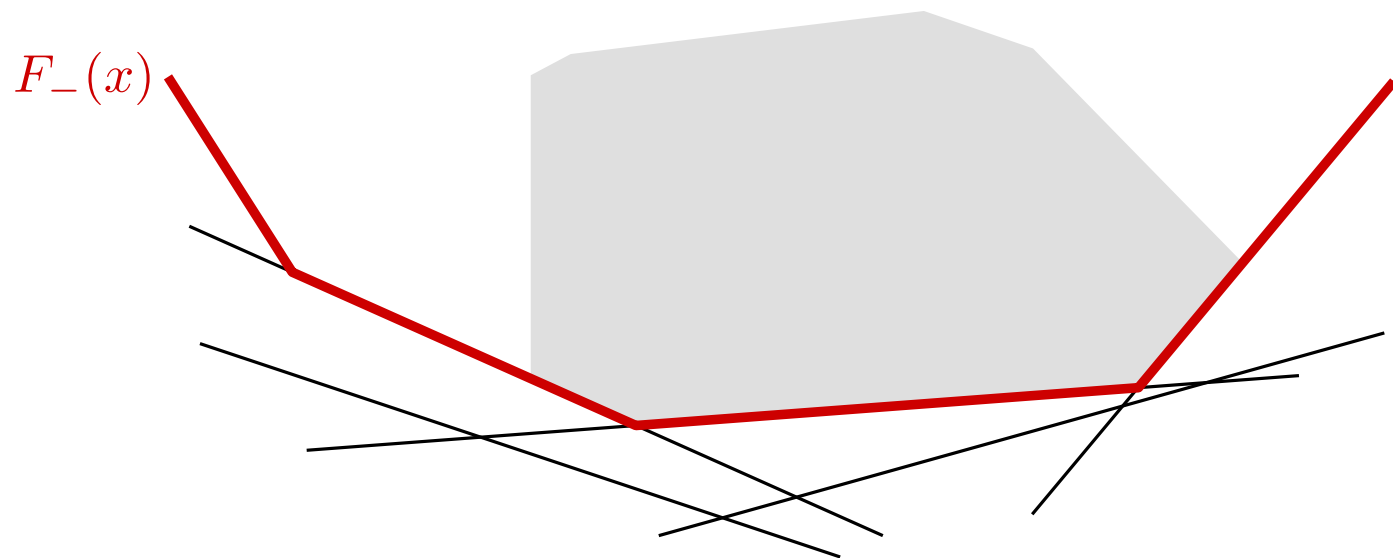
Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.

$I_+ = \{i \mid b_i > 0\}$, corresponds to restrictions of the form $y \leq d_i x + e_i$.

Let's define $F_+(x) = \min\{d_i x + e_i \mid i \in I_+\}$.

$I_- = \{i \mid b_i < 0\}$, corresponds to restrictions of the form $y \geq d_i x + e_i$.

Let's define $F_-(x) = \max\{d_i x + e_i \mid i \in I_-\}$.



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

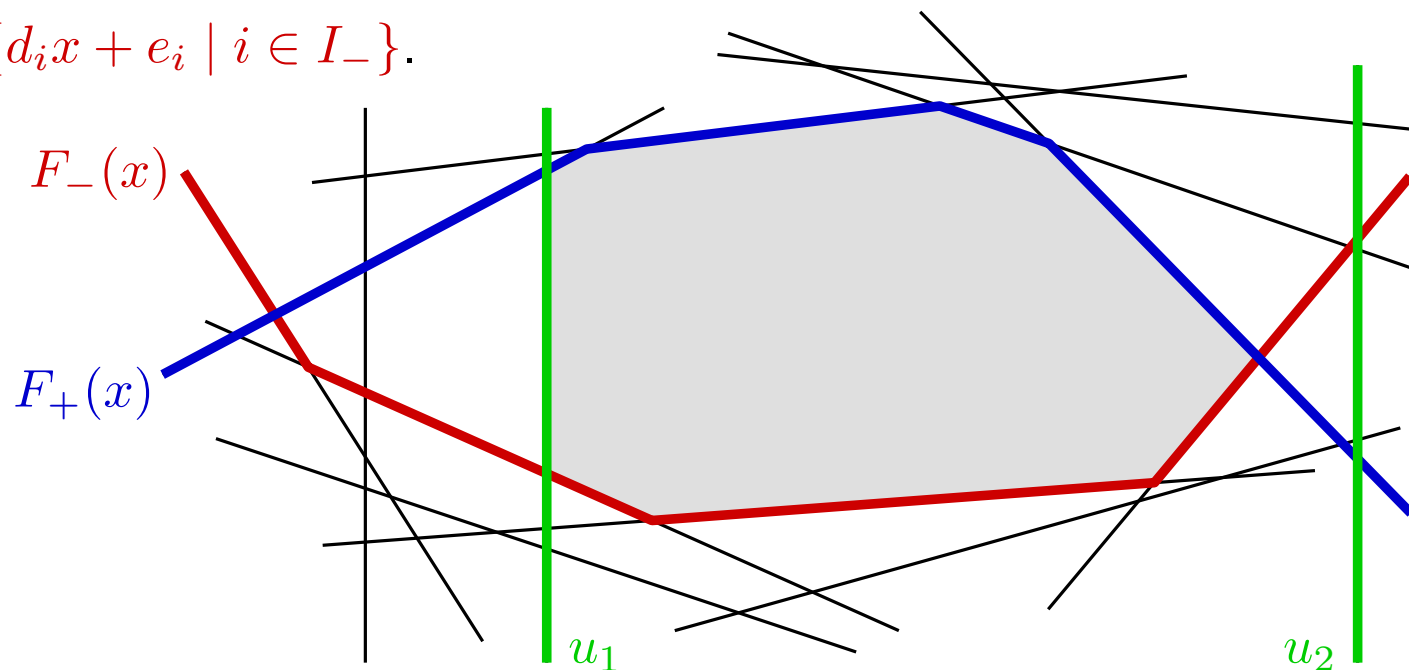
Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.

$I_+ = \{i \mid b_i > 0\}$, corresponds to restrictions of the form $y \leq d_i x + e_i$.

Let's define $F_+(x) = \min\{d_i x + e_i \mid i \in I_+\}$.

$I_- = \{i \mid b_i < 0\}$, corresponds to restrictions of the form $y \geq d_i x + e_i$.

Let's define $F_-(x) = \max\{d_i x + e_i \mid i \in I_-\}$.



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.

$I_+ = \{i \mid b_i > 0\}$, corresponds to restrictions of the form $y \leq d_i x + e_i$.

Let's define $F_+(x) = \min\{d_i x + e_i \mid i \in I_+\}$.

$I_- = \{i \mid b_i < 0\}$, corresponds to restrictions of the form $y \geq d_i x + e_i$.

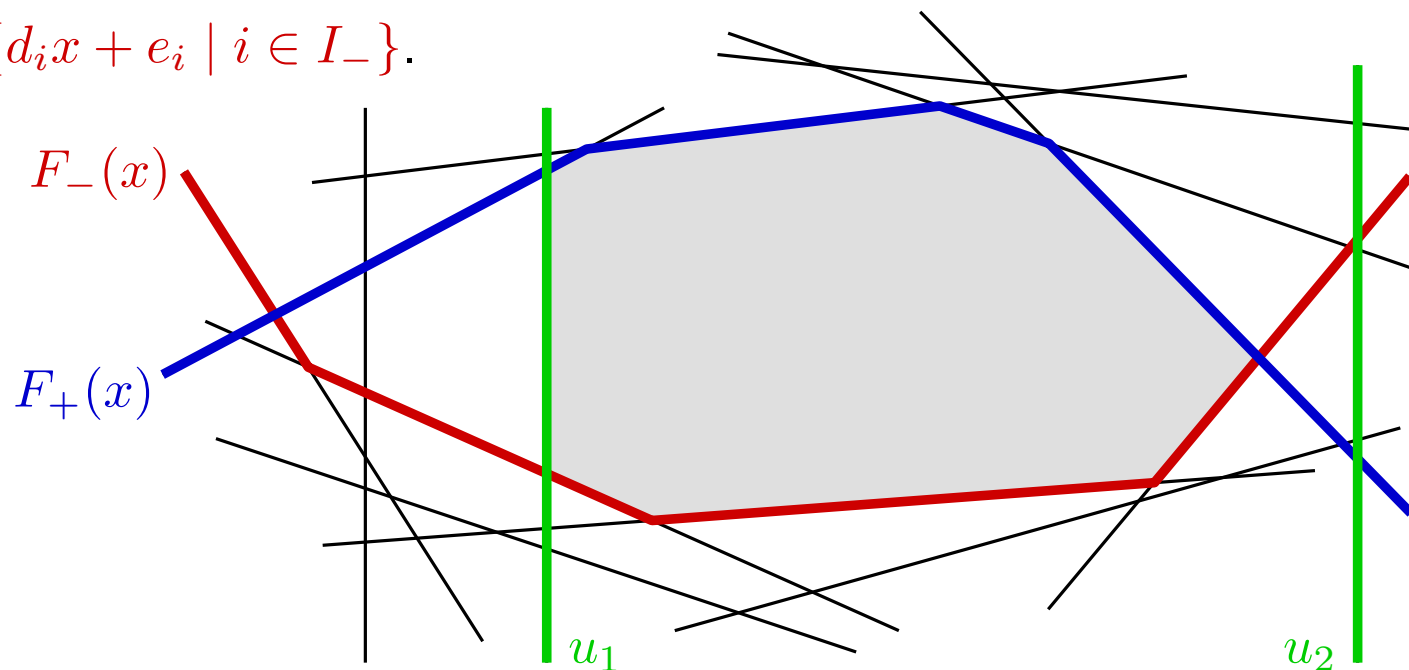
Let's define $F_-(x) = \max\{d_i x + e_i \mid i \in I_-\}$.

Minimize $F_-(x)$

restricted to

$$u_1 \leq x \leq u_2$$

$$F_-(x) \leq F_+(x)$$



SOLVING LINEAR PROGRAMS

Minimize y

restricted to $a_i x + b_i y + c_i \leq 0$, with $i = 1, \dots, n$.

Partition $I = \{1, \dots, n\}$ into three sets:

$I_0 = \{i \mid b_i = 0\}$, corresponds to vertical restrictions $a_i x + c_i \leq 0$.

Let's define $u_1 = \max\{-c_i/a_i \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_i \mid i \in I_0, a_i \geq 0\}$.

$I_+ = \{i \mid b_i > 0\}$, corresponds to restrictions of the form $y \leq d_i x + e_i$.

Let's define $F_+(x) = \min\{d_i x + e_i \mid i \in I_+\}$.

$I_- = \{i \mid b_i < 0\}$, corresponds to restrictions of the form $y \geq d_i x + e_i$.

Let's define $F_-(x) = \max\{d_i x + e_i \mid i \in I_-\}$.

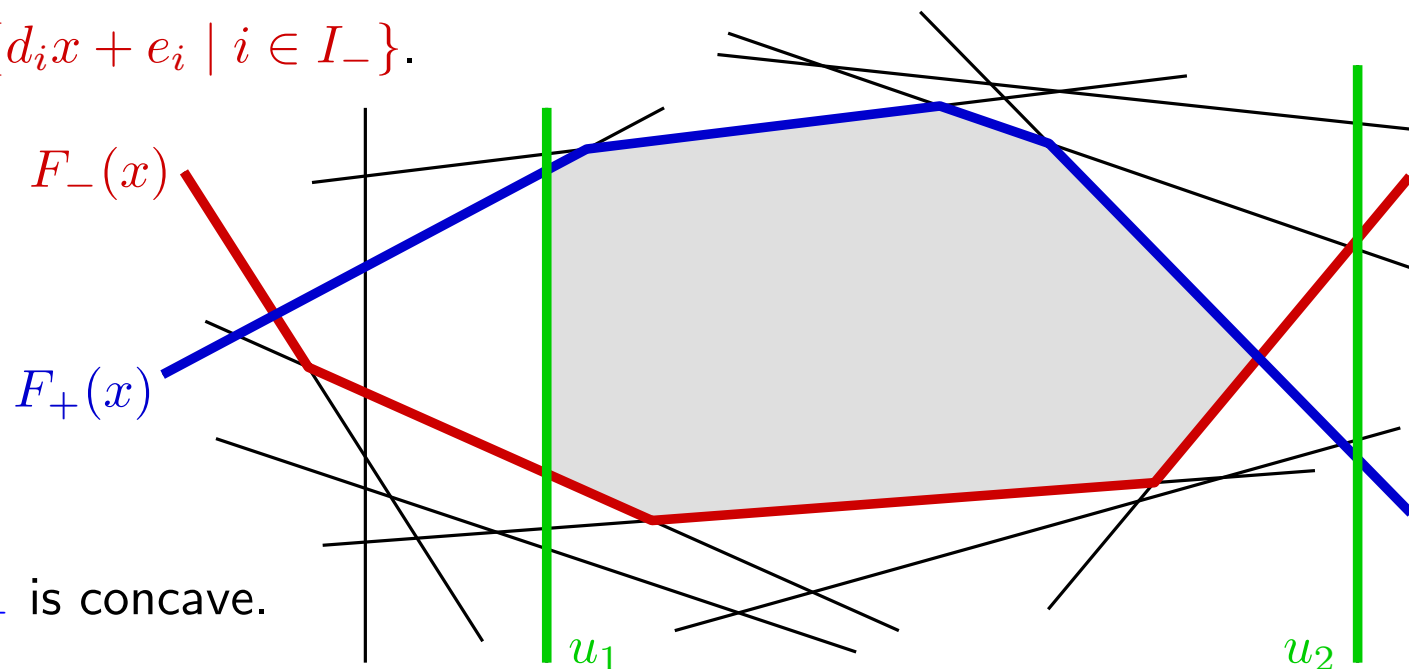
Minimize $F_-(x)$

restricted to

$$u_1 \leq x \leq u_2$$

$$F_-(x) \leq F_+(x)$$

Notice that F_- is convex and F_+ is concave.



SOLVING LINEAR PROGRAMS

The search

SOLVING LINEAR PROGRAMS

The search

Given $x' \in [u_1, u_2]$, in $O(n)$ time it is possible to get to one of the following conclusions:

1. x' is infeasible, and the problem has no solution.
2. x' is infeasible, and the solution of the problem does not lie to its left (right).
3. x' is feasible, and the solution of the problem lies to its right (left).
4. x' is feasible, and minimizing $F_-(x)$.

SOLVING LINEAR PROGRAMS

The search

Given $x' \in [u_1, u_2]$, in $O(n)$ time it is possible to get to one of the following conclusions:

1. x' is infeasible, and the problem has no solution.
2. x' is infeasible, and the solution of the problem does not lie to its left (right).
3. x' is feasible, and the solution of the problem lies to its right (left).
4. x' is feasible, and minimizing $F_-(x)$.

This is done by analyzing the values of F_+ , F_- and their slopes $f_+^l, f_+^r, f_-^l, f_-^r$ in x' , where the slopes are defined as follows:

- If $F_-(x')$ (resp. $F_+(x')$) is defined by a unique index $i \in I_-$ (resp. I_+), then $f_-^l(x') = f_-^r(x') = d_i$ (resp. $f_+^l(x') = f_+^r(x') = d_i$).
- If $F_-(x')$ (resp. $F_+(x')$) is defined by two indexes $i, j \in I_-$ (resp. I_+), then $f_-^l(x') = \min(d_i, d_j)$ and $f_-^r(x') = \max(d_i, d_j)$ (resp. $f_+^l(x') = \max(d_i, d_j)$ and $f_+^r(x') = \min(d_i, d_j)$).

SOLVING LINEAR PROGRAMS

The search

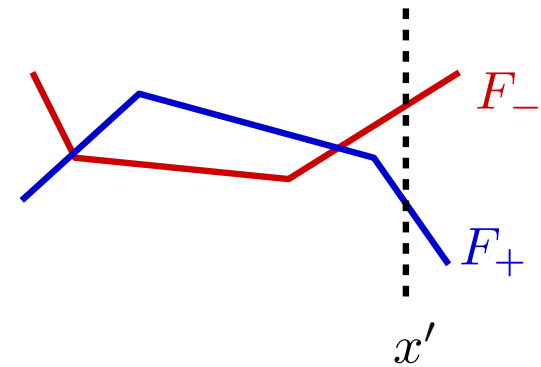
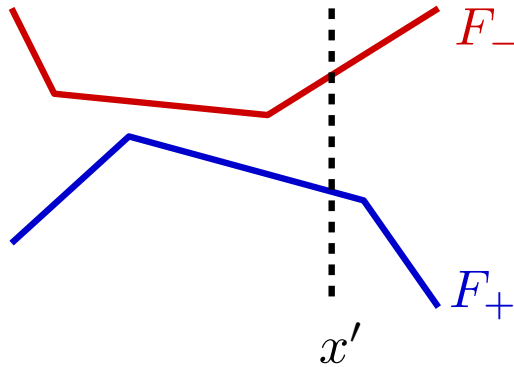
Case 1. $F_-(x') > F_+(x')$ (x' is infeasible)

SOLVING LINEAR PROGRAMS

The search

Case 1. $F_-(x') > F_+(x')$ (x' is infeasible)

If $f_-^l(x') > f_+^l(x')$,
then search to the left of x' .

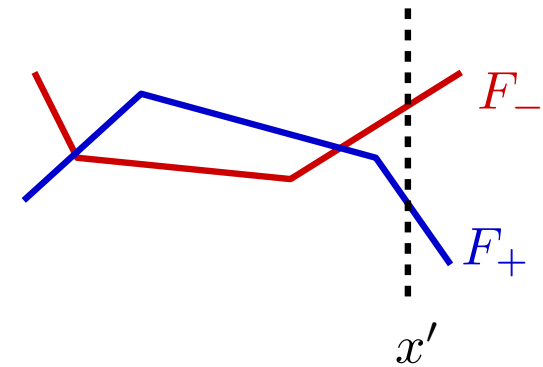
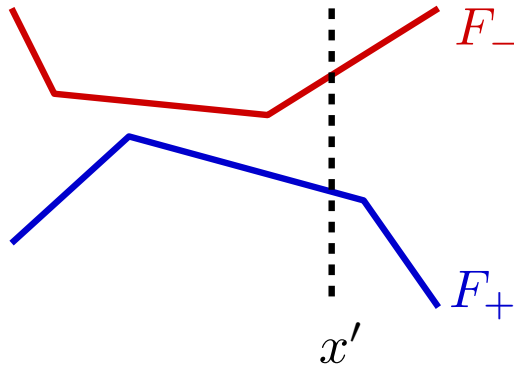


SOLVING LINEAR PROGRAMS

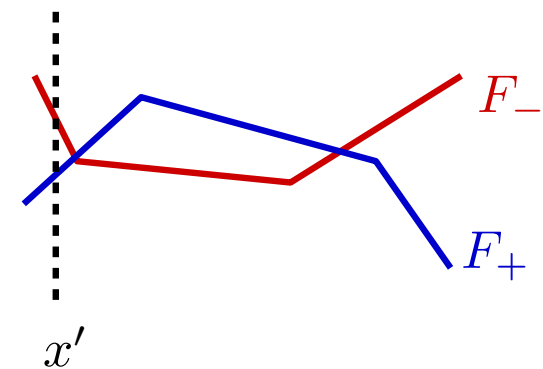
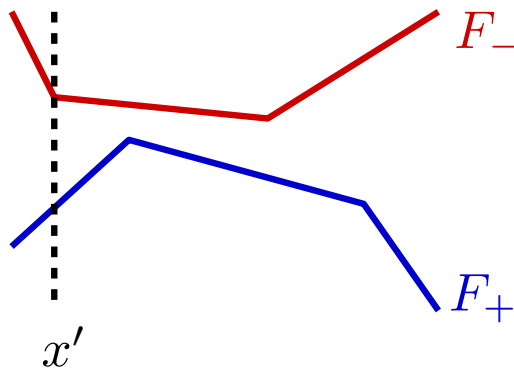
The search

Case 1. $F_-(x') > F_+(x')$ (x' is infeasible)

If $f_-^l(x') > f_+^l(x')$,
then search to the left of x' .



If $f_-^r(x') < f_+^r(x')$,
then search to the right of x' .

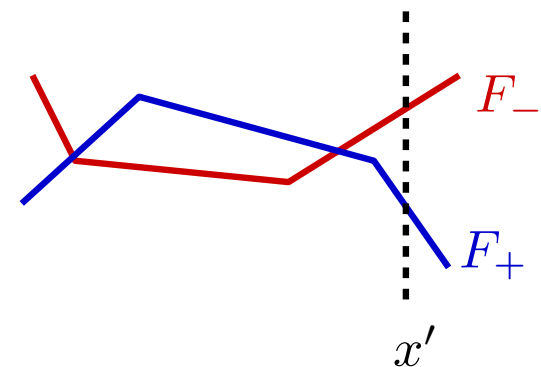
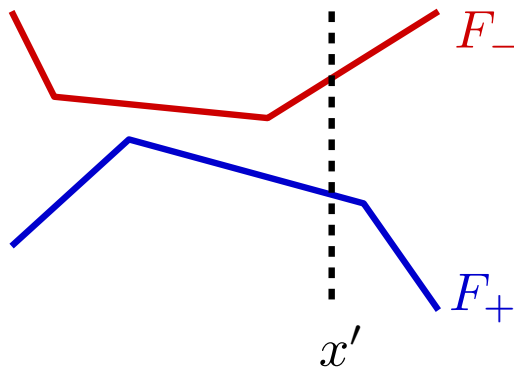


SOLVING LINEAR PROGRAMS

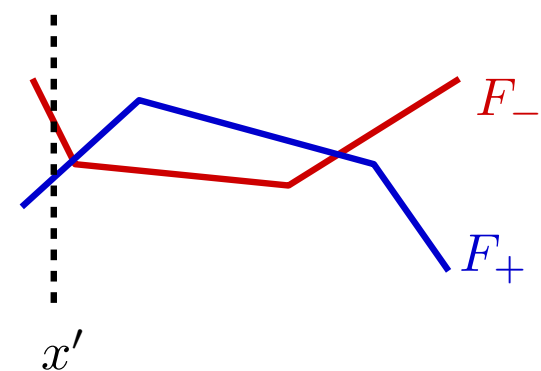
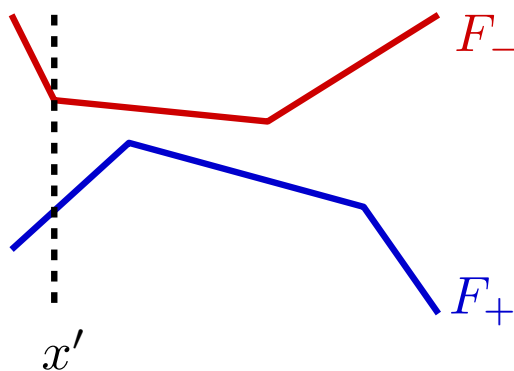
The search

Case 1. $F_-(x') > F_+(x')$ (x' is infeasible)

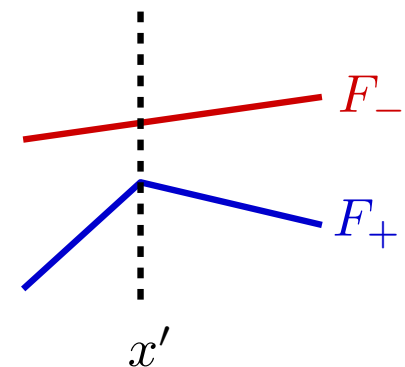
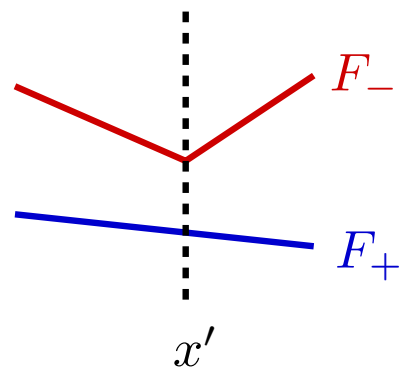
If $f_-^l(x') > f_+^l(x')$,
then search to the left of x' .



