Describing and implementing basic geometric objects

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Basic geometric objects

- Points: $p = (x, y) \in \mathbb{R}^2$ and $(x, y, z) \in \mathbb{R}^3$
- Lines and line segments: p_1, p_2
- Halflines: p_1, p_2 or p, $\overrightarrow{v}$
- Polygonal lines and polygons: $p_1, p_2, \ldots, p_n$
- Circles and discs: p_1, p_2, p_3 non collinear
- Balls and spheres: p_1, p_2, p_3, p_4 non coplanar

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- Planar graphs such as planar decompositions and polyhedra:
  - For each vertex v:
    - Its coordinates (x, y, z)
    - An edge $e(v)$ incident to it
  - For each face f:
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    - Its starting and ending vertices $v_S(e), v_E(e)$
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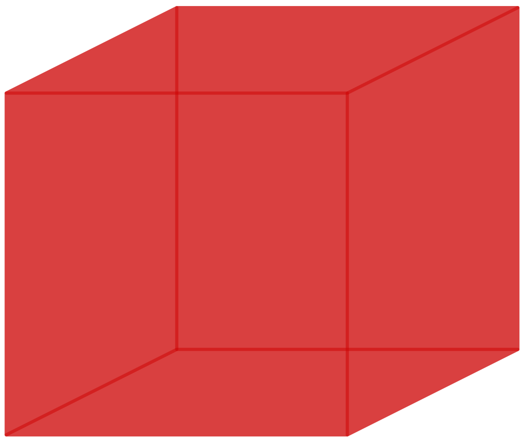

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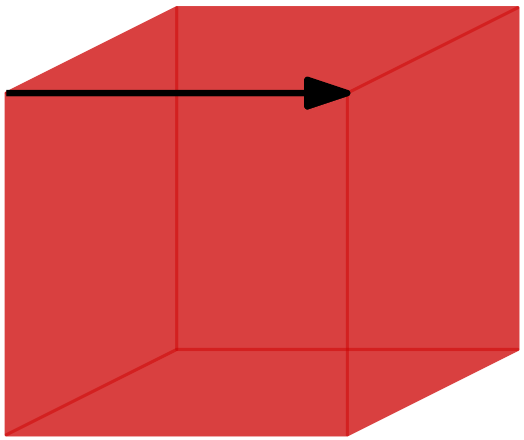

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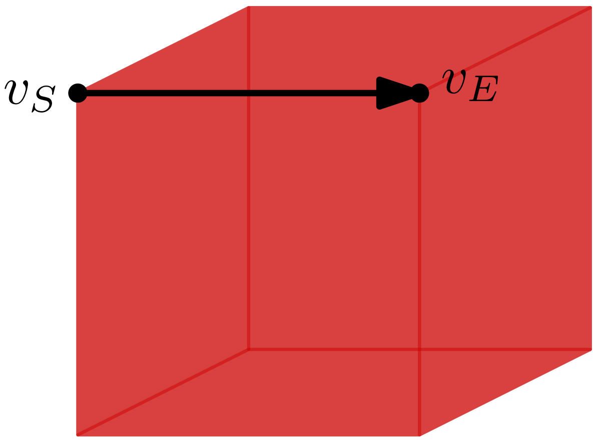

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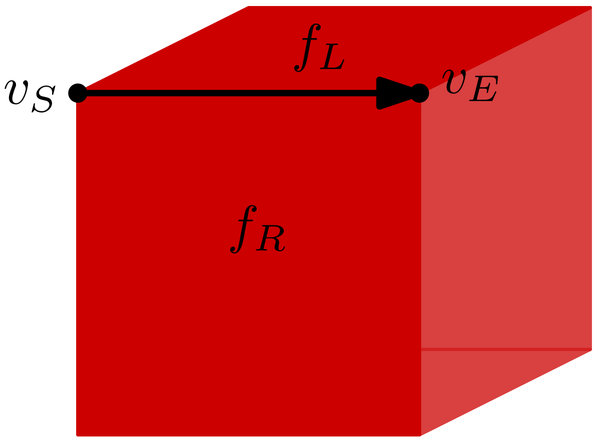

