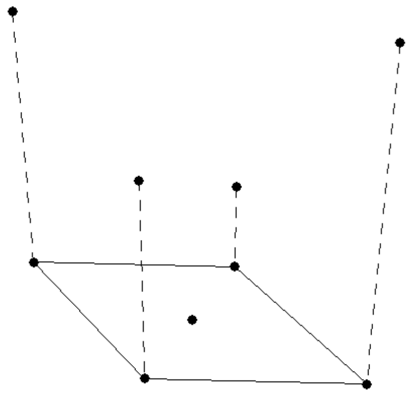
DELAUNAY TRIANGULATION OF POINT SETS

Vera Sacristán

Seminar: Geometric Algorithms (MIRI)
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

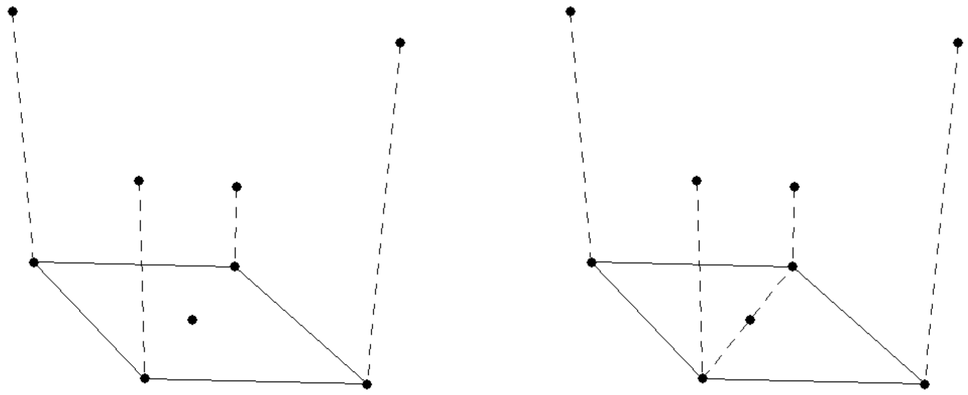
A TOOL FOR INTERPOLATION

A TOOL FOR INTERPOLATION



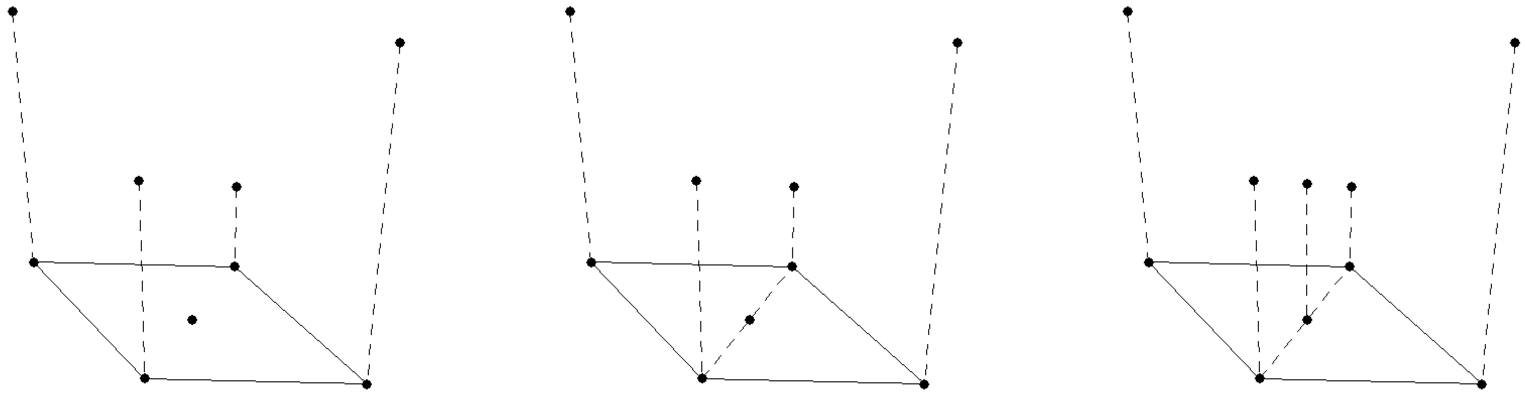
DELAUNAY TRIANGULATION

A TOOL FOR INTERPOLATION

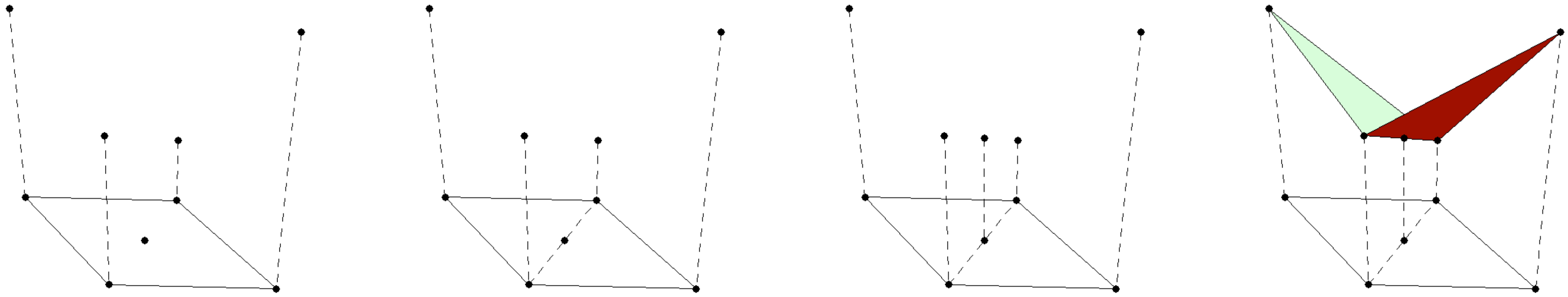


DELAUNAY TRIANGULATION

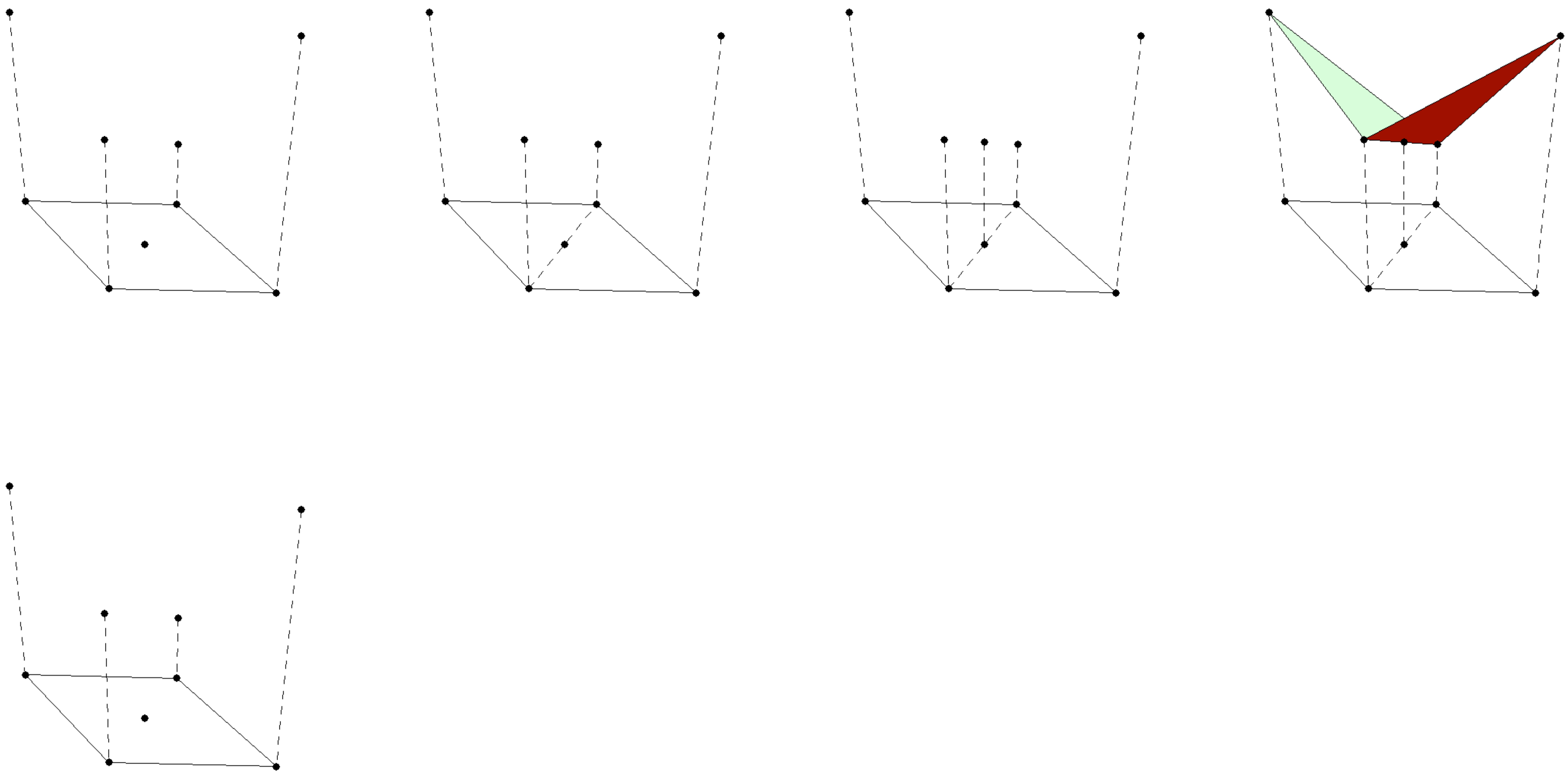
A TOOL FOR INTERPOLATION



A TOOL FOR INTERPOLATION