If $f_-^r(x') < f_+^r(x')$,
then search to the right of x' .



If $f_-^l(x') \leq f_+^l(x')$ and $f_-^r(x') \geq f_+^r(x')$
then there is no solution.



SOLVING LINEAR PROGRAMS

The search

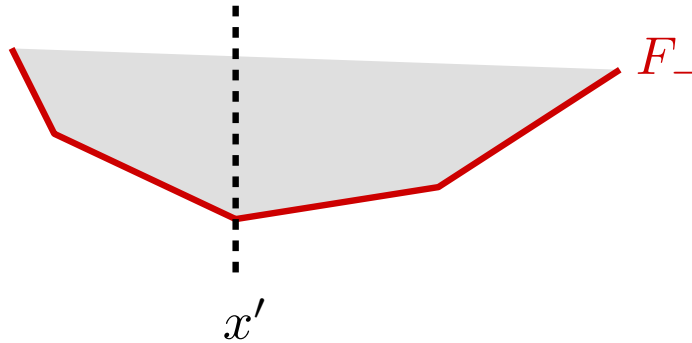
Case 2. $F_-(x') \leq F_+(x')$ (x' is feasible)

SOLVING LINEAR PROGRAMS

The search

Case 2. $F_-(x') \leq F_+(x')$ (x' is feasible)

If $f_-^l(x') < 0 < f_-^r(x')$,
then x' is the solution.

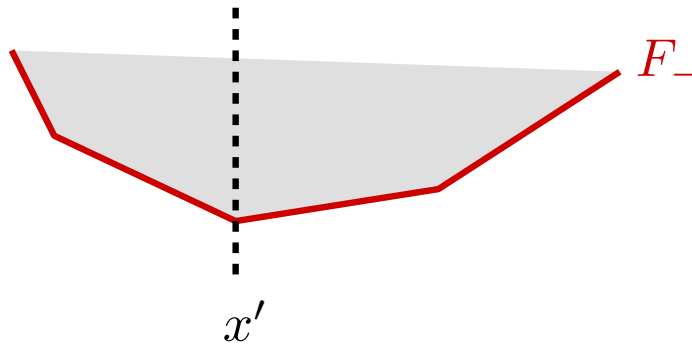


SOLVING LINEAR PROGRAMS

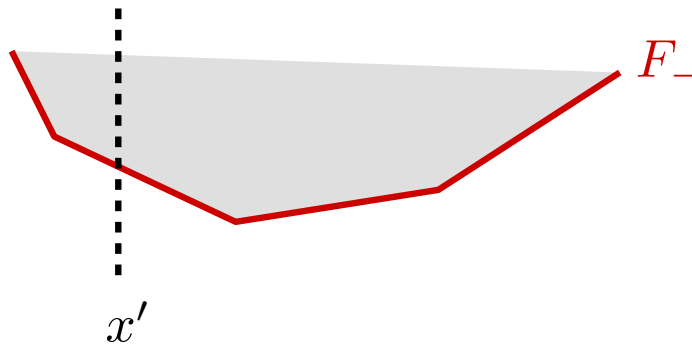
The search

Case 2. $F_-(x') \leq F_+(x')$ (x' is feasible)

If $f_-^l(x') < 0 < f_-^r(x')$,
then x' is the solution.



If $f_-^r(x') < 0$,
then search to the right of x' .

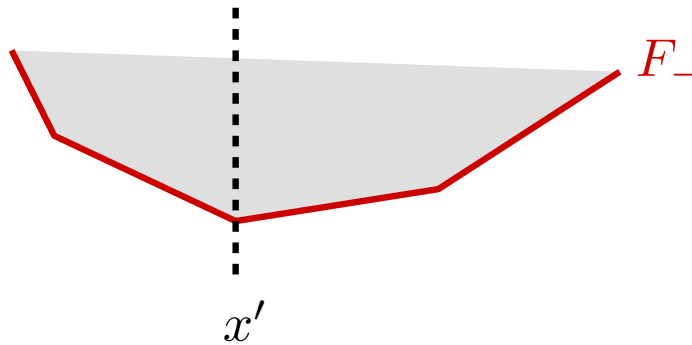


SOLVING LINEAR PROGRAMS

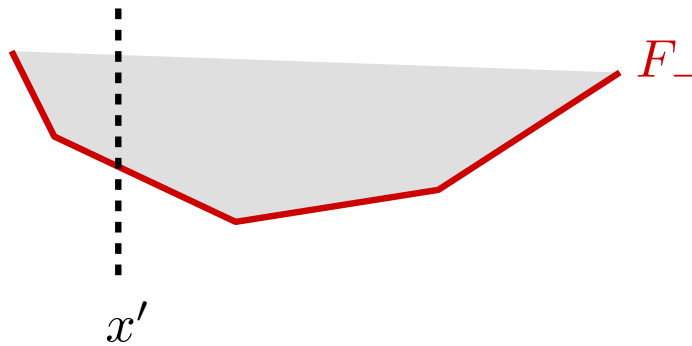
The search

Case 2. $F_-(x') \leq F_+(x')$ (x' is feasible)

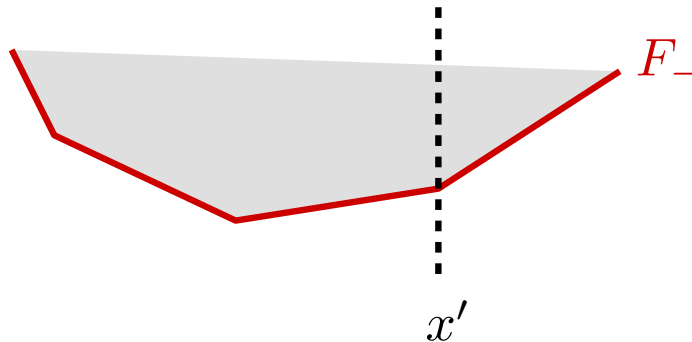
If $f_-^l(x') < 0 < f_-^r(x')$,
then x' is the solution.



If $f_-^r(x') < 0$,
then search to the right of x' .



If $f_-^l(x') > 0$,
then search to the left of x' .



SOLVING LINEAR PROGRAMS

The algorithm

SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist.

SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist.

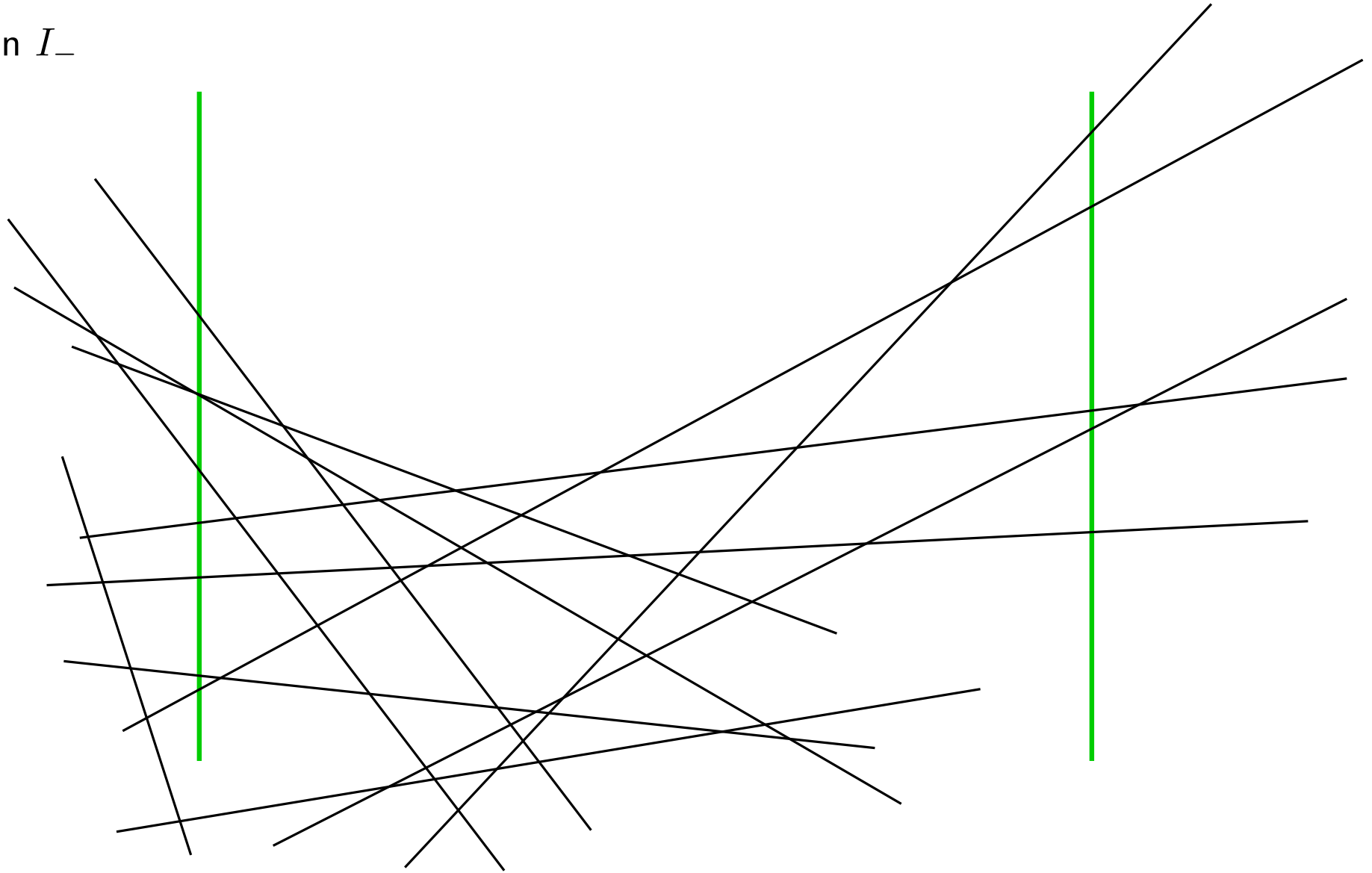
Advance. Repeat as many times as necessary:

1. Pair up all restrictions corresponding to I_- (possibly but one). Analogously for I_+ . Compute the abscissa of the intersection point of each pair of lines, if it exists.
2. For all pairs $i, j \in I_-$ (analogously for I_+), do:
 - If $d_i = d_j$, eliminate the line corresponding to $\min(e_i, e_j)$ (resp. $\max(e_i, e_j)$).
 - If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
 - If $x_{ij} < u_1$, eliminate the line corresponding to $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).
 - If $x_{ij} > u_2$, eliminate the line corresponding to $\max(d_i, d_j)$ (resp. $\min(d_i, d_j)$).
3. Compute the median value x' of the surviving x_{ij} .
4. Search: Apply the search procedure to x' . If the answer is x' , return x' .
5. Prune: If the answer is “search to the left” then, for each $x_{ij} > x'$ eliminate the line corresponding to $\max(d_i, d_j)$ (resp. $\min(d_i, d_j)$). If the answer is “search to the right” then, for each $x_{ij} < x'$ eliminate the line corresponding to $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).

SOLVING LINEAR PROGRAMS

The algorithm

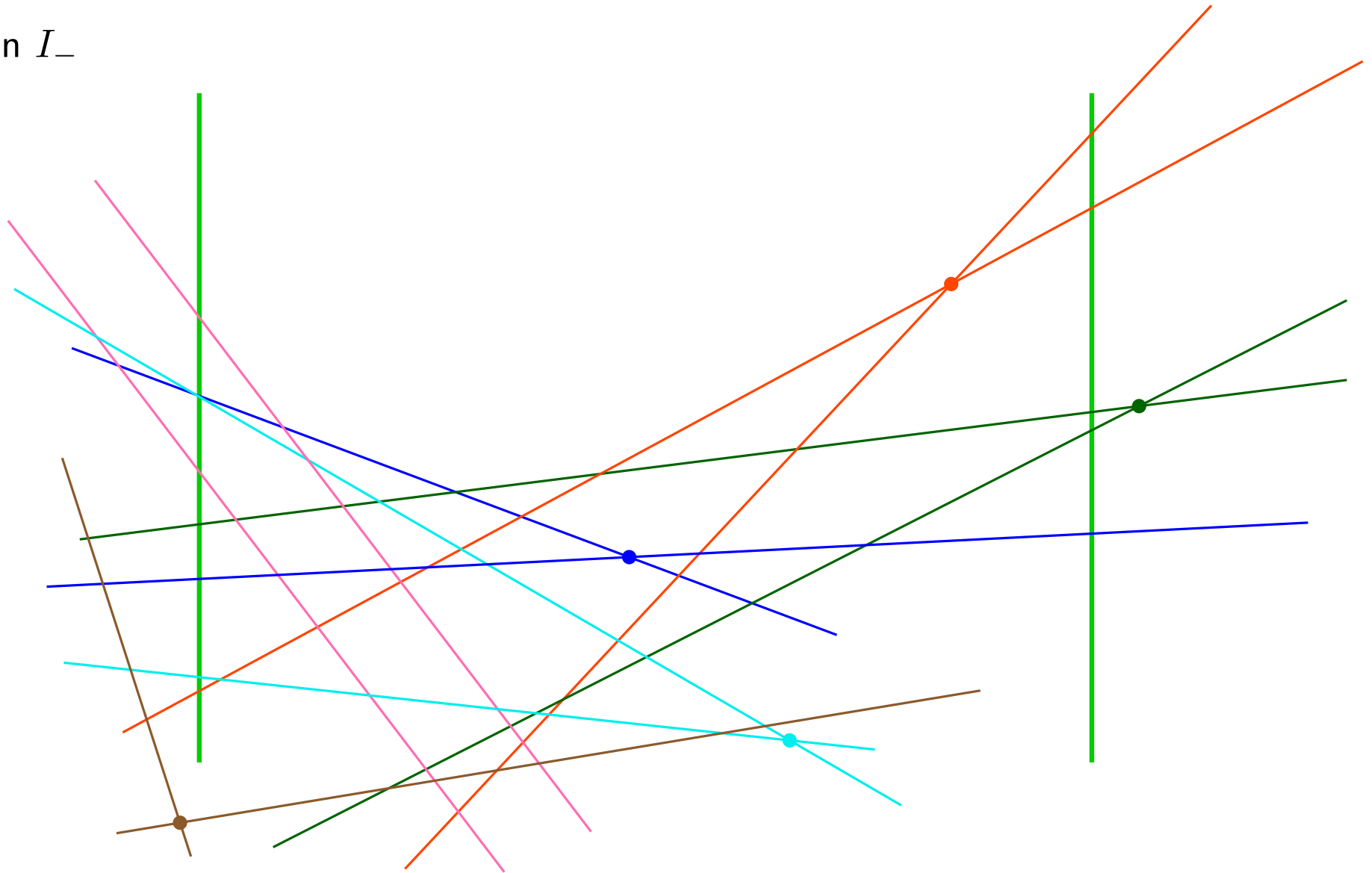
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

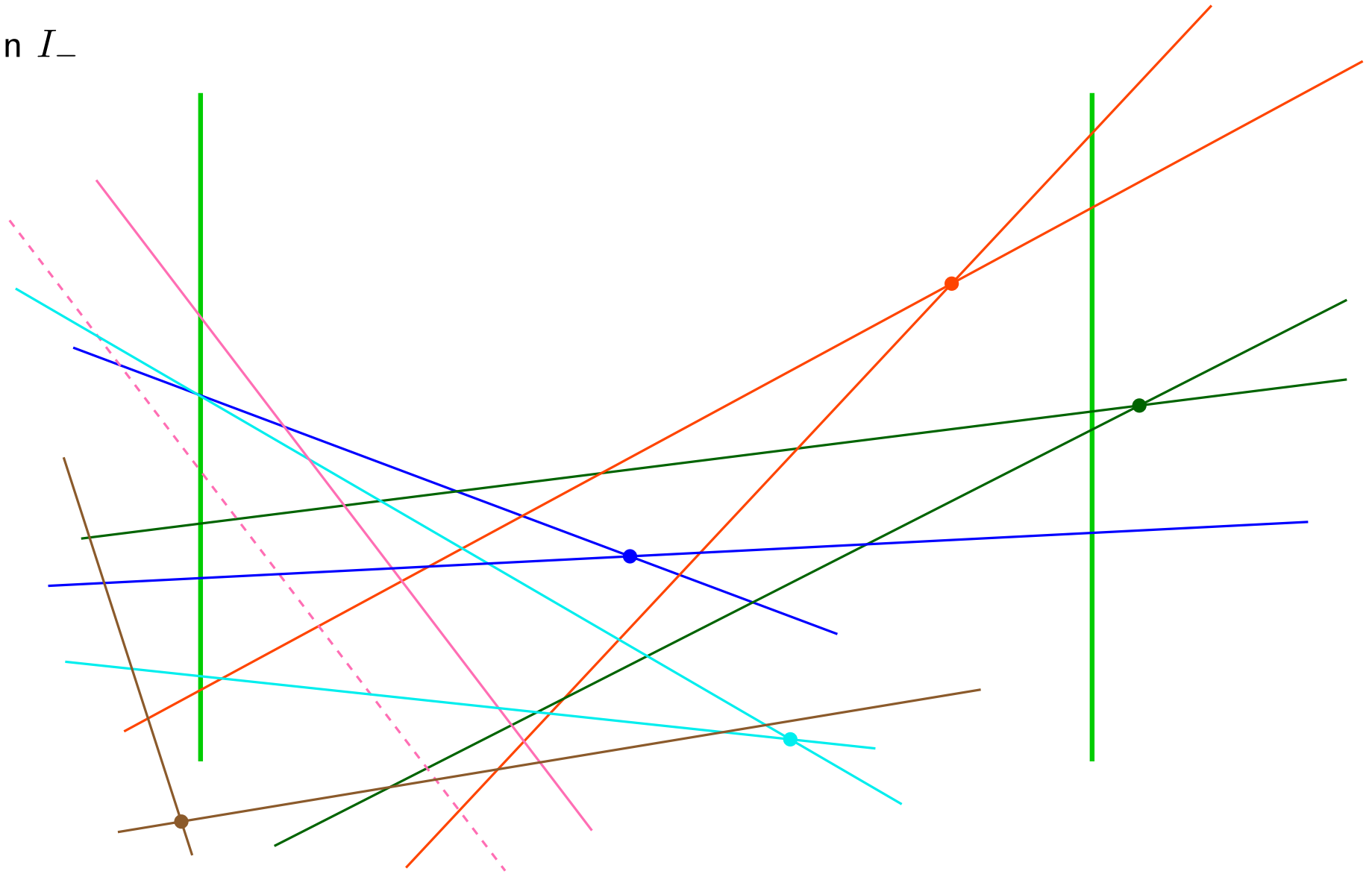
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

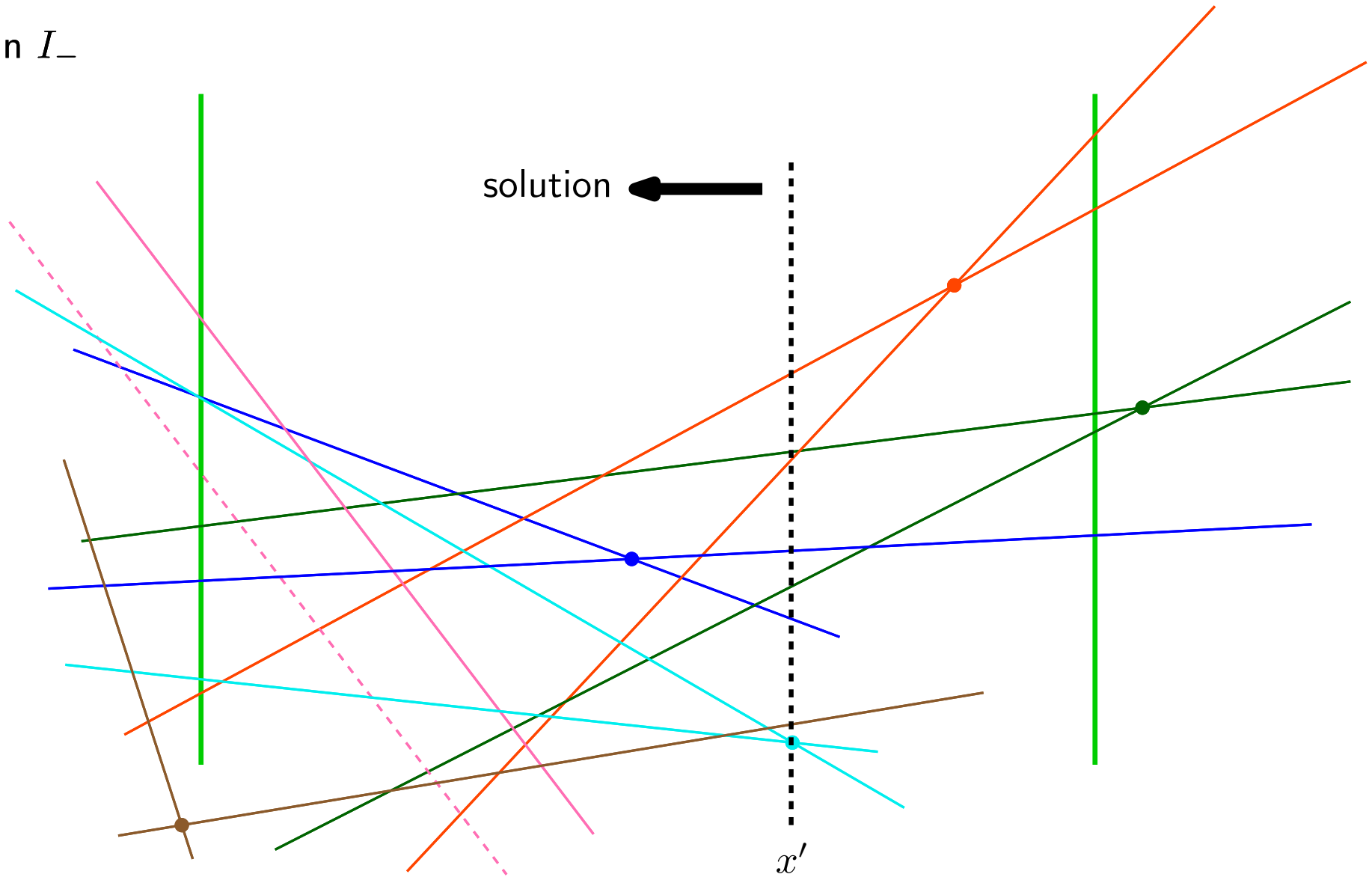
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