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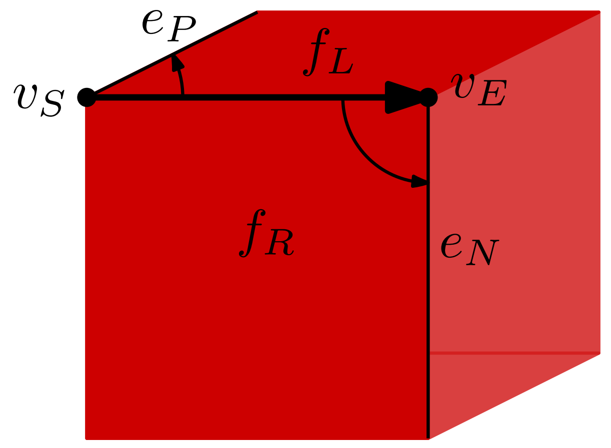

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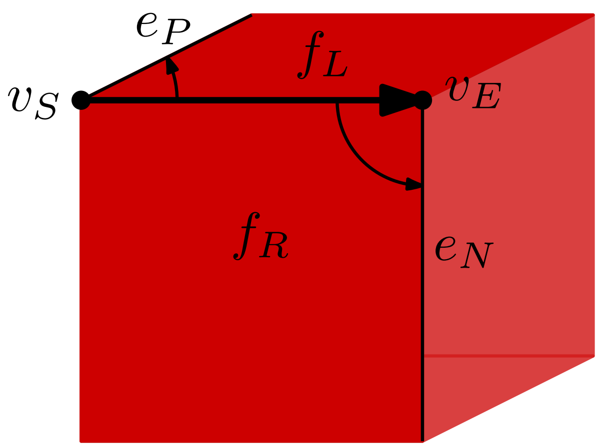
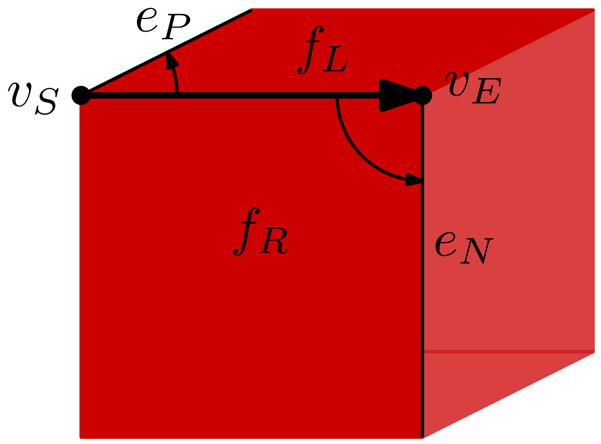


Why do we use a DCEL?

- Time:
  - It allows to obtain the sorted lists of vertices and edges of any face in time proportional to the size of the face.
  - It allows to obtain the sorted lists of edges and faces indicent to any vertex in time proportional to the degree of the vertex.



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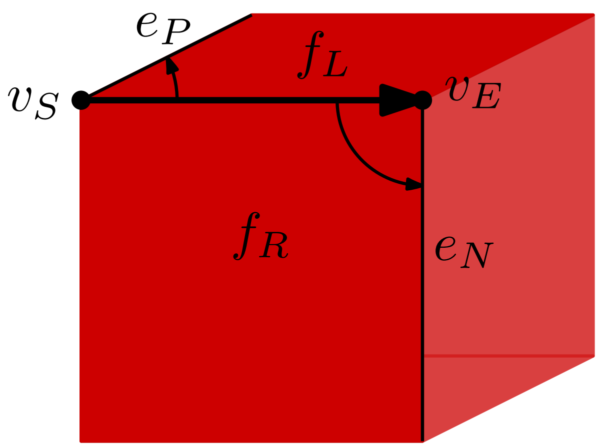

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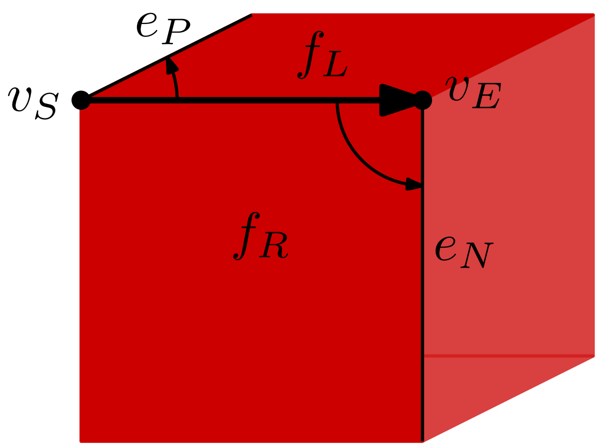