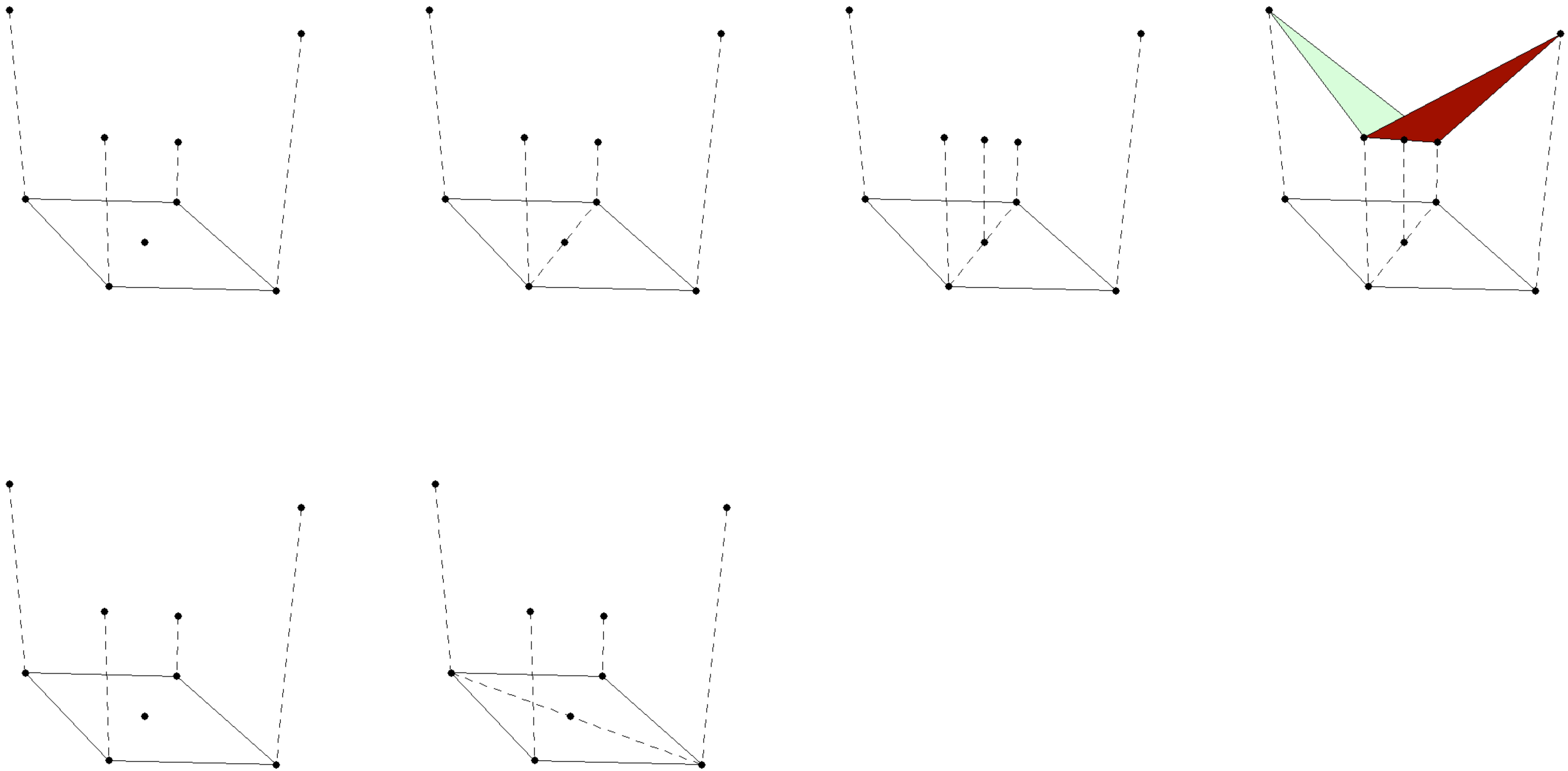


A TOOL FOR INTERPOLATION

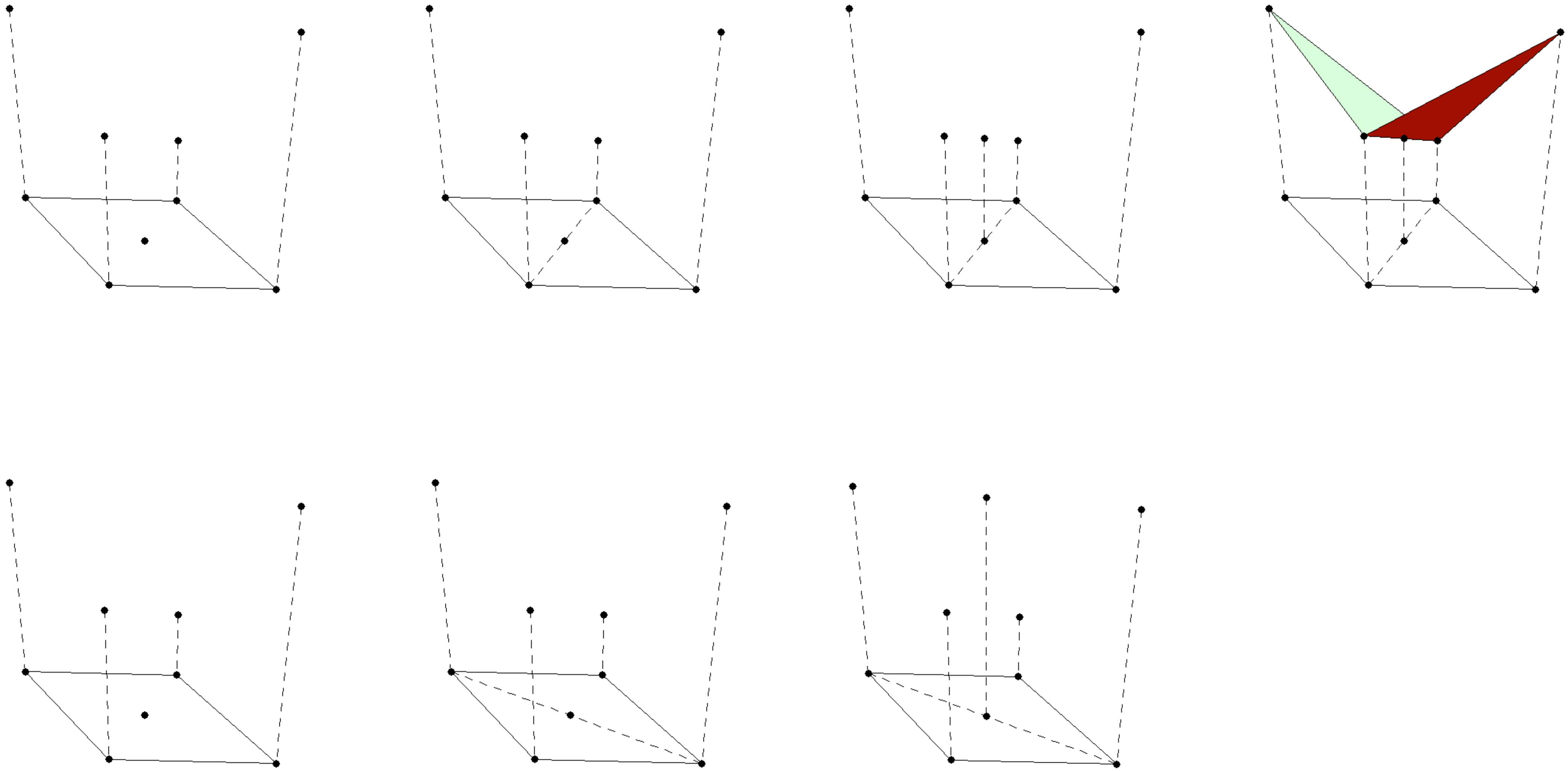


DELAUNAY TRIANGULATION

A TOOL FOR INTERPOLATION

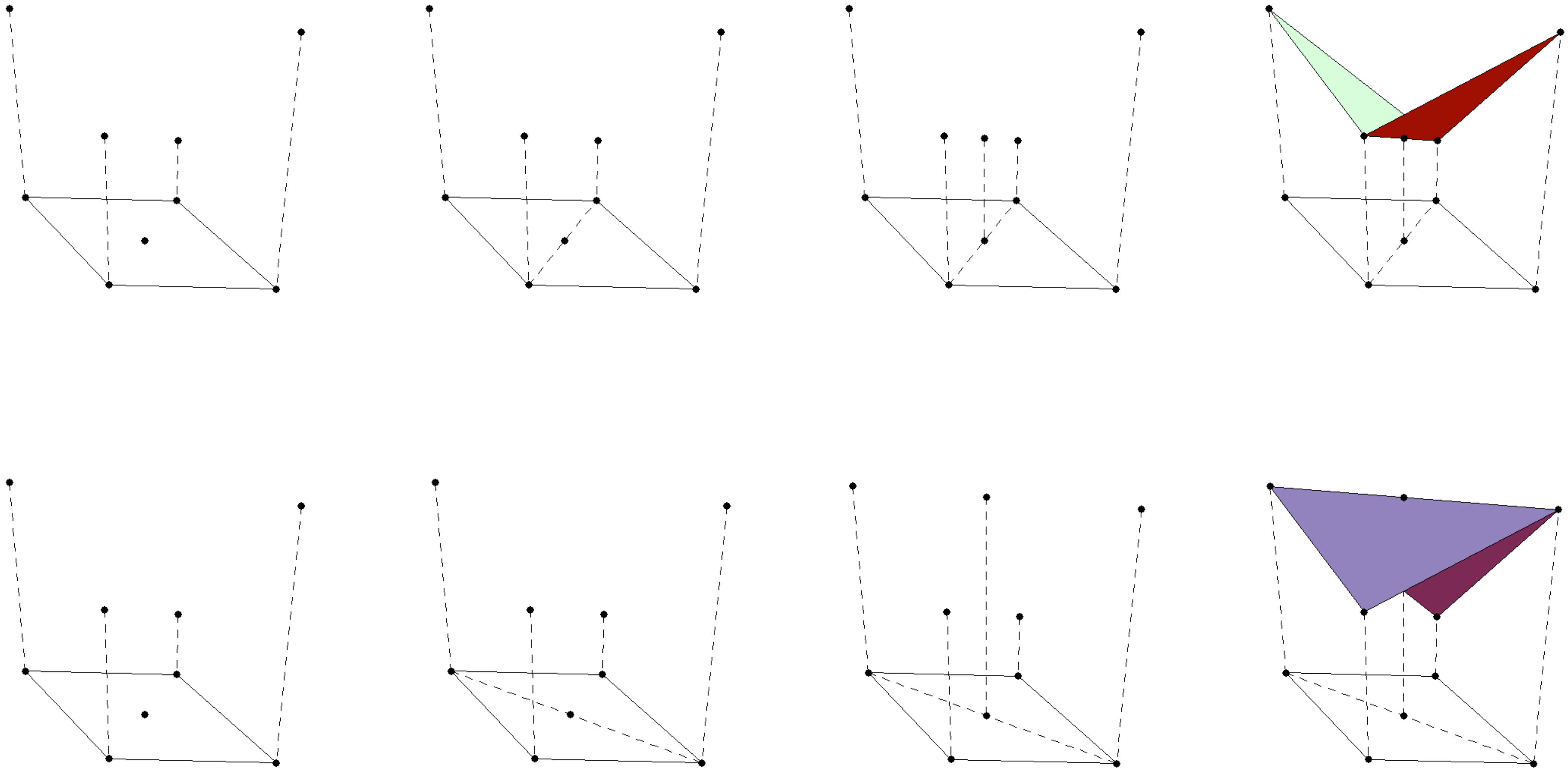


A TOOL FOR INTERPOLATION



DELAUNAY TRIANGULATION

A TOOL FOR INTERPOLATION



DELAUNAY TRIANGULATION

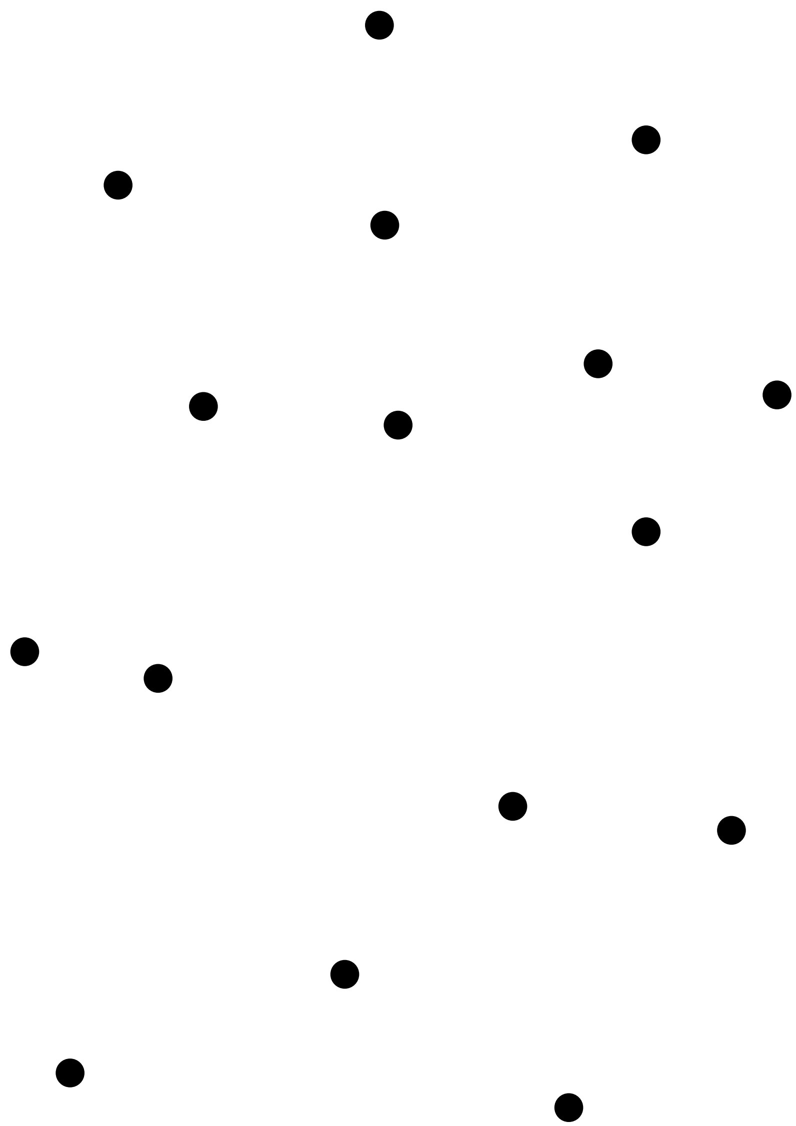
DEFINITION AND PROPERTIES

DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

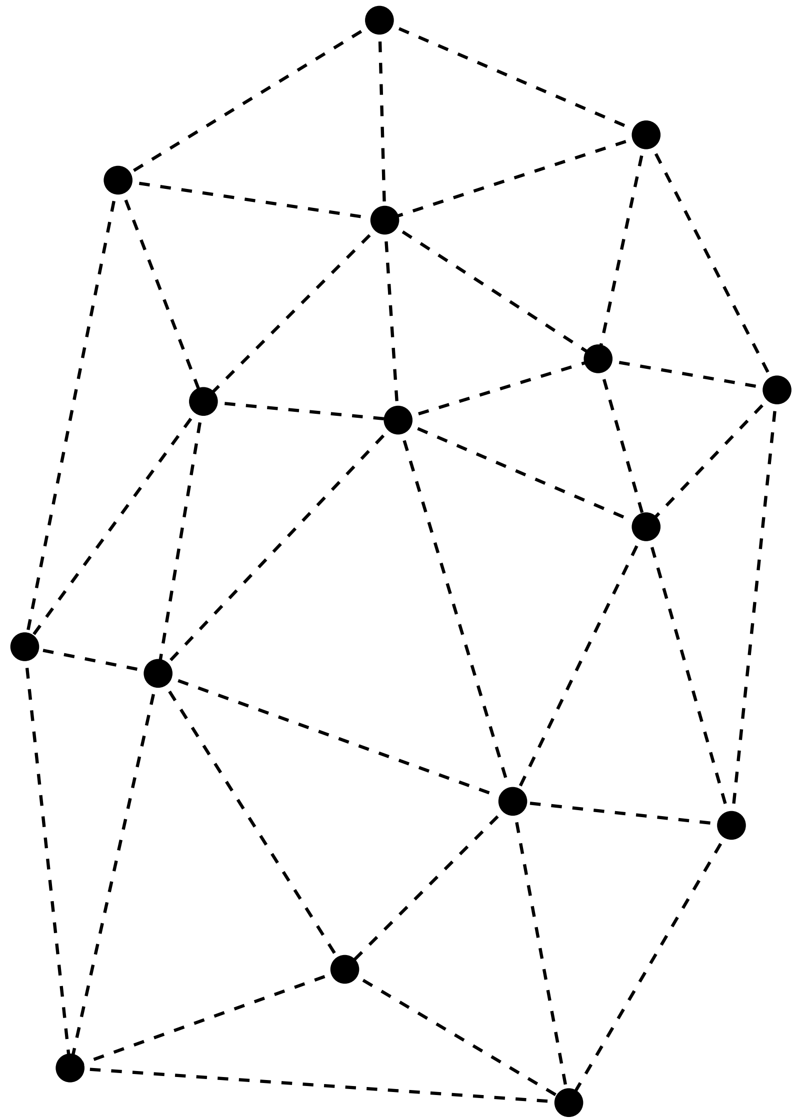
Given a set P with n points in the plane...



