TRIANGULATING POINT SETS

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DEFINITION

A triangulation of A set P of n points in the plane is a graph having P as set of vertices which is rectilinear, planar, and maximal in the number of edges.

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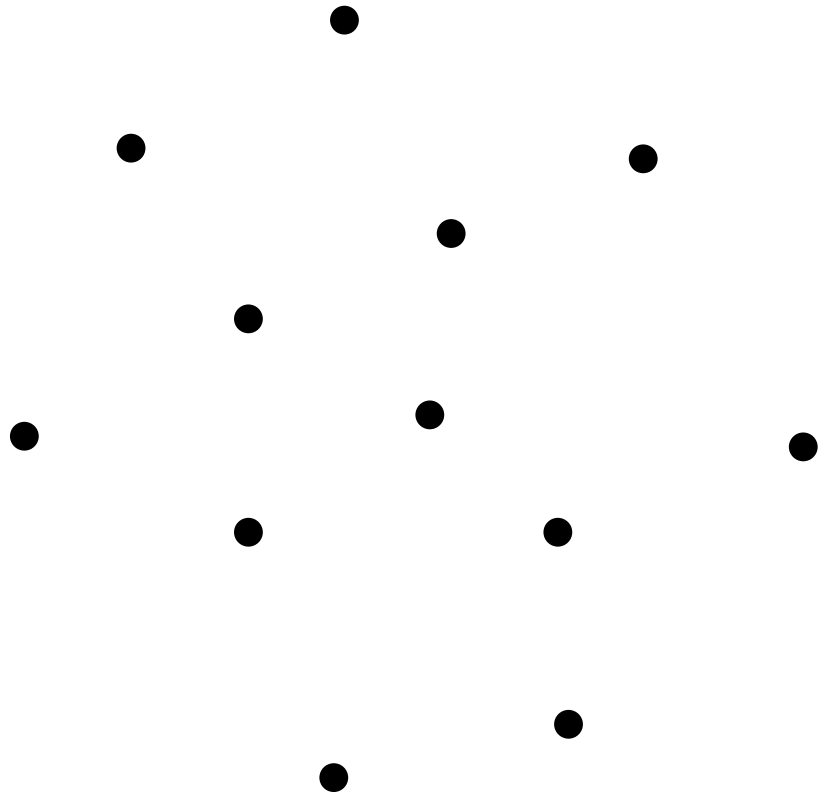

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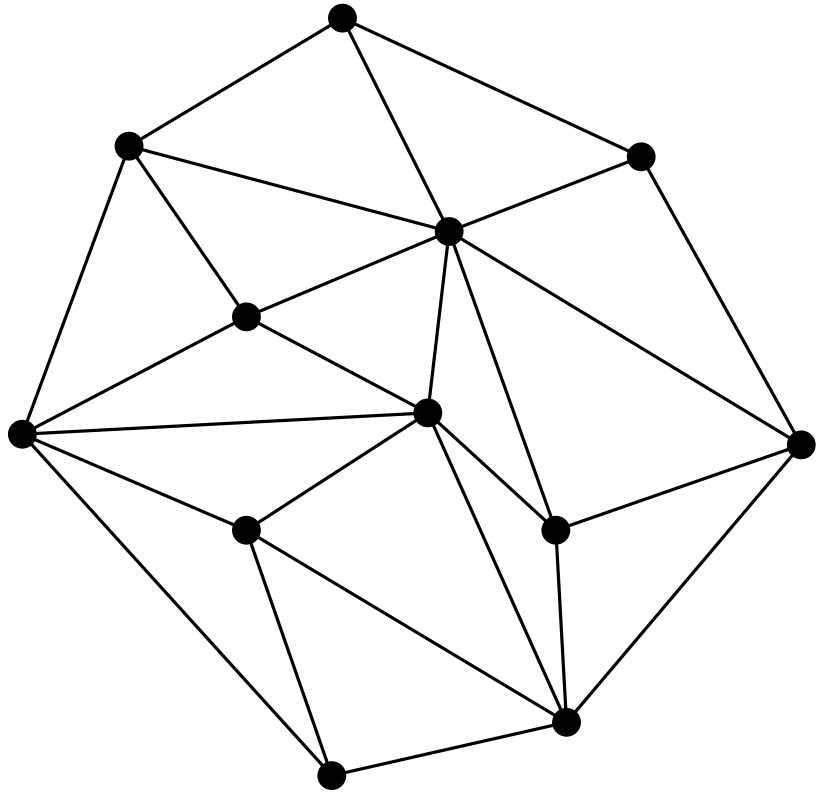

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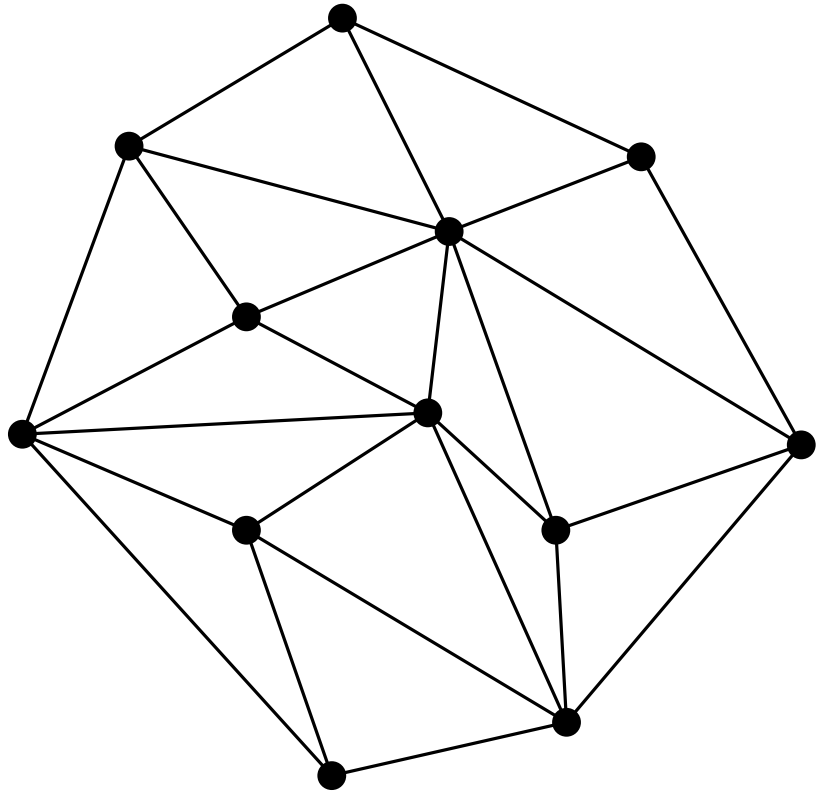
Corollary. All the faces of such a graph are triangles, except for the unbounded one, which is the exterior of the convex hull of P.



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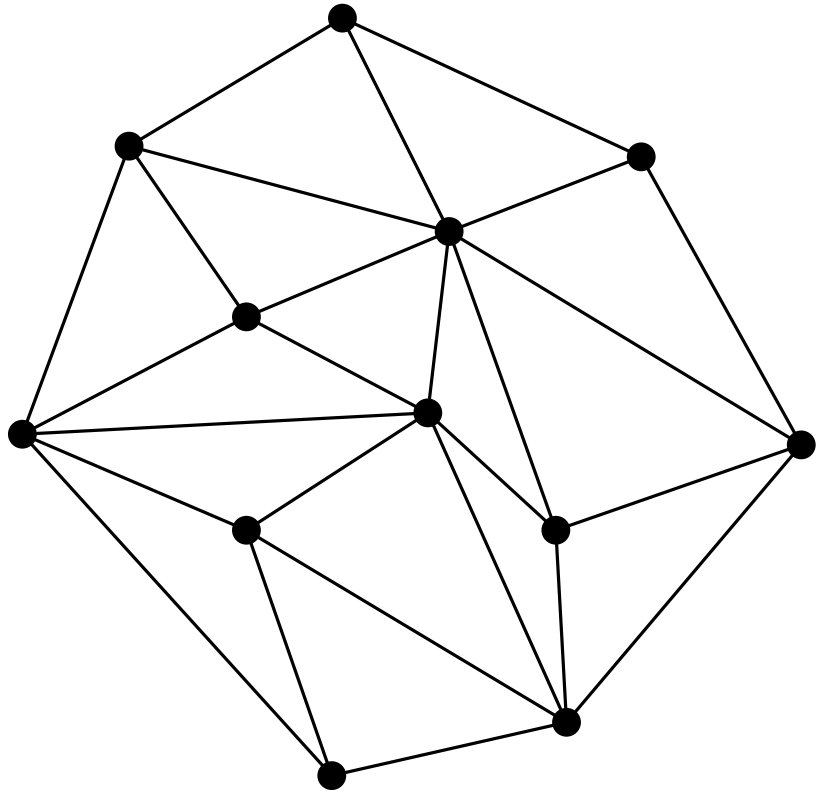
COMPLEXITY

Every triangulation of any set P of n points has:

$2n - h - 2$ triangles

$3n - h - 3$ edges

where h is the number of vertices of $ch(P)$.



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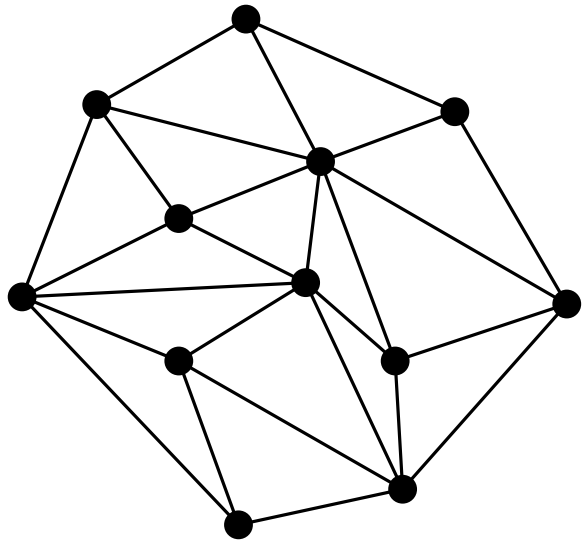

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Proof. Each triangle has exactly 3 edges. Each internal edge belongs to exactly 2 triangles. Each external edge belongs to exactly 1 triangle. Therefore, $3t = 2(e - h) + h = 2e - h$.
According to Euler's formula: $n + (t + 1) = v + f = e + 2$.
Combining both equations:

$$e = n + t - 1 \Rightarrow 3e = 3n + 3t - 3 = 3n + 2e - h - 3 \Rightarrow e = 3n - h - 3$$
$$3t = 2e - h = 6n - 2h - 6 - h = 6n - 3h - 6 \Rightarrow t = 2n - h - 2$$

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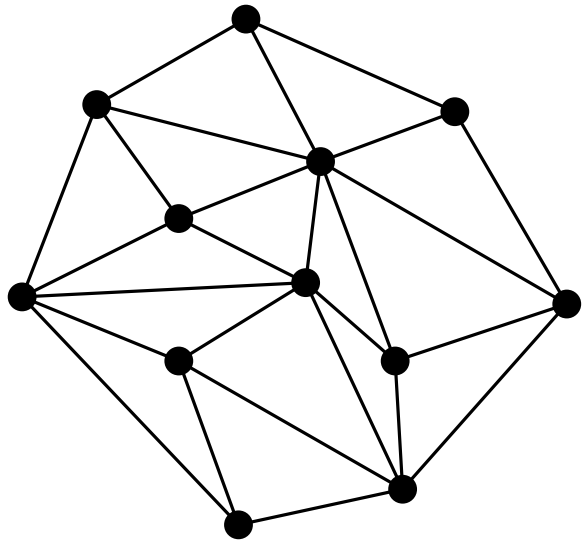

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DEGENERACIES

As you may have noticed, we are assuming that the set P does not contain three or more points on a line. The assumption is hold along the entire presentation.

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DATA STRUCTURE

We want to answer the most usual questions for any decomposition of the plane:

- For any given triangle, report its edges/vertices.
- For any given vertex, report the sorted list of edges/triangles incident to it.
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Advantage: small memory usage.

Disadvantage: it suffices to draw the diagram, but it does not contain the proximity information. For example, given a site p_i, finding its neighbors or reporting the vertices and edges of its Voronoi region is too expensive.

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Advantage: allows to quickly recover neighborhood information.

Disadvantage: the stored data is redundant and it uses more space than required.

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For each site p_i, storing the sorted list of vertices and edges of its Voronoi region, as well as the sorted list of its neighbors, etc.

The data structure which is most frequently used to store triangulations is the DCEL (doubly connected edge list).

The DCEL is also used to store plane partitions, polyhedra, Voronoi diagrams, etc.

DCEL

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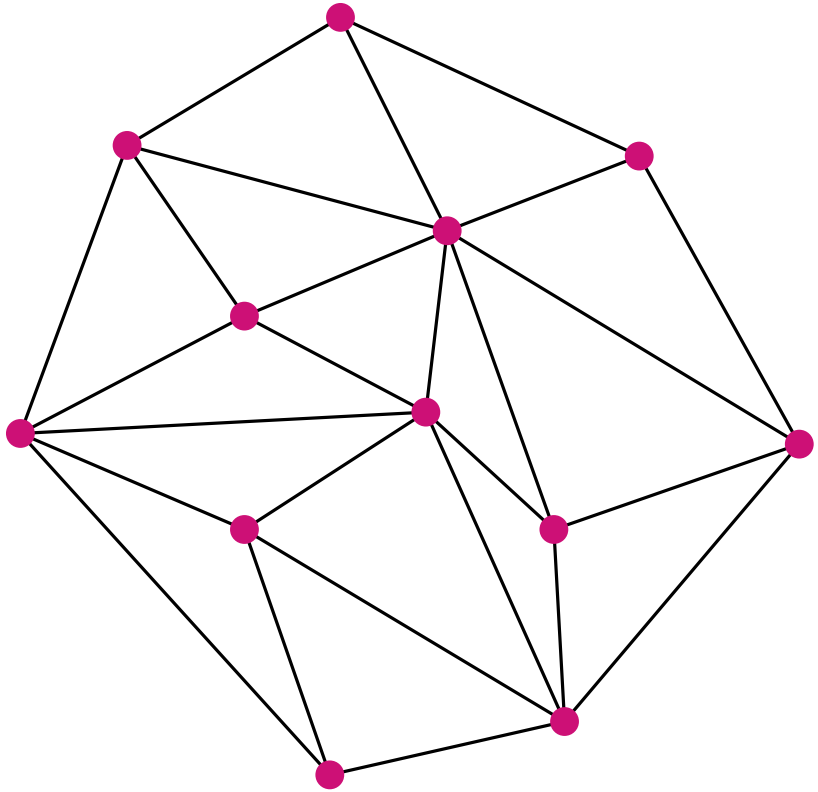

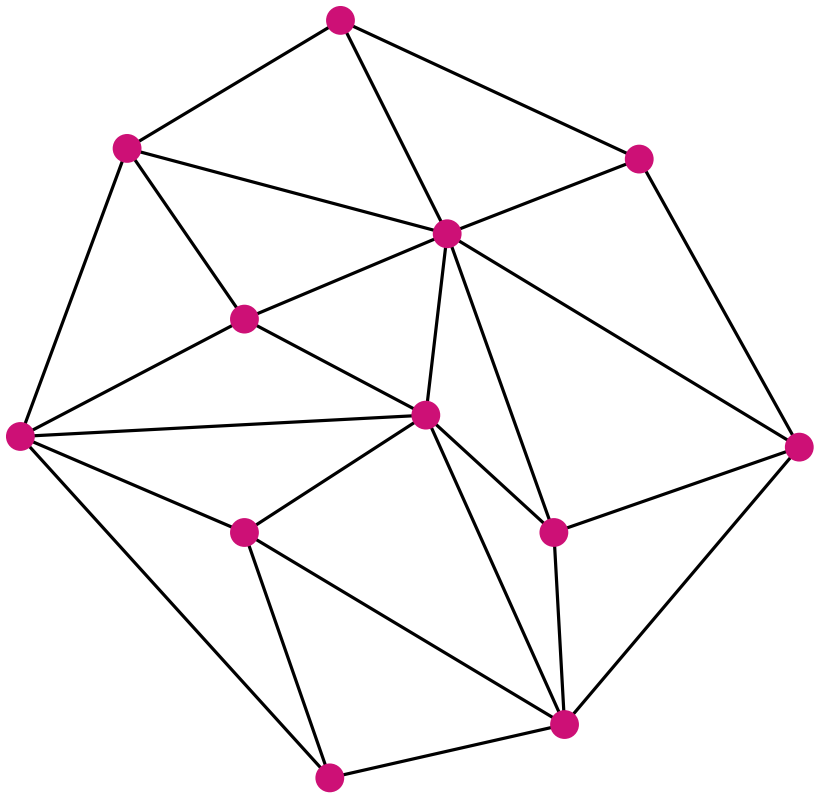

DCEL

Table of vertices

v	x	y	e

Table of faces

t	e

DCEL

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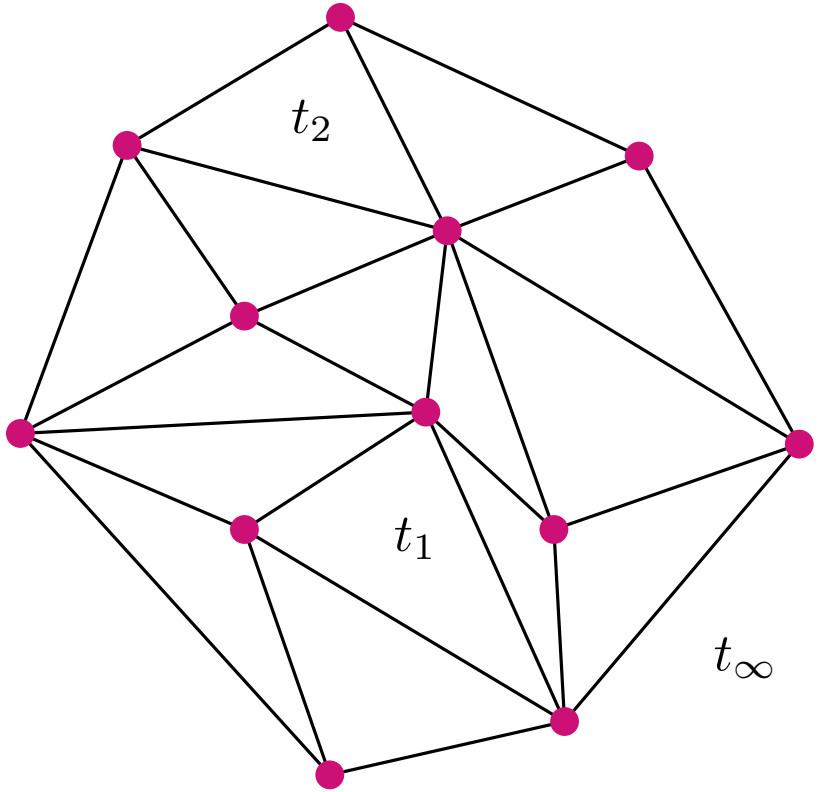

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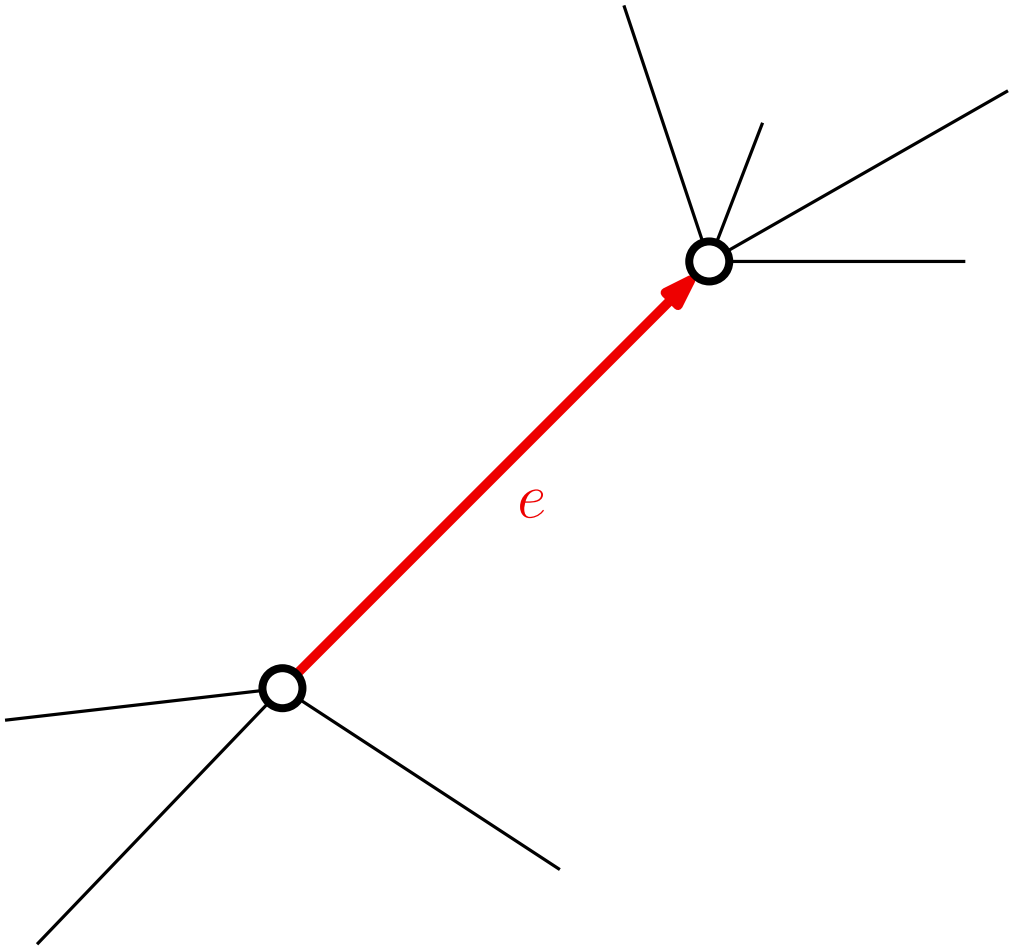

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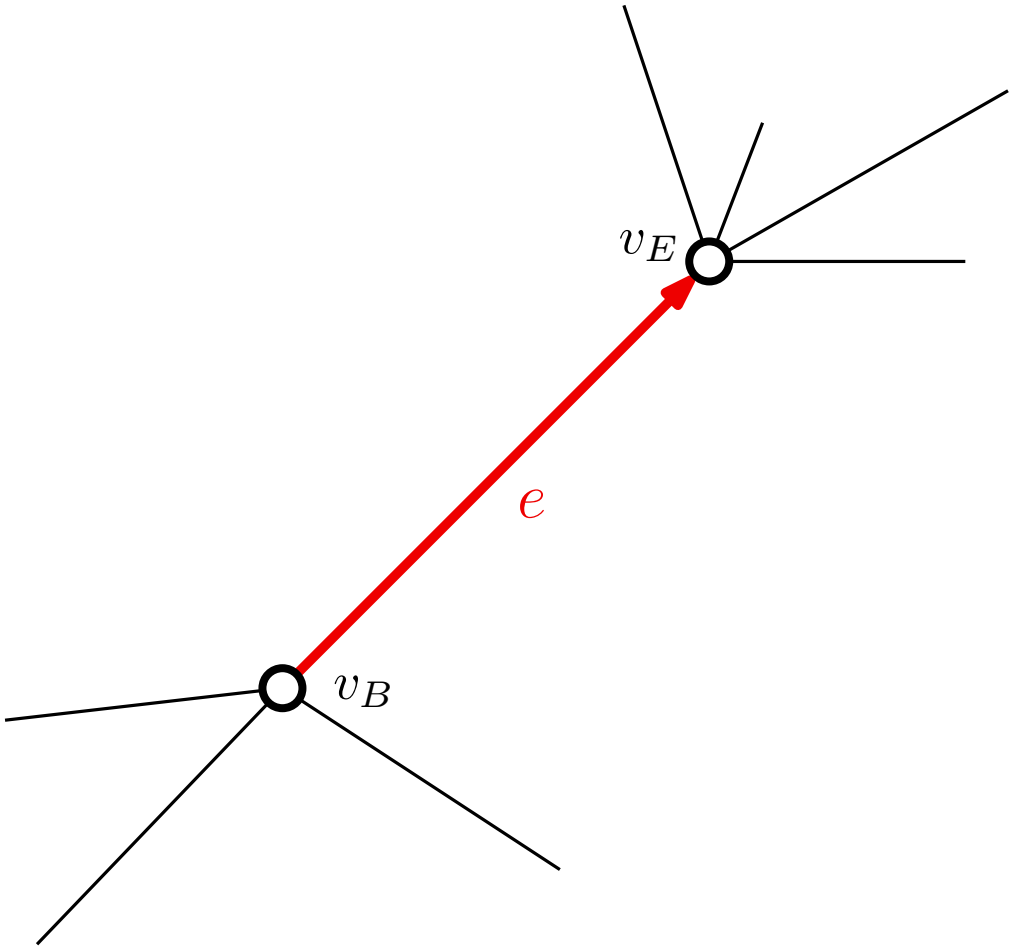

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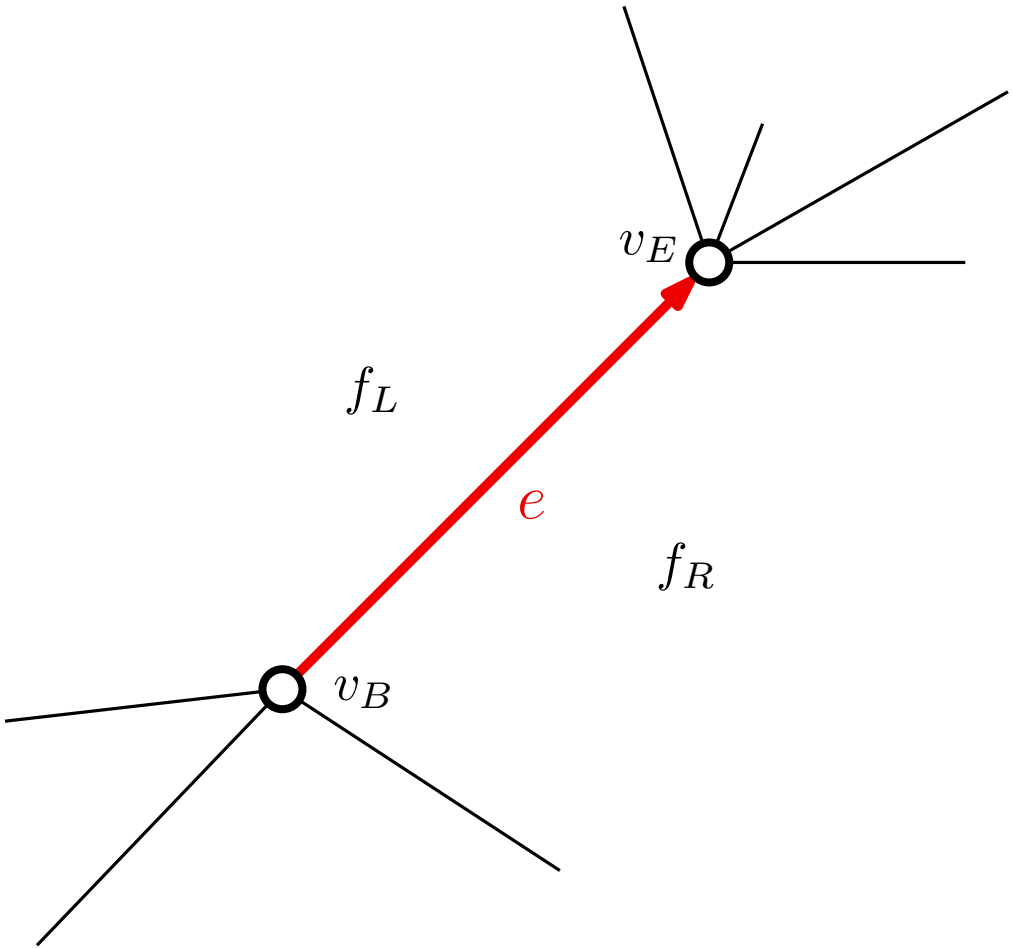