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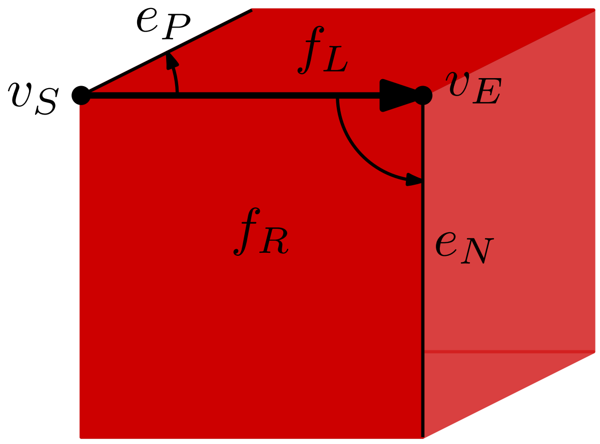

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Proof: Since the graph is planar, $V + F = E + 2$.
Since the faces are n-gons (with $n \geq 3$), $2E \geq 3F$.

Therefore:

$$2V + 2F = 2E + 4 \geq 3F + 4 \Longrightarrow F \leq 2V - 4.$$

$$3E + 6 = 3V + 3F \leq 3V + 2E \Longrightarrow E \leq 3V - 6.$$

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$\mathbb{Q}$ is $\mathbb{Z} \times \mathbb{Z}/\sim$, where $(a, b) \sim (a', b') \iff ab' = a'b$.

Algebraic numbers in $\mathbb{R}$ can be symbollically treated, but trascendental cannot.

$\mathbb{C}$ can be identified with $\mathbb{R}^2$.

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DESCRIBING AND IMPLEMENTING BASIC GEOMETRIC OBJECTS

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Recommended libraries

The GNU multiple-precision arithmetic library: https://gmplib.org/

The GNU multiple-precision floating-point library: http://www.mpfr.org/

The Computational Geometry Algorithms Library: http://www.cgal.org/

Homogeneous coordinates

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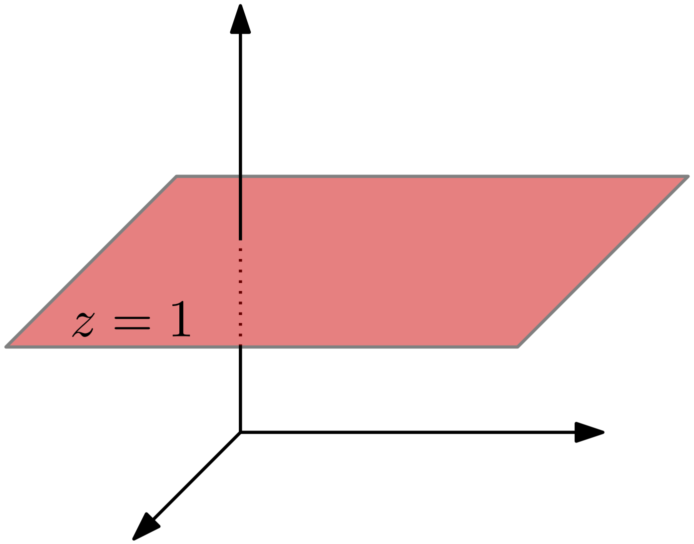
Embed the plane in 3-dimensional space as plane $z = 1$.



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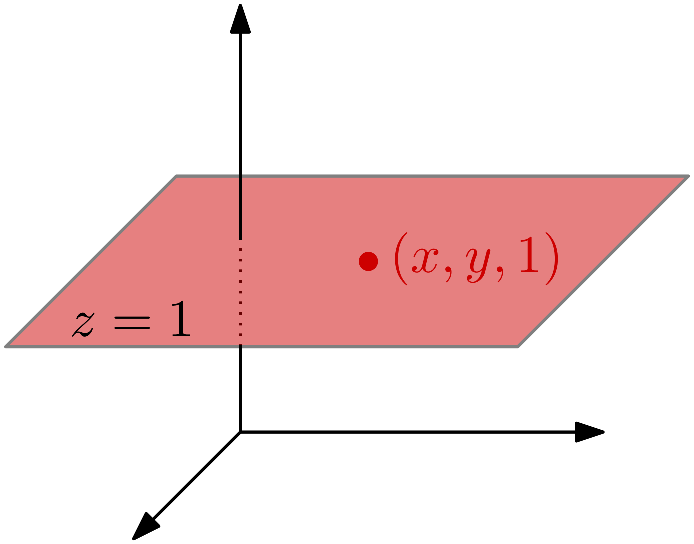