DEFINITION AND PROPERTIES

Definition

Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

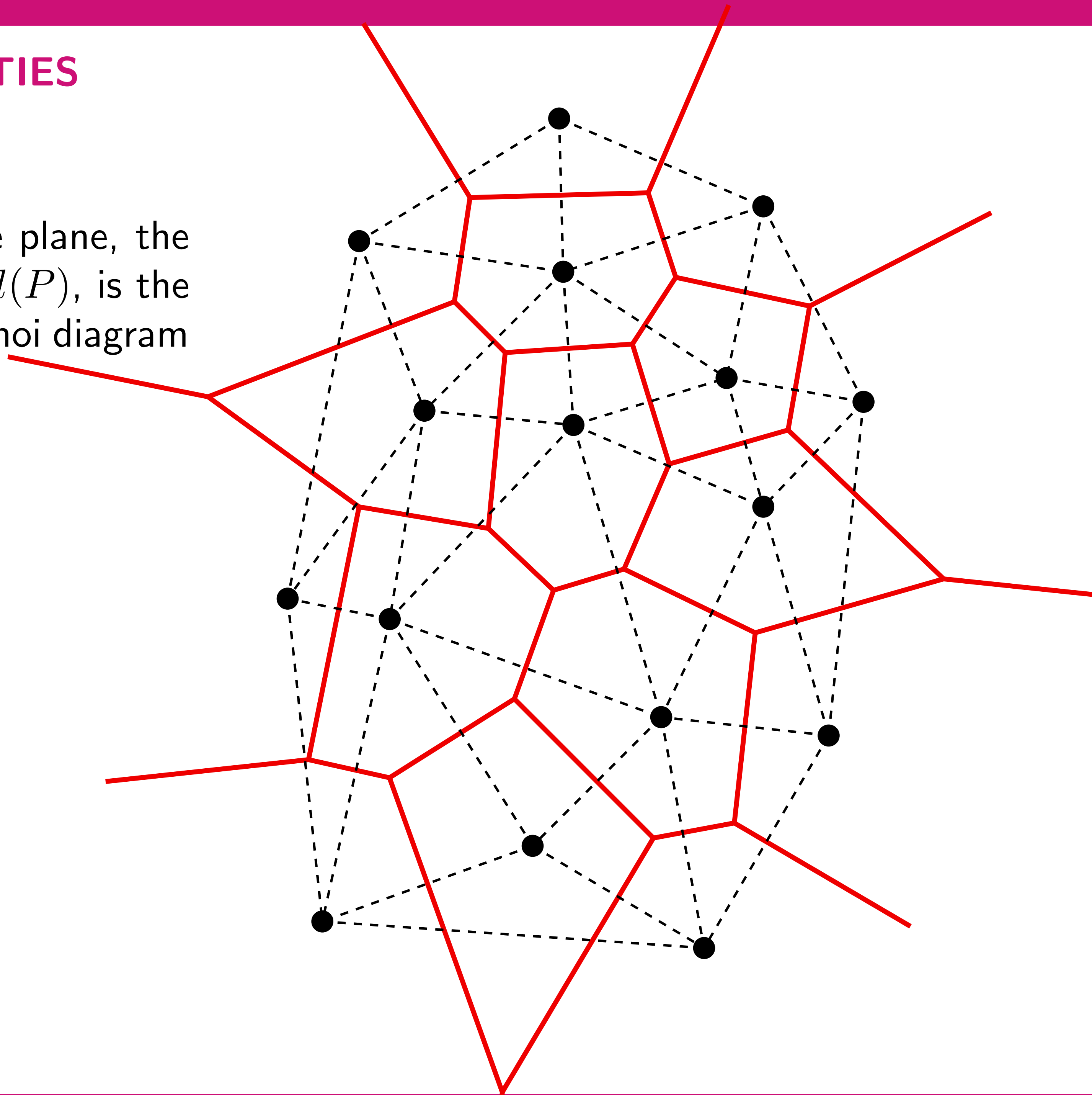


DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

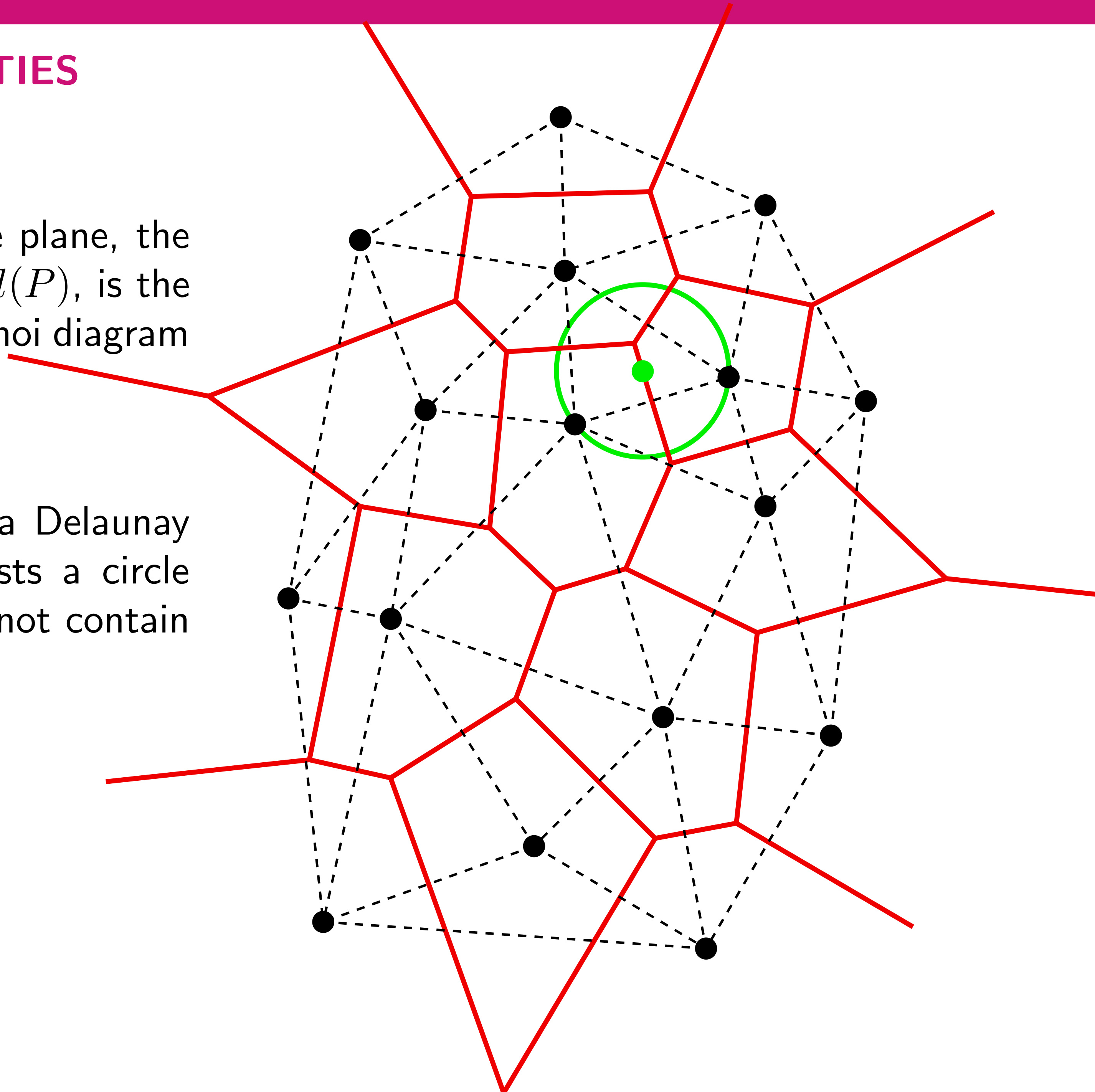
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.



DEFINITION AND PROPERTIES

Definition

Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.



Characterization

- Two points $p_i, p_j \in P$ form a Delaunay edge if and only if there exists a circle through p_i and p_j wich does not contain any point of P in its interior.

DEFINITION AND PROPERTIES

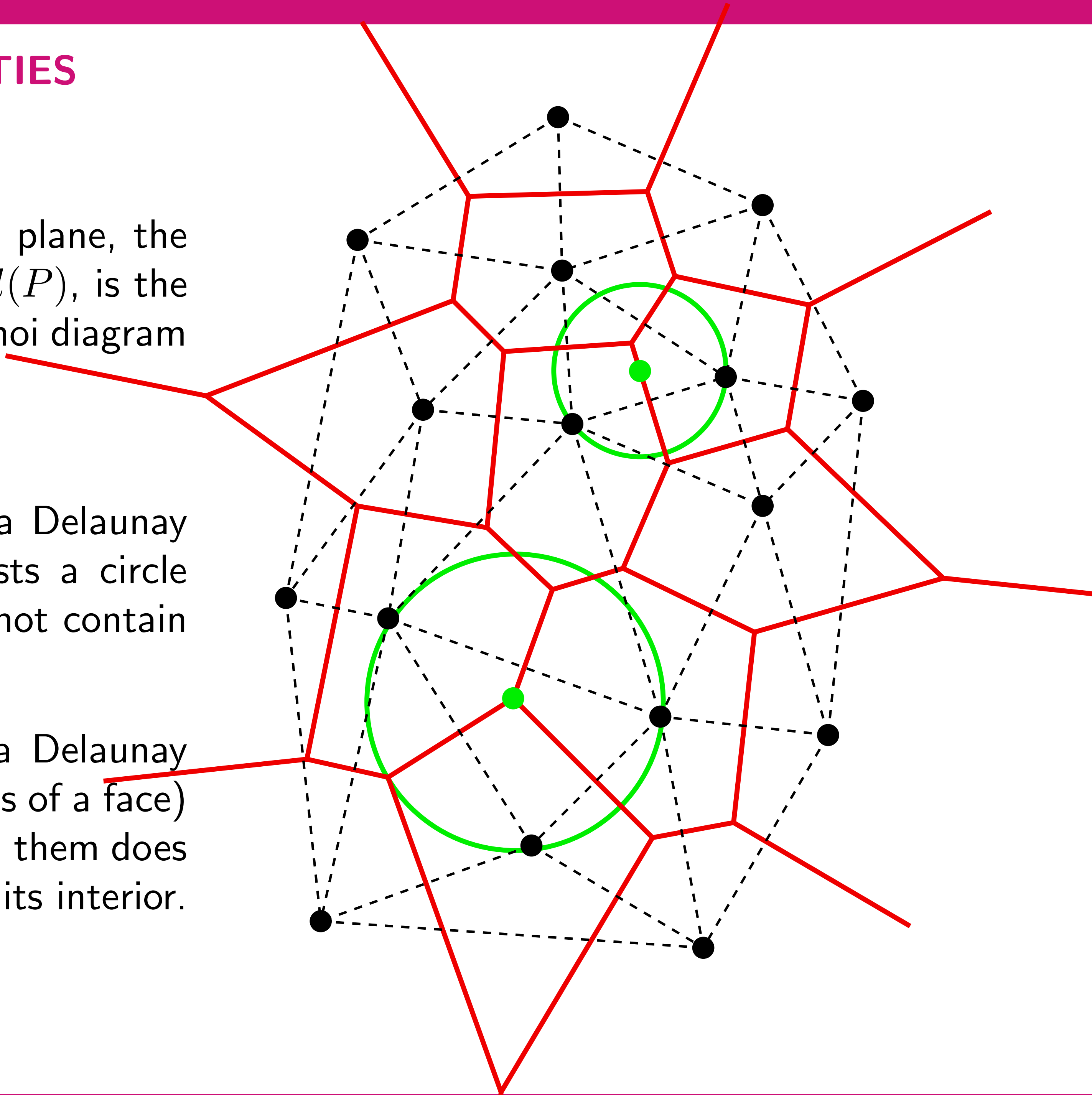
Definition

Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Characterization

- Two points $p_i, p_j \in P$ form a Delaunay edge if and only if there exists a circle through p_i and p_j wich does not contain any point of P in its interior.

- Three points p_i, p_j, p_k form a Delaunay triangle (in general, are vertices of a face) if and only if the circle through them does not contain any point of P in its interior.



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

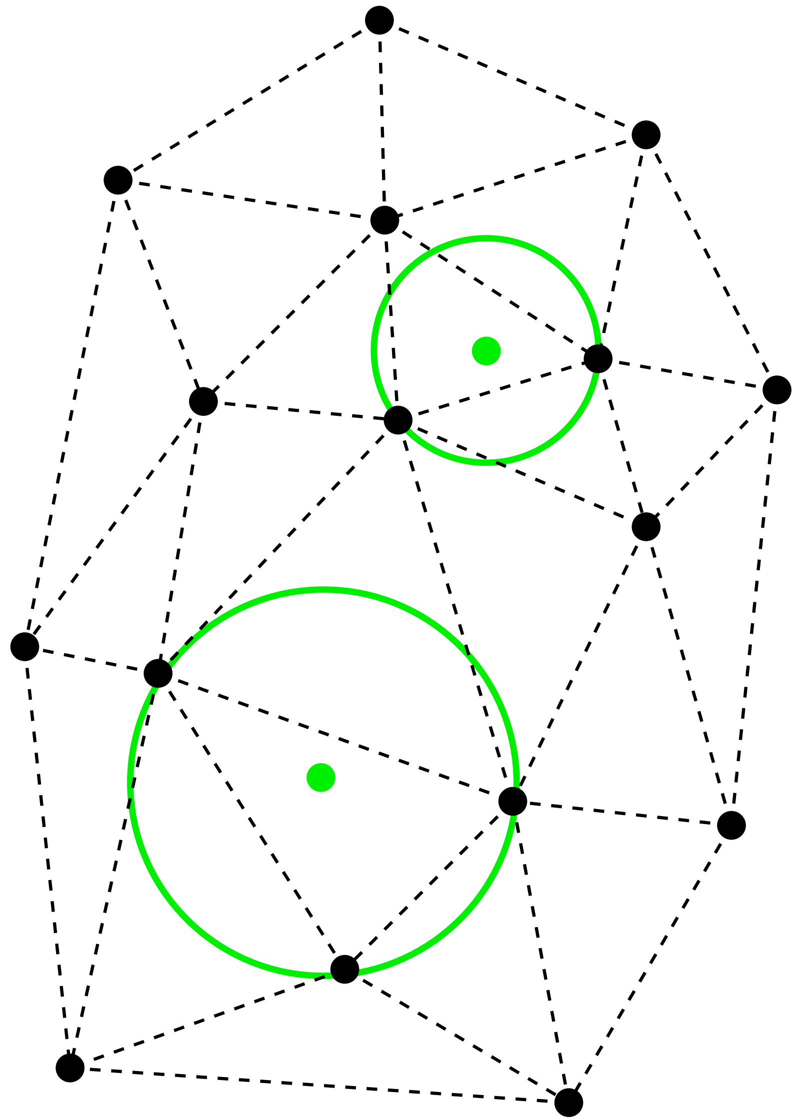
Definition

Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Characterization

- Two points $p_i, p_j \in P$ form a Delaunay edge if and only if there exists a circle through p_i and p_j wich does not contain any point of P in its interior.

- Three points p_i, p_j, p_k form a Delaunay triangle (in general, are vertices of a face) if and only if the circle through them does not contain any point of P in its interior.



