INTERSECTING LINE-SEGMENTS

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Input: n line-segments in the plane, $s_i = (p_i, q_i),\ i = 1 \ldots n$.
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Some applications

Geographic information systems
Detecting the intersections among the elements of the different layers of information (cities, roads, services, ...)

Realistic visualization
Eliminating the hidden portions of a scene

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Brute force solution

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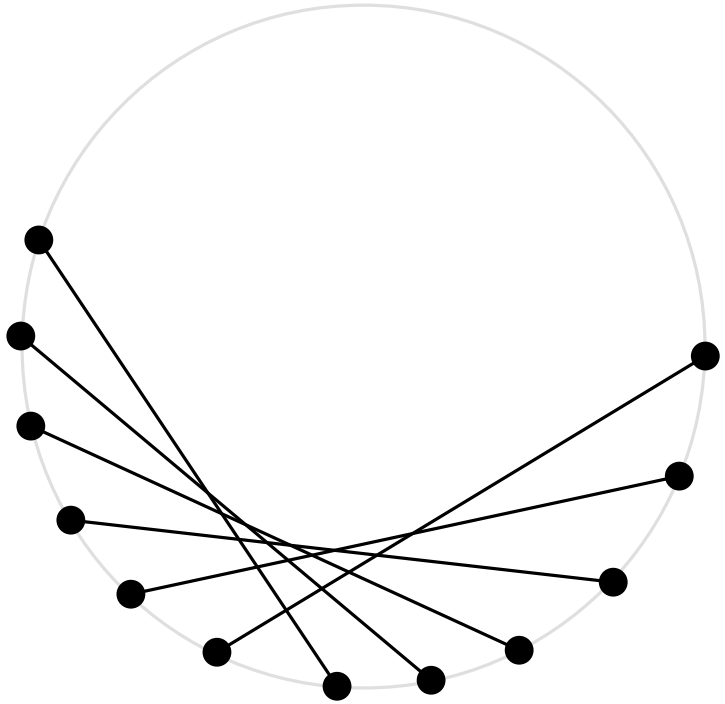

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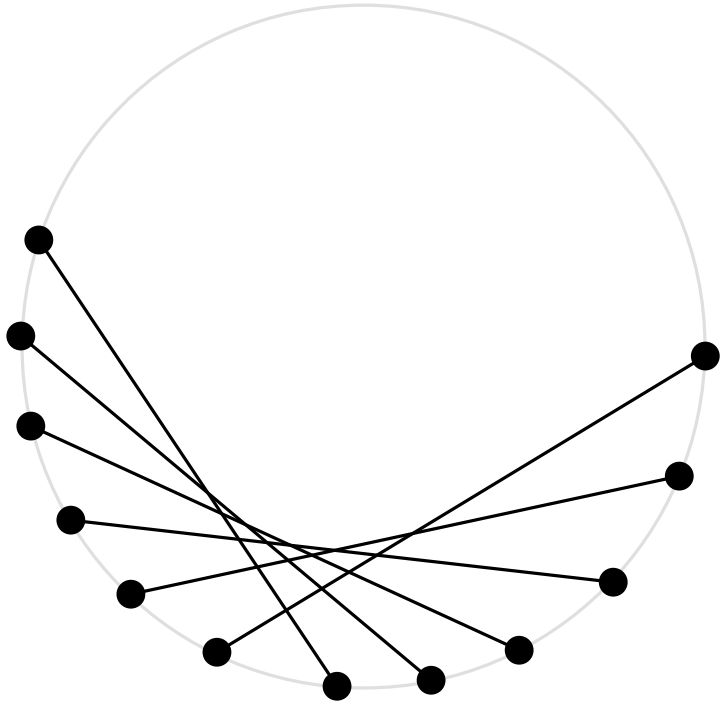

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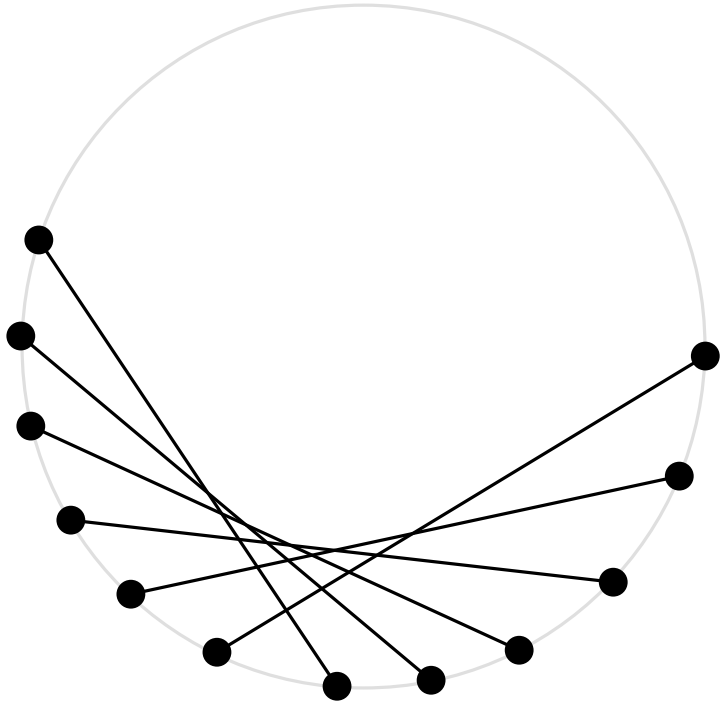

Output-sensitive solution

Algorithm whose running time depends on the number of intersections to be reported.

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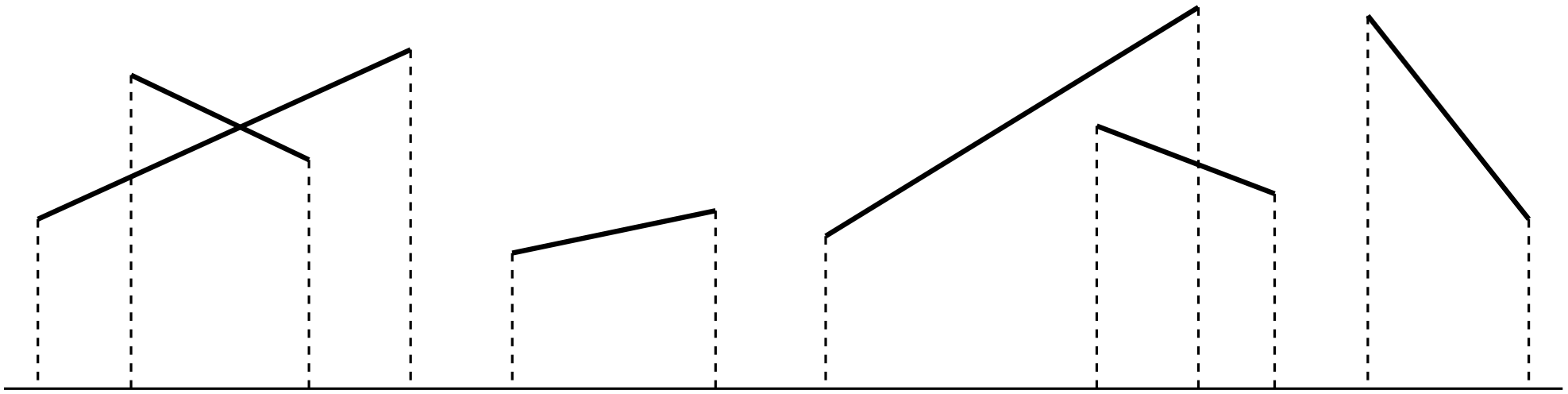

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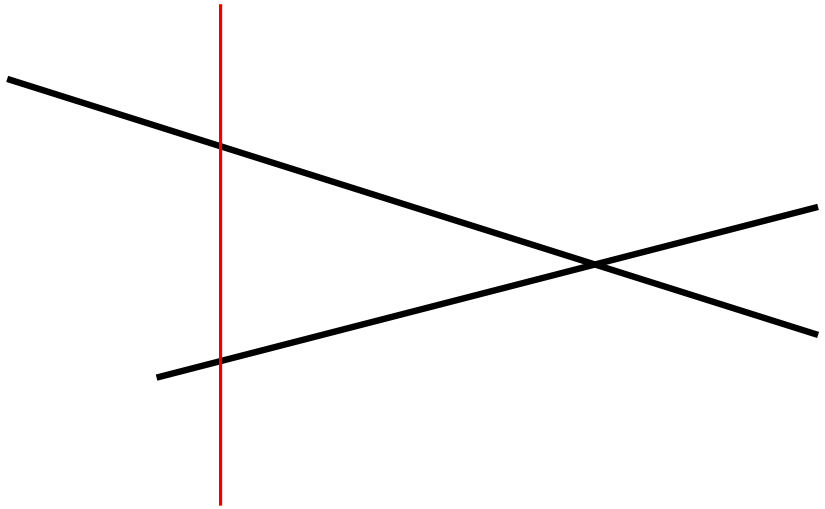

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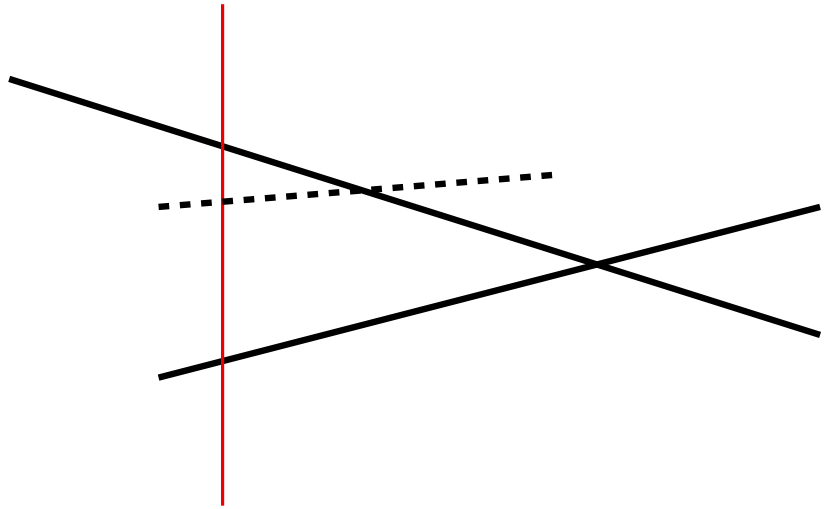

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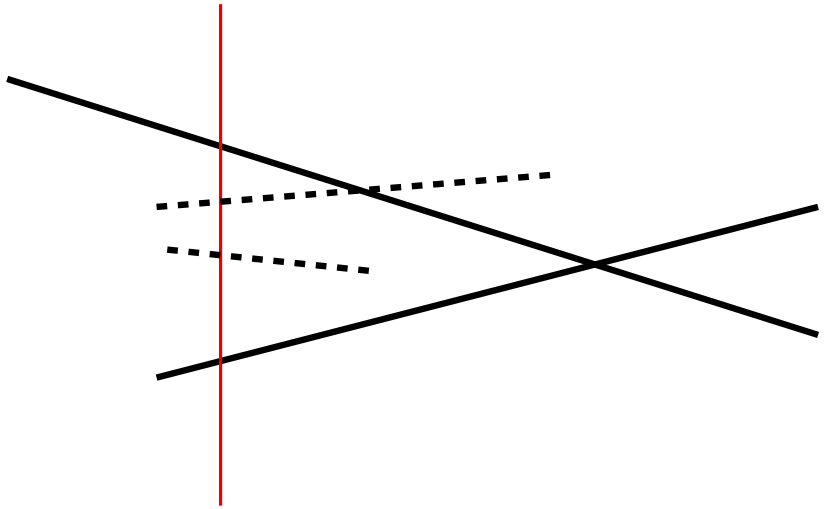

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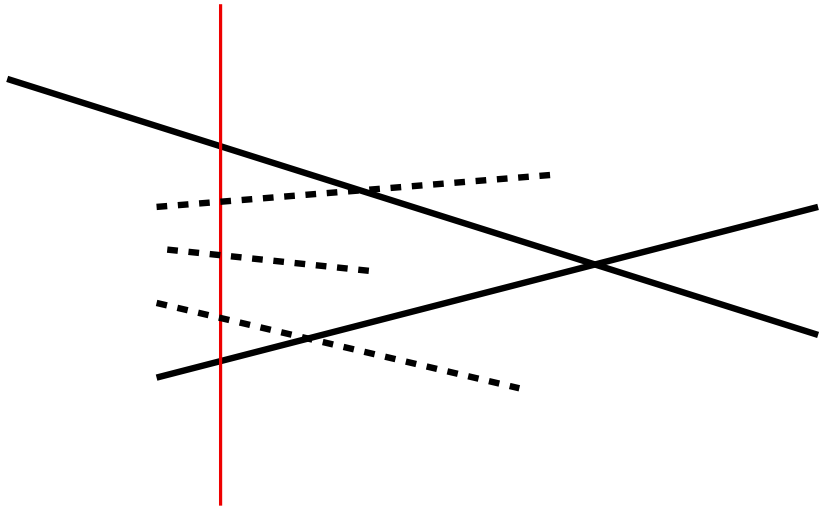

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Bentley-Ottman's algorithm

Sweep-line algorithm

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Sweep-line algorithm

A straight line (vertical, in this case) scans the scene (the line-segments) and allows detecting and constructing the desired elements (intersections), leaving the problem solved behind it. The sweeping process is discretized.

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Sweep-line algorithm

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Essential elements of a sweep line algorithm:

- Sweep line

  Data structure storing the information of the portion of the scene intersected by the sweep line. It stays updated at all times. In our problem, it will contain the information of the line-segments intersected by the sweep line, sorted by y-coordinate.

- Events queue

  Priority queue keeping the information of the locations where the sweep line changes and needs to be updated. In our problem, the events will be the endpoints and the intersection points of the line-segments, sorted by their x-coordinate. Notice that not all events are known in advance!

- Output data structure

Bentley-Ottman's algorithm

Sweep-line algorithm

Hypothesis (to be eliminated later on)

1. There are no repeated abscissae, i.e.: there are no vertical line-segments, and no two endpoints of two line-segments, no two intersection points of line-segments, no endpoint and intersection point, lie in the same vertical line.

2. At each intersection point, only two line-segments intersect.

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Sweep line (L)

Stabbed line-segments, in vertical order.

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Sweep line (L)

Stabbed line-segments, in vertical order.

Events queue (E)

- All endpoints of the line-segments (known a priori).
- All intersection points of line-segments (found on the fly).

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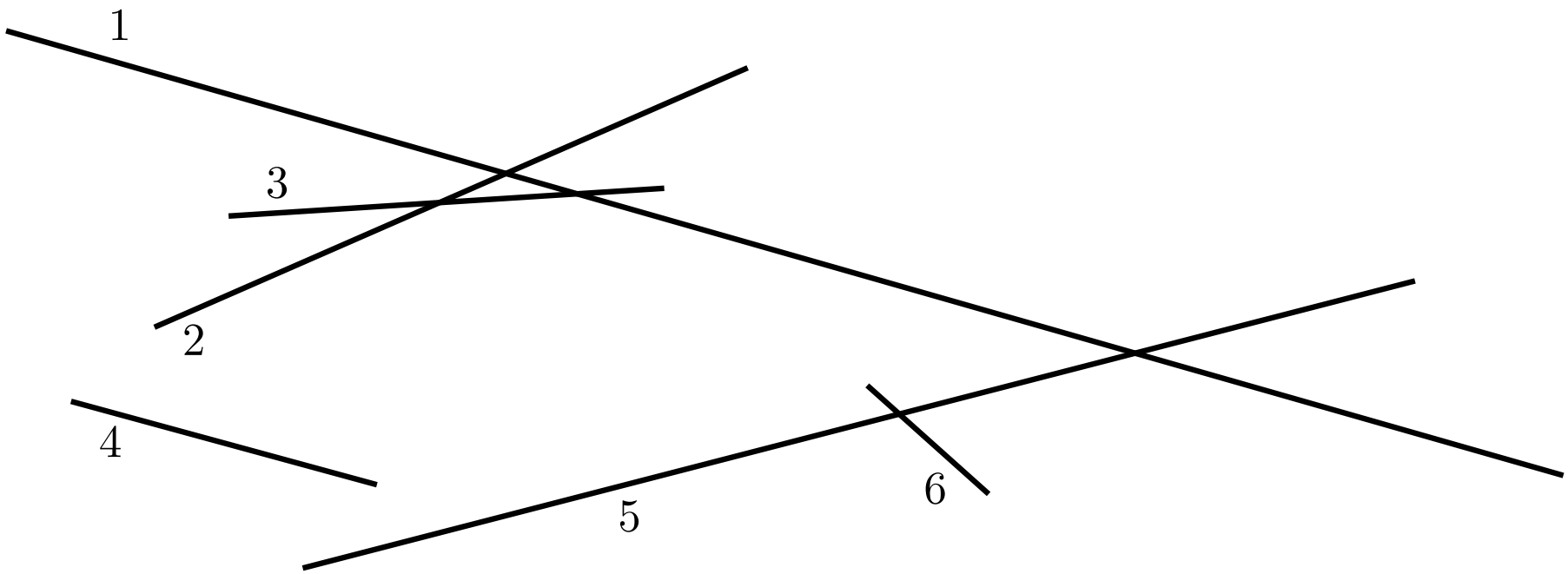
Initialization:
- Sort the $2n$ endpoints by abscissa and store the information in E.
- Line L starts empty.

Advance

While $E \neq \emptyset$ do:

1. $p = \min E$
2. If $p = \text{start}(s)$, then:
   - Insert s in L
   - If s^- and s intersect to the right of p, insert their intersection point in E and report it (if needed). Do the same for s^+.
3. If $p = \text{end}(s)$, then:
   - If s^- and s^+ intersect to the right of p, insert their intersection point in E and report it (if needed).
   - Delete s from L
4. If $p = s_1 \cap s_2$ with $s_1 <_L s_2$, then:
   - If s_1^- and s_2 intersect to the right of p, insert their intersection point in E and report it (if needed). Do the same for s_2^+ and s_1.
   - Transpose s_1 and s_2 in L
5. Delete p from E

Bentley-Ottman's algorithm: simulation



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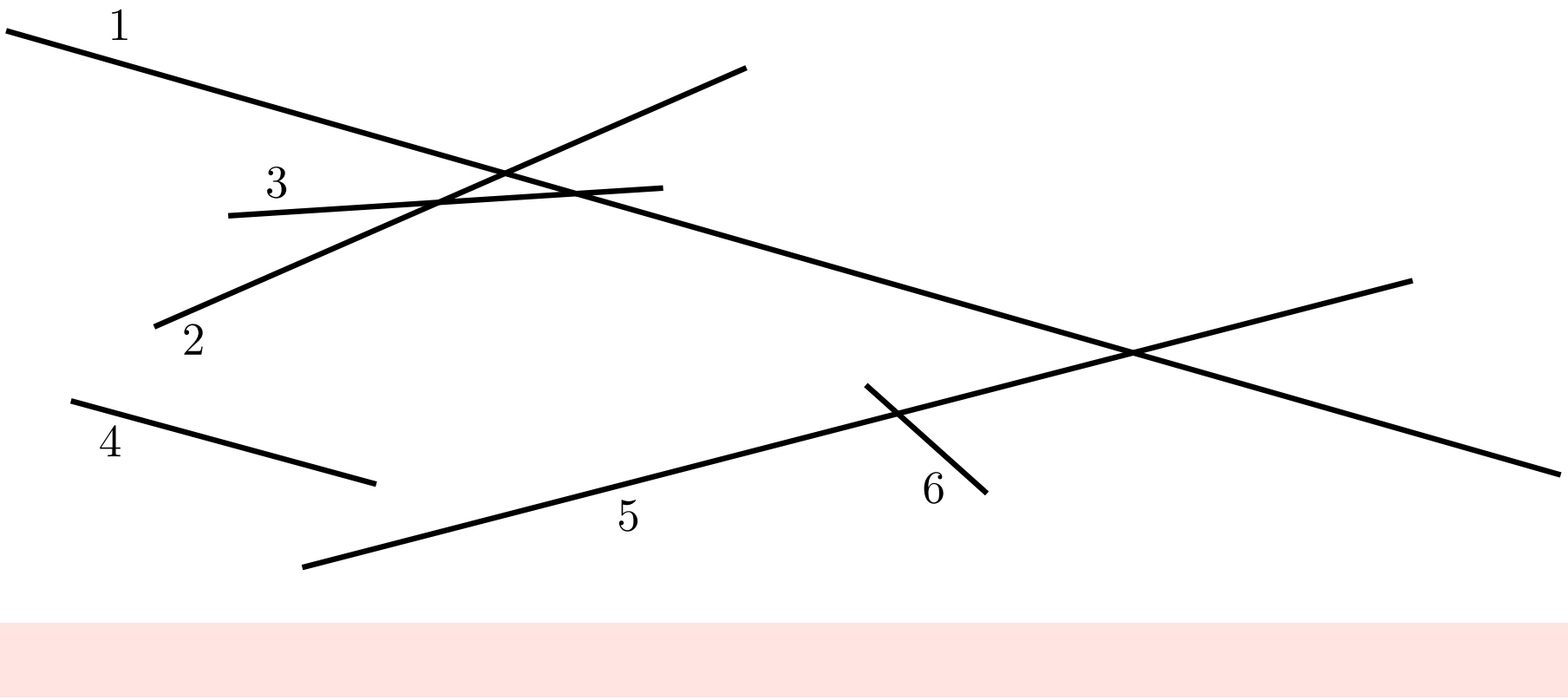

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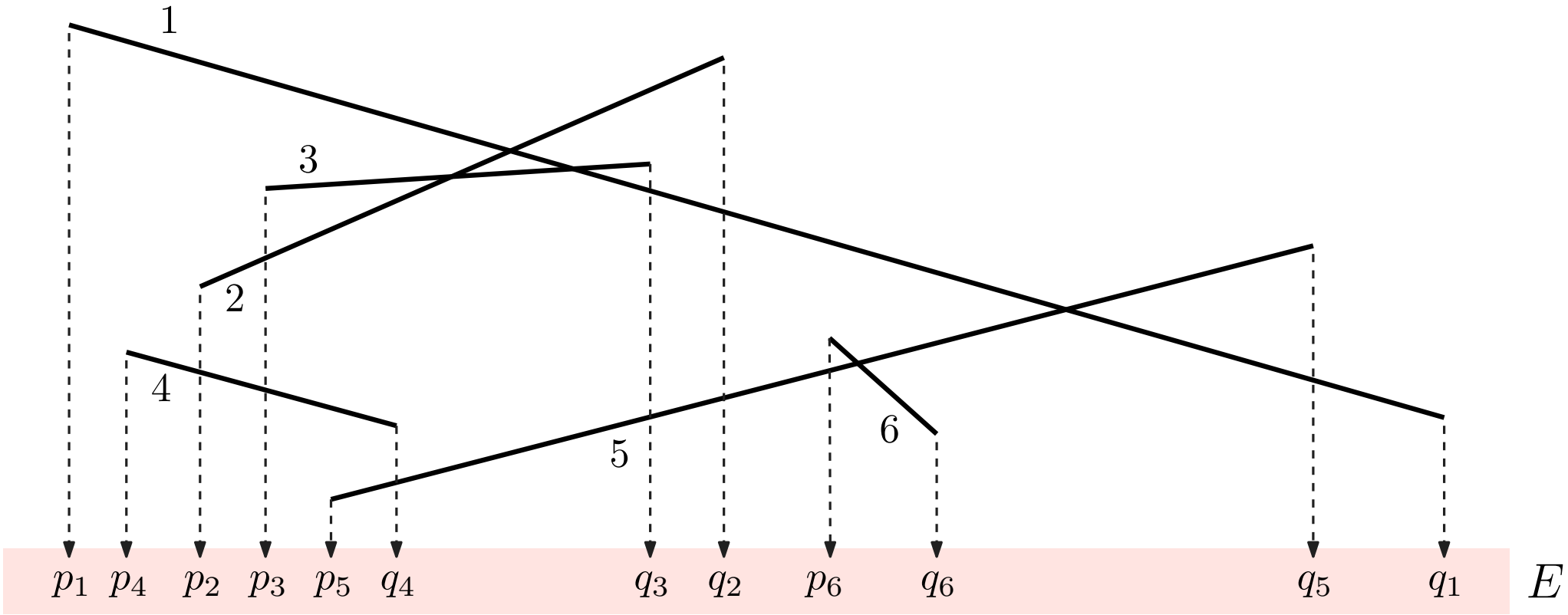

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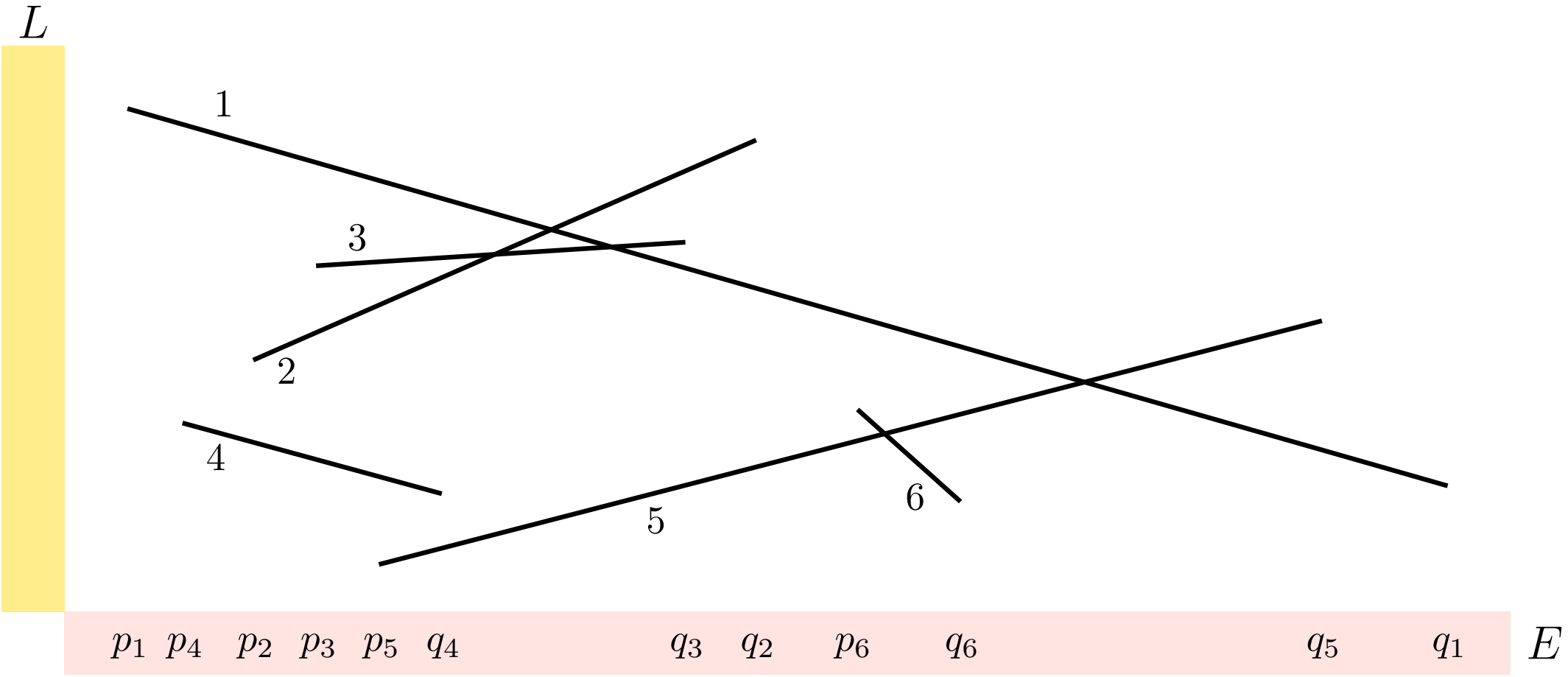

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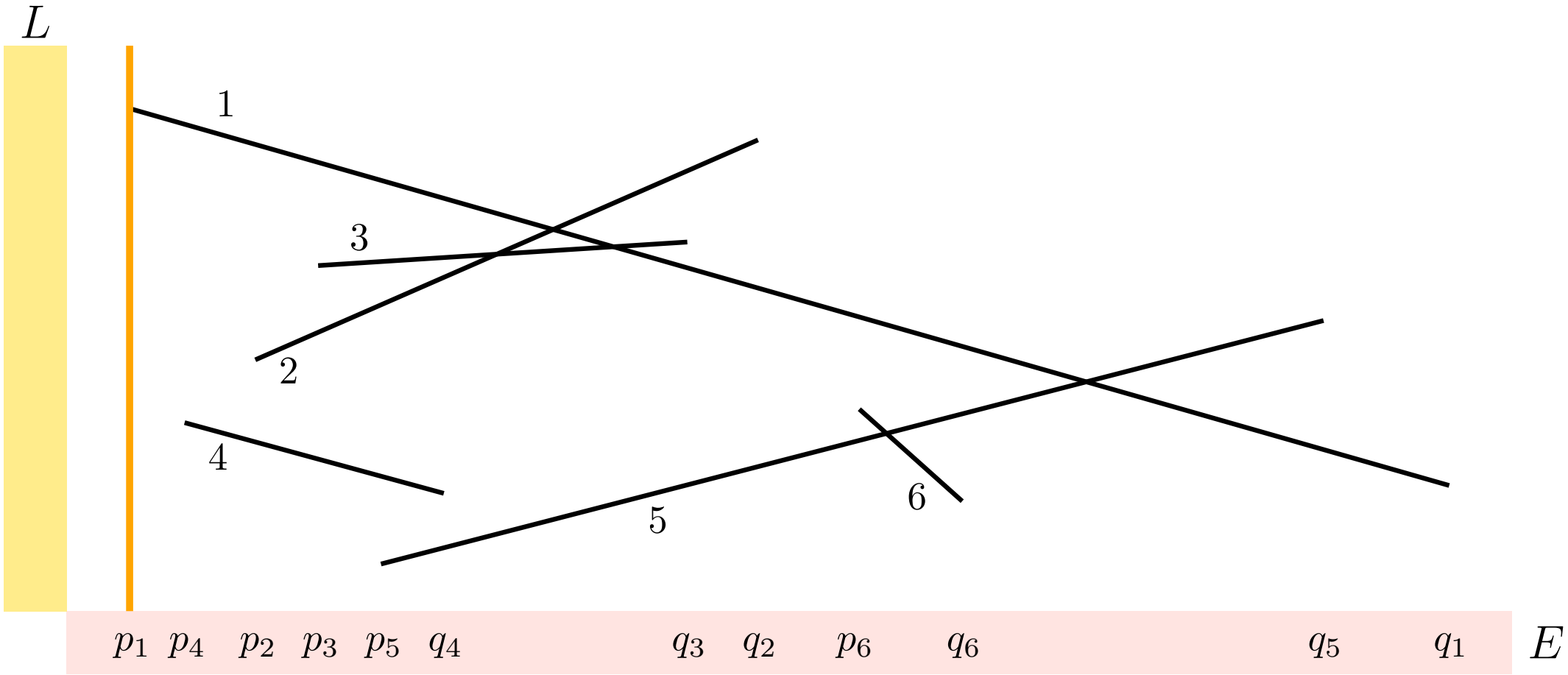

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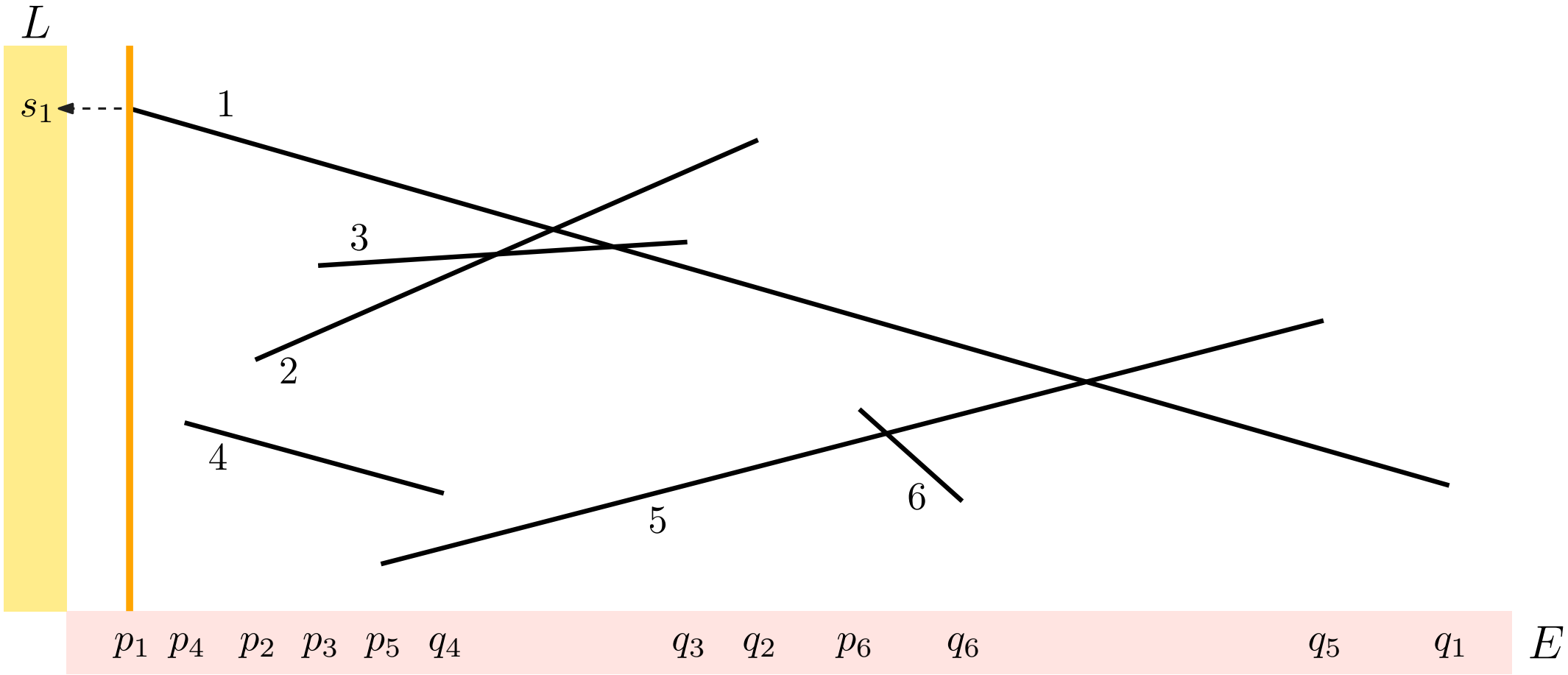

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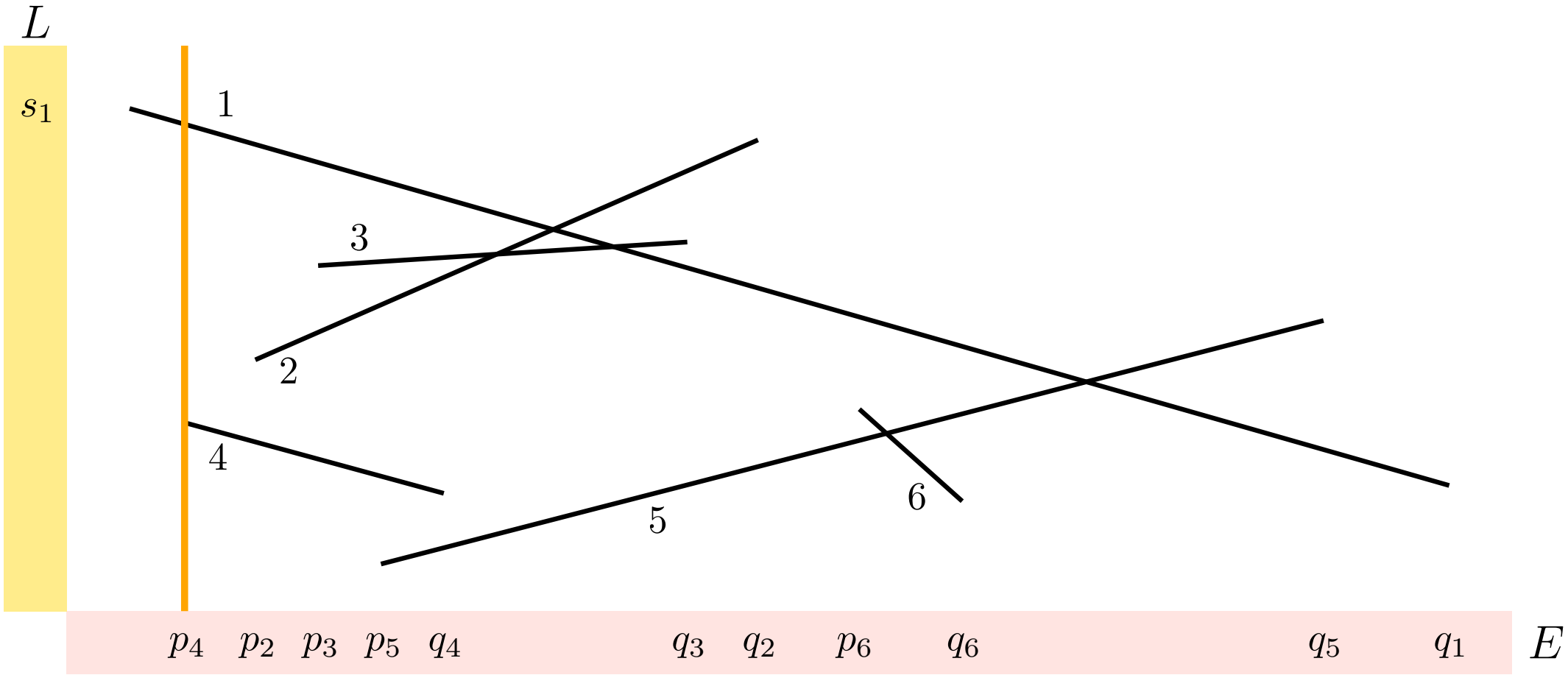