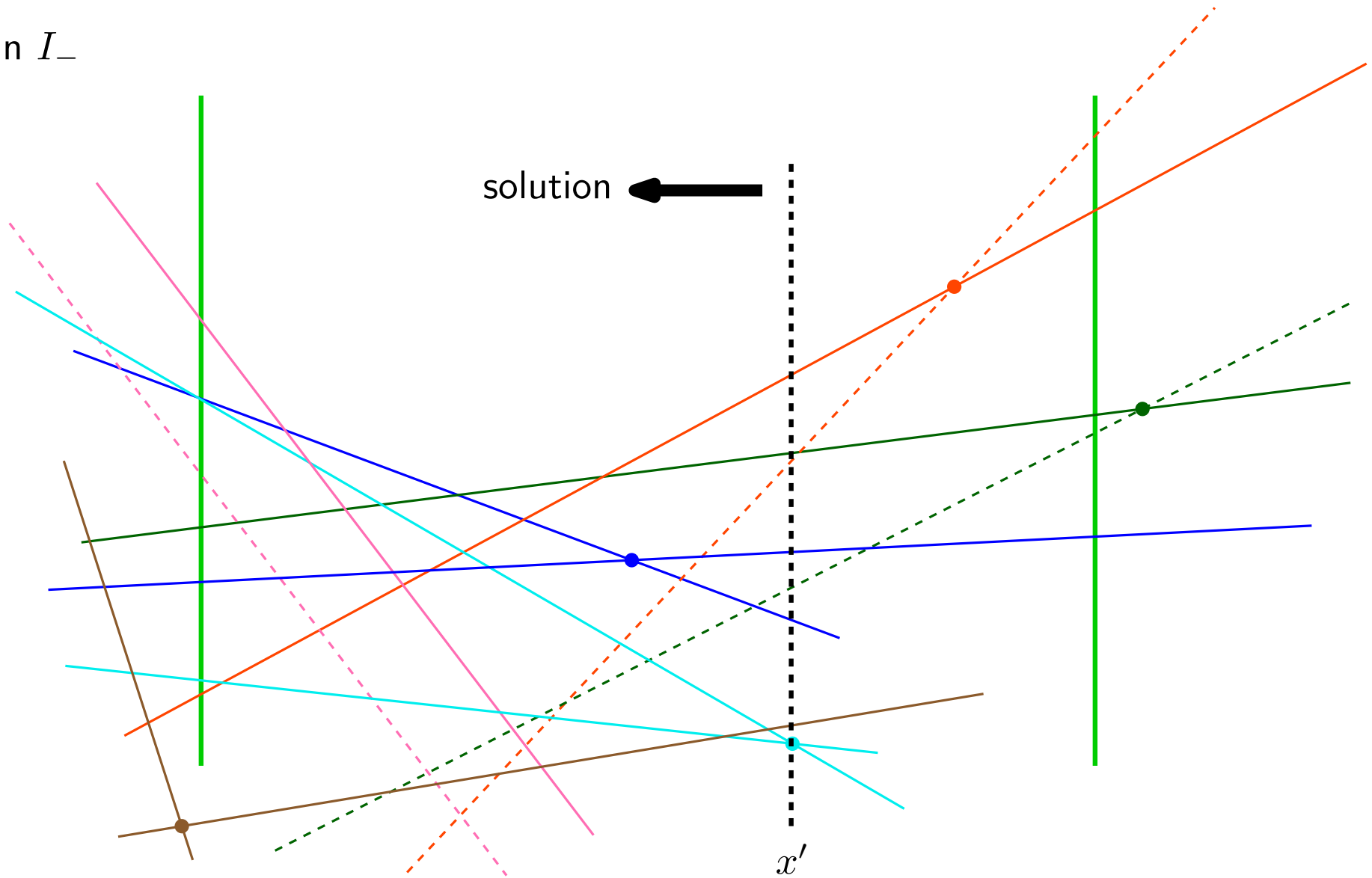
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

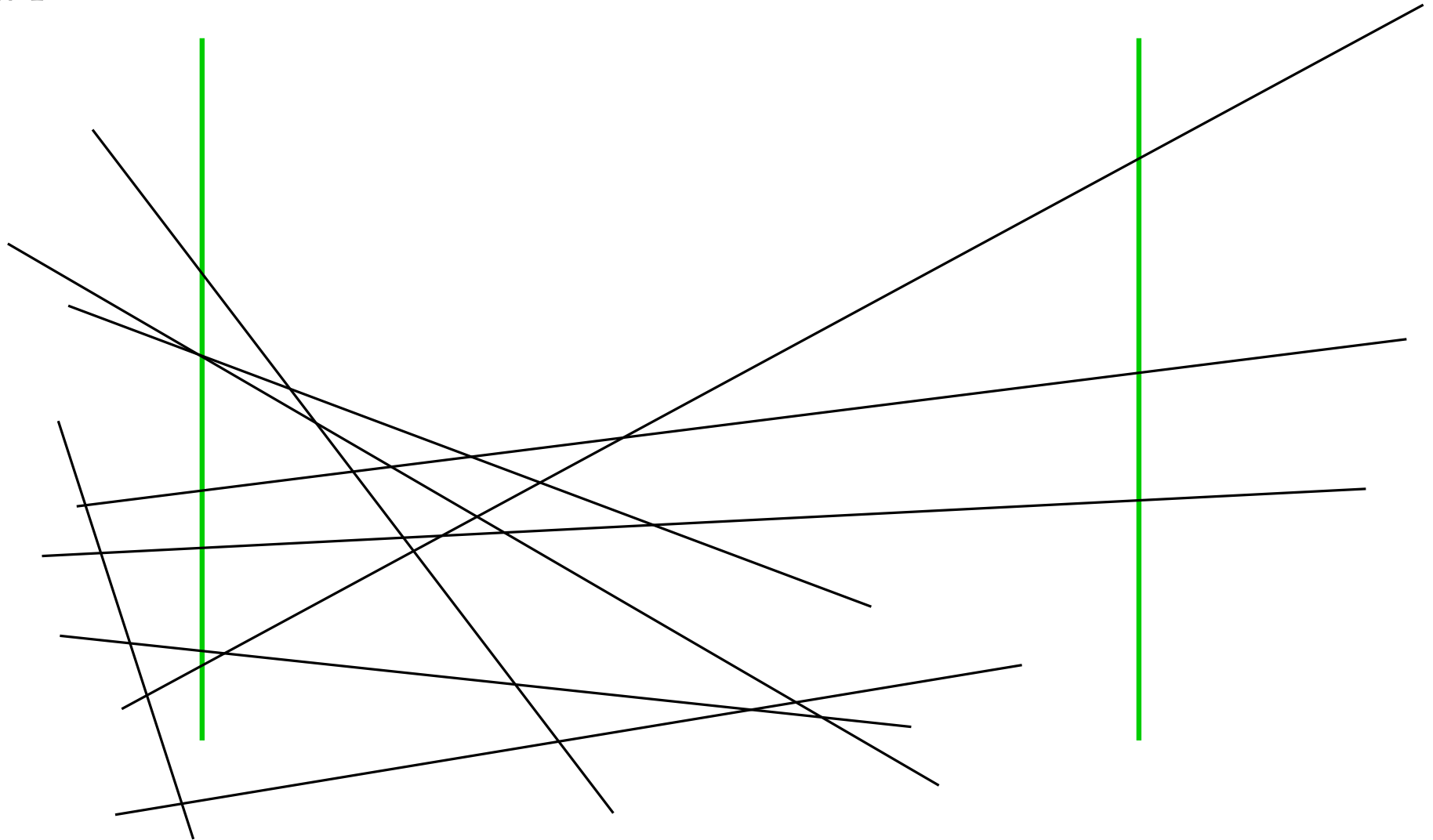
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

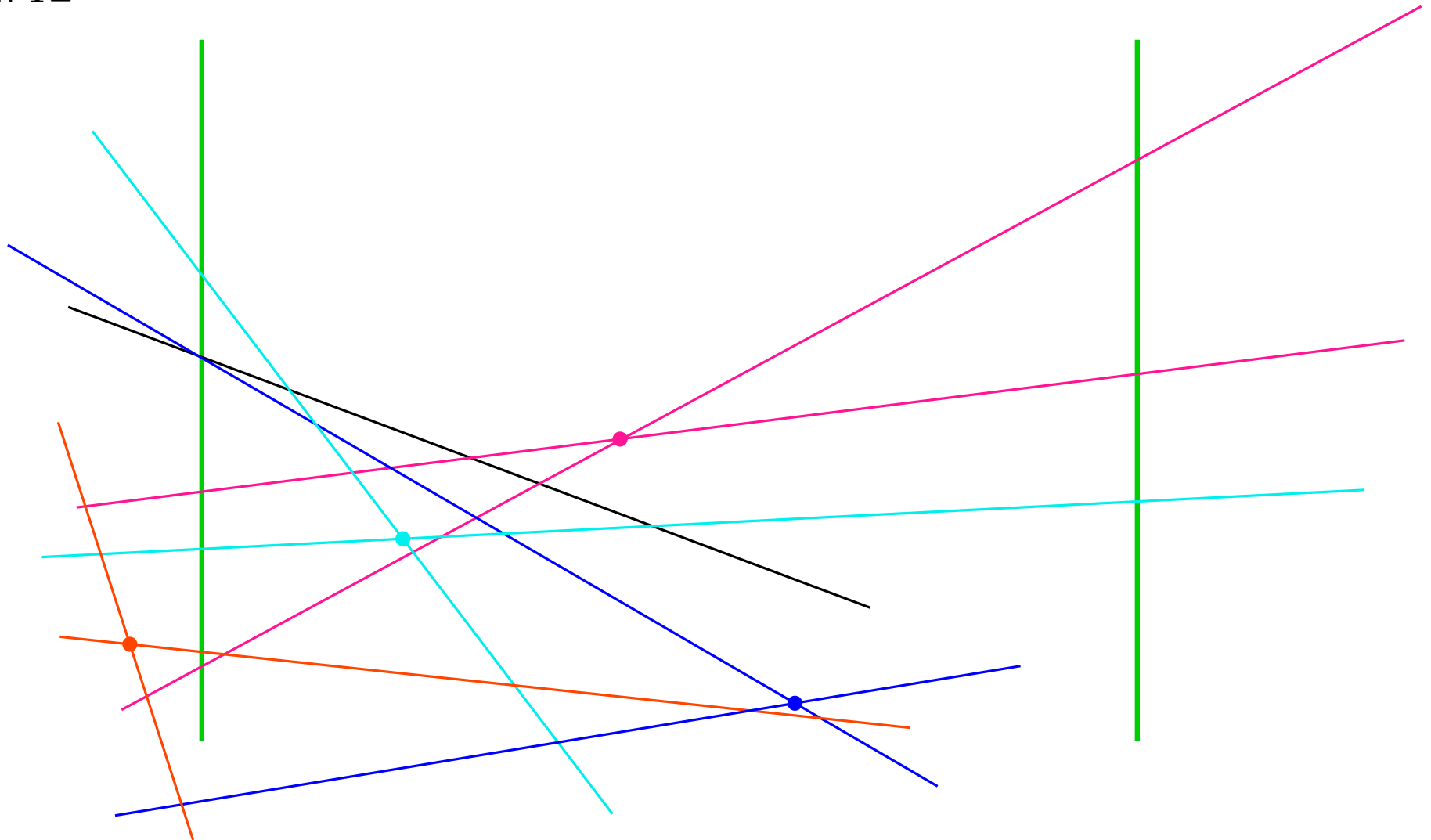
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

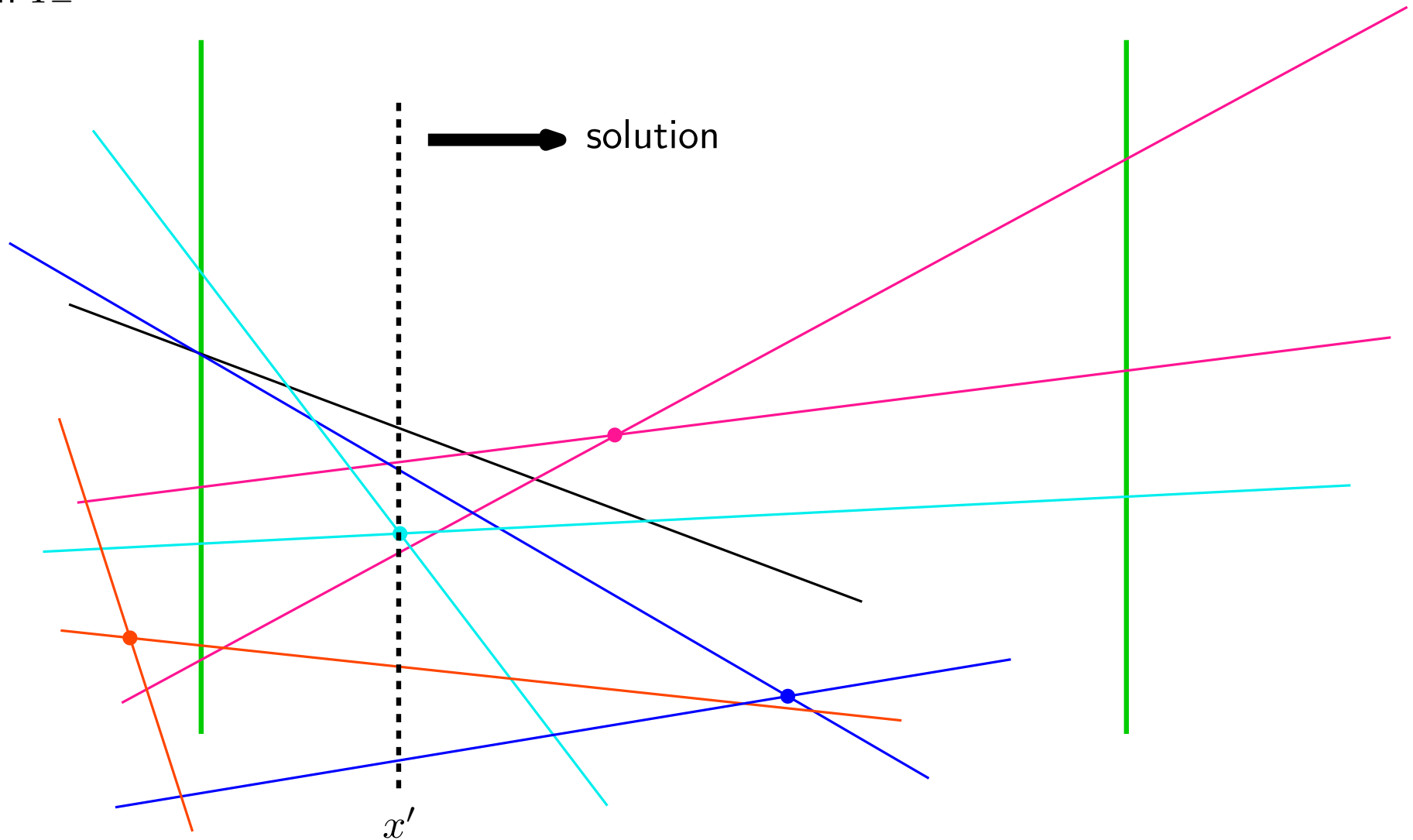
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

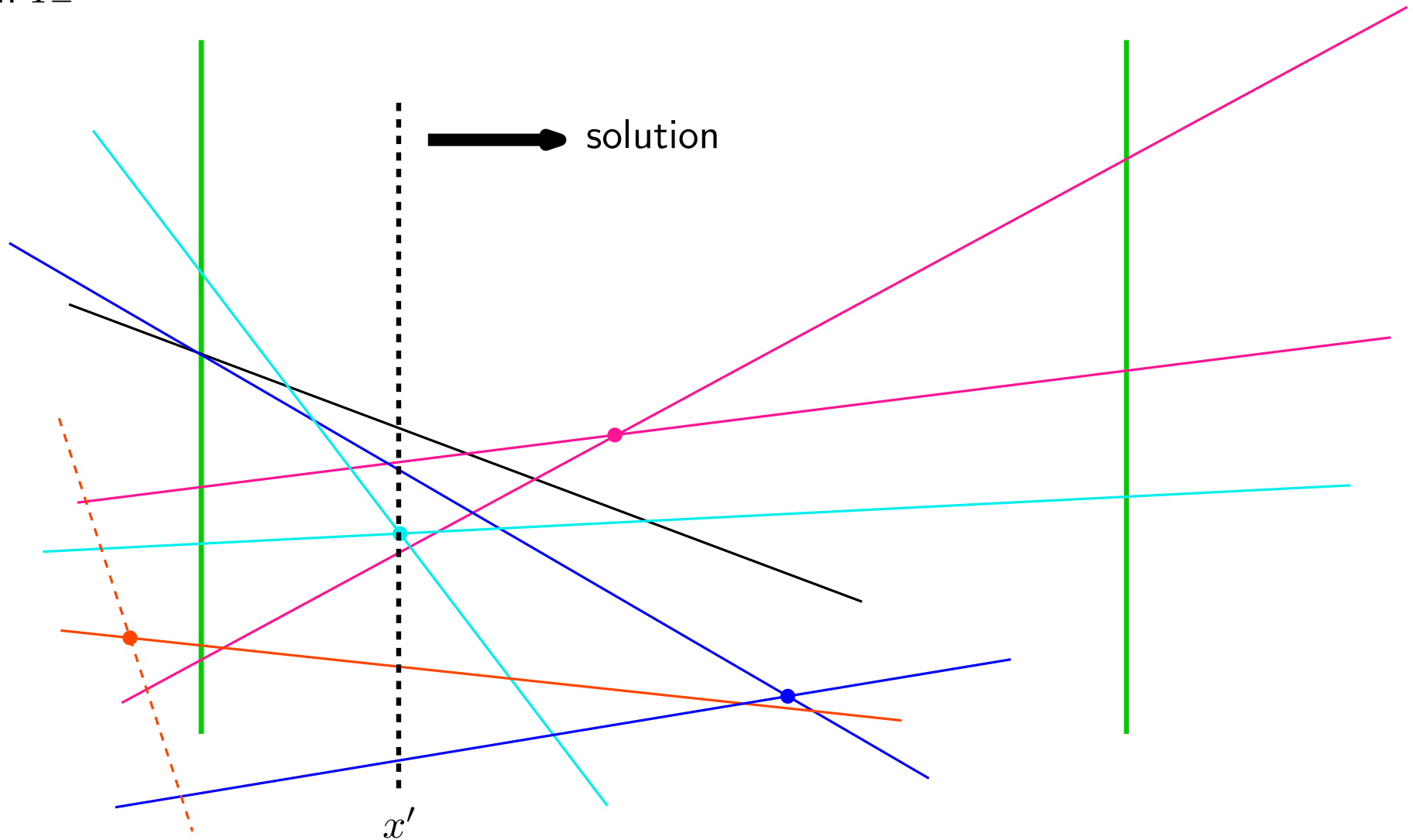
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

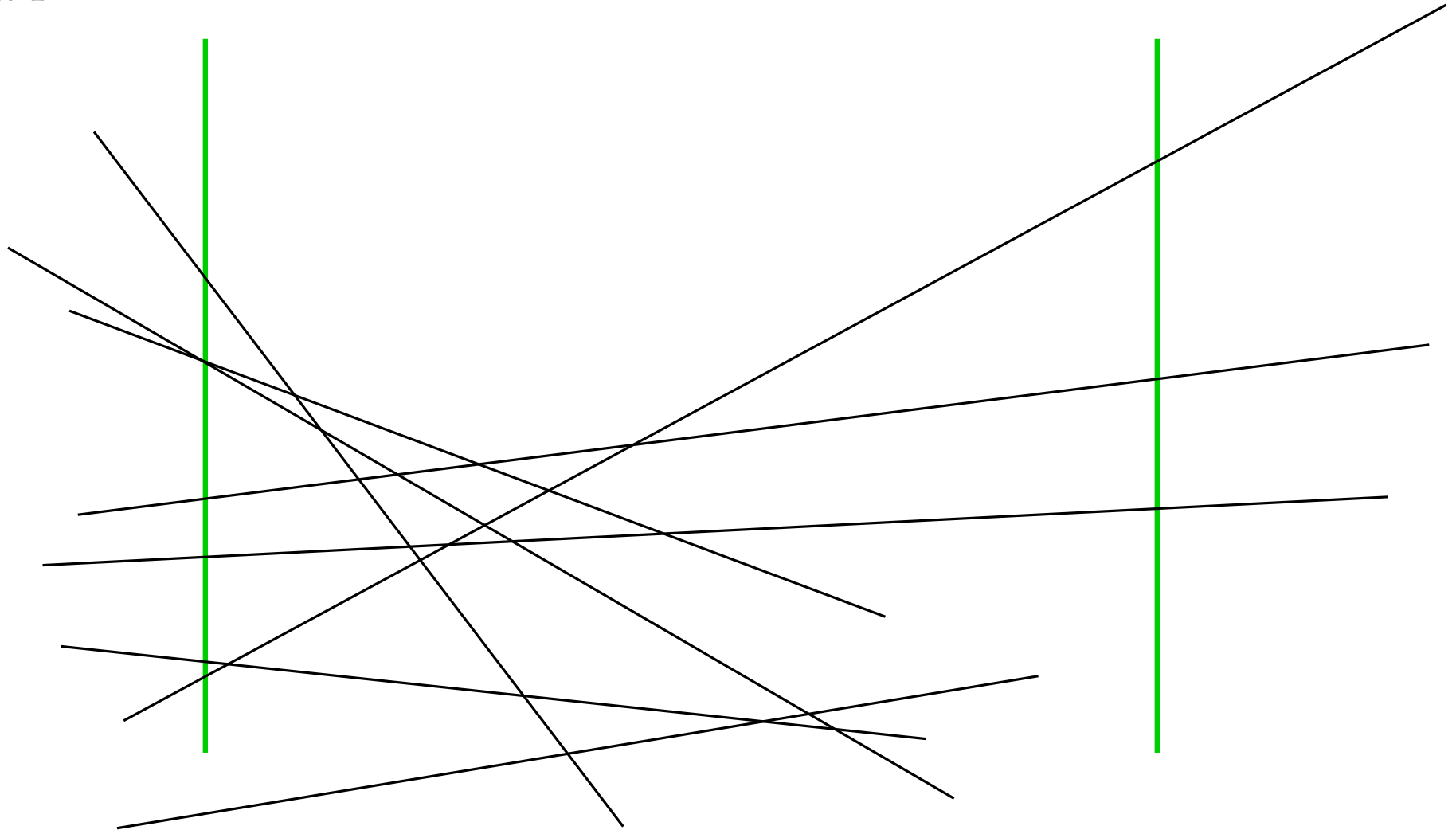
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