DELAUNAY TRIANGULATION

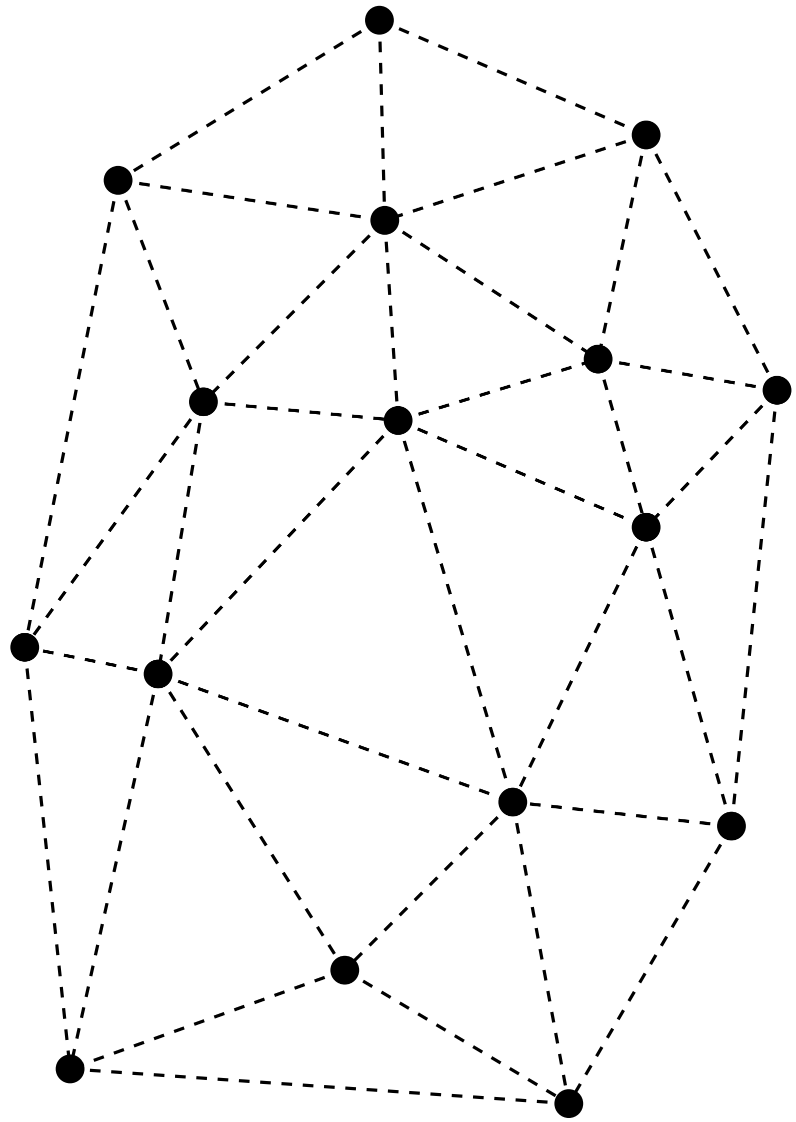
DEFINITION AND PROPERTIES

Definition

Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph



DEFINITION AND PROPERTIES

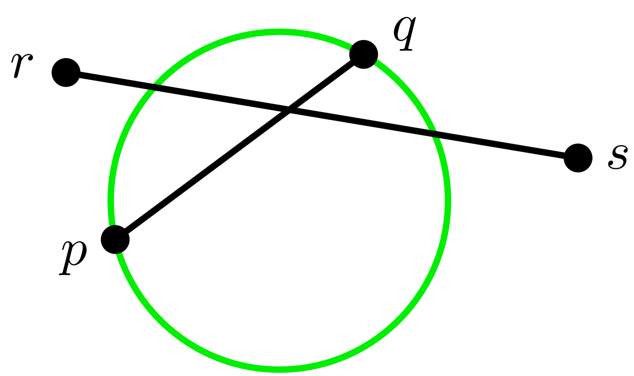
Definition

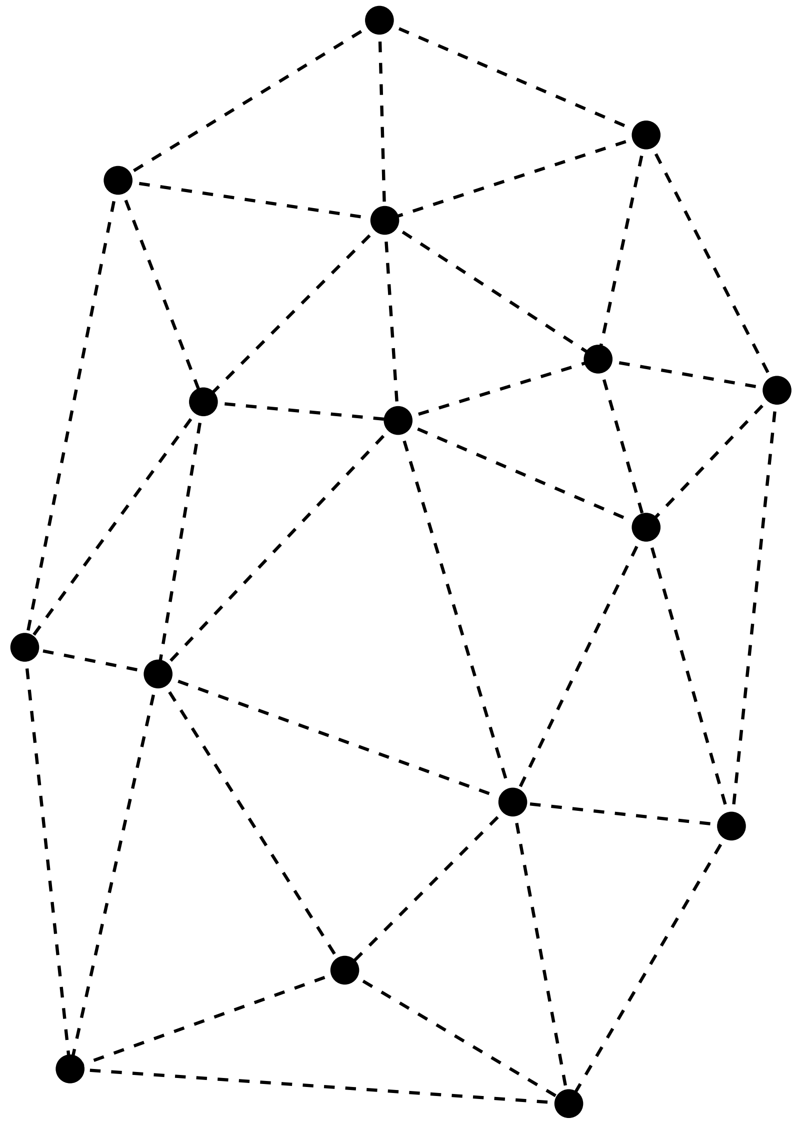
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

If $\overline{pq}$ is a Delaunay edge, there exists an empty circle through p and q. If a segment $\overline{rs}$ intersects $\overline{pq}$, then every circle through r and s contains at least one of p or q.





DEFINITION AND PROPERTIES

Definition

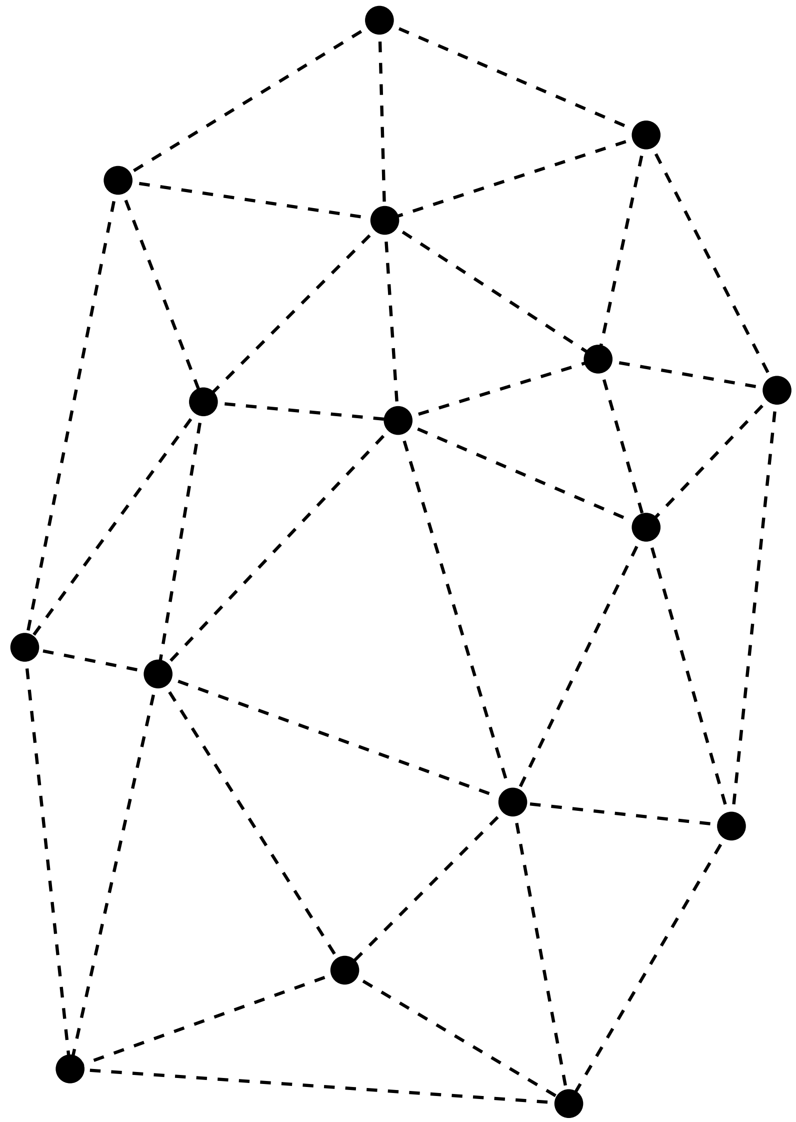
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

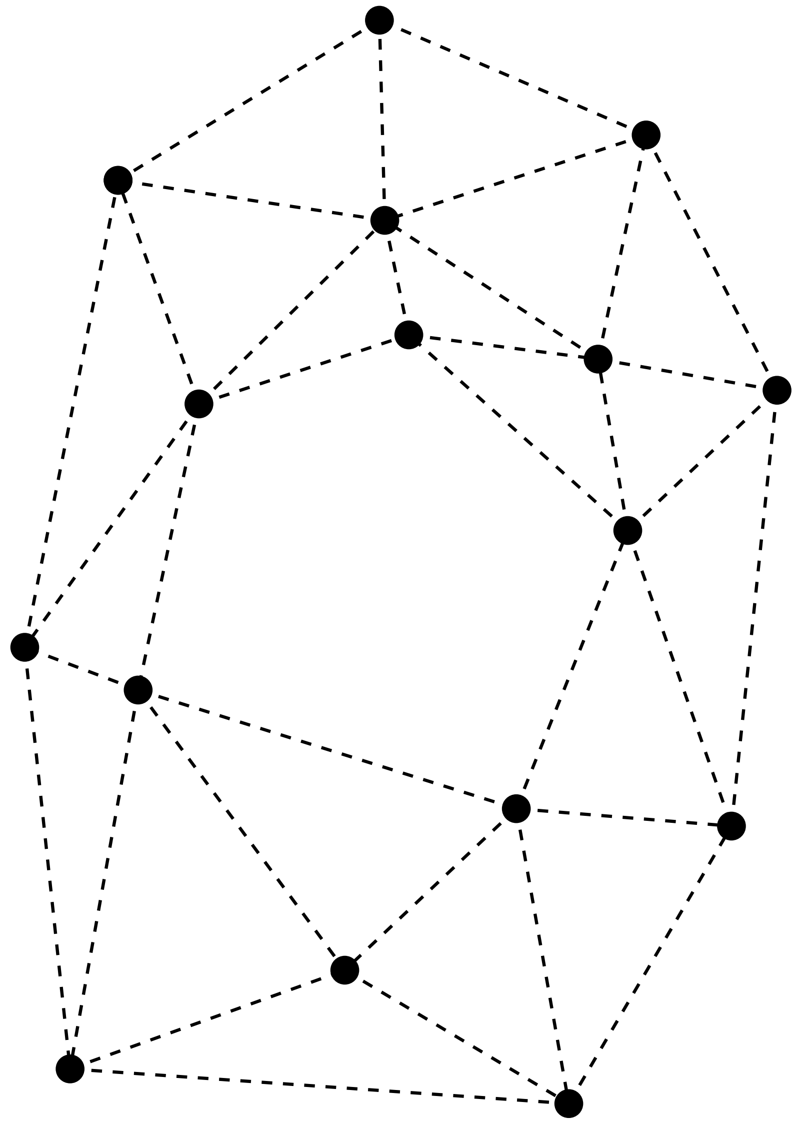
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

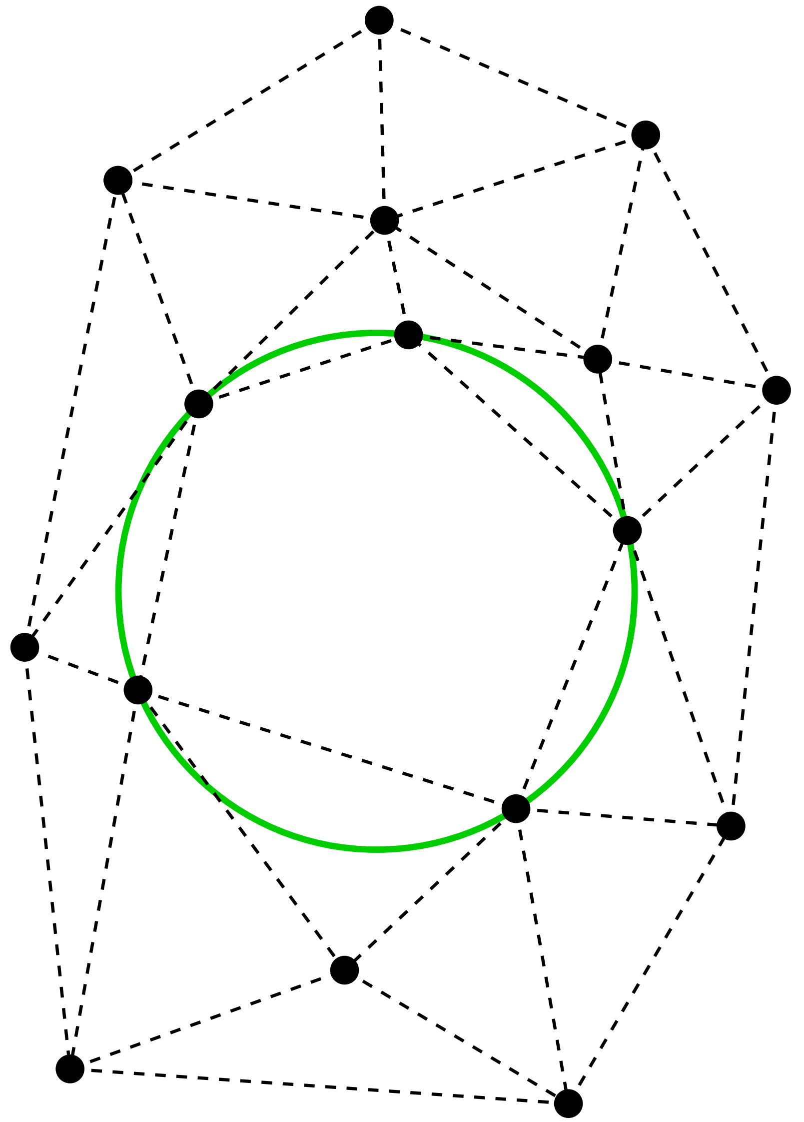
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DEFINITION AND PROPERTIES

Definition

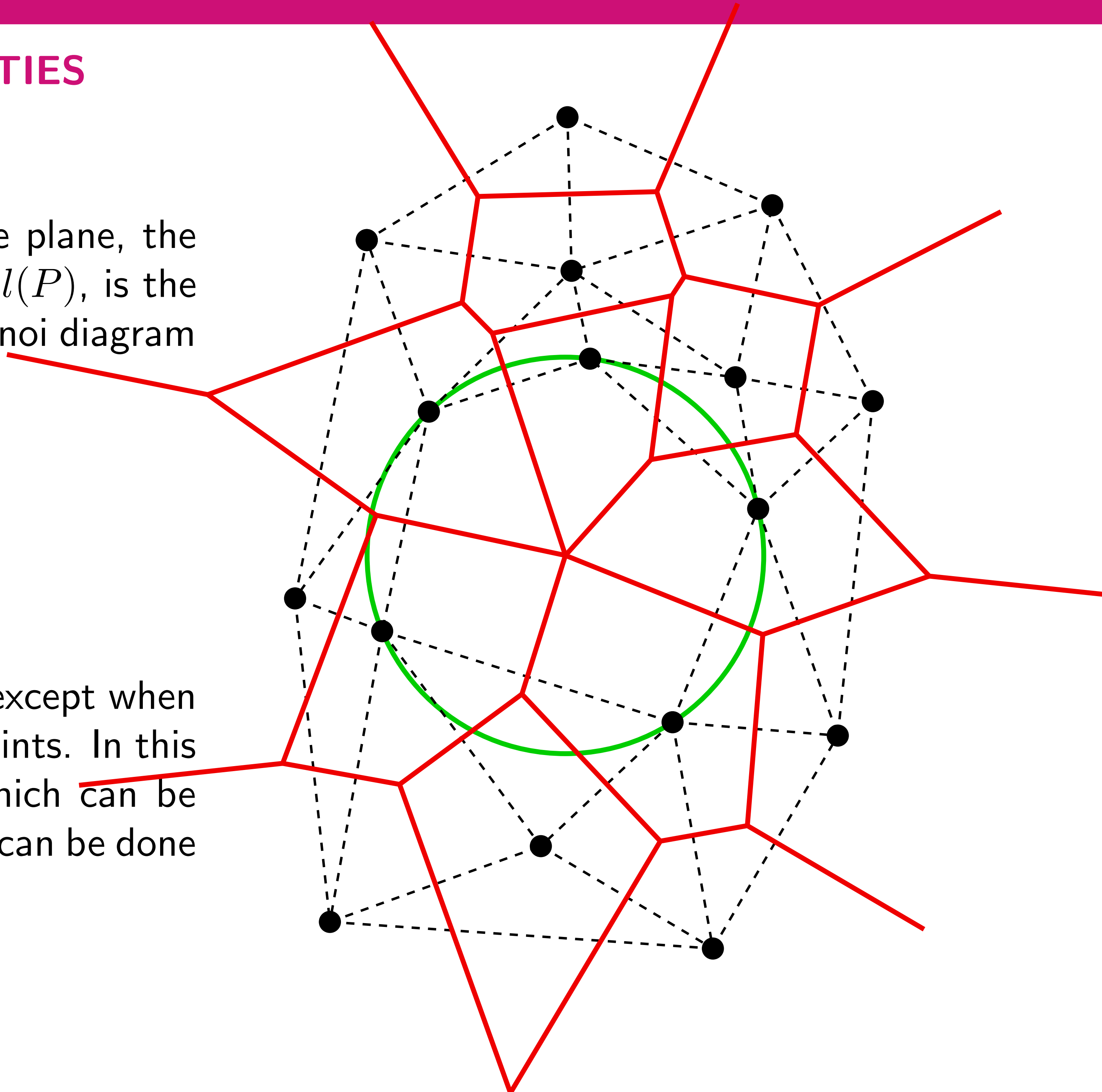
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