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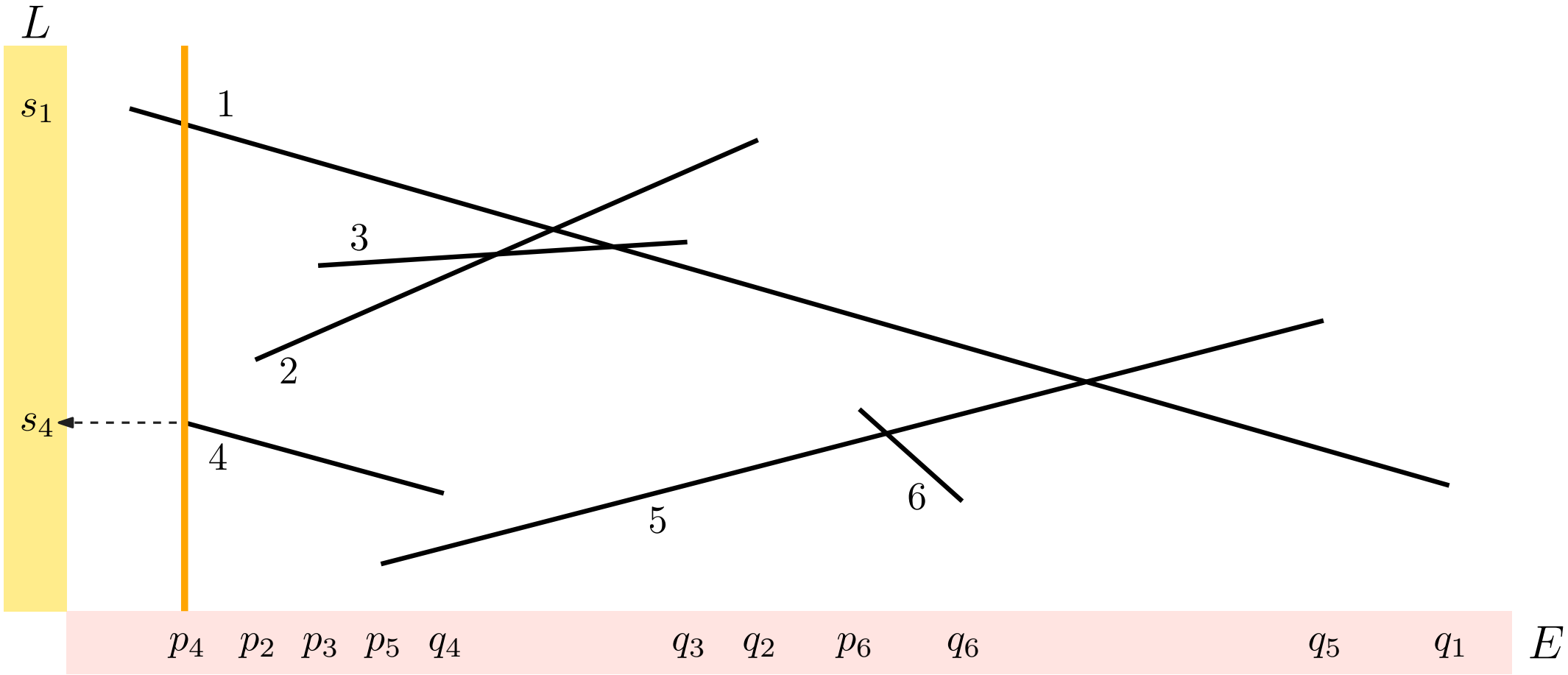

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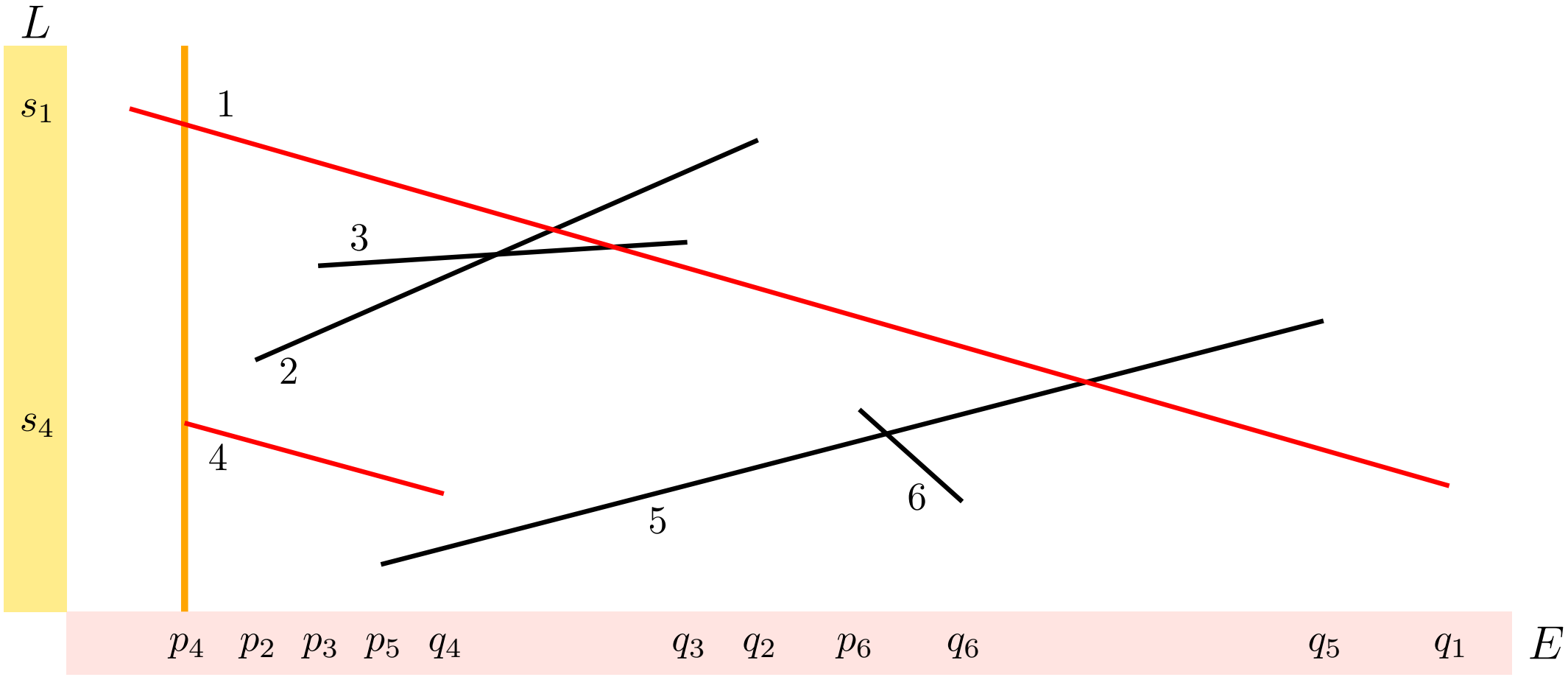

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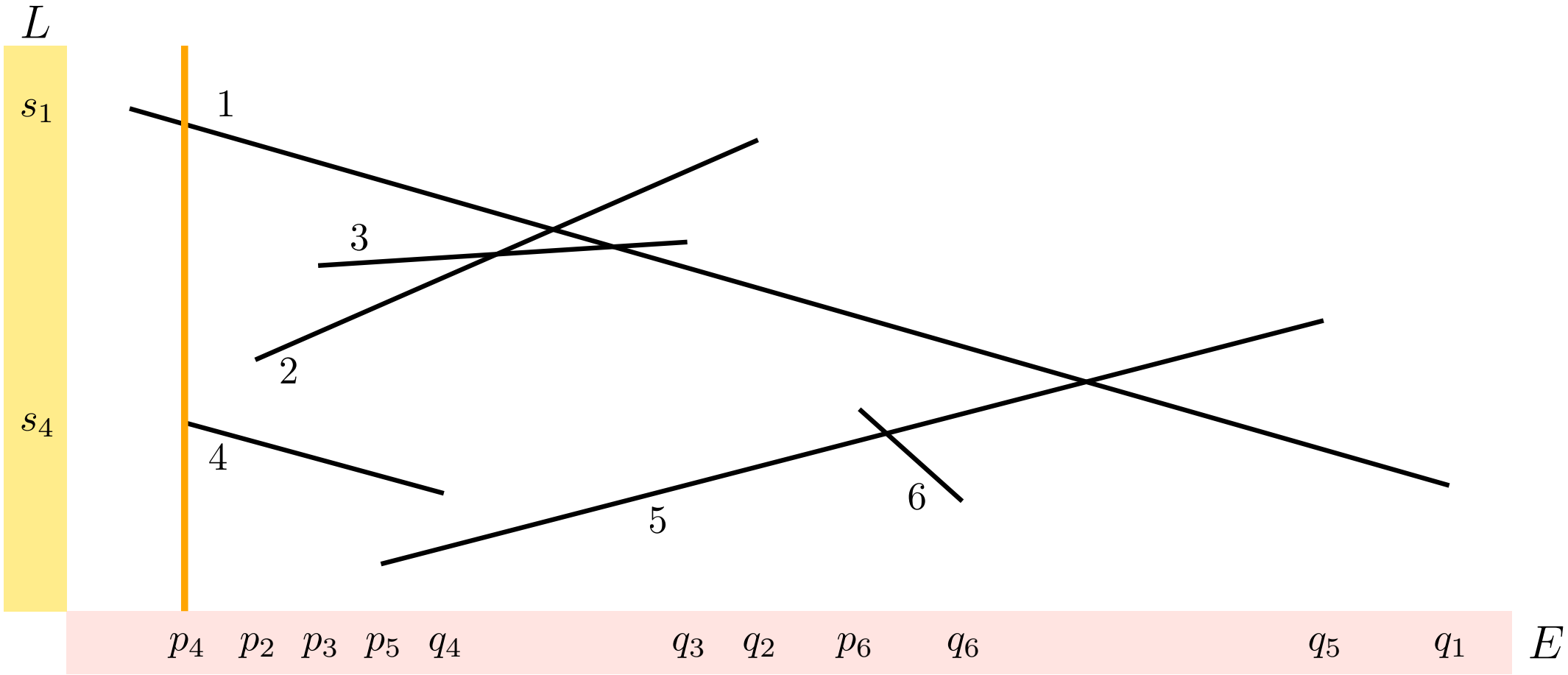

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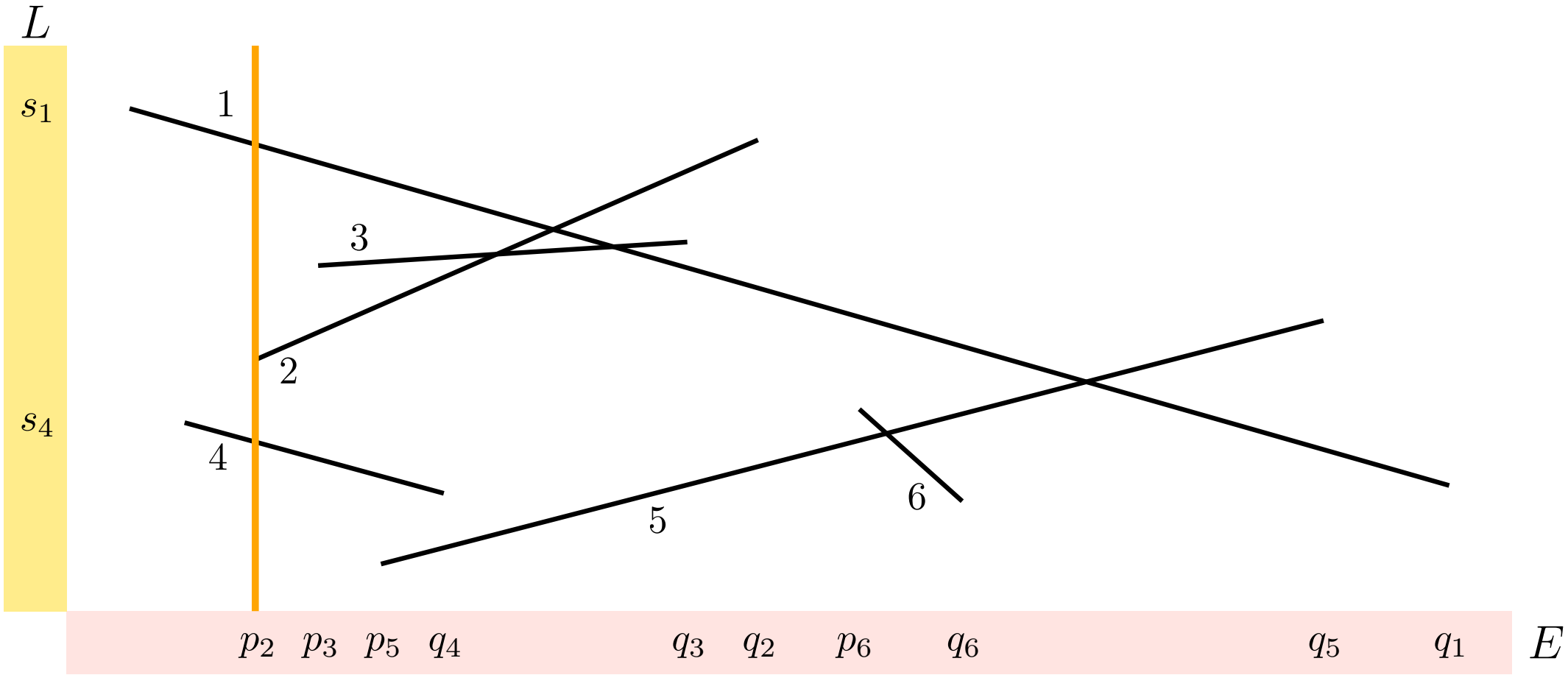

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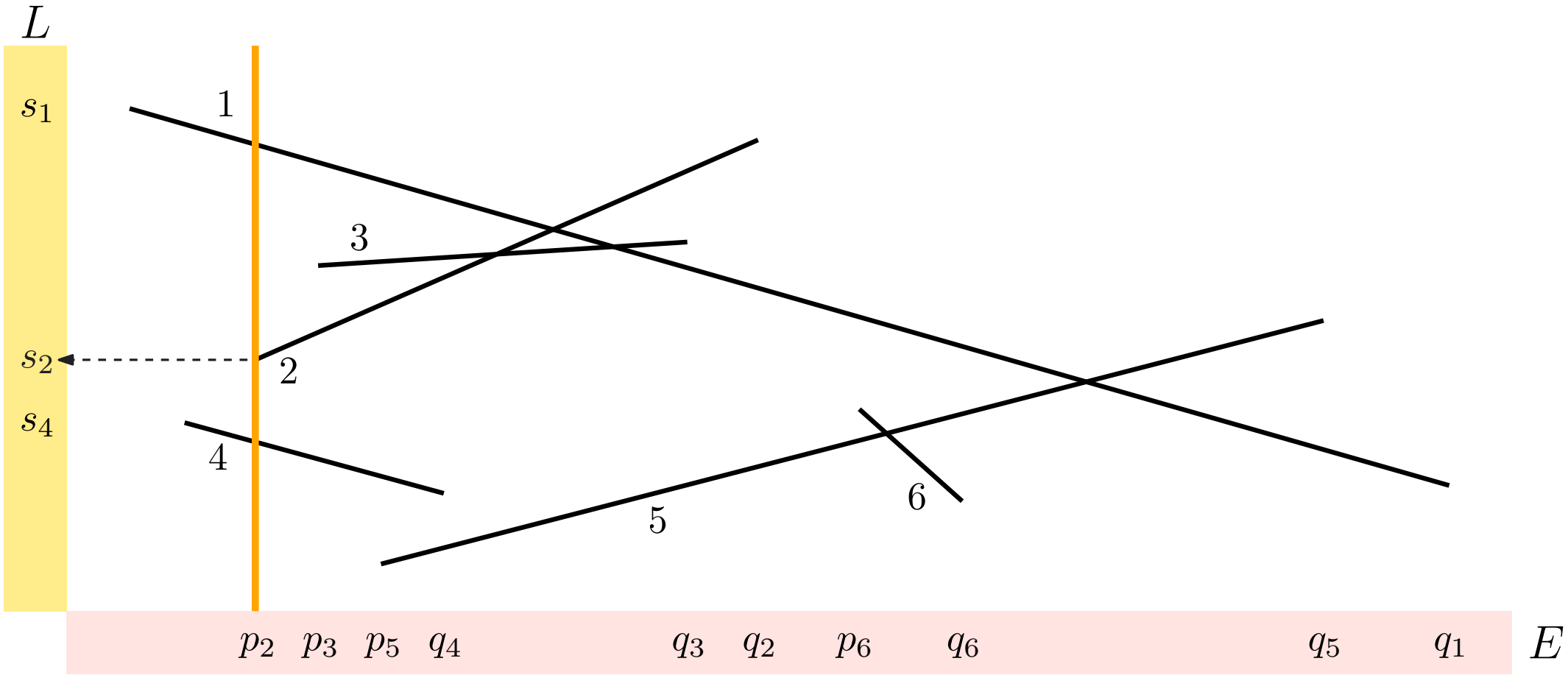

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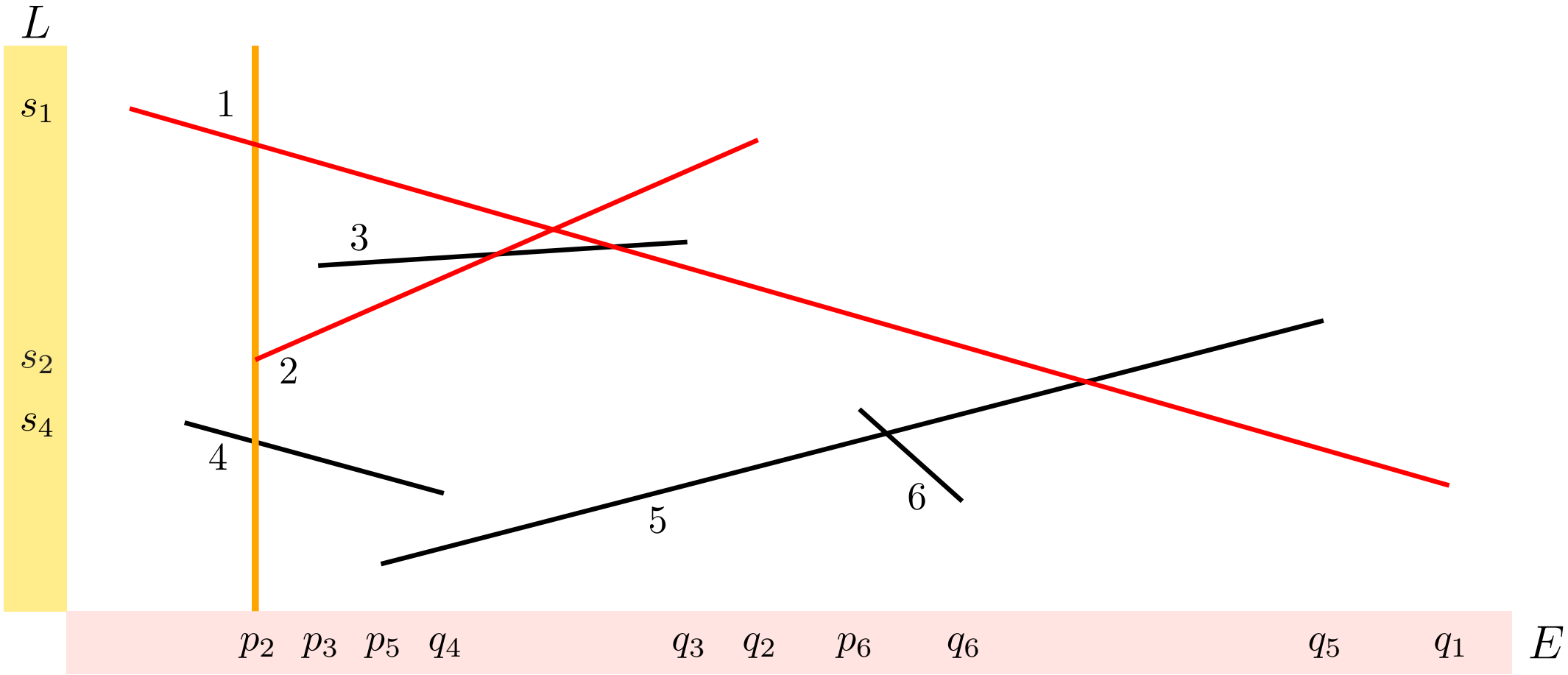

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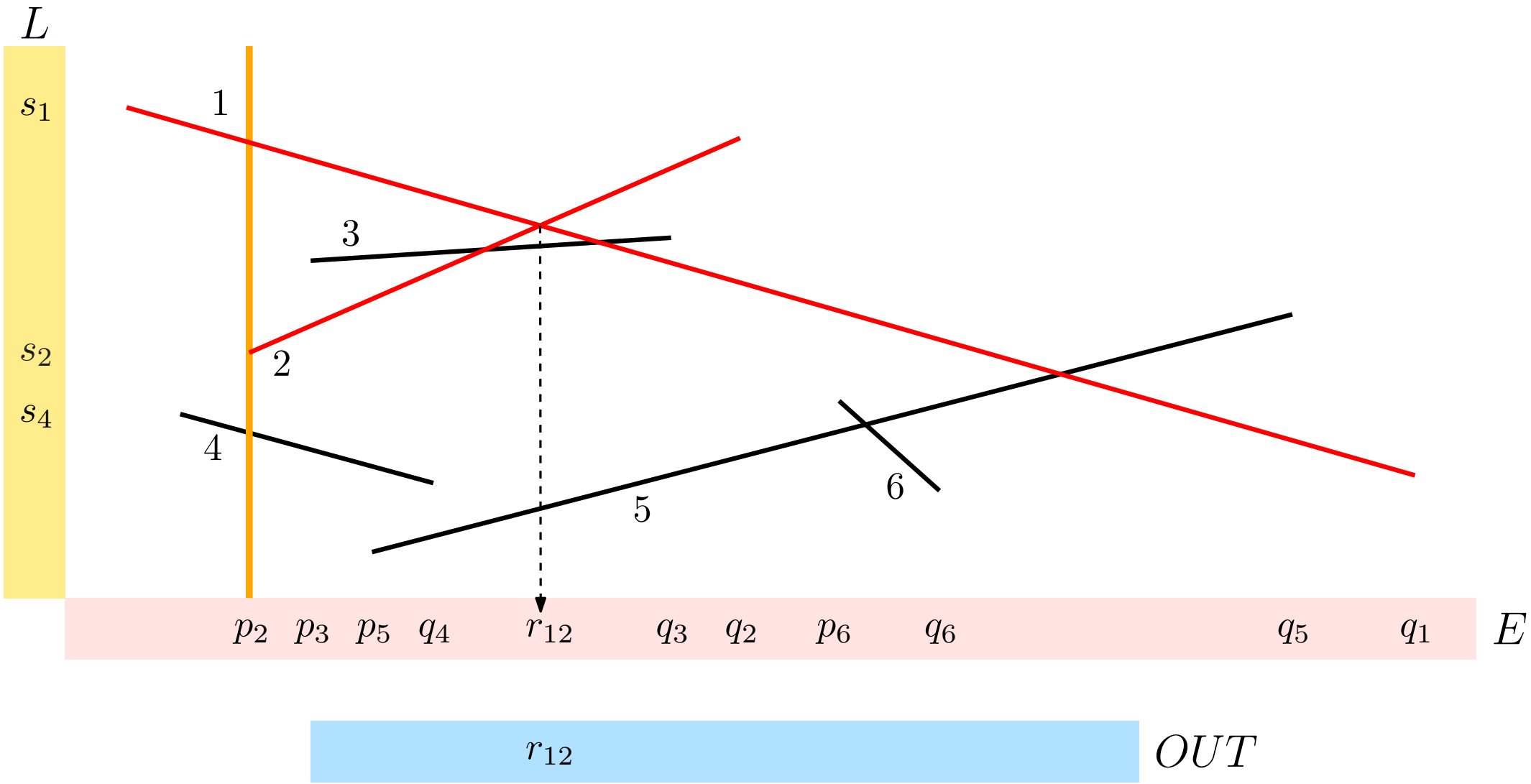

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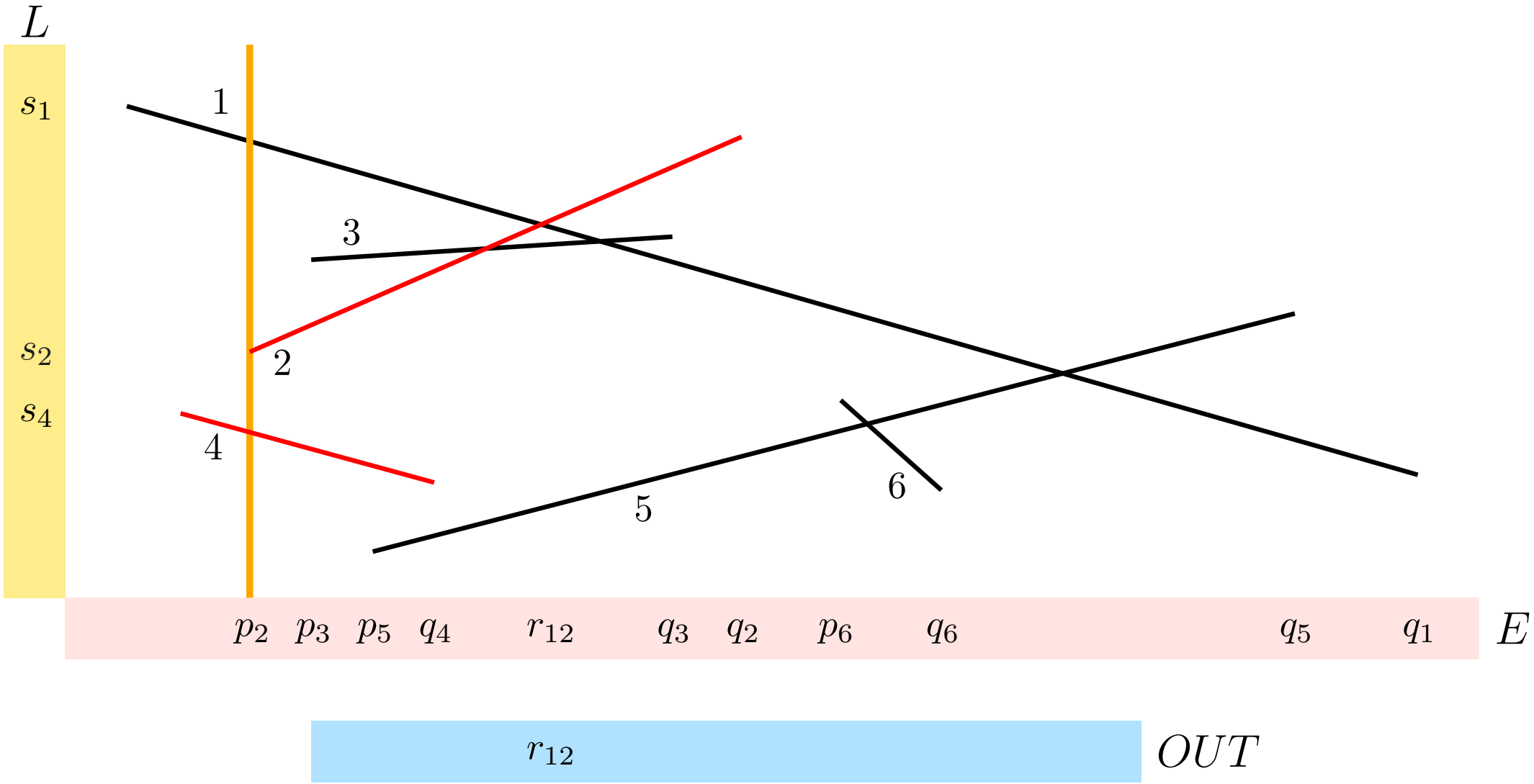

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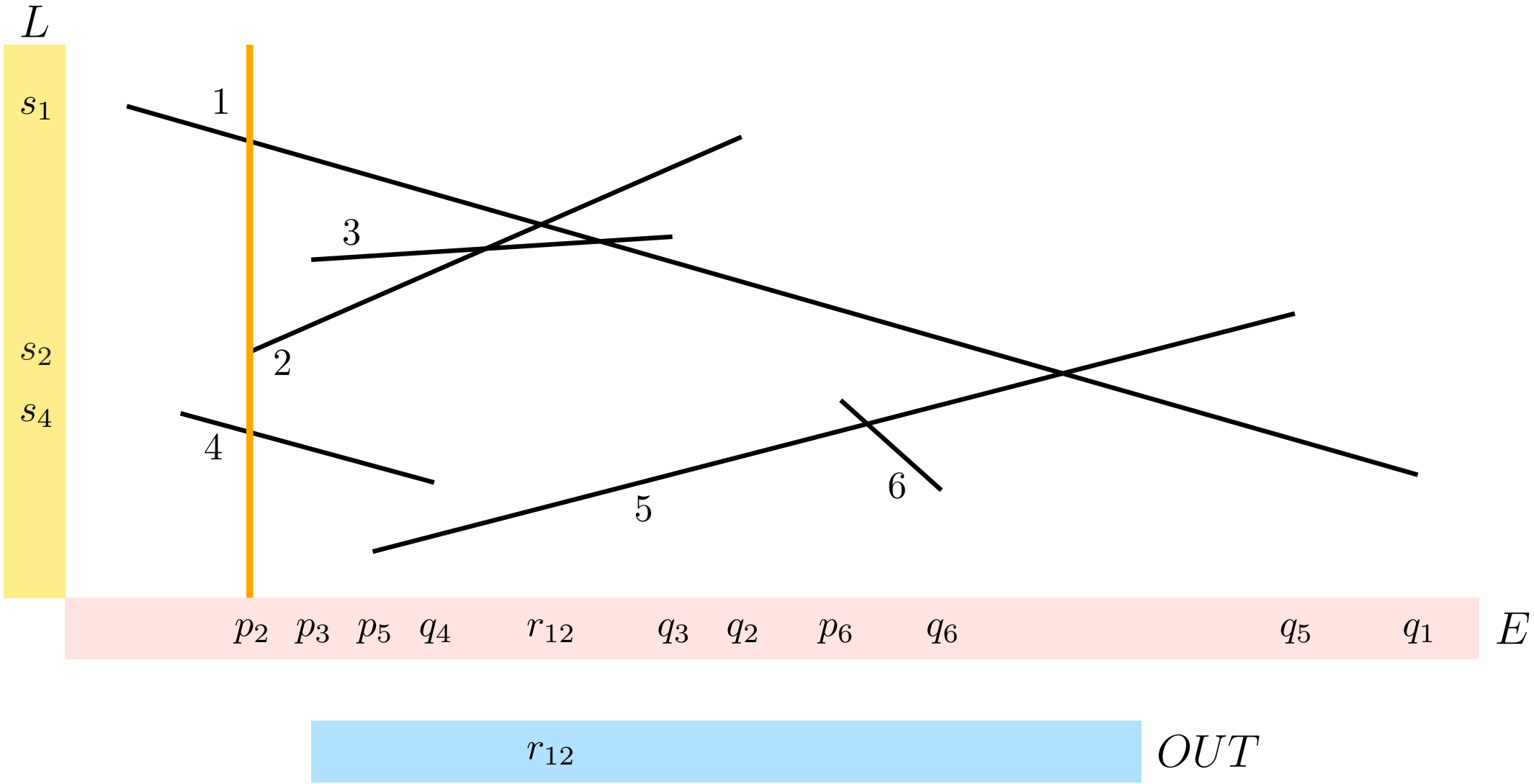

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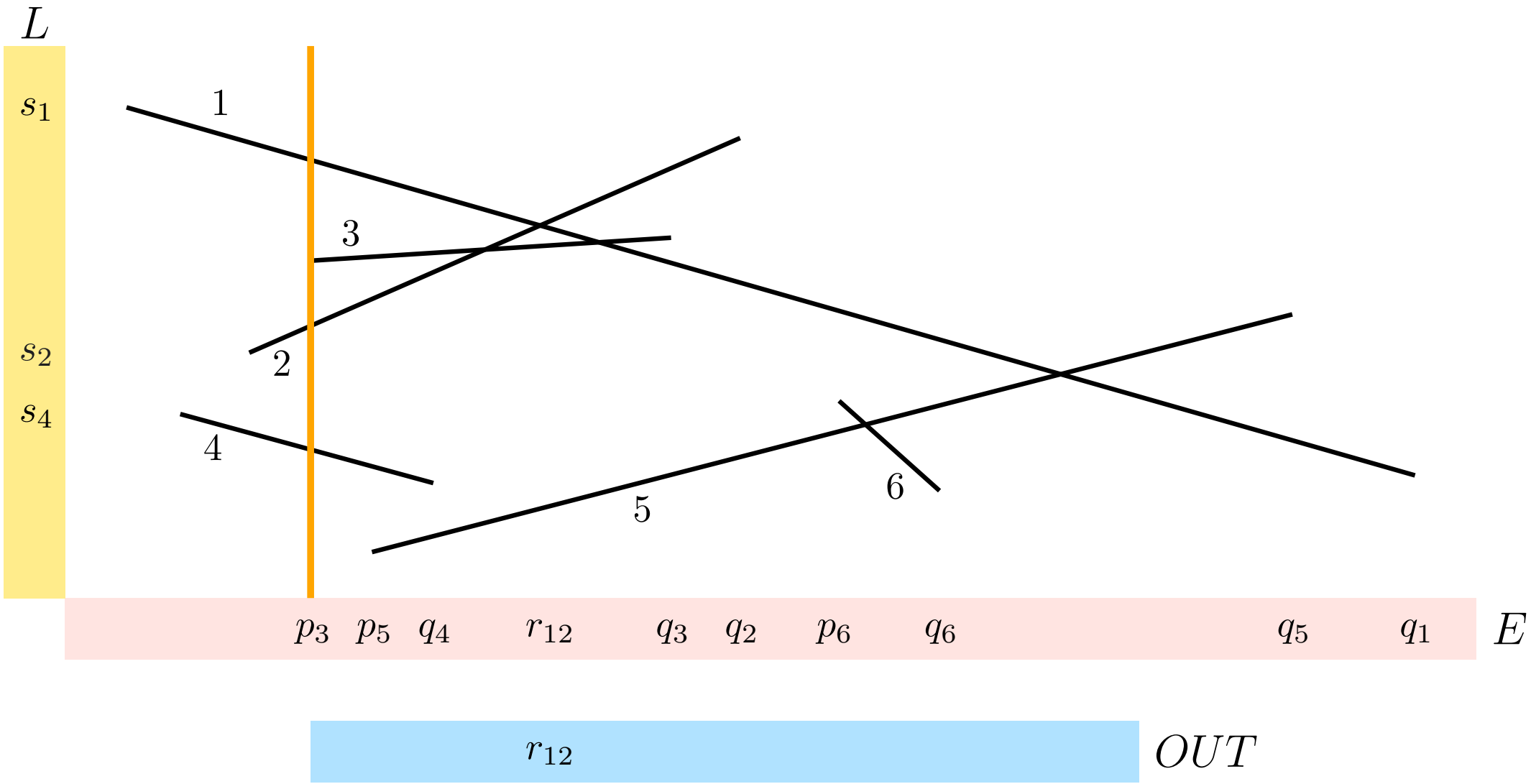

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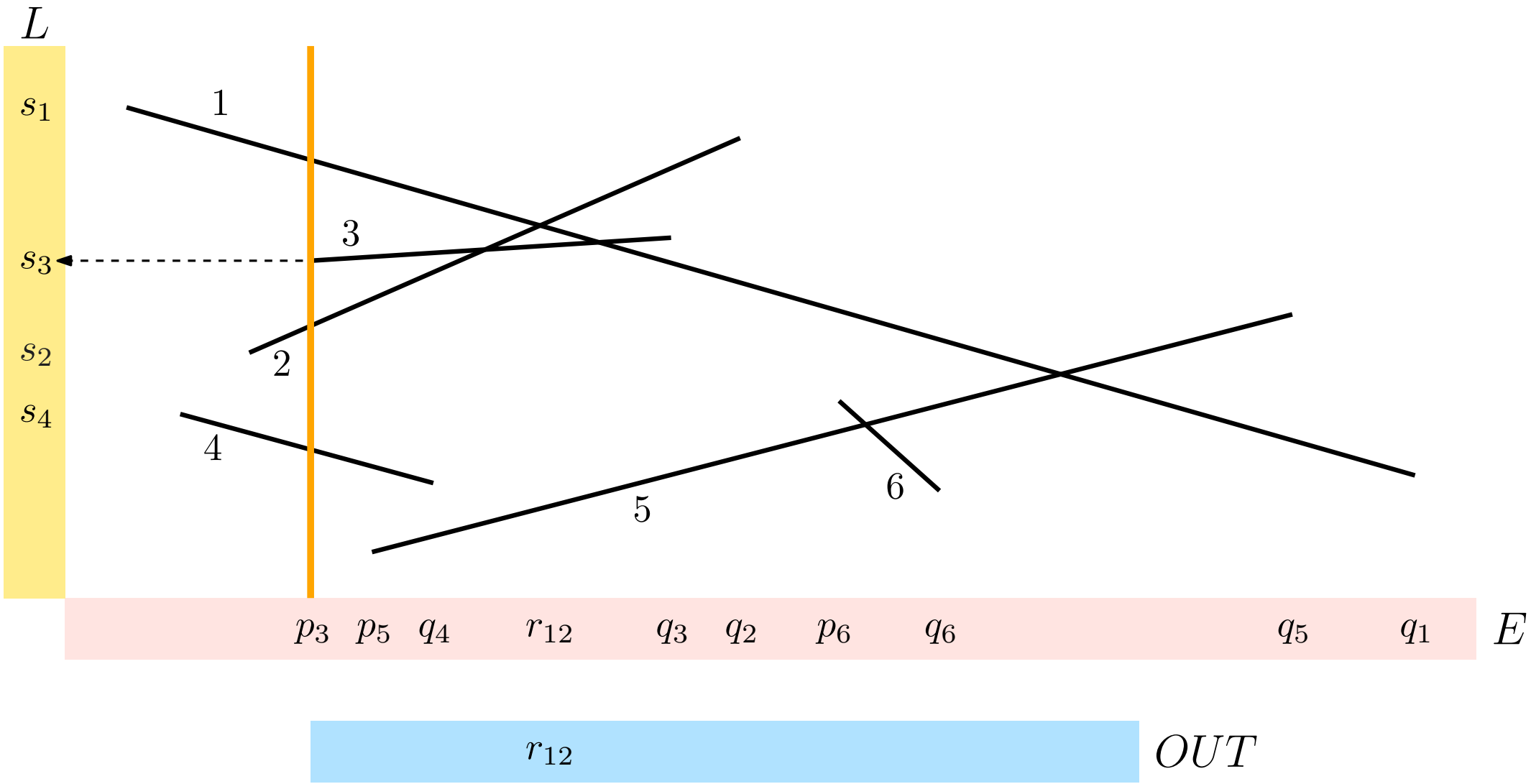