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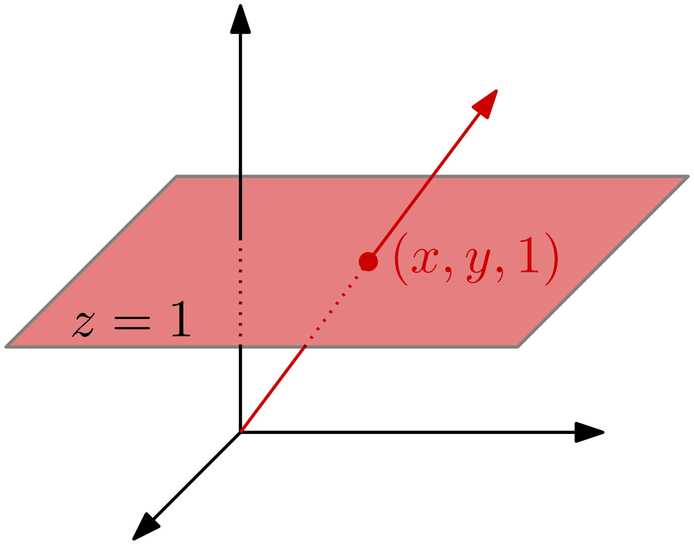
Then identify point $(x, y, 1)$ with all the points in the ray connecting it to the origin:

$(x, y) \sim (x, y, 1) \sim (\lambda x : \lambda y : \lambda)$, where $\lambda > 0$.



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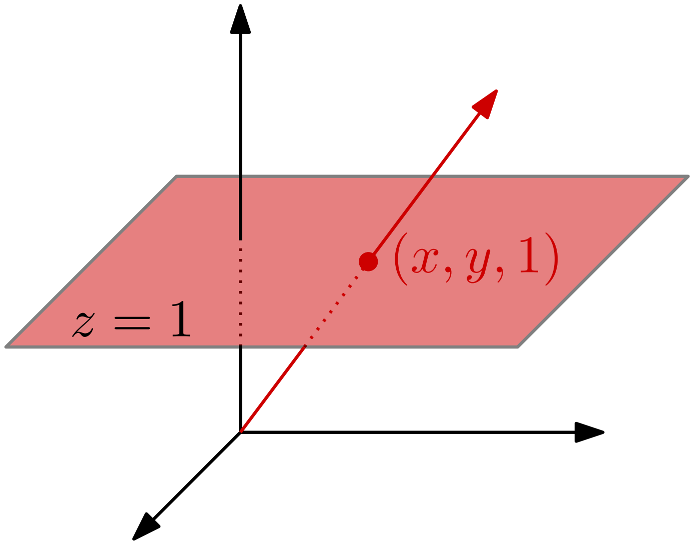

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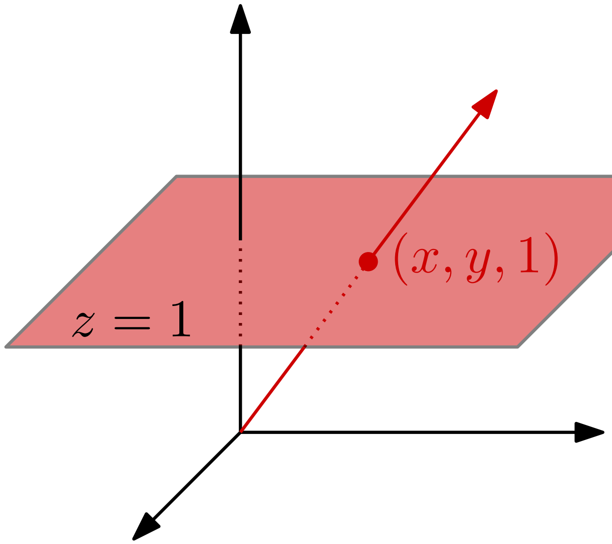

What about vectors?

- Vectors have coordinates $(x : y : 0)$
  because they are parallel to the plane $z = 1$.
- They correspond to horizontal rays, i.e., to points at infinity.

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$$\iff p \perp \ell$$

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Intersecting two lines

$$\left.\begin{array}{l} p \text{ lies in } \ell_1 \Longrightarrow p \perp \ell_1 \\ p \text{ lies in } \ell_2 \Longrightarrow p \perp \ell_2 \end{array}\right\} \Longrightarrow p = \ell_1 \times \ell_2$$

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FURTHER READING

Read about CGAL!

- At this cours web page (topic 13)
- At CGAL web page