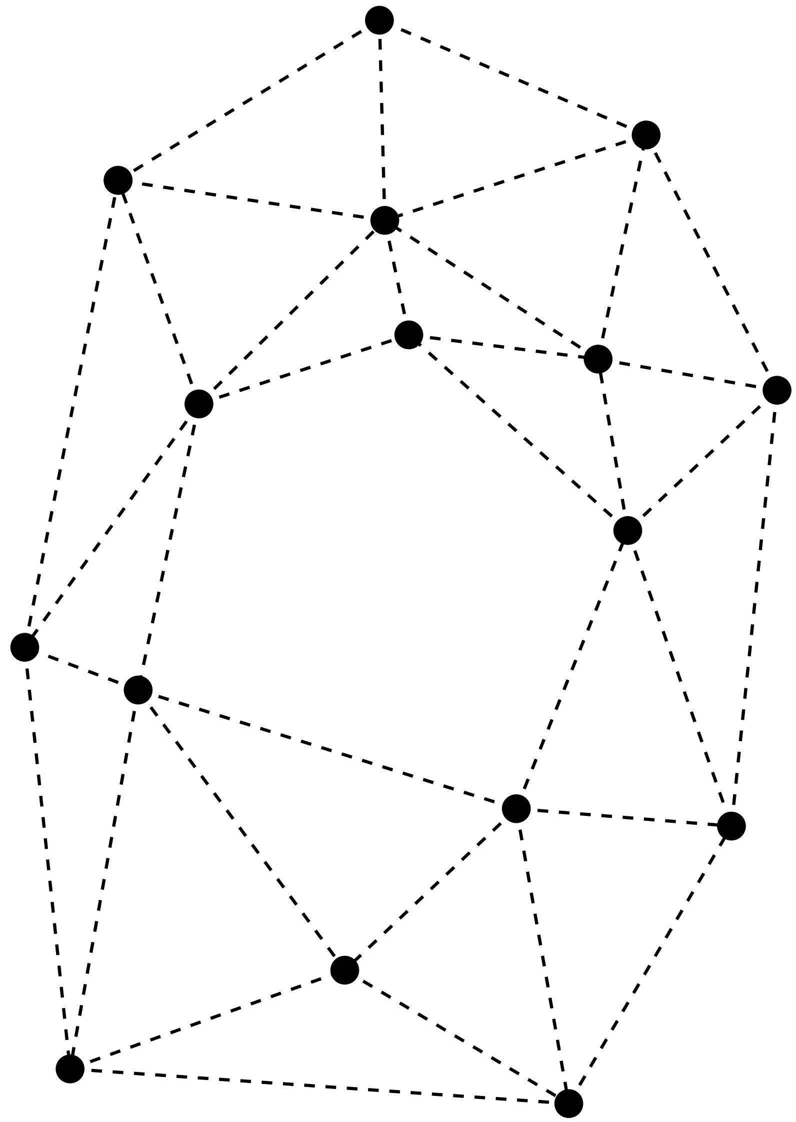
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

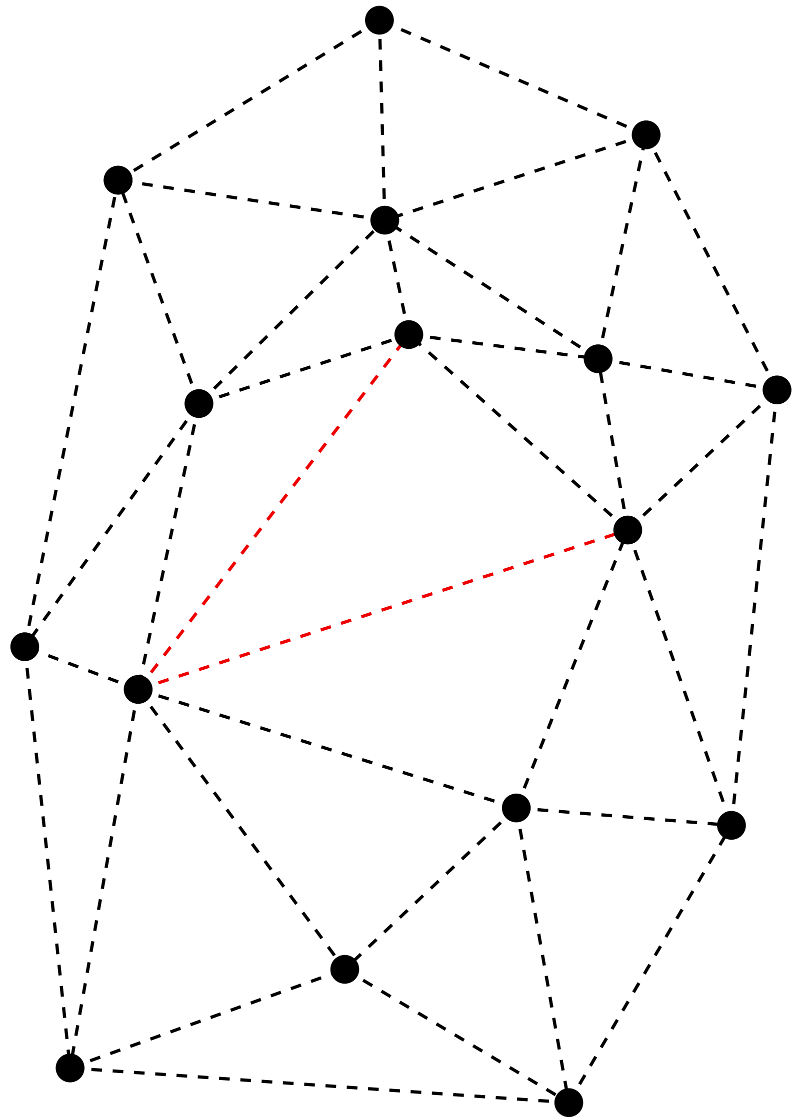
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

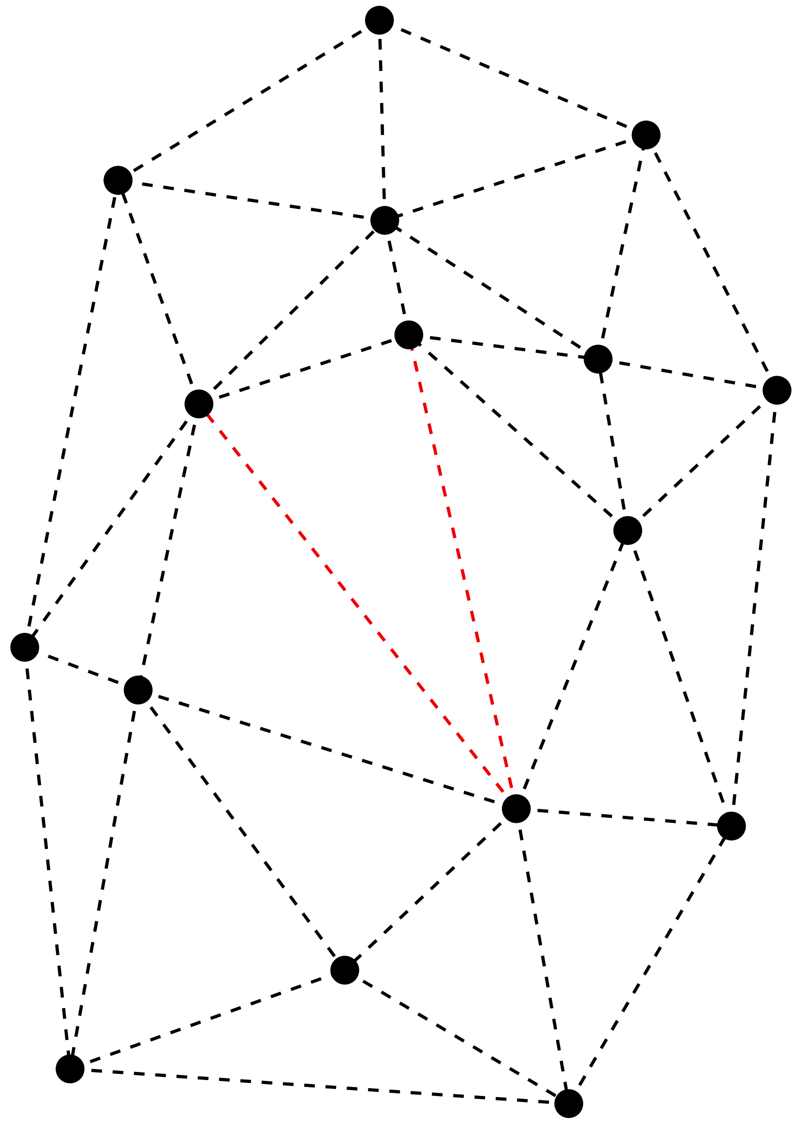
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

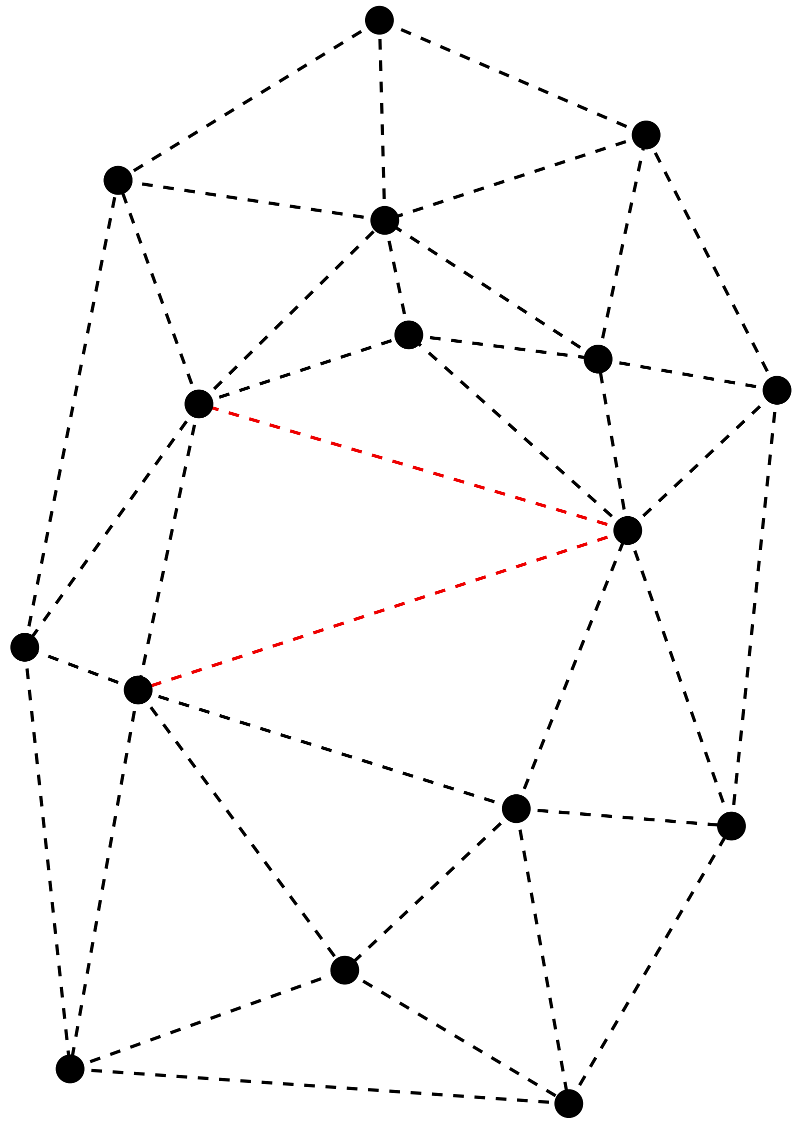
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

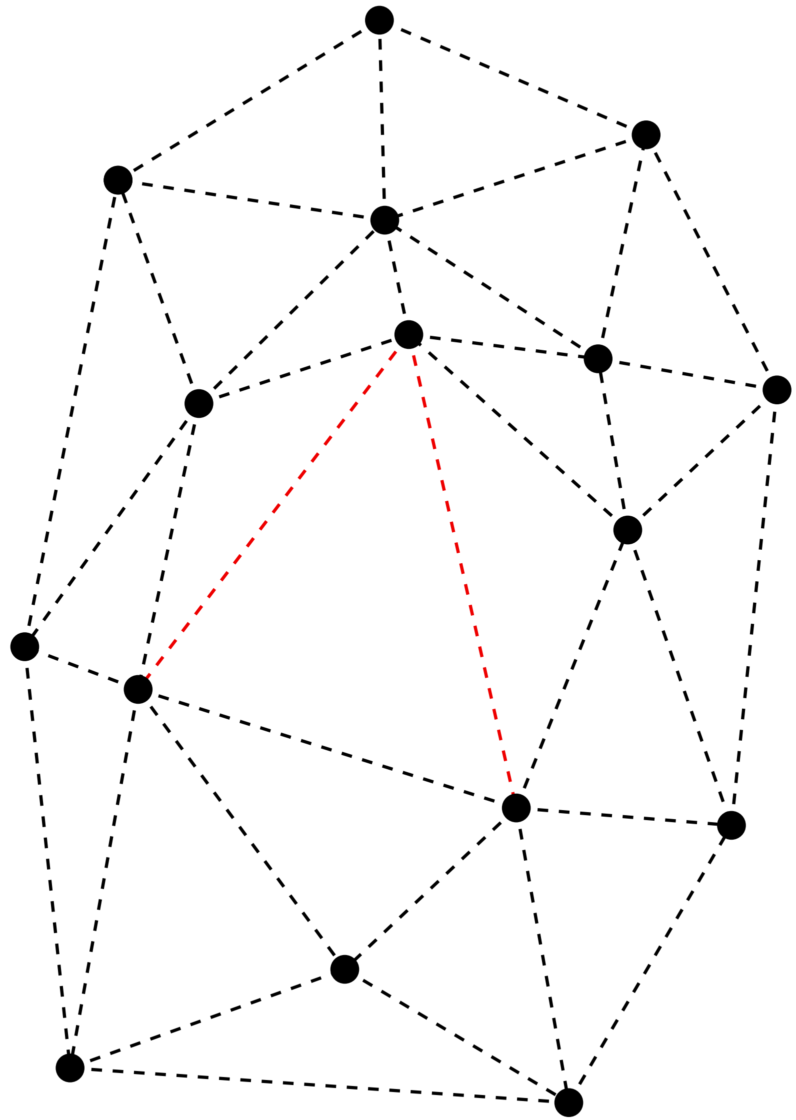
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