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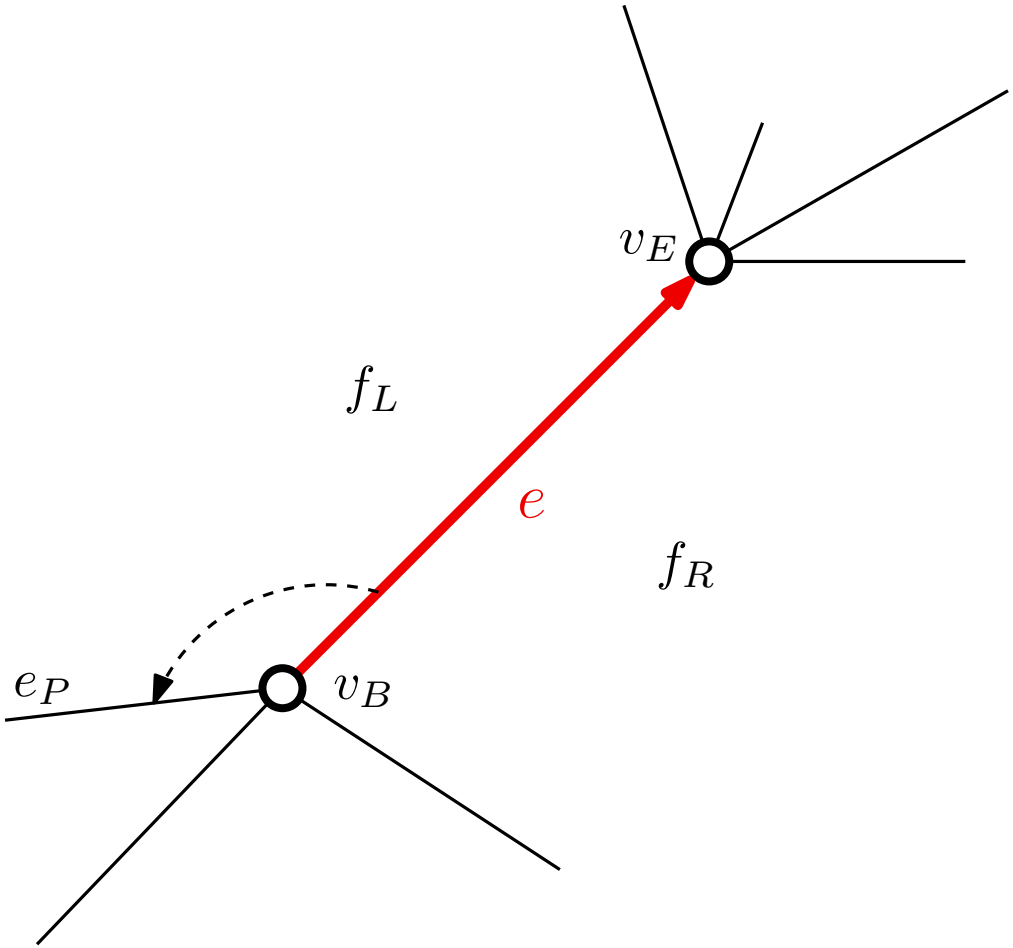

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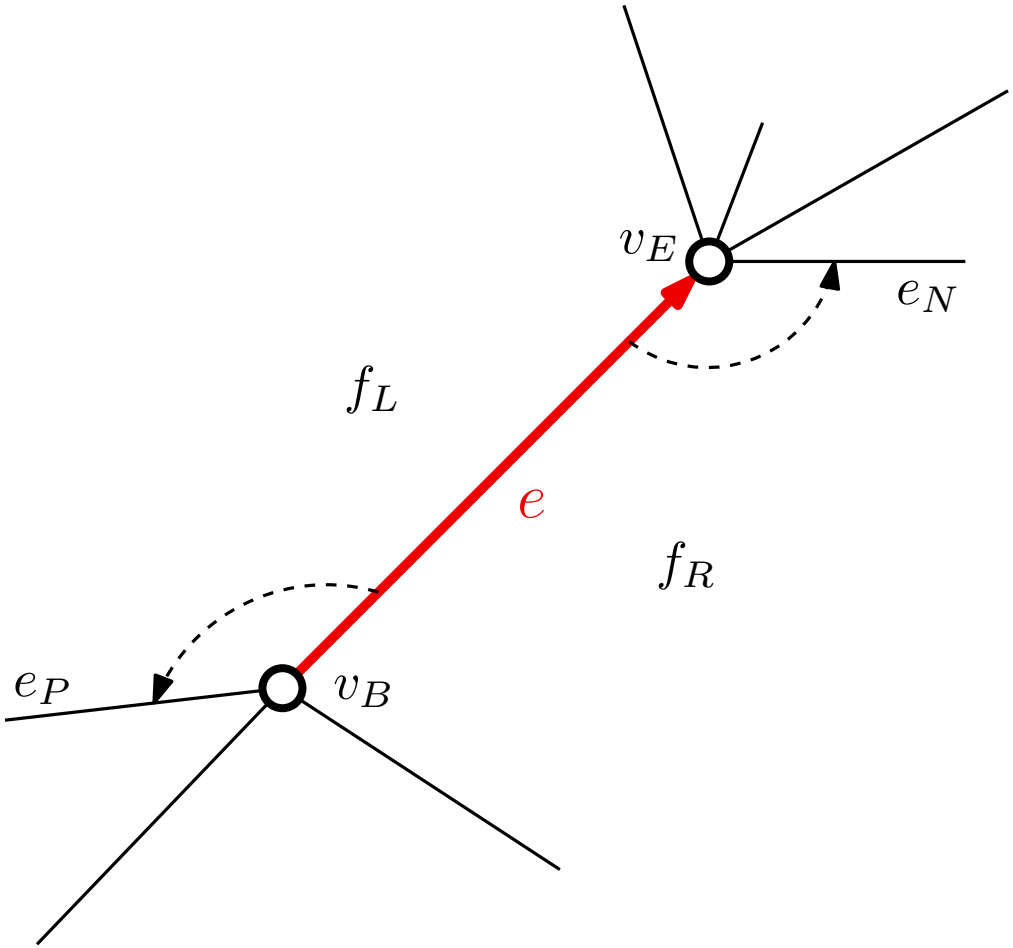

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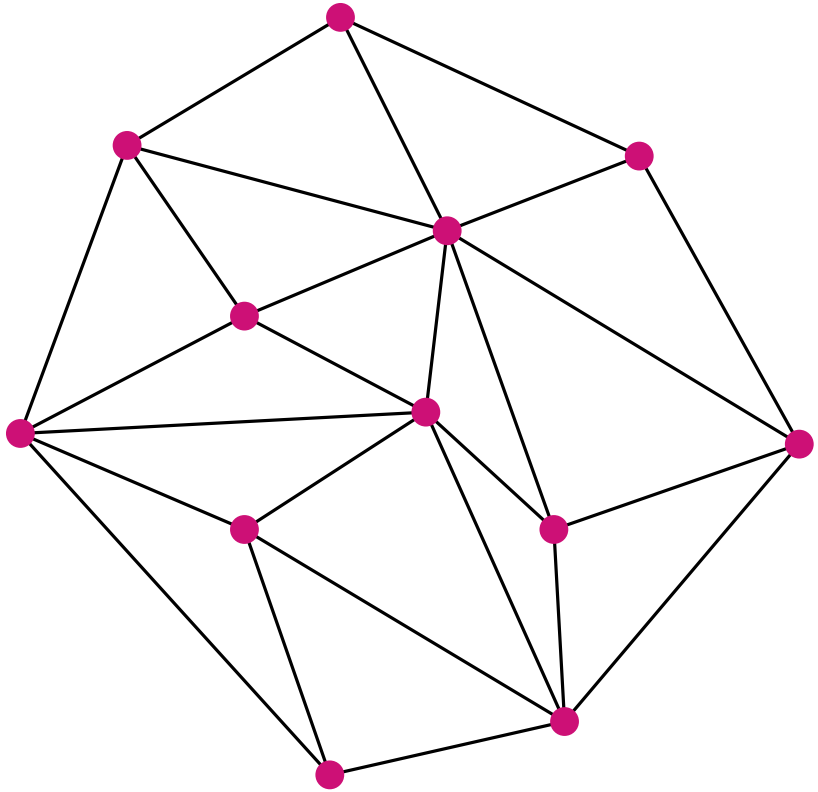

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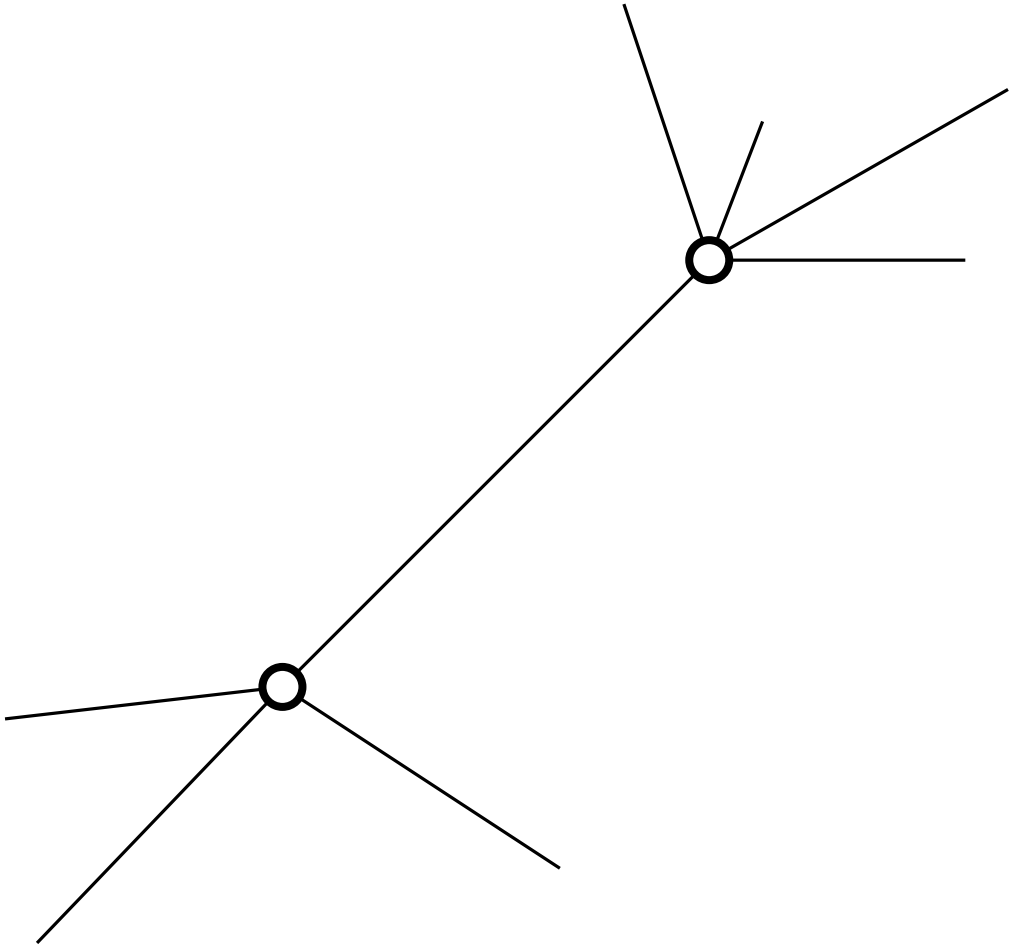
Storage space

- For each face: 2 coordinates + 1 pointer.
- For each vertex: 2 coordinates + 1 pointer + 1 bit.
- For each edge: 6 pointers.

In total, the storage space is $O(n)$.

DCEL

There are other DCEL variants, as for example:



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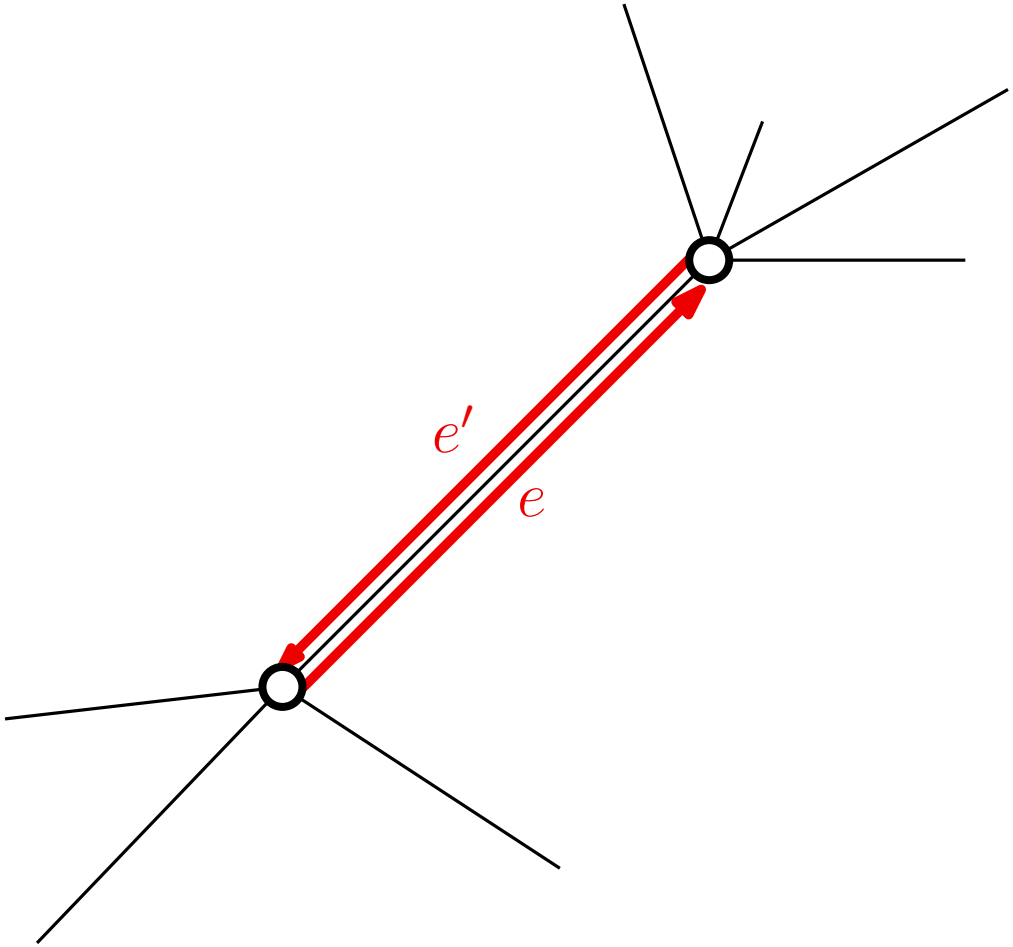

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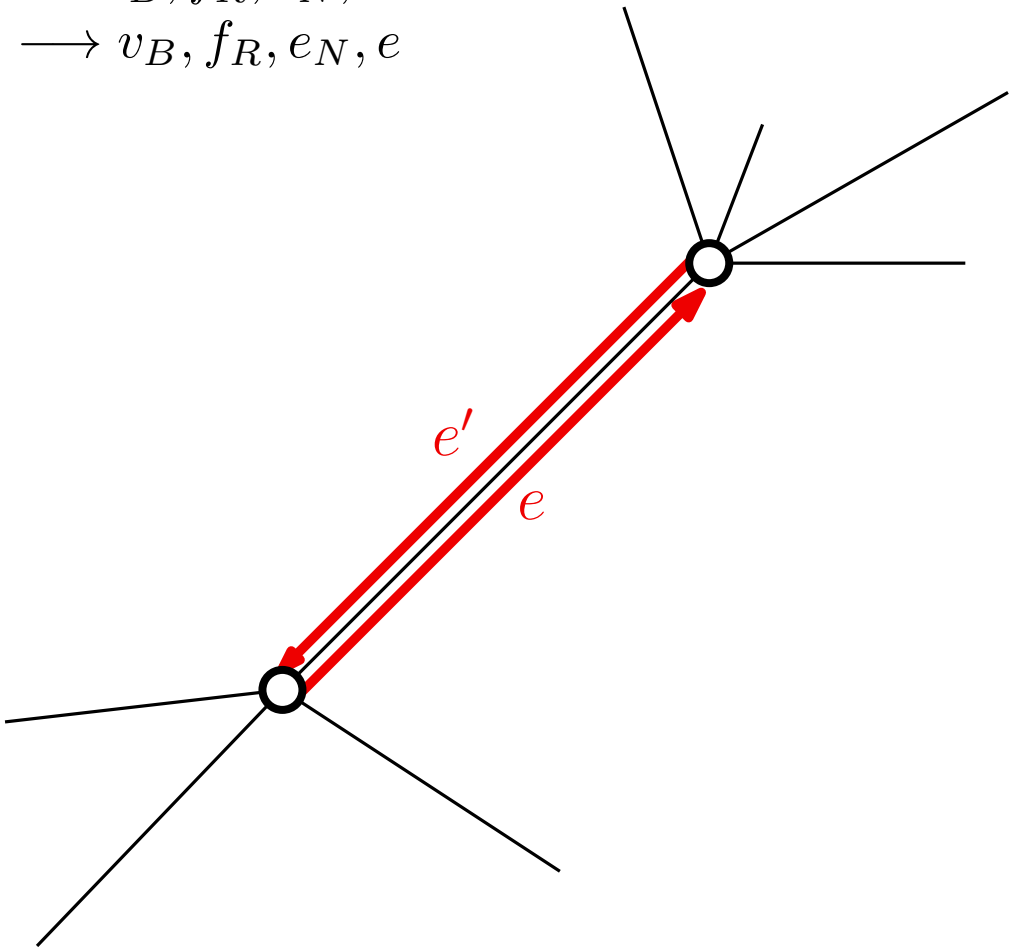

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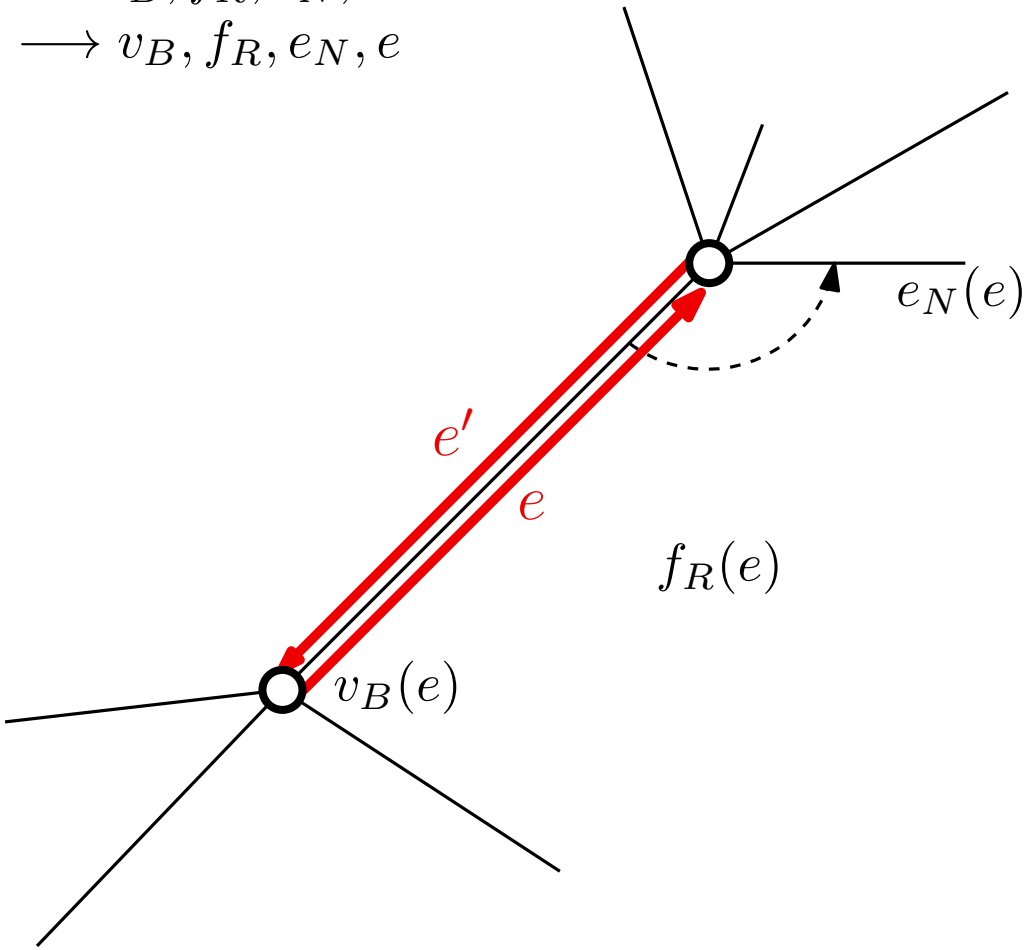

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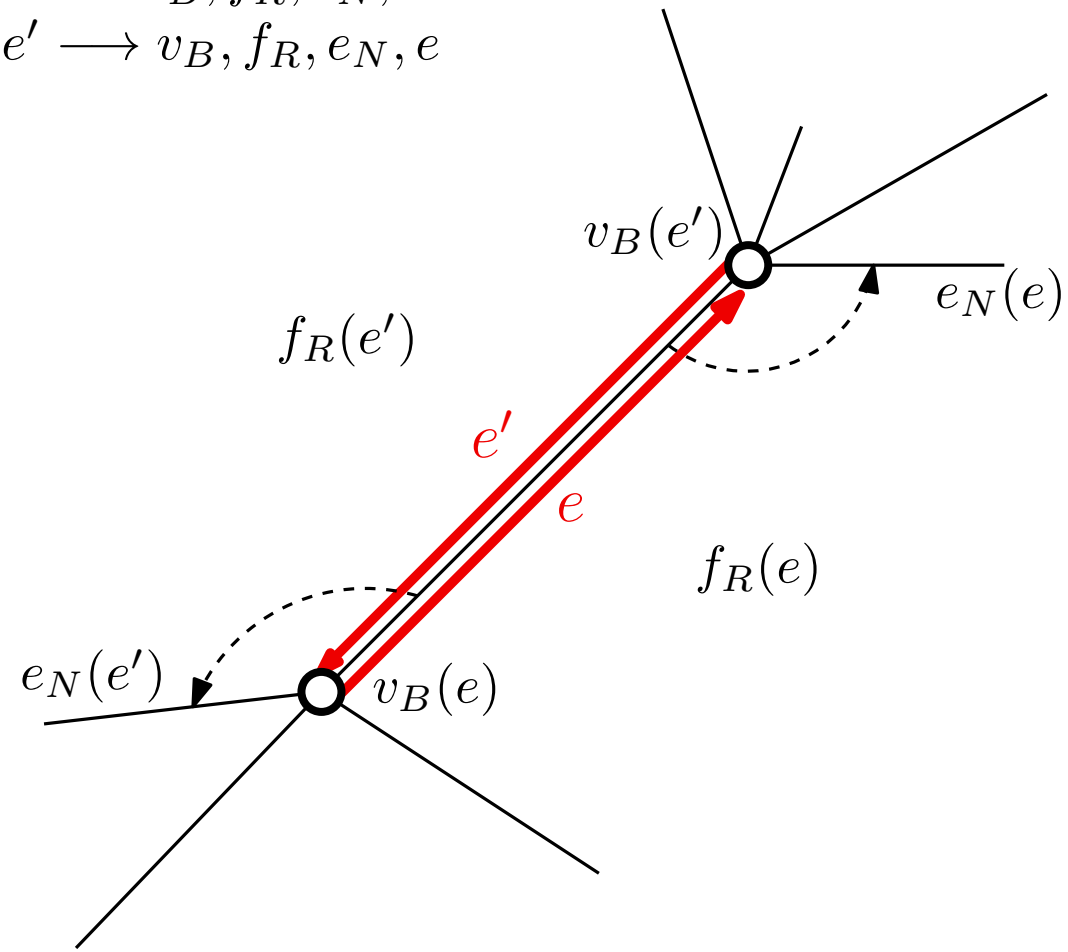

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1. Incremental algorithms

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   1.1. Without sorting

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For each i, detect whether p_i lies in the interior or the exterior of $ch(p_1, \ldots, p_{i-1})$. If it is external, compute the supporting lines from p_i to $ch(p_1, \ldots, p_{i-1})$ and add all the intermediate diagonals to the triangulation. If it is internal, detect the triangle T containing p_i and partition T into 3 triangles.