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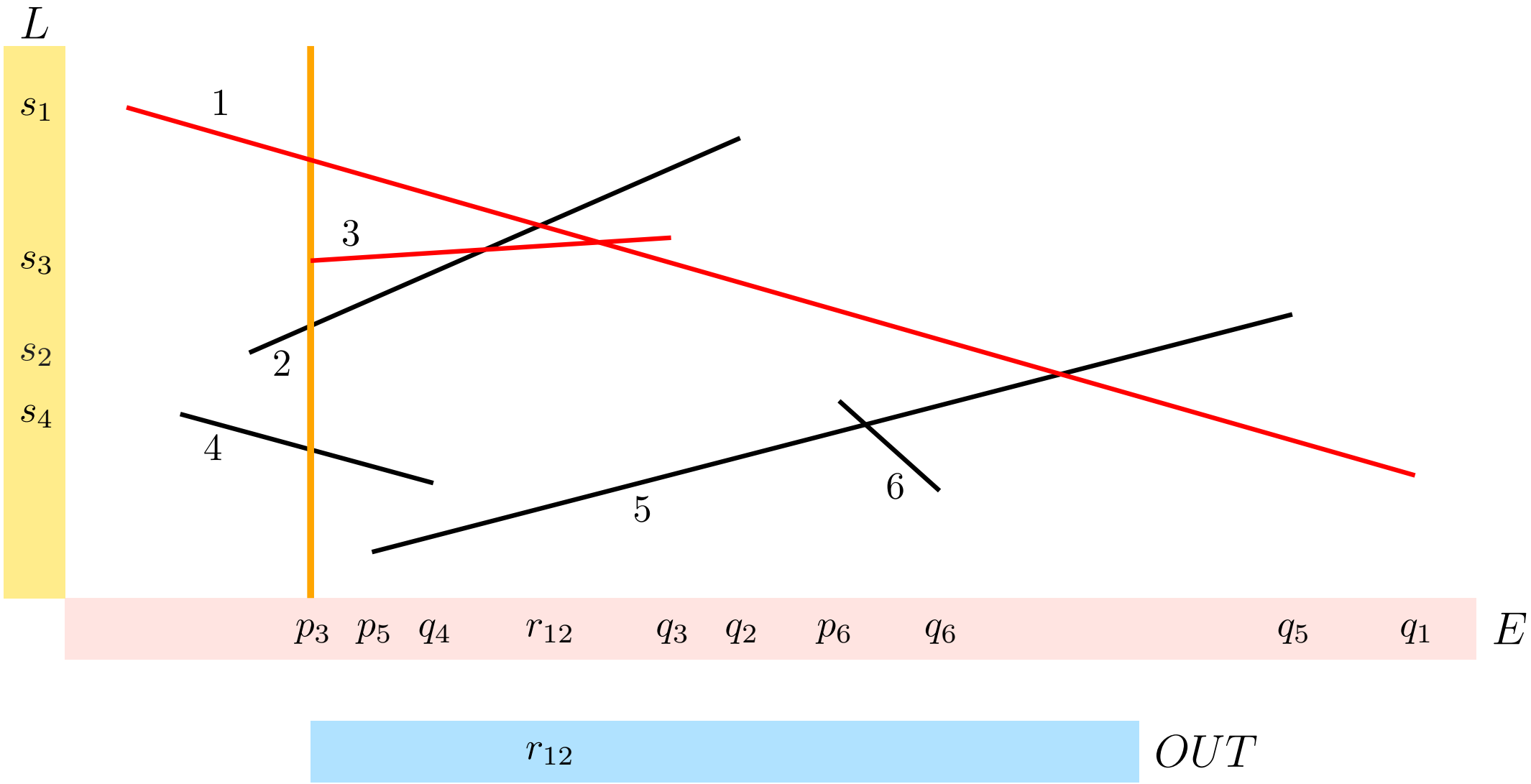

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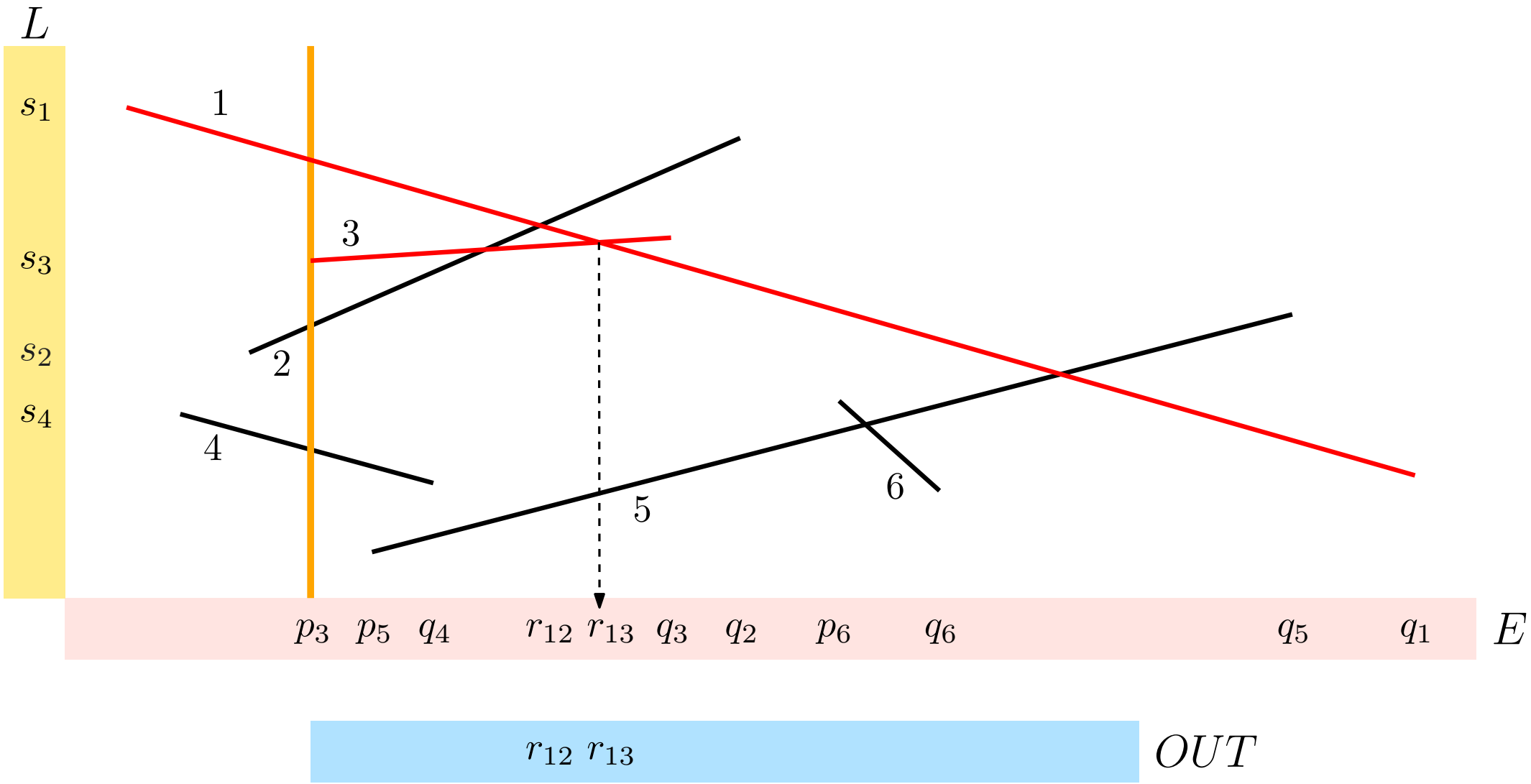

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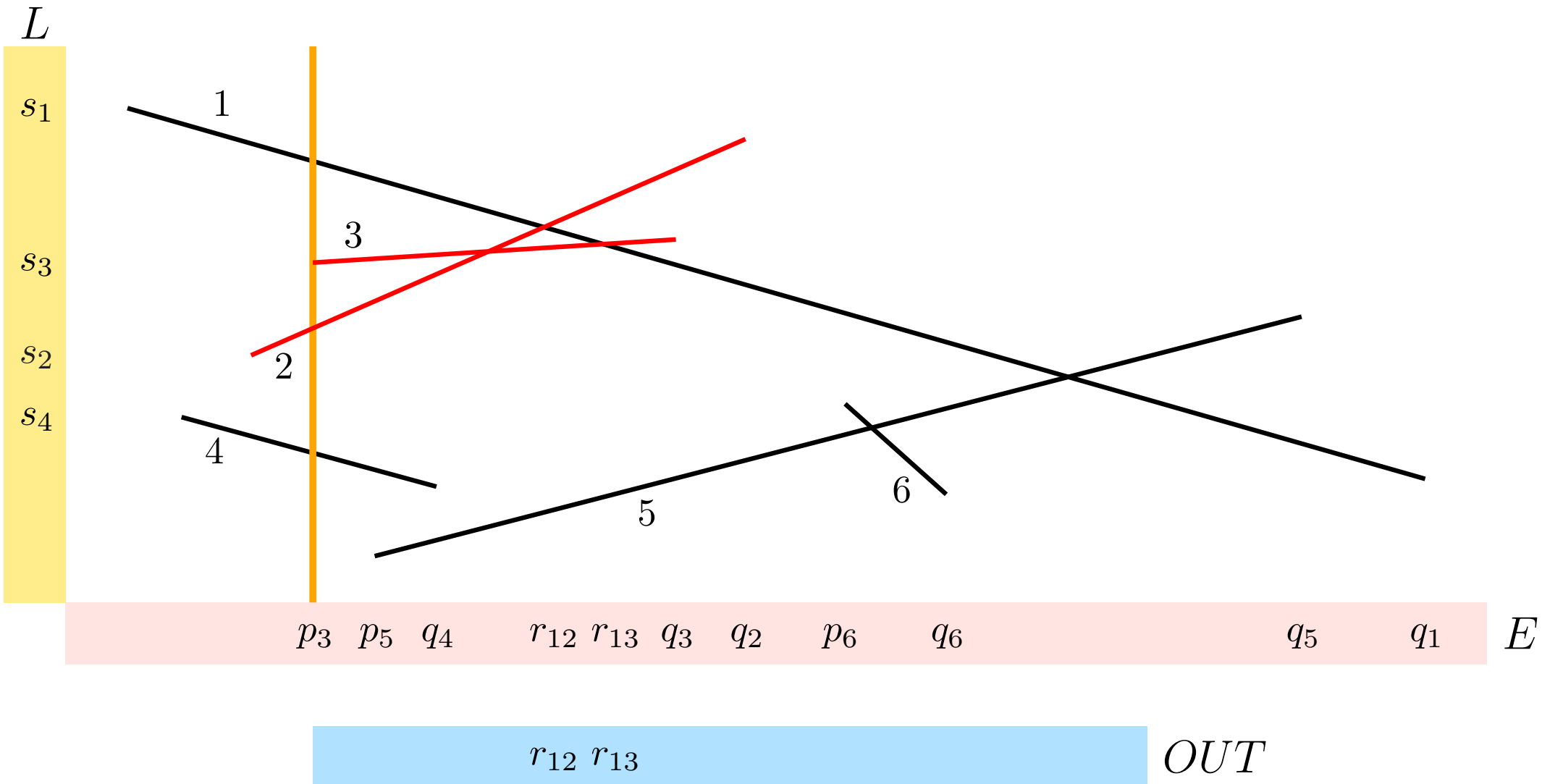

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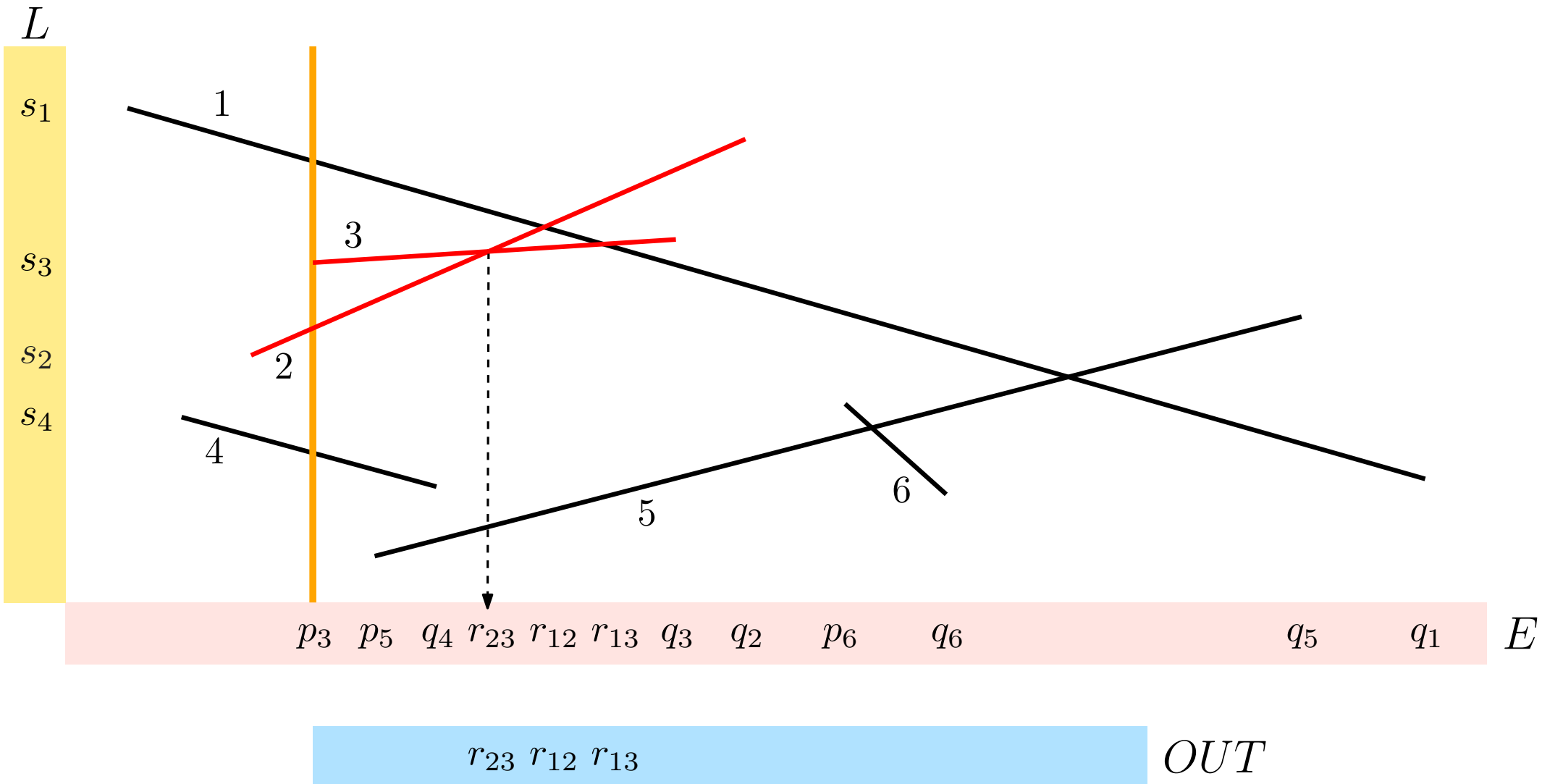

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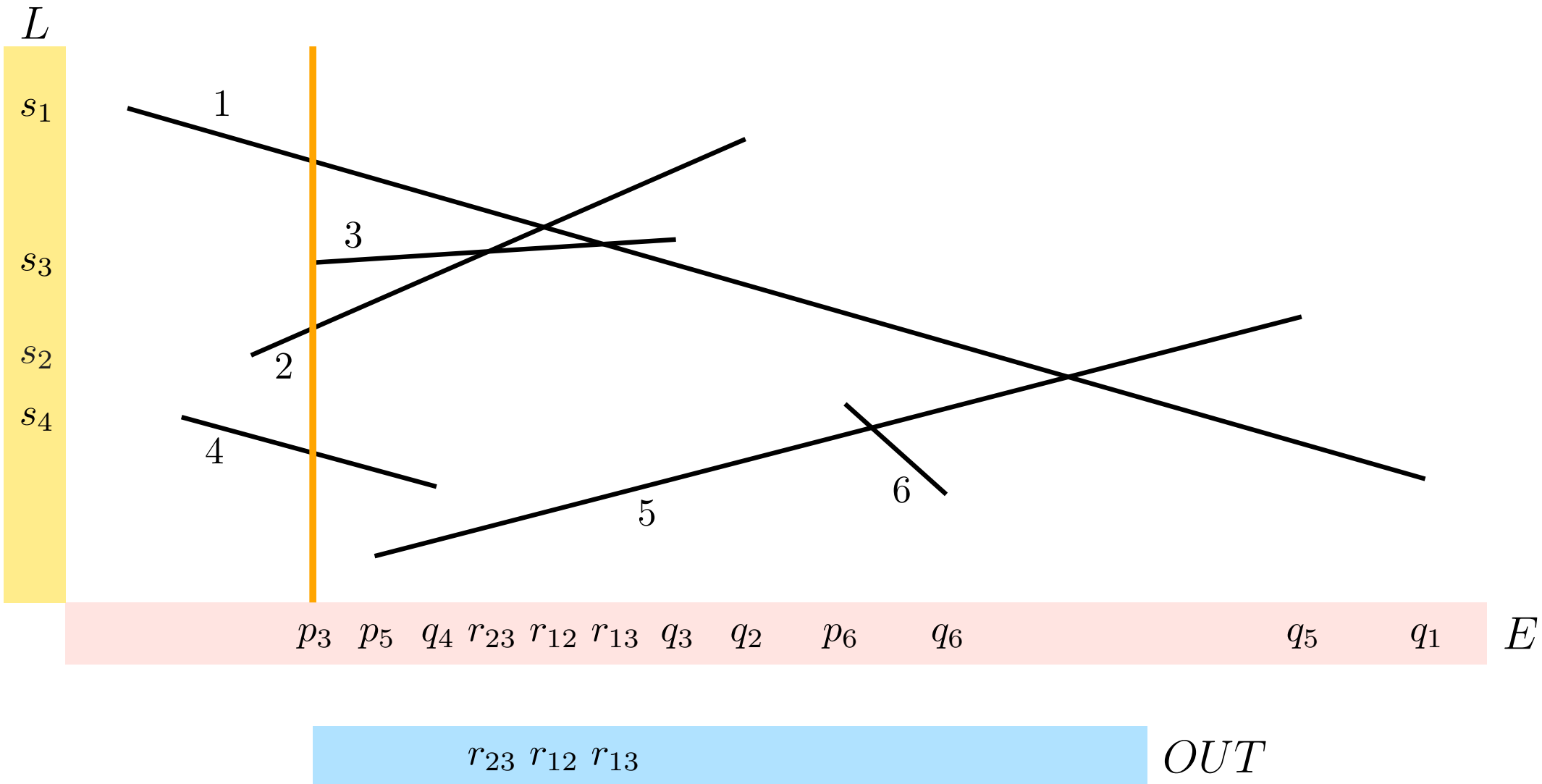

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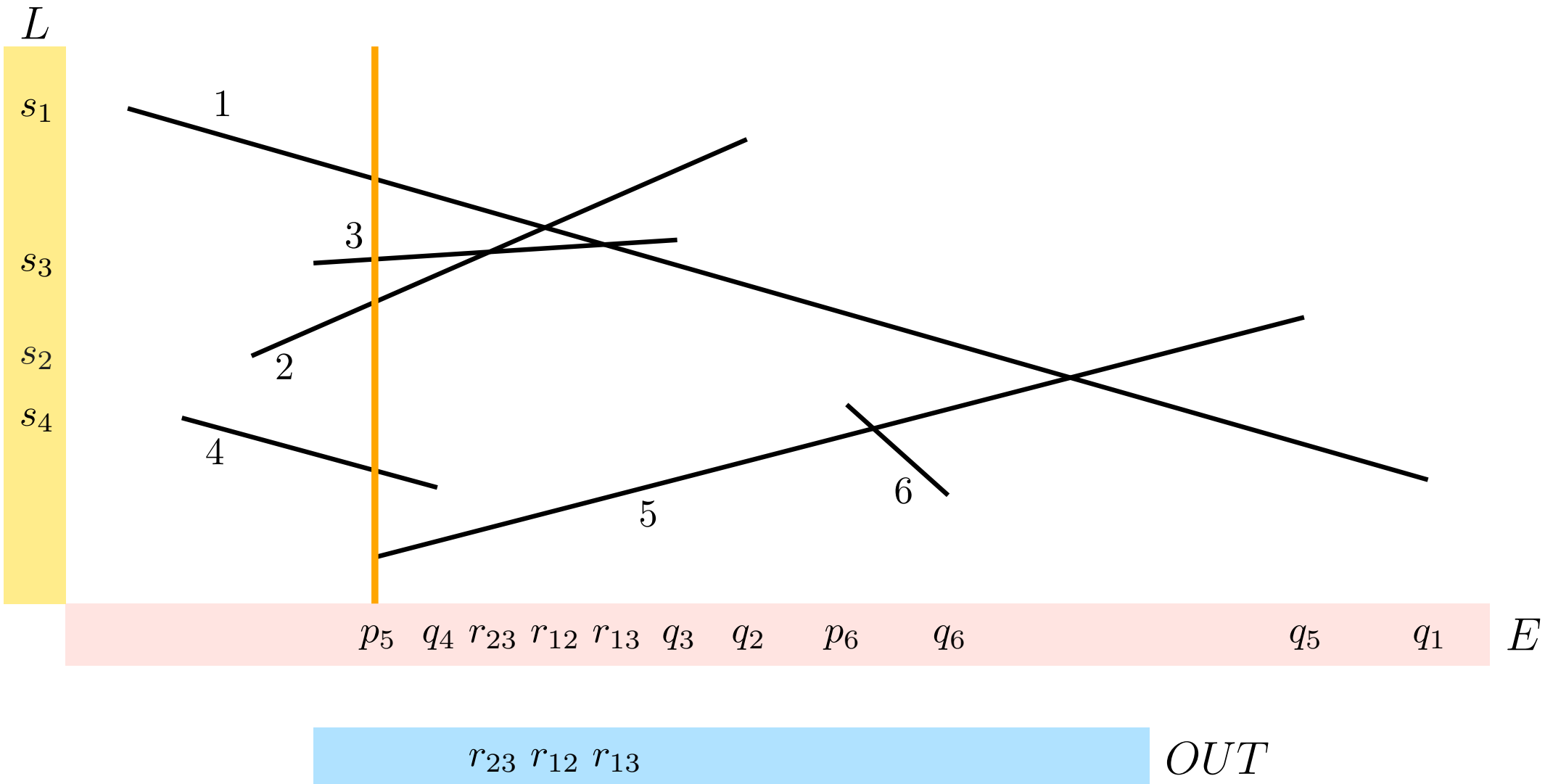

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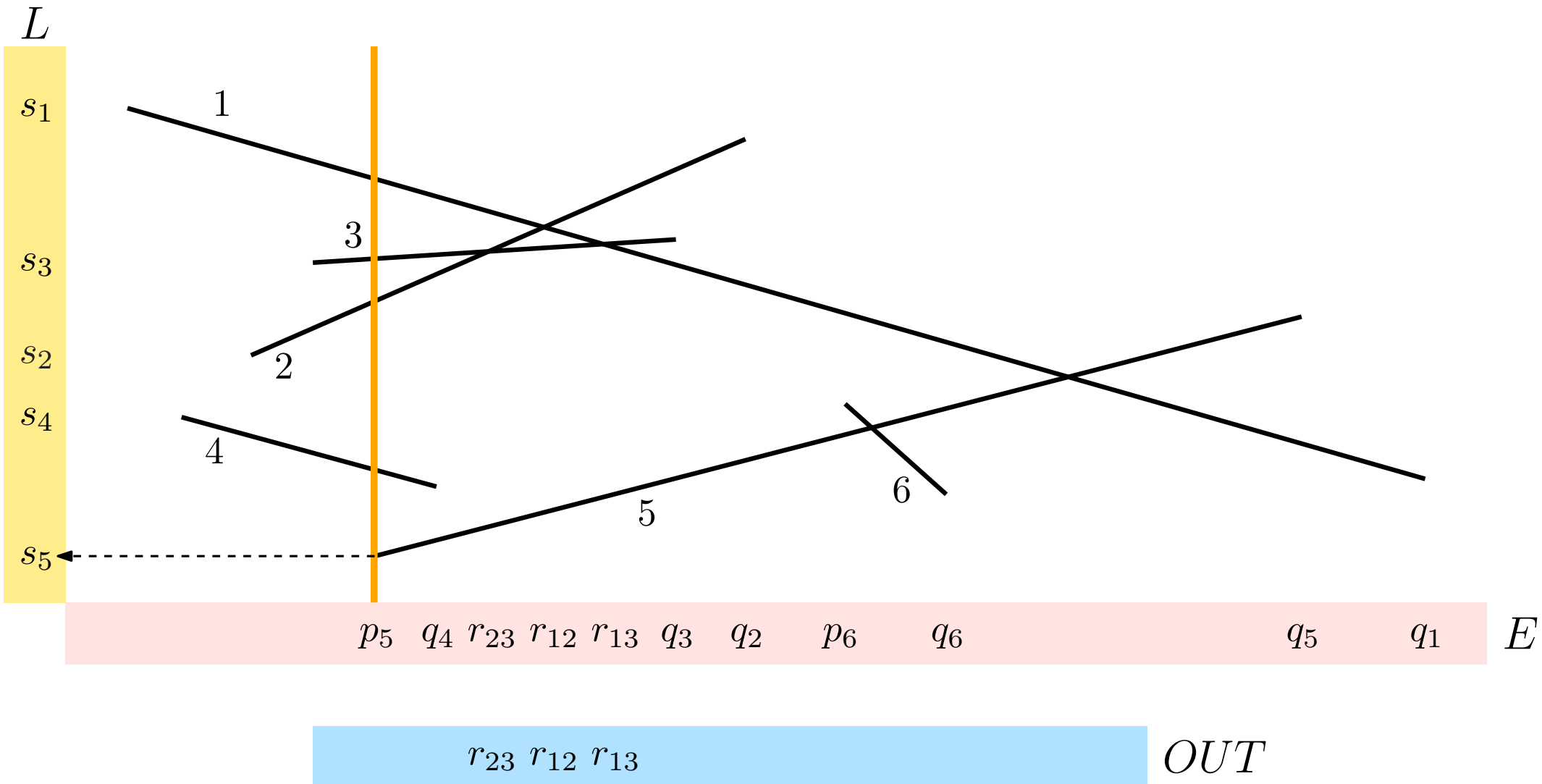

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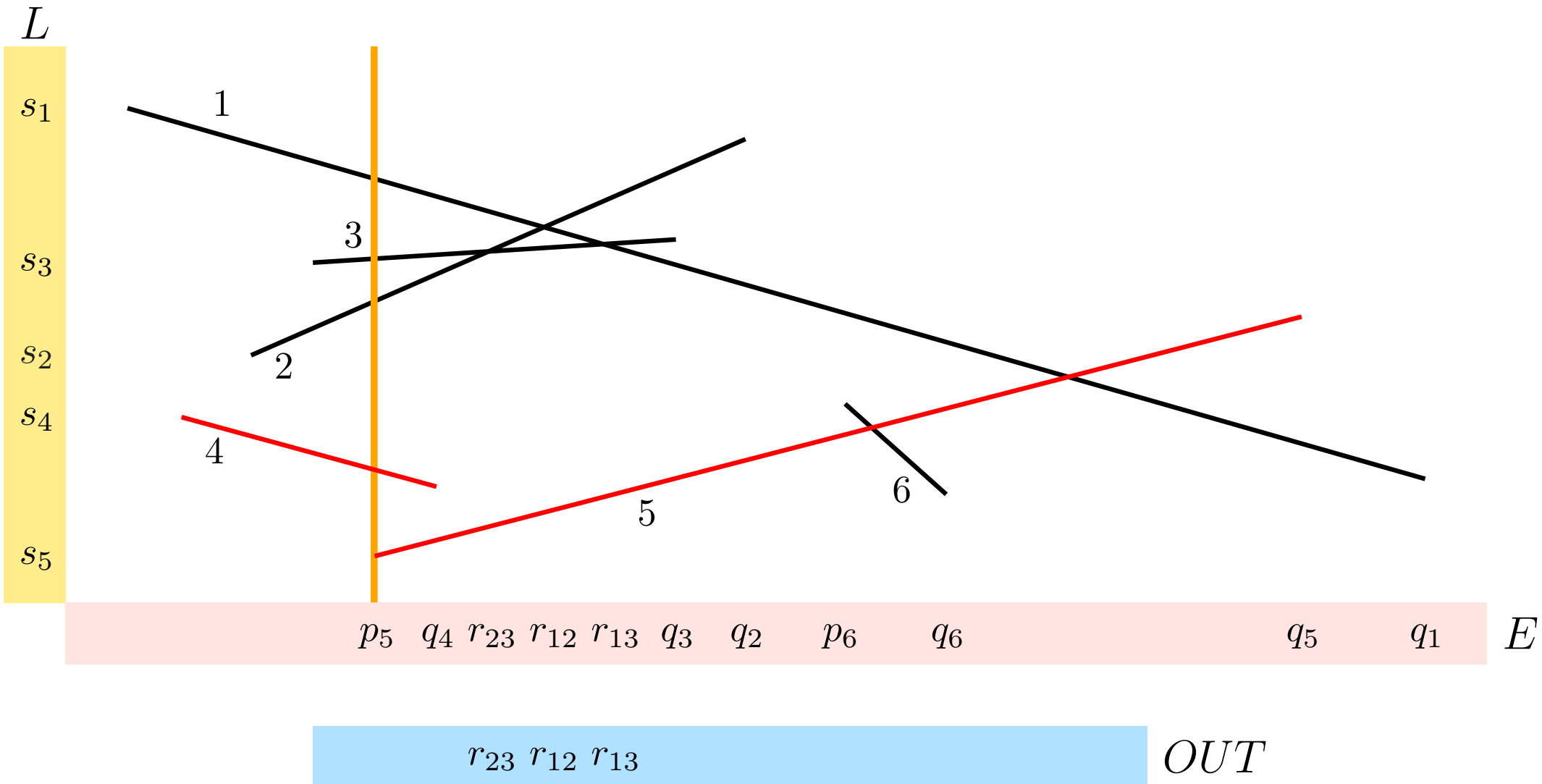

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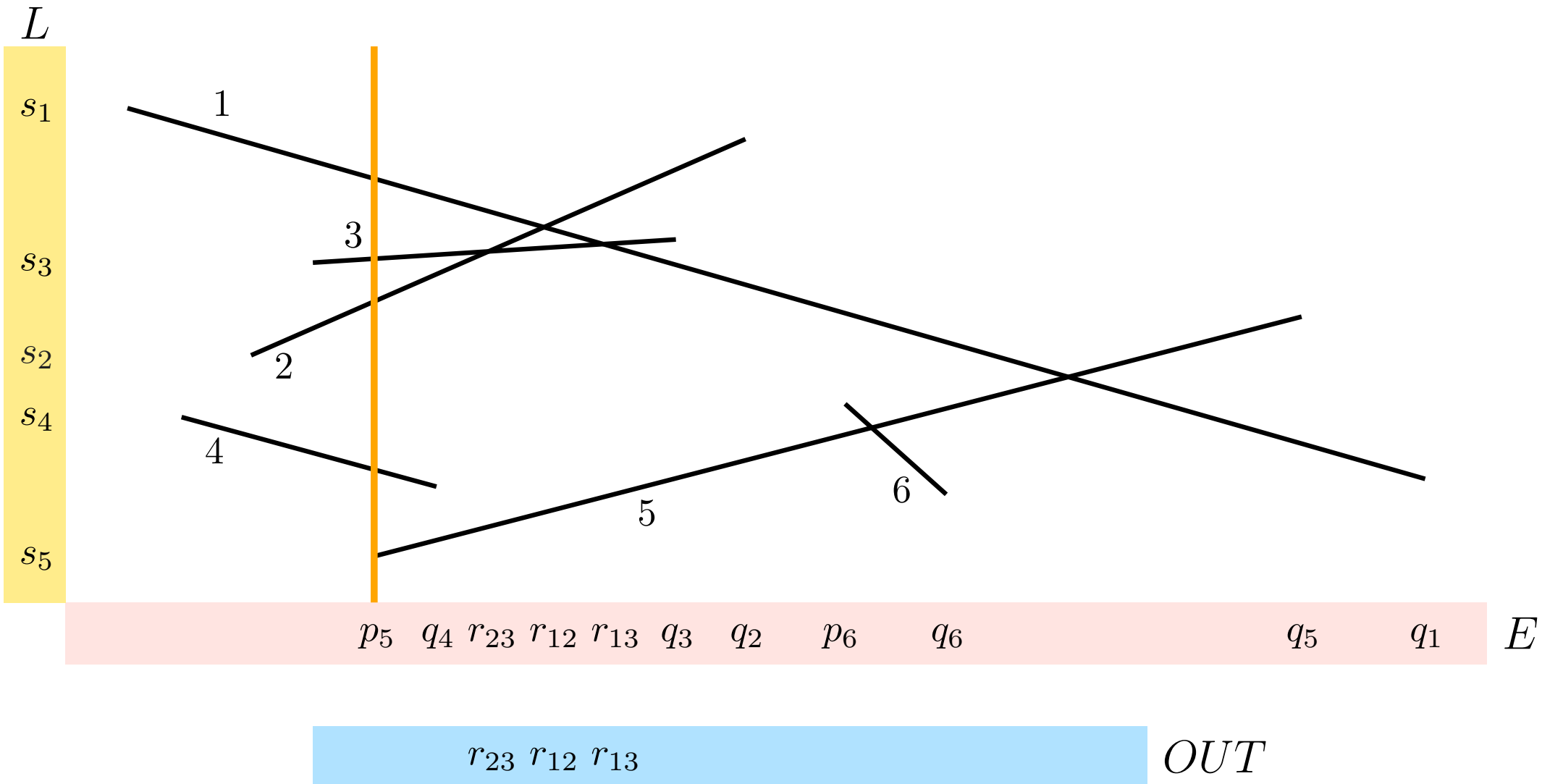

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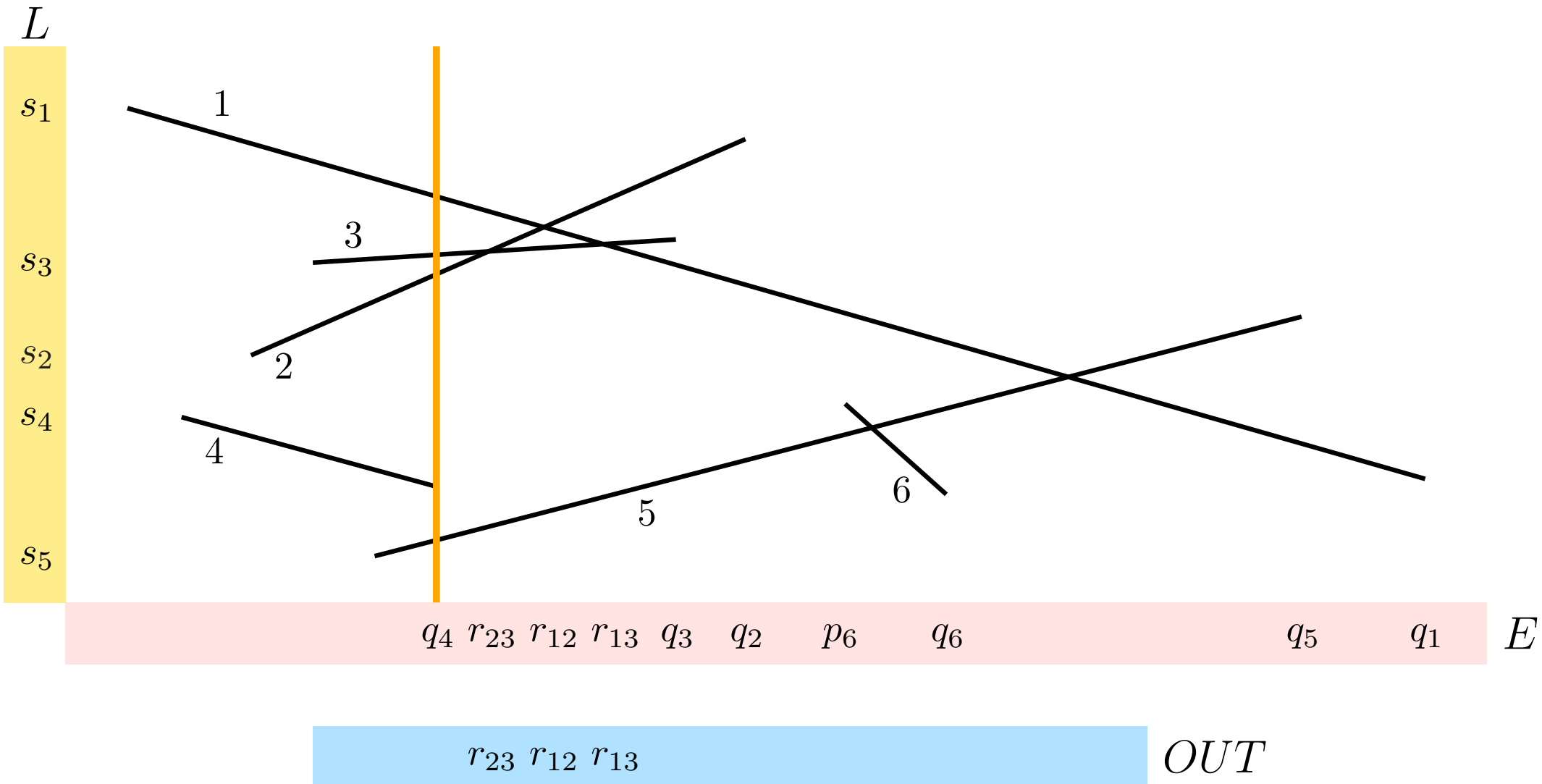