DELAUNAY TRIANGULATION

DEFINITION AND PROPERTIES

Definition

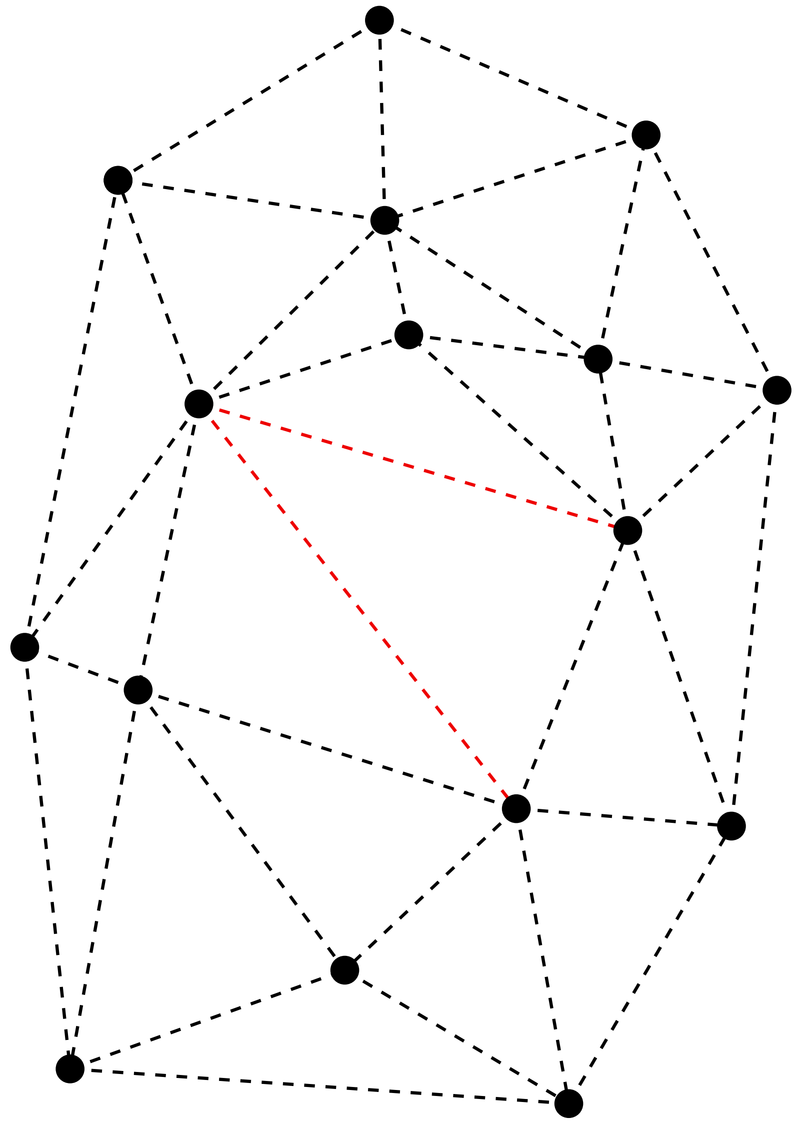
Given a set P of n points in the plane, the Delaunay triangulation of P, $Del(P)$, is the rectilinear dual graph of the Voronoi diagram $Vor(P)$.

Property

$Del(P)$ is a plane graph

Property

$Del(P)$ is a triangulation of P, except when P has three or more concyclic points. In this case, it is a pre-triangulation which can be trivally completed (although this can be done in several different ways).



GLOBAL CHARACTERIZATION

Theorem

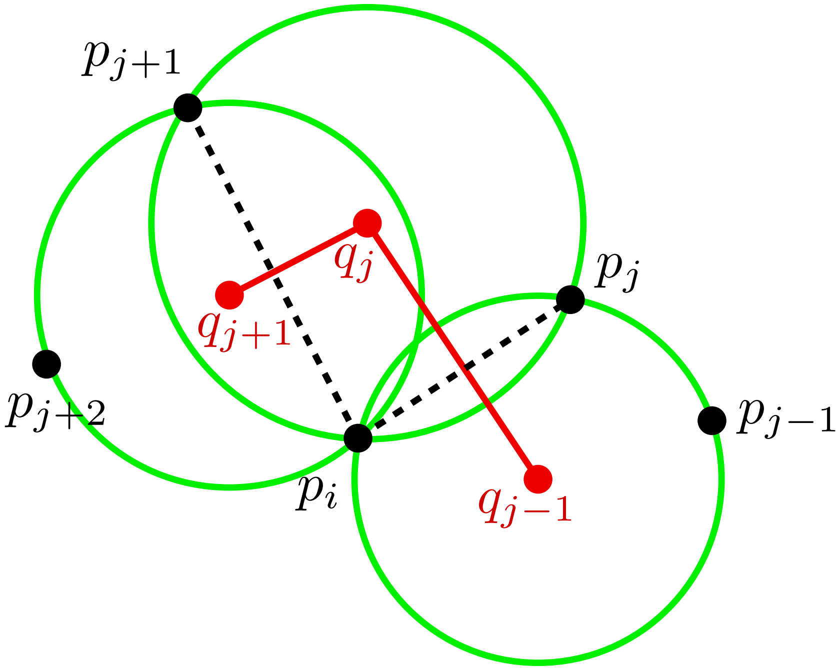
$T(P) = Del(P)$ iff the circumcircles of the triangles of $T(P)$ are empty of points of P.

GLOBAL CHARACTERIZATION

Theorem

$T(P) = Del(P)$ iff the circumcircles of the triangles of $T(P)$ are empty of points of P.

Let $p_i \in P$. Let $p_1, \ldots, p_k$ be the vertices of the triangles of $T(P)$ incident to p_i, sorted in counterclockwise order, $C_1, \ldots, C_k$ be their circumcircles, and $q_1, \ldots, q_k$ their centers (q_j denotes the center of C_j, the circumcircle of p_i, p_j, p_{j+1}). We will prove that the polygon $Q = \{q_1, \ldots, q_k\}$ coincides with $Vor(p_i)$.



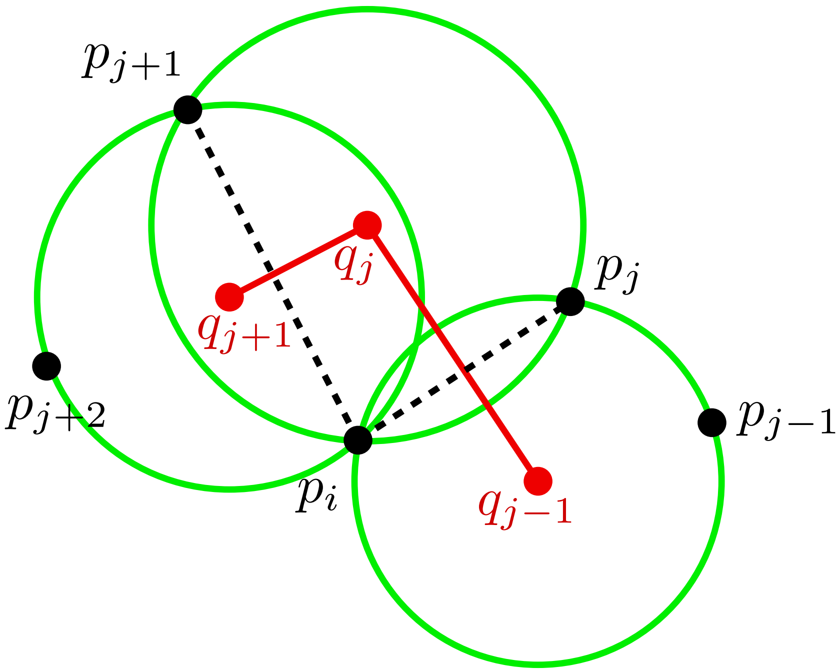
GLOBAL CHARACTERIZATION

Theorem

$T(P) = Del(P)$ iff the circumcircles of the triangles of $T(P)$ are empty of points of P.

Let $p_i \in P$. Let $p_1, \ldots, p_k$ be the vertices of the triangles of $T(P)$ incident to p_i, sorted in counterclockwise order, $C_1, \ldots, C_k$ be their circumcircles, and $q_1, \ldots, q_k$ their centers (q_j denotes the center of C_j, the circumcircle of p_i, p_j, p_{j+1}). We will prove that the polygon $Q = \{q_1, \ldots, q_k\}$ coincides with $Vor(p_i)$.

$$\overline{q_{j-1}q_j} \perp \overline{p_ip_j} \Longrightarrow Q = \bigcap_{j=1}^{k} H_{ij}$$

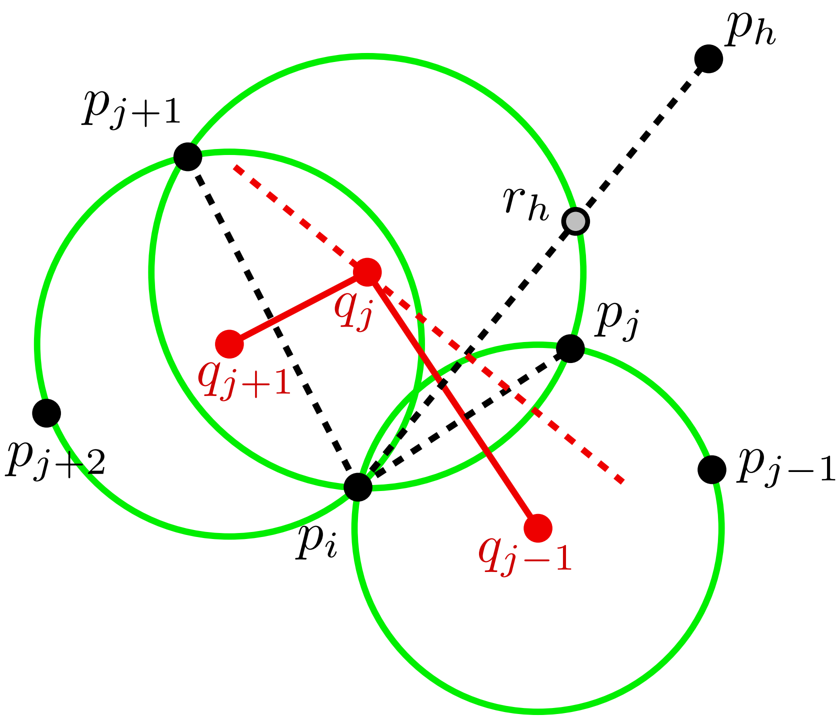


GLOBAL CHARACTERIZATION

Theorem

$T(P) = Del(P)$ iff the circumcircles of the triangles of $T(P)$ are empty of points of P.

Let $p_i \in P$. Let $p_1, \ldots, p_k$ be the vertices of the triangles of $T(P)$ incident to p_i, sorted in counterclockwise order, $C_1, \ldots, C_k$ be their circumcircles, and $q_1, \ldots, q_k$ their centers (q_j denotes the center of C_j, the circumcircle of p_i, p_j, p_{j+1}). We will prove that the polygon $Q = \{q_1, \ldots, q_k\}$ coincides with $Vor(p_i)$.



$$\overline{q_{j-1}q_j} \perp \overline{p_i p_j} \Longrightarrow Q = \bigcap_{j=1}^{k} H_{ij}$$

If $h \neq 1, \ldots, k$ then $q_j \in b(p_i, r_h)$ and, therefore,

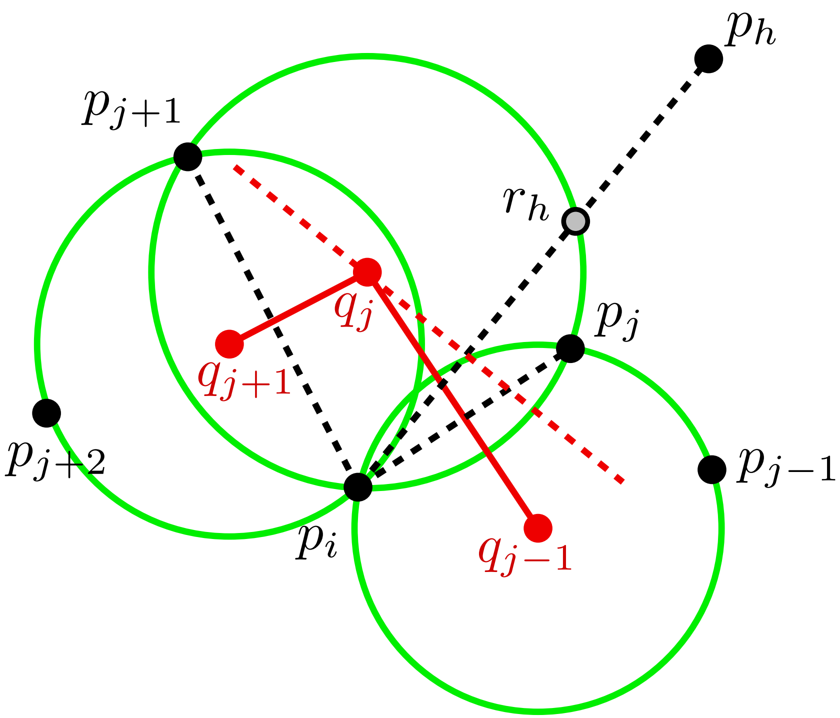
$$\bigcap_{j=1}^{k} H_{ij} \subset H(p_i, r_h) \subset H_{ih}$$

GLOBAL CHARACTERIZATION

Theorem

$T(P) = Del(P)$ iff the circumcircles of the triangles of $T(P)$ are empty of points of P.