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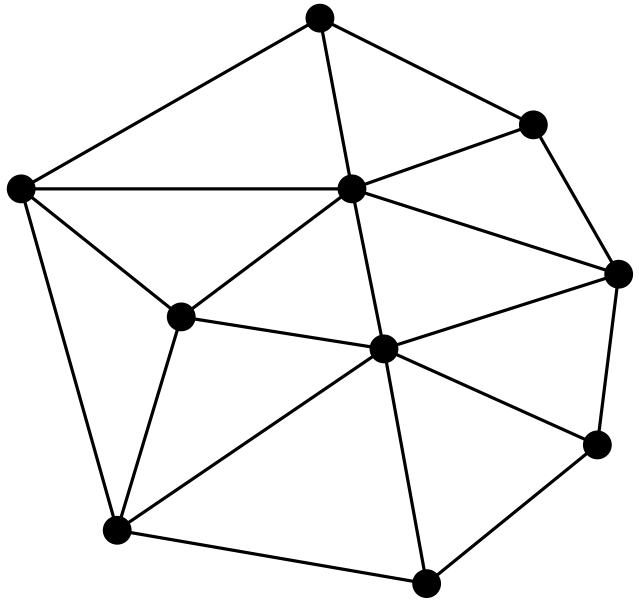

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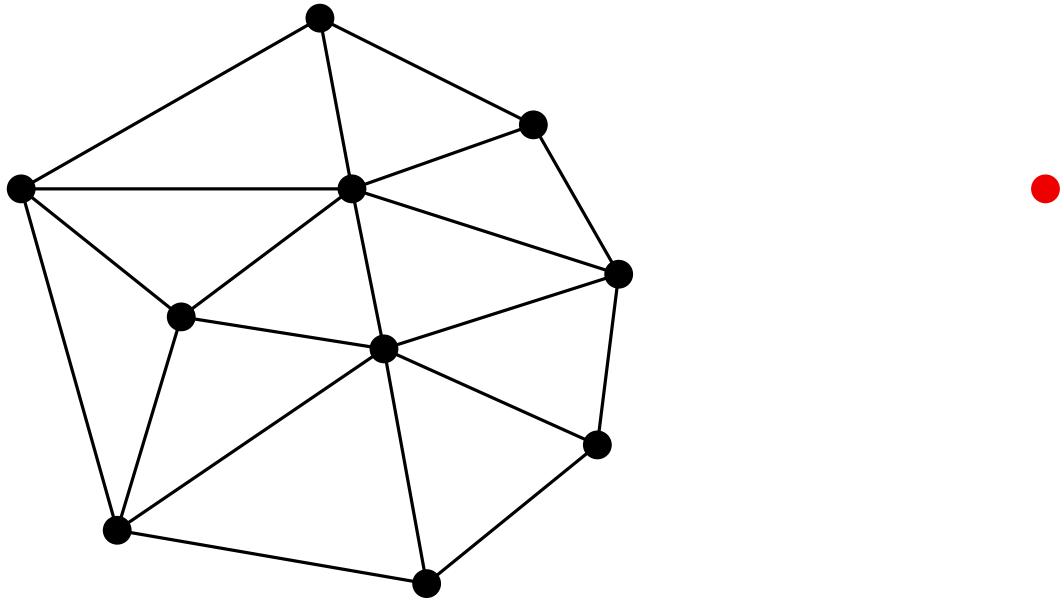

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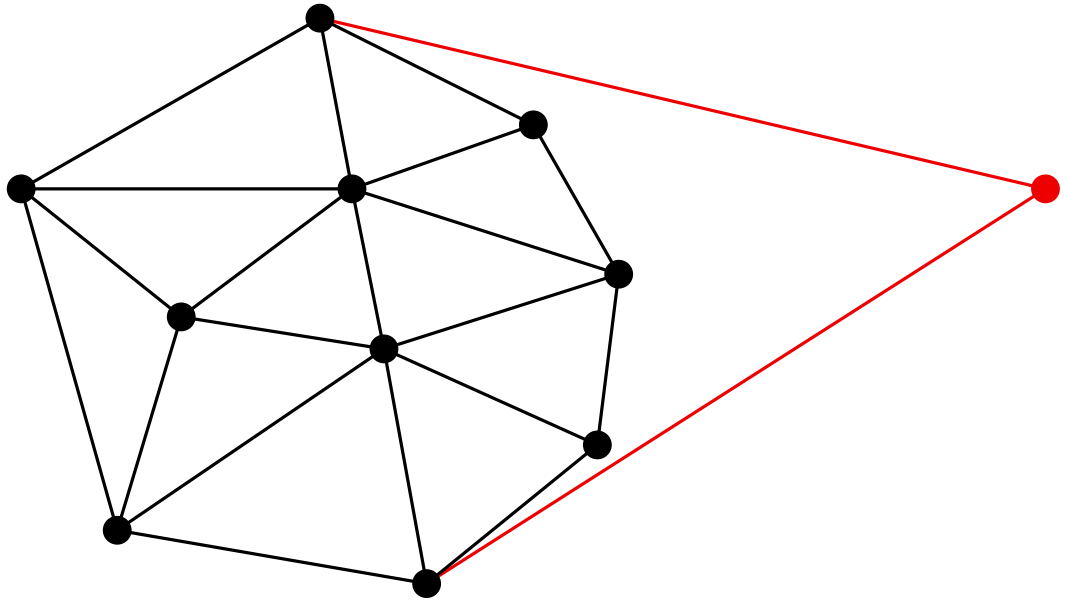

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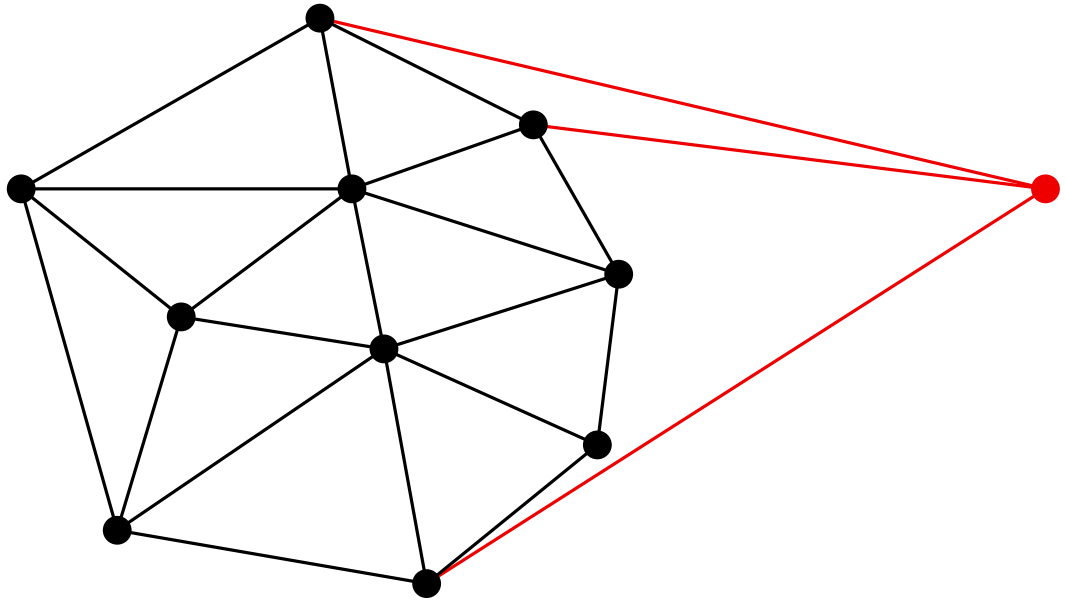

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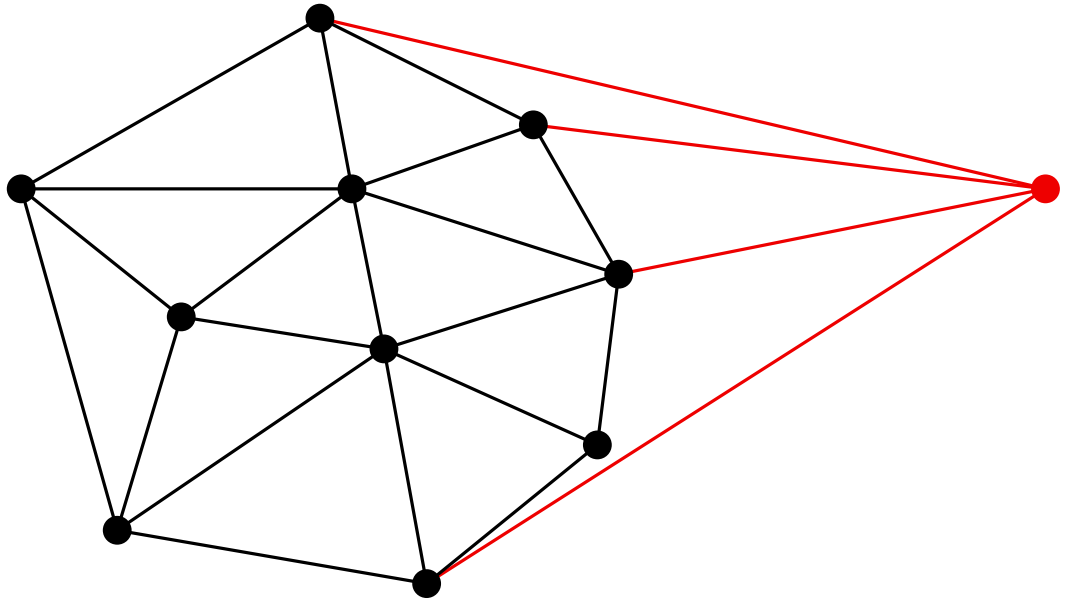

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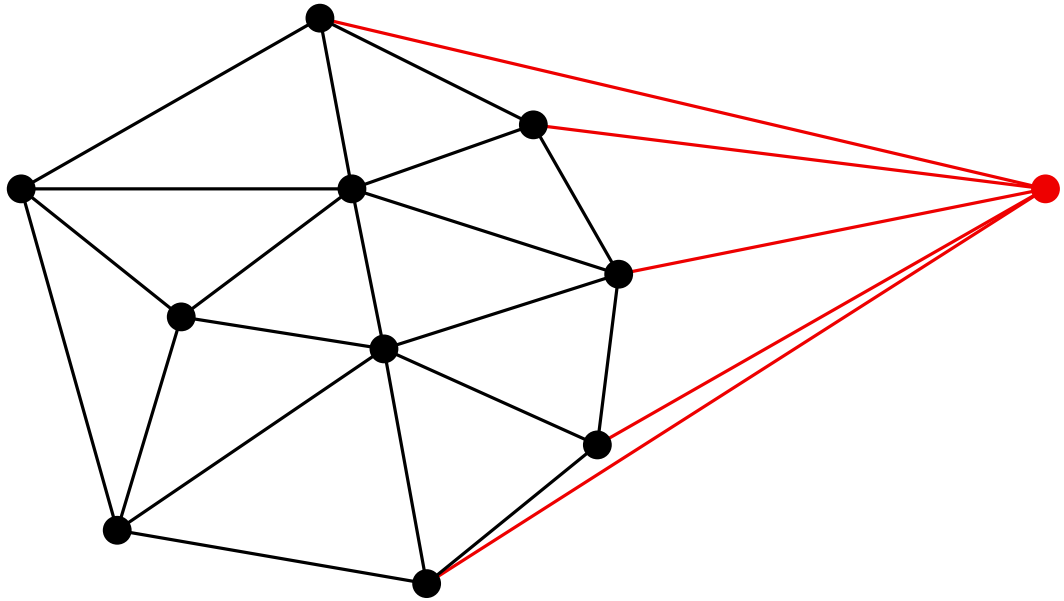

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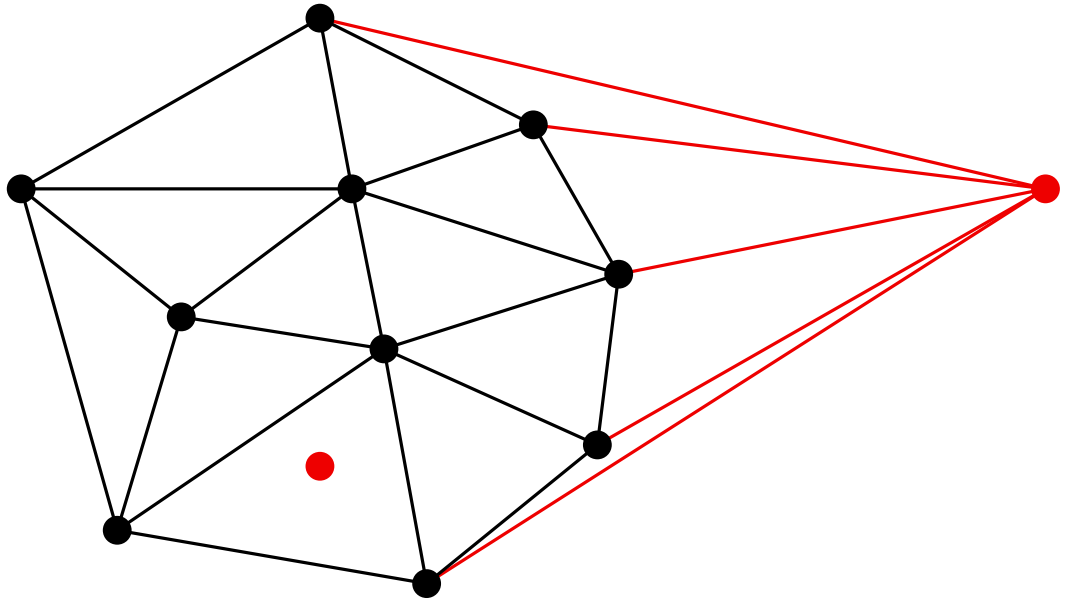

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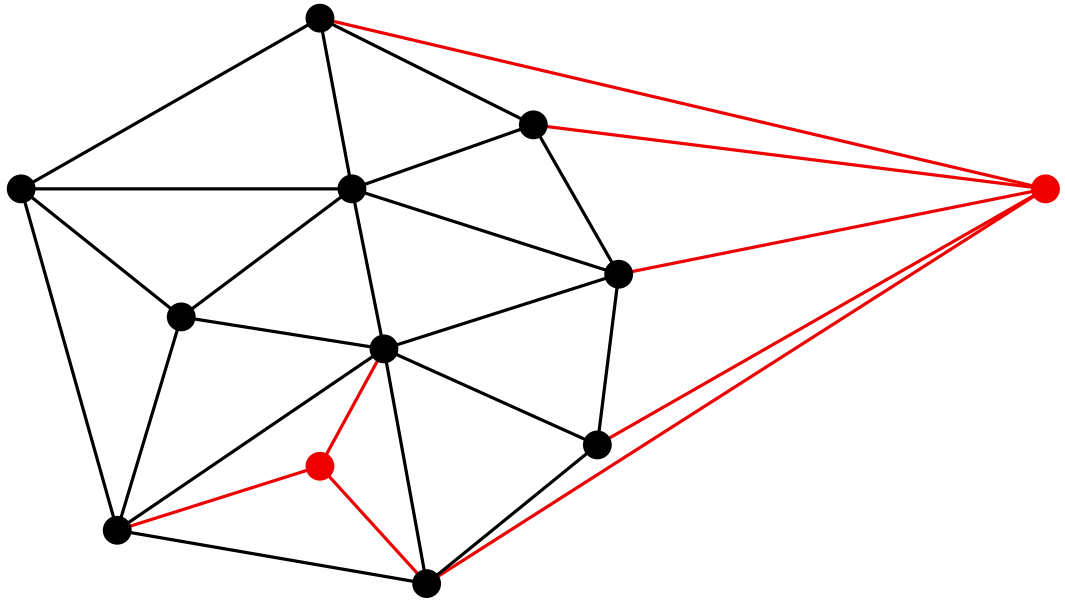

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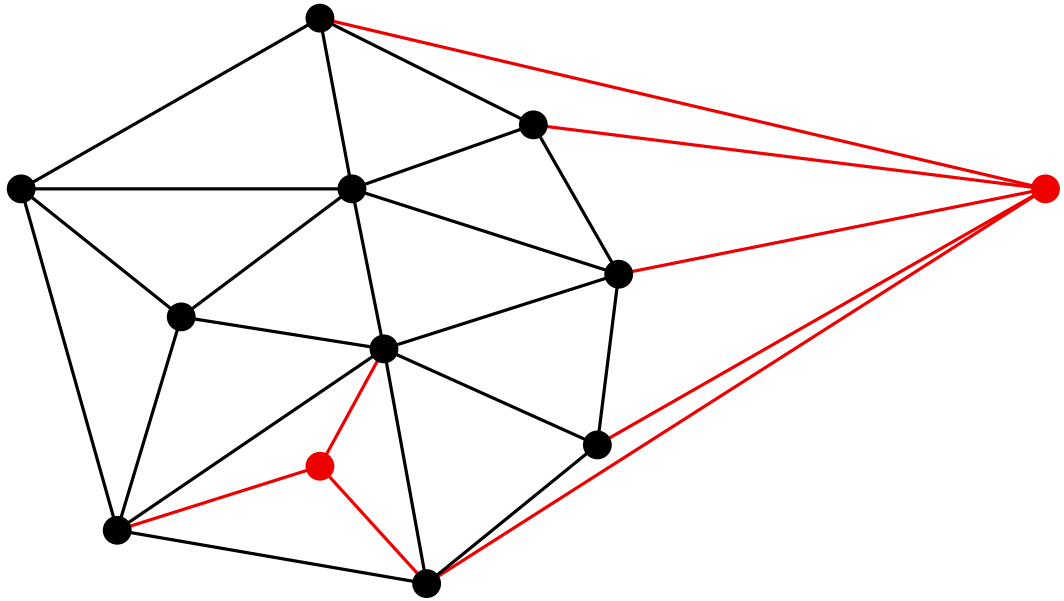

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ALGORITHMS

1. Incremental algorithms
   - **1.1.** Without sorting
   - **1.2.** With sorting

Running time: $O(n^2)$

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1.1. Without sorting **Running time:** $O(n^2)$

1.2. With sorting

Start by sorting the points in lexicographical order in $O(n \log n)$ time. The information of the sorted order of the points allows to add the i diagonals in $O(i)$ time, so that the amortized cost of the insertion of all diagonals is done in $O(n)$ time.

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1.3. With hierarchical structure

Using an auxiliary enclosing triangle and a hierarchy of triangles: each time a new point is added, a triangle gets subdivided into three children.

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ALGORITHMS

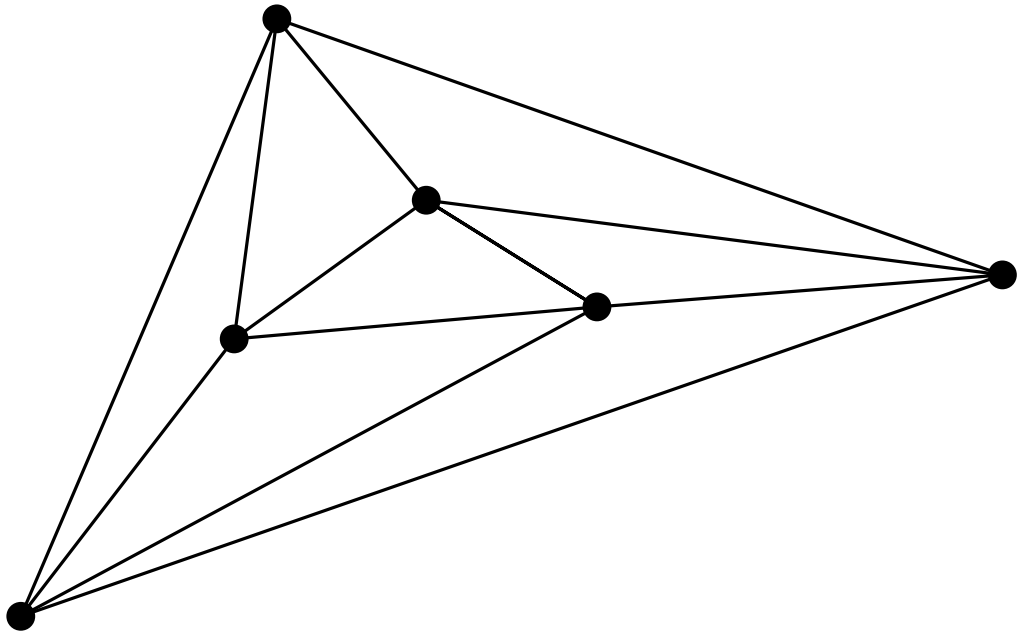
1. Incremental algorithms

1.1. Without sorting — **Running time:** $O(n^2)$

1.2. With sorting — **Running time:** $O(n \log n)$

1.3. With hierarchical structure

Using an auxiliary enclosing triangle and a hierarchy of triangles: each time a new point is added, a triangle gets subdivided into three children.



TRIANGULATING POINT SETS

ALGORITHMS

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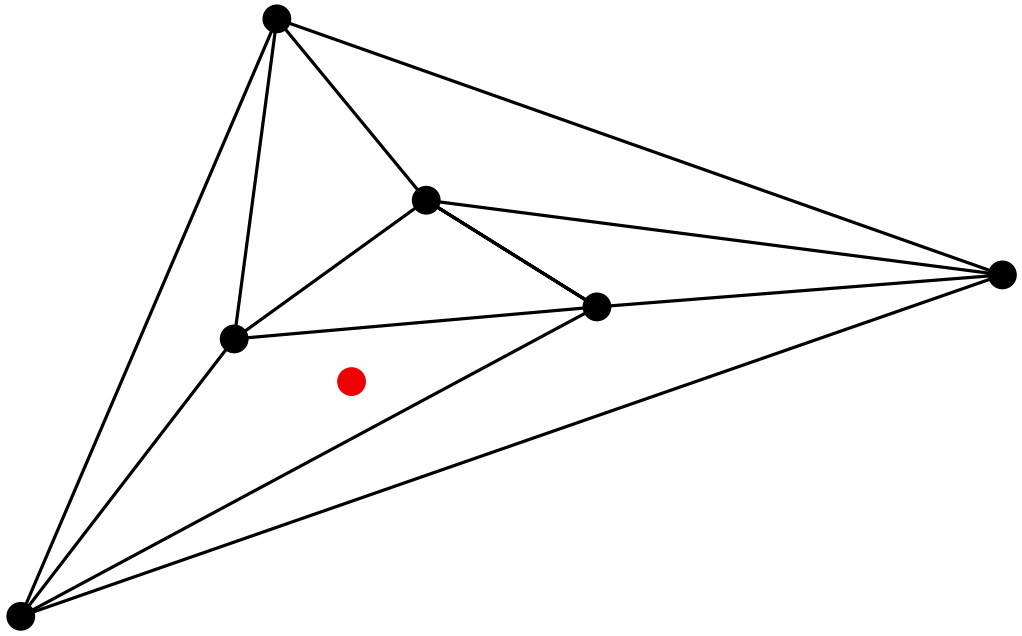

TRIANGULATING POINT SETS

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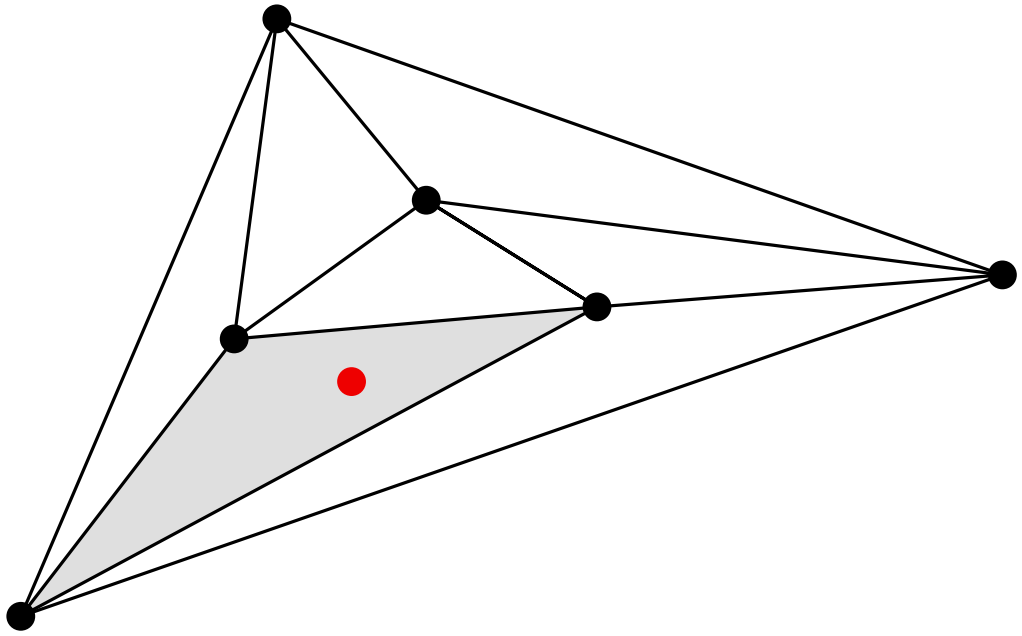

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TRIANGULATING POINT SETS

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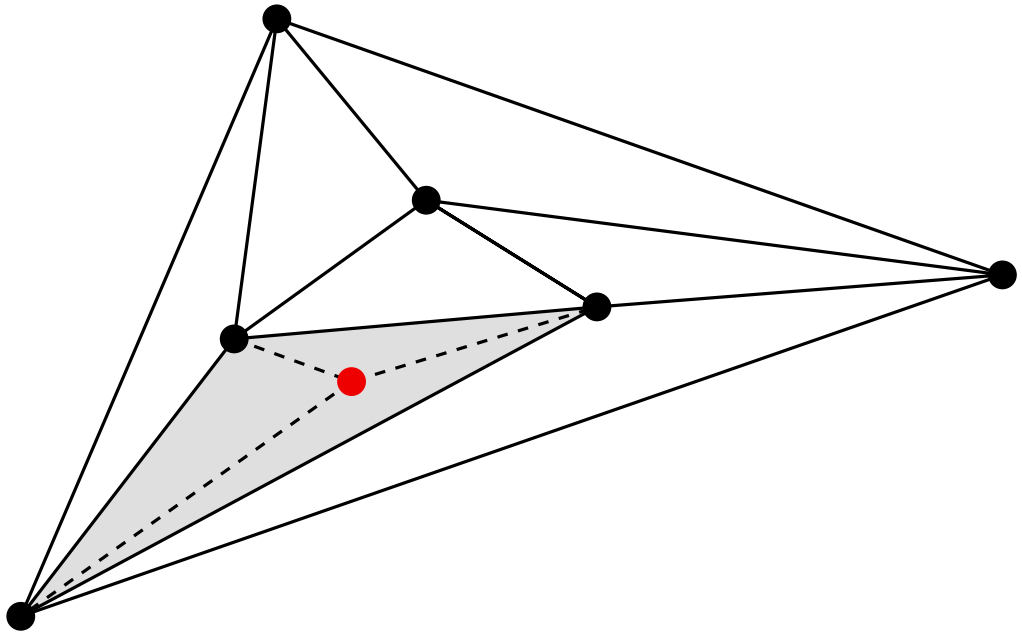

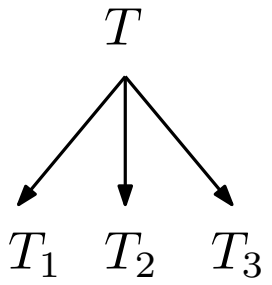

TRIANGULATING POINT SETS

ALGORITHMS

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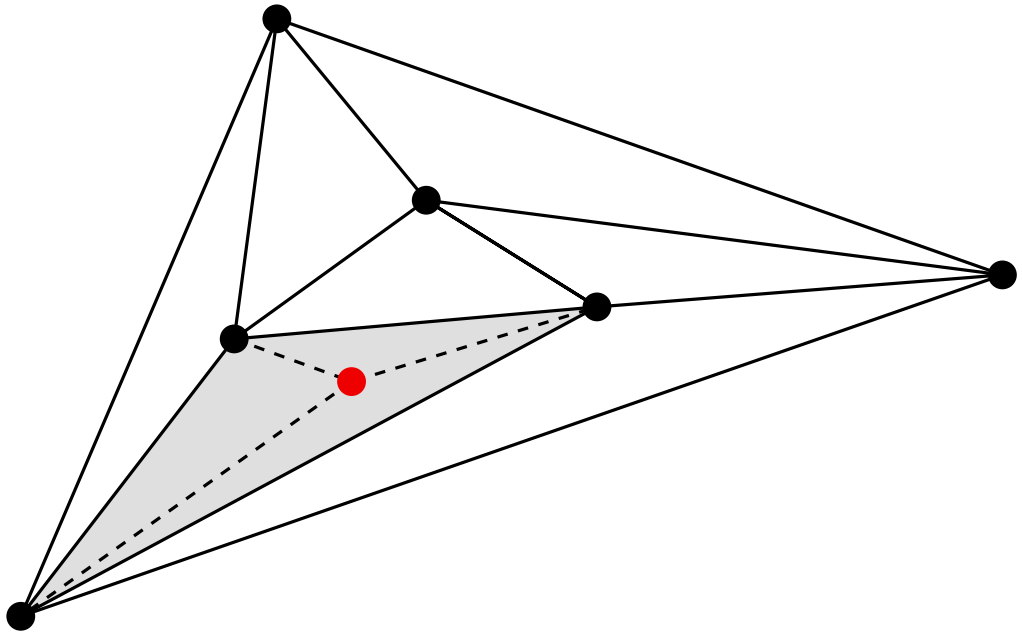

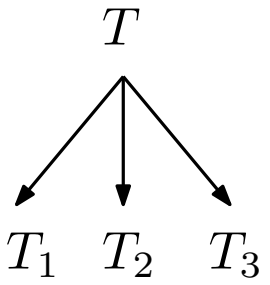

TRIANGULATING POINT SETS

ALGORITHMS

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TRIANGULATING POINT SETS

ALGORITHMS

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TRIANGULATING POINT SETS

ALGORITHMS

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- **1.5.** With auxiliary point(s)