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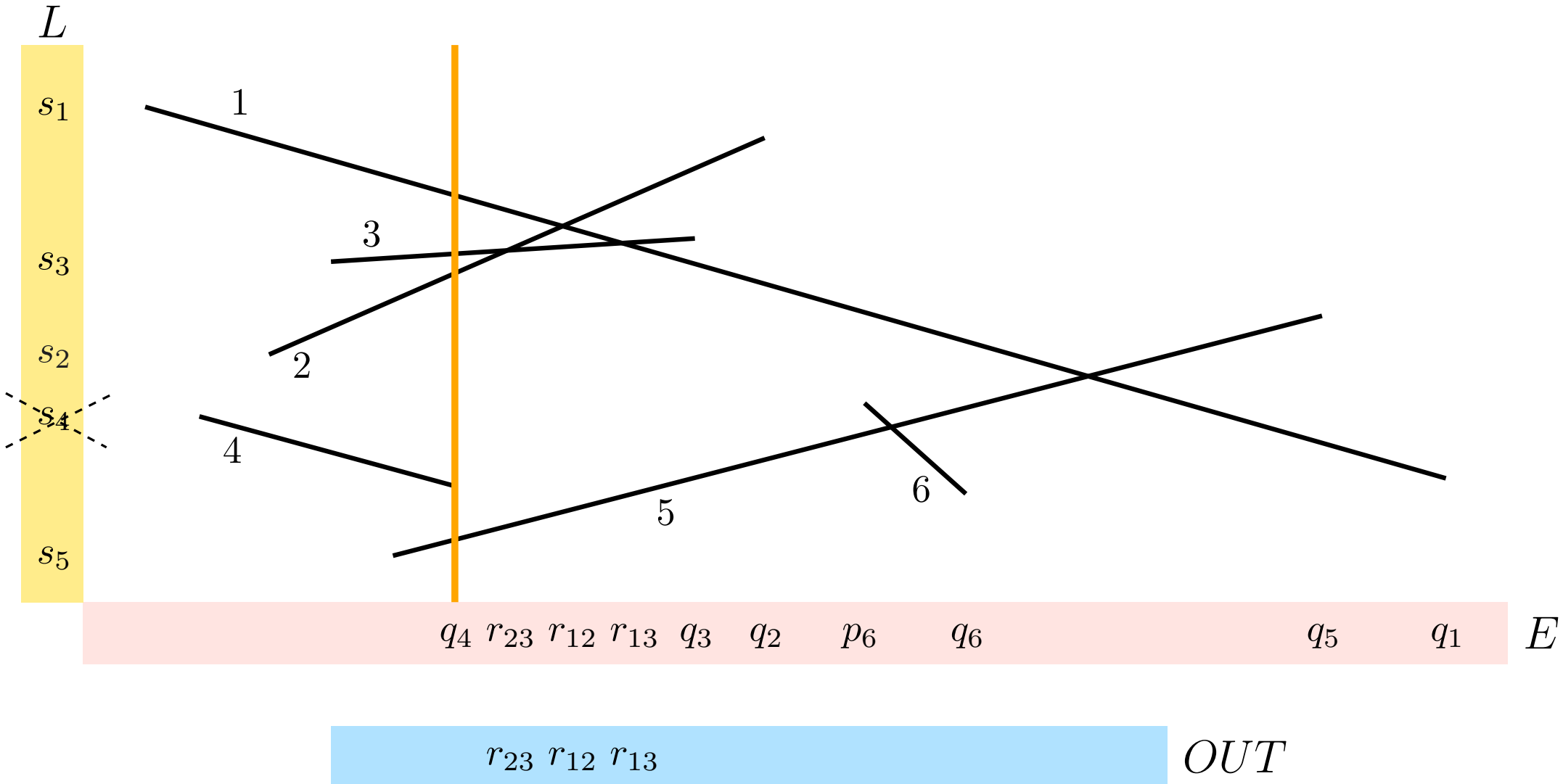

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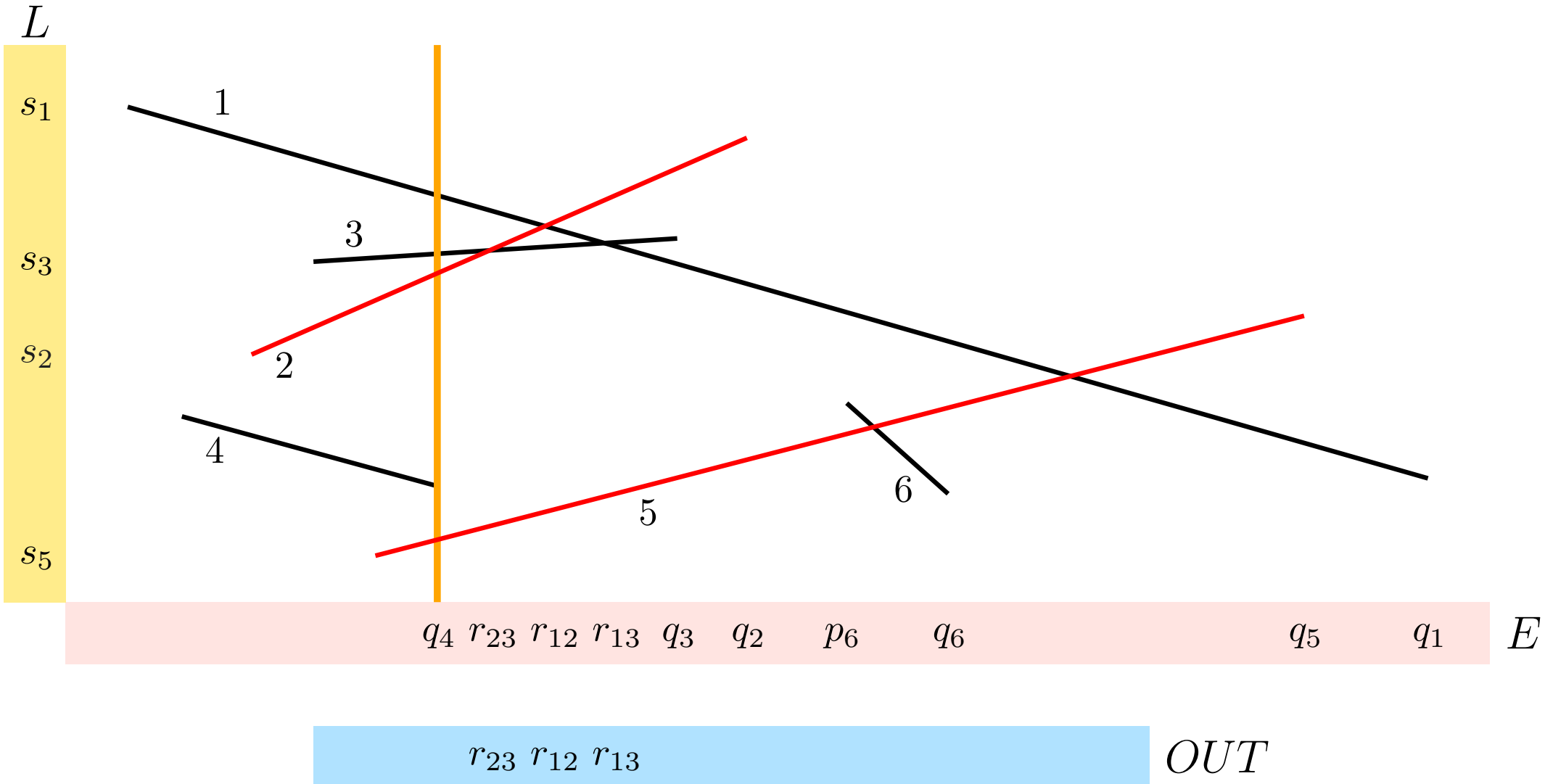

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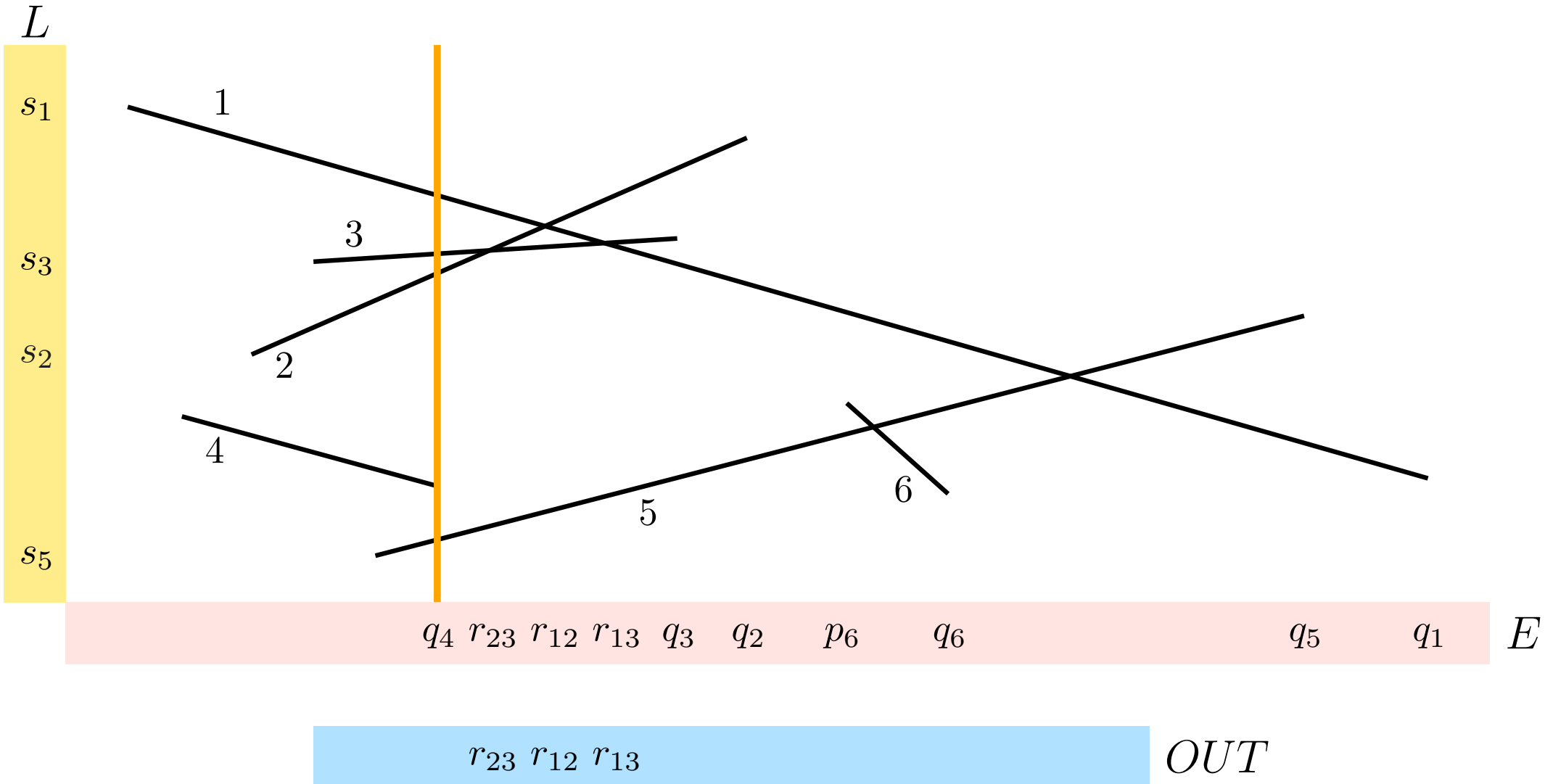

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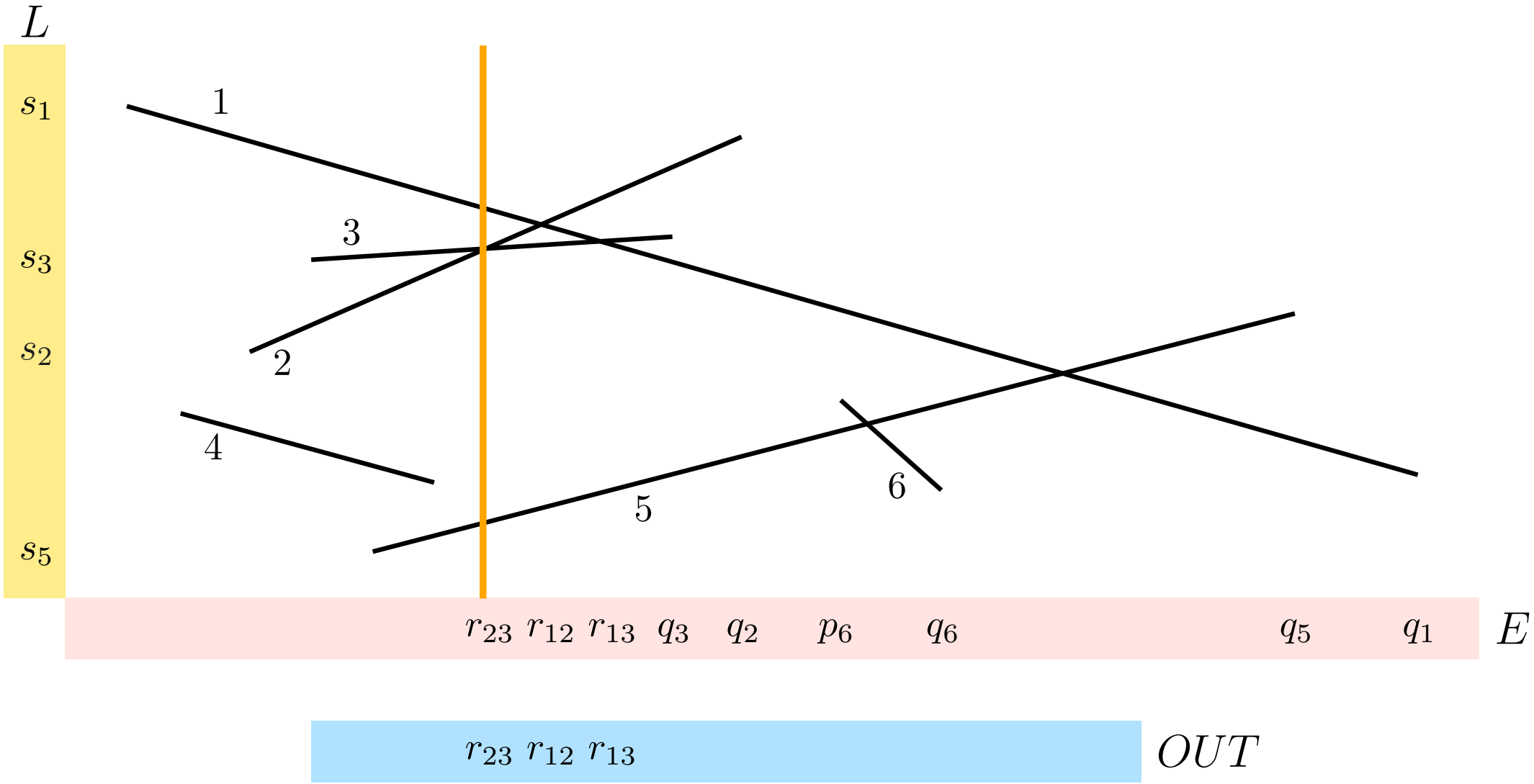

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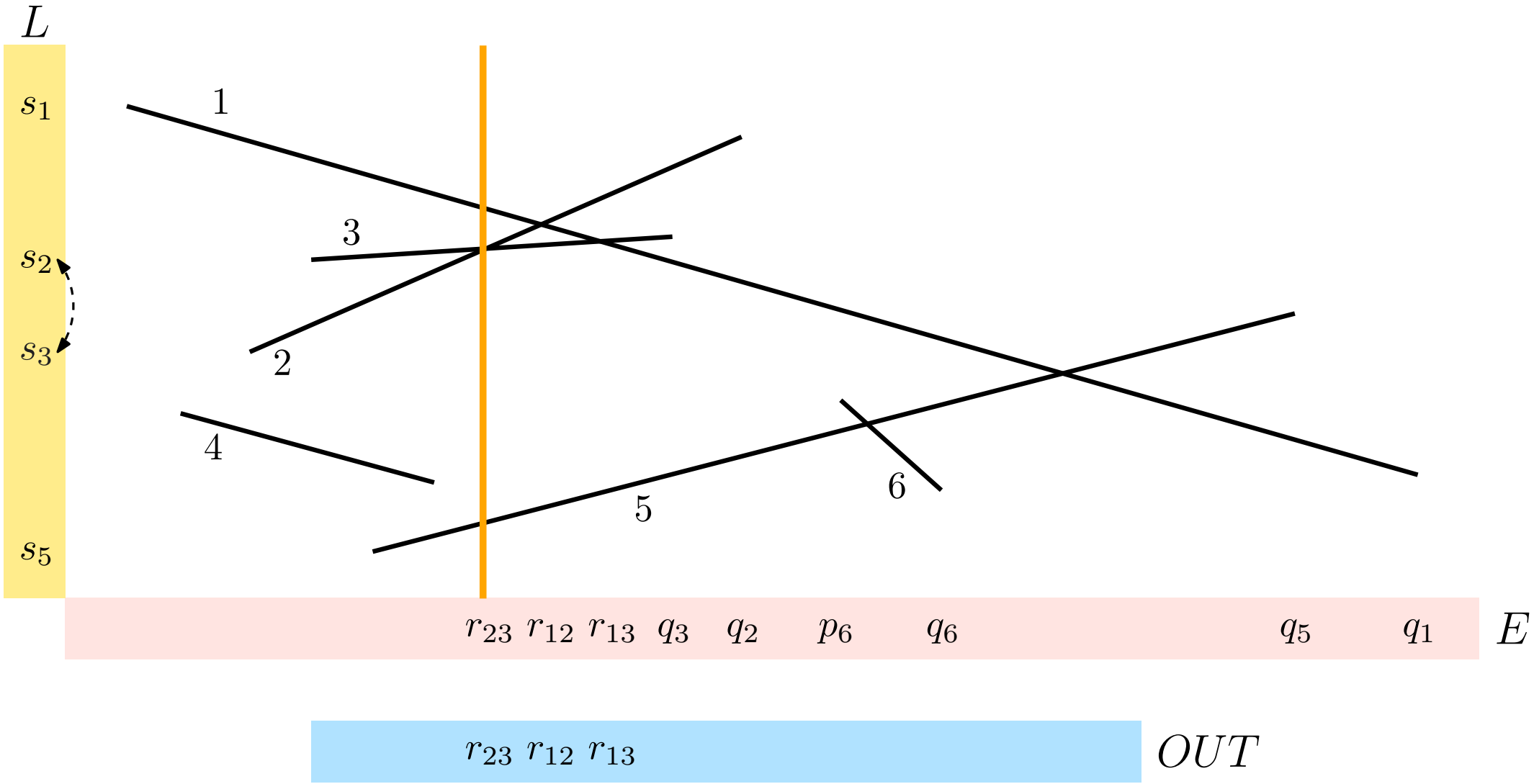

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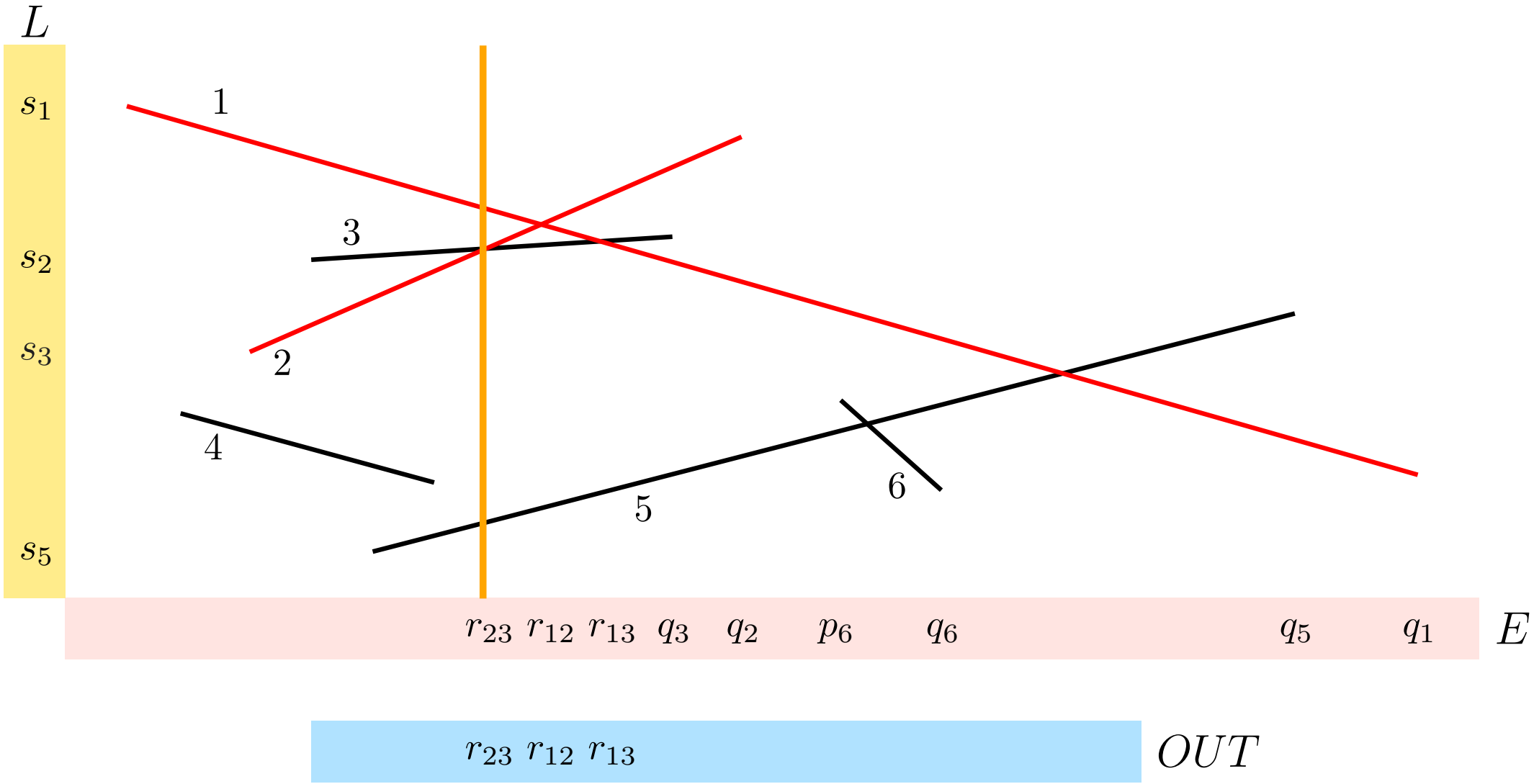

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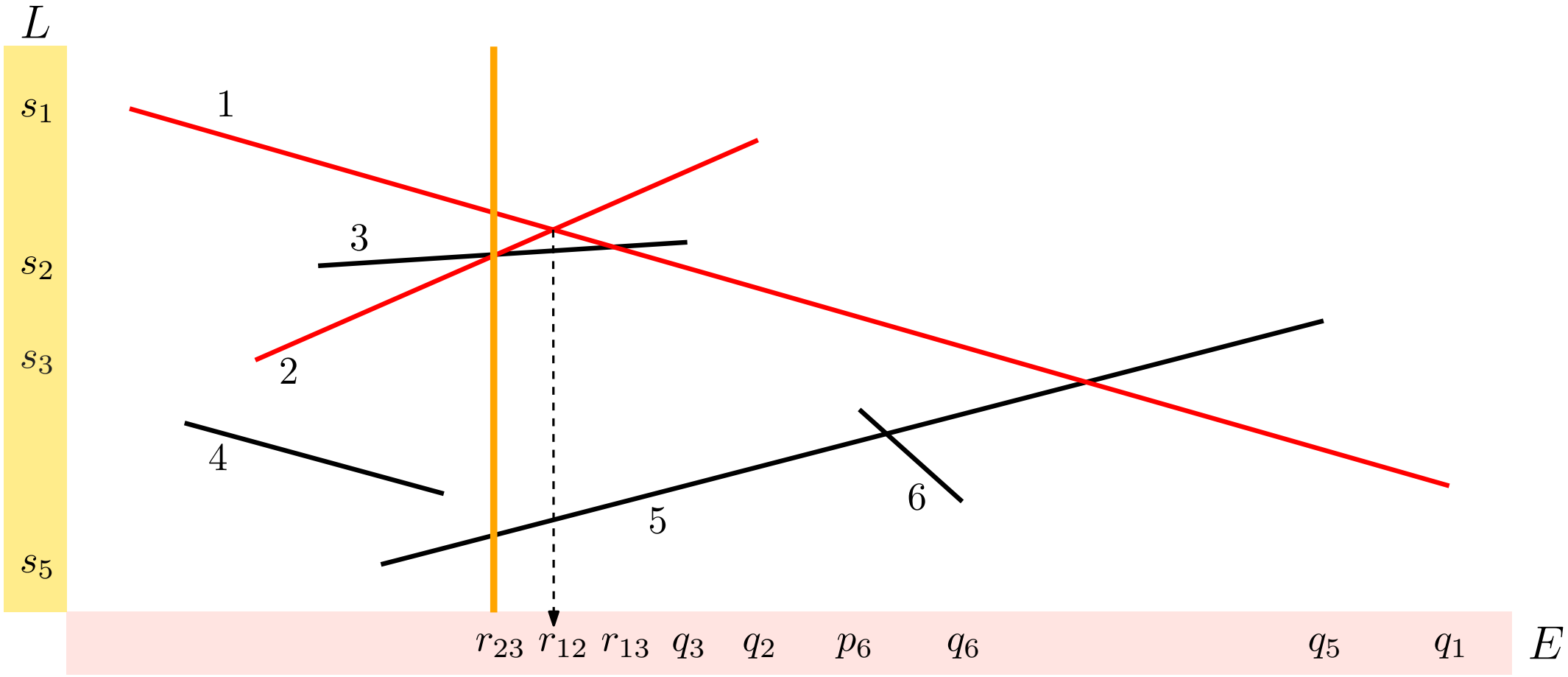
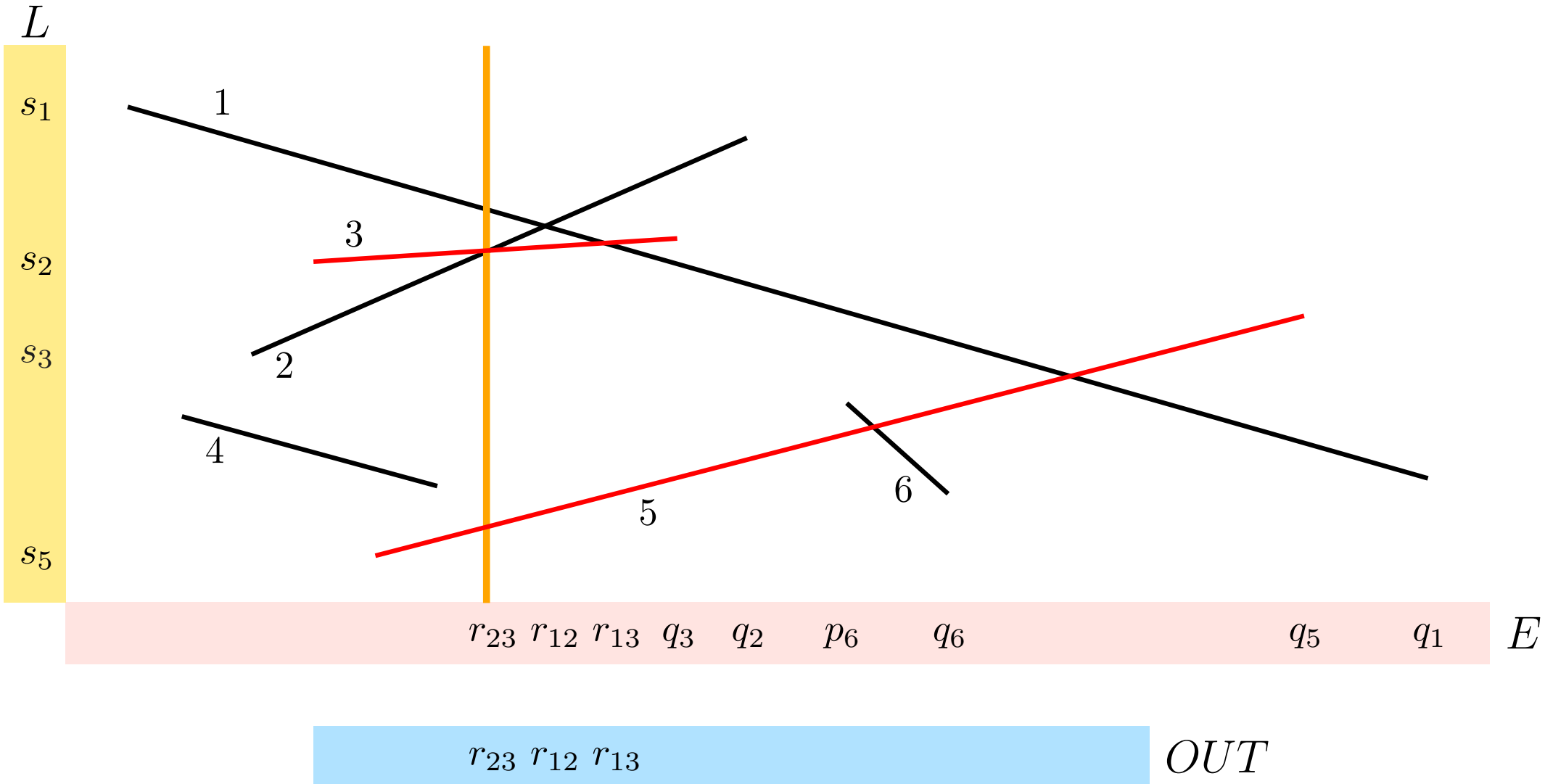


r_{23} r_{12} r_{13} OUT

Bentley-Ottman's algorithm: simulation



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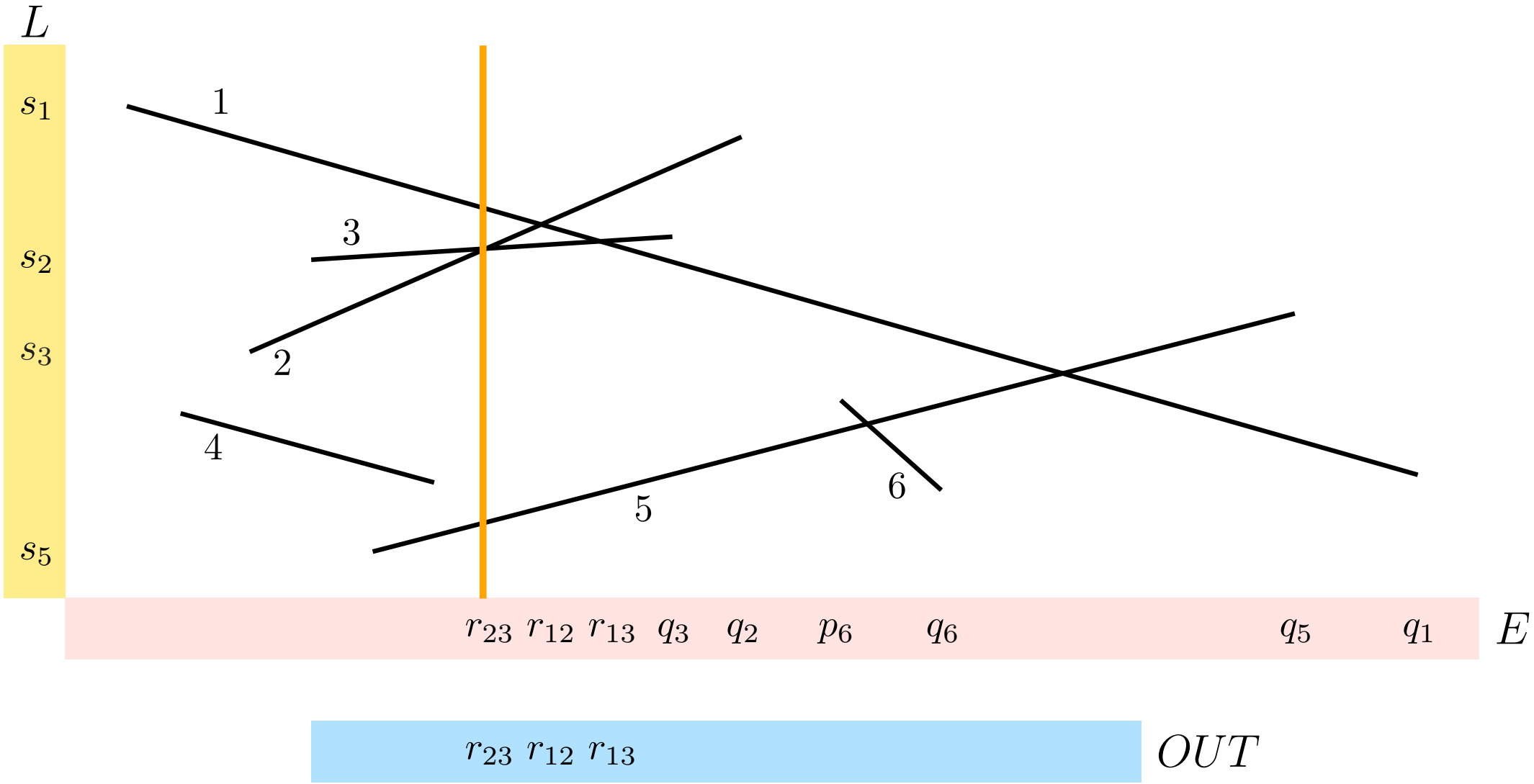

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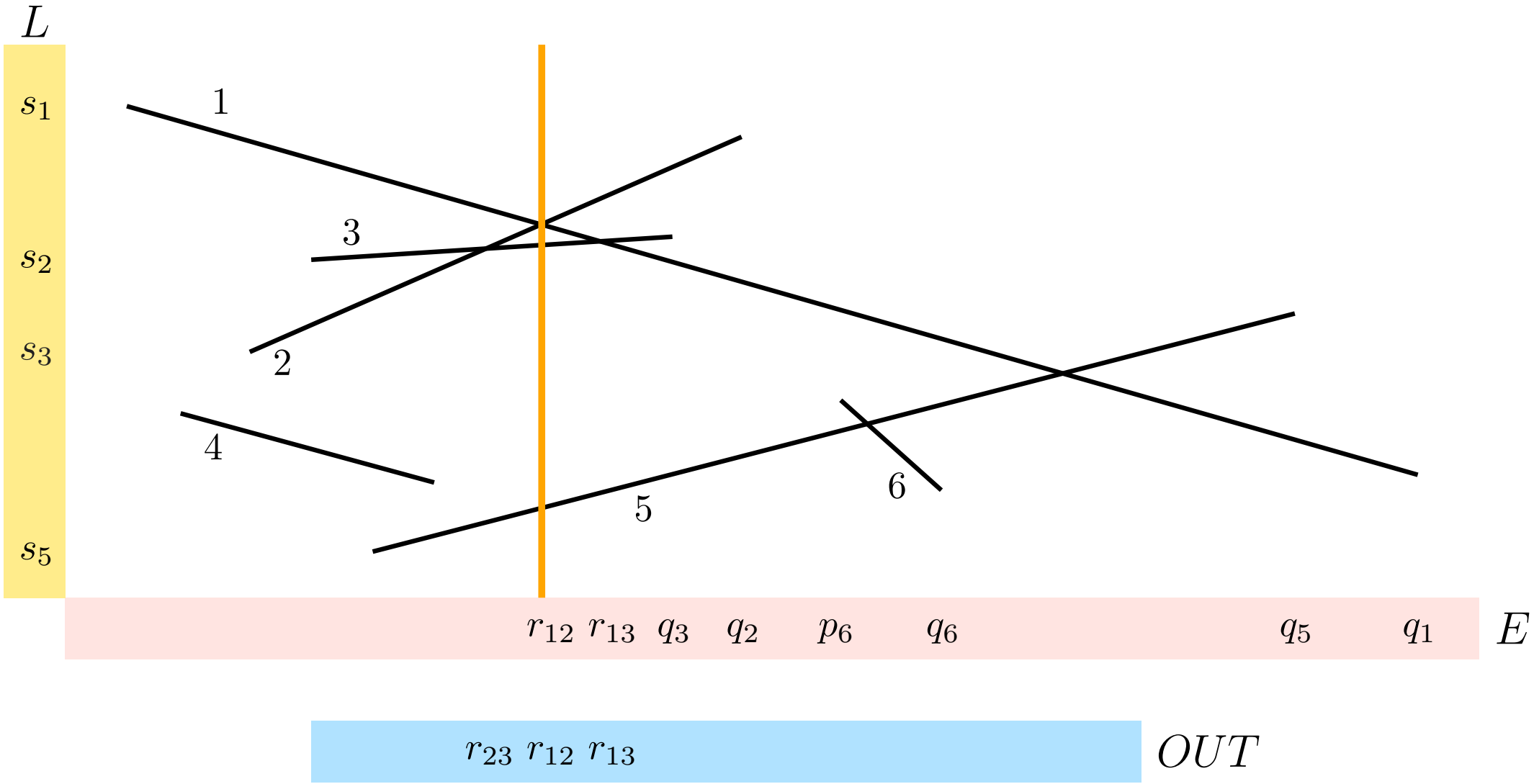

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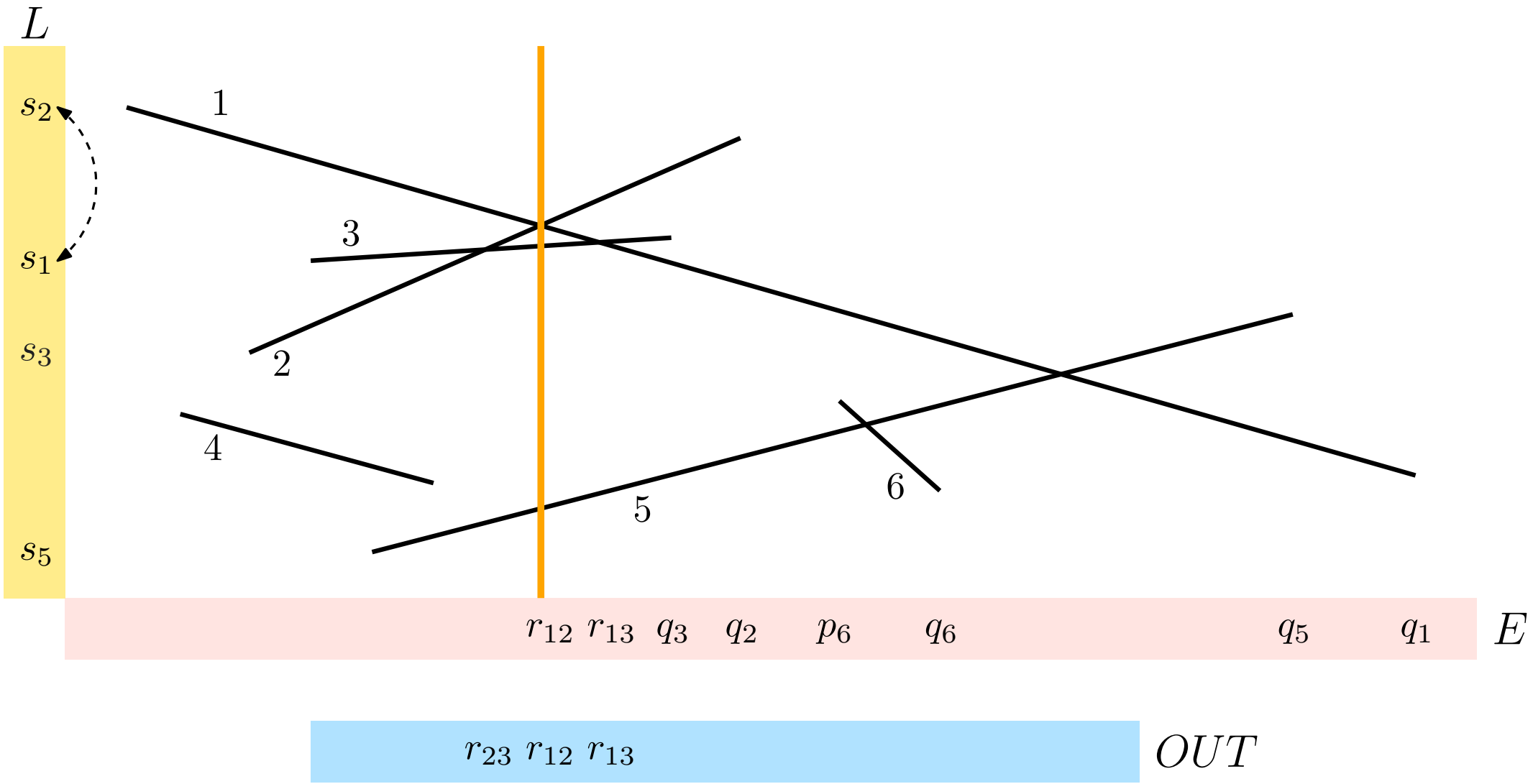