Let $p_i \in P$. Let $p_1, \ldots, p_k$ be the vertices of the triangles of $T(P)$ incident to p_i, sorted in counterclockwise order, $C_1, \ldots, C_k$ be their circumcircles, and $q_1, \ldots, q_k$ their centers (q_j denotes the center of C_j, the circumcircle of p_i, p_j, p_{j+1}). We will prove that the polygon $Q = \{q_1, \ldots, q_k\}$ coincides with $Vor(p_i)$.



$$\overline{q_{j-1}q_j} \perp \overline{p_ip_j} \Longrightarrow Q = \bigcap_{j=1}^{k} H_{ij}$$

If $h \neq 1, \ldots, k$ then $q_j \in b(p_i, r_h)$ and, therefore,

$$\bigcap_{j=1}^{k} H_{ij} \subset H(p_i, r_h) \subset H_{ih}$$

Hence,

$$Q = \bigcap_{j=1}^{k} H_{ij} = \bigcap_{j \neq i} H_{ij} = Vor(p_i)$$

DELAUNAY TRIANGULATION AND 3D CONVEX HULL

DELAUNAY TRIANGULATION AND 3D CONVEX HULL

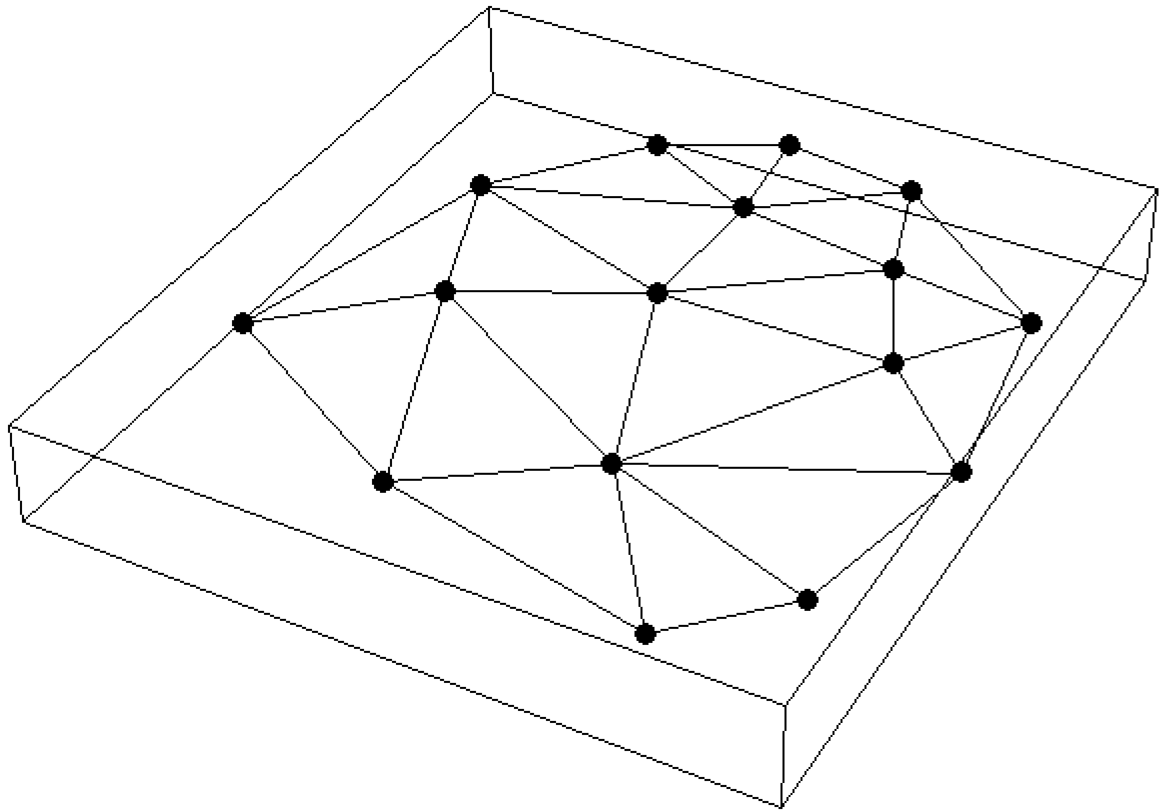
Theorem

Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.

DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

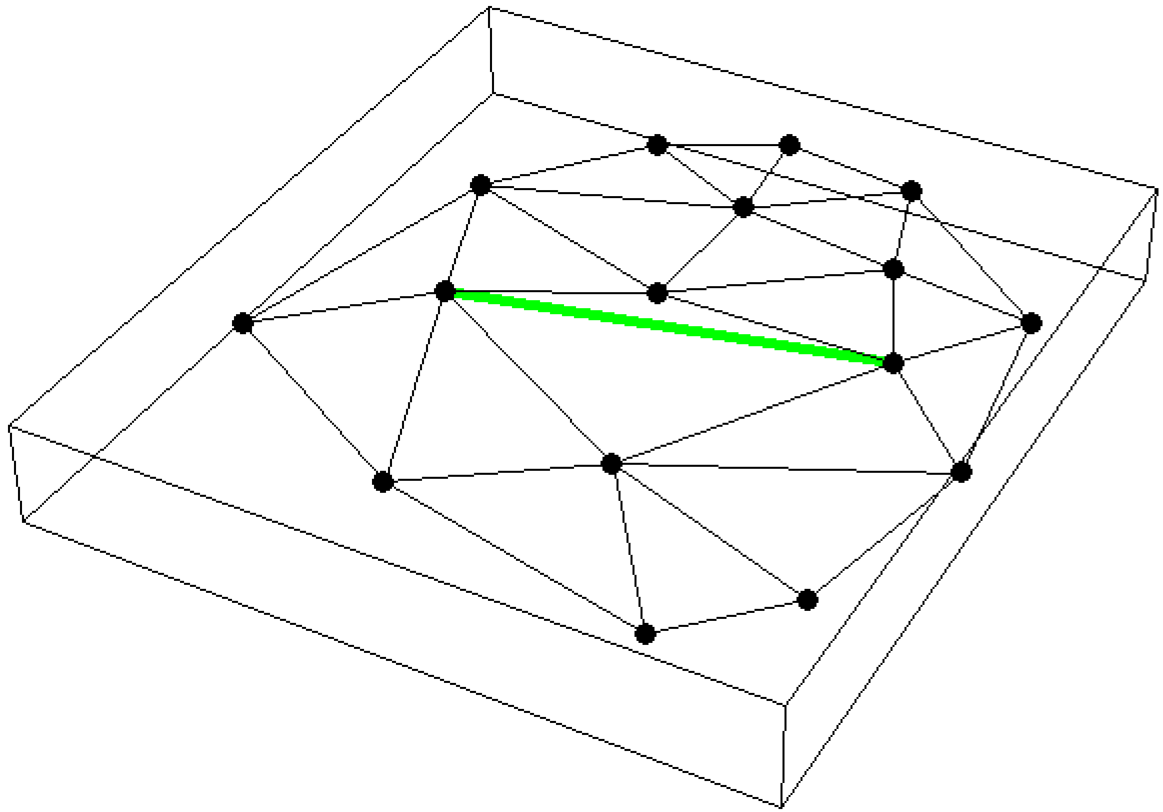
Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

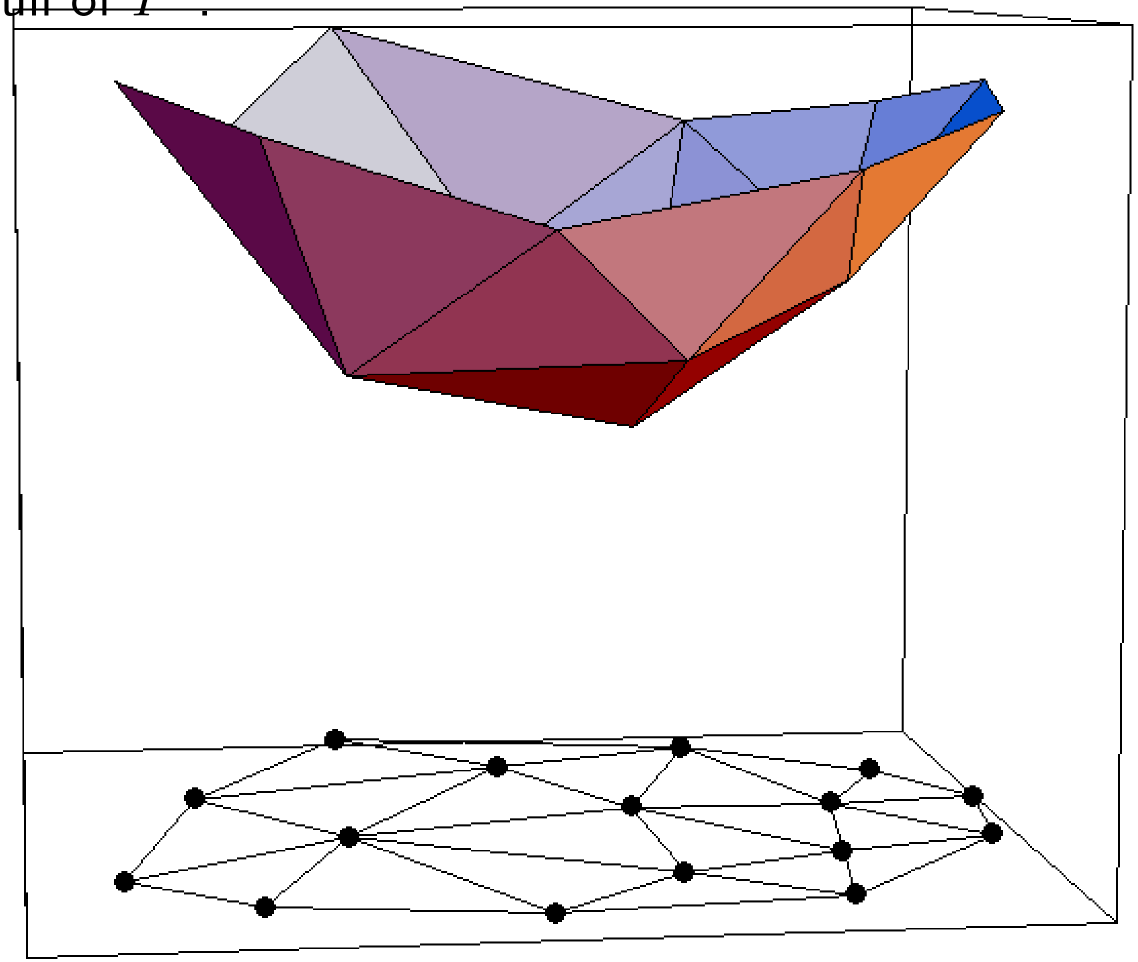
Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

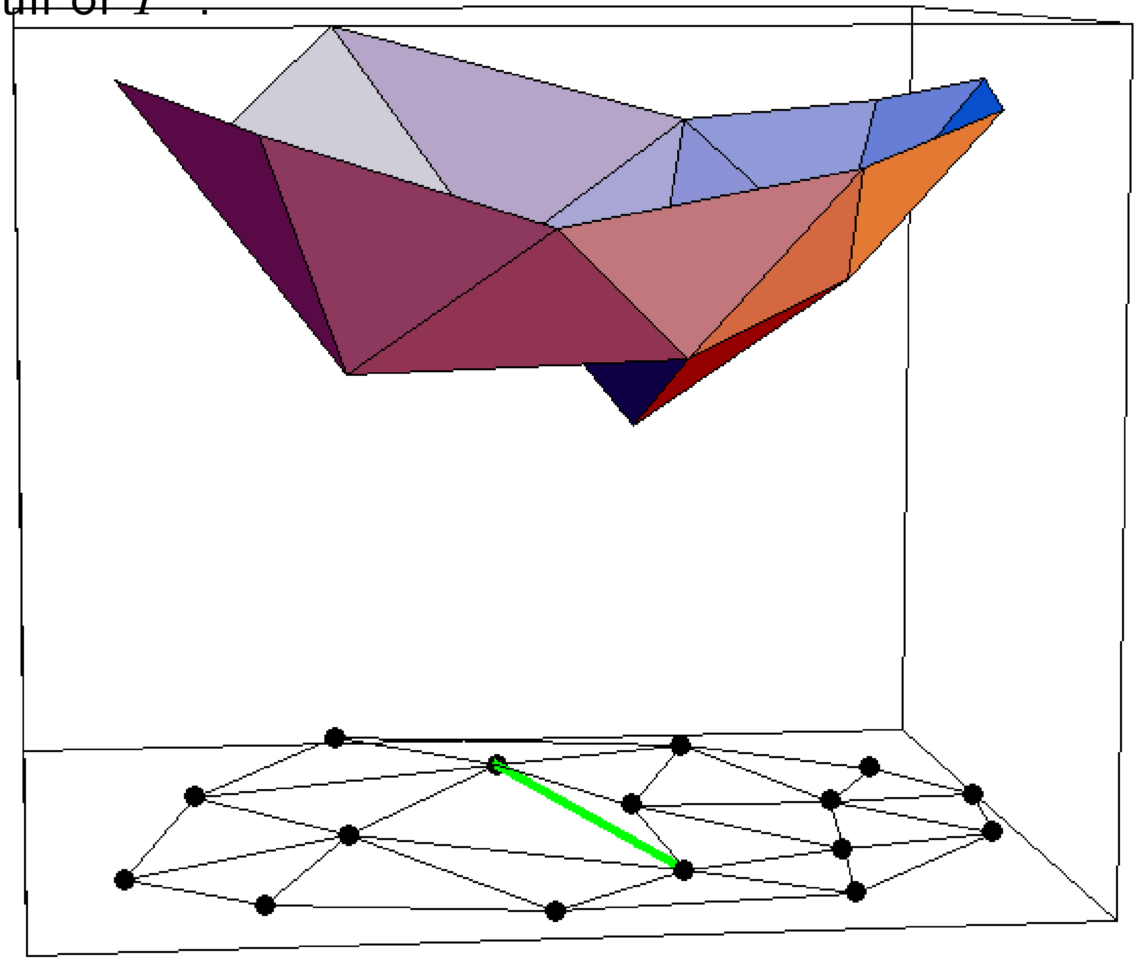
Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

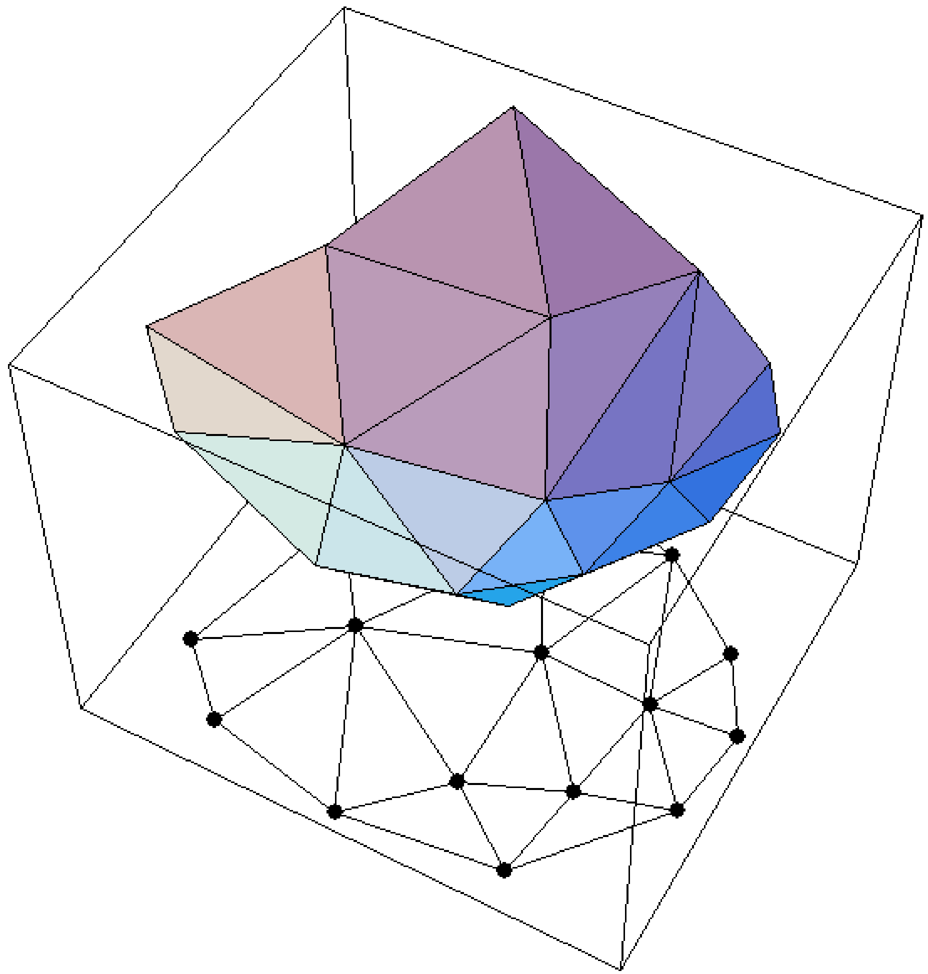
Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