TRIANGULATING POINT SETS

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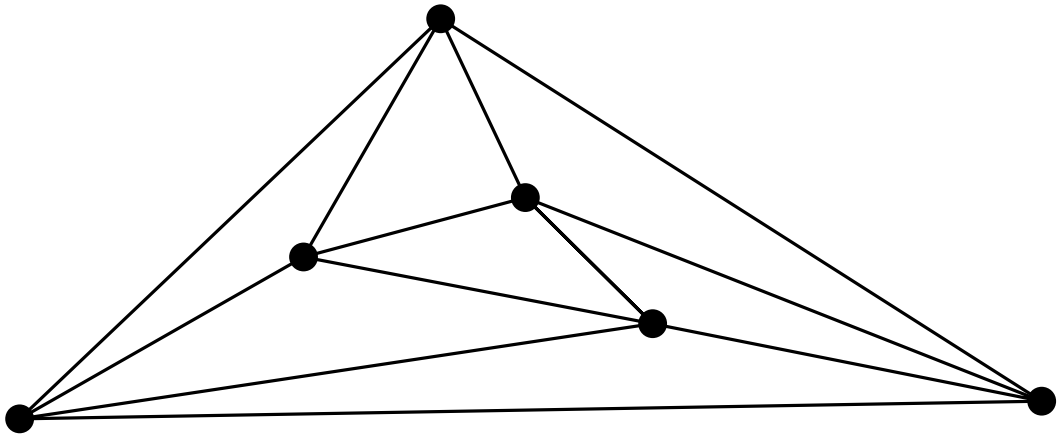

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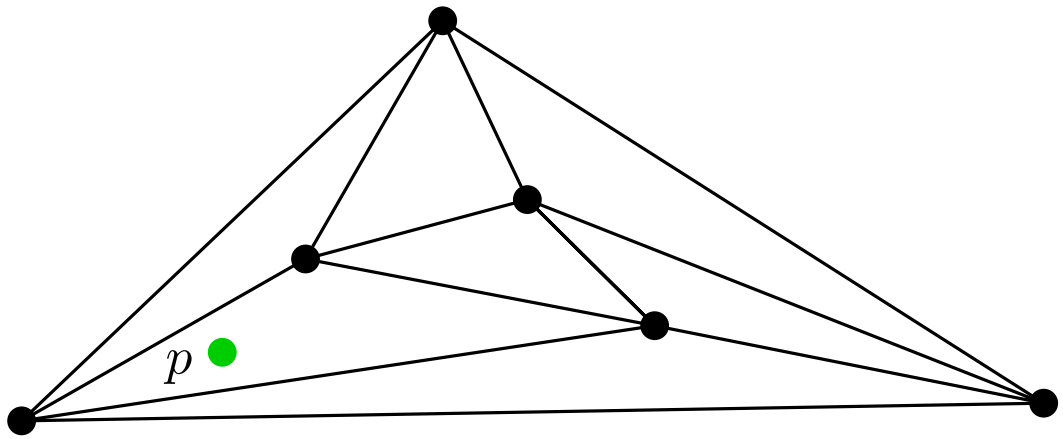

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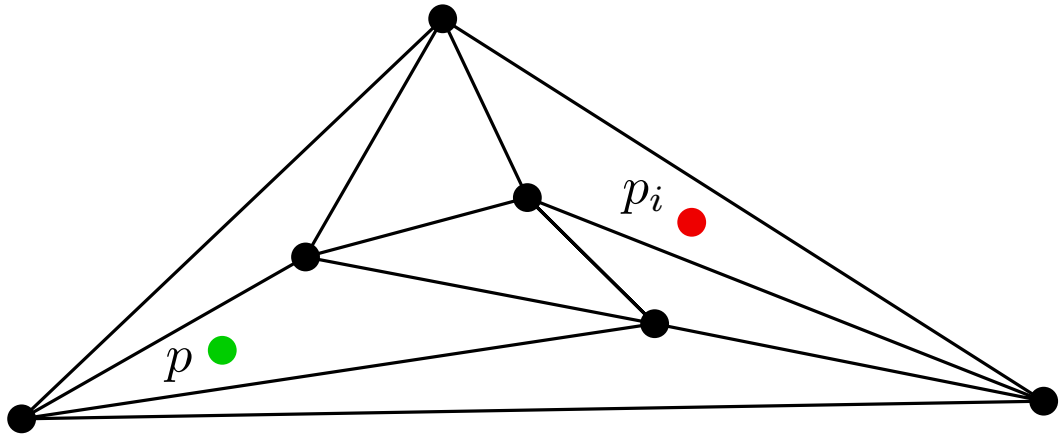

TRIANGULATING POINT SETS

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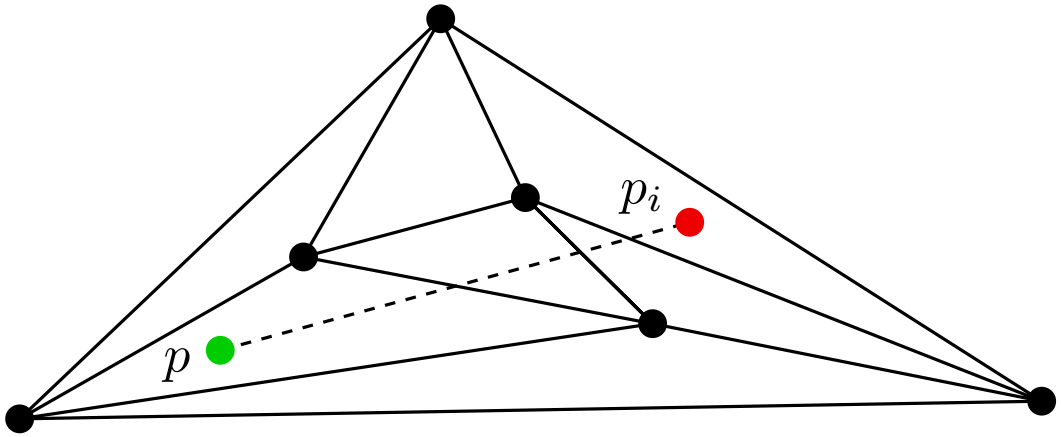

TRIANGULATING POINT SETS

ALGORITHMS

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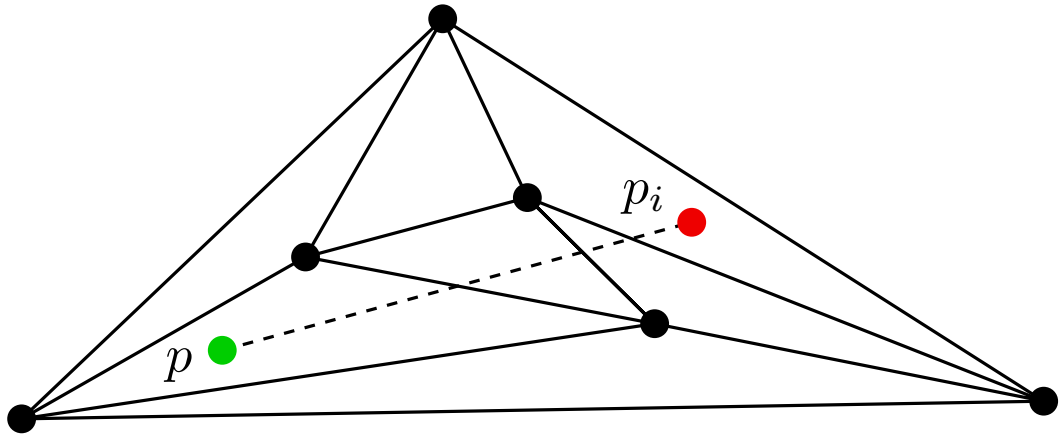

TRIANGULATING POINT SETS

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TRIANGULATING POINT SETS

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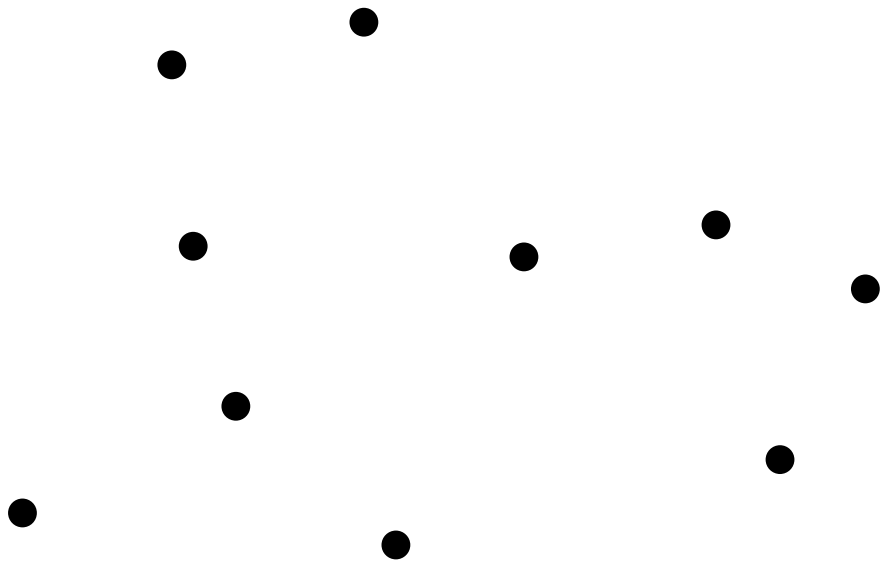

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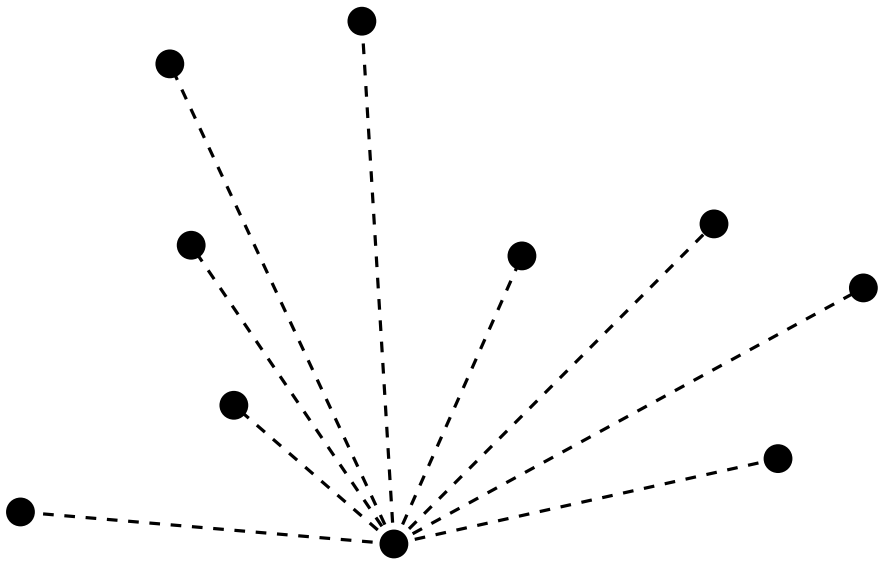

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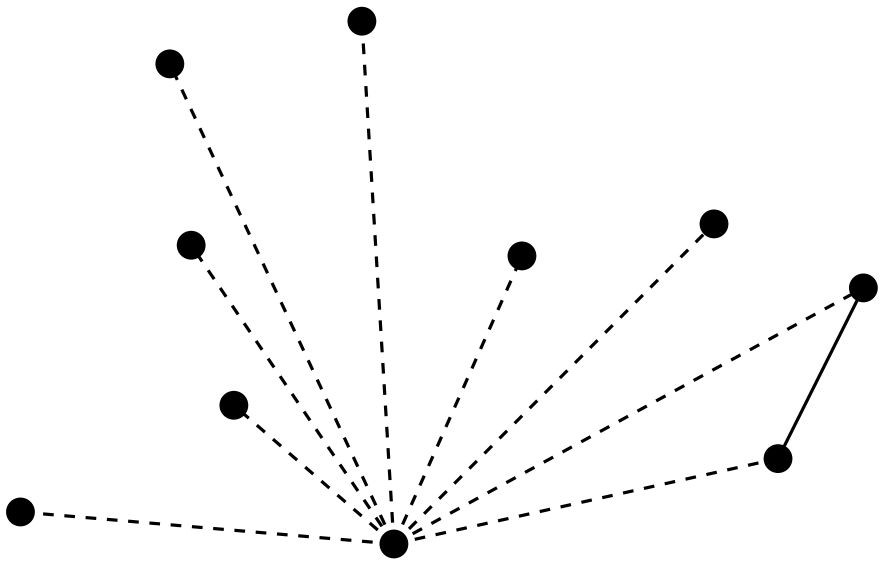

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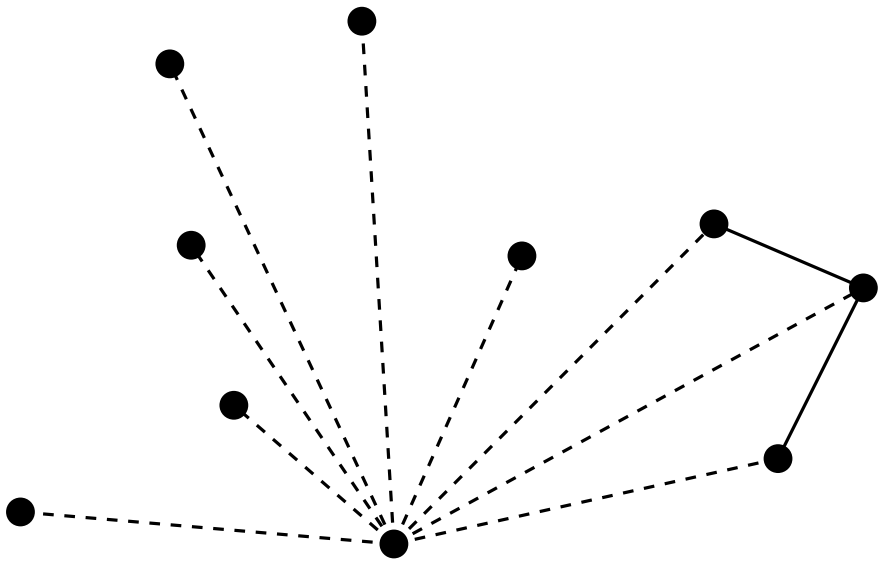

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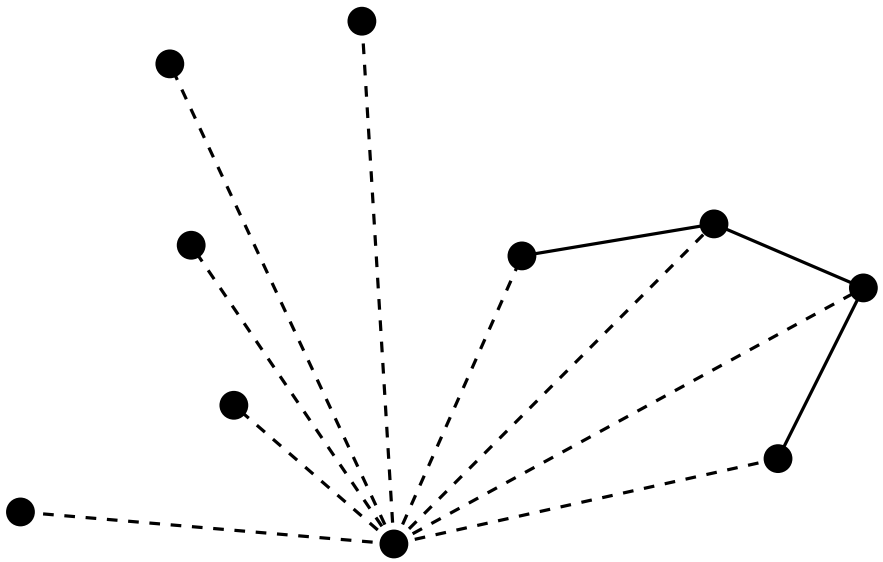

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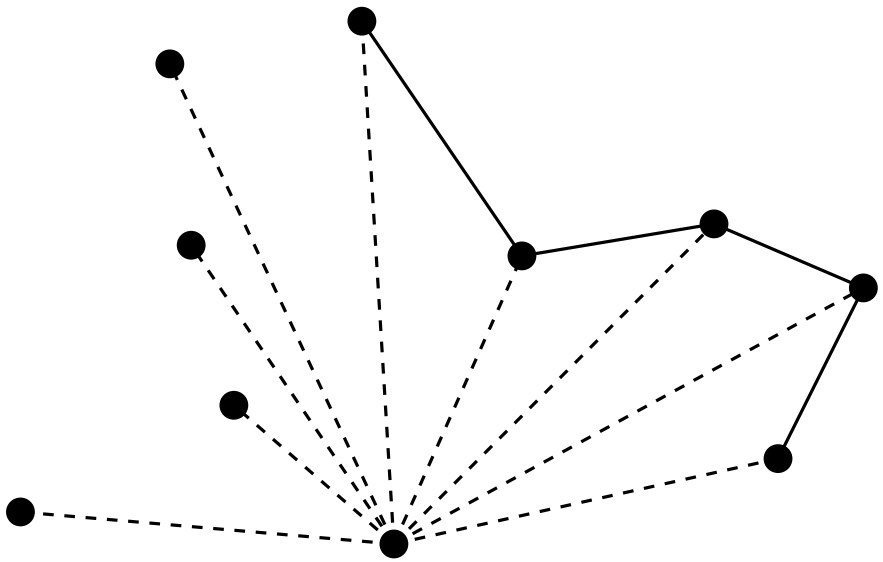

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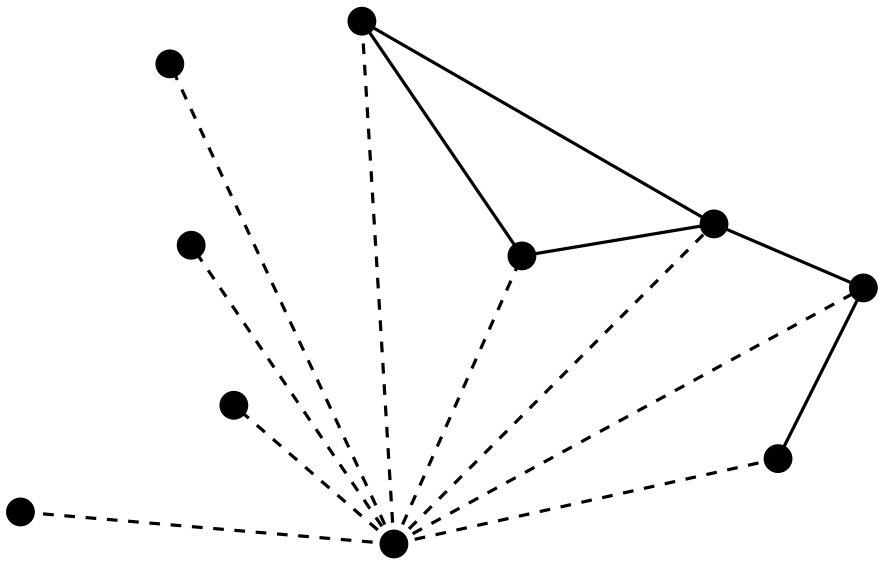

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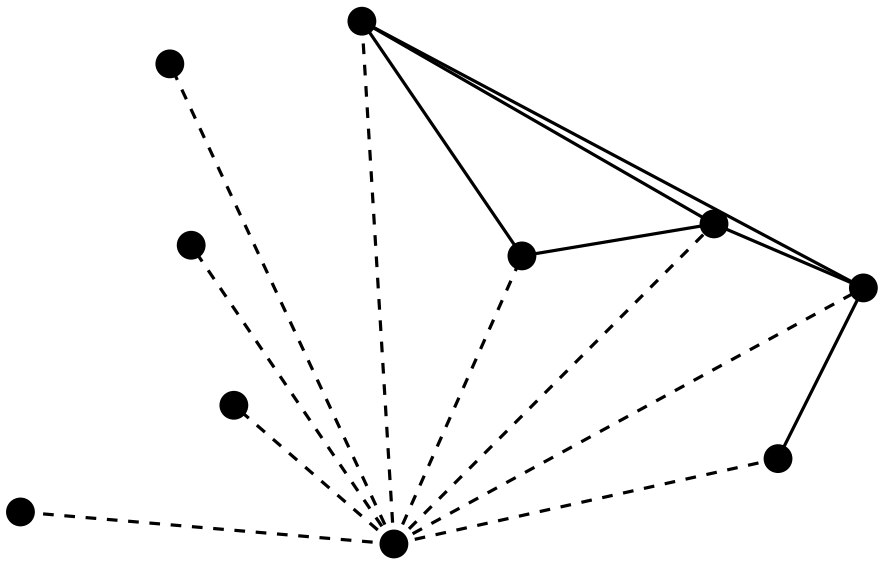

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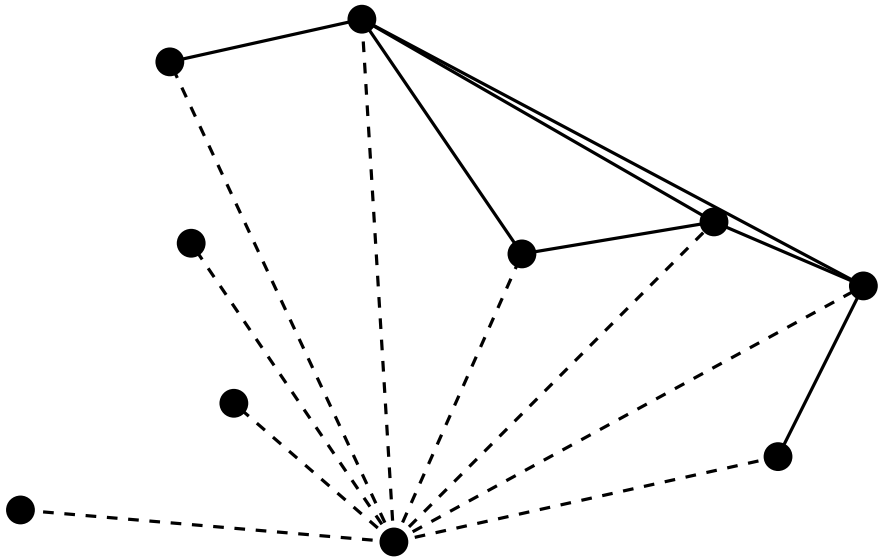

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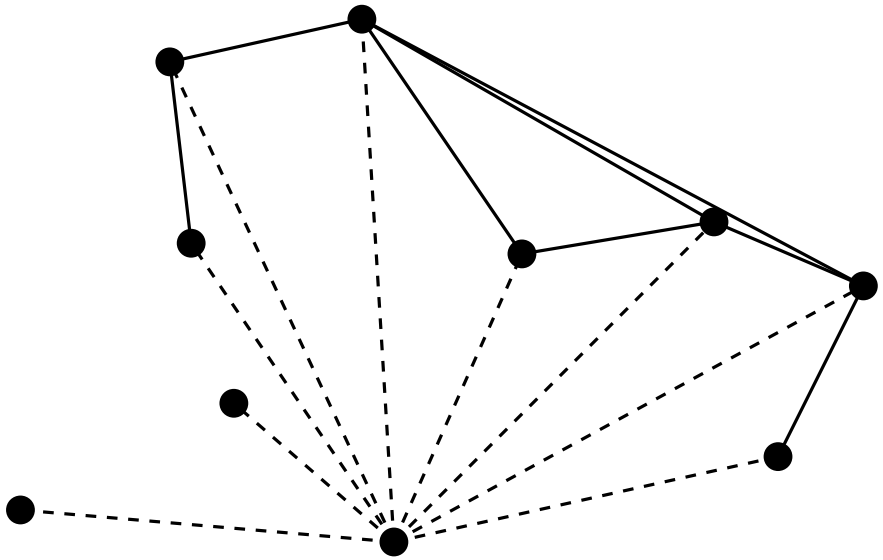

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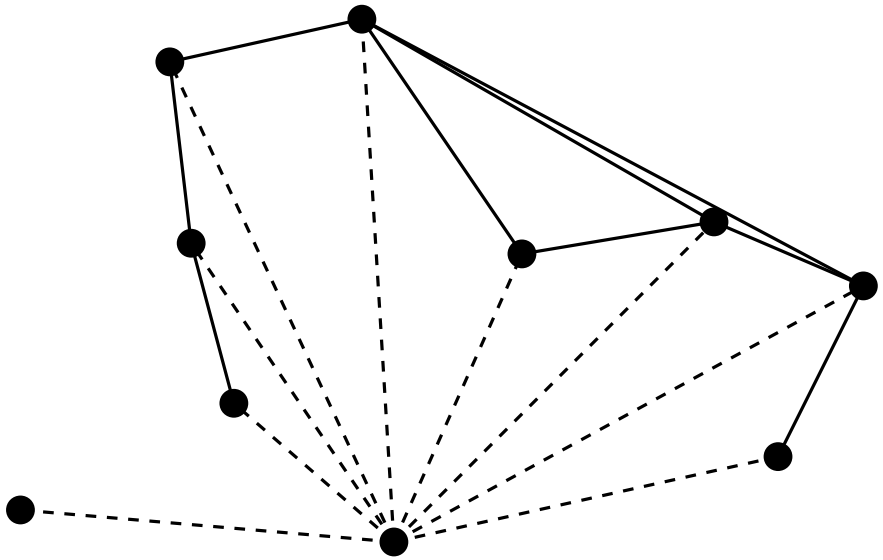

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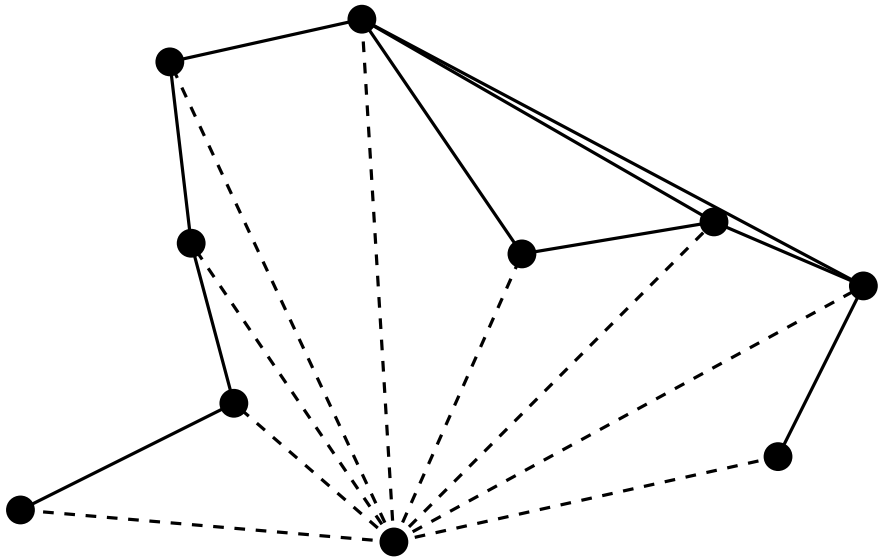

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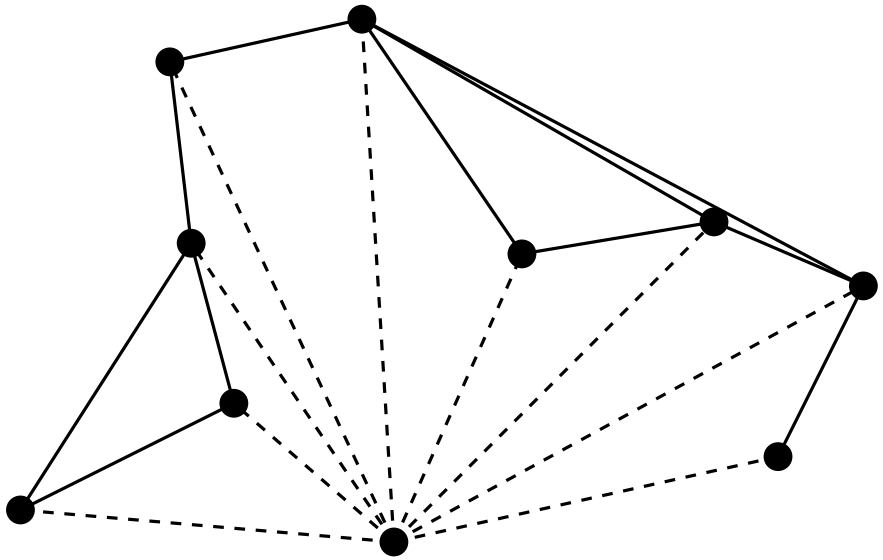

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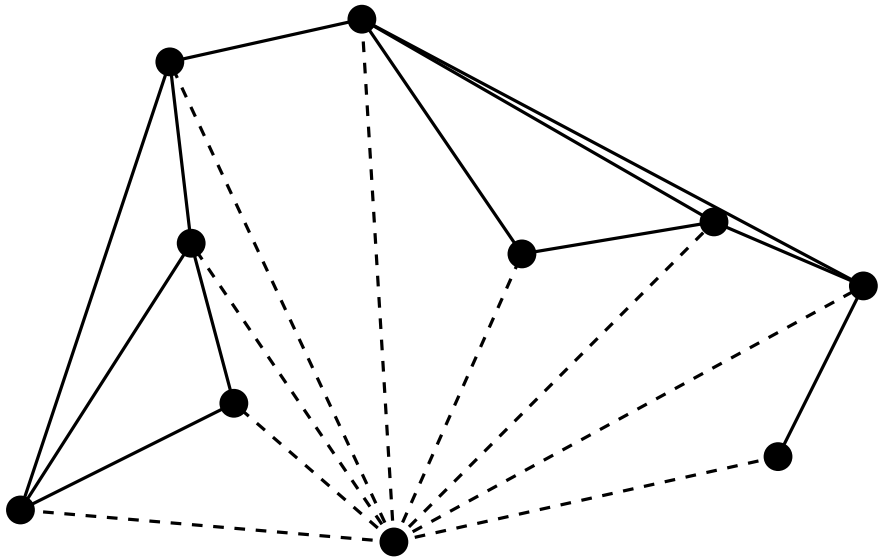

TRIANGULATING POINT SETS

ALGORITHMS

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   - **1.4.** Randomized — **Running time:** $O(n \log n)$ expected
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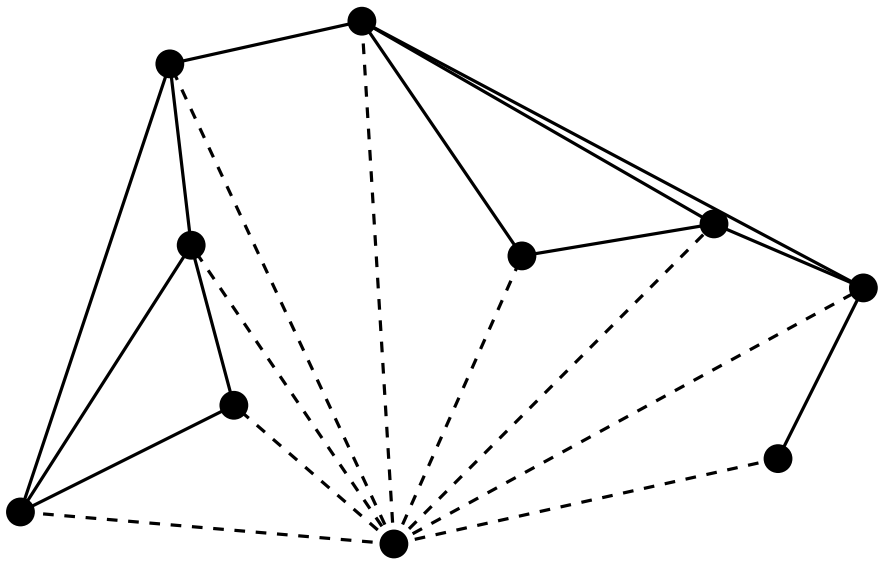
2. Graham's algorithm — **Running time:** $O(n \log n)$



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Initialization

Sort the points by abscissa

Advance

- Partition: divide the points into roughly two vertically separated halfs
- Recursion: recursively triangulate each half
- Fusion: compute the external common tangents and triangulate the intermediate space

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3. Dividide and conquer — **Running time:** $O(n \log n)$

LOWER BOUND

This problem has an $\Omega(n \log n)$ lower bound, since the convex hull of the set of points can be trivially obtained in $O(n)$ time from the triangulation.