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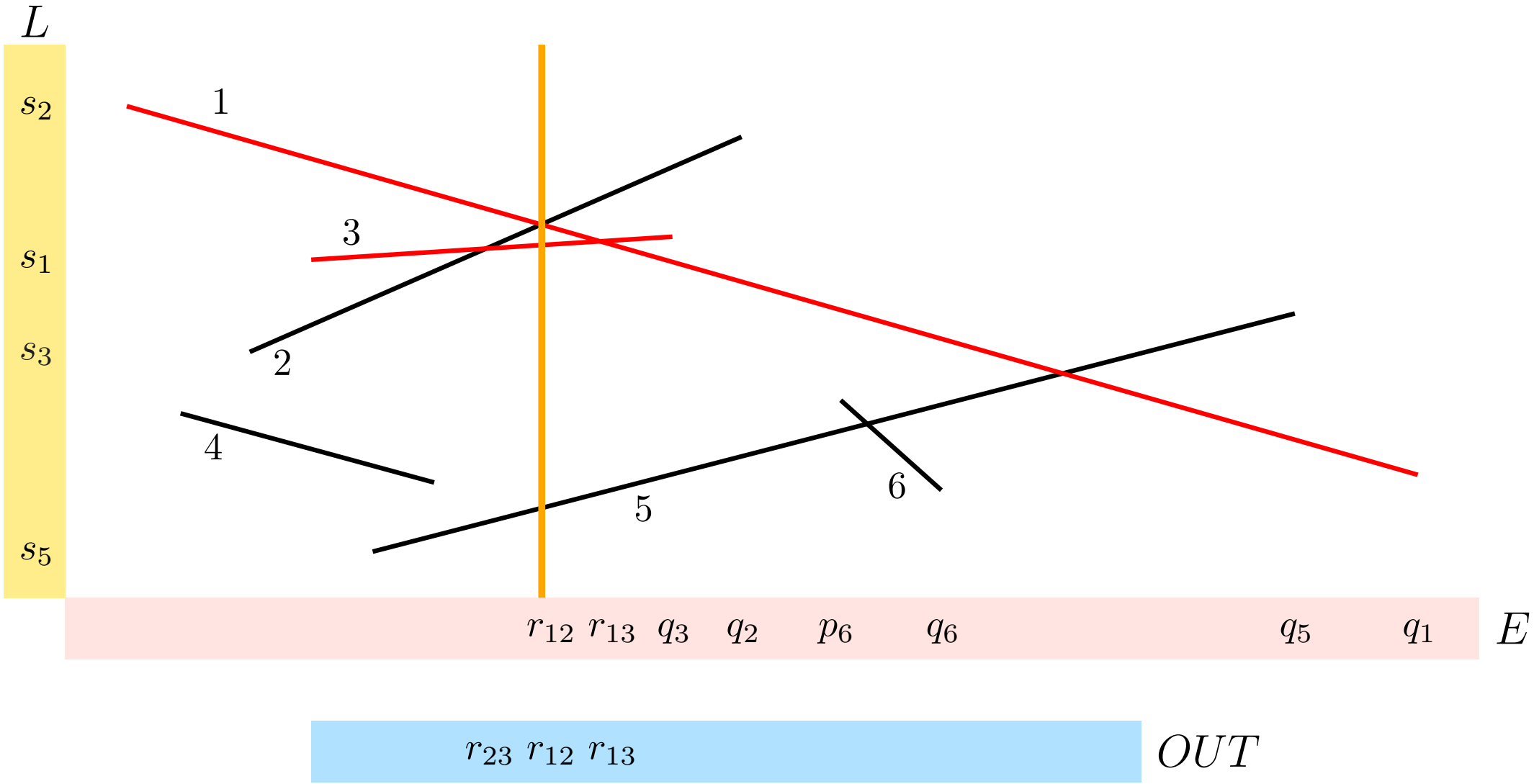

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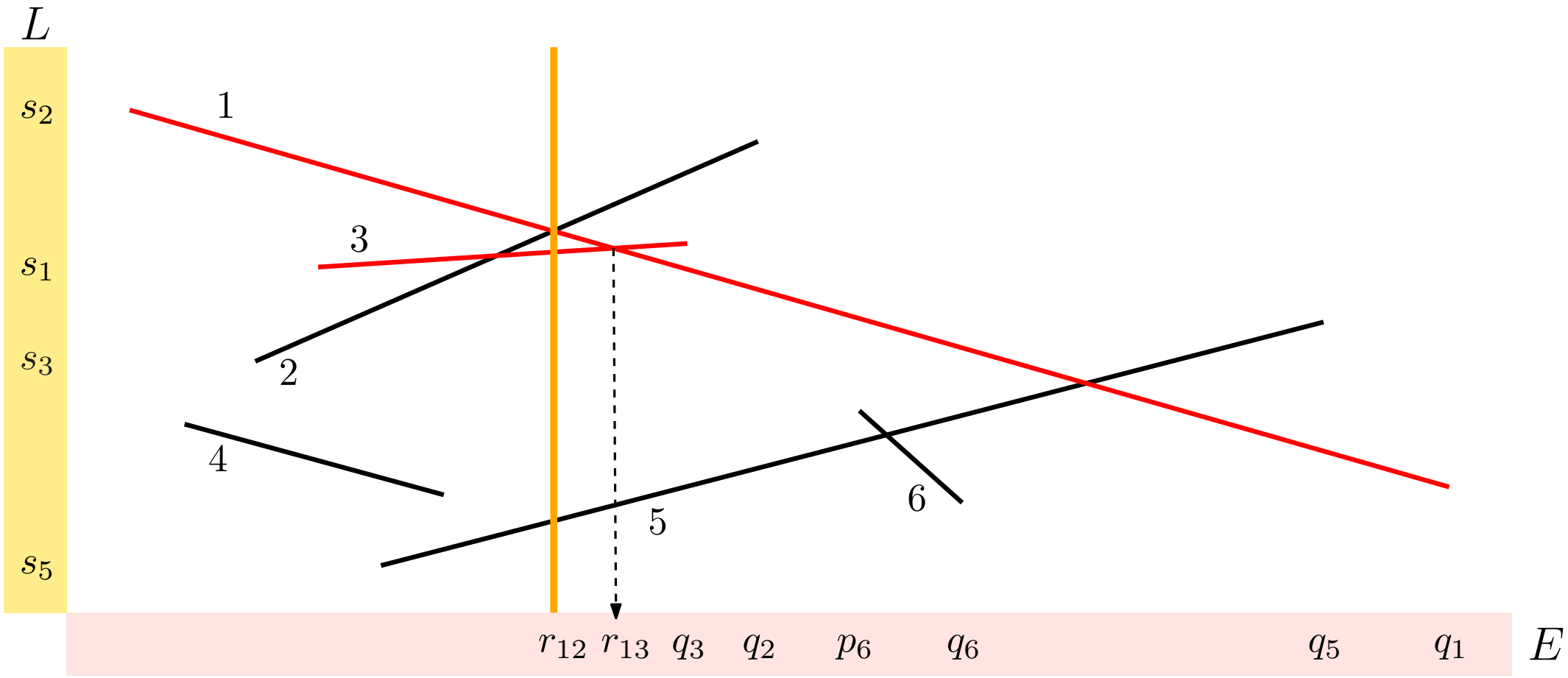

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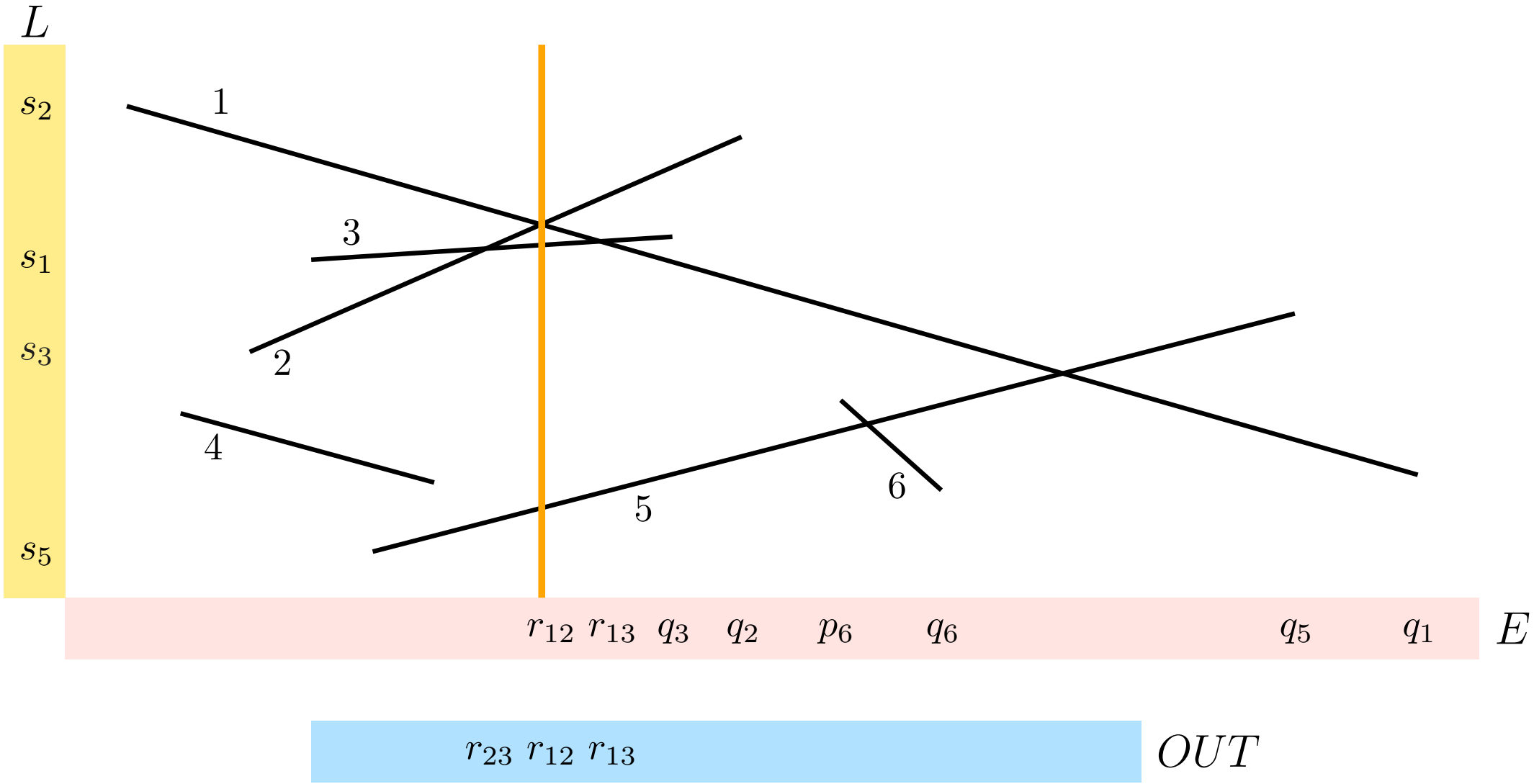

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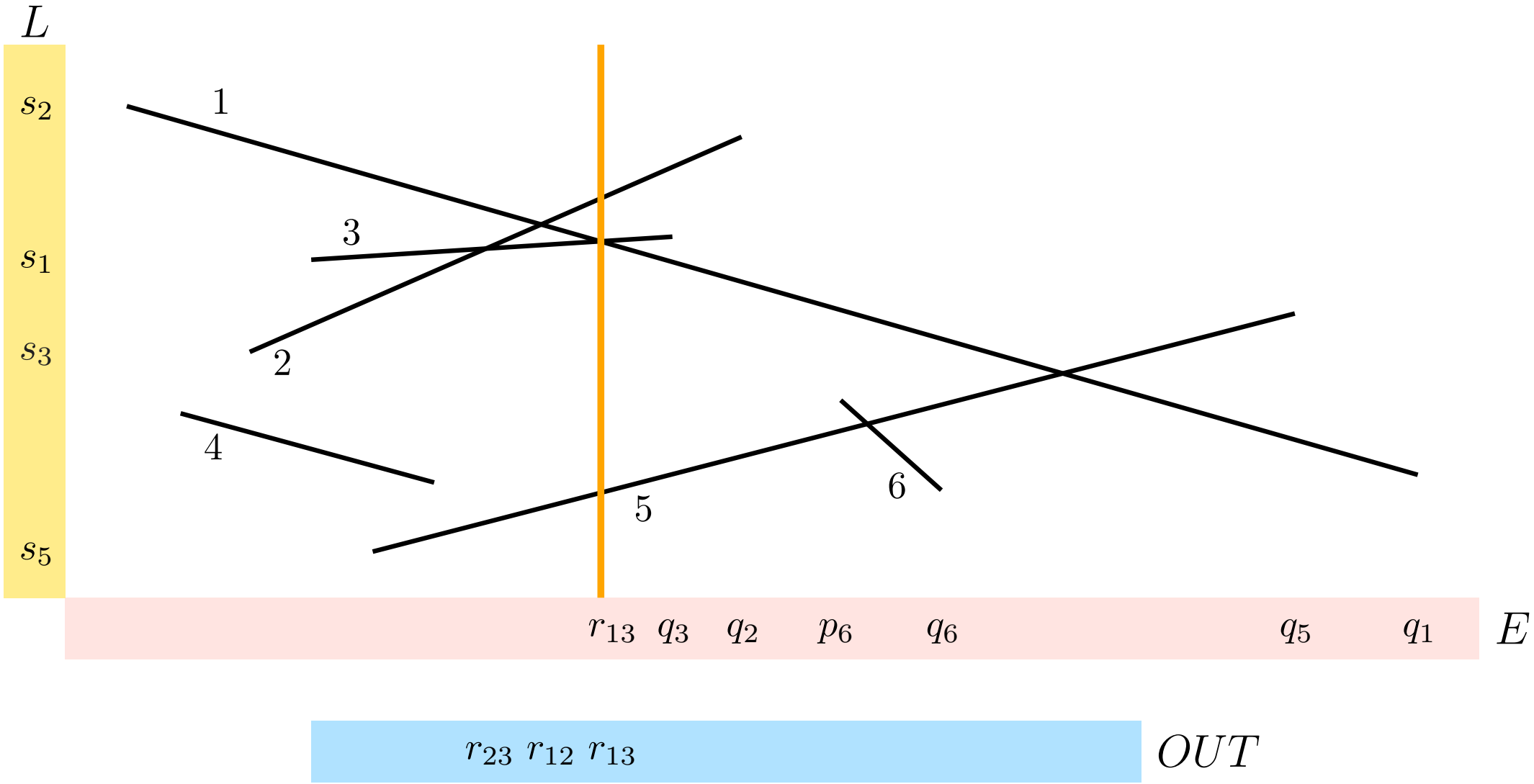

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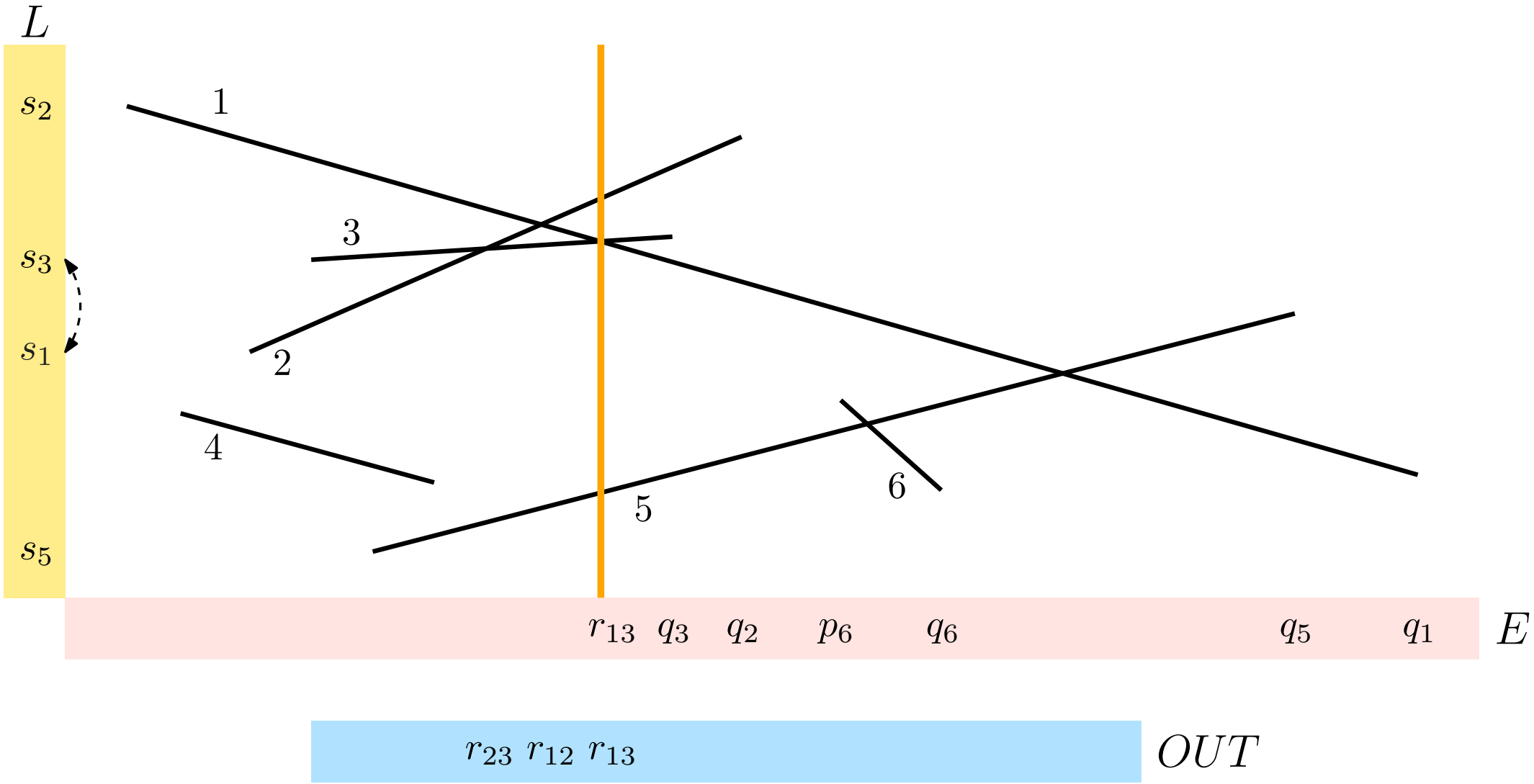

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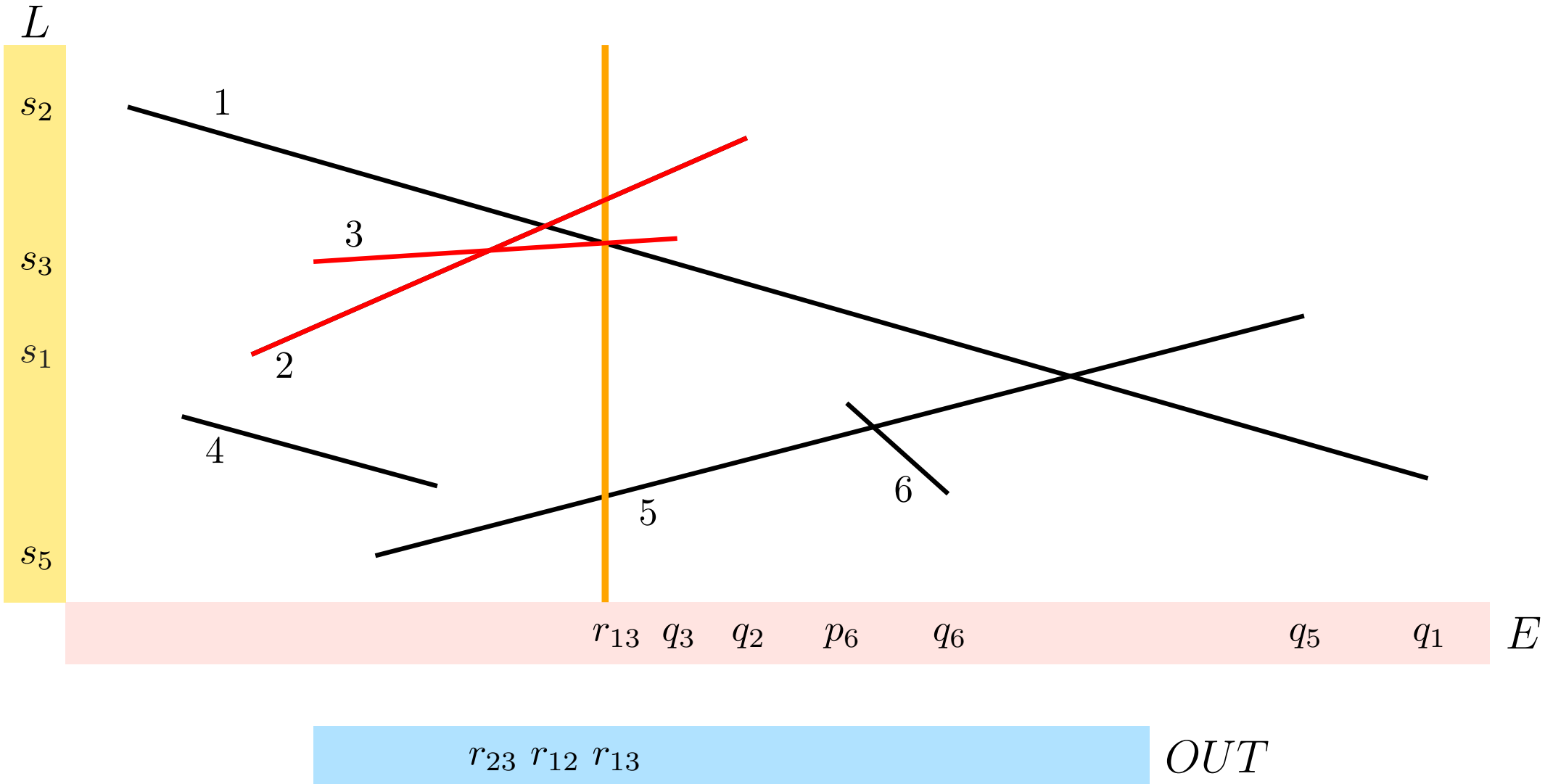

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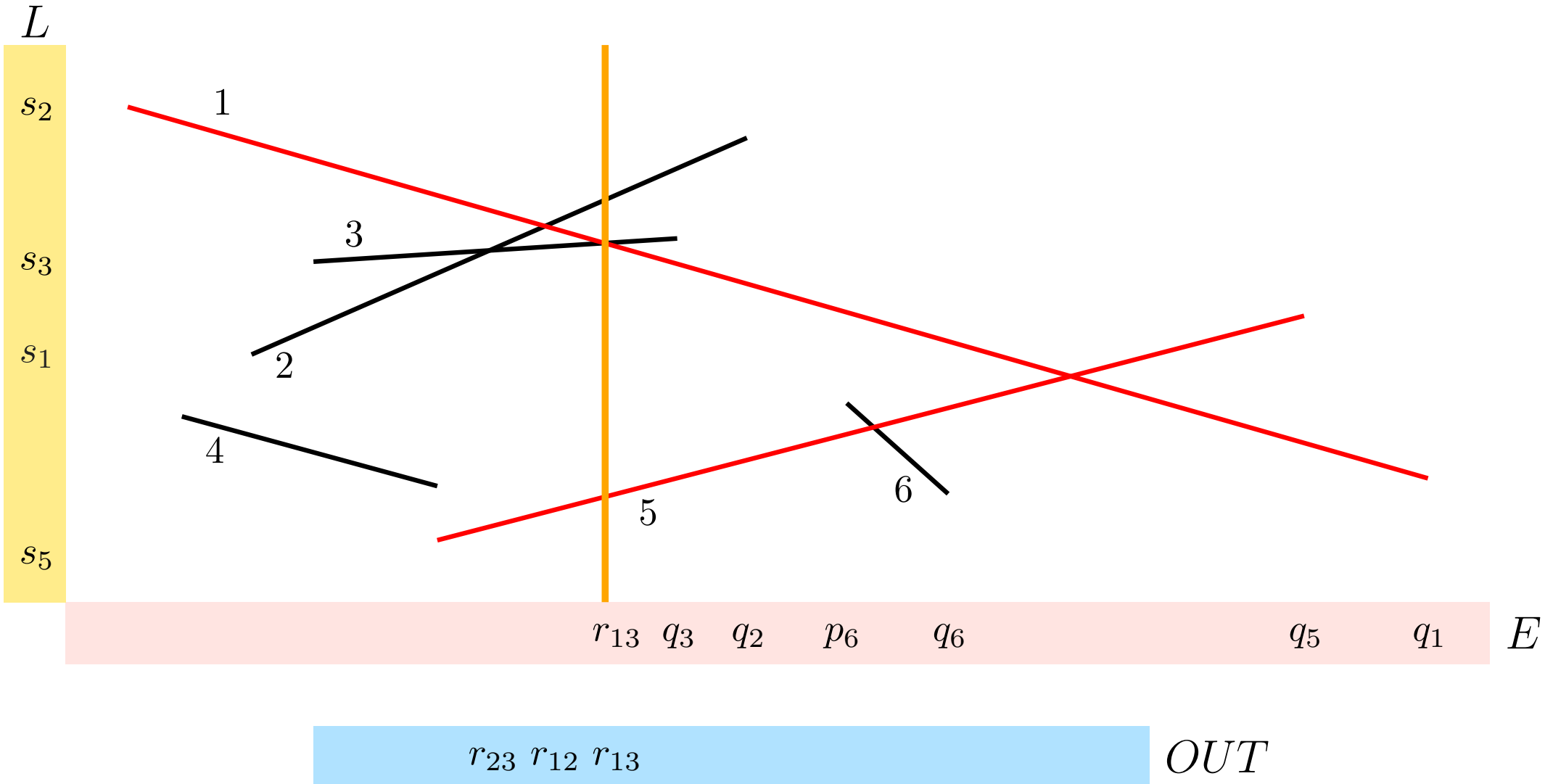

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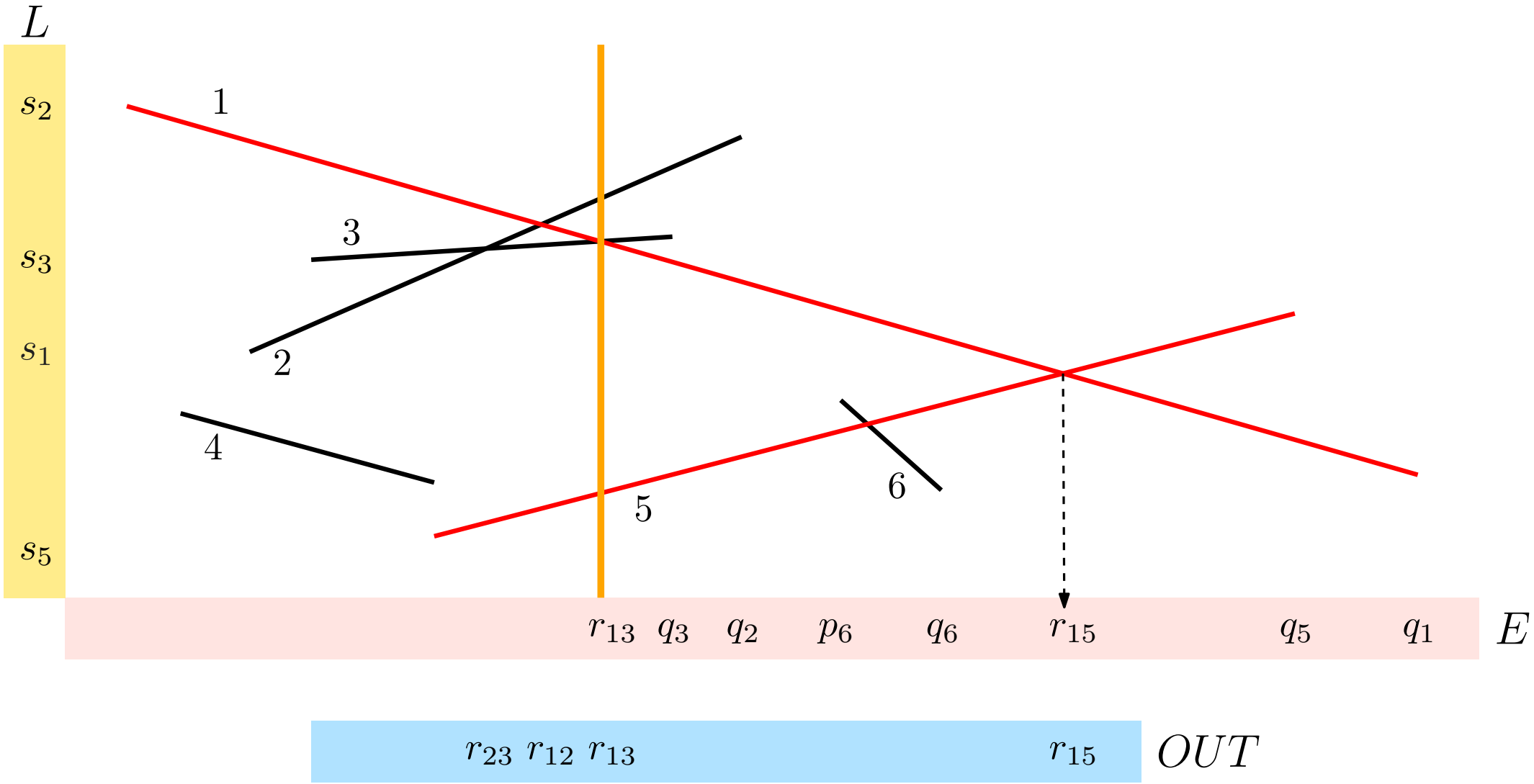

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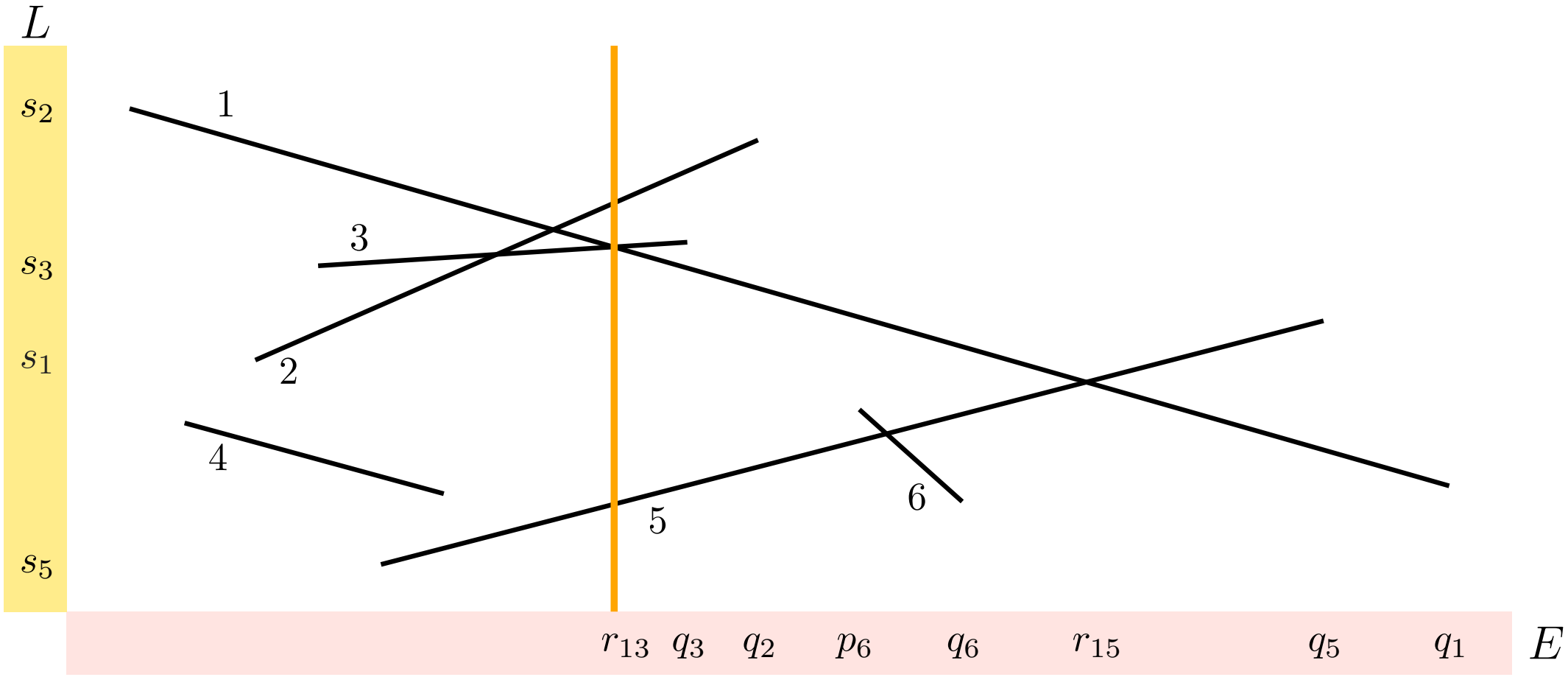

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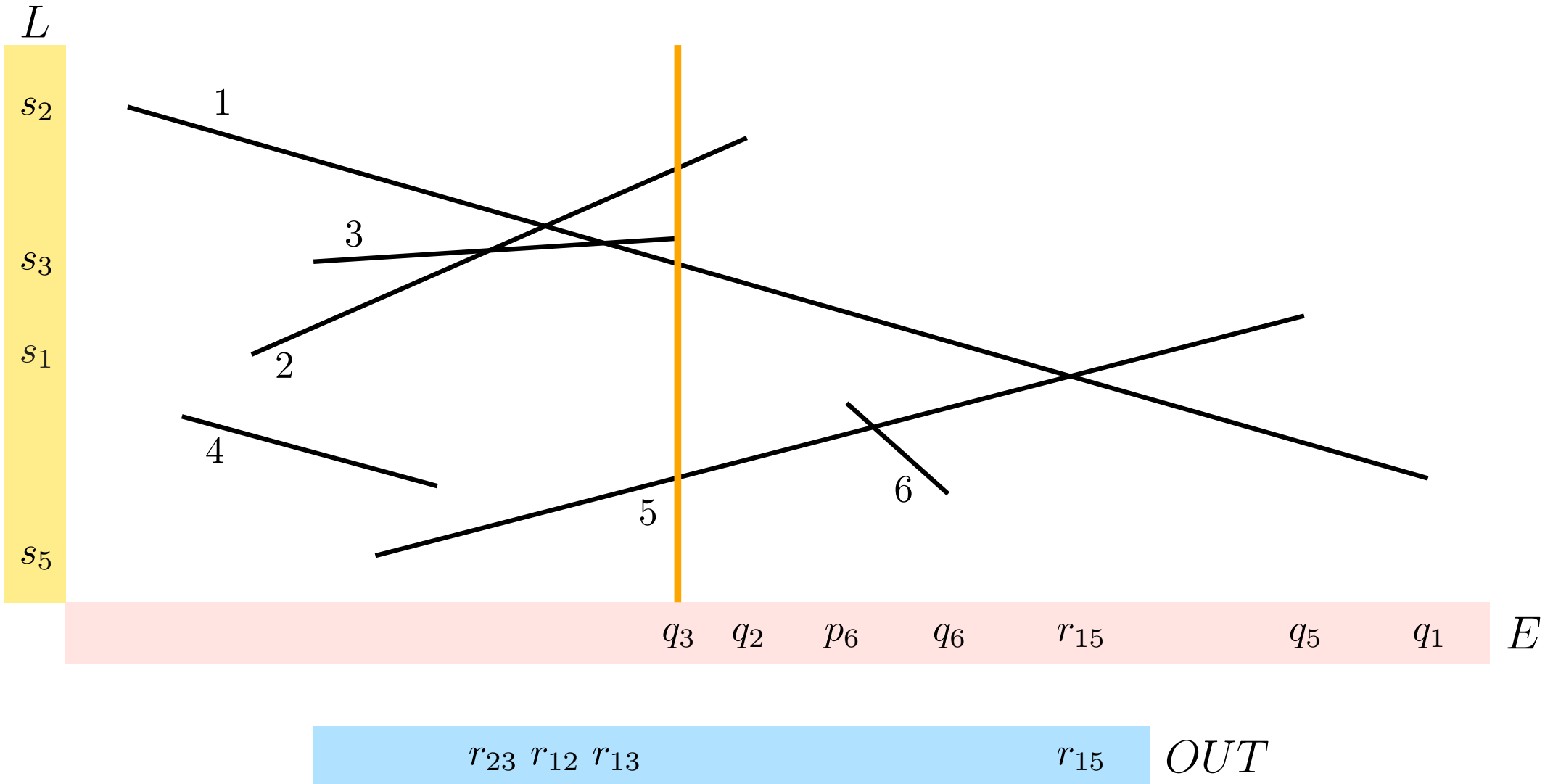