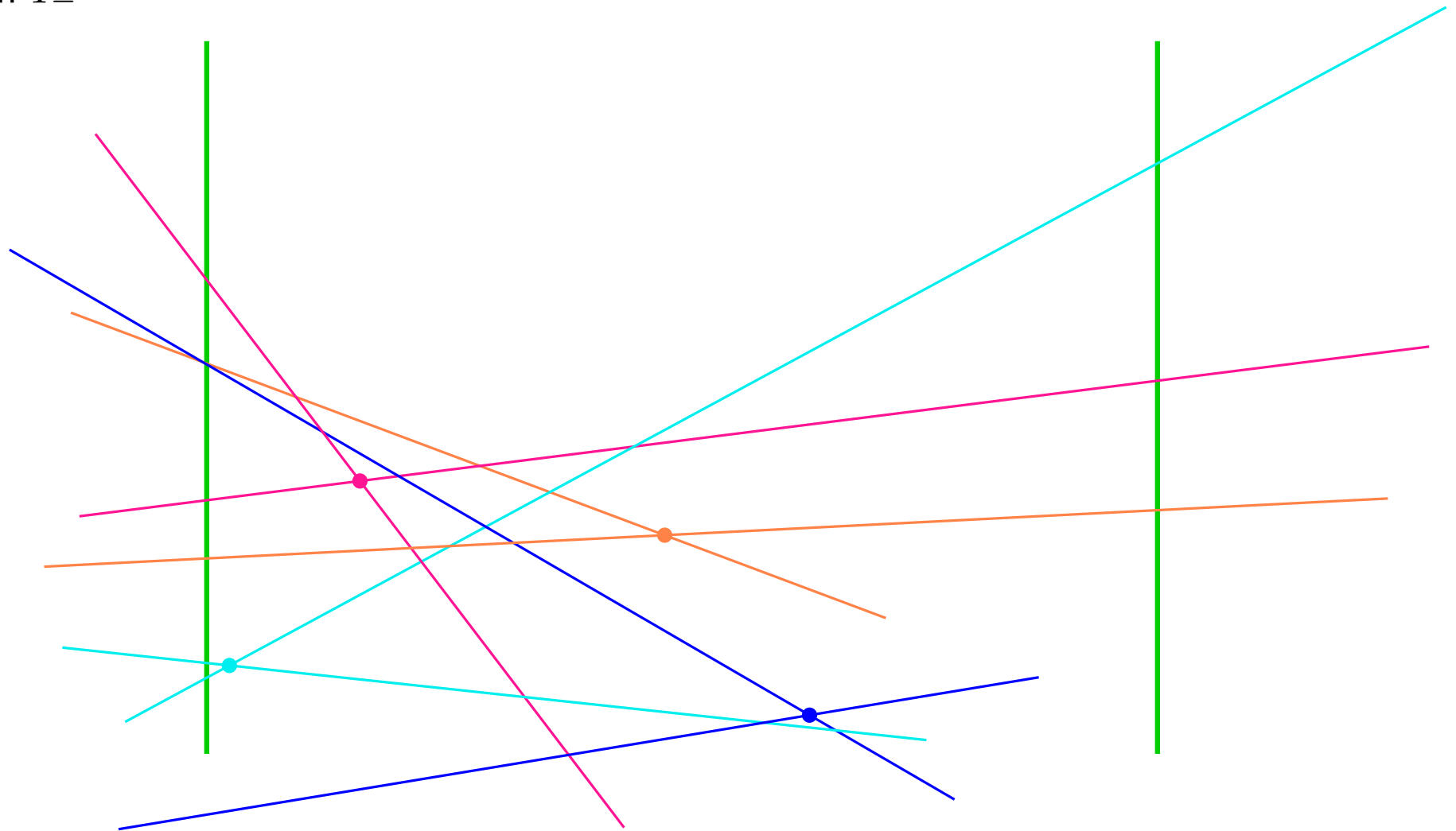
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

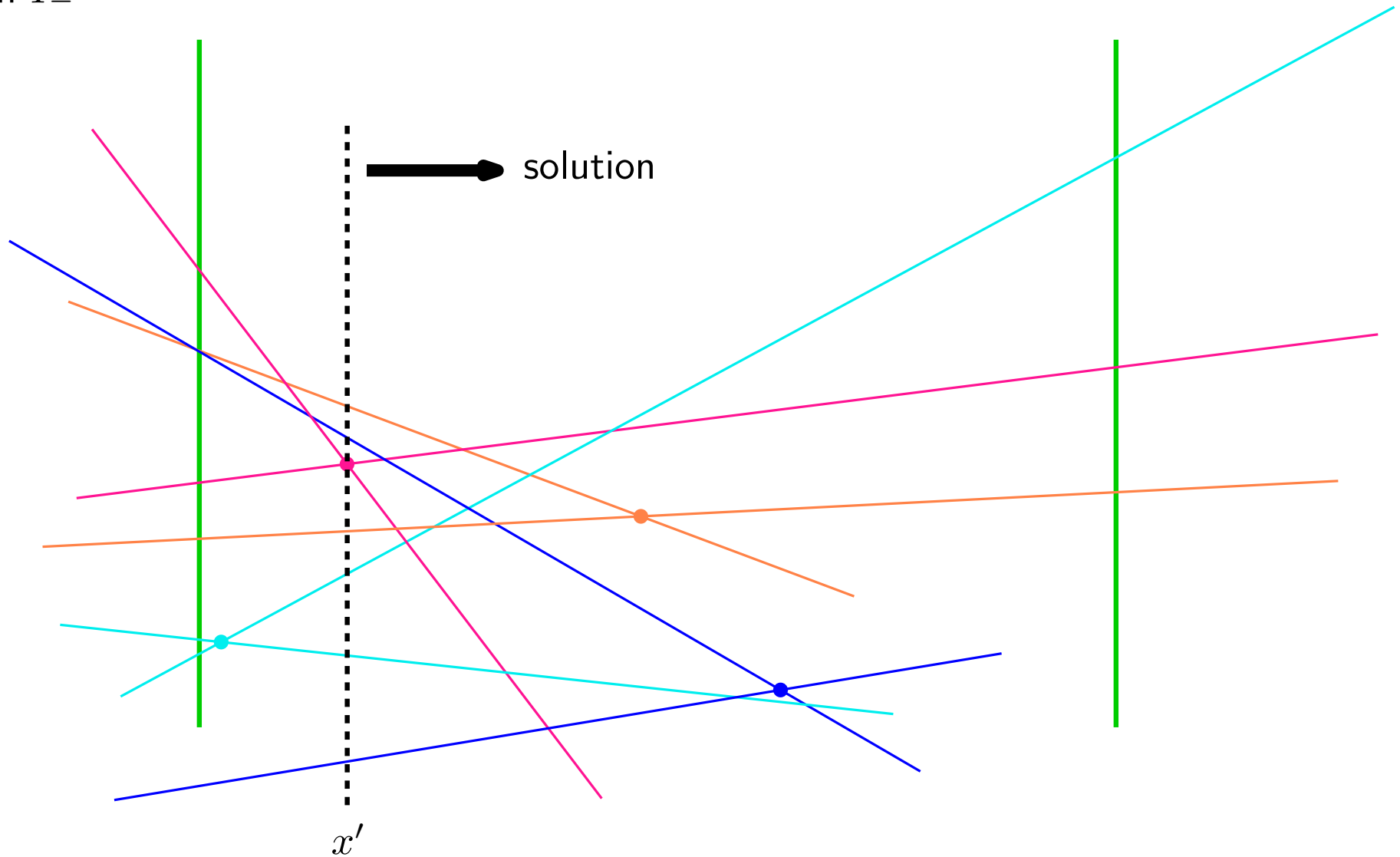
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

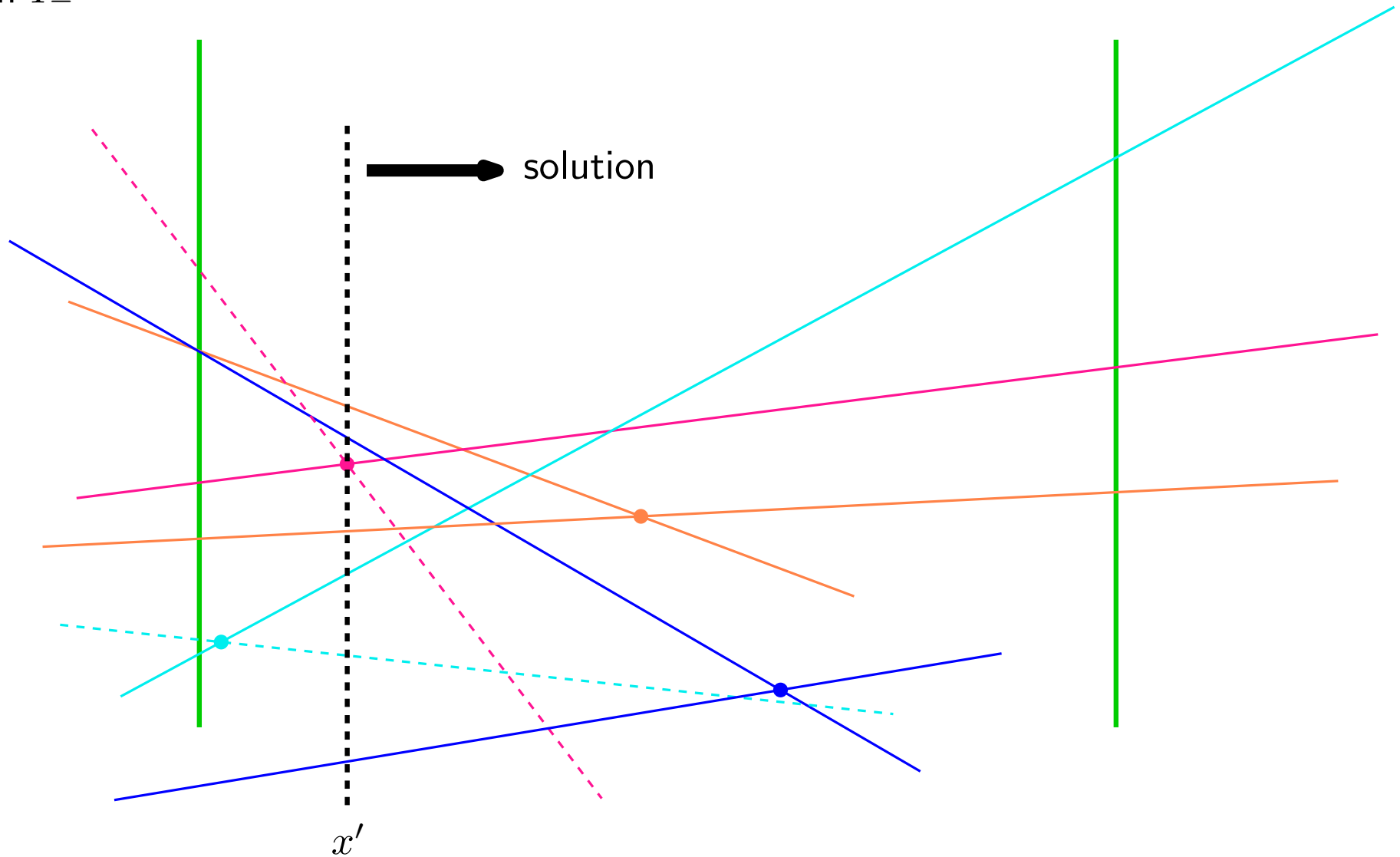
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

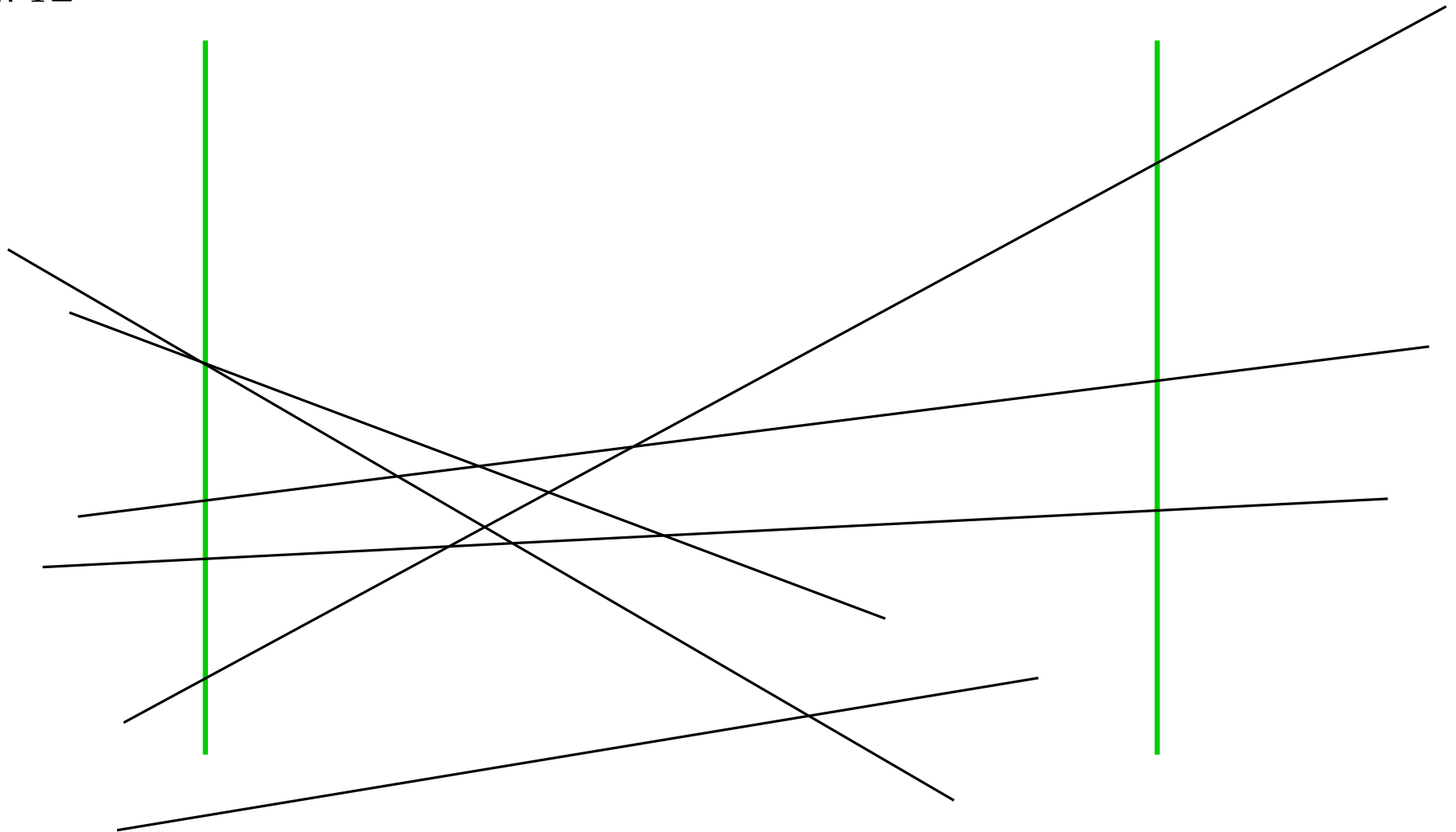
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

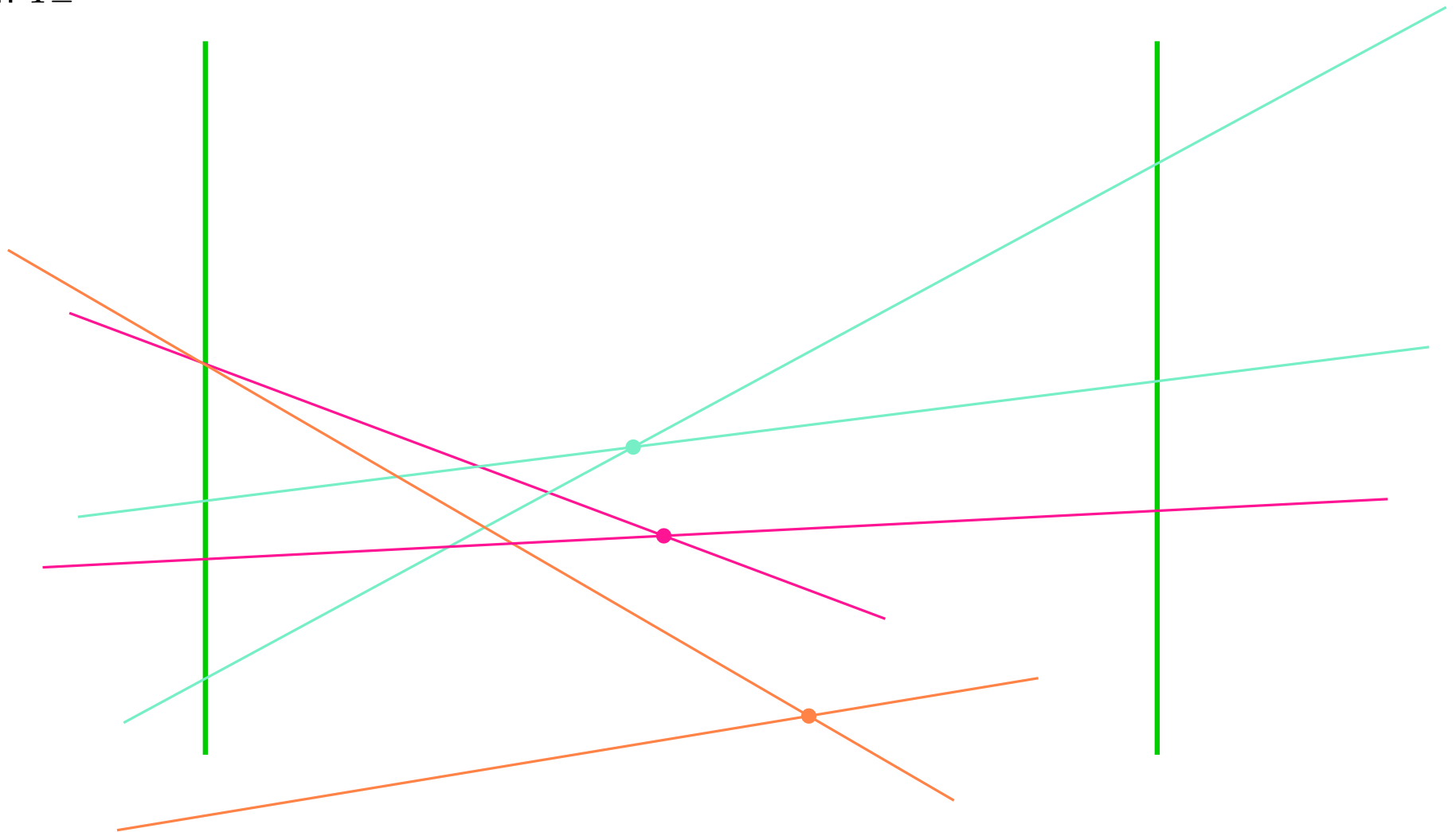
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

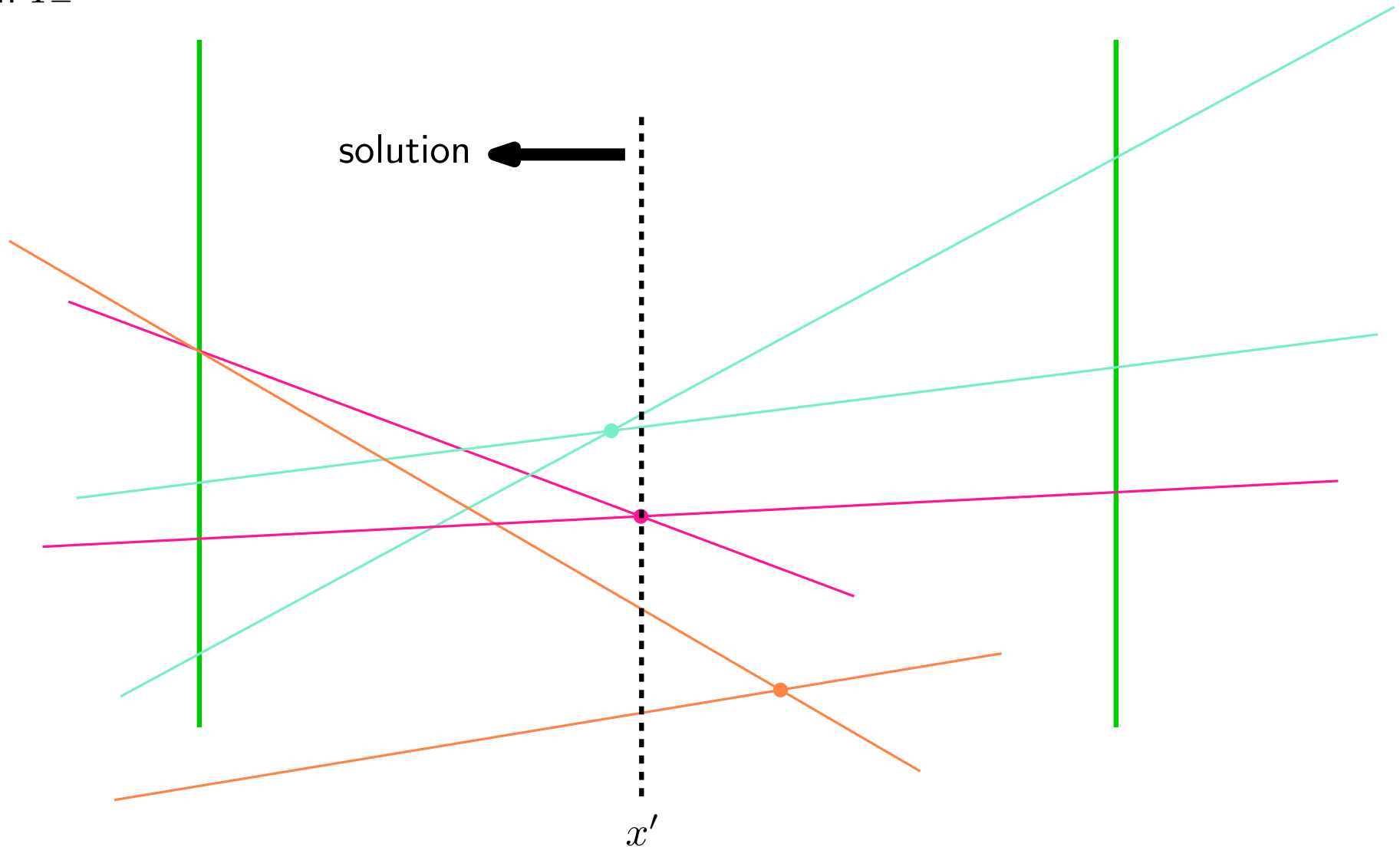
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

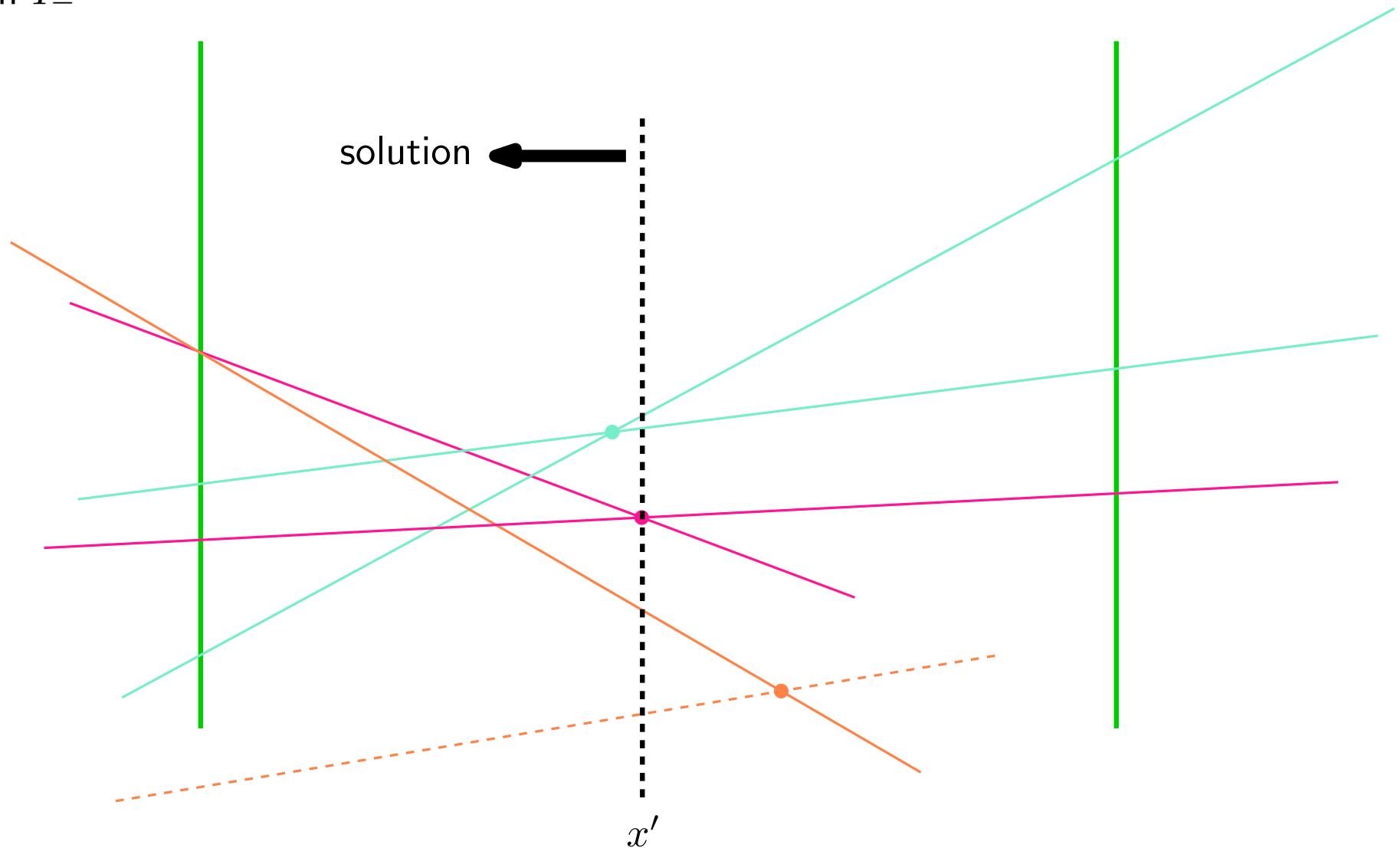
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