QUALITY OF A TRIANGULATION

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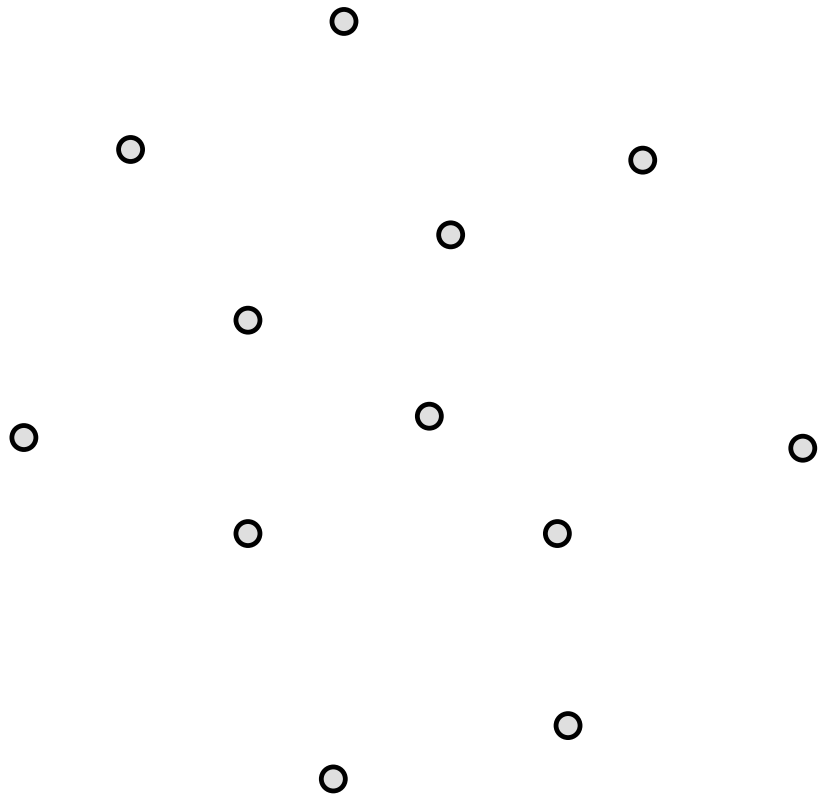

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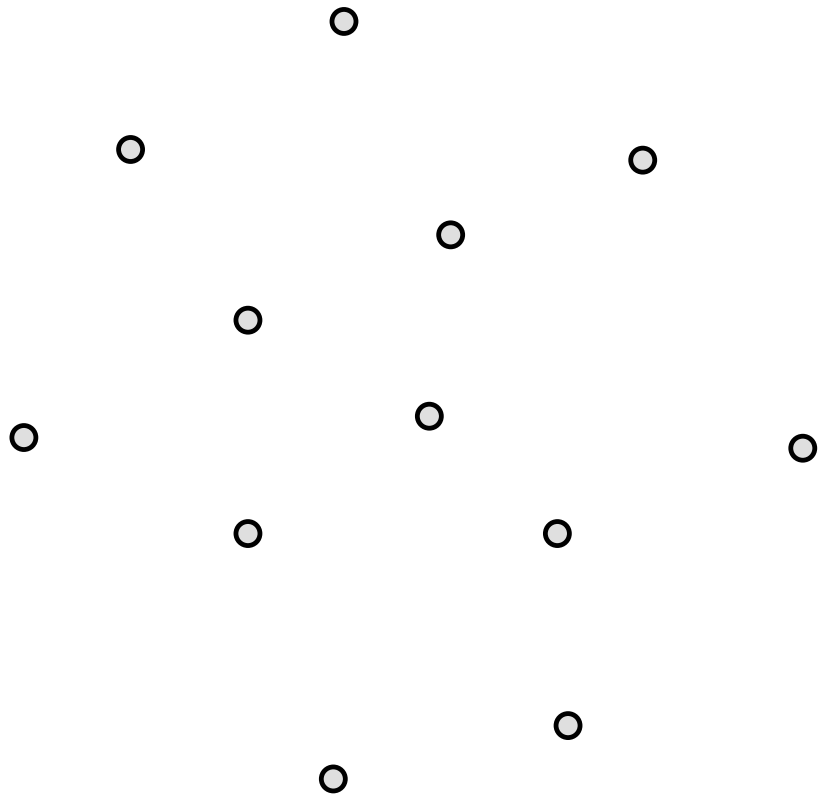
Incremental, without sorting



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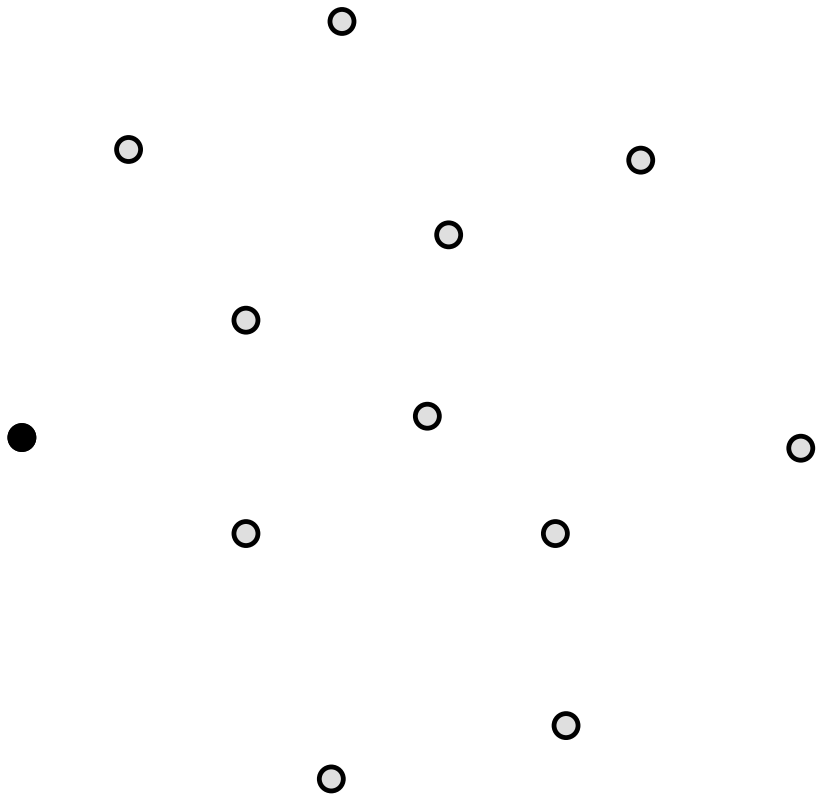

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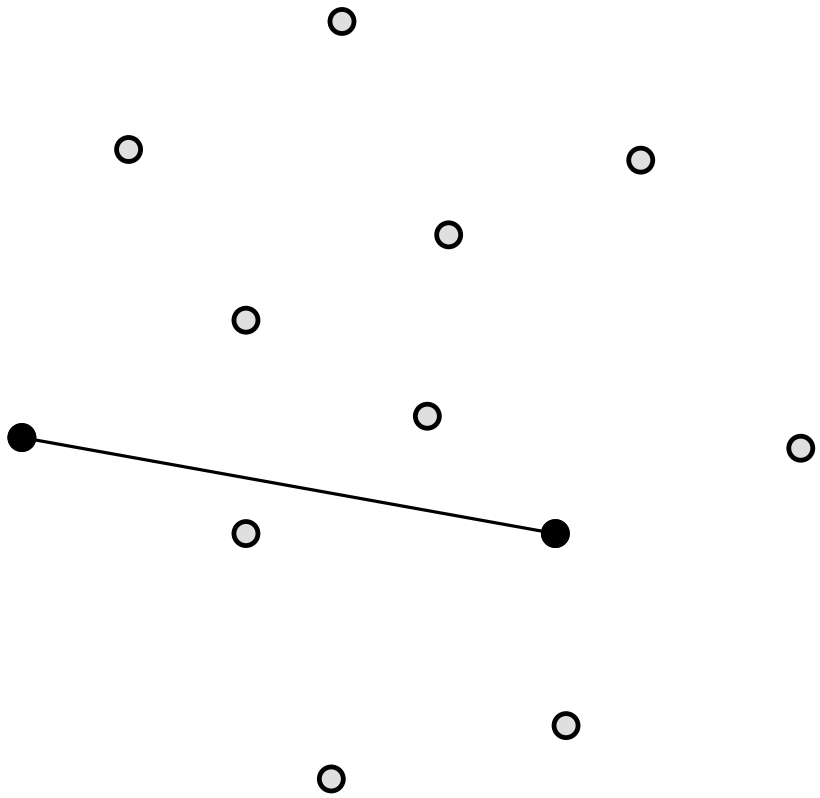
Incremental, without sorting



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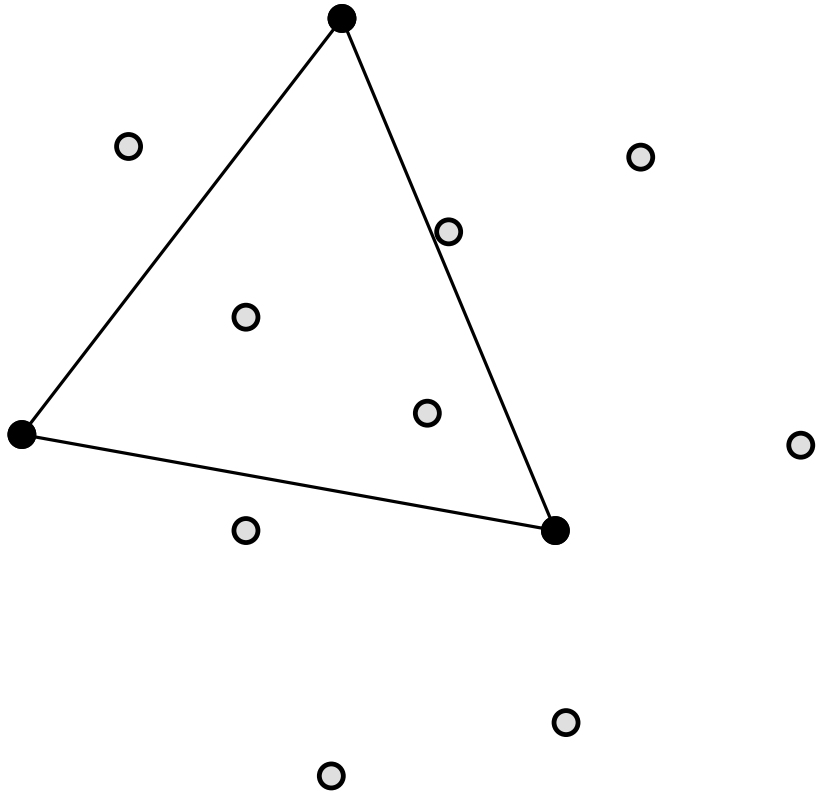

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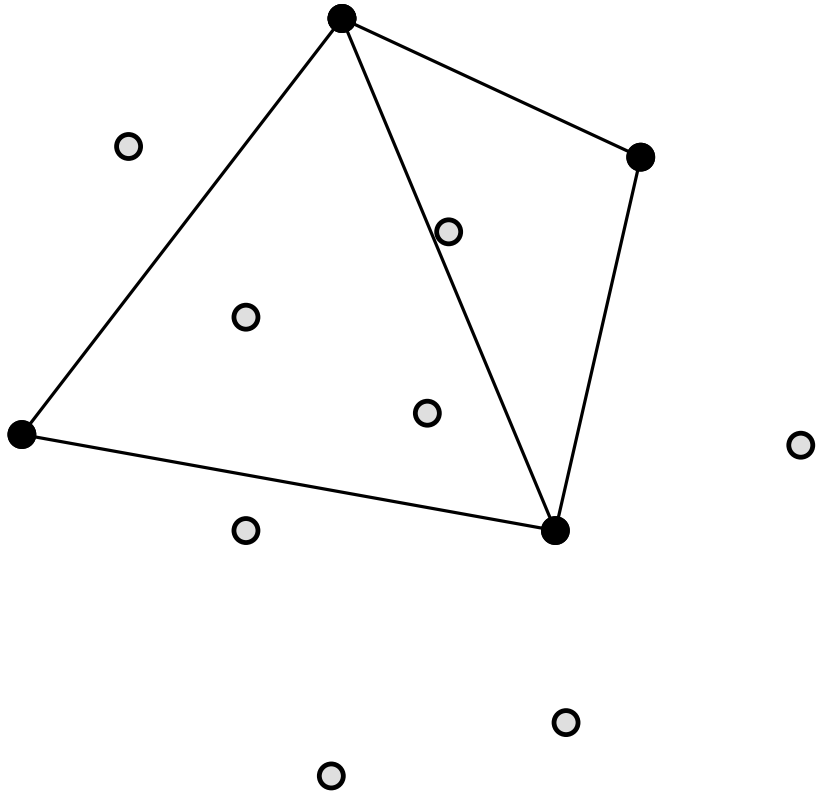

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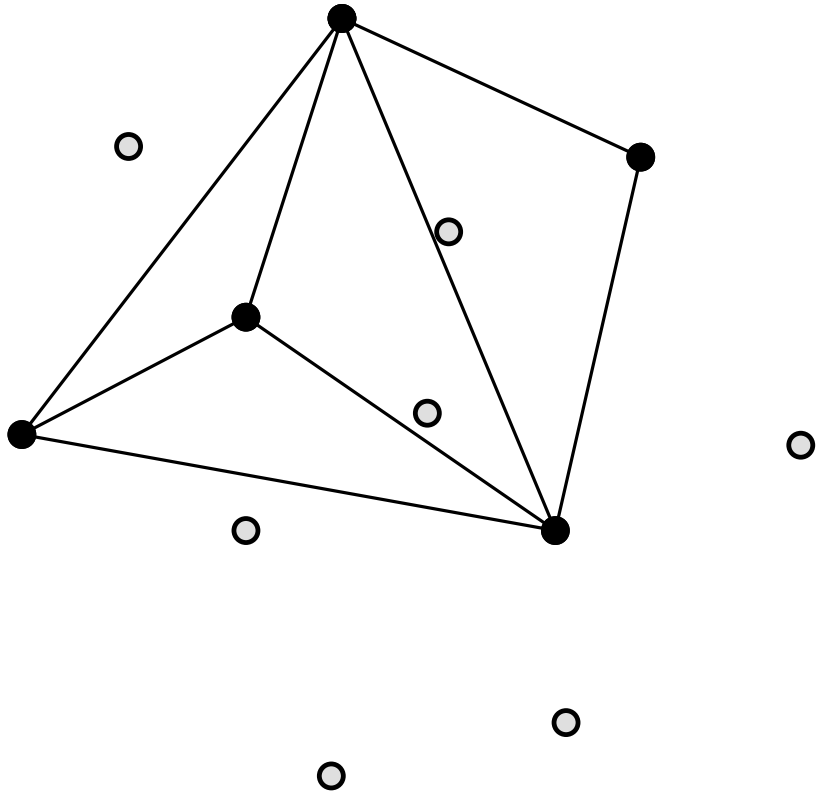

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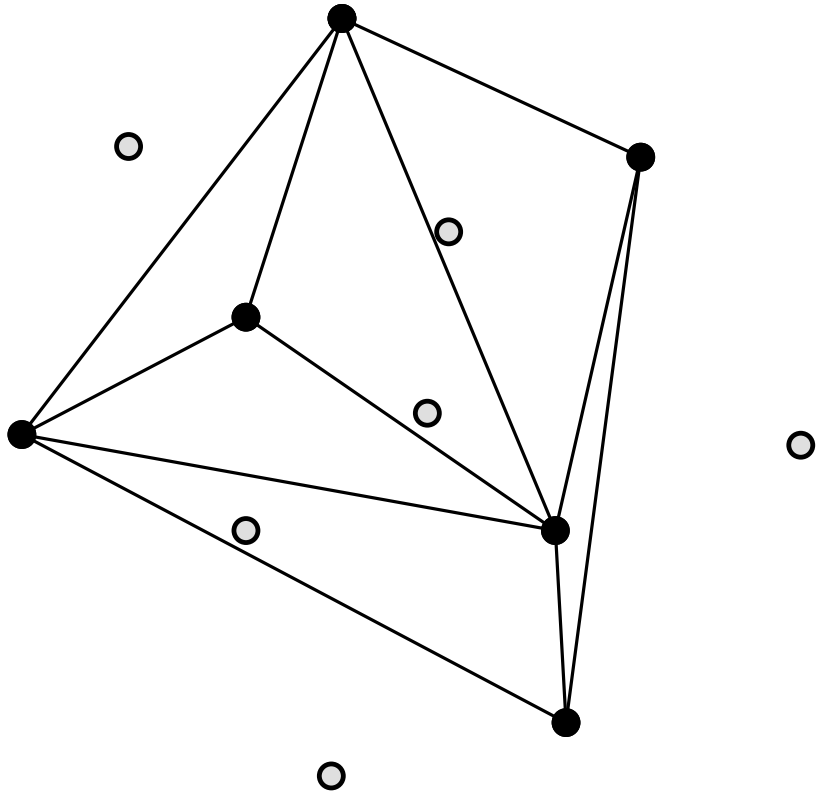

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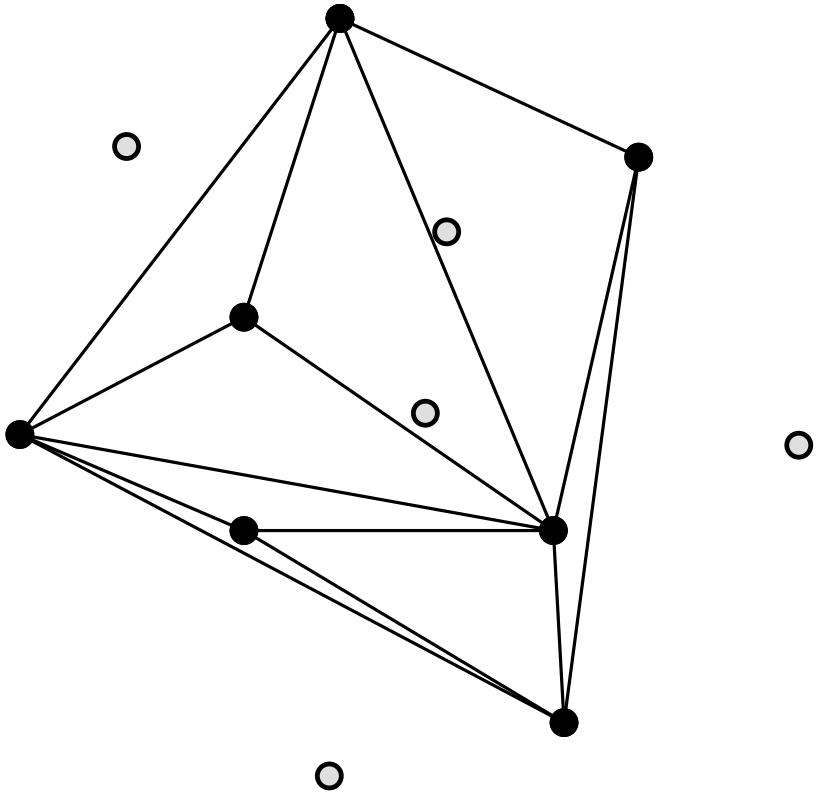

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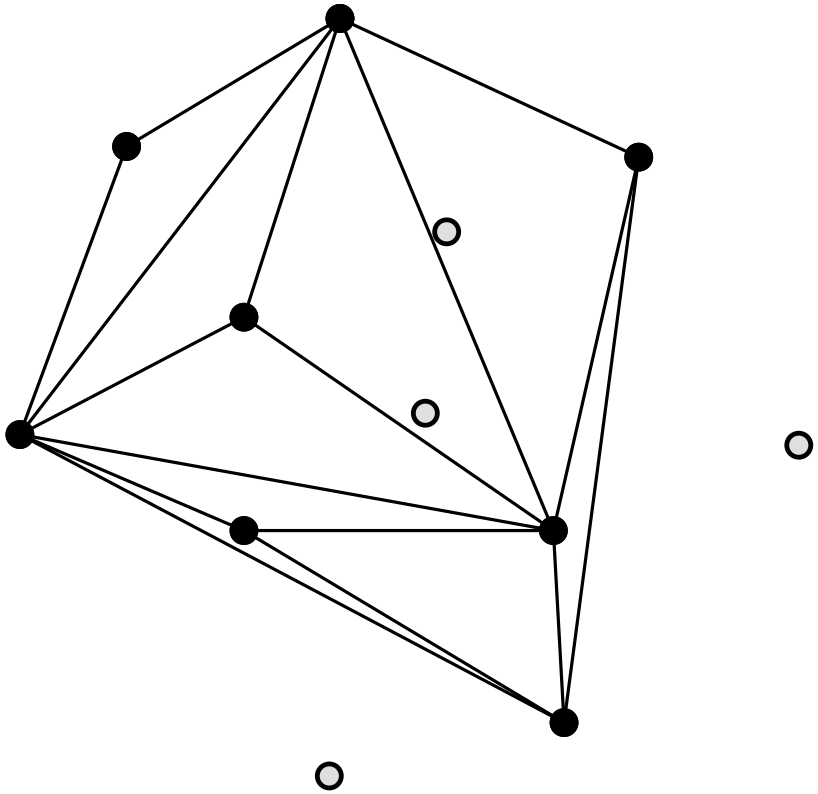

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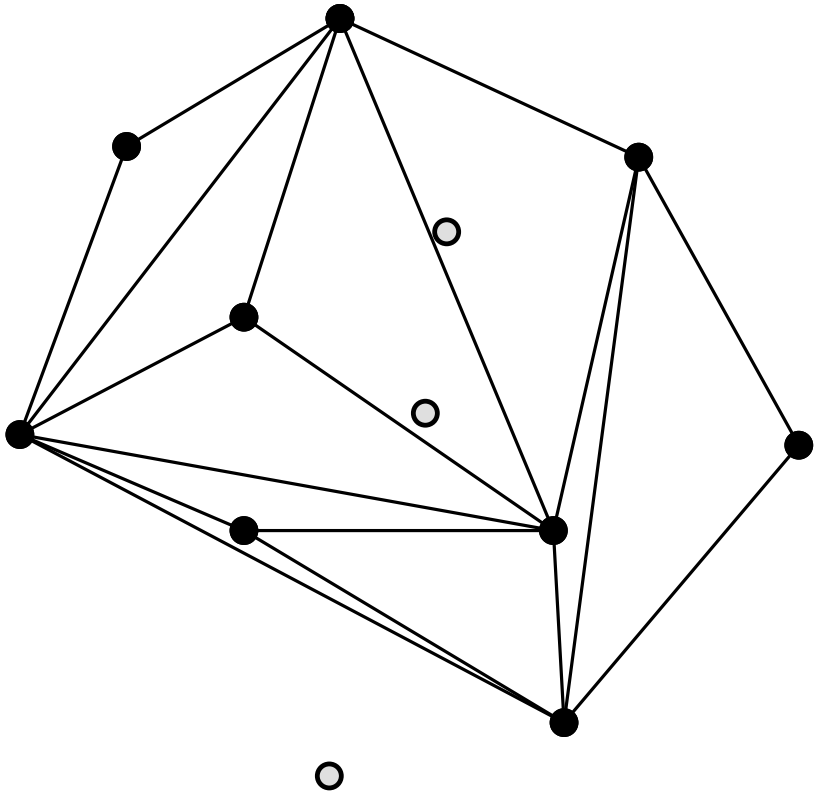