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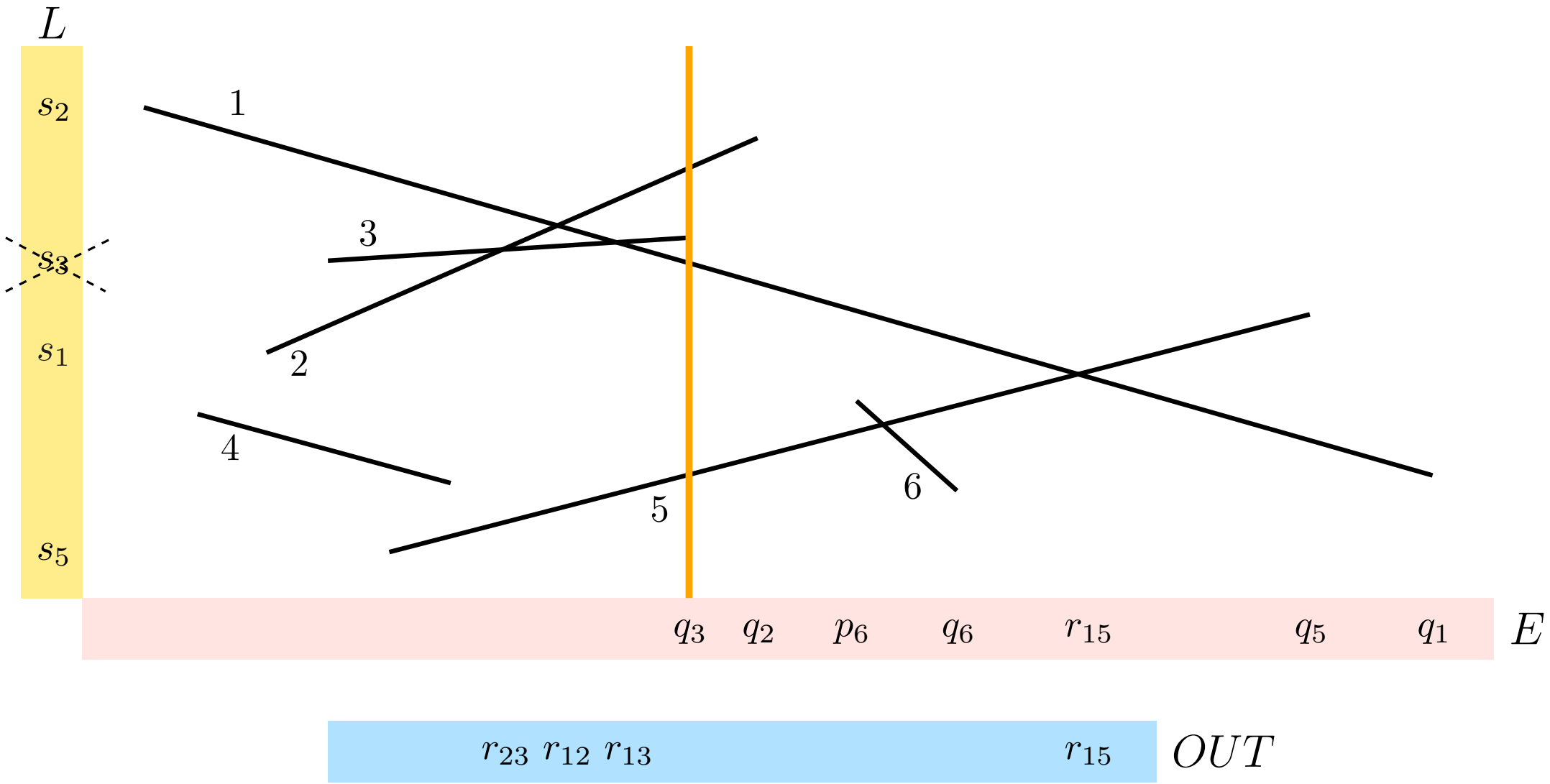

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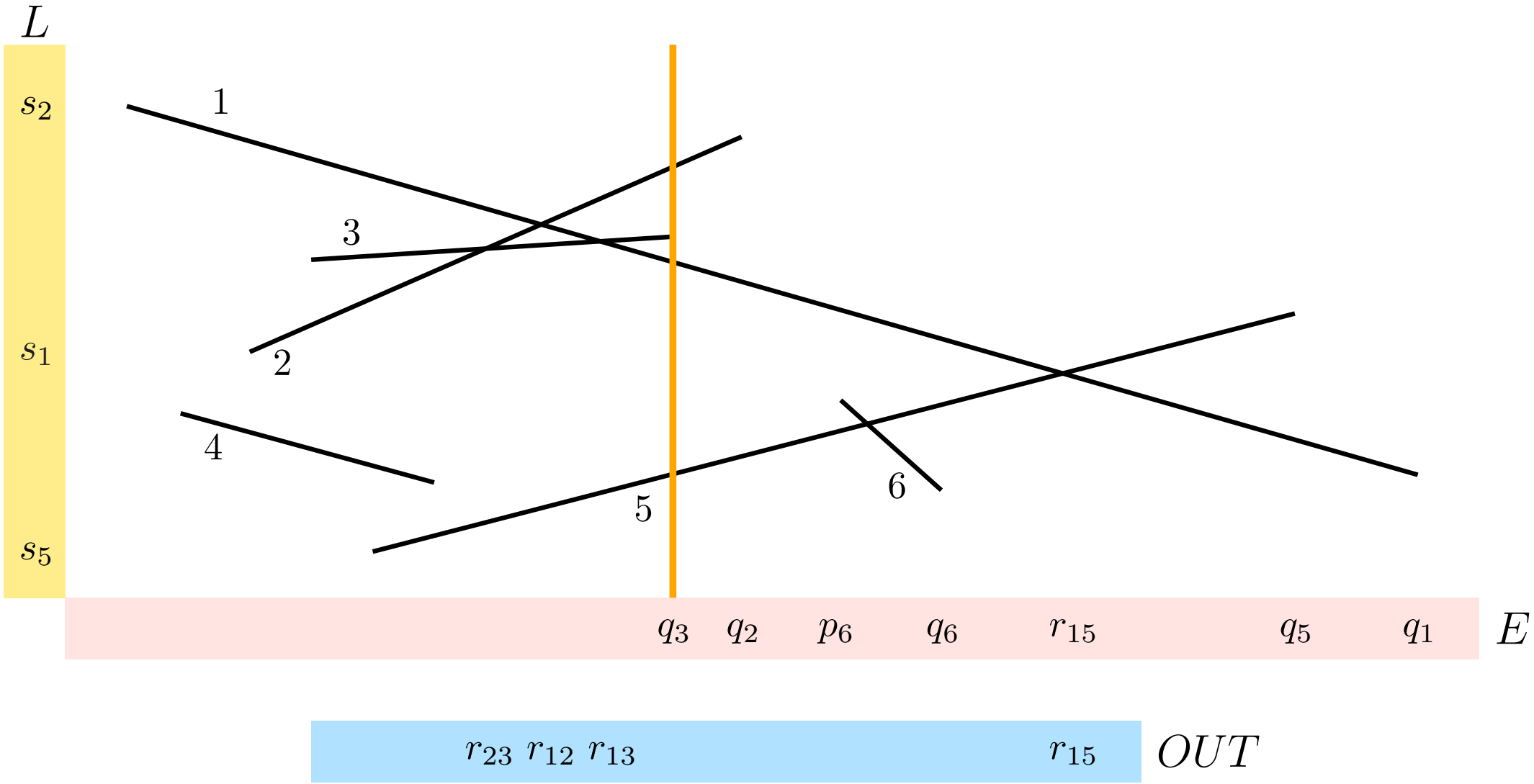

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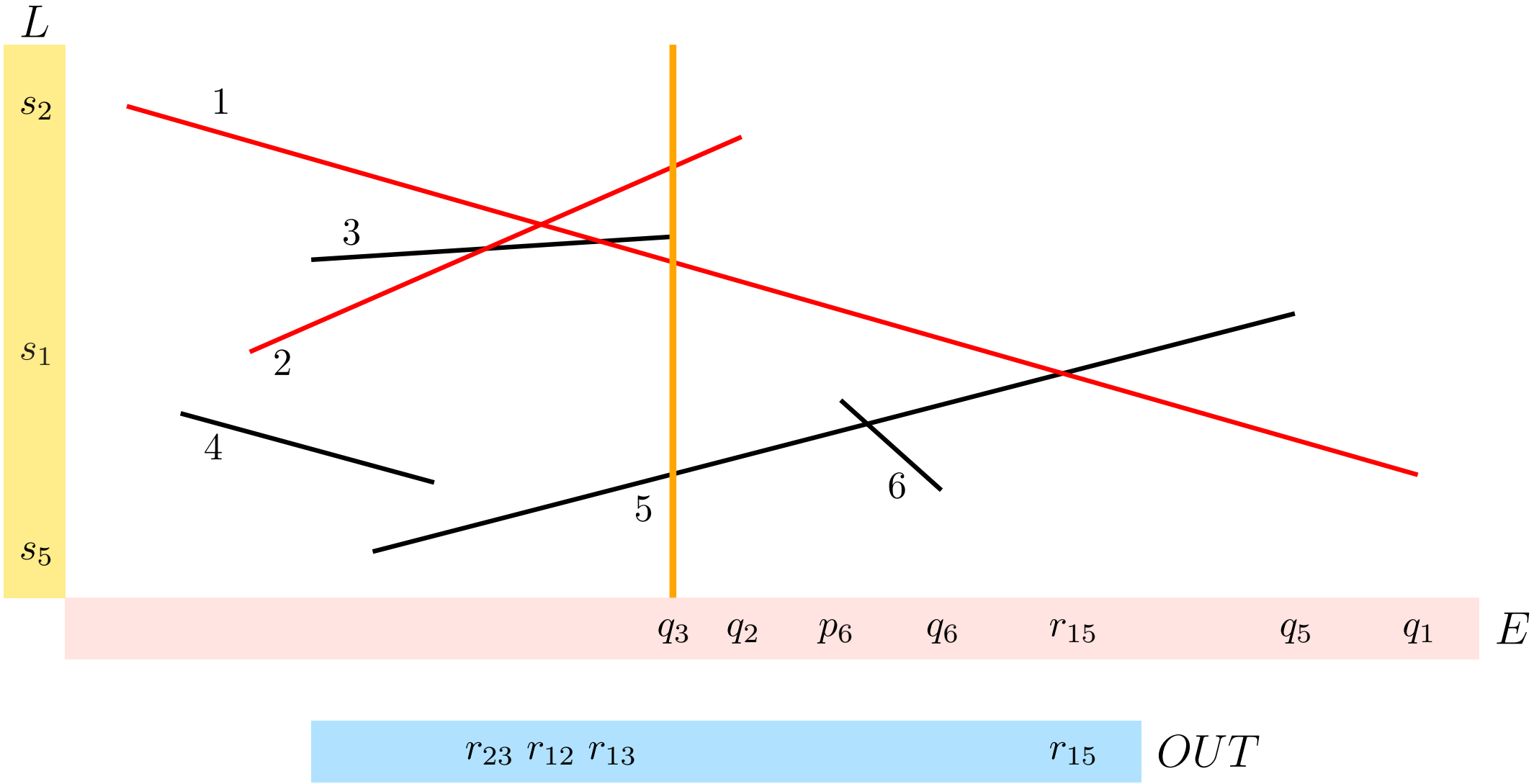

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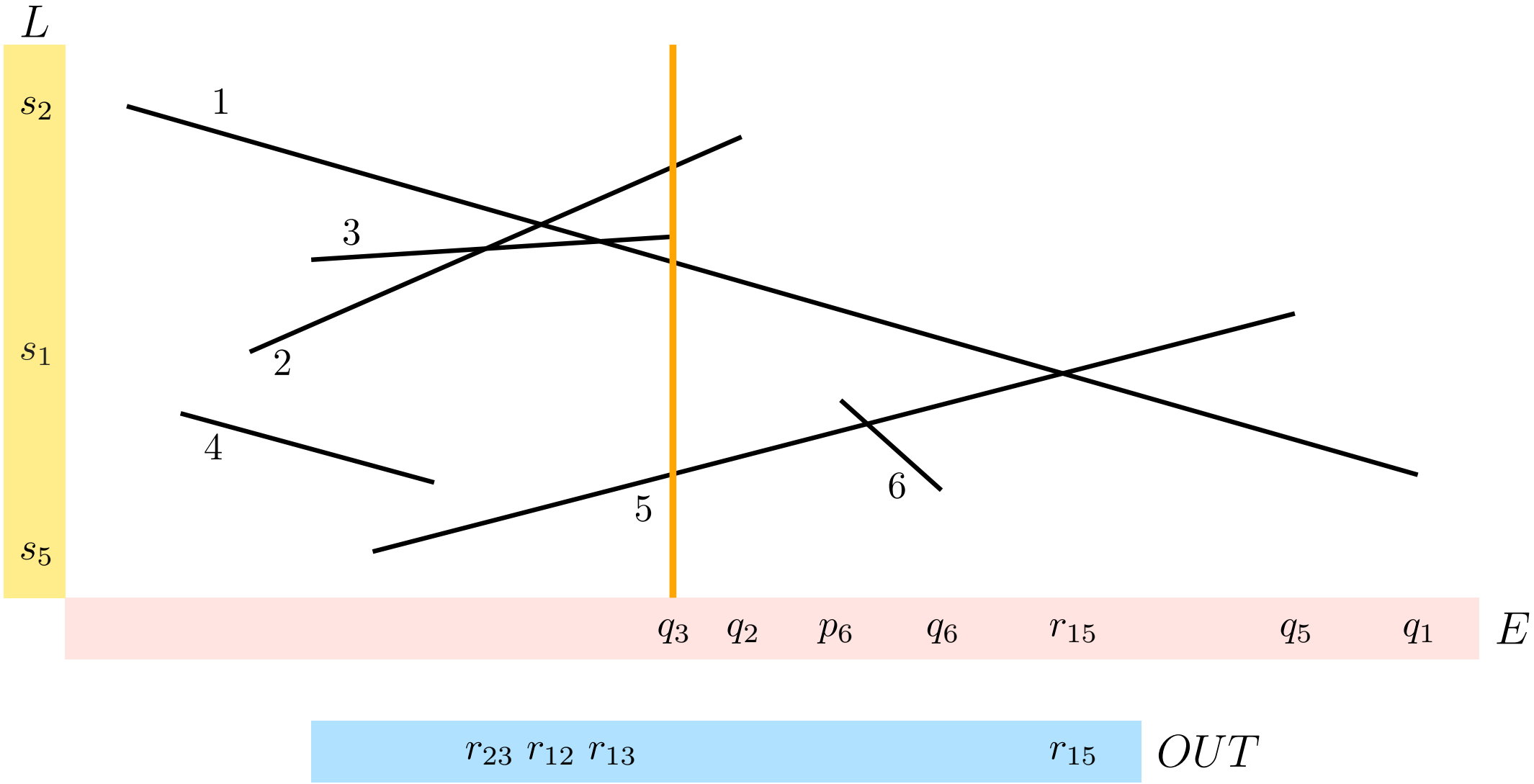

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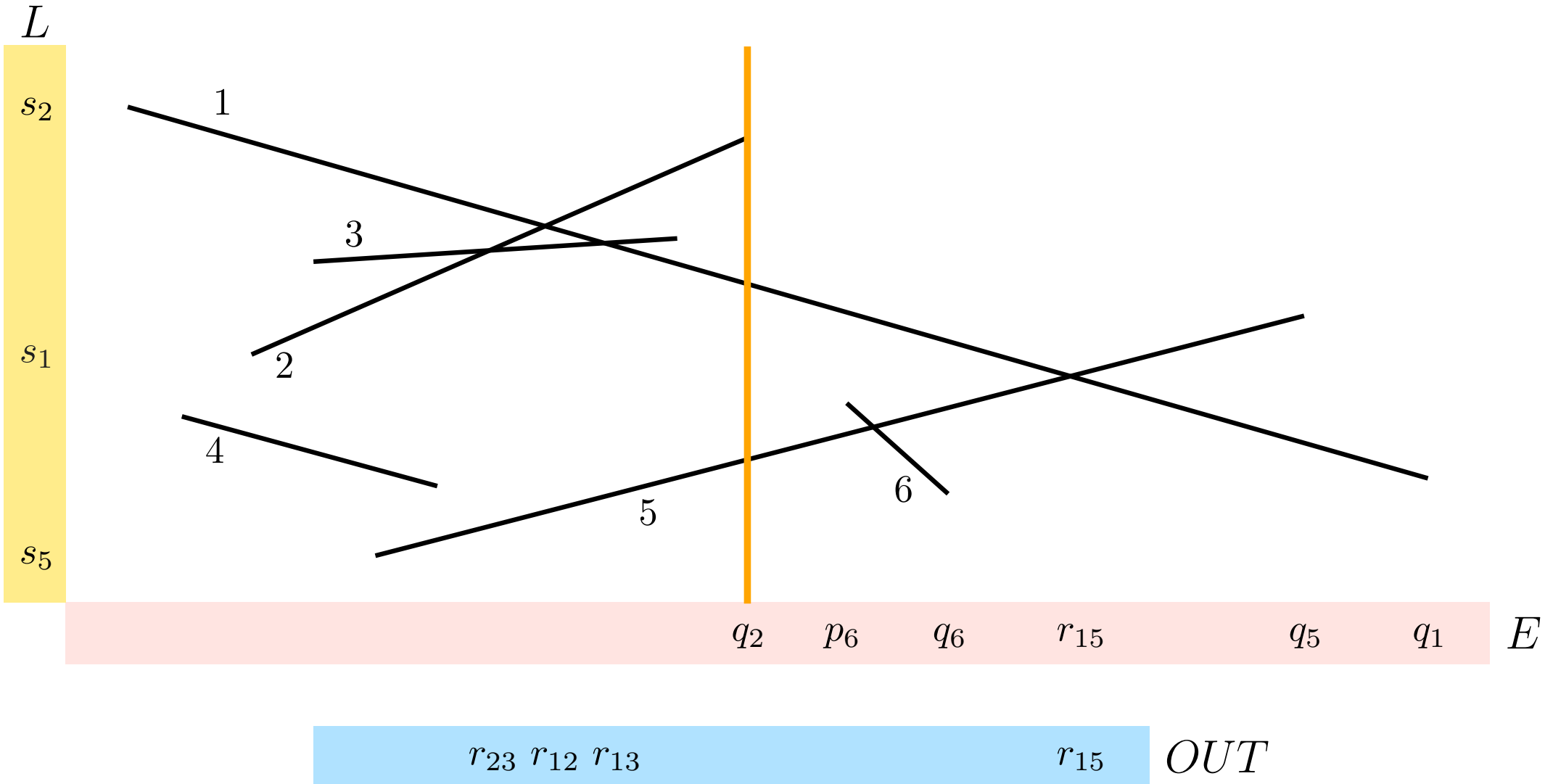

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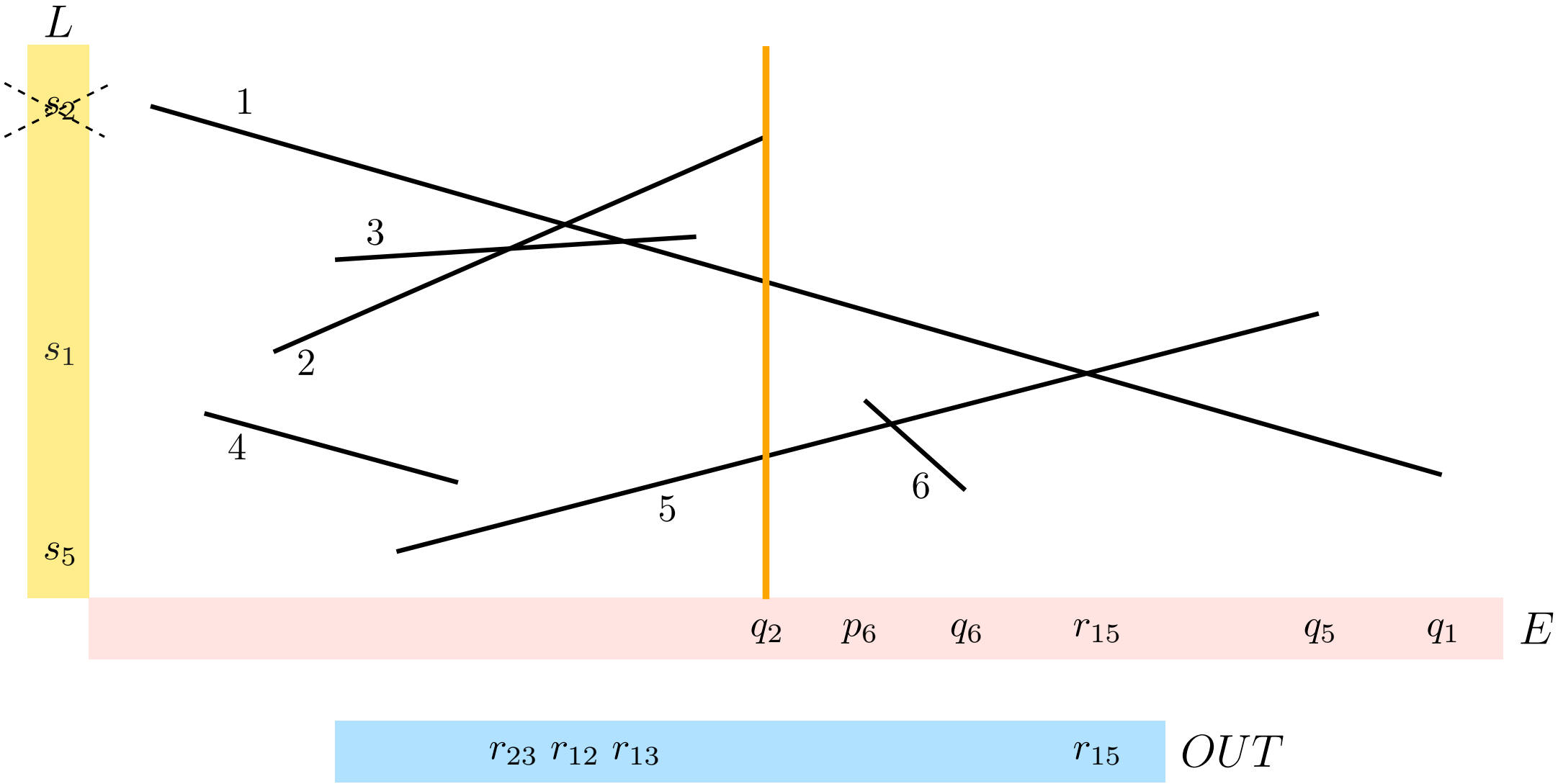

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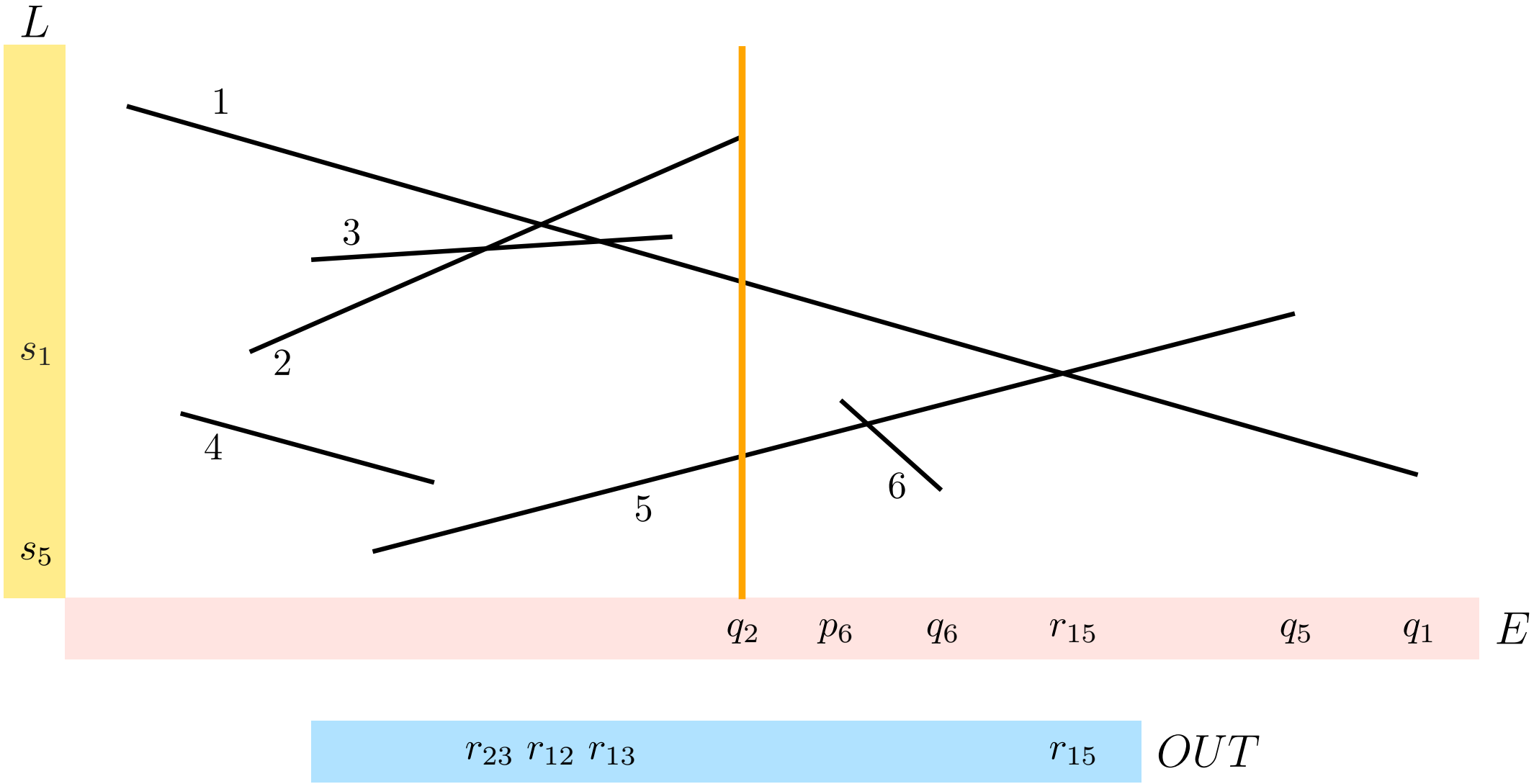

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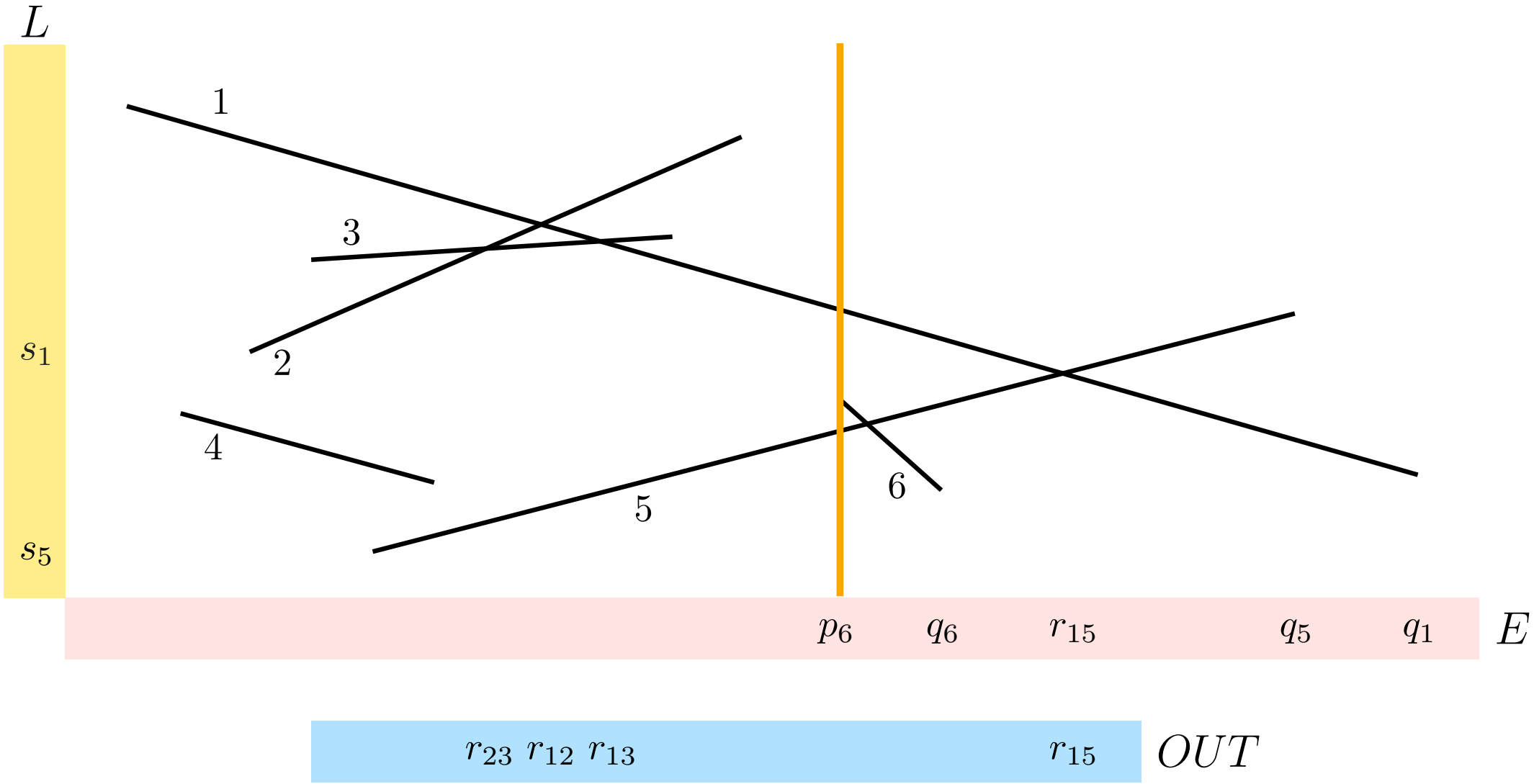

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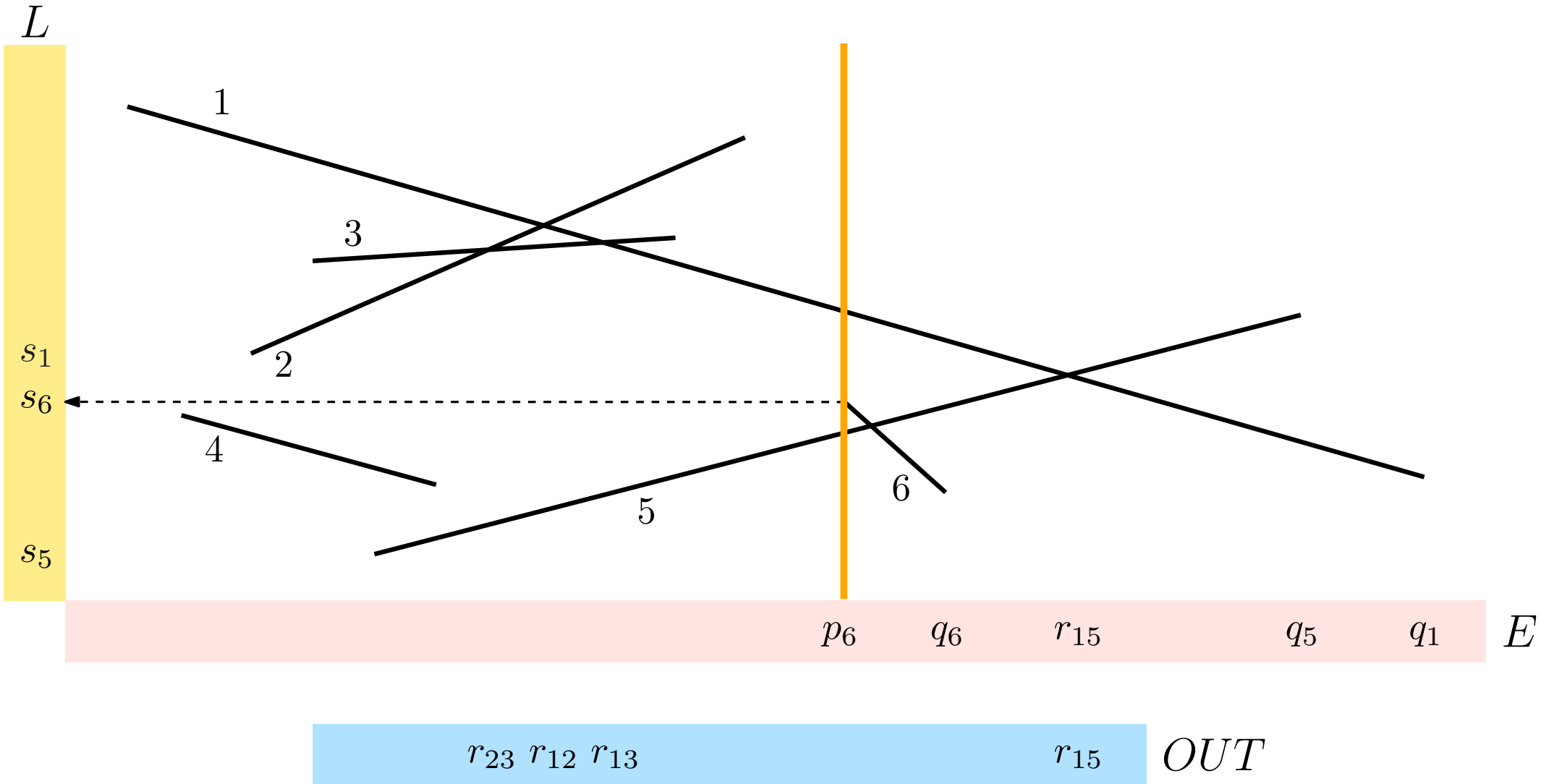

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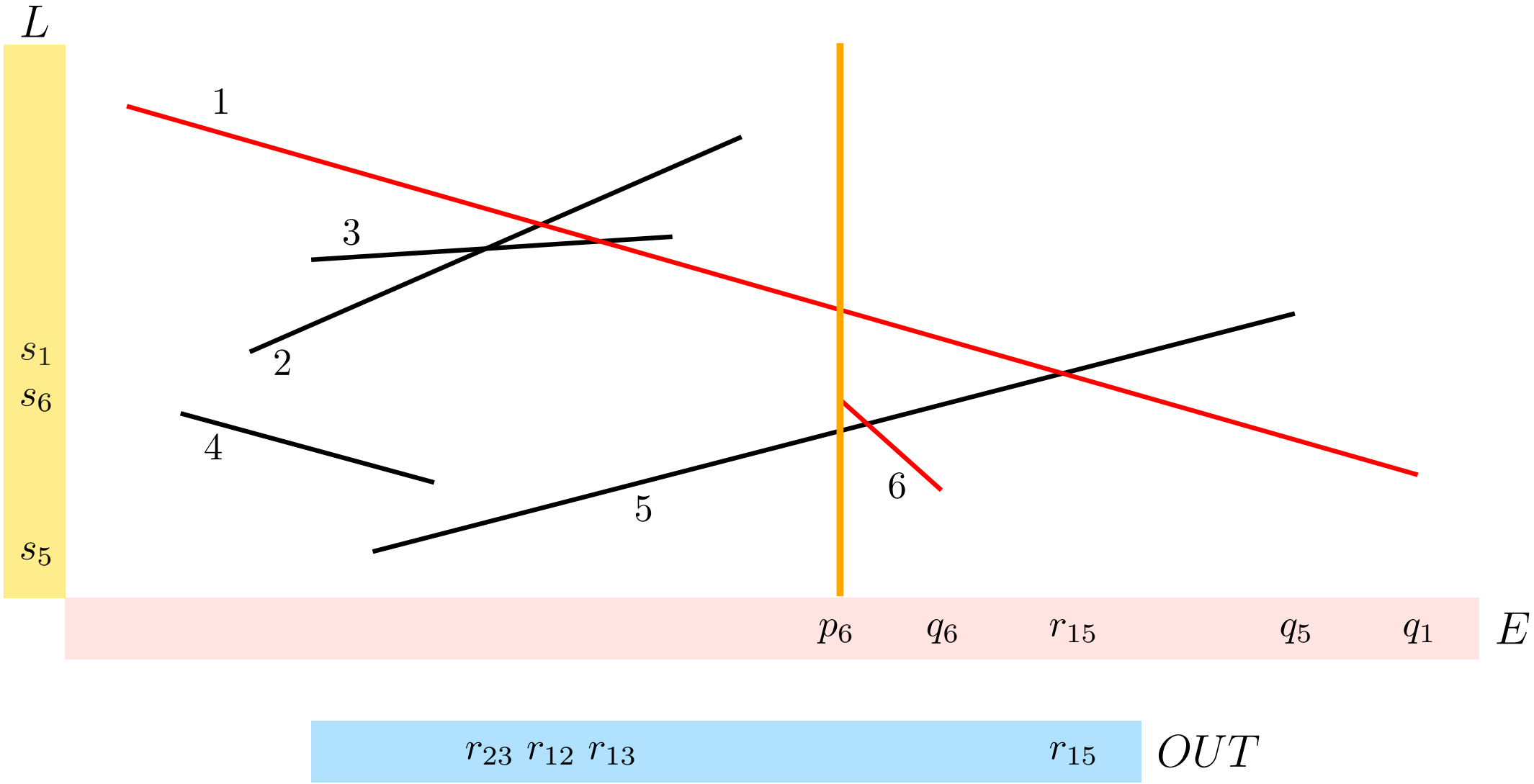

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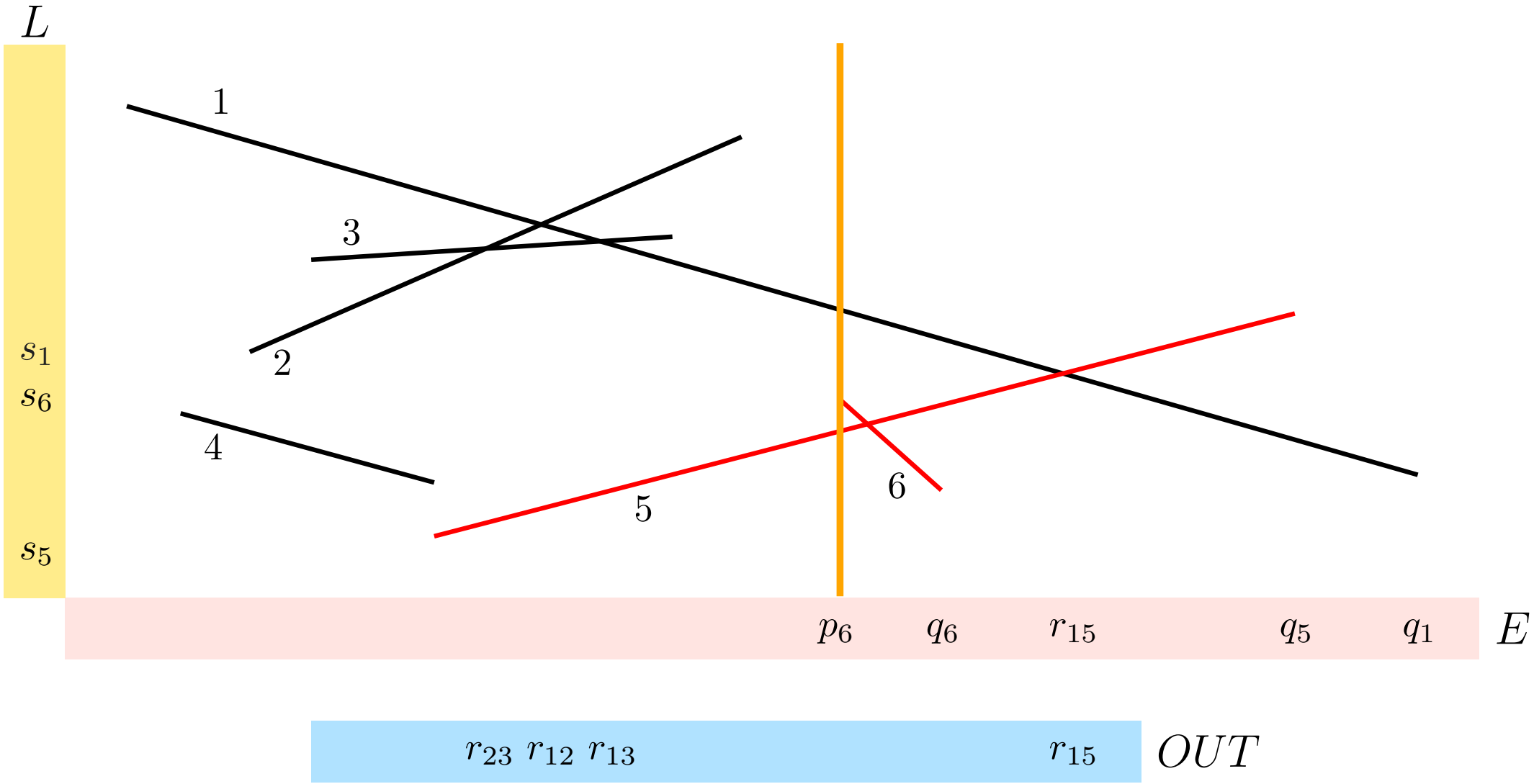