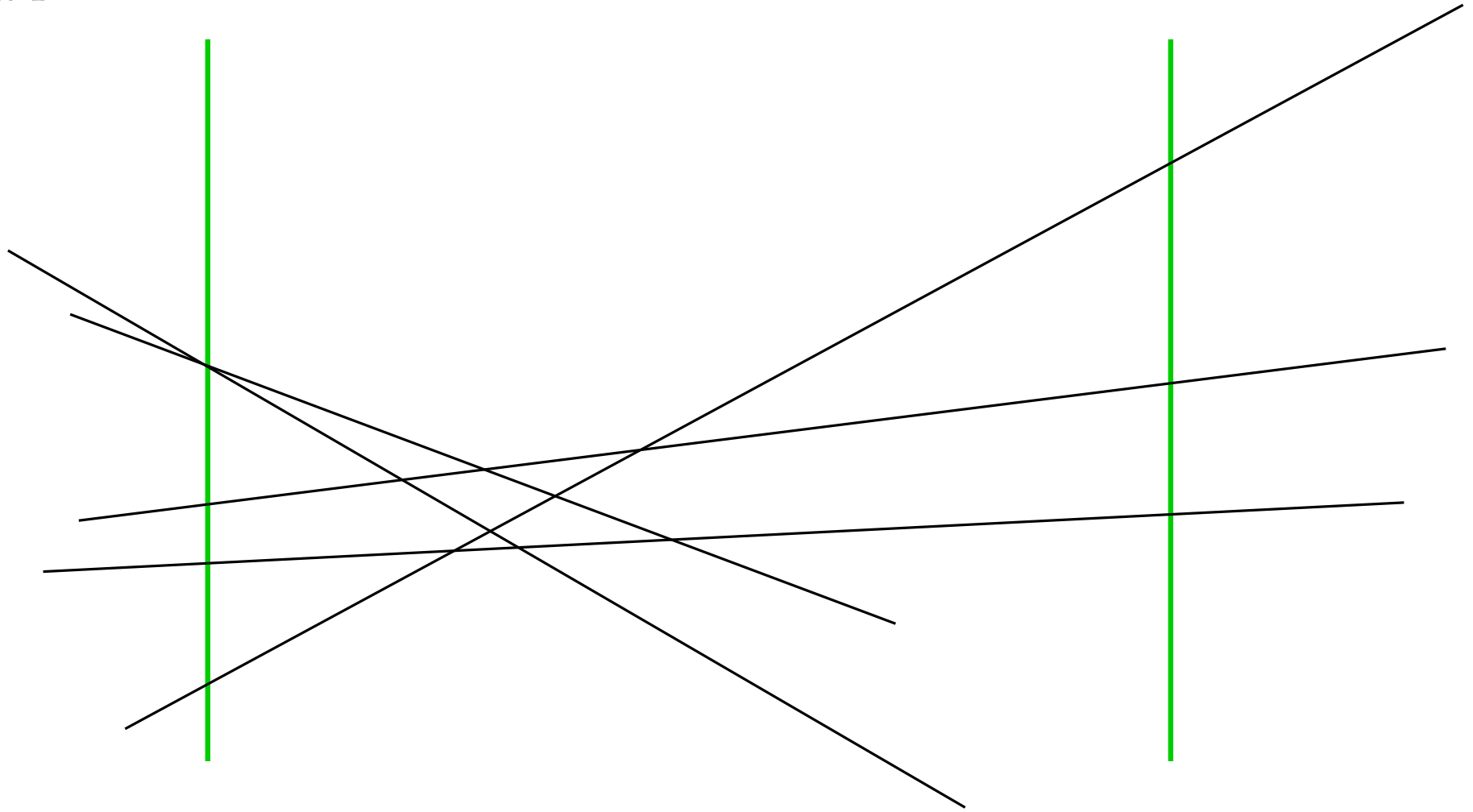
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

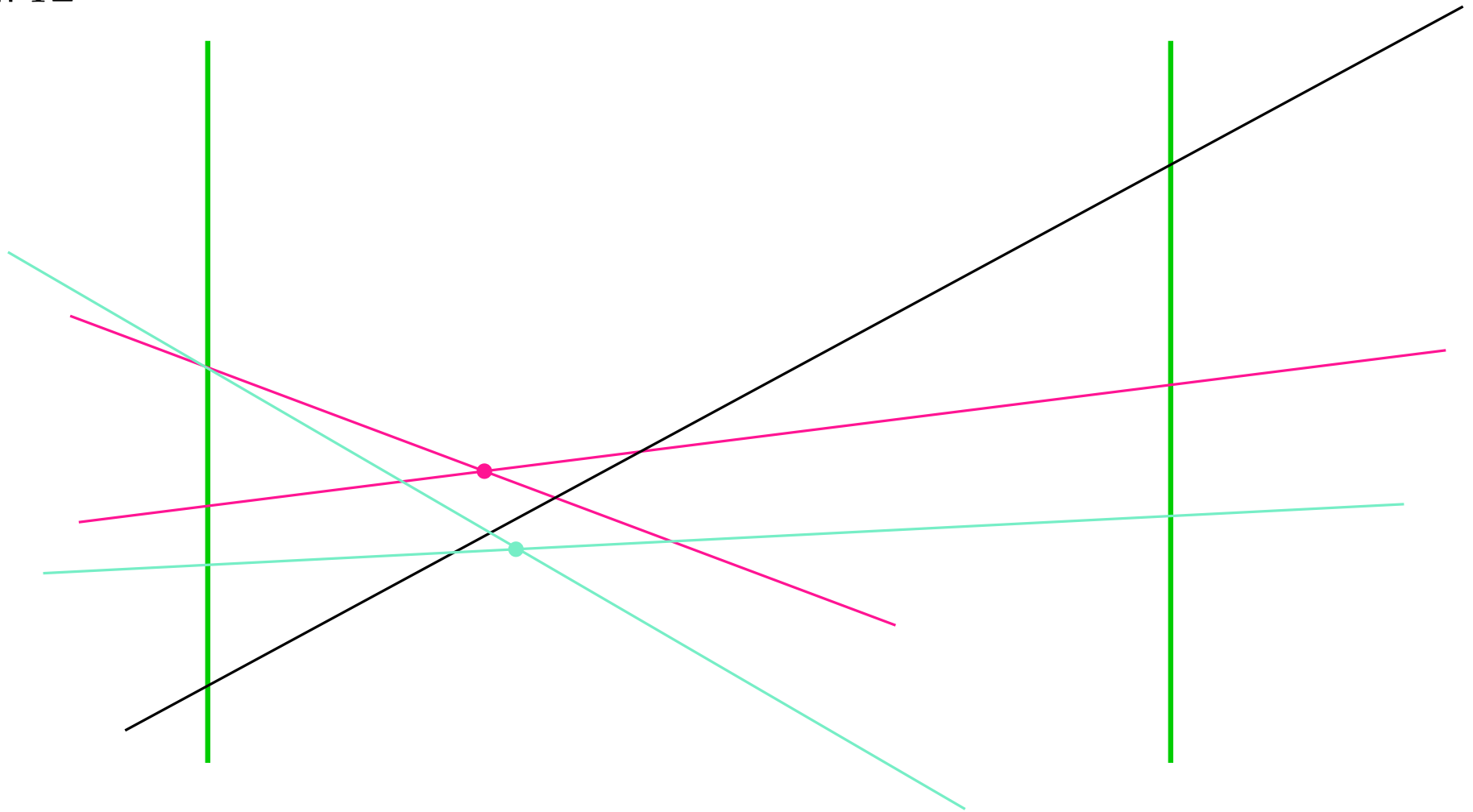
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

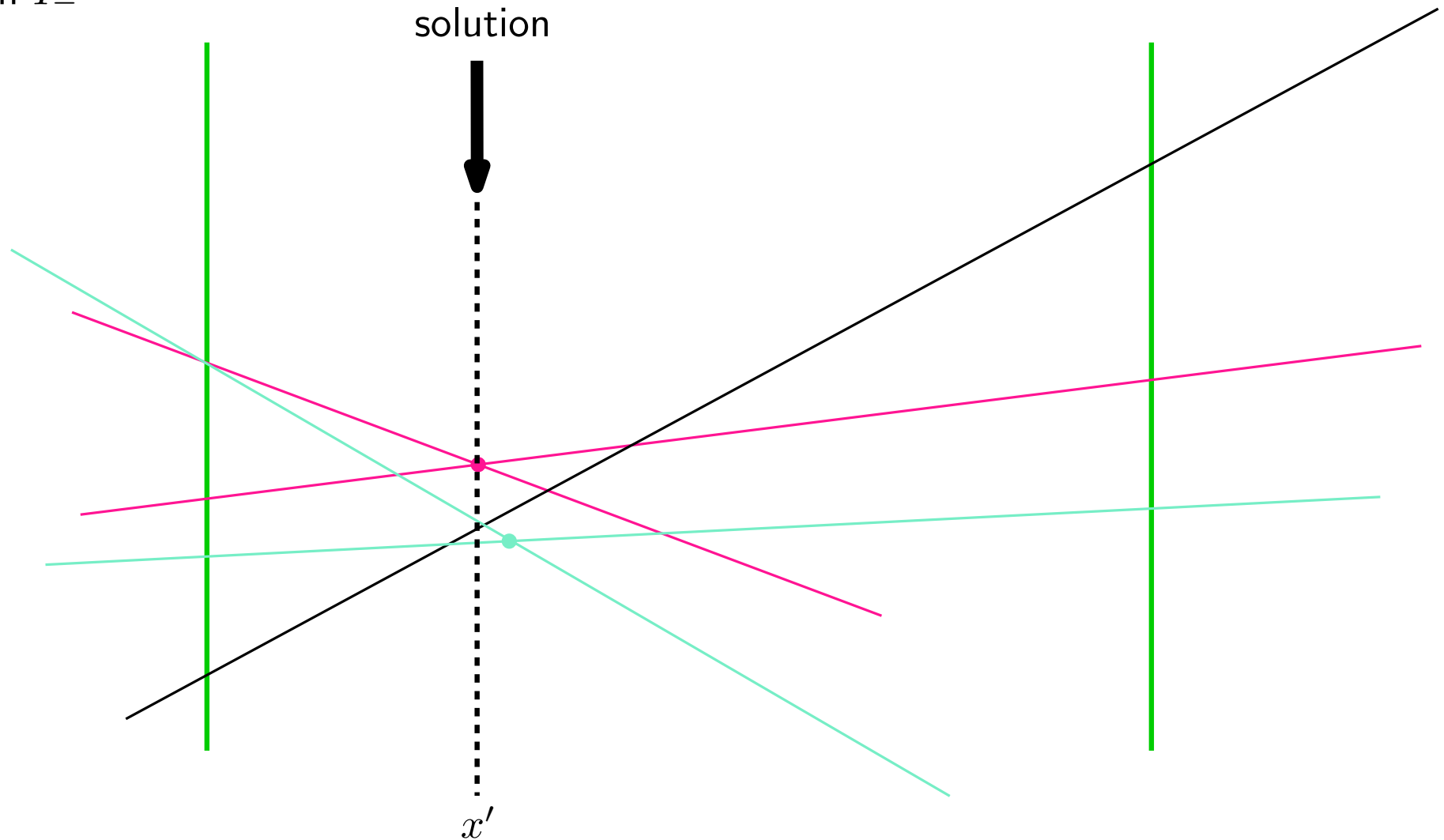
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

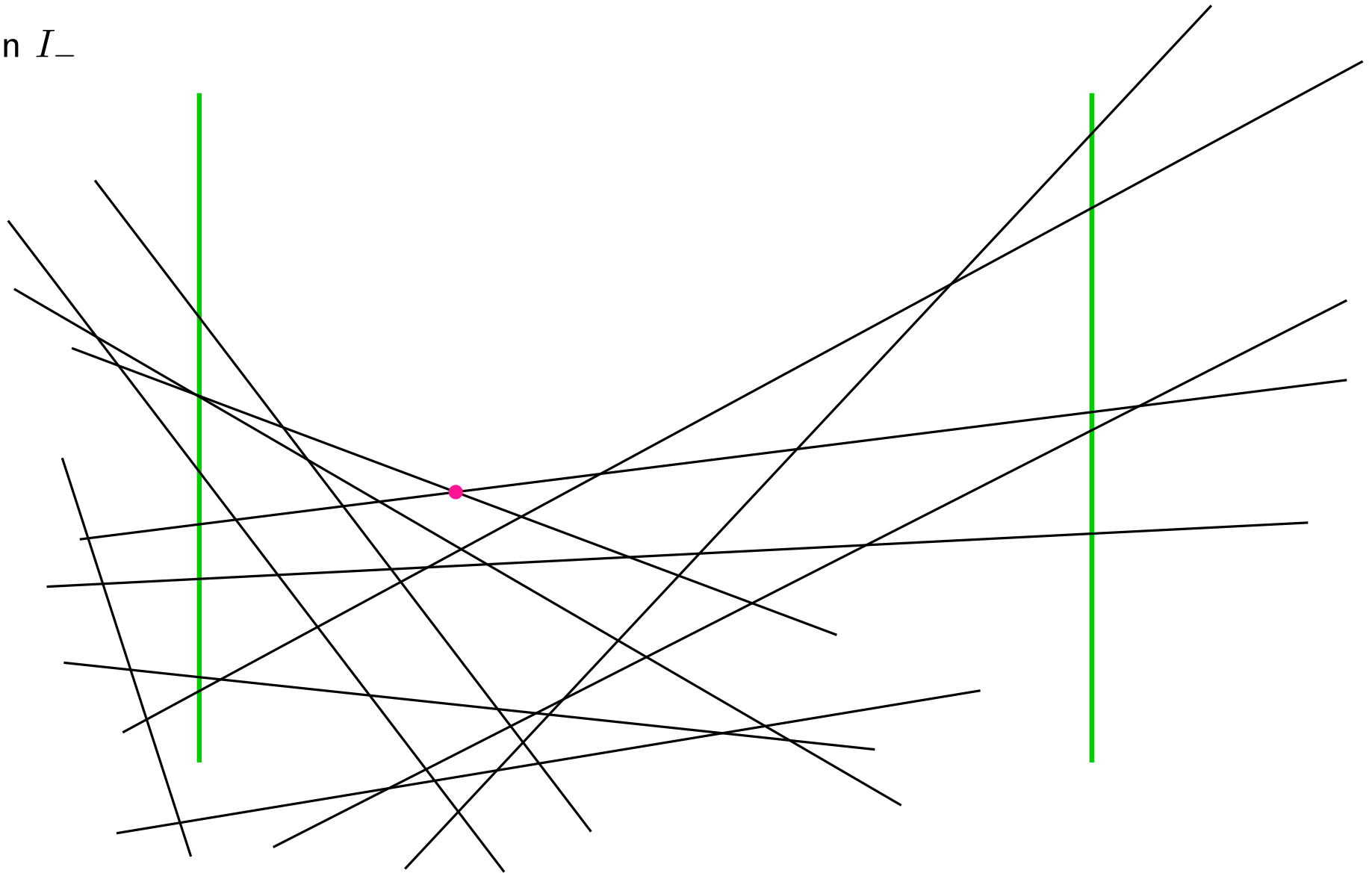
Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

Pruning in I_-



SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist. $O(n)$

Advance. Repeat as many times as necessary:

1. Pair up restrictions and compute the abscissa of the intersection points. $O(n)$
2. For all pairs, do: $O(n)$
 - If $d_i = d_j$, eliminate one of the two.
 - If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
 - If $x_{ij} < u_1$, eliminate one of the two.
 - If $x_{ij} > u_2$, eliminate one of the two.
3. Compute the median value x' of the surviving x_{ij} . $O(n)$ (see the appropriate reference)
4. Search: $O(n)$
5. Prune: $O(n)$

SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist. $O(n)$

Advance. Repeat as many times as necessary:

1. Pair up restrictions and compute the abscissa of the intersection points. $O(n)$
2. For all pairs, do: $O(n)$
 - If $d_i = d_j$, eliminate one of the two.
 - If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
 - If $x_{ij} < u_1$, eliminate one of the two.
 - If $x_{ij} > u_2$, eliminate one of the two.
3. Compute the median value x' of the surviving x_{ij} . $O(n)$ (see the appropriate reference)
4. Search: $O(n)$
5. Prune: $O(n)$

How many restrictions are pruned at each step?

SOLVING LINEAR PROGRAMS

The algorithm

Initialization Eliminate all $i \in I_0$ except those corresponding to u_1 and u_2 , if they exist. $O(n)$

Advance. Repeat as many times as necessary:

1. Pair up restrictions and compute the abscissa of the intersection points. $O(n)$
2. For all pairs, do: $O(n)$
 - If $d_i = d_j$, eliminate one of the two.
 - If $d_i \neq d_j$ and x_{ij} is the abscissa of the intersection point of the two lines, then do:
 - If $x_{ij} < u_1$, eliminate one of the two.
 - If $x_{ij} > u_2$, eliminate one of the two.
3. Compute the median value x' of the surviving x_{ij} . $O(n)$ (see the appropriate reference)
4. Search: $O(n)$
5. Prune: $O(n)$

How many restrictions are pruned at each step? $O(n/4)$

SOLVING LINEAR PROGRAMS

The algorithm

Exact counting:

- The initial number of vertical restrictions is k , where $k \geq 0$.
- The vertical restrictions prune step eliminates all of them but at most 2.
- The initial number of pairs of non vertical lines (i.e., points x_{ij}) is $\frac{m}{2}$, $\frac{m-1}{2}$ or $\frac{m}{2} - 1$, depending on the parities of $|I_-|$ and $|I_+|$, where $m + k = n$.
- The prune step eliminates roughly half of them.

Alltogether, the eliminated restrictions are, at least:

$$k + \left\lfloor \frac{\frac{m}{2} - 1}{2} \right\rfloor = \left\lfloor \frac{m}{4} + k - \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{4} + \frac{3}{4}k - \frac{1}{2} \right\rfloor > \frac{n}{4} - 1.$$

SOLVING LINEAR PROGRAMS

The algorithm

Exact counting:

- The initial number of vertical restrictions is k , where $k \geq 0$.
- The vertical restrictions prune step eliminates all of them but at most 2.
- The initial number of pairs of non vertical lines (i.e., points x_{ij}) is $\frac{m}{2}$, $\frac{m-1}{2}$ or $\frac{m}{2} - 1$, depending on the parities of $|I_-|$ and $|I_+|$, where $m + k = n$.
- The prune step eliminates roughly half of them.

Alltogether, the eliminated restrictions are, at least:

$$k + \left\lfloor \frac{\frac{m}{2} - 1}{2} \right\rfloor = \left\lfloor \frac{m}{4} + k - \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{4} + \frac{3}{4}k - \frac{1}{2} \right\rfloor > \frac{n}{4} - 1.$$

Running time

Hence, the running time of the algorithm is

$$cn + c\frac{3}{4}n + c\left(\frac{3}{4}\right)^2 n + c\left(\frac{3}{4}\right)^3 n + \dots = cn \sum_{k=0}^{\lceil \log_{3/4} n \rceil} \left(\frac{3}{4}\right)^k < cn \frac{1}{1 - \frac{3}{4}} = 4cn \in O(n).$$