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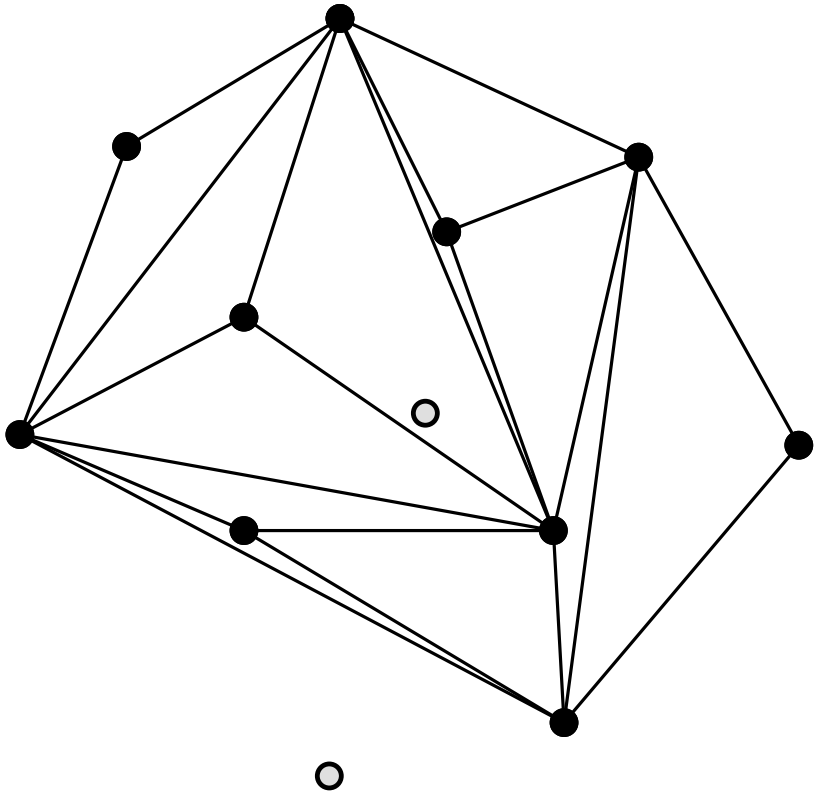

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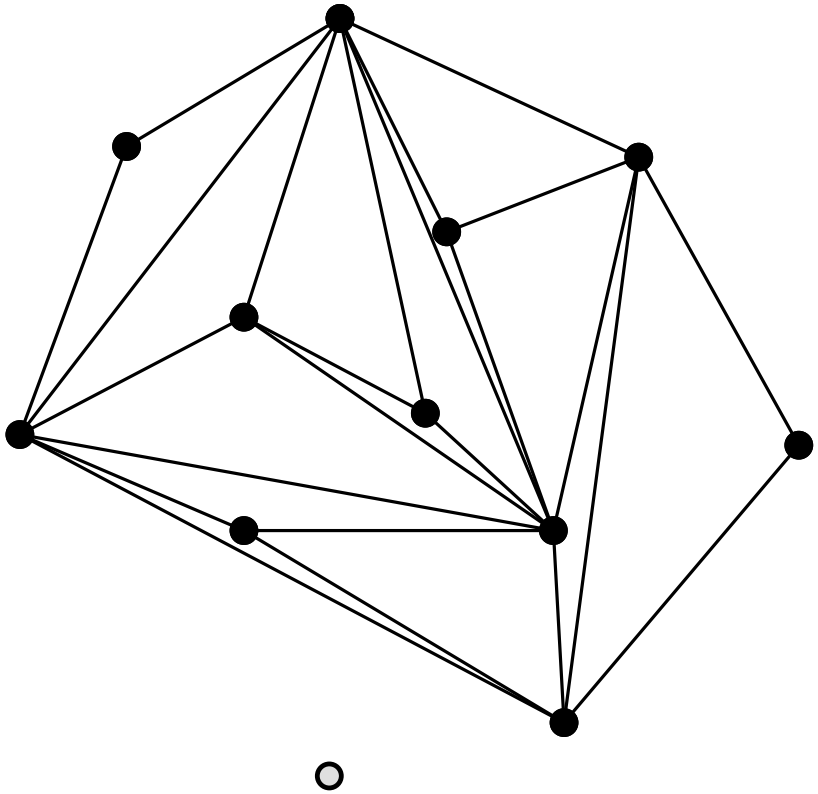

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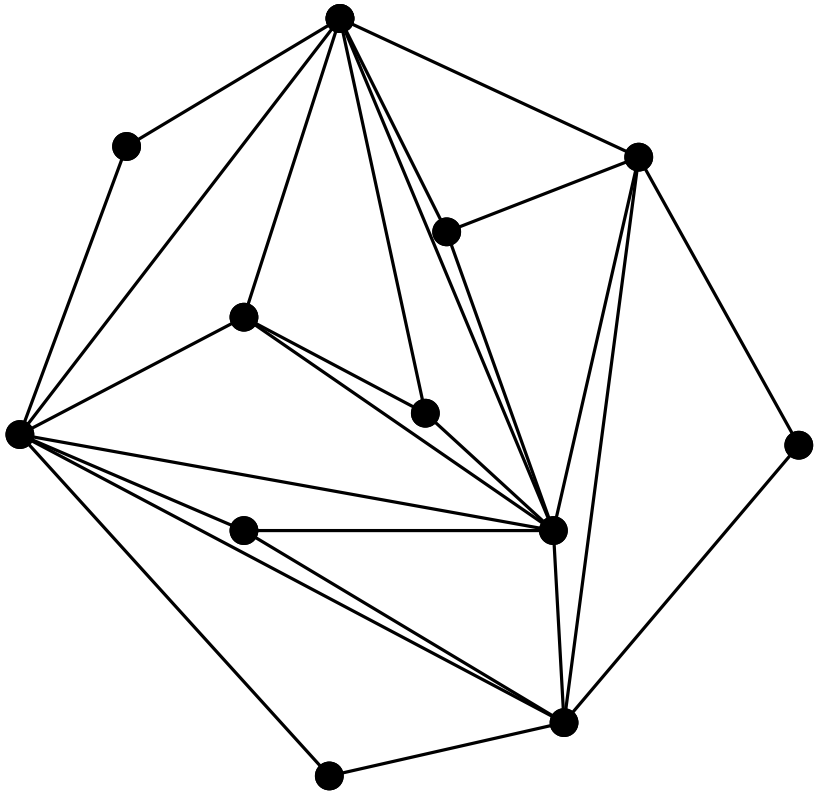

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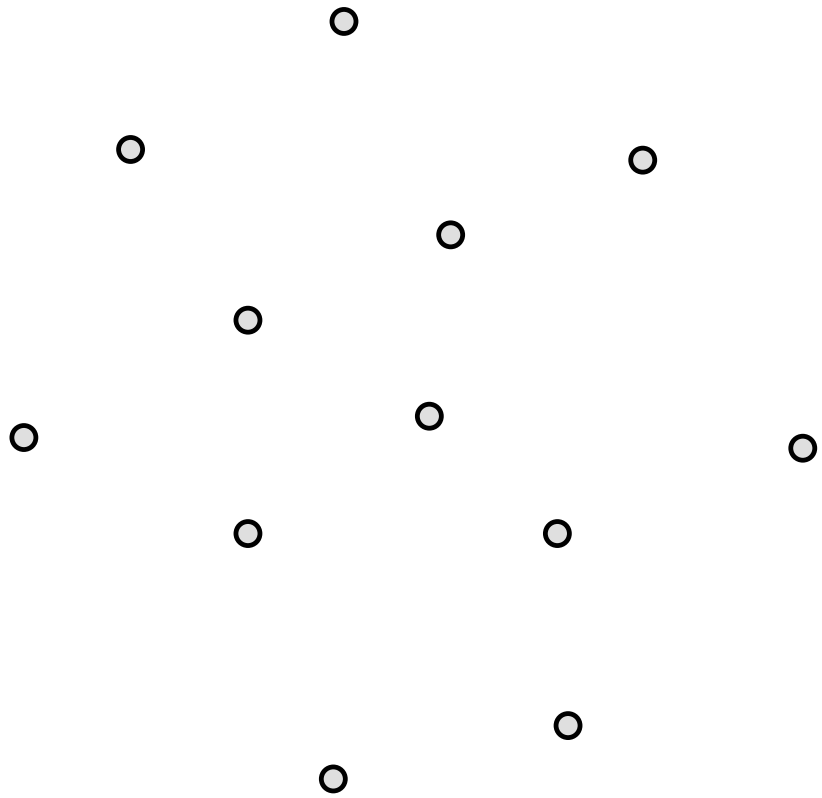

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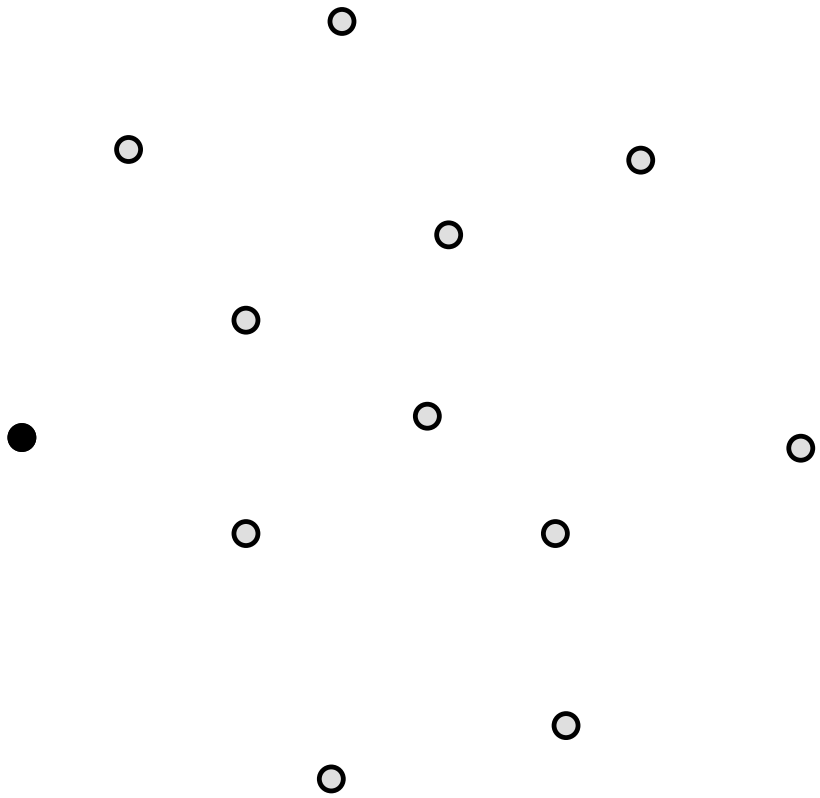

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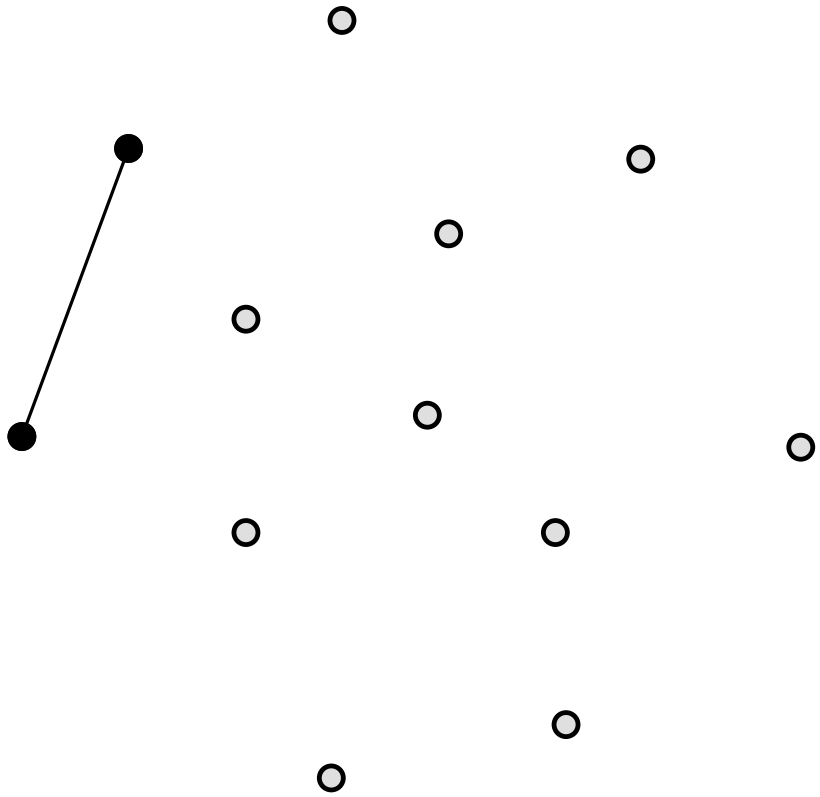

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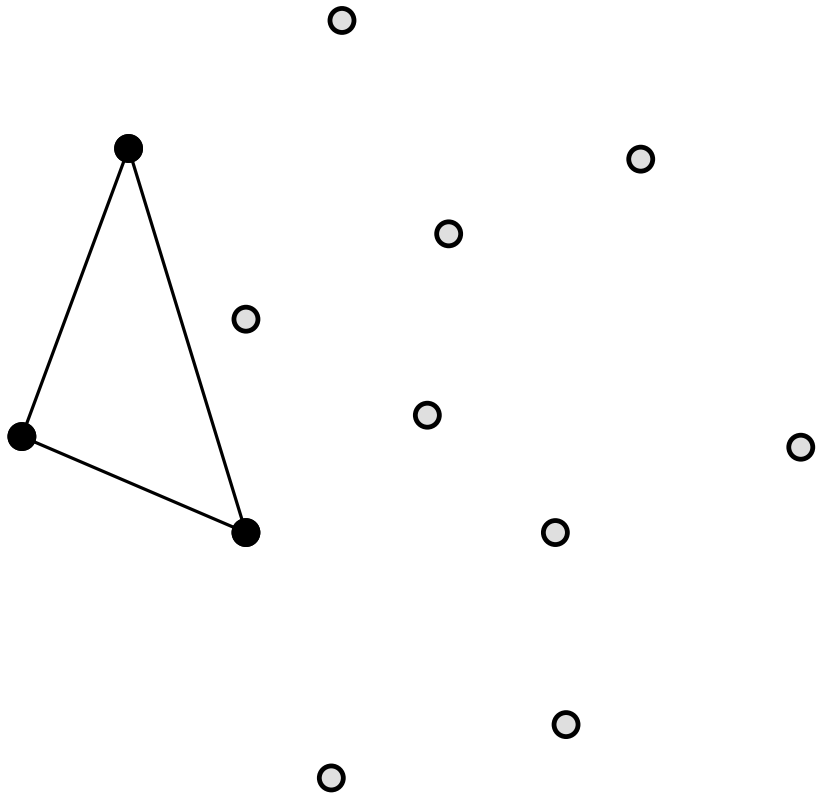

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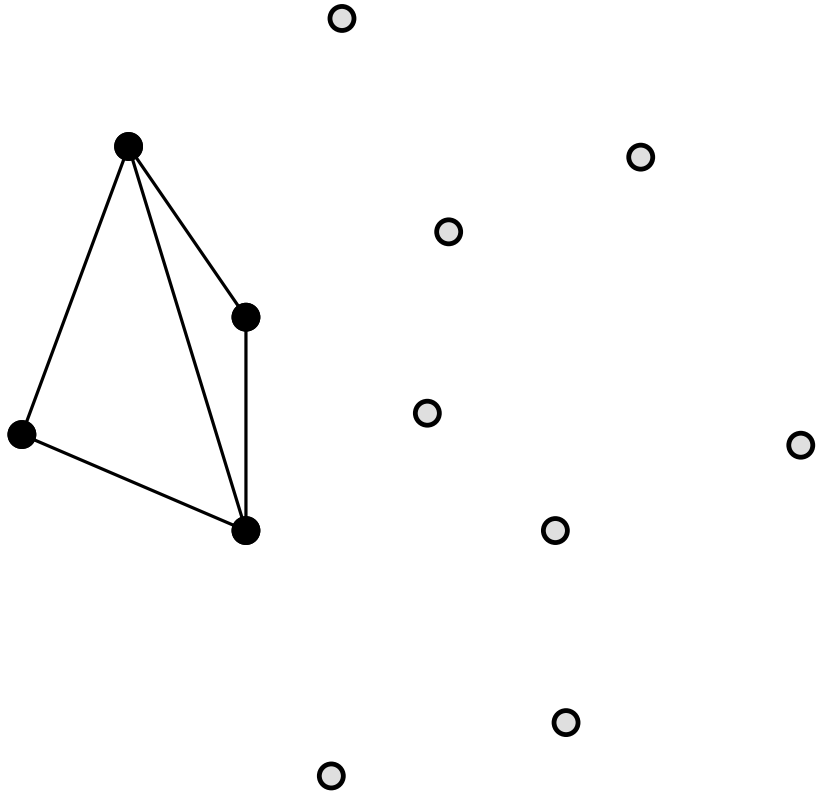

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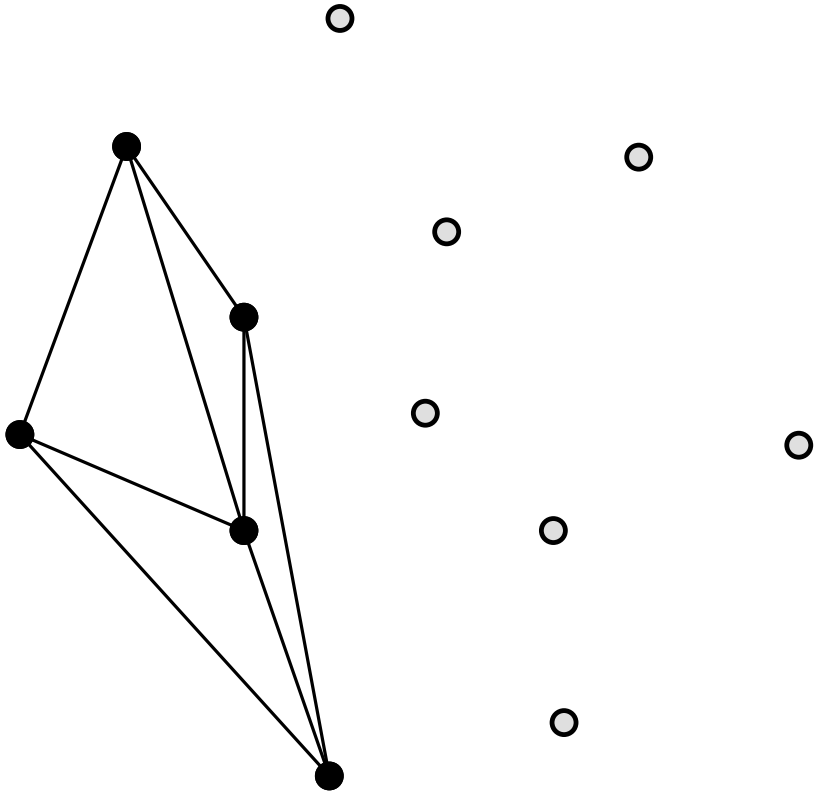

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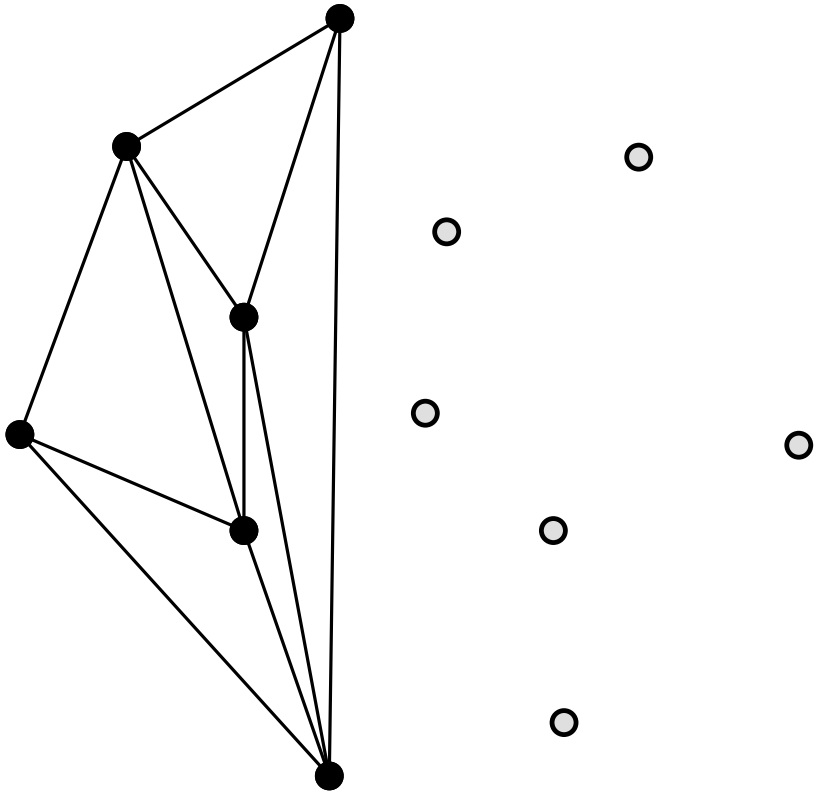

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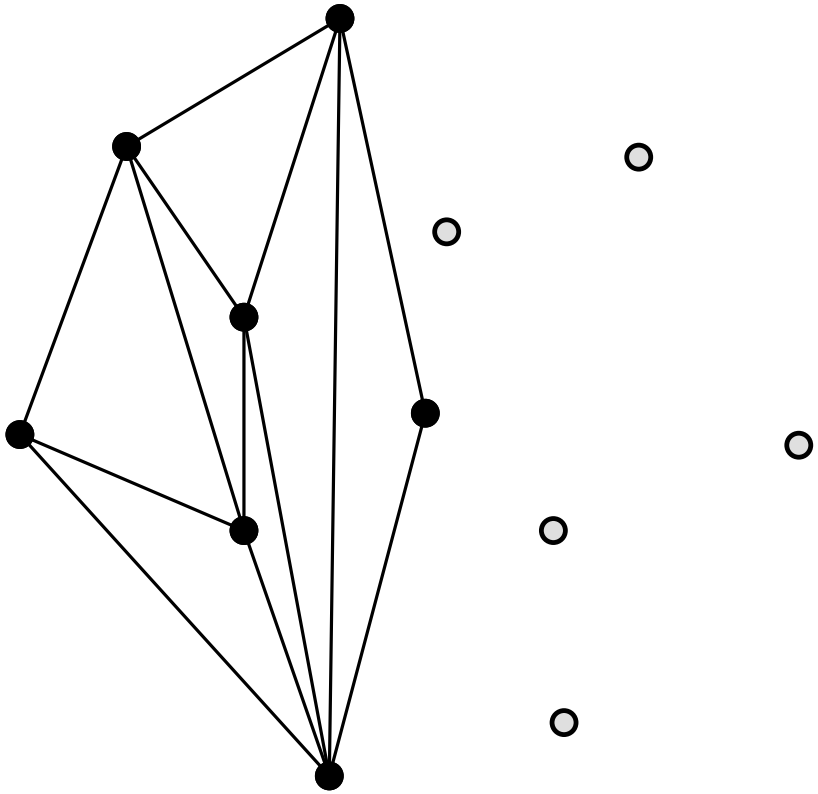

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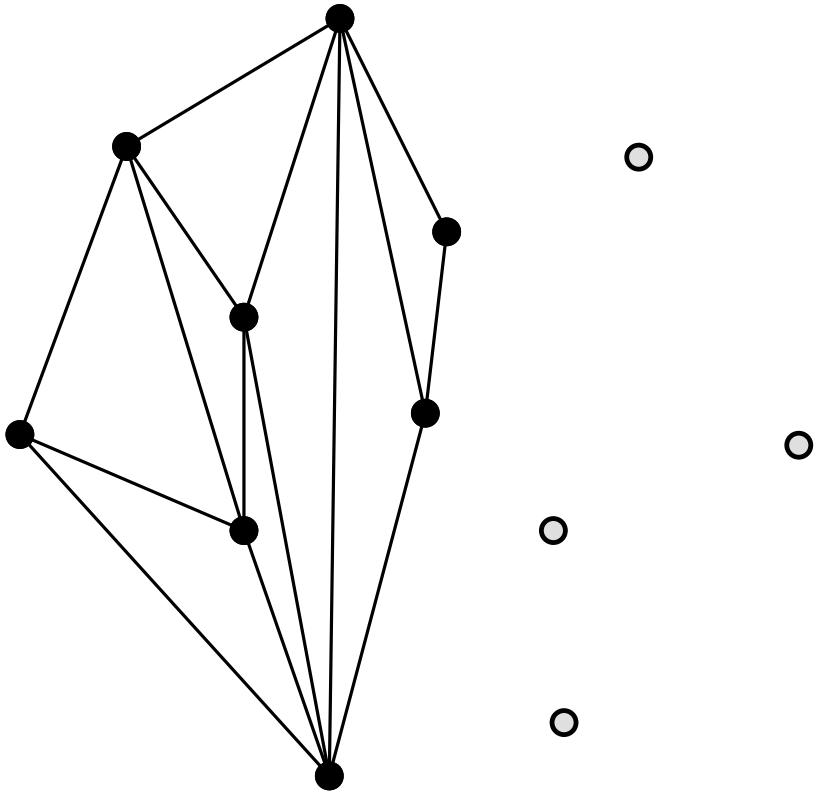

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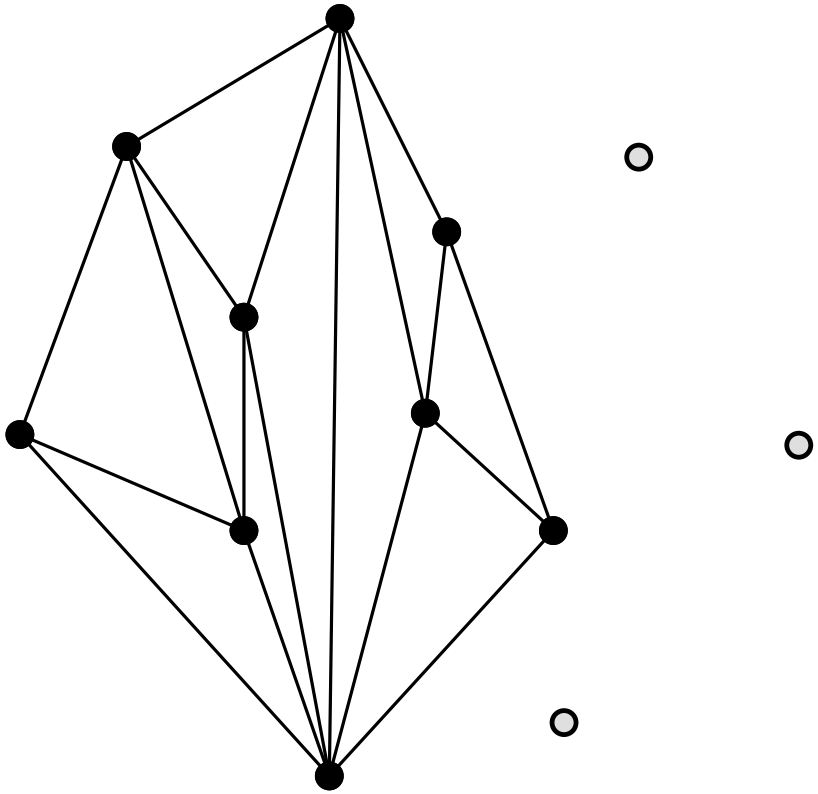