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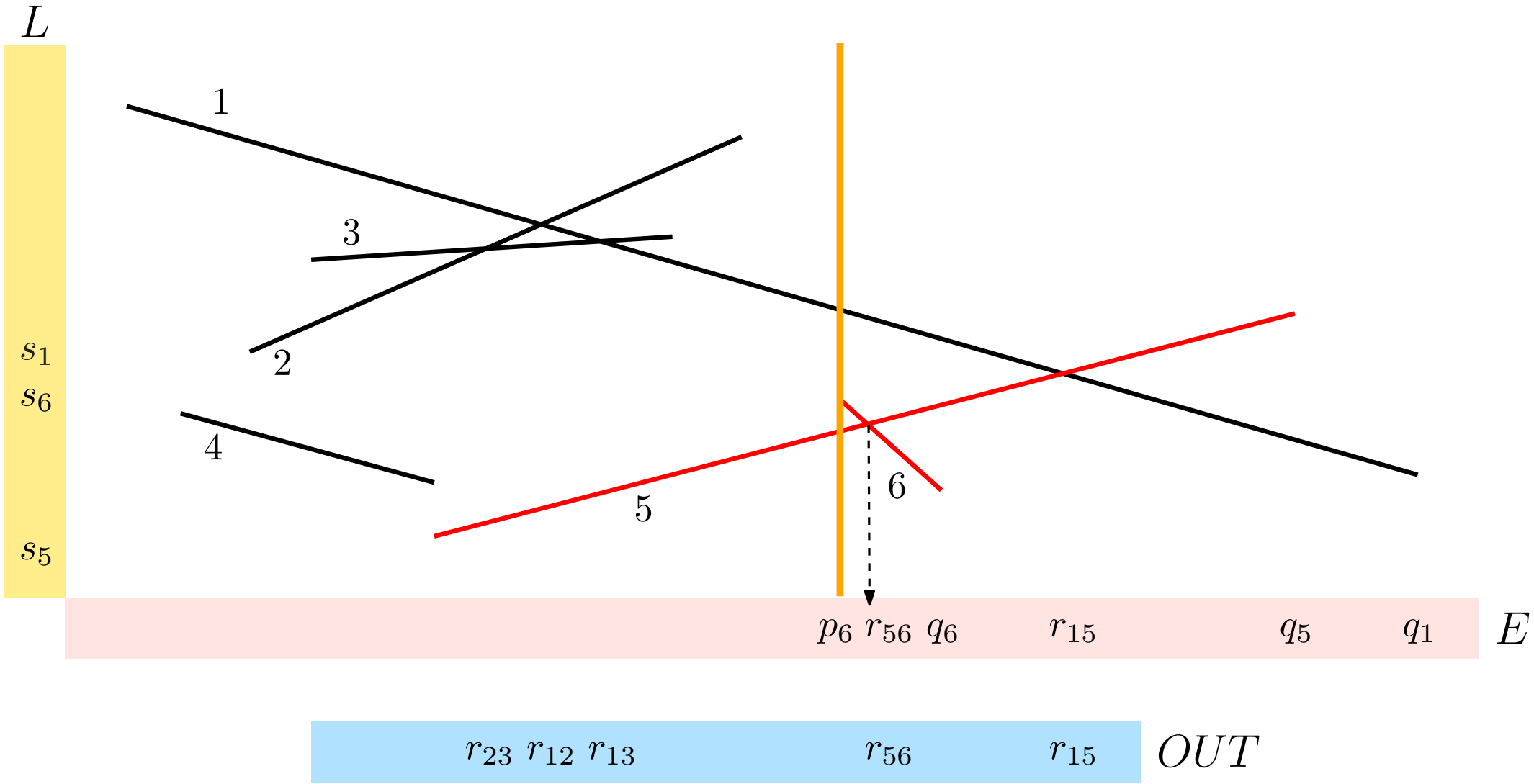

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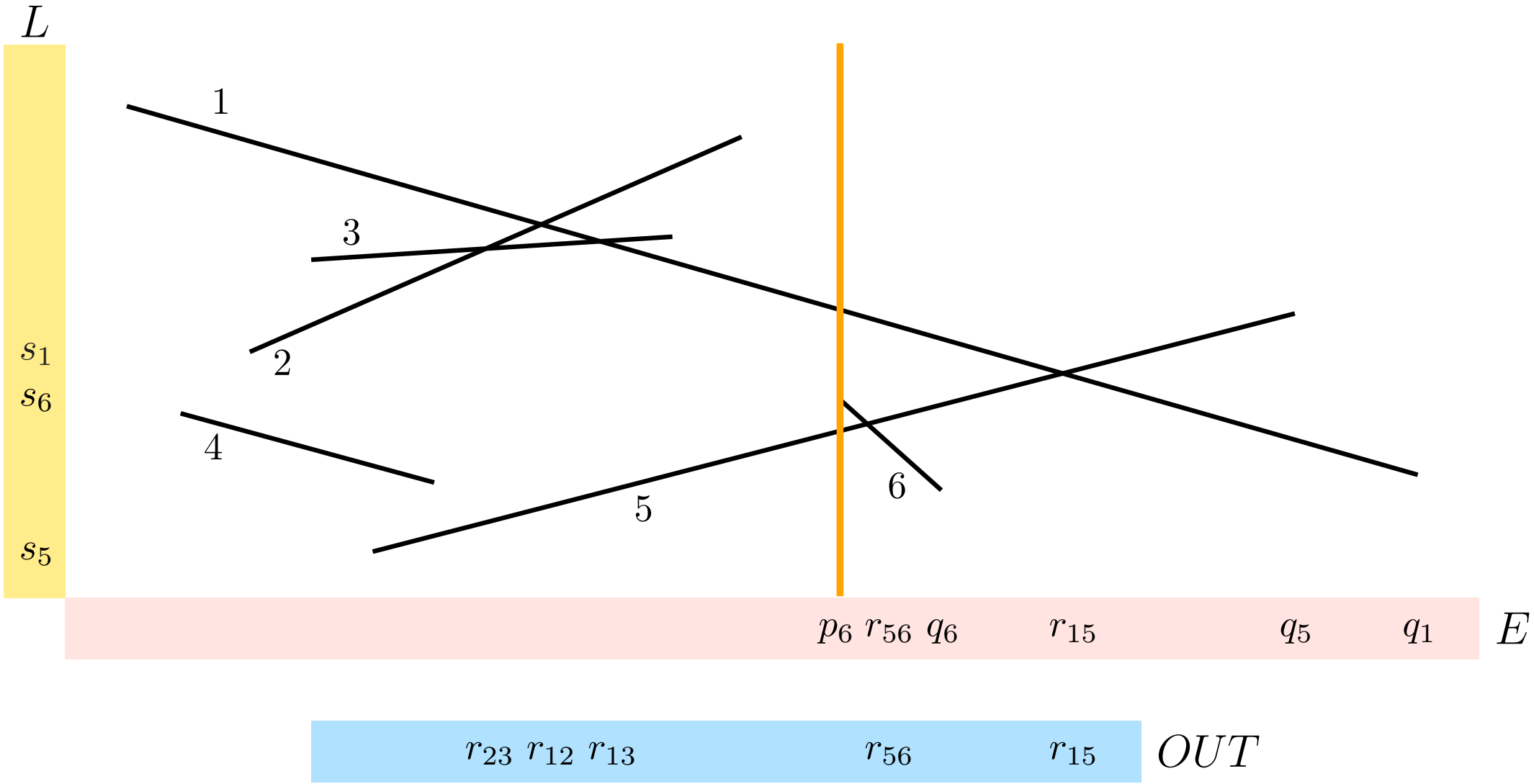

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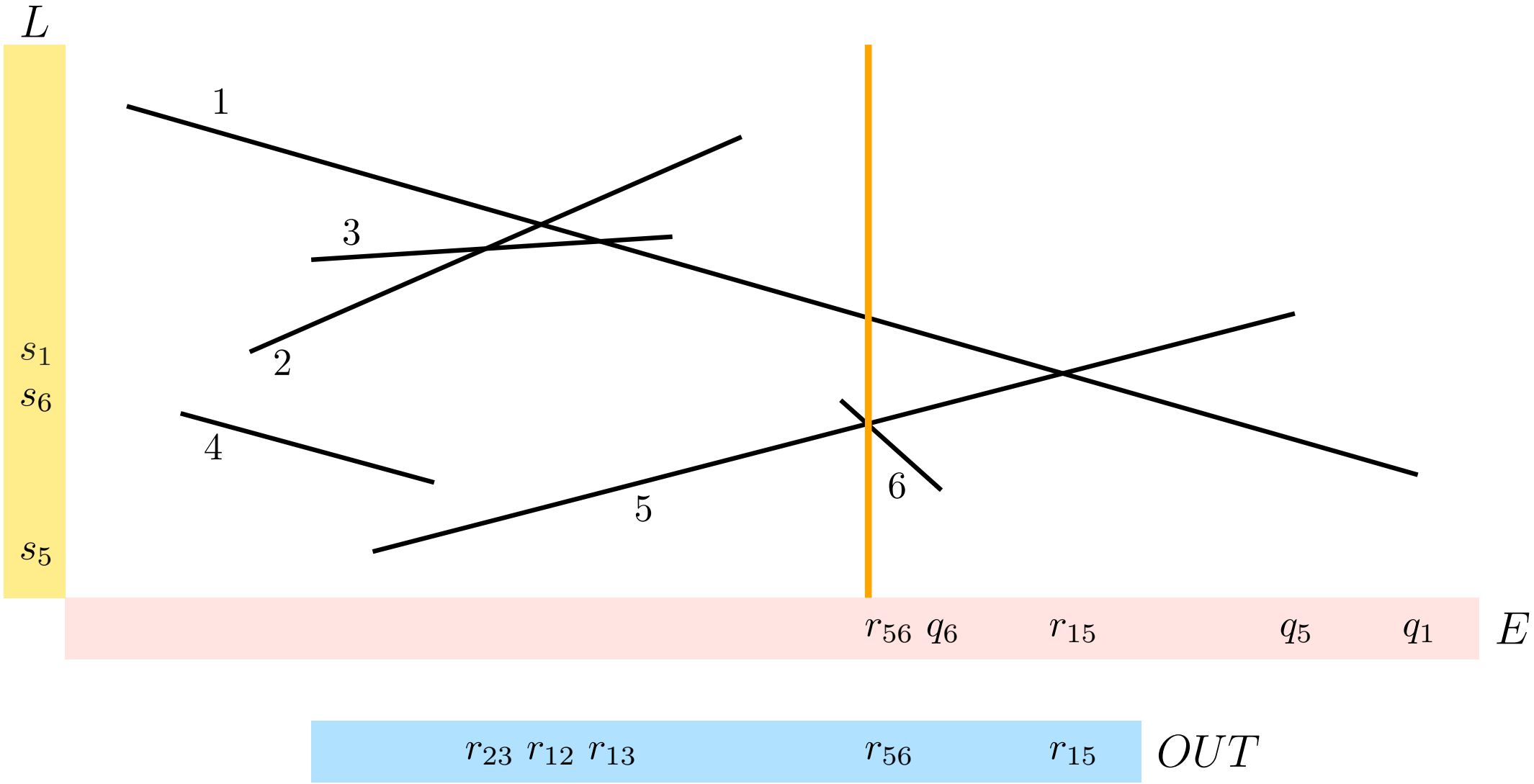

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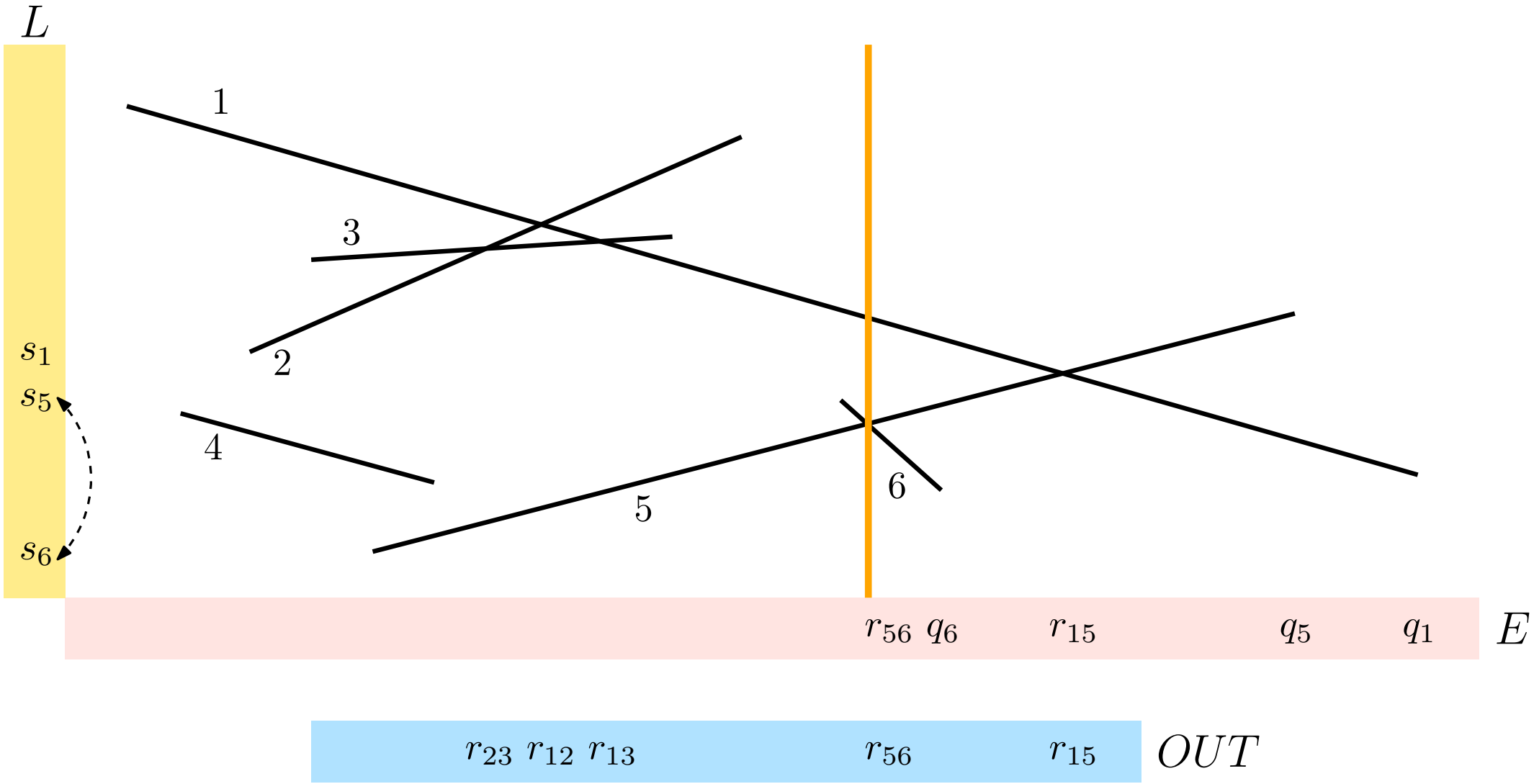

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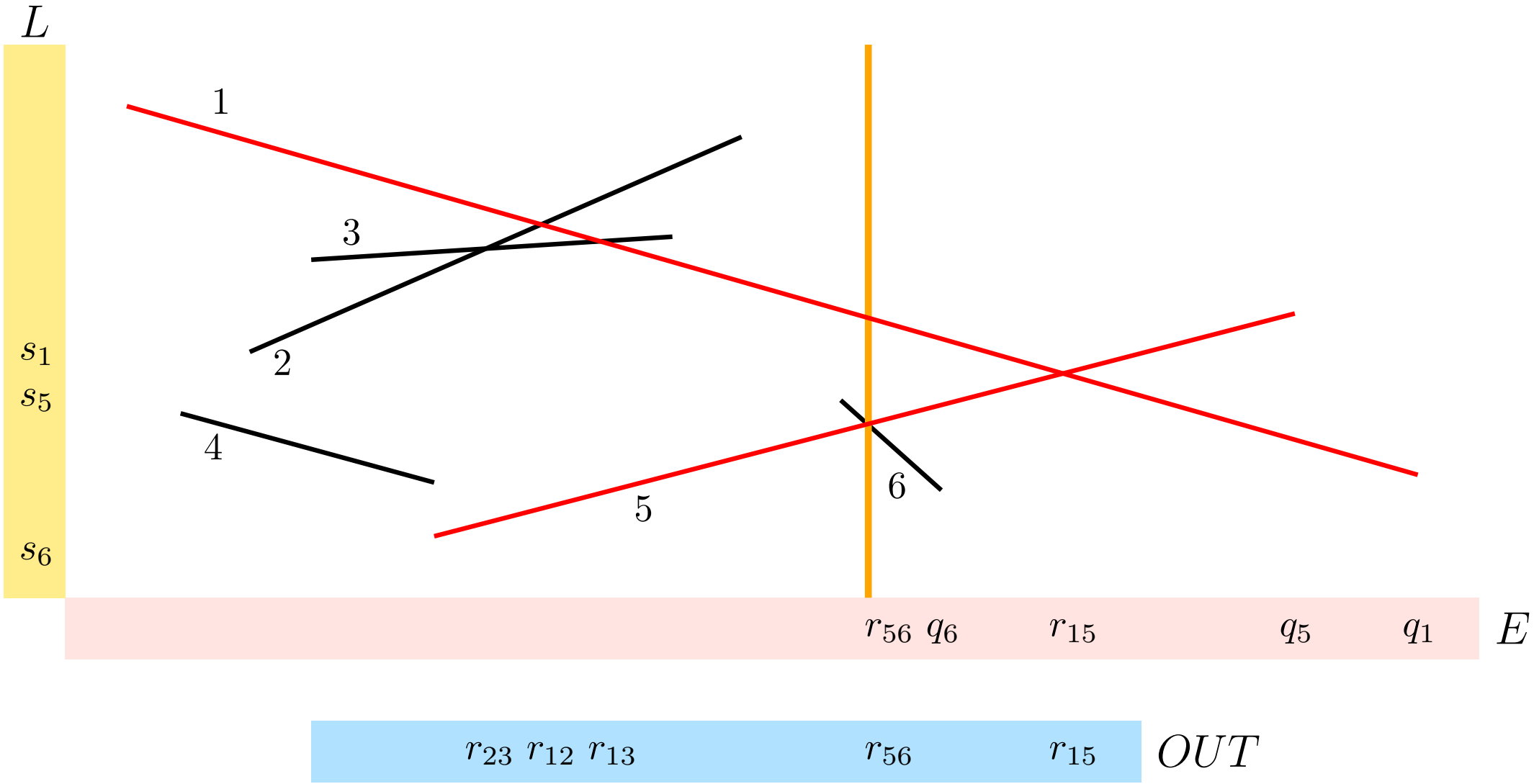

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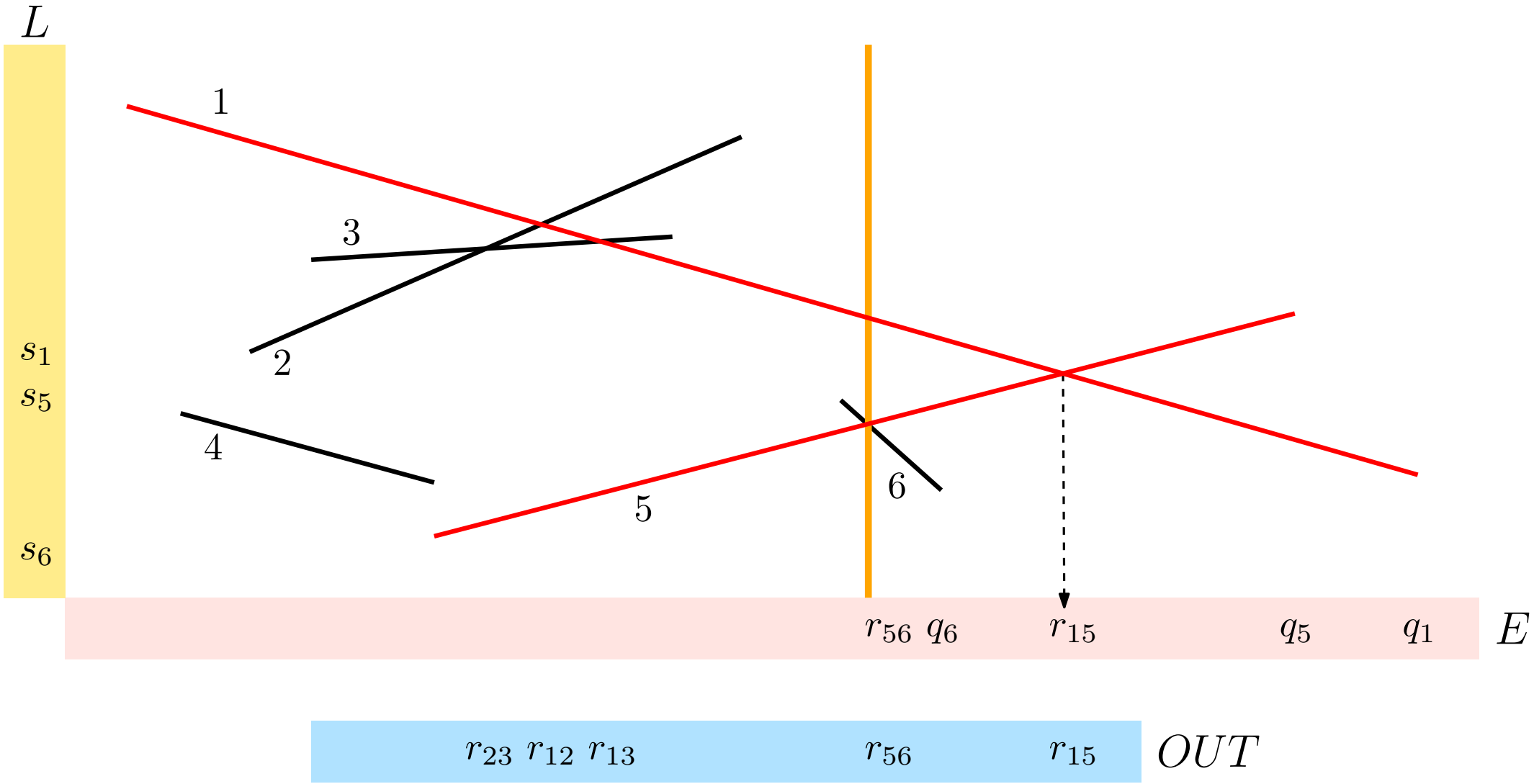

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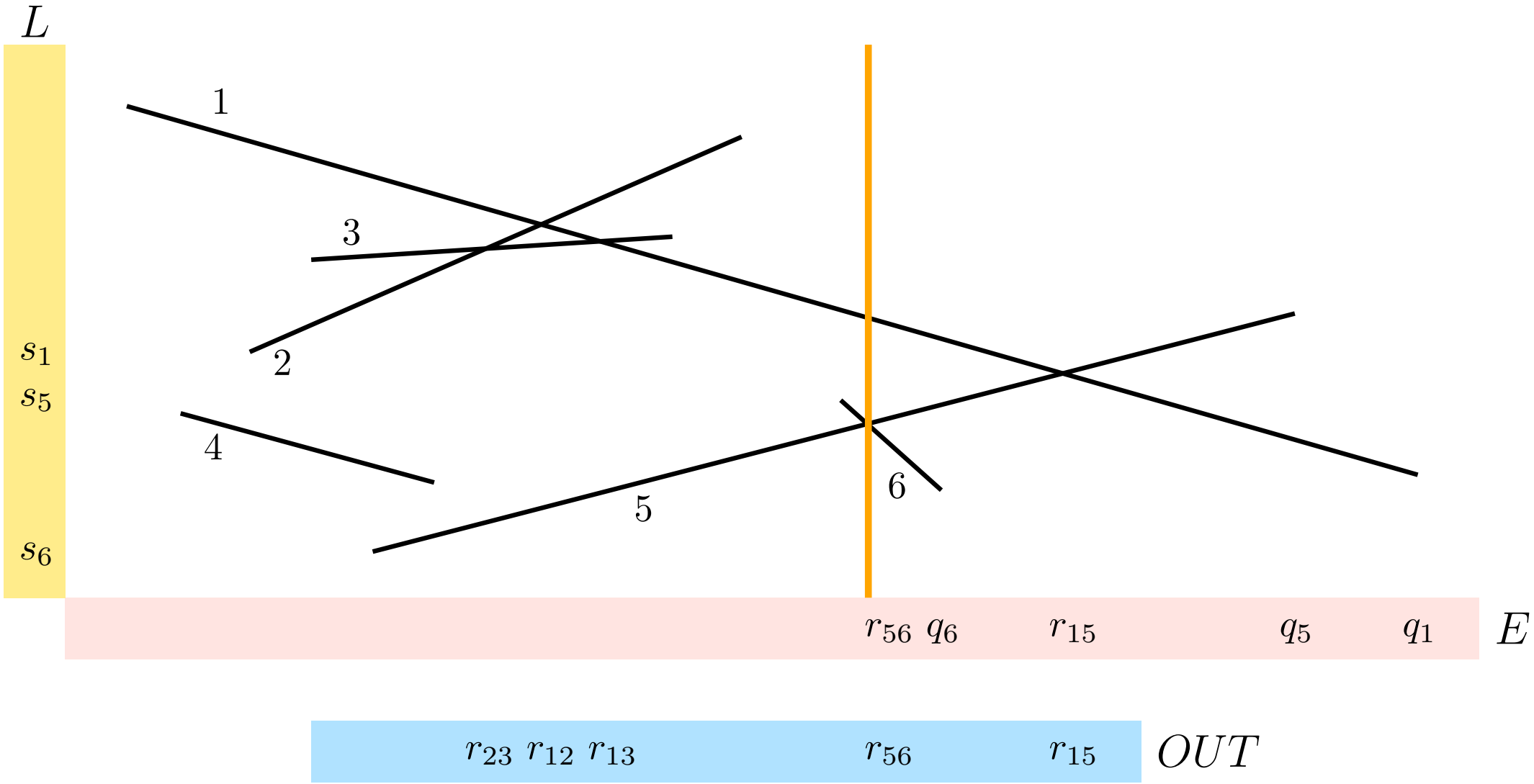

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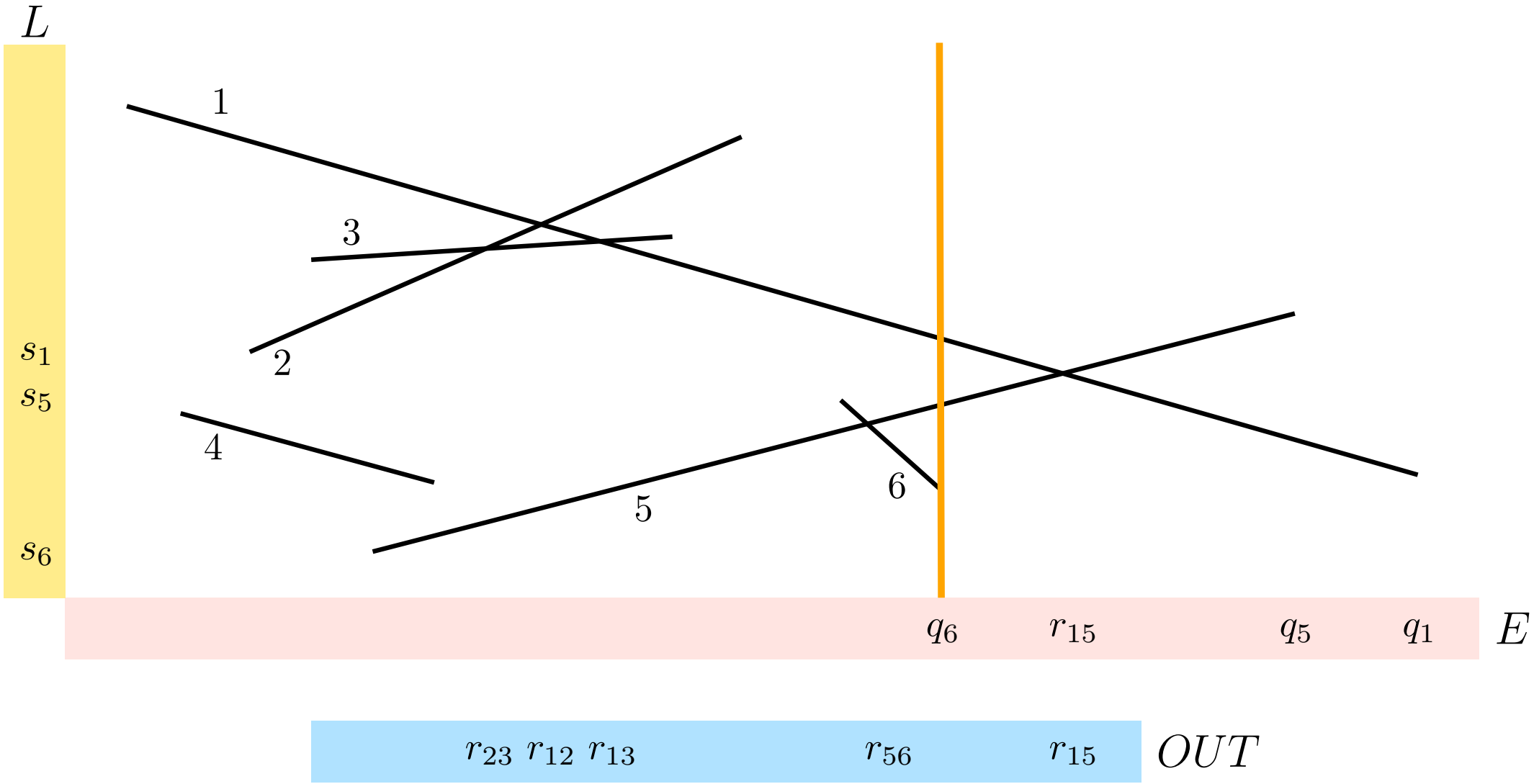

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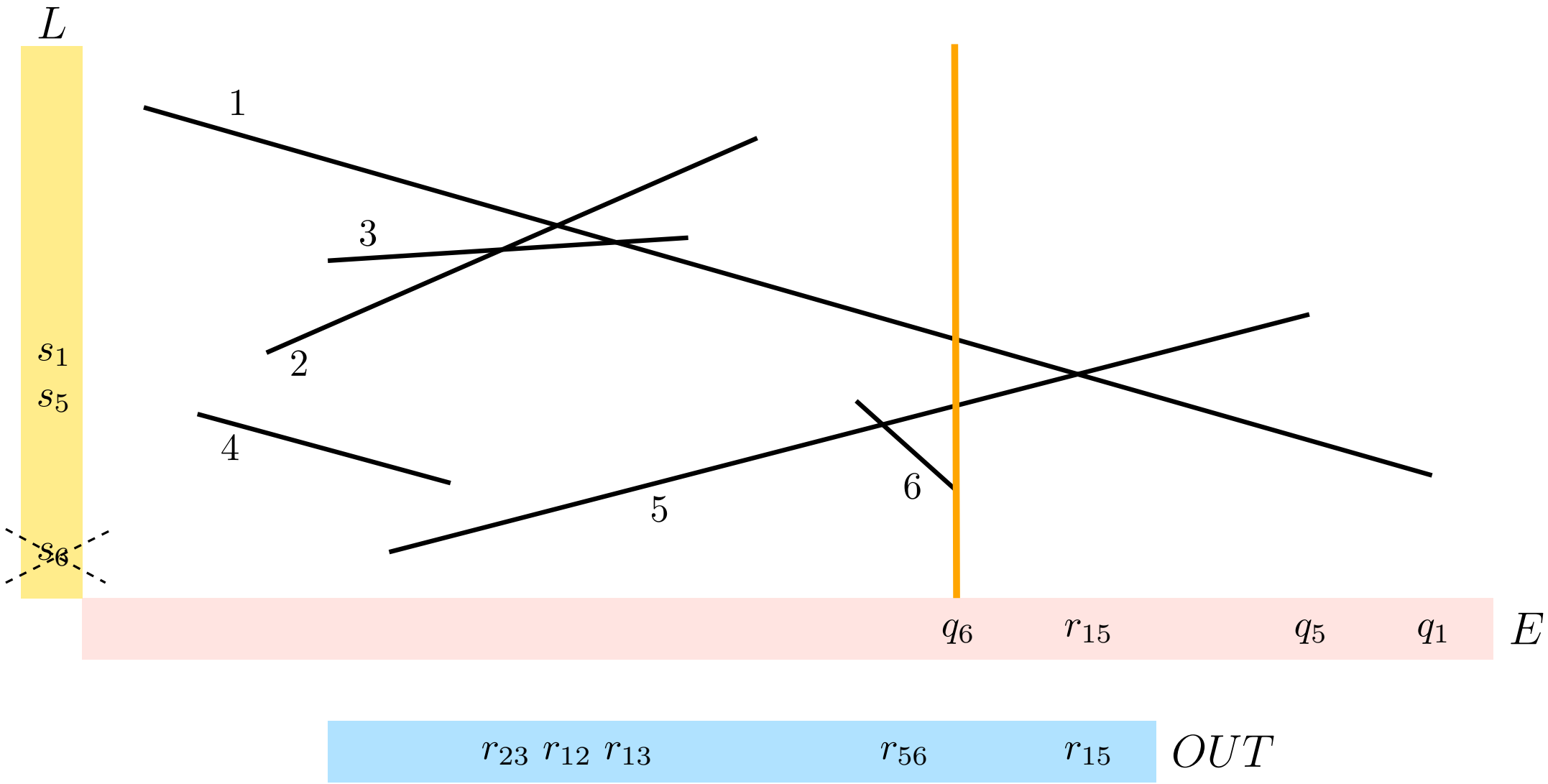

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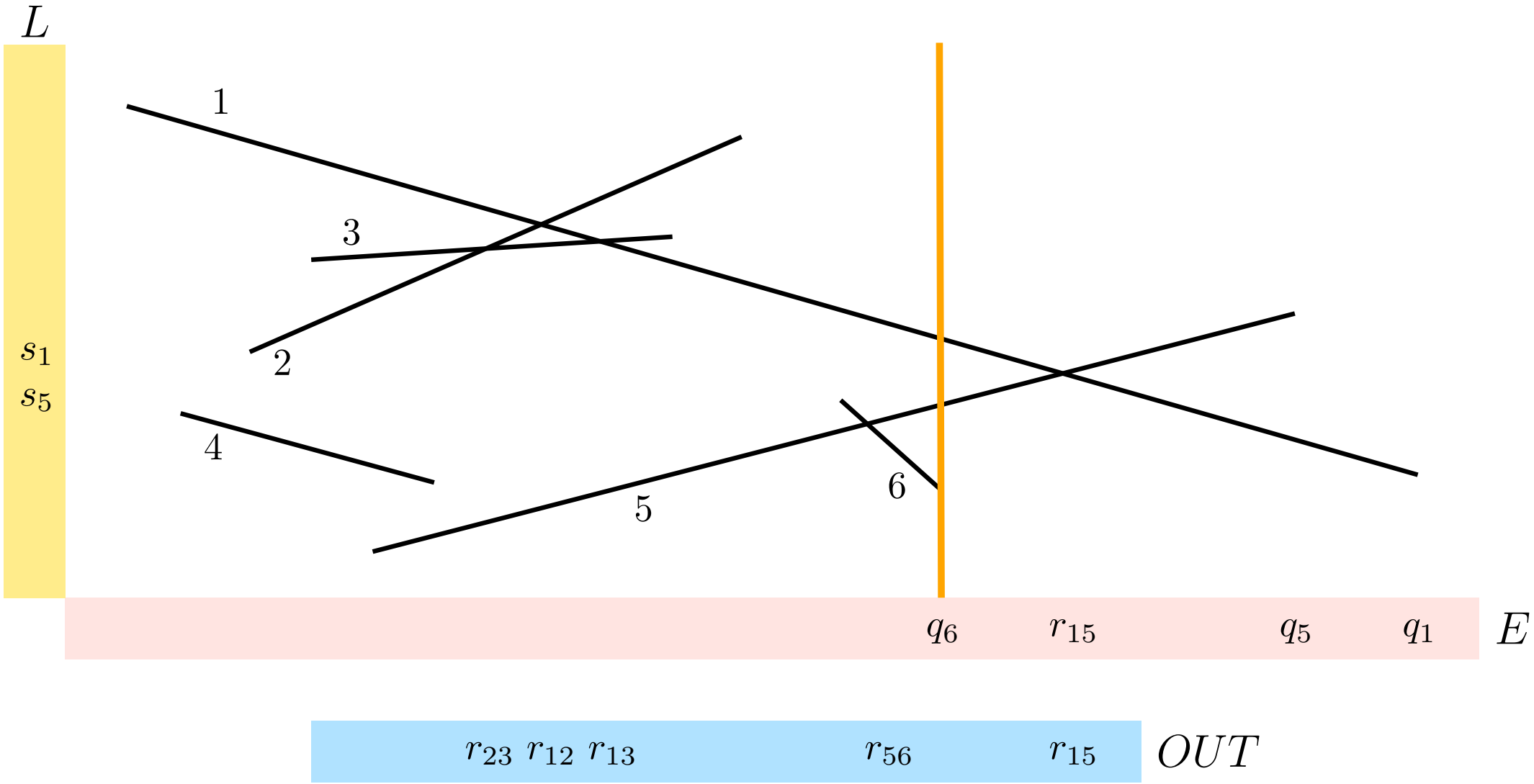