COMPUTING THE MINIMUM SPANNING CIRCLE

COMPUTING THE MINIMUM SPANNING CIRCLE

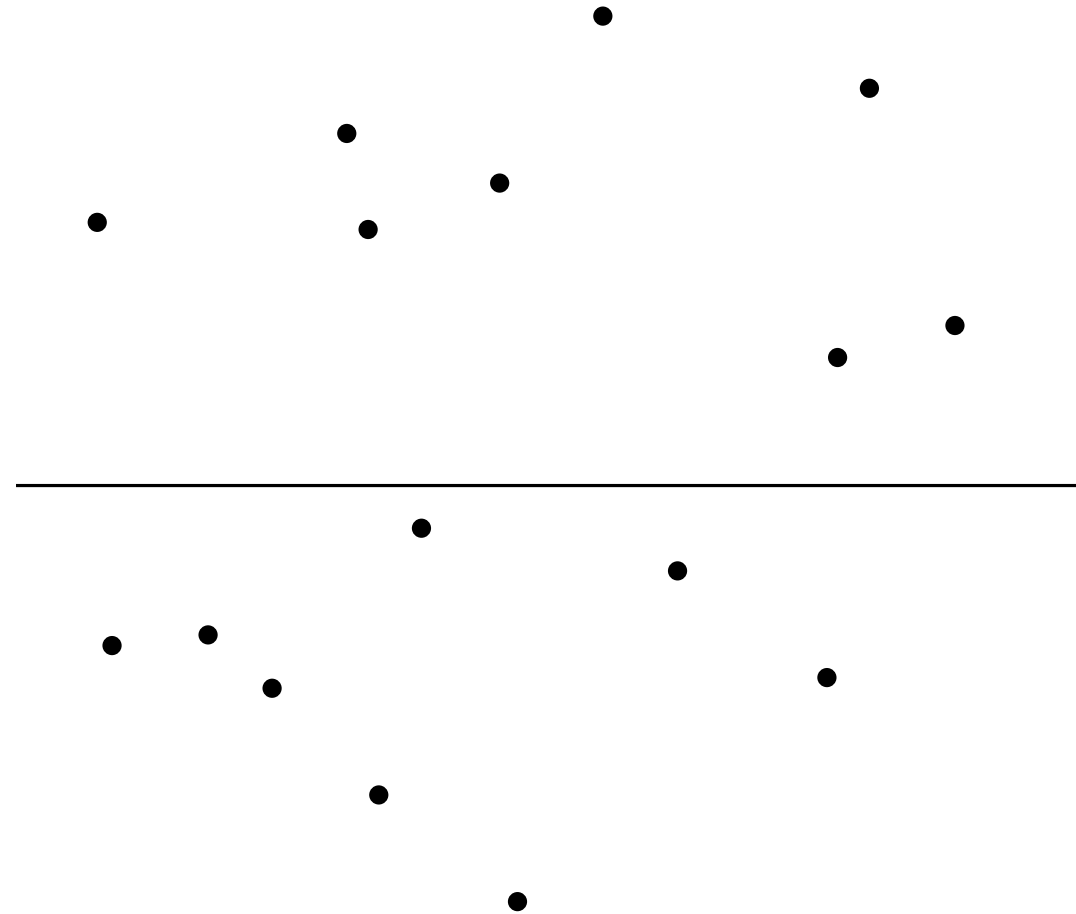
MSC restricted to a line

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

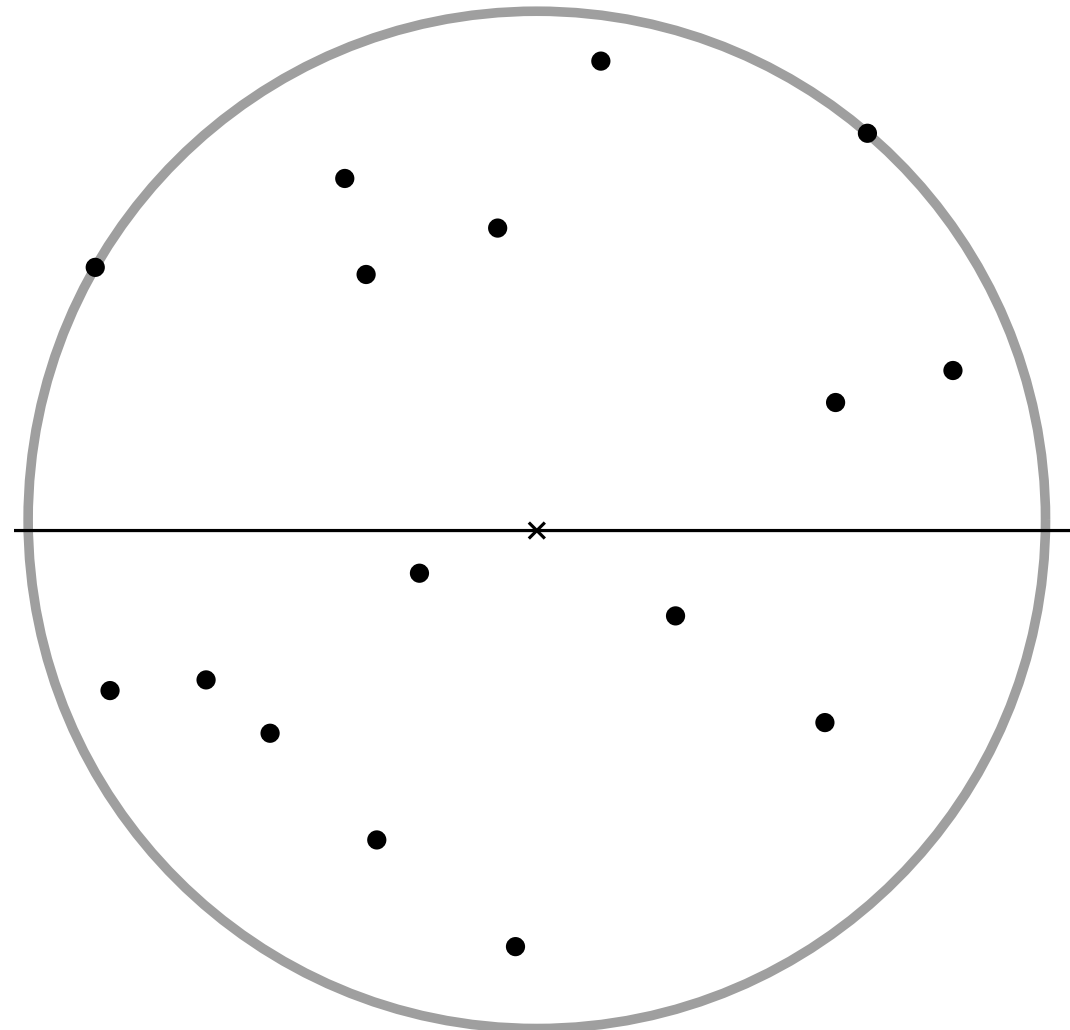


COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.



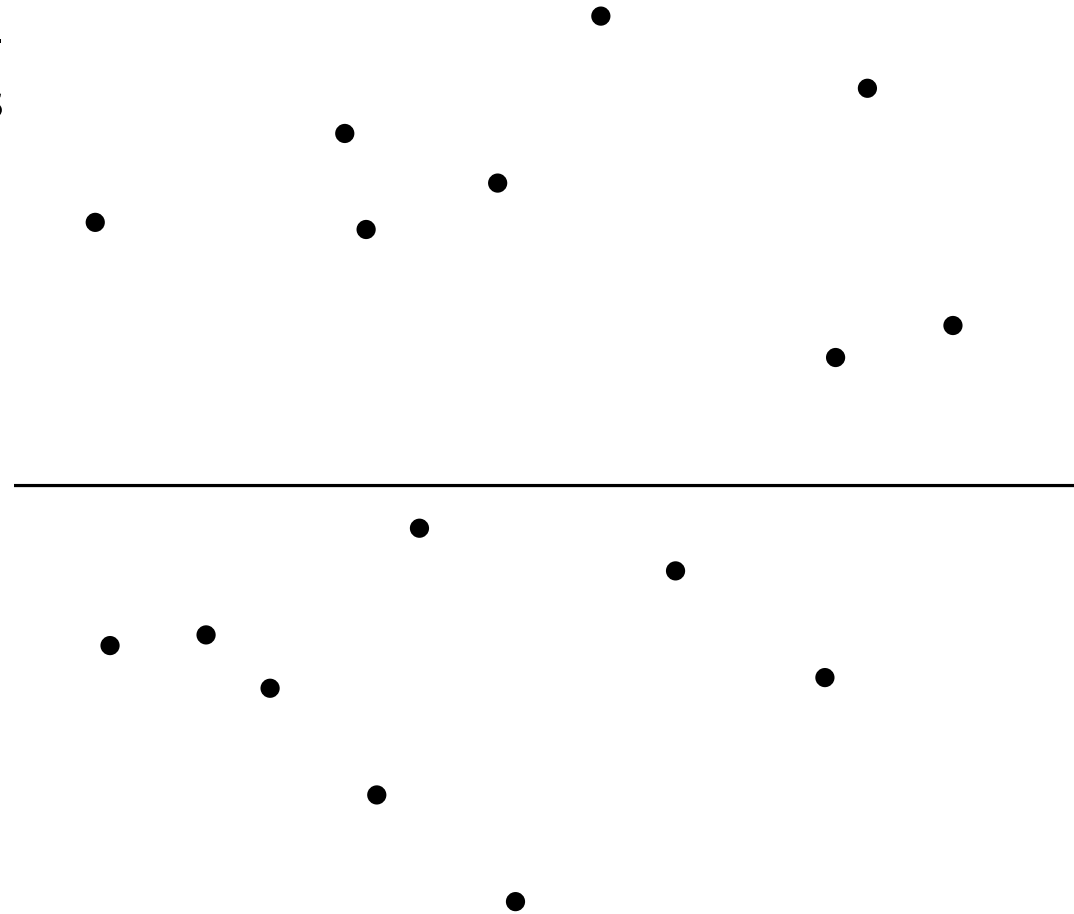
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



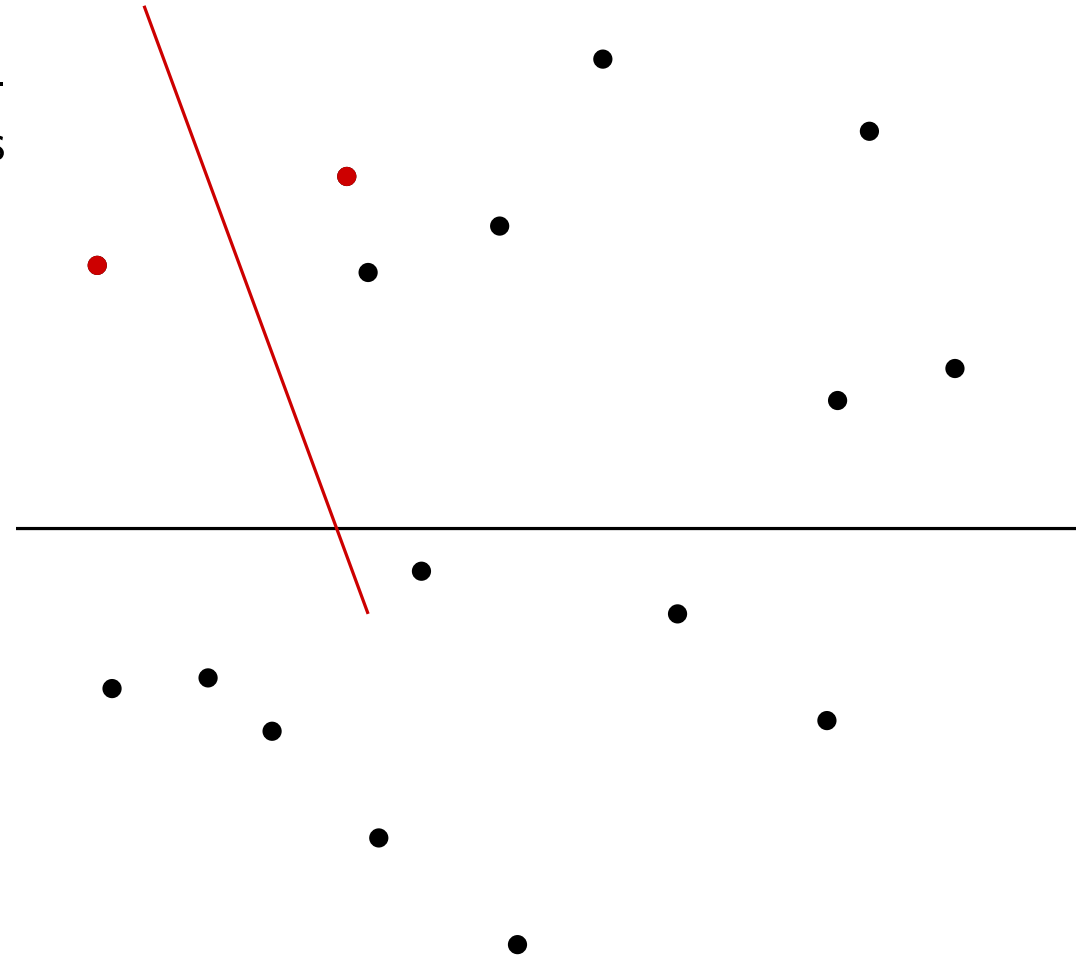
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



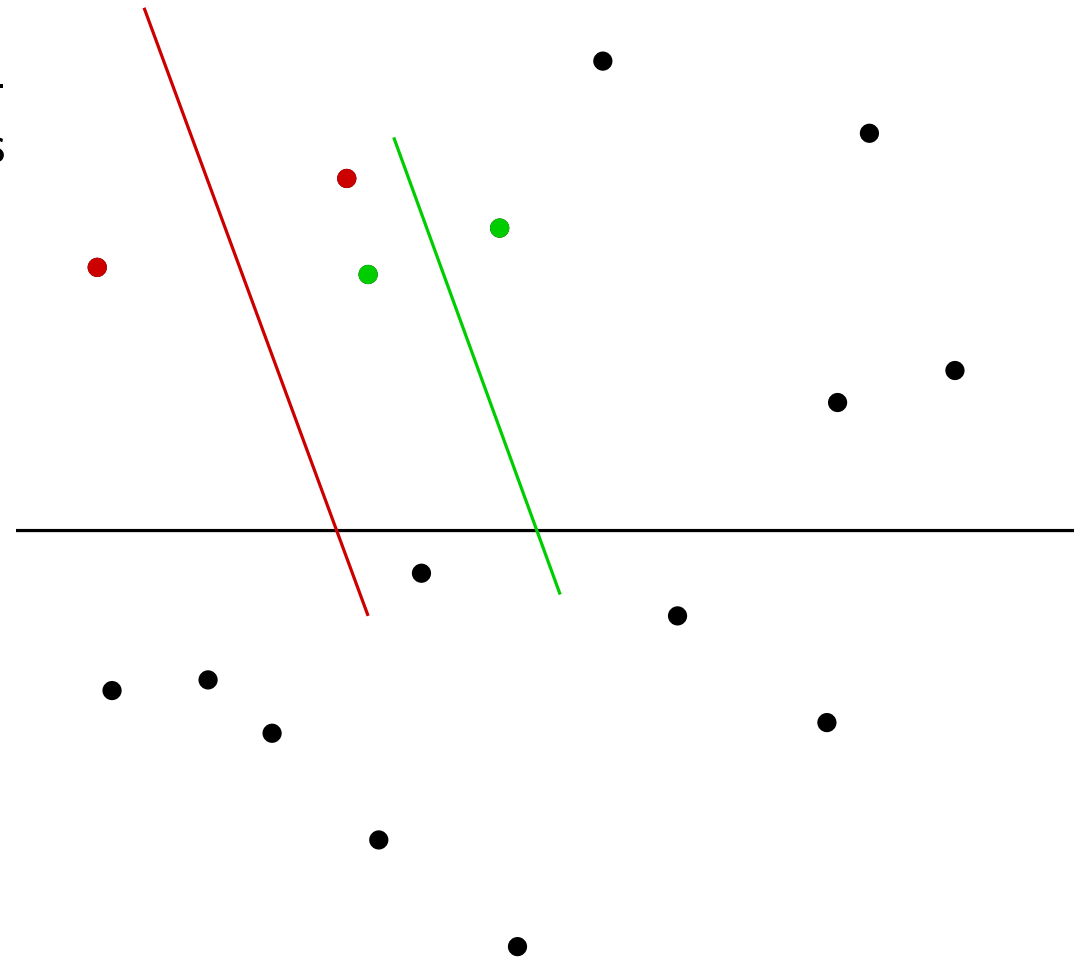
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



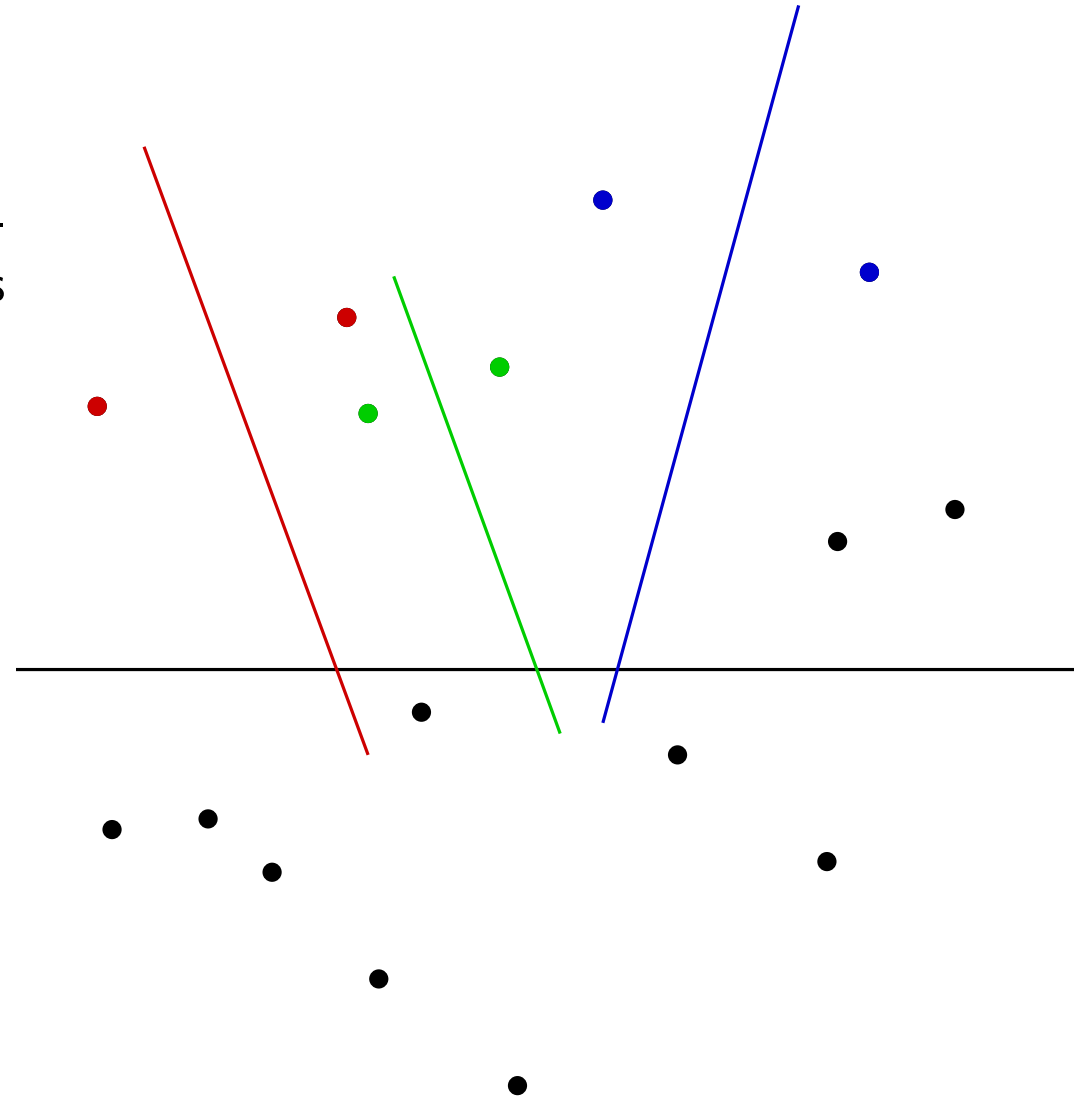
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



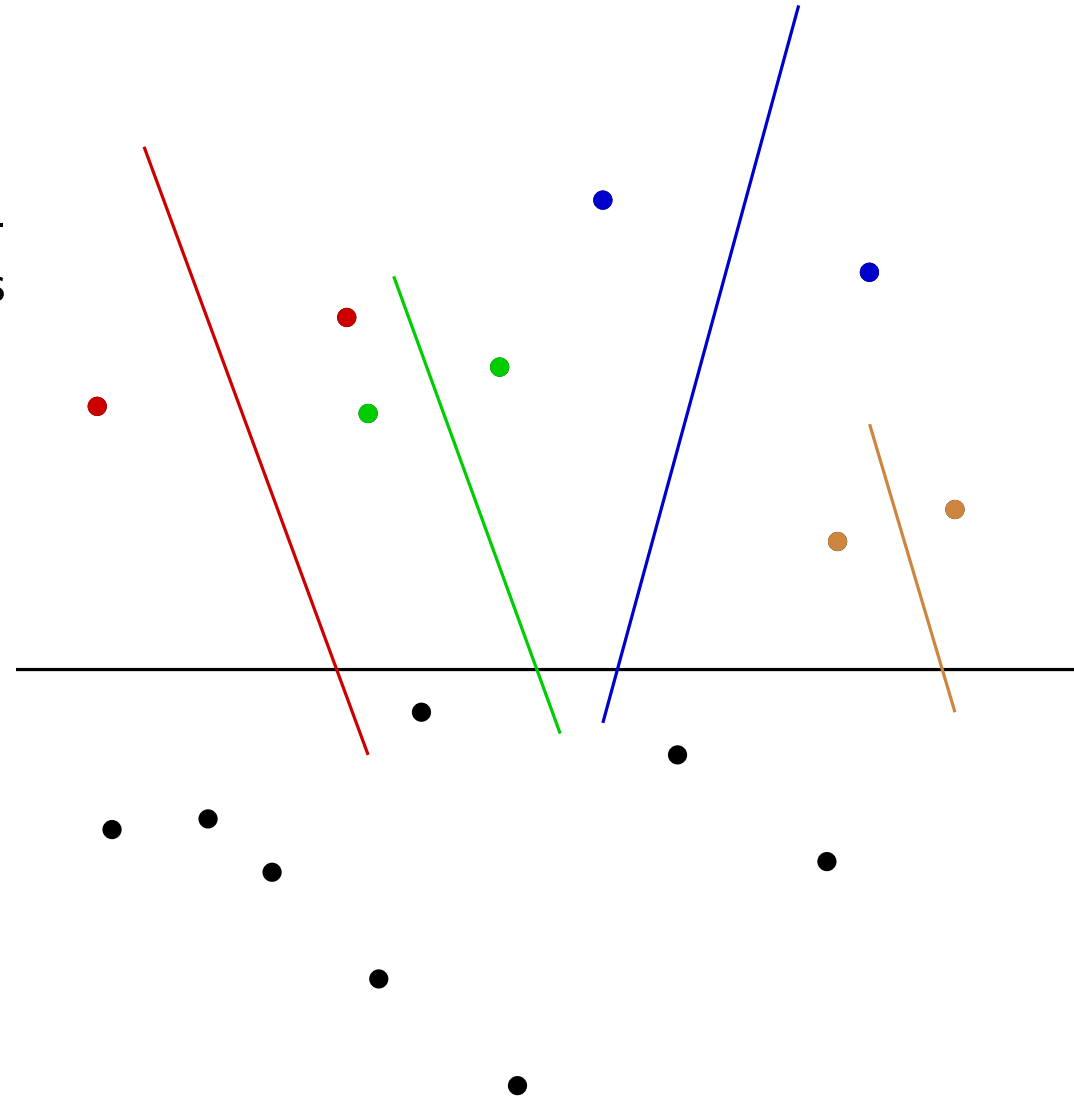
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



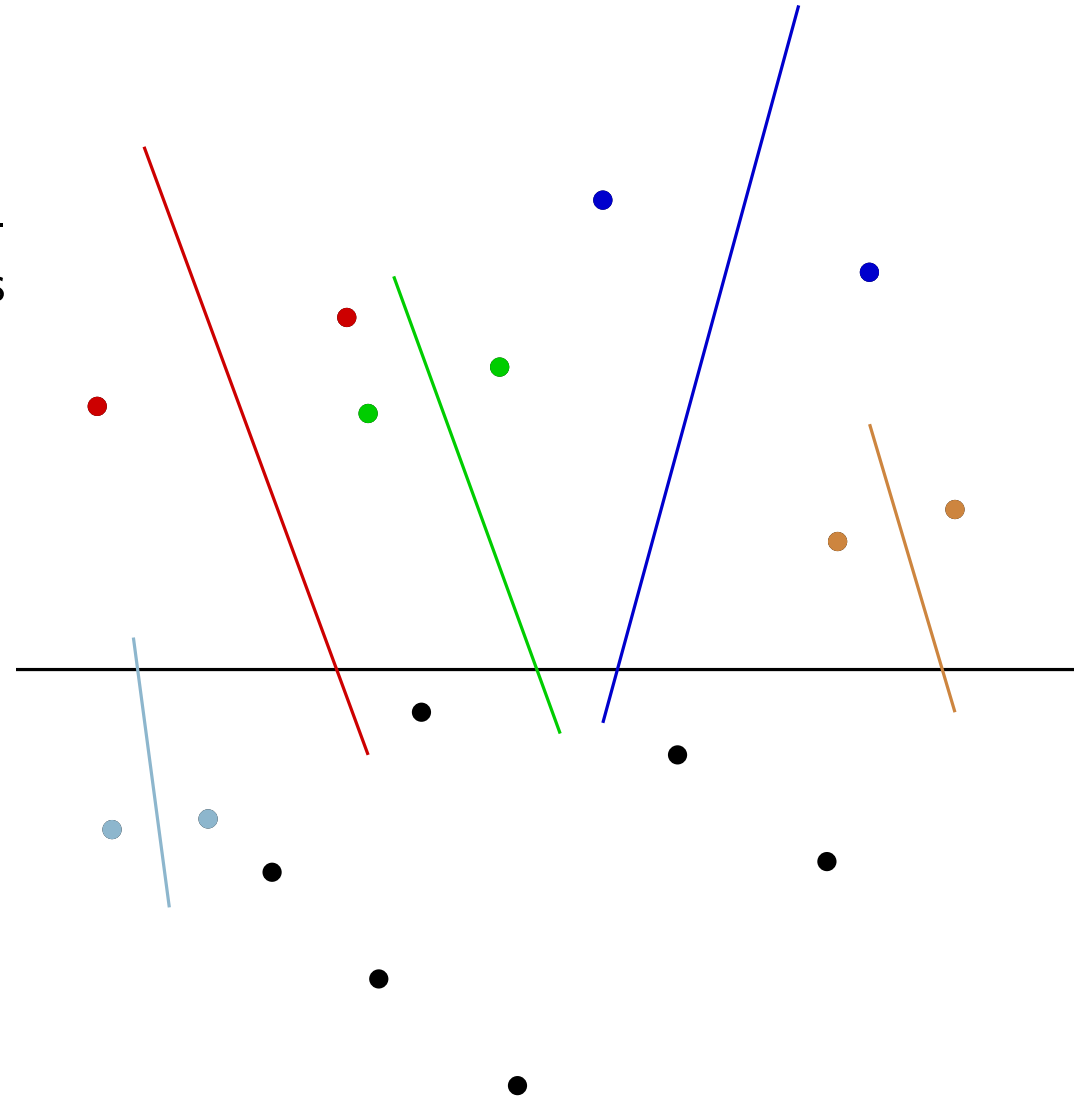
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