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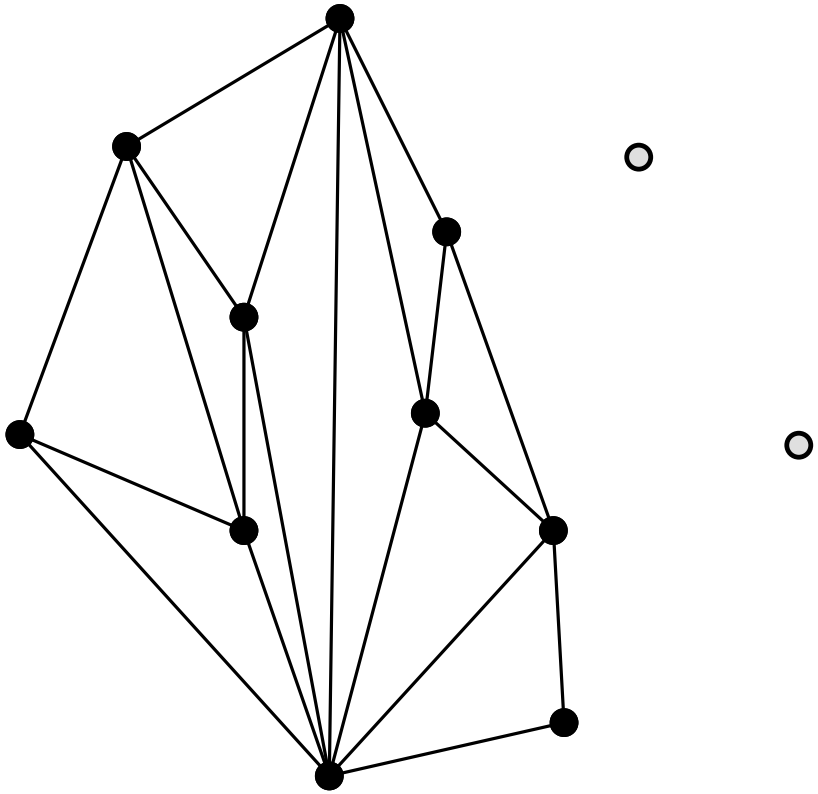

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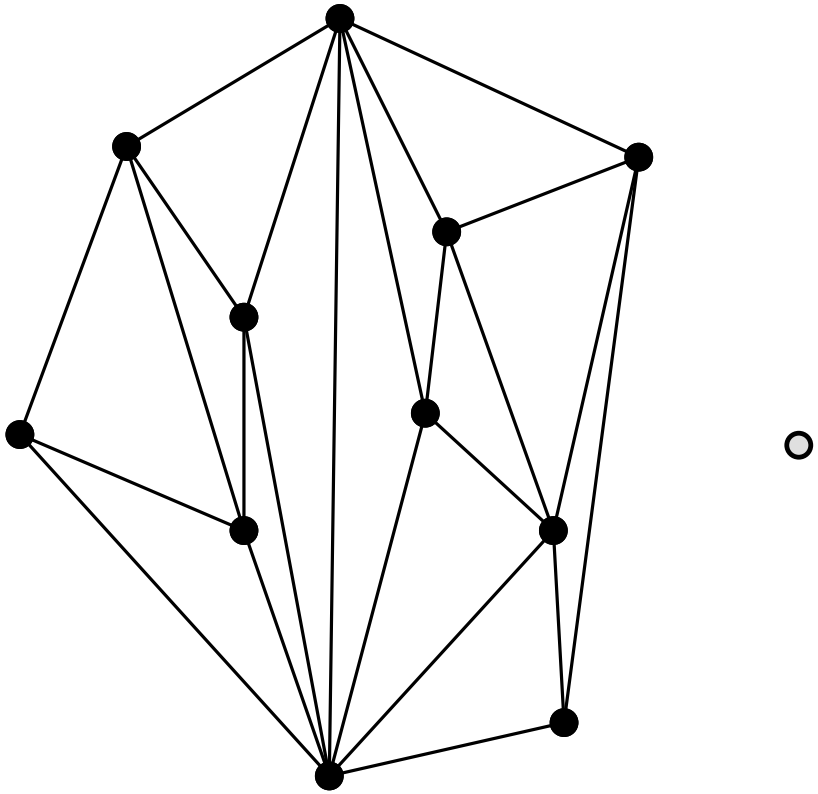

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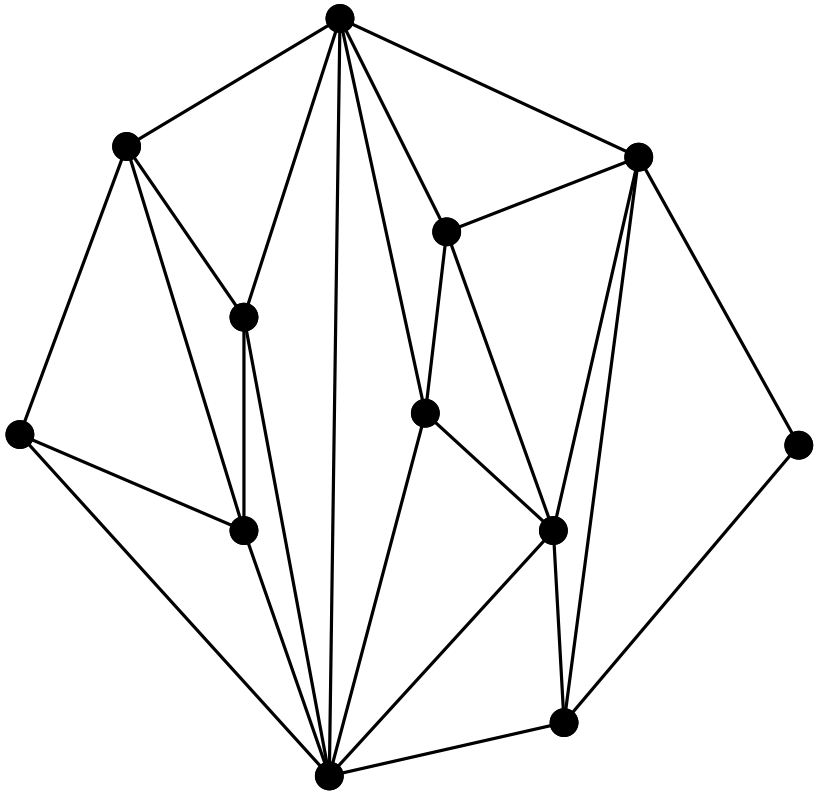

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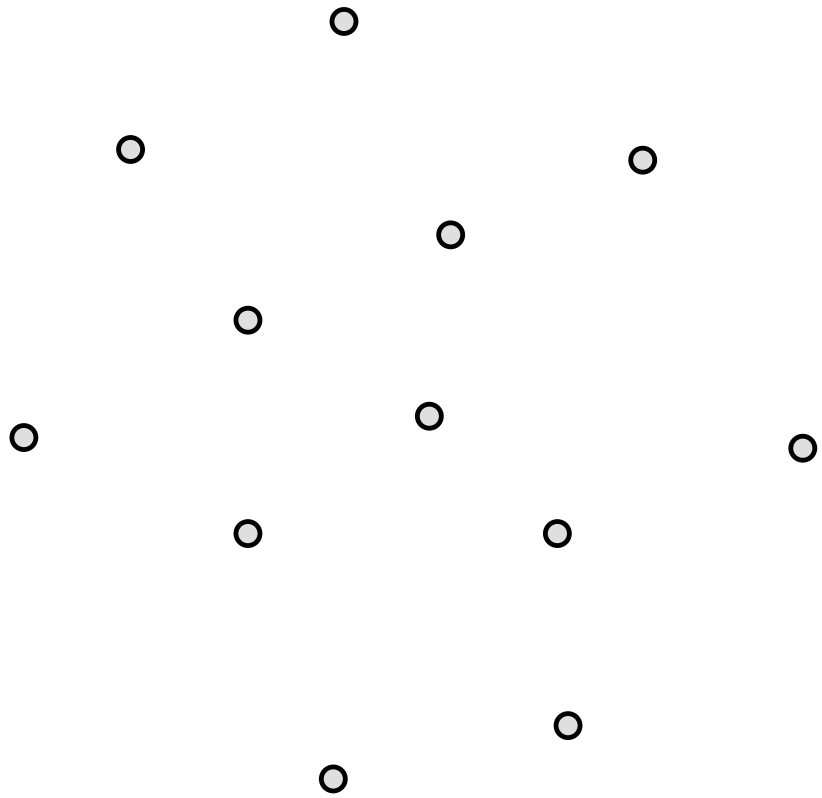

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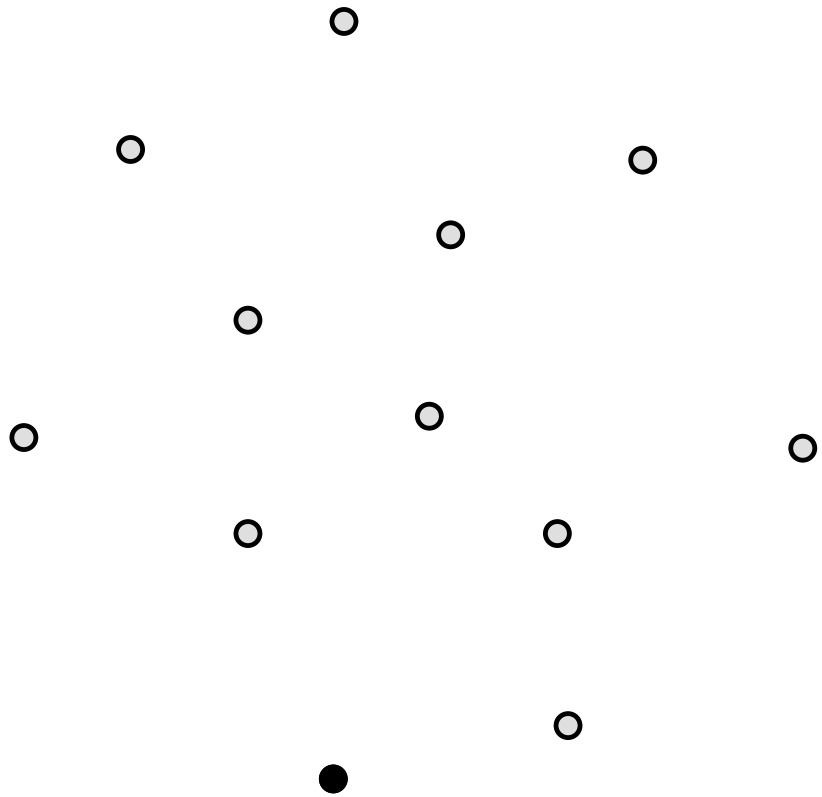

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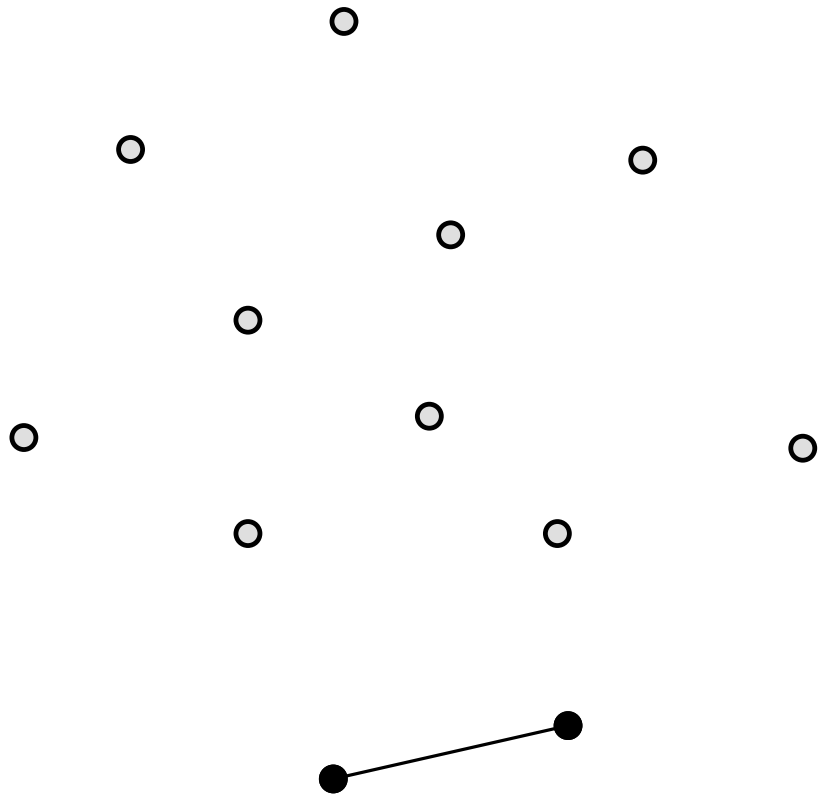

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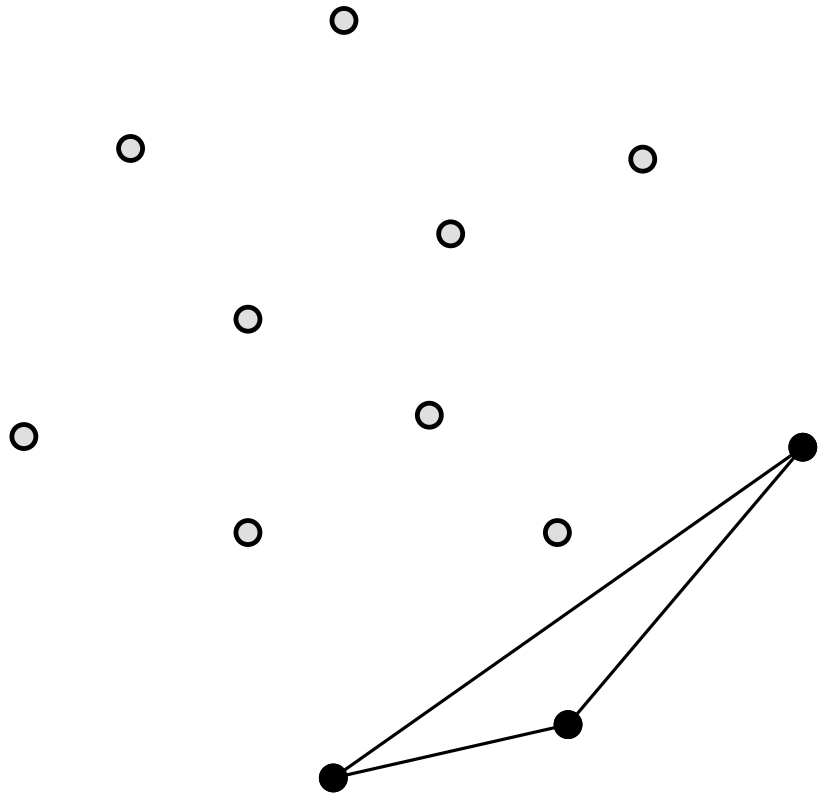

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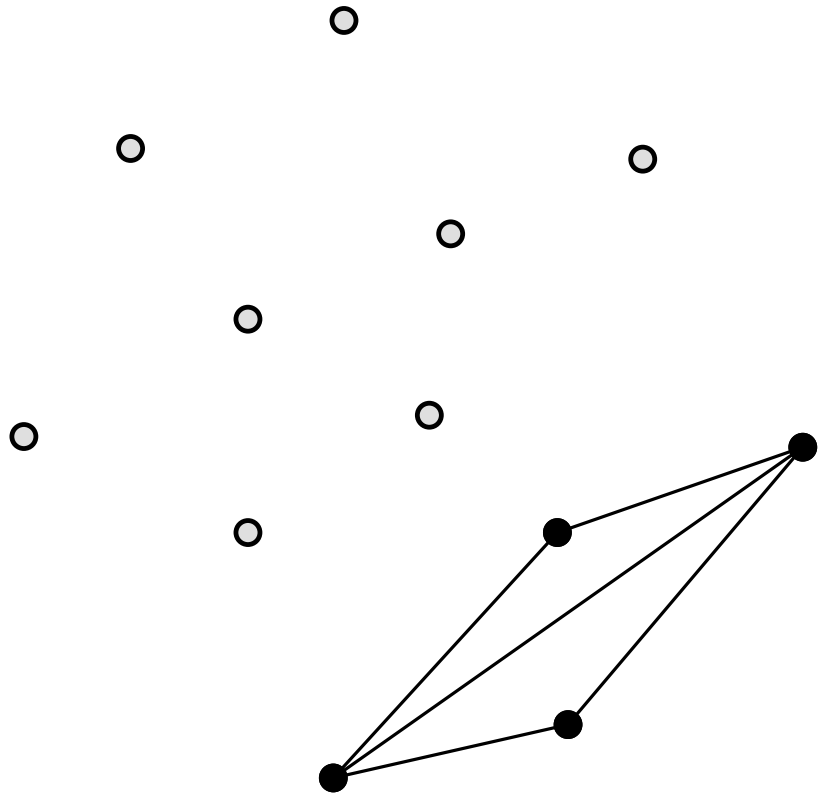

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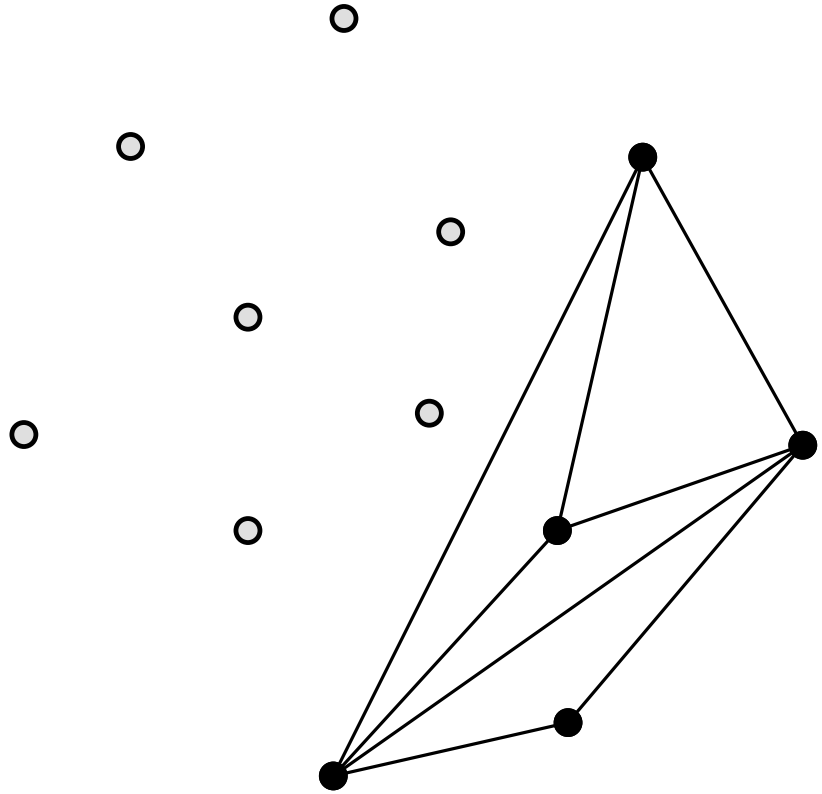

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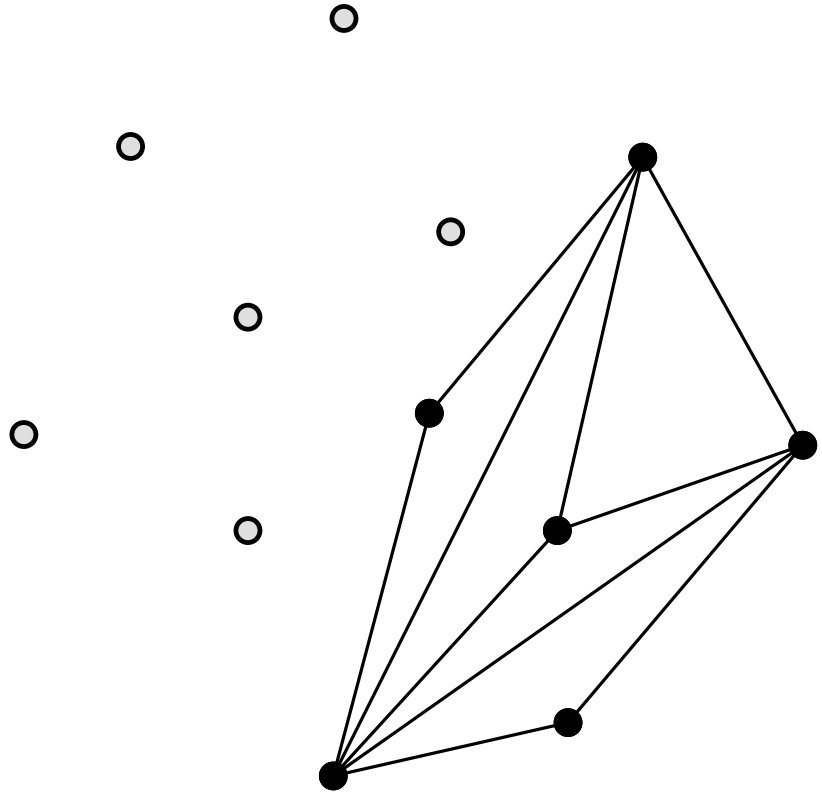

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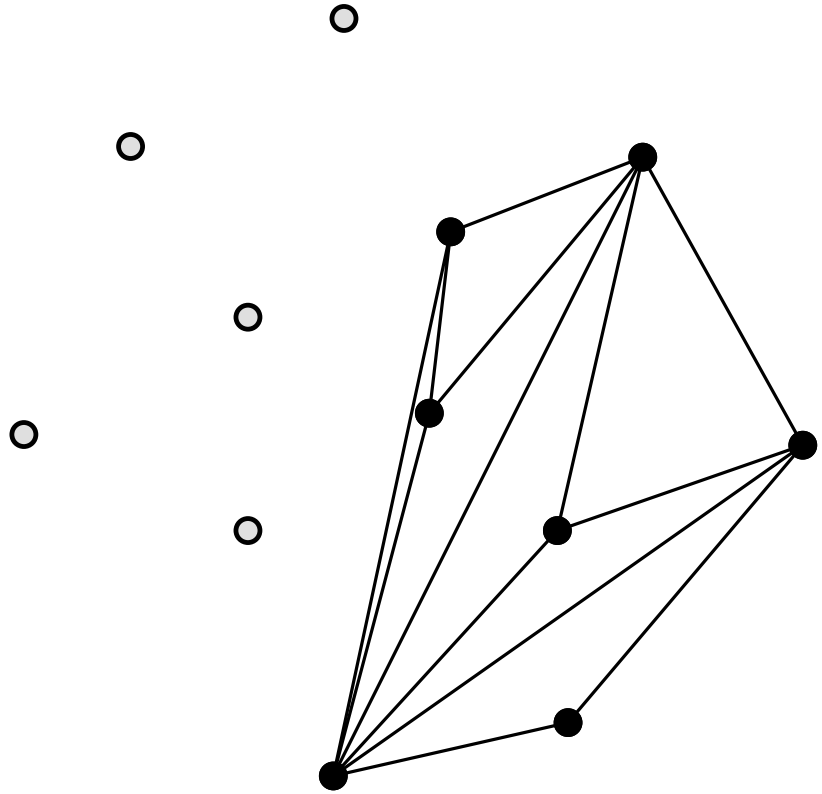

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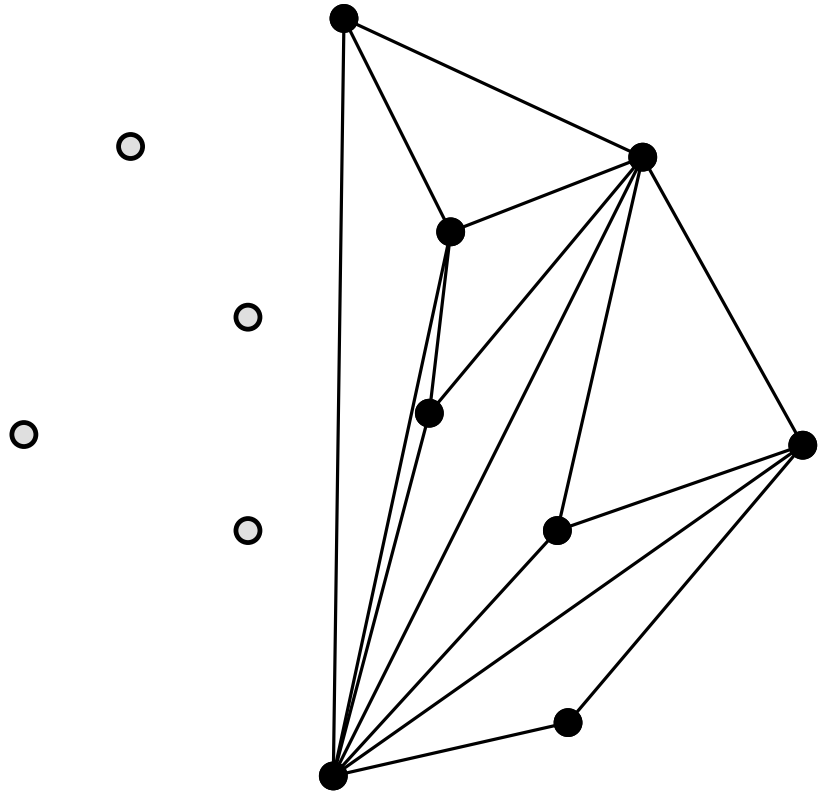

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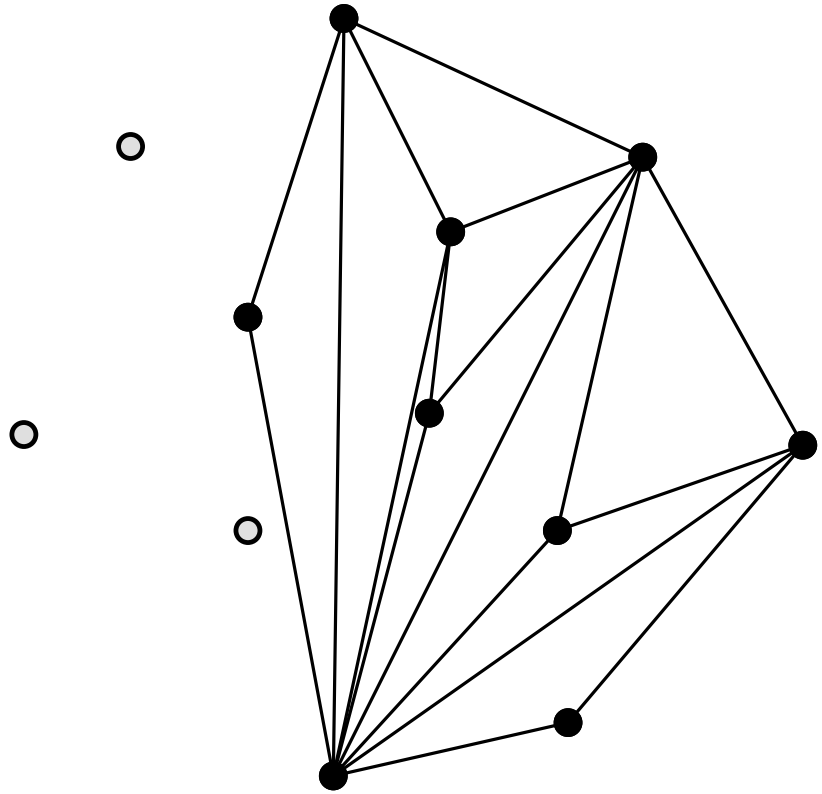