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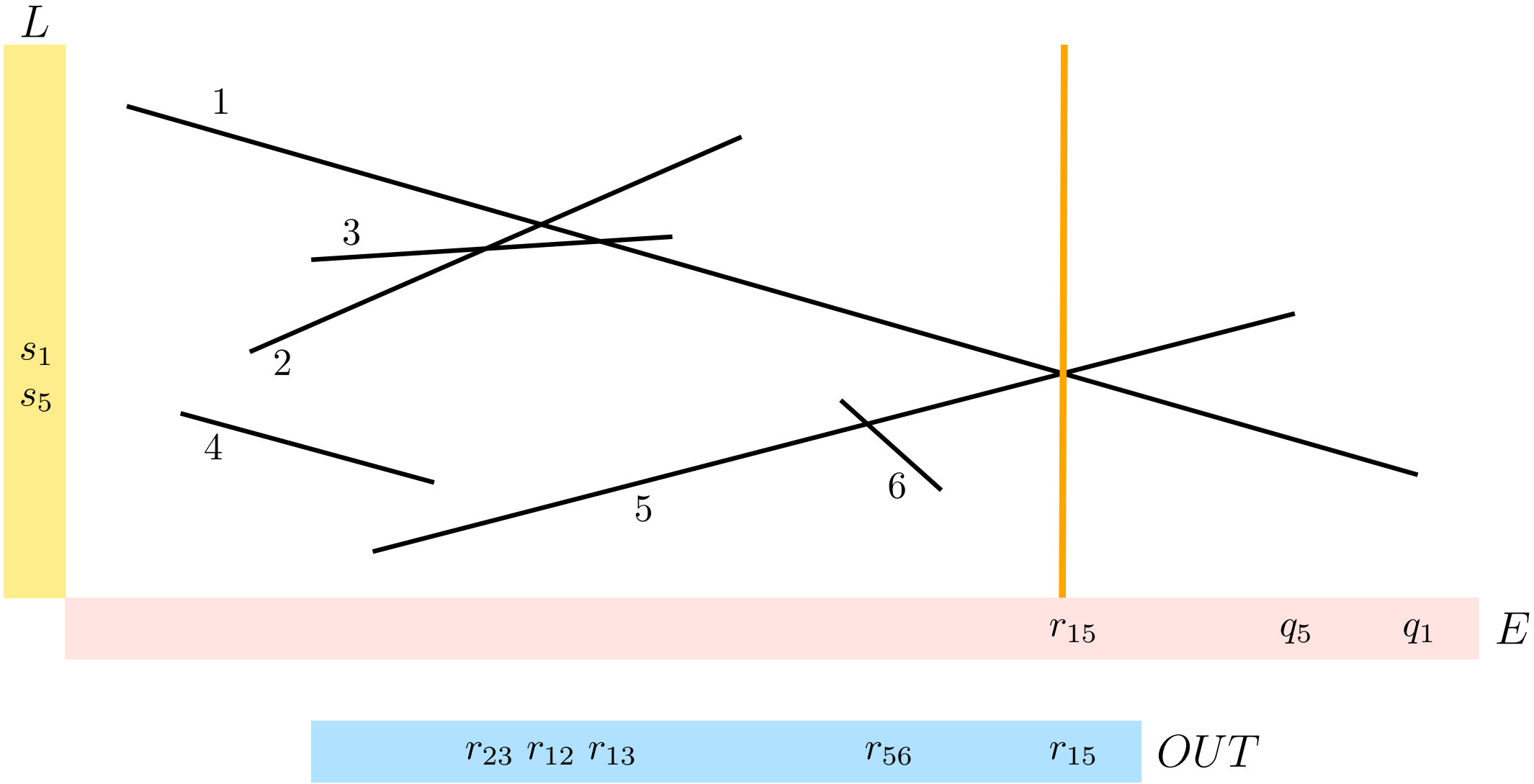

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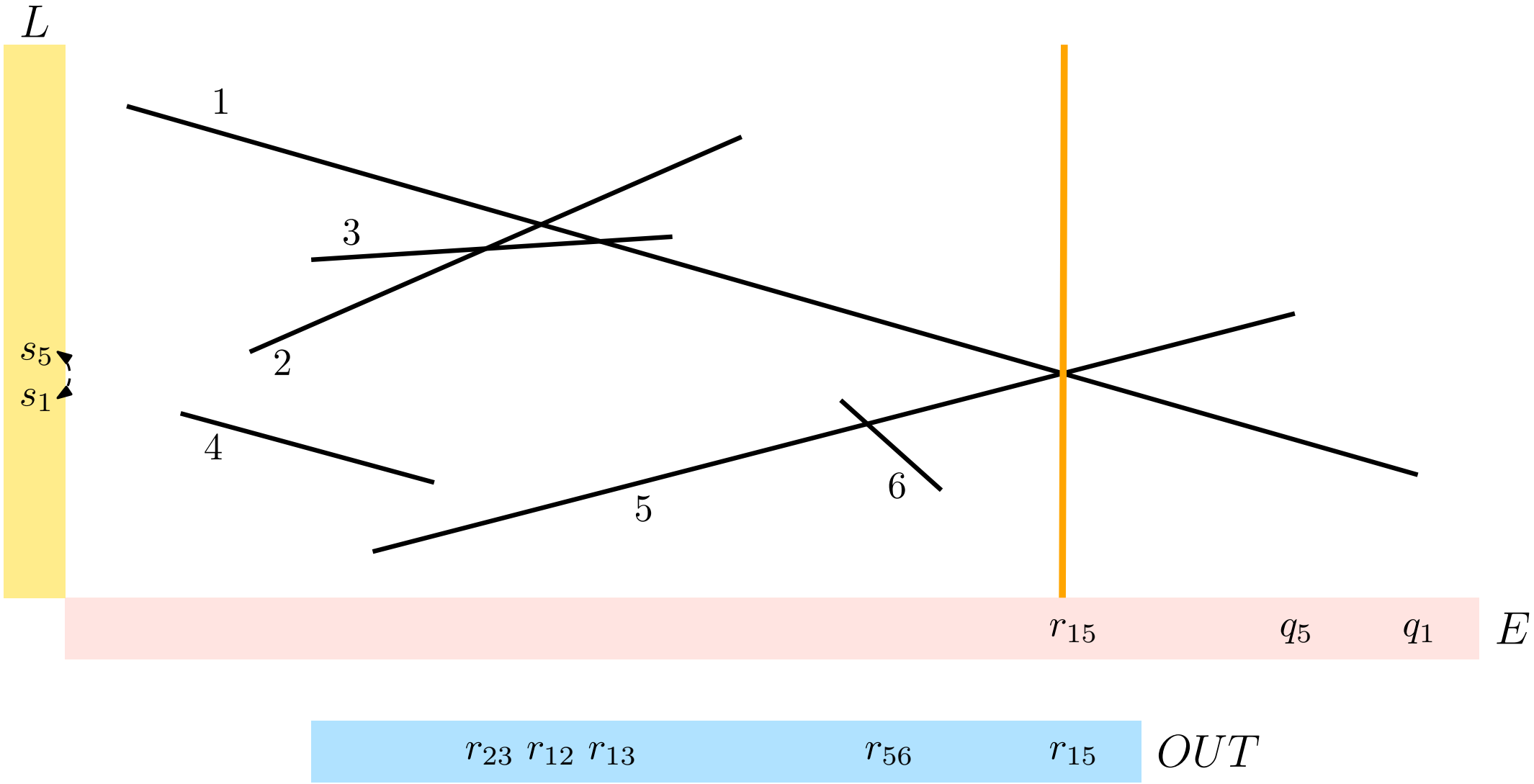

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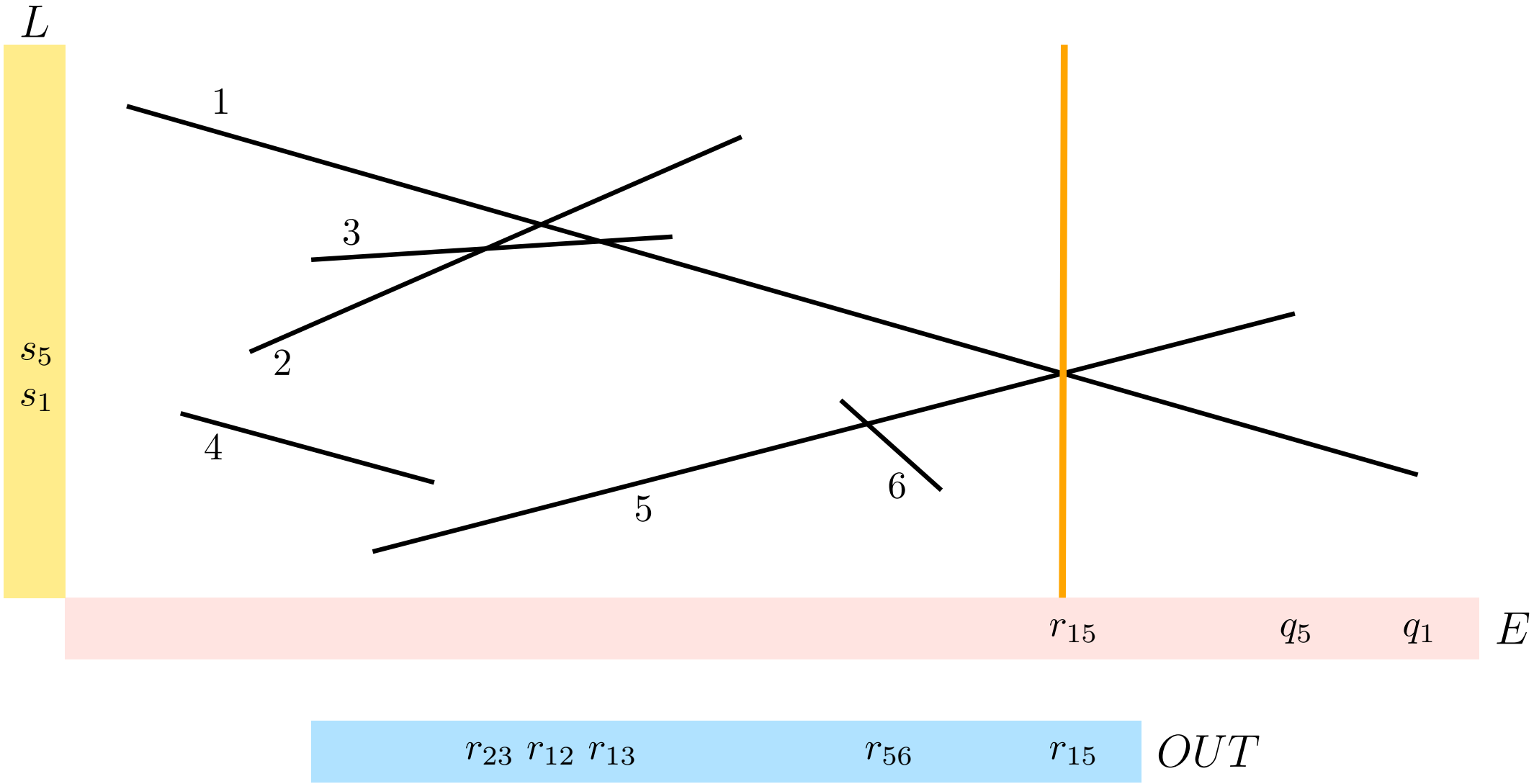

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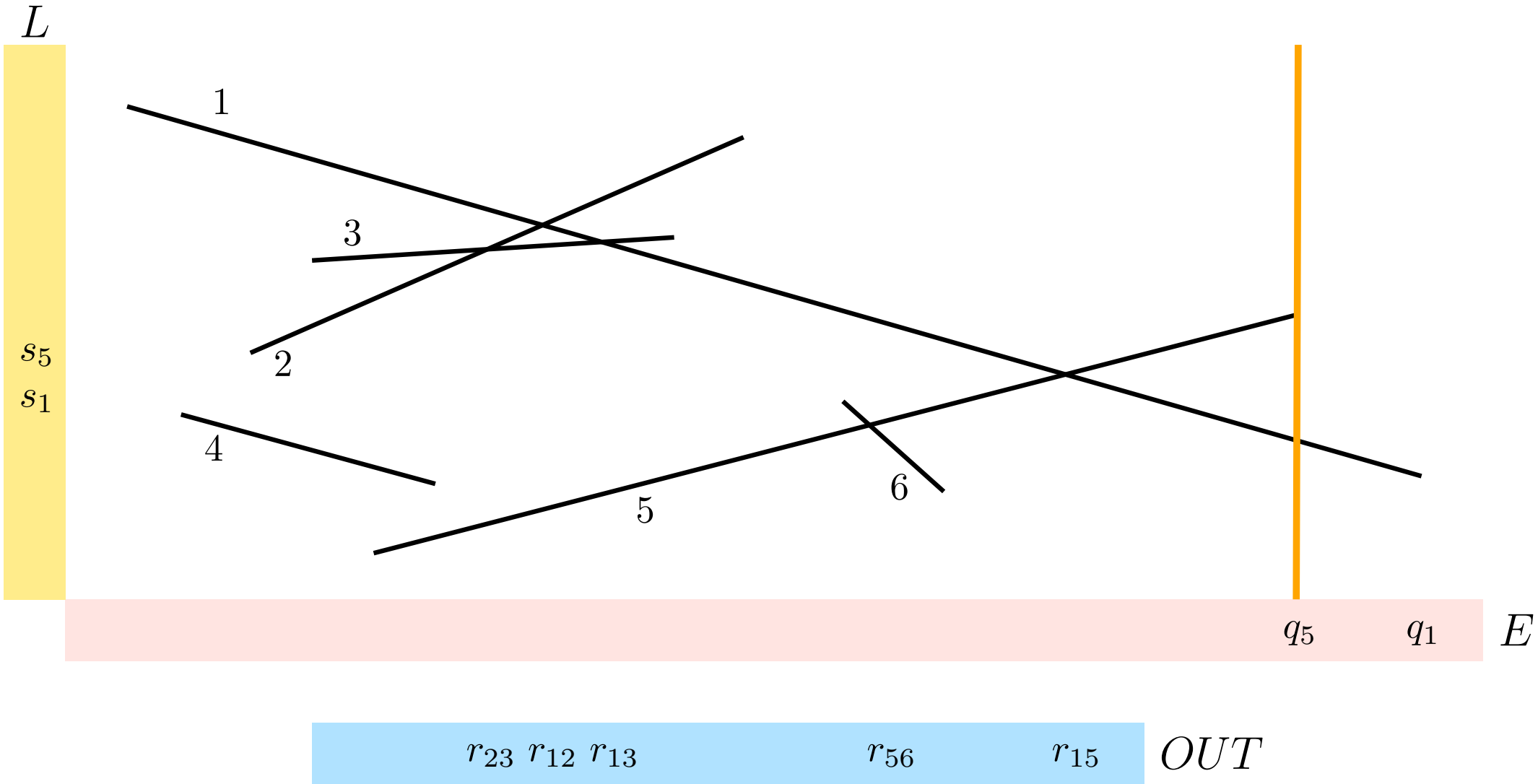

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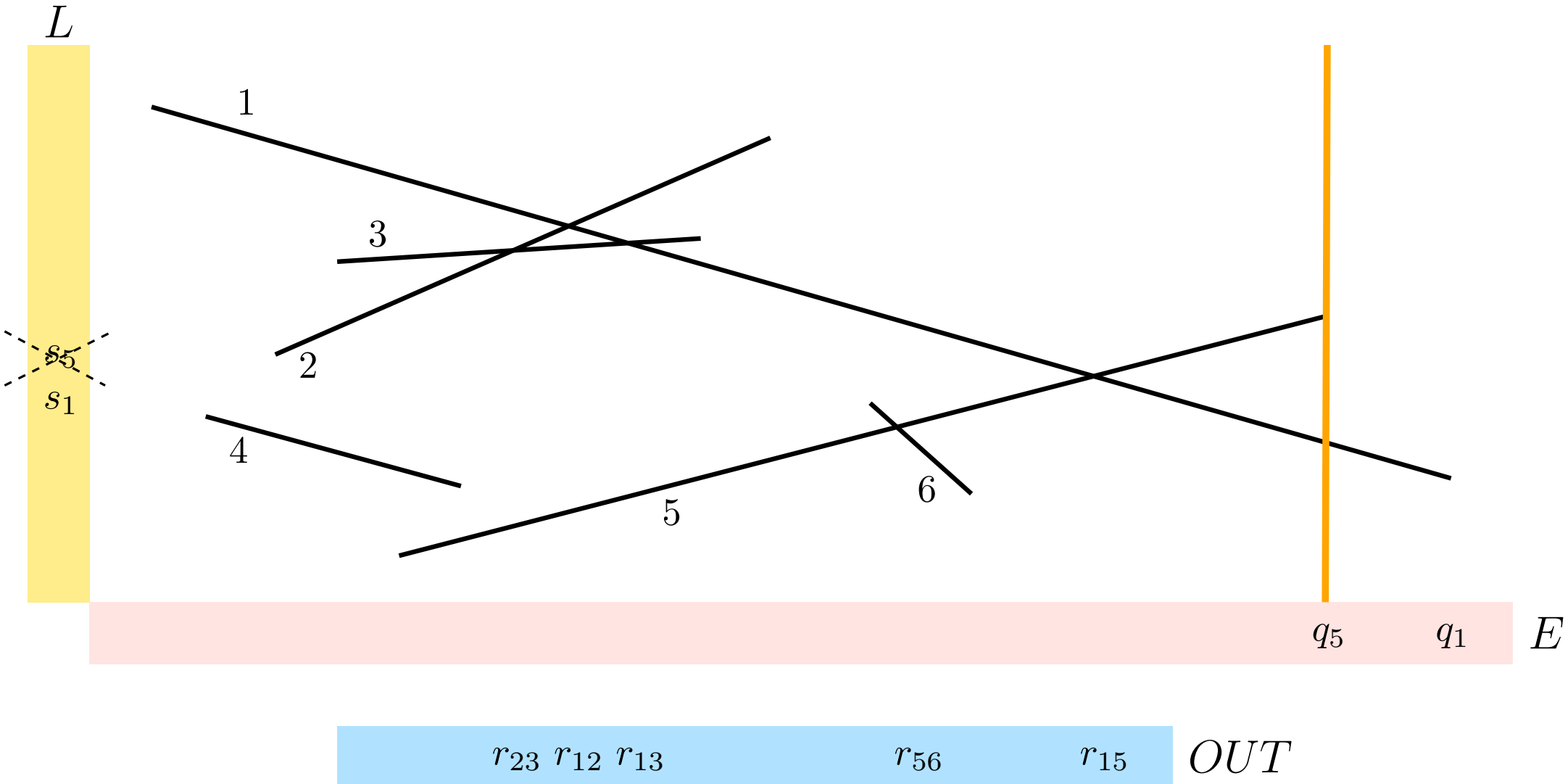

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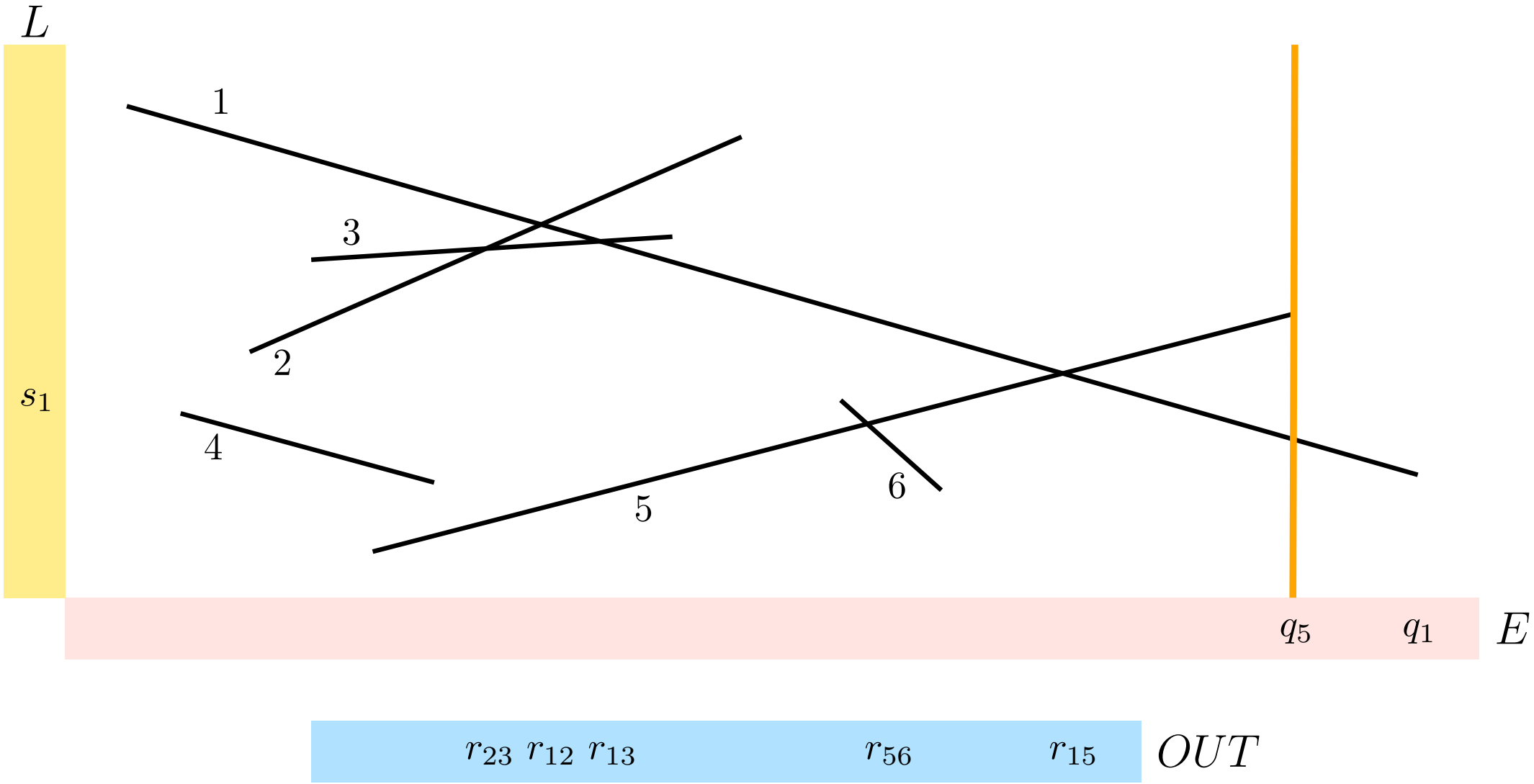

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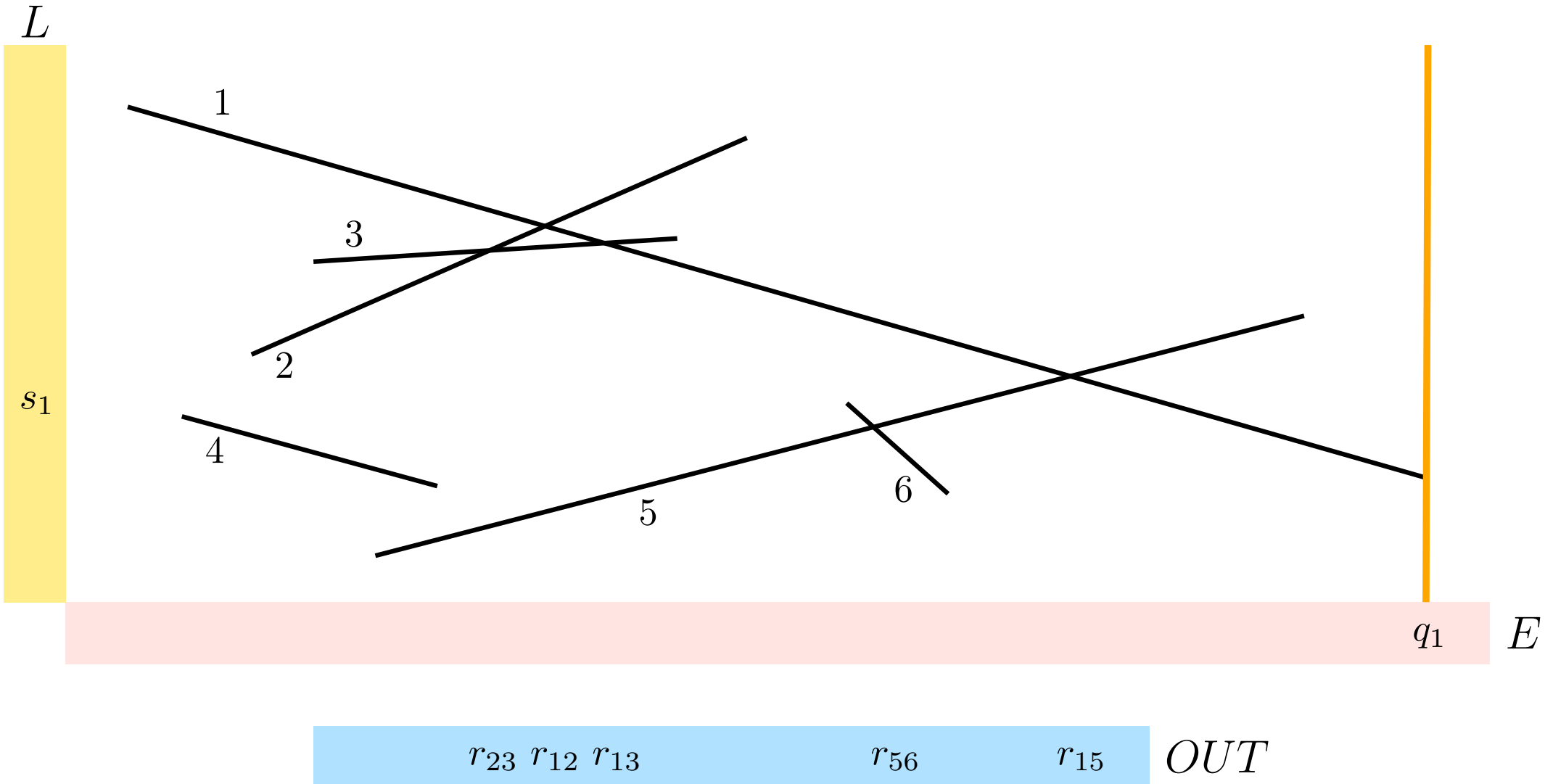

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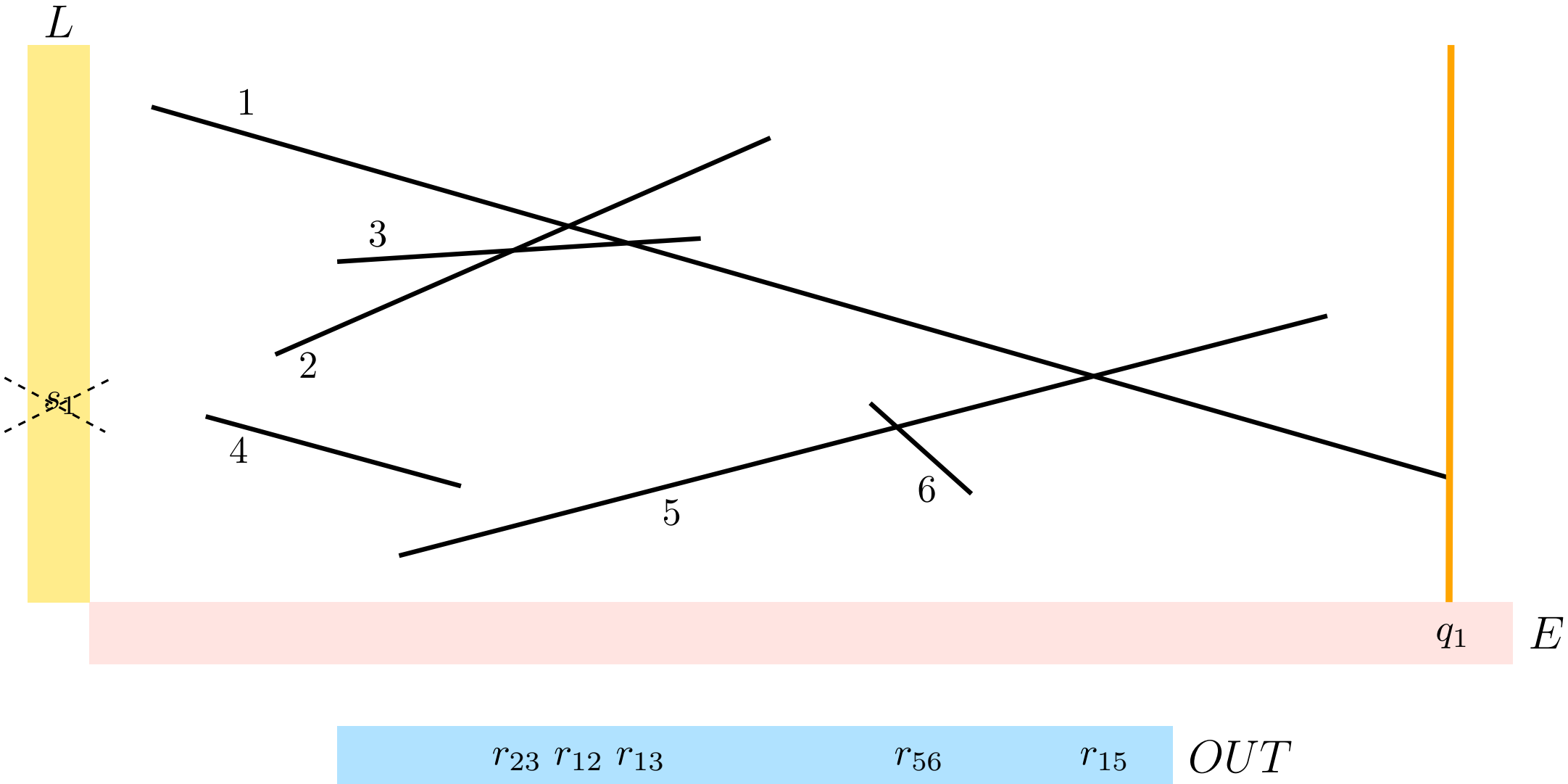

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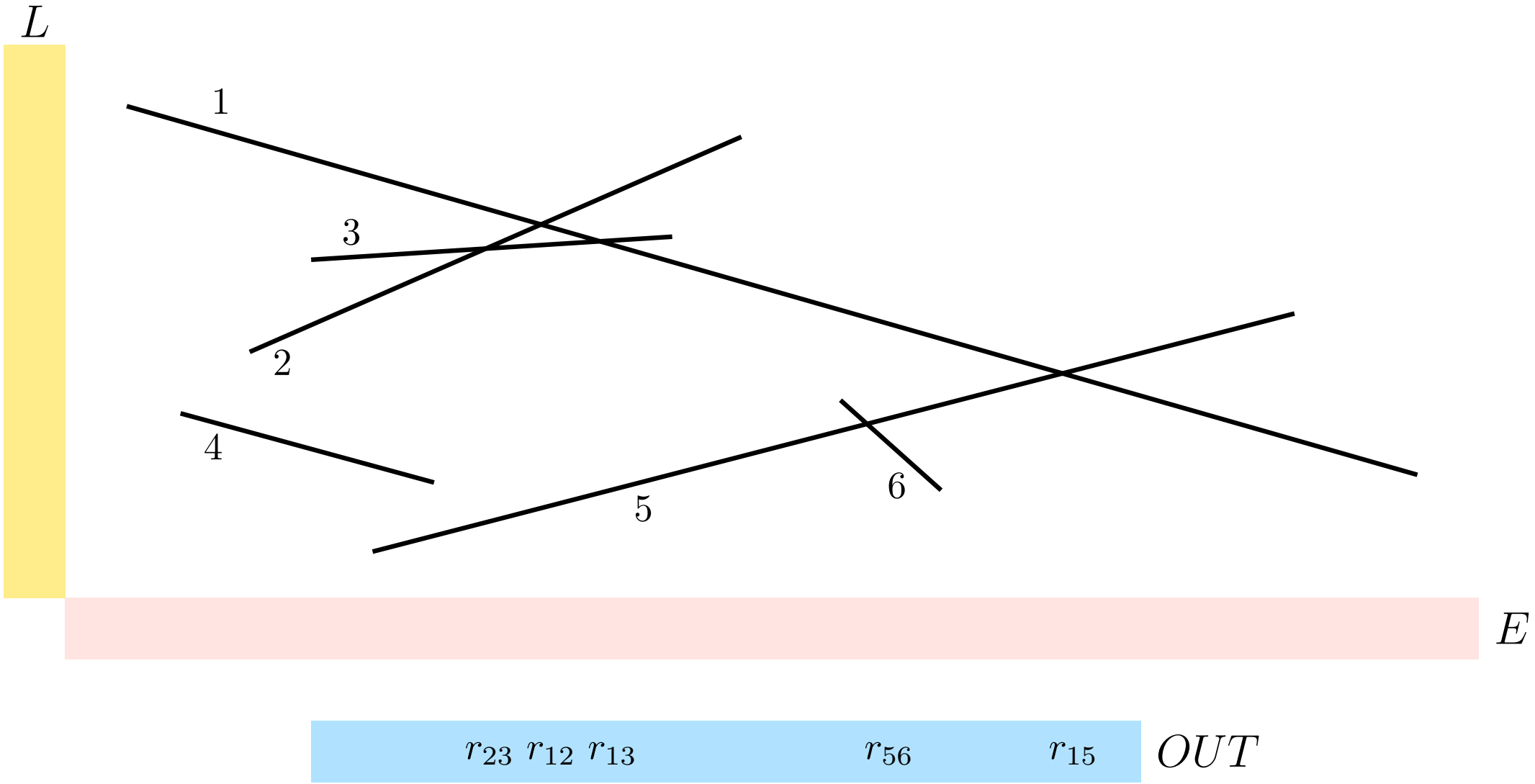

INTERSECTING LINE-SEGMENTS

Bentley-Ottman's Algorithm

Bentley-Ottman's Algorithm

Correctness

- The algorithm finds all intersections (due to Observation 2).
- The algorithm does not find any faked intersection (all intersections are checked).

Bentley-Ottman's Algorithm

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Dealing with degenerate cases

- In order to deal with input data containing more than one point sharing the same abscissa, the event queue E must store the points in lexicographical order (and not only by abscissae).
- The algorithm can trivially detect whether two or more line-segments intersect in more than one point (i.e., intersect in a line-segment), since it stops at their endpoints.
- A slight modification also allows to deal with input data in which three or more line-segments intersect at the same point: in this case, the algorithm inverts their order in the sweep line at the intersection point event.

Bentley-Ottman's Algorithm

Data structures

Bentley-Ottman's Algorithm

Data structures

Sweep line, L:

Keeps the total order of the stabbed line-segments and supports:

- insert(s)
- delete(s)
- transpose(s_1, s_2)
- previous(s)
- next(s)

A dictionary (balanced binary tree) allows to perform each of these operations in $O(\log n)$ time.

Bentley-Ottman's Algorithm

Data structures

Events queue, E:

Keeps the total order of the events and supports:

- minimum (report and extract)
- insert(p)
- memberQ(p)

A priority queue (balanced binary tree) allows to perform each of these operations in $O(\log n)$ time.

Bentley-Ottman's Algorithm

Complexity (time)

Bentley-Ottman's Algorithm

Complexity (time)

Initialization (sort endpoints): $O(n \log n)$

Advance (performed $2n + k$ times):
- Step 1 (find nex event): $O(\log n)$
- Steps 2, 3, or 4 (process event): $O(\log n)$
- Step 5 (delete event): $O(\log n)$

Overall running time: $O((n + k) \log n)$

Bentley-Ottman's Algorithm

Complexity (time)

Initialization (sort endpoints): $O(n \log n)$

Advance (performed $2n + k$ times):
- Step 1 (find nex event): $O(\log n)$
- Steps 2, 3, or 4 (process event): $O(\log n)$
- Step 5 (delete event): $O(\log n)$

Overall running time: $O((n + k) \log n)$

The previous counting corresponds to the non degenerated case.
When each intersection point, v_i, may correspond to more than two intersecting line-segments, the total running time of the advance step of the algorithm is $O((\sum_{i=1}^{k} \text{degree}(v_i)) \log n)$.
However, considering the points v_i as vertices of the graph:

$$\sum_{i=1}^{k} \text{degree}(v_i) \leq 2e = O(e) = O(v) = O(2n + k) = O(n + k).$$

Bentley-Ottman's Algorithm

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In the formulation exposed so far, the events queue may at some point store all intersection points, which are $k = O(n^2)$.

However, a slight modification allows the events queue to store, at each step of the algorithm, at most $n-1$ intersection events. This can be achieved if, at each step, E only stores intersection points of line-segments adjacent in L, and the intersection points are deleted from E whenever the intersecting segments become non adjacent.

The decision problem

INTERSECTIN LINE-SEGMENTS

The decision problem

Input: n line-segments in the plane, $s_i = (p_i, q_i),\ i = 1 \ldots n$.
Output: there is / there is not a pair of intersecting line-segments, and report a witness.

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Proof: by reduction from unicity of integers.

Given $x_1, \dots, x_n \in \mathbb{N}$, compute $p_i = (x_i, 0)$, $q_i = (x_i, 1)$ and $s_i = (p_i, q_i)$.
There exists a pair of intersecting line-segments if and only if there exist duplicate numbers in the original set.

If you don't like degeneracies, consider the following points:
$p_i = (x_i - \frac{1}{2i}, 0)$ and $q_i = (x_i + \frac{1}{2i}, 1)$.

INTERSECTING LINE-SEGMENTS

The problem of reporting all intersections

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Corollary

The problem of reporting all intersection has complexity $\Omega(k + n \log n)$, because

- Reporting requires $\Omega(k)$ time
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An algorithm by Chazelle and Edelsbrunner solves this problem in $\Theta(k + n \log n)$ time.

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Consequences

- Deciding whether a polygon is simple can be solved in $O(n \log n)$ time.
- Deciding whether two simple polygons intersect can be solved in $O(n \log n)$ time.

FURTHER READING

- F. Preparata, M. Shamos, *Computational Geometry: An introduction*, Springer.
- M. de Berg, O. Cheong, M. van Kreveld, M. Overmars: *Computational Geometry: Alhorithms and Applications*, Springer.
- J-D. Boissonnat, M. Yvinec, *Algorithmic Geometry*, Cambridge University Press.