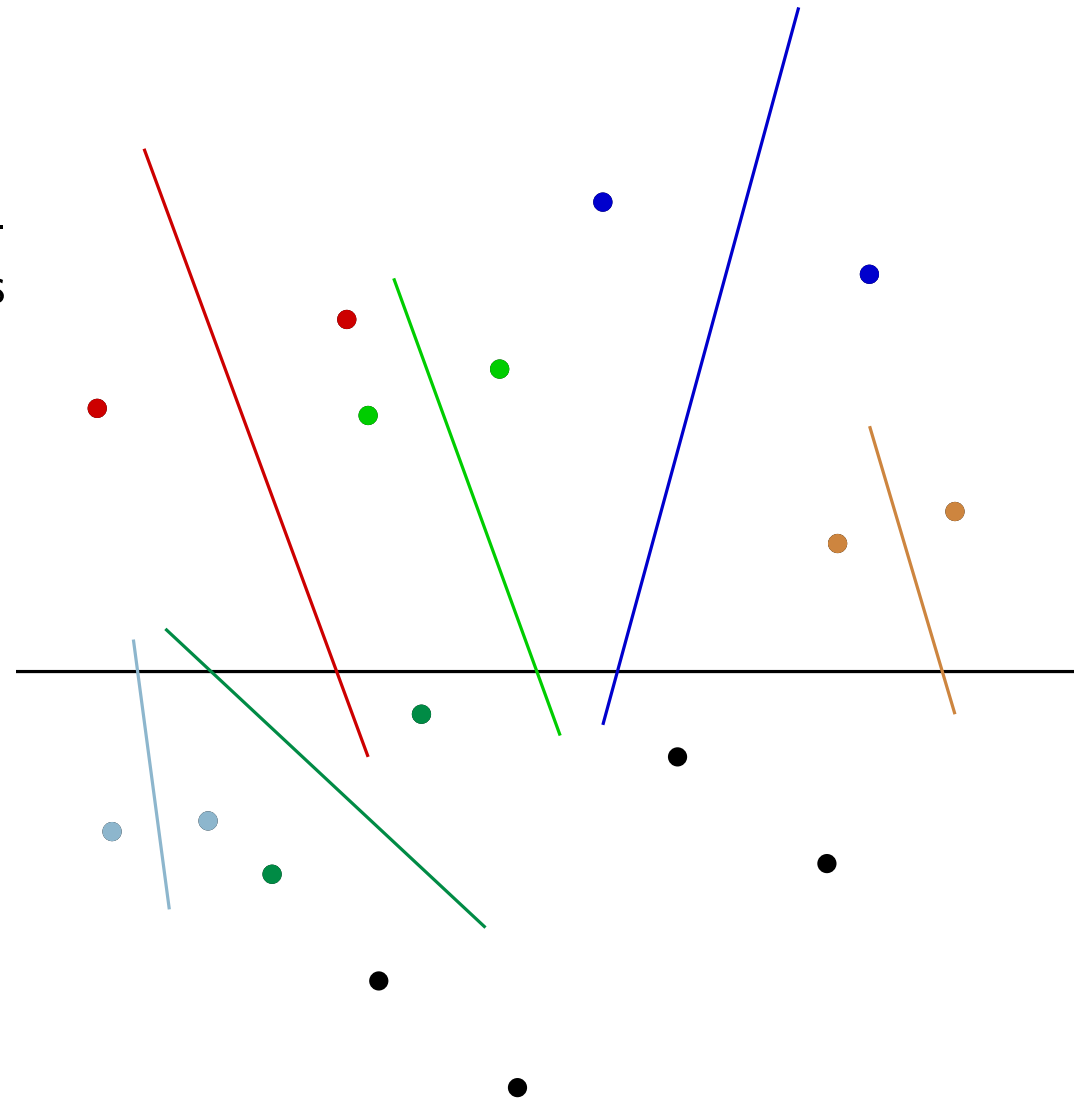
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



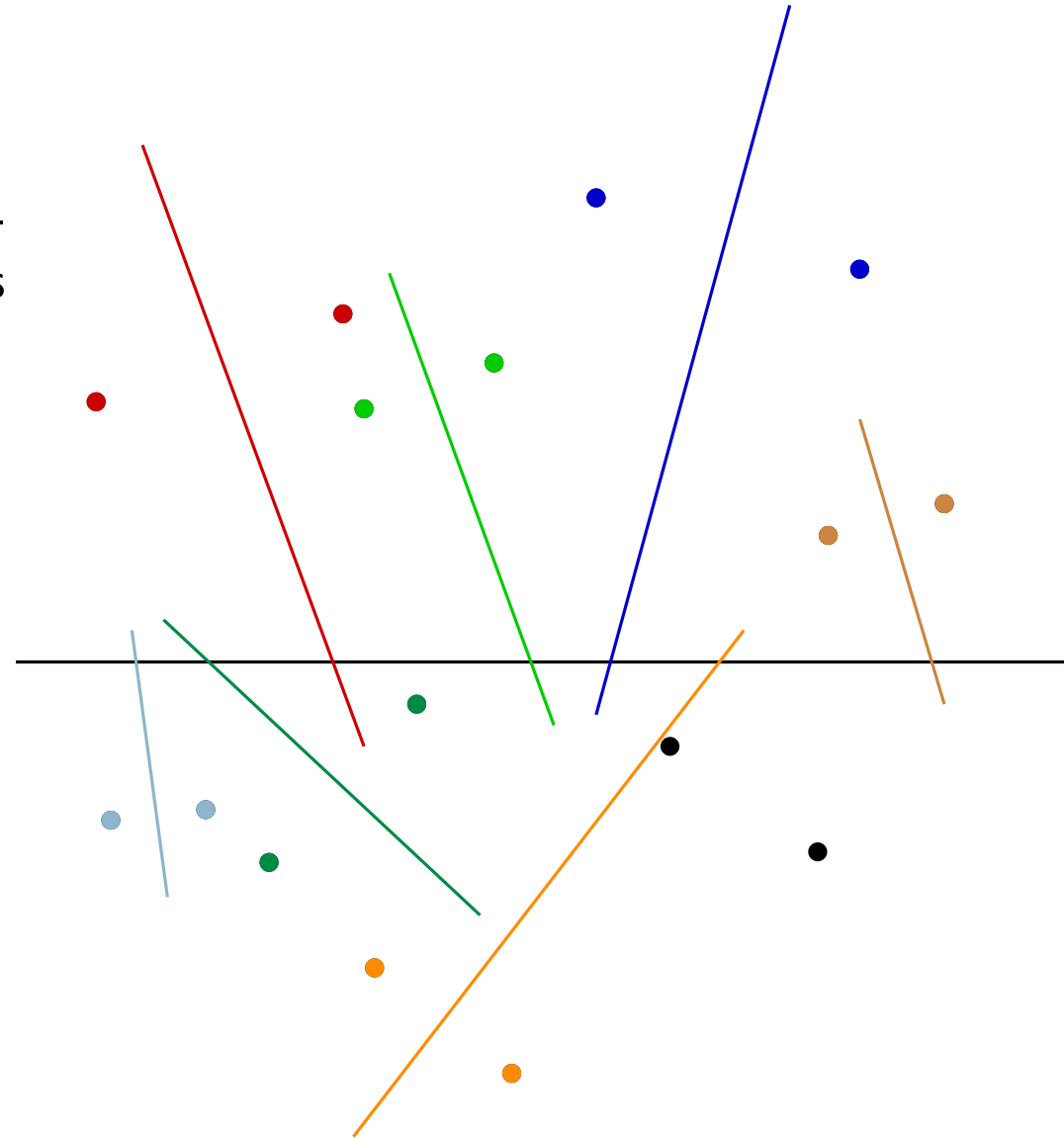
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



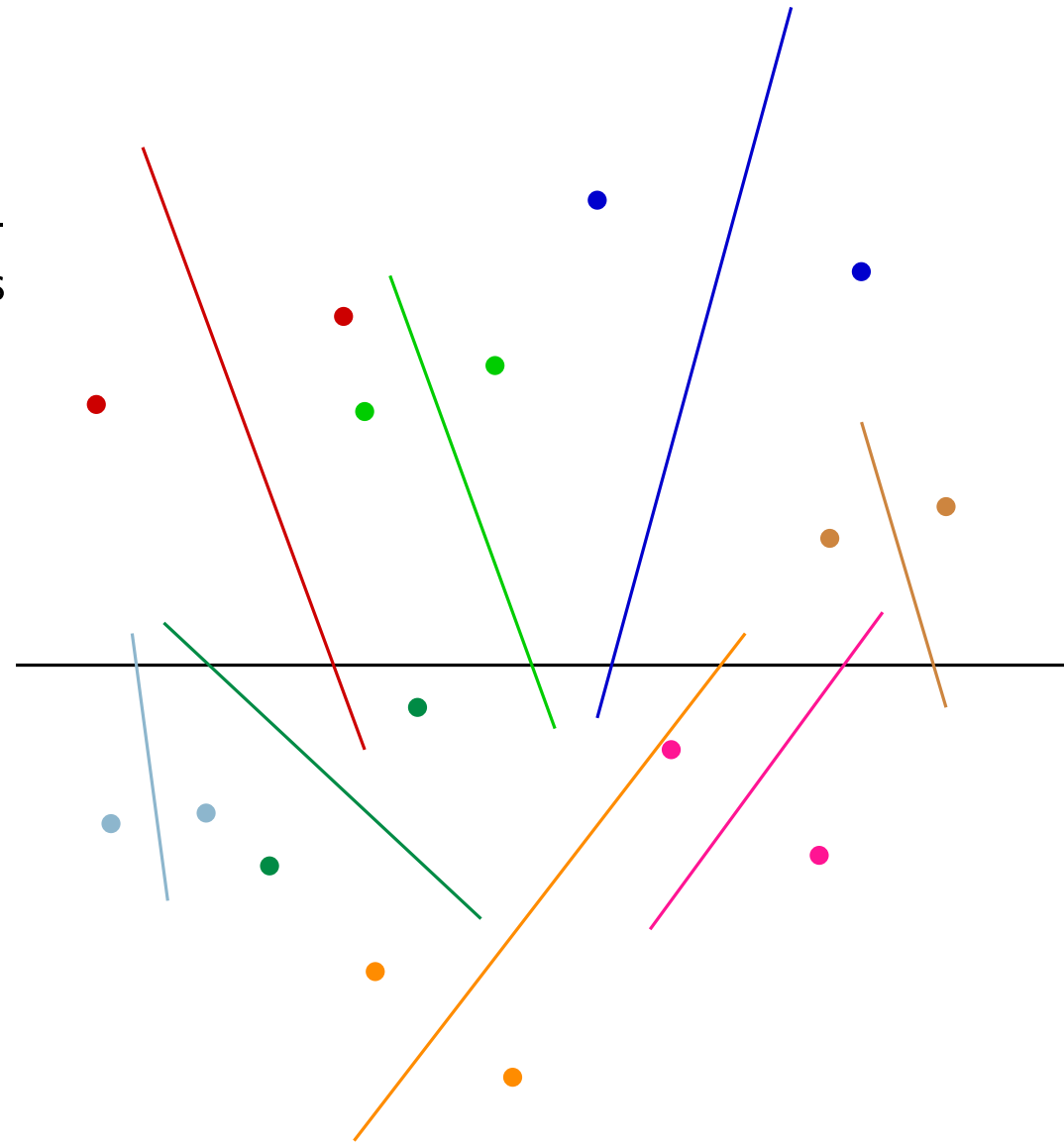
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.



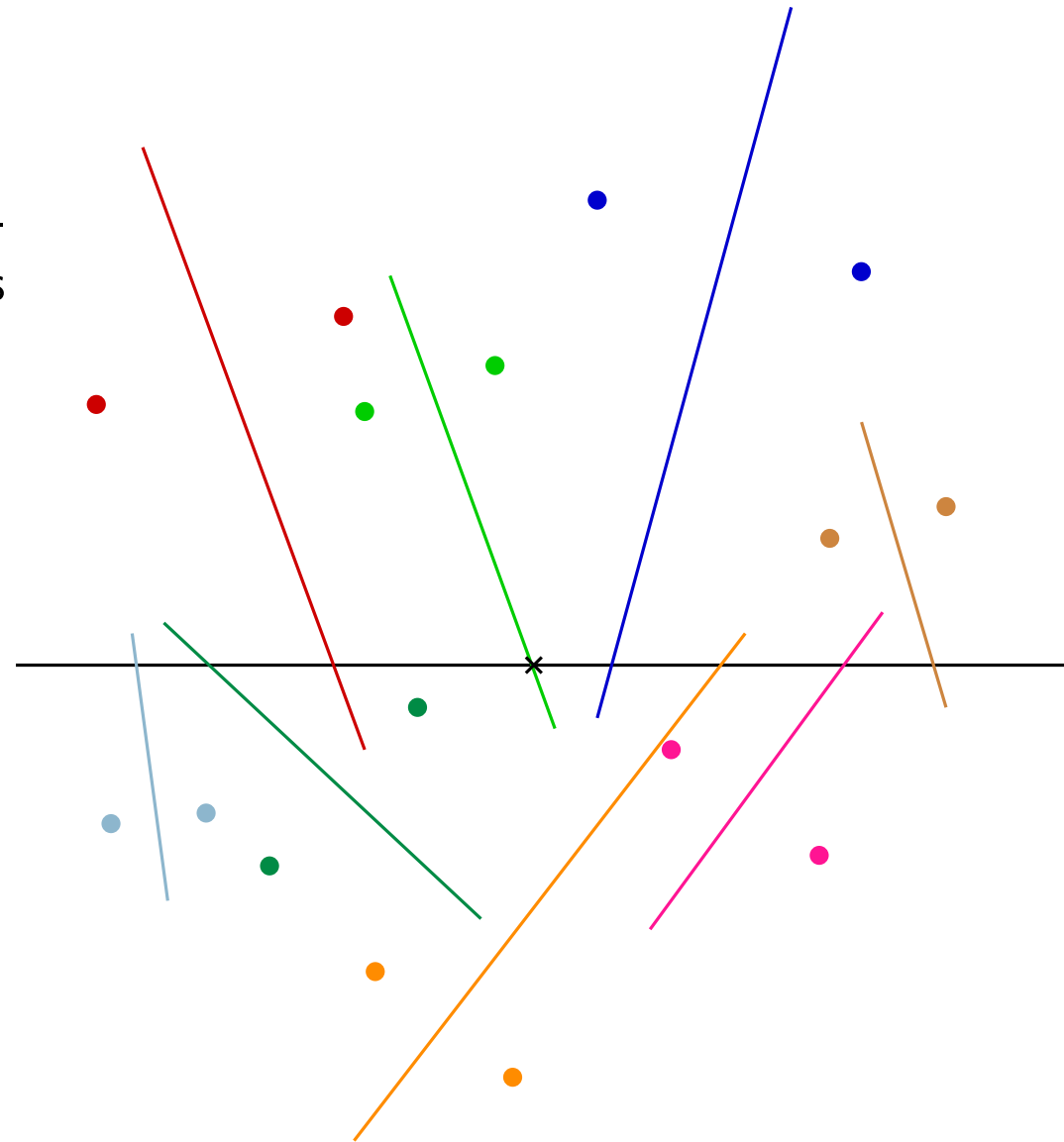
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m , the median value of the x_{ij} .



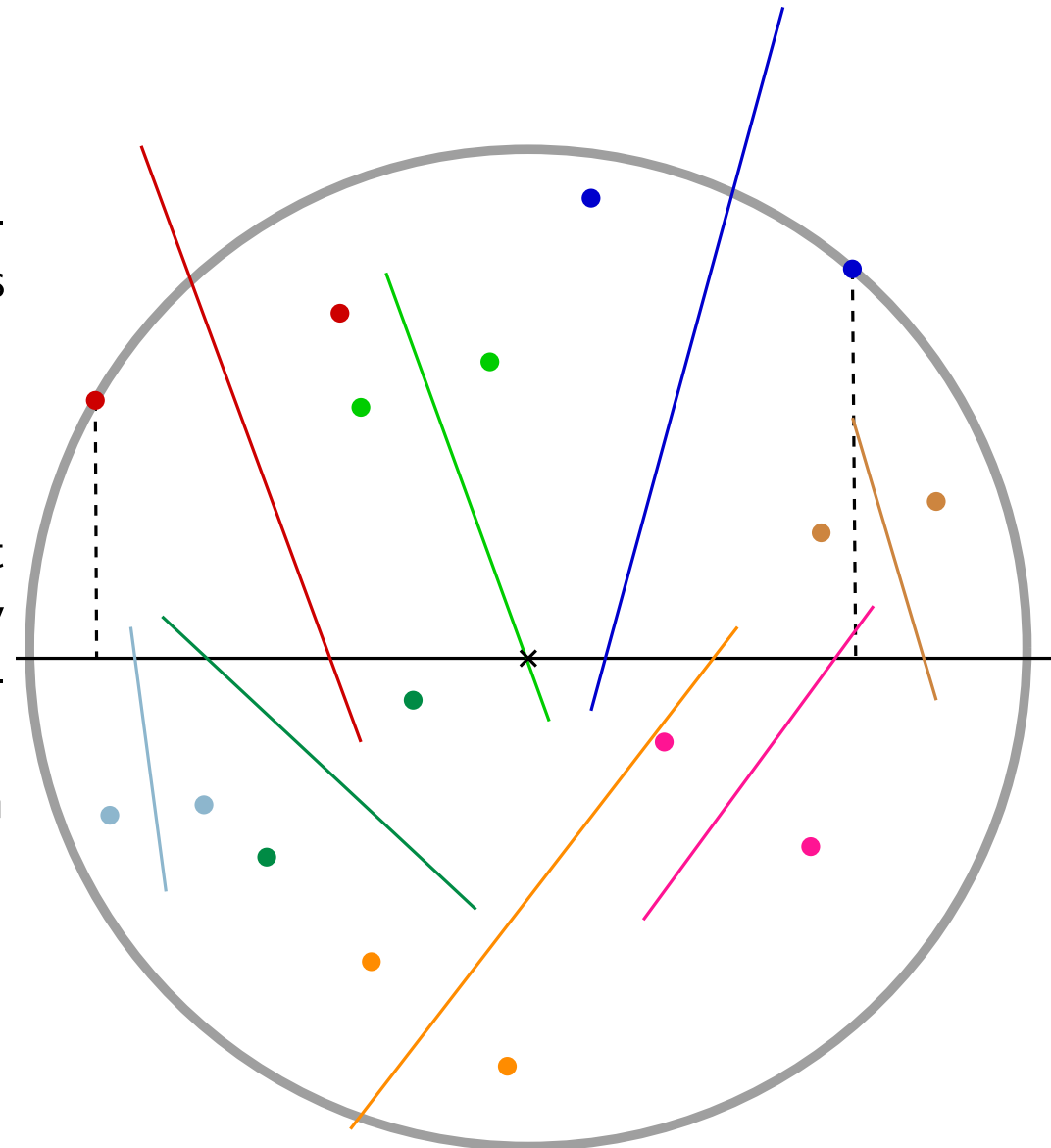
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m , the median value of the x_{ij} .
3. Search: Compute C_m , the MSC centered at x_m . If the points p_i lying in the boundary of C_m project orthogonally onto $y = 0$ to different sides of x_m , then x_m is the solution. If they all project onto the same side then search on that side.



COMPUTING THE MINIMUM SPANNING CIRCLE

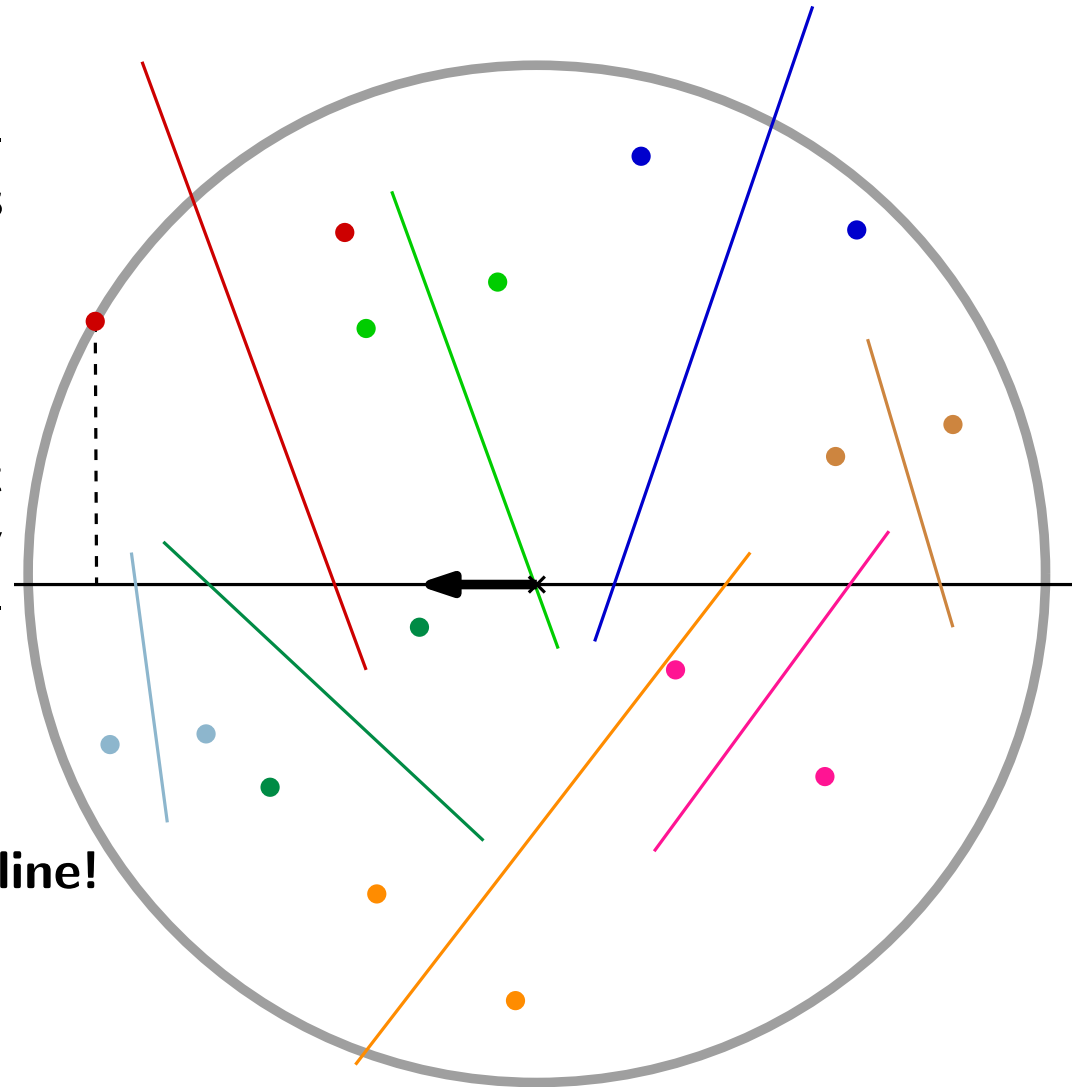
MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m , the median value of the x_{ij} .
3. Search: Compute C_m , the MSC centered at x_m . If the points p_i lying in the boundary of C_m project orthogonally onto $y = 0$ to different sides of x_m , then x_m is the solution. If they all project onto the same side then search on that side.