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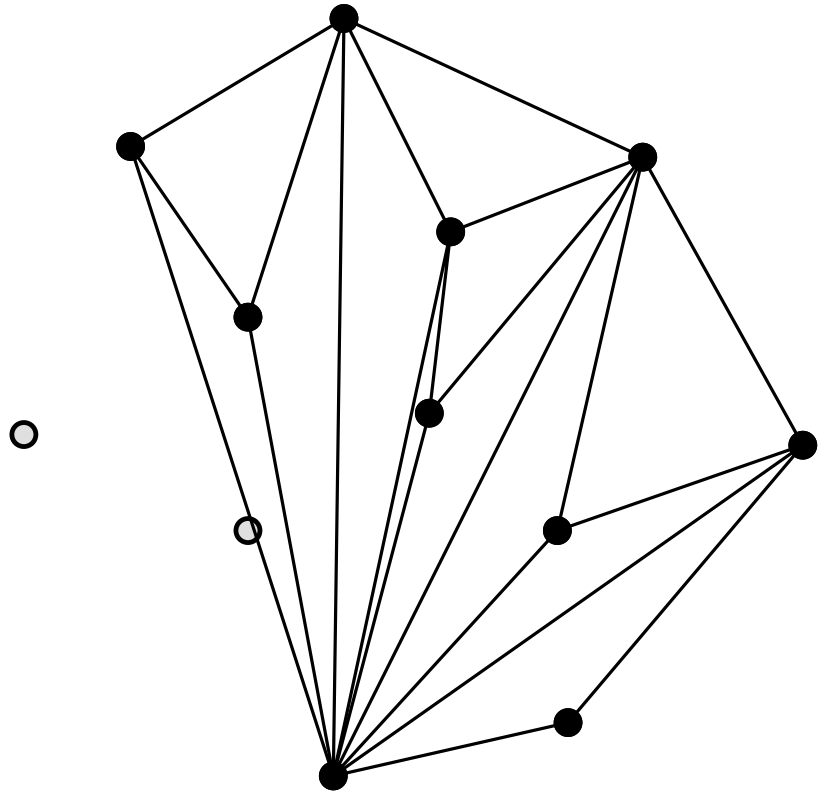

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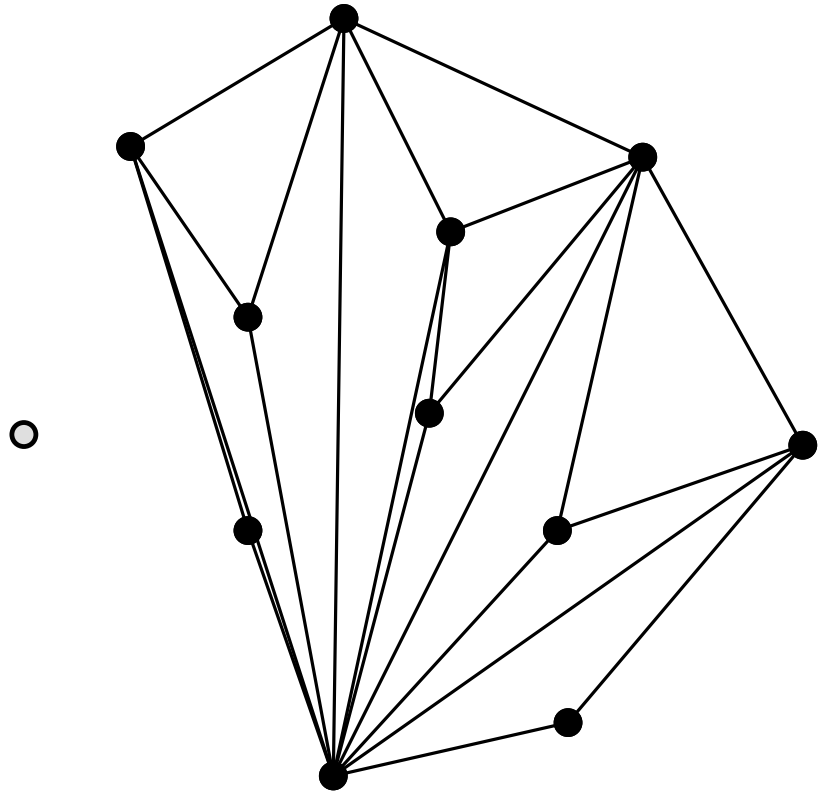

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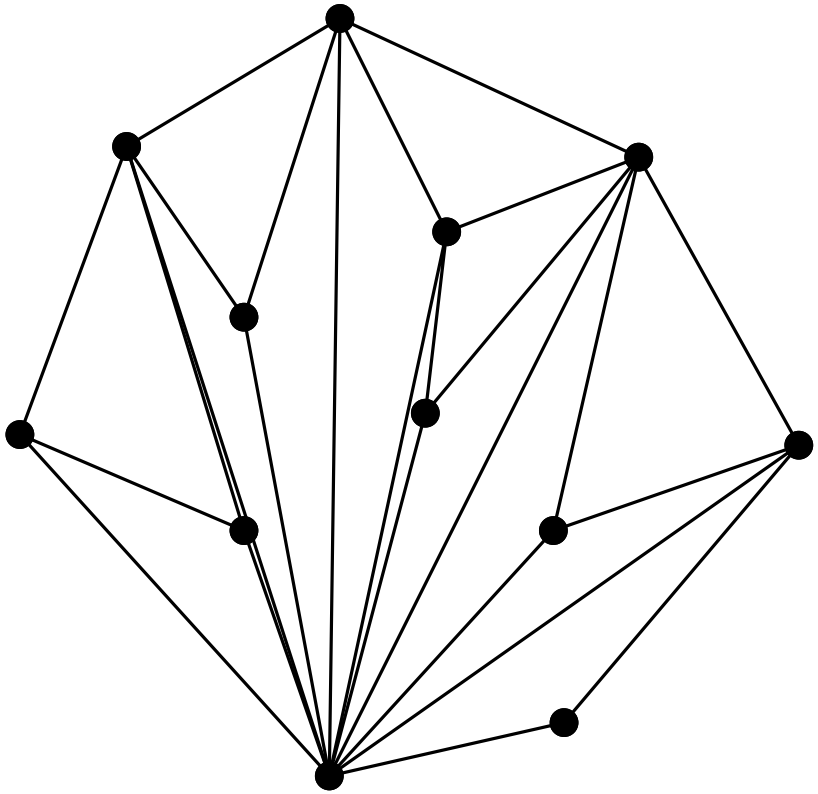
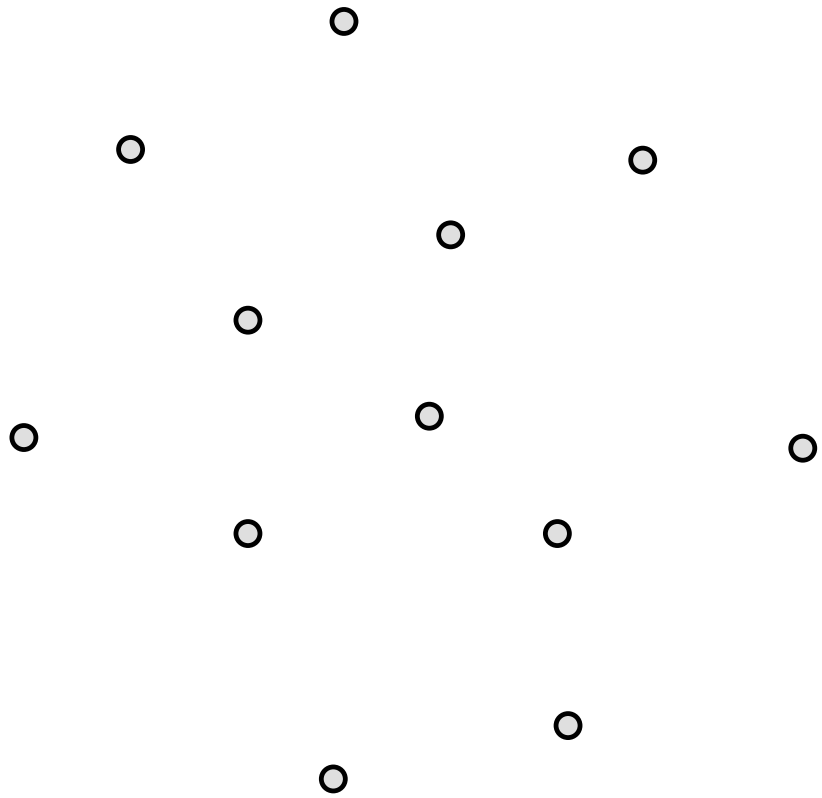


QUALITY OF A TRIANGULATION

Divide and conquer



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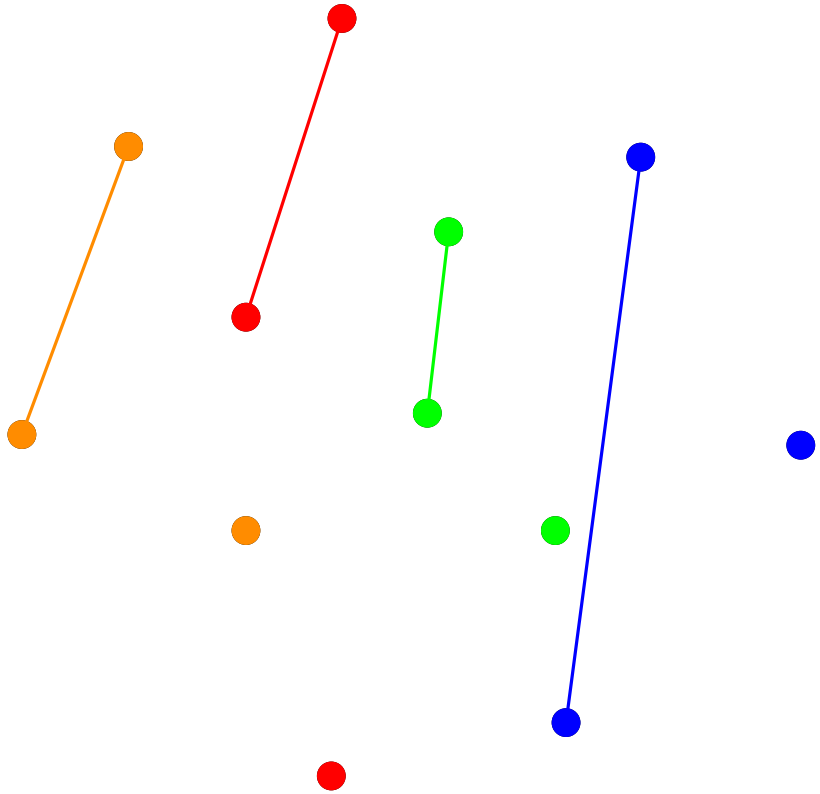

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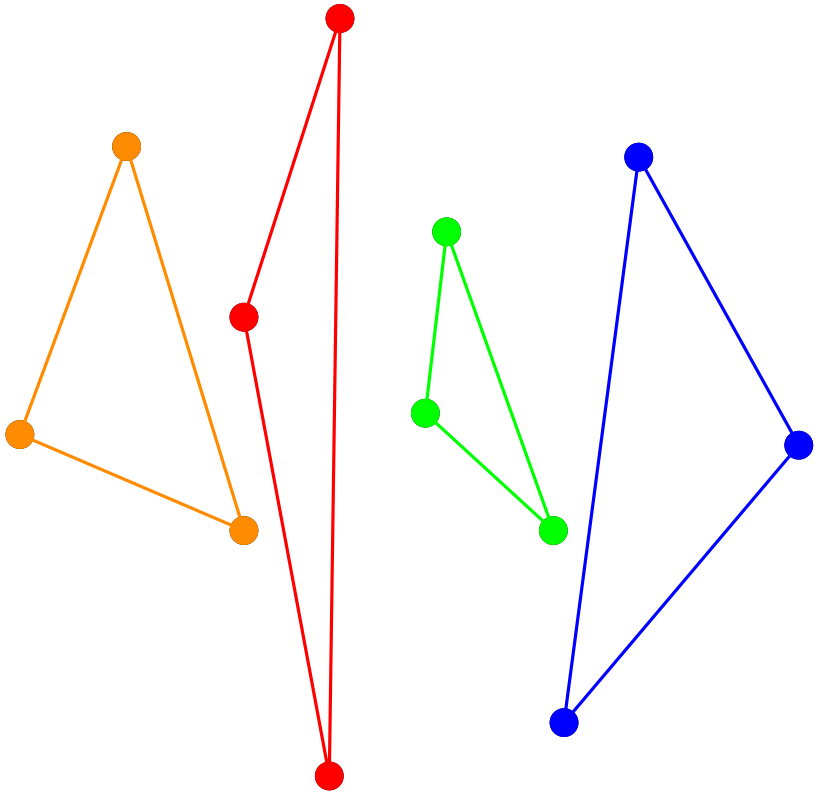

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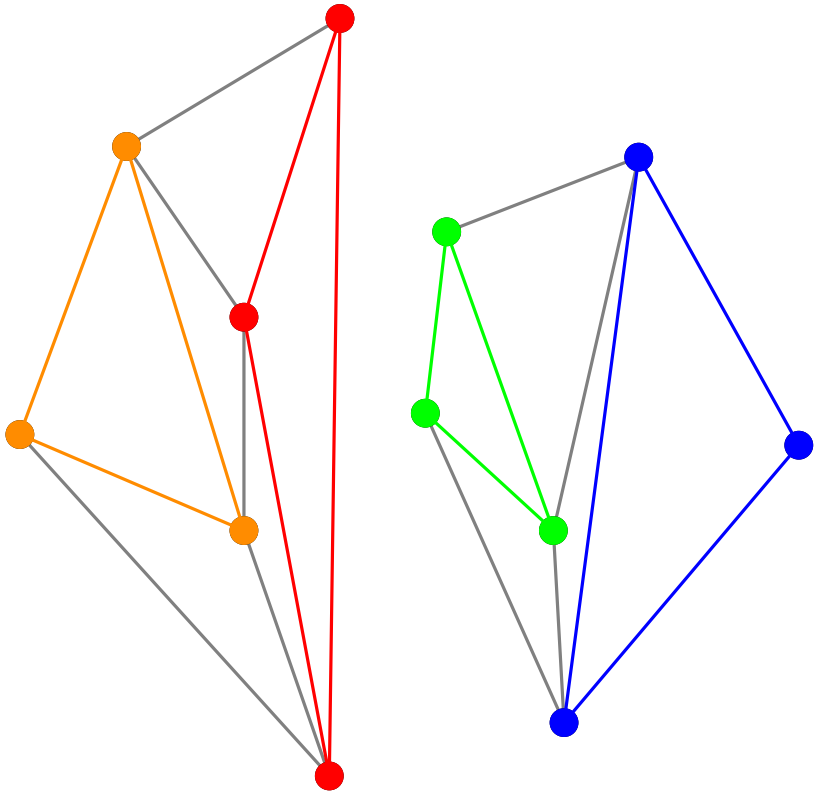

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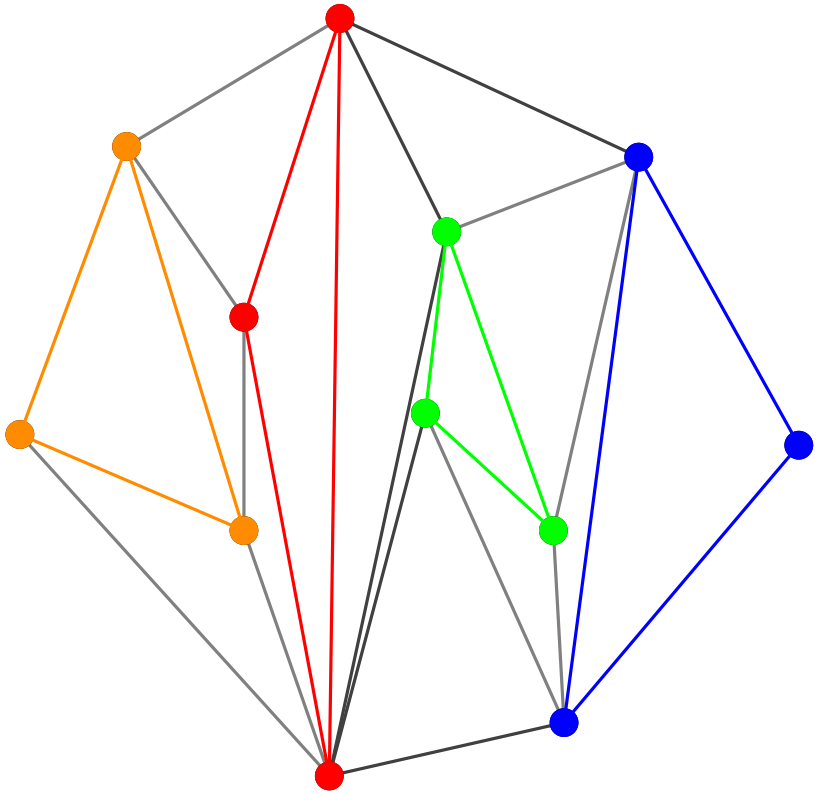

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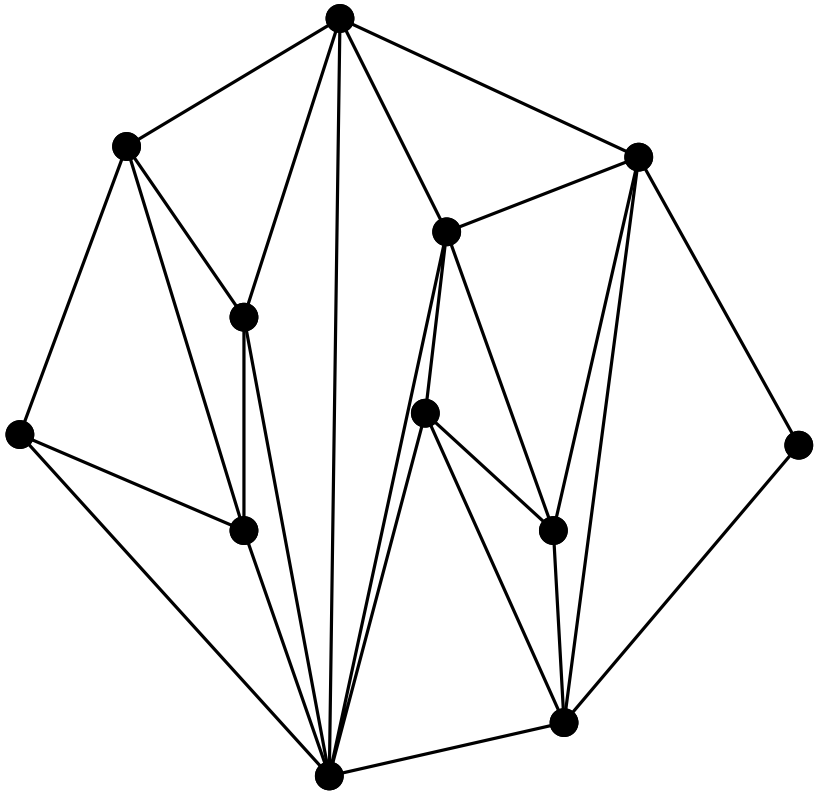
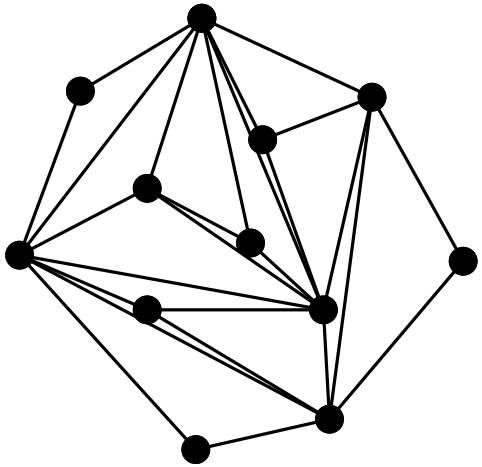
Divide and conquer

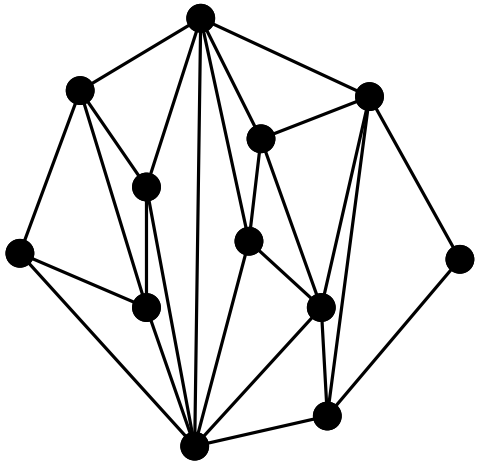


TRIANGULATING POINT SETS

QUALITY OF A TRIANGULATION



Incremental, unsorted



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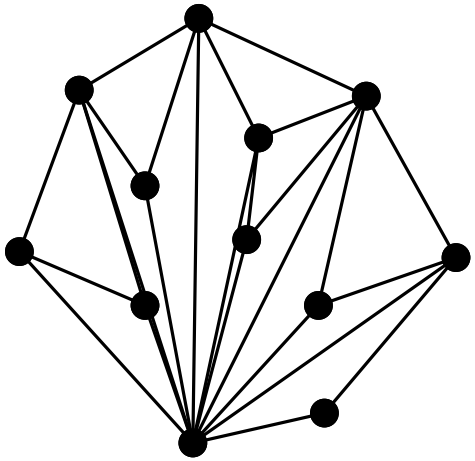
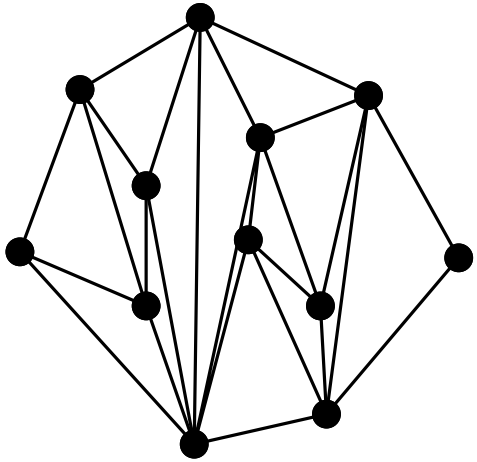


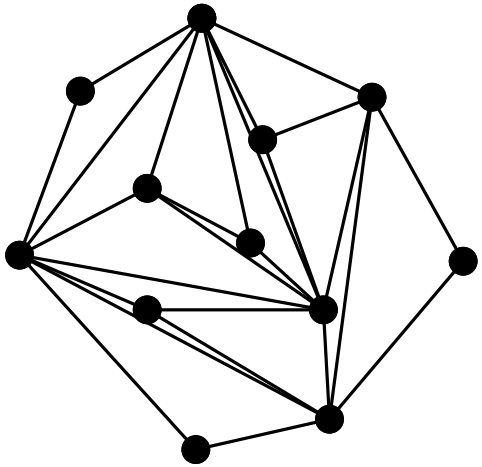
Graham's scan



Divide and conquer

TRIANGULATING POINT SETS

QUALITY OF A TRIANGULATION



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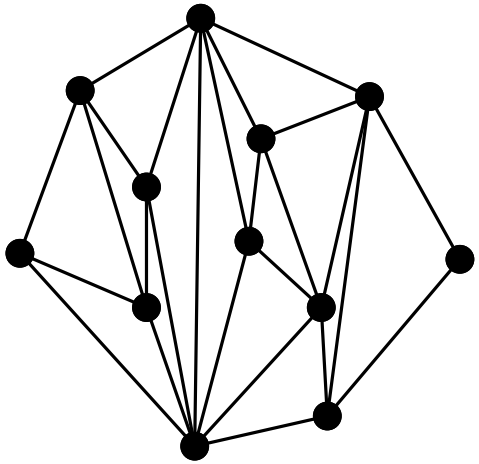

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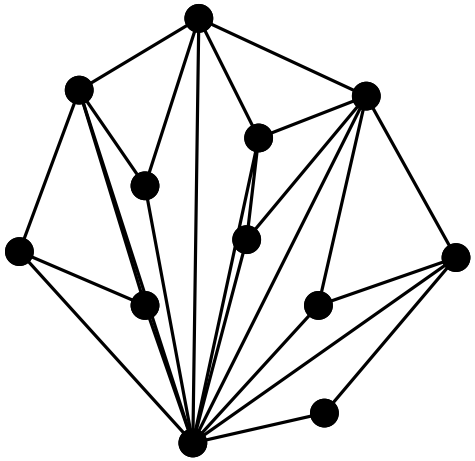
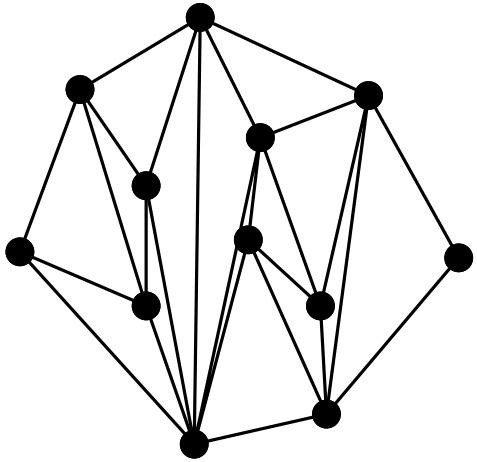
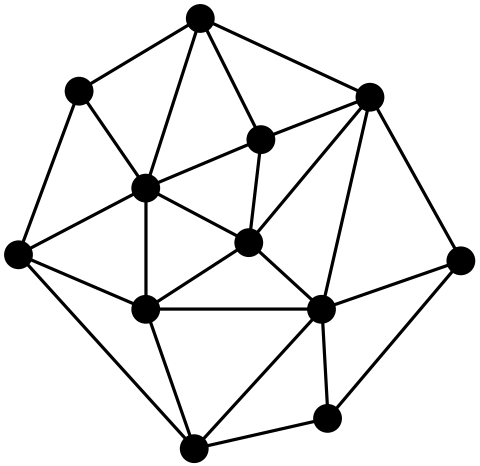


Graham's scan



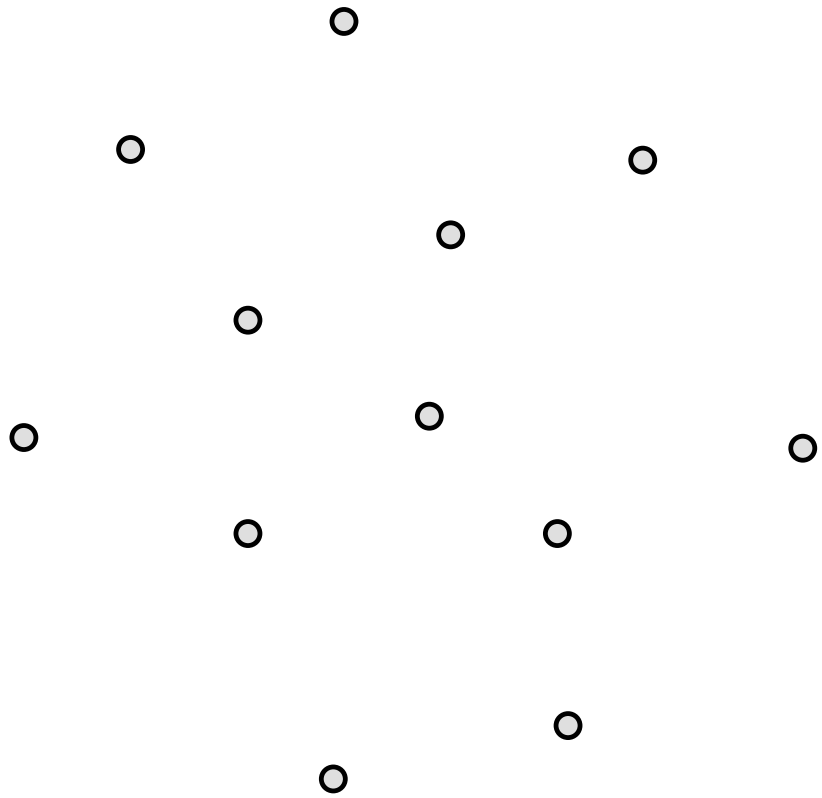
Divide and conquer



Delaunay triangulation

QUALITY OF A TRIANGULATION

Delaunay



QUALITY OF A TRIANGULATION

Delaunay