Notice that the radius is unimodal along the line!



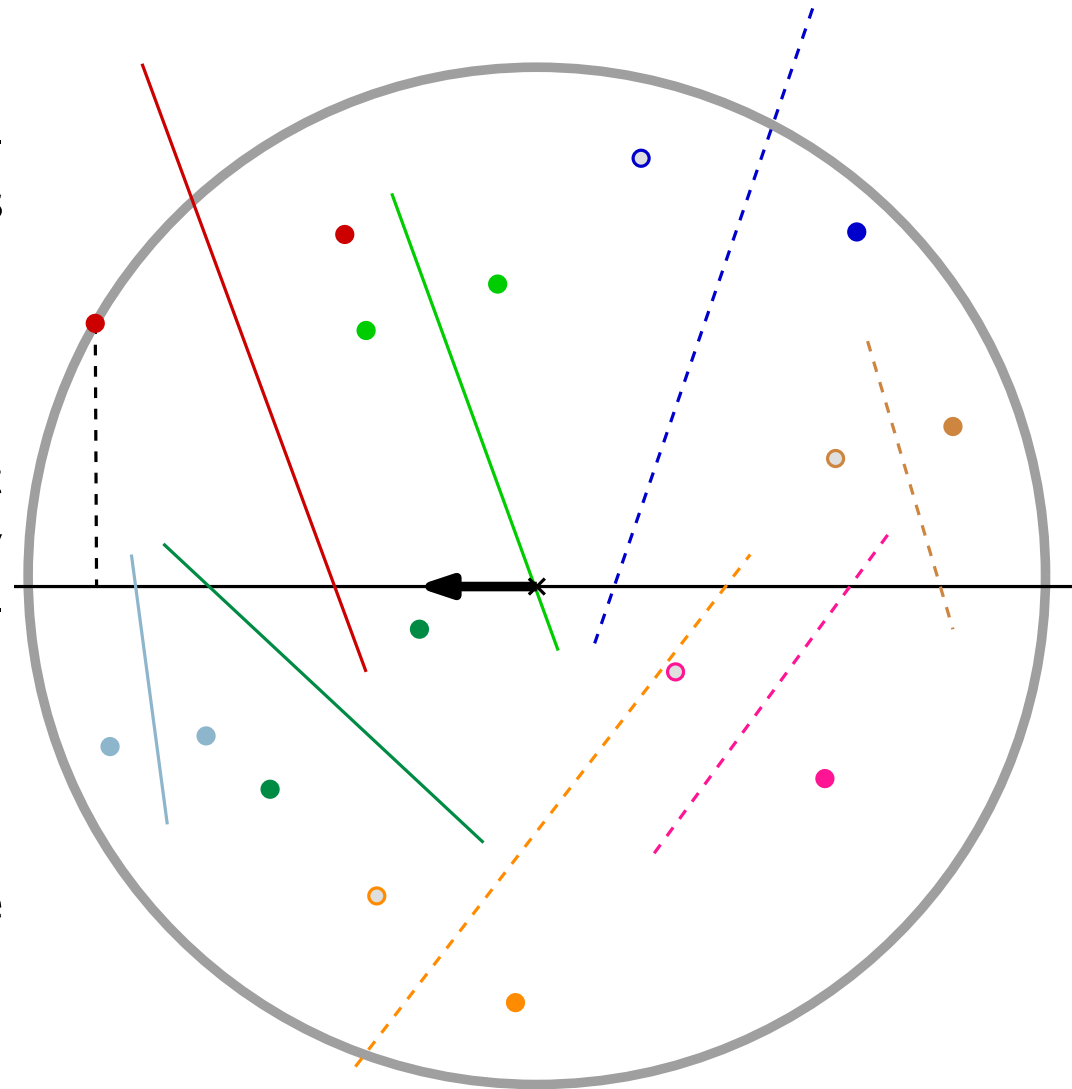
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair p_i, p_j , compute its perpendicular bisector b_{ij} and find its intersection x_{ij} with the line $y = 0$.
2. Compute x_m , the median value of the x_{ij} .
3. Search: Compute C_m , the MSC centered at x_m . If the points p_i lying in the boundary of C_m project orthogonally onto $y = 0$ to different sides of x_m , then x_m is the solution. If they all project onto the same side then search on that side.
4. Prune: For all x_{ij} located in the side opposite to the solution, eliminate the point p_i or p_j which is closest to x_m .



COMPUTING THE MINIMUM SPANNING CIRCLE

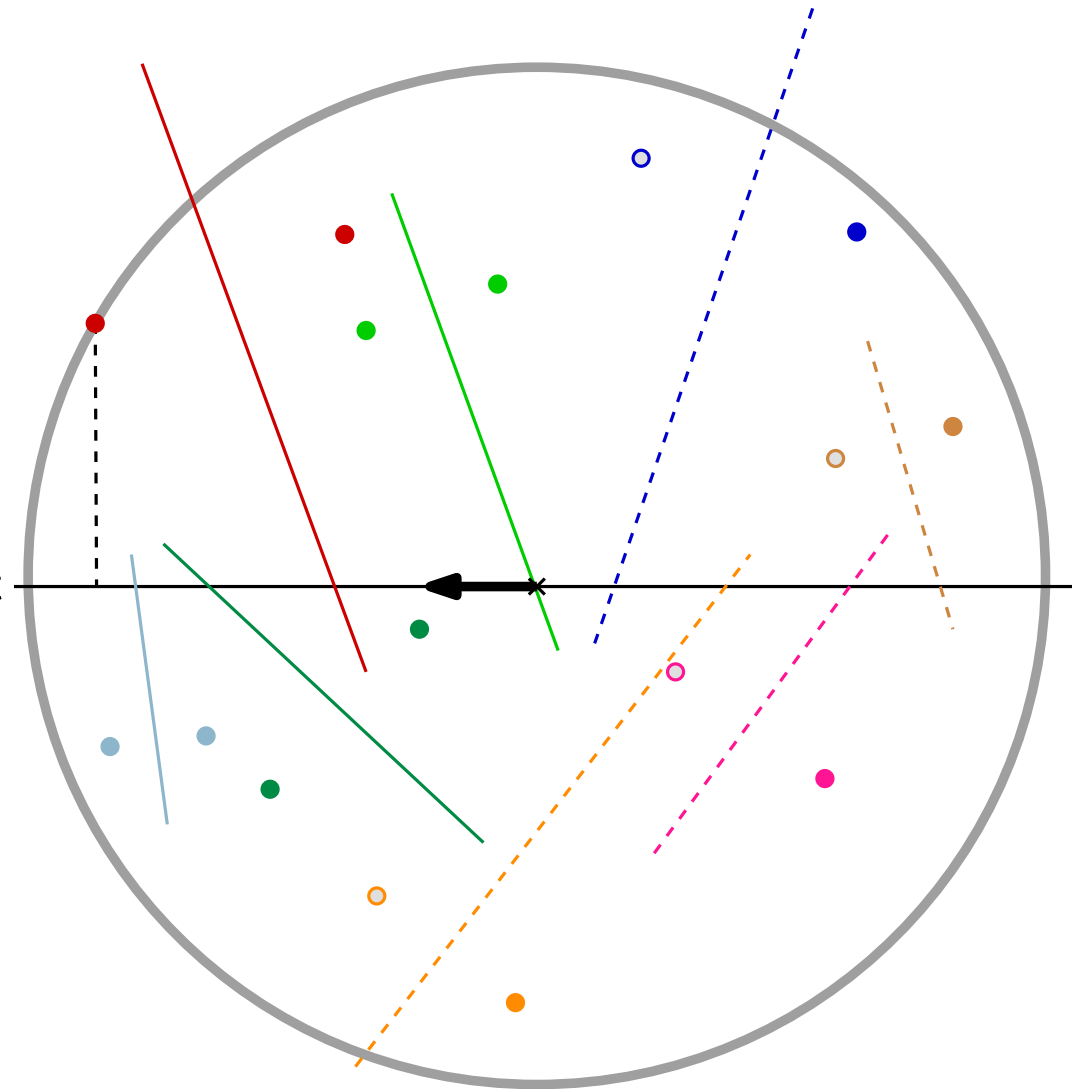
MSC restricted to a line

Input: a set of n points (a_i, b_i) , $i = 1, \dots, n$.

Output: $\min_{x \in \mathbb{R}} \max_{i=1 \dots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points, compute the perpendicular bisector of each pair, and find its intersection with the line $y = 0$. $O(n)$
2. Compute a median value. $O(n)$
3. Search: $O(n)$
4. Prune: In $O(n)$ time, at least $1/4$ of the input is pruned.

The problem is solved in $O(n)$ time.



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

COMPUTING THE MINIMUM SPANNING CIRCLE

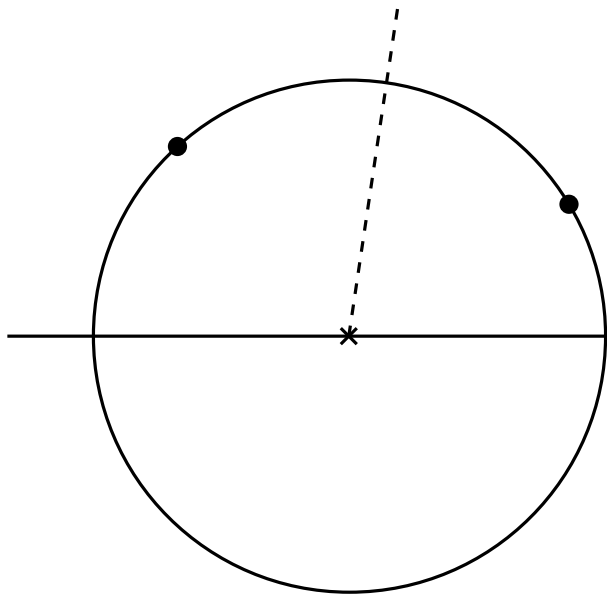
MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair.
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal.
3. Pair up the non-vertical non-horizontal bisectors, always pairing up one positive slope bisector with one negative slope bisector. Compute the intersection point of each pair.
4. First search: compute the median value x_m of the x -coordinate of the intersection points. Solve the MSC problem restricted to the line $x = x_m$. Use the solution to decide whether the unrestricted solution has been found or it lies to the left/right of the vertical line $x = x_m$.
5. Second search: if the solution has not been found, compute the median value y_m of the y -coordinate of the intersection points lying in the opposite halfplane. Solve the MSC problem restricted to the line $y = y_m$. Use the solution to decide whether the unrestricted solution has been found or it lies above/below the horizontal line $y = y_m$.
6. Prune: if the solution has not been found, then each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated. Analogously, for each vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated, and the same happens with the horizontal bisectors.

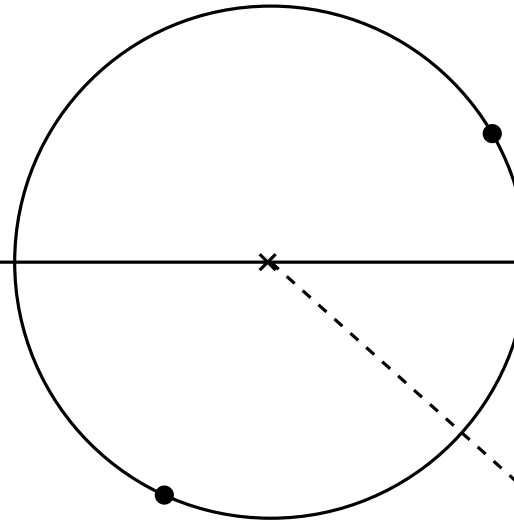
COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

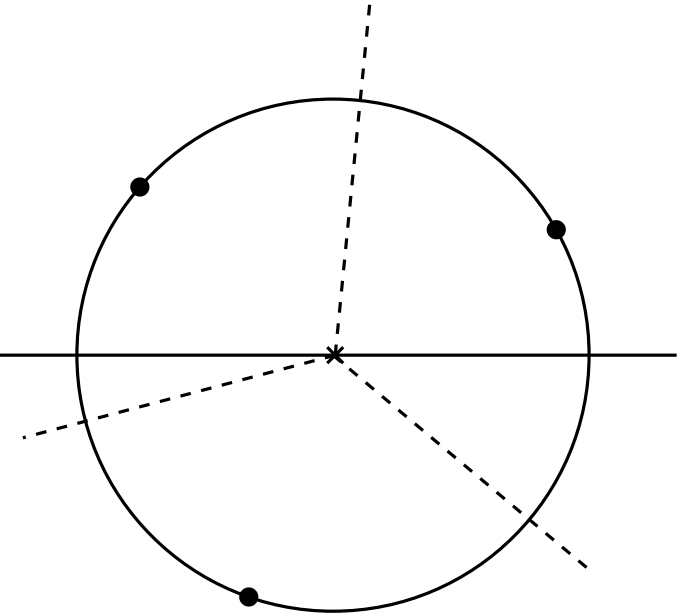
Search step



The solution
lies above



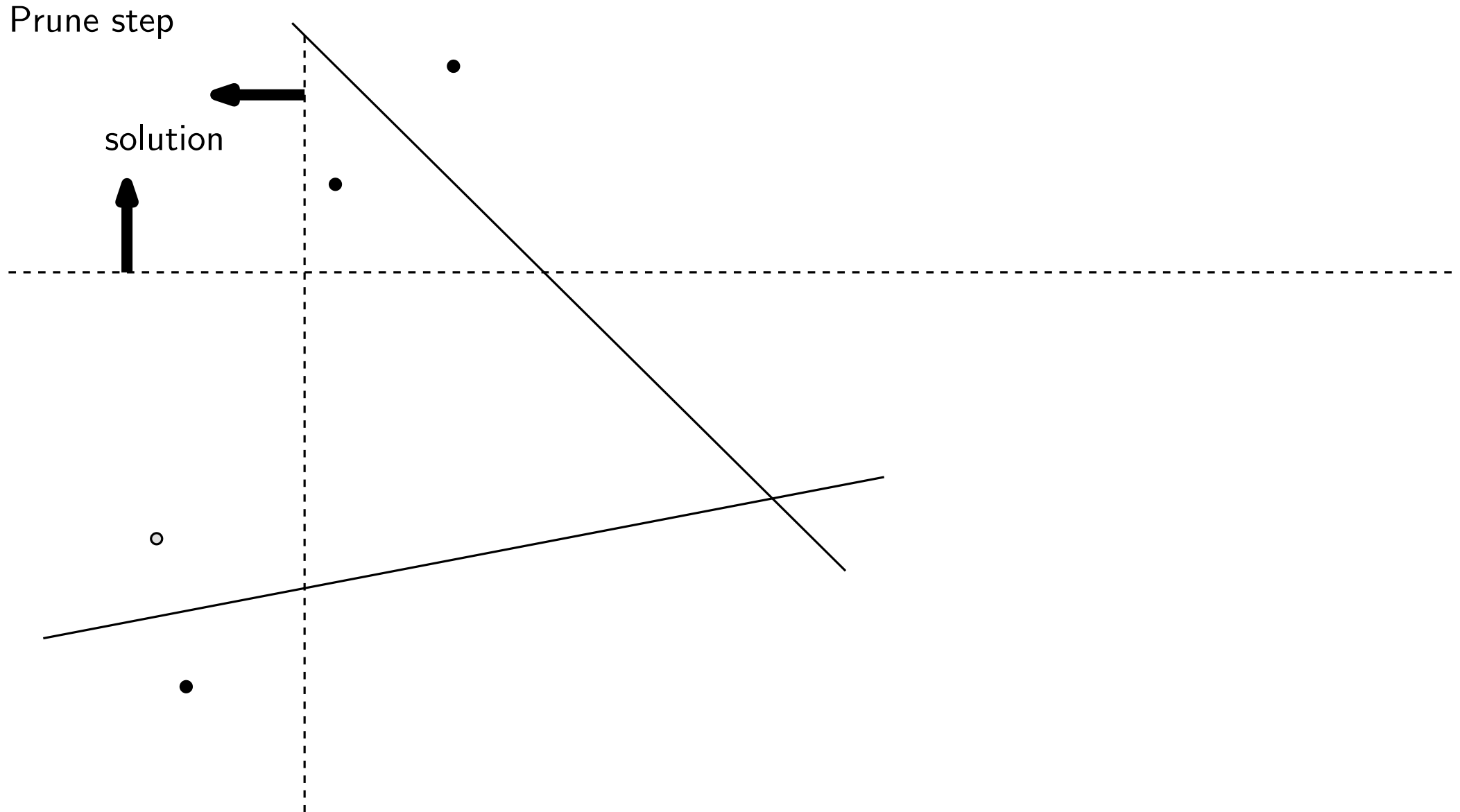
The solution
lies below



The solution
has been found

COMPUTING THE MINIMUM SPANNING CIRCLE

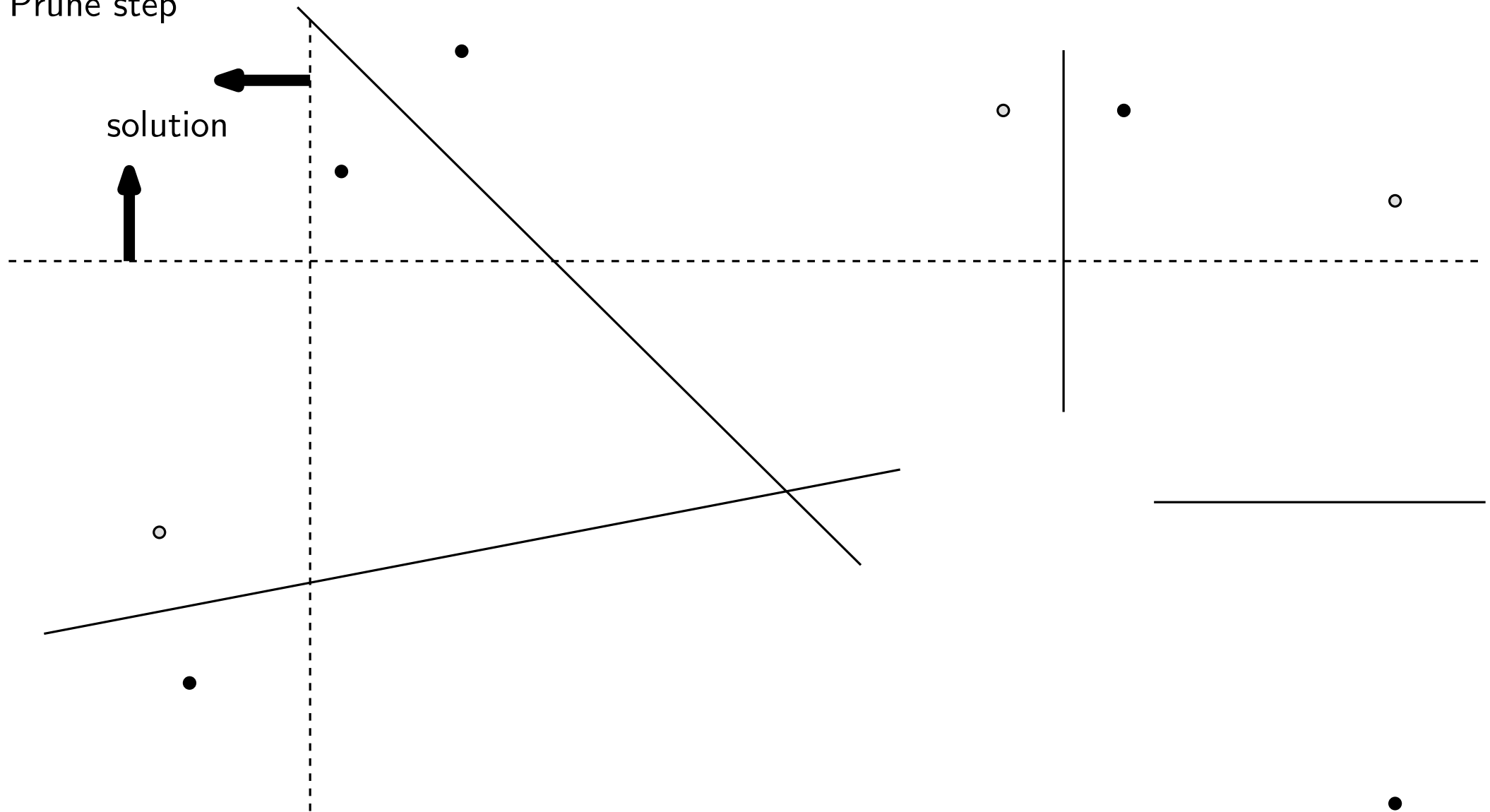
MSC in the plane



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

Prune step



COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair. $O(n)$
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal. $O(n)$
3. Pair up the non-vertical non-horizontal bisectors, and compute the intersection point of each pair. $O(n)$
4. First search: $O(n)$
5. Second search: $O(n)$
6. Prune: In $O(n)$ time, at least $1/16$ of the input is pruned.

COMPUTING THE MINIMUM SPANNING CIRCLE

MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair. $O(n)$
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal. $O(n)$
3. Pair up the non-vertical non-horizontal bisectors, and compute the intersection point of each pair. $O(n)$
4. First search: $O(n)$
5. Second search: $O(n)$
6. Prune: In $O(n)$ time, at least $1/16$ of the input is pruned.

Conclusion

Given n points in the plane, the min-max facility location problem can be solved in linear time.

TO LEARN MORE

- F. P. Preparata and M. I. Shamos, **Computational Geometry: an Introduction** (revised ed.). Springer-Verlag, 1993.
- N. Megiddo, Linear-time algorithms for linear programming in \mathbb{R}^3 and related problems, *SIAM Journal on Computing*, Vol. 12, N. 4, pp. 759-776, 1983.
- M. E. Dyer, On a multidimensional search technique and its application to the euclidean one-centre problem, *SIAM Journal on Computing*, Vol. 15, N. 3, pp. 725-738, 1986.
- F. Hurtado, V. Sacristán, G. Toussaint, Some Constrained Minimax and Maximin Location Problems, *Studies in Locational Analysis*, Vol. 15, pp. 17-35, 2000.
- F. Gómez, F. Hurtado, S. Ramaswami, V. Sacristán, G. Toussaint, Implicit Convex Polygons, *Journal of Mathematical Modelling and Algorithms*, Vol. 1, N. 1, pp. 57-85, 2002.