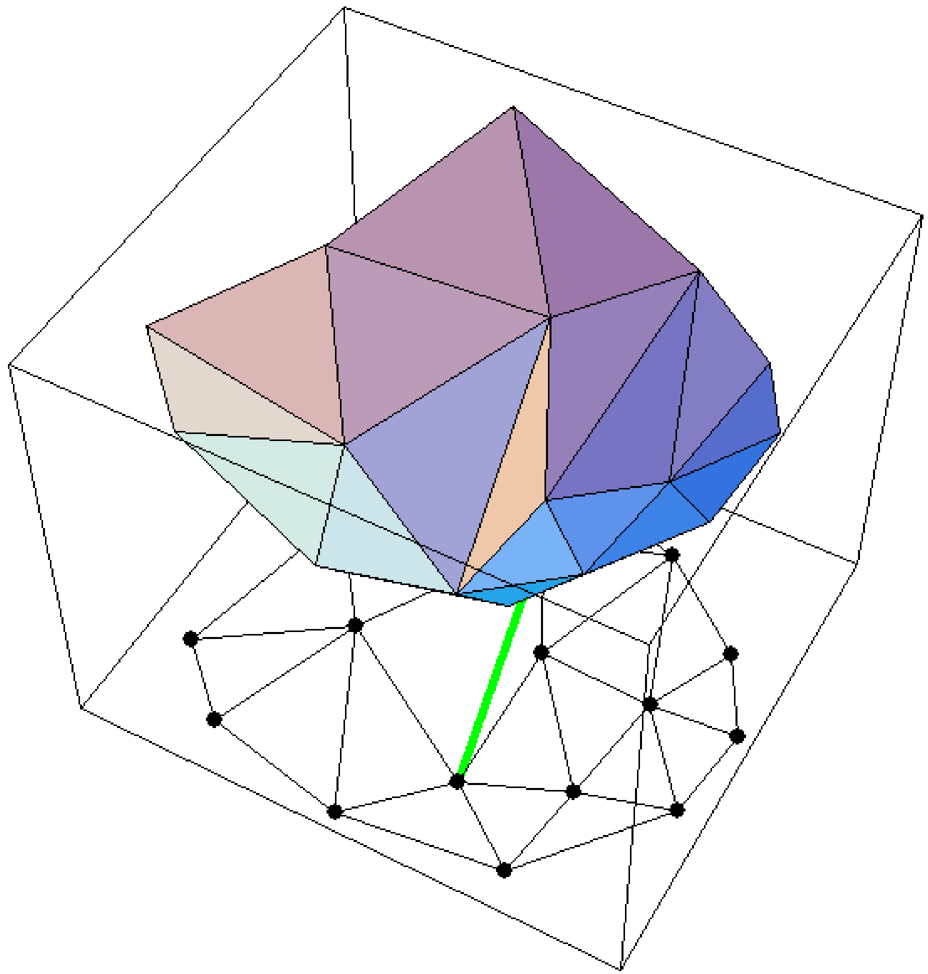
Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

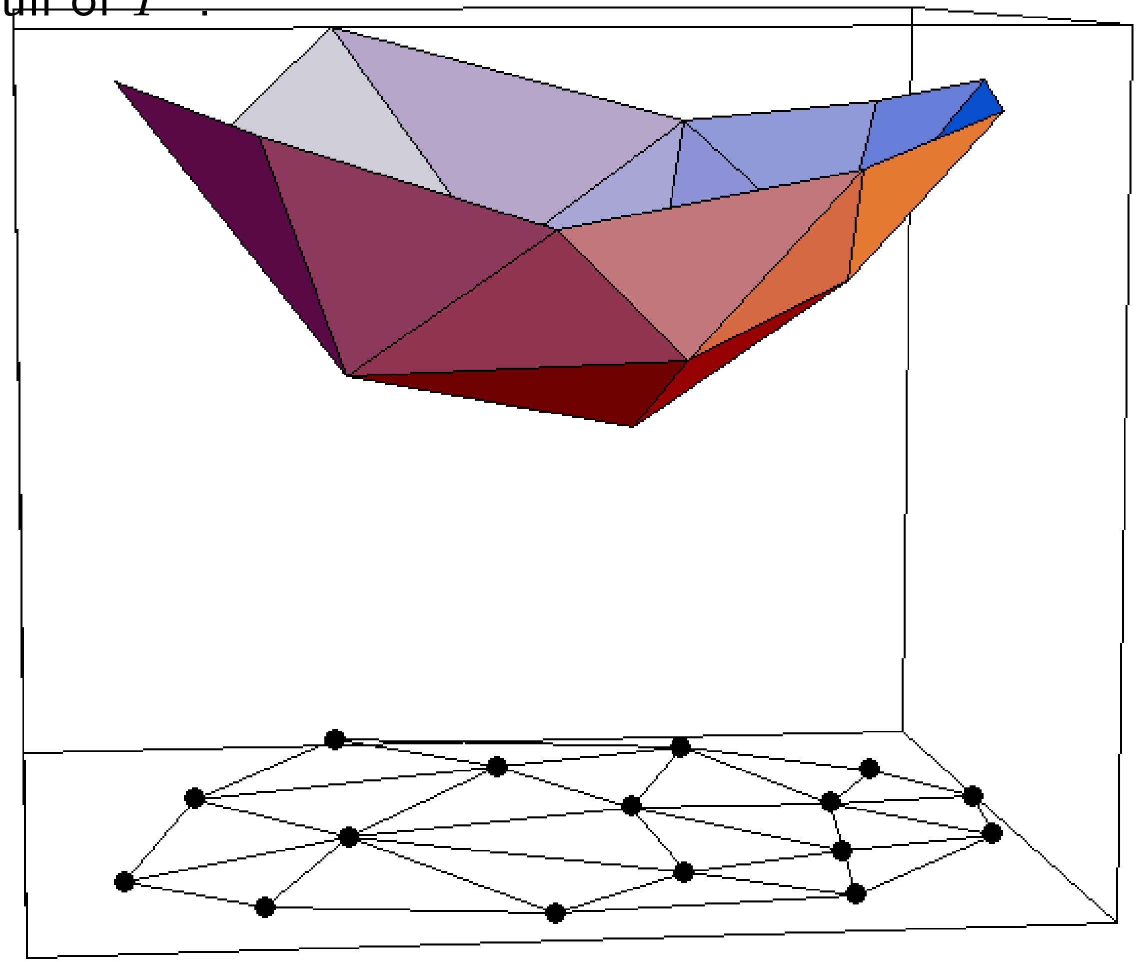
Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.

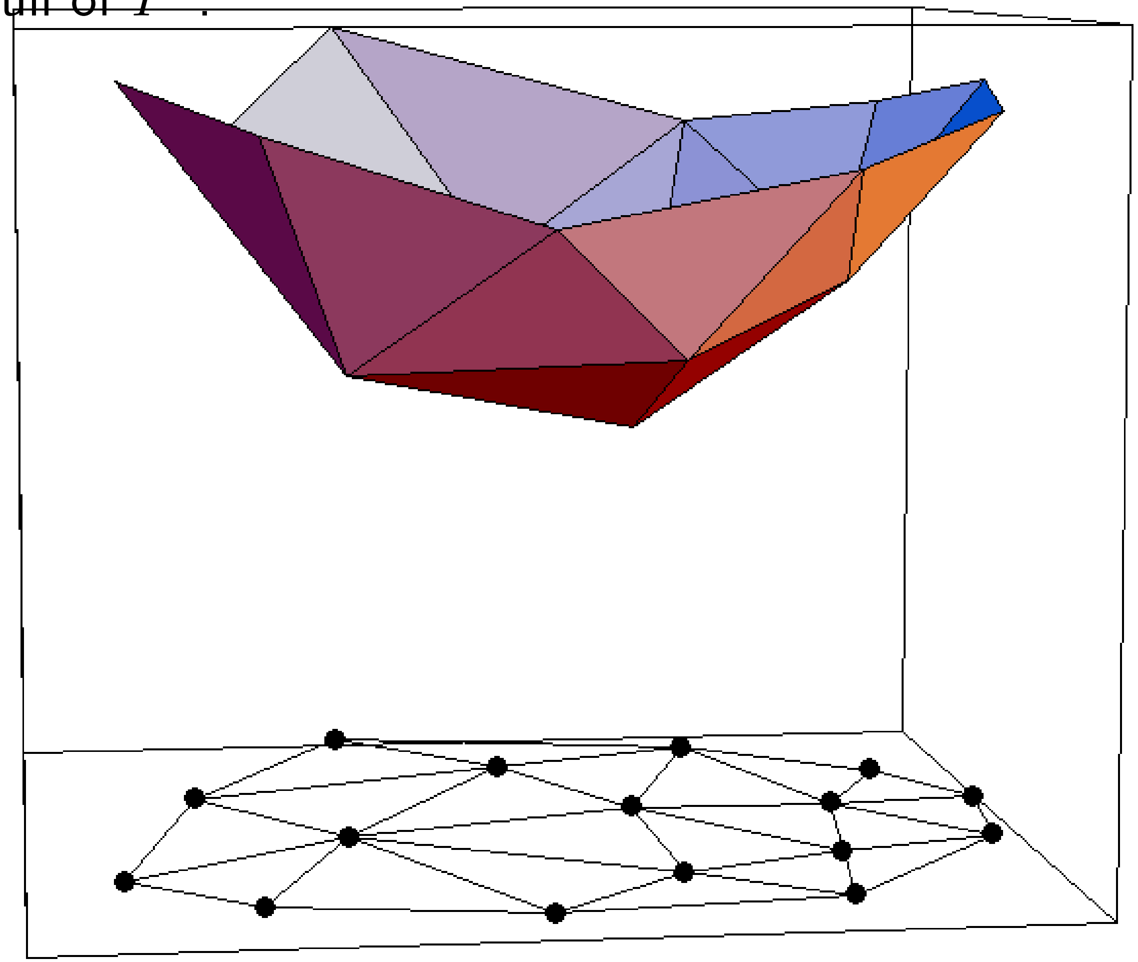


DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.

p_i^*, p_j^*, p_k^* form a (triangular) face of the lower convex hull of P^*



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

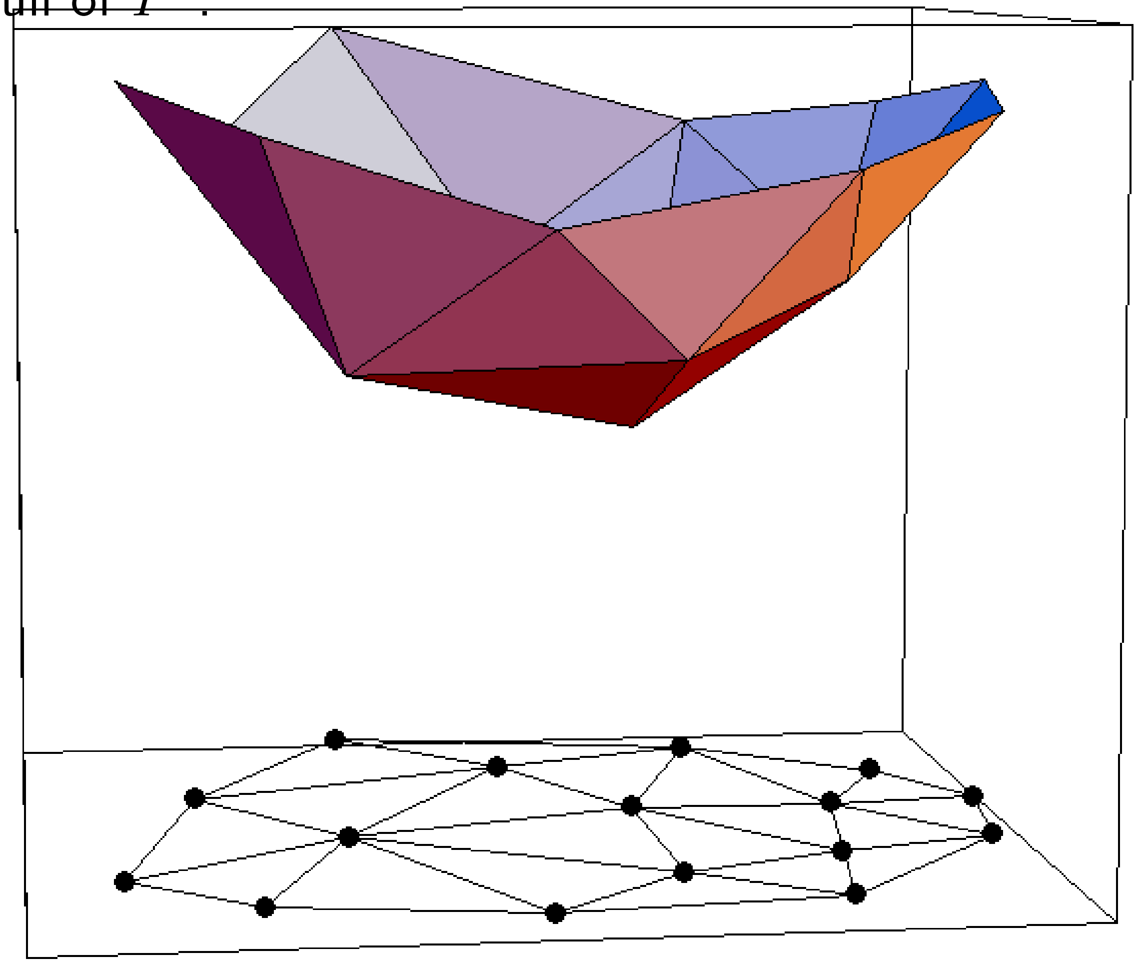
Theorem

Let $P = \{p_1, \dots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.

p_i^*, p_j^*, p_k^* form a (triangular) face of the lower convex hull of P^*

$\Updownarrow$

The plane through p_i^*, p_j^*, p_k^* leaves all the remaining points of P^* above it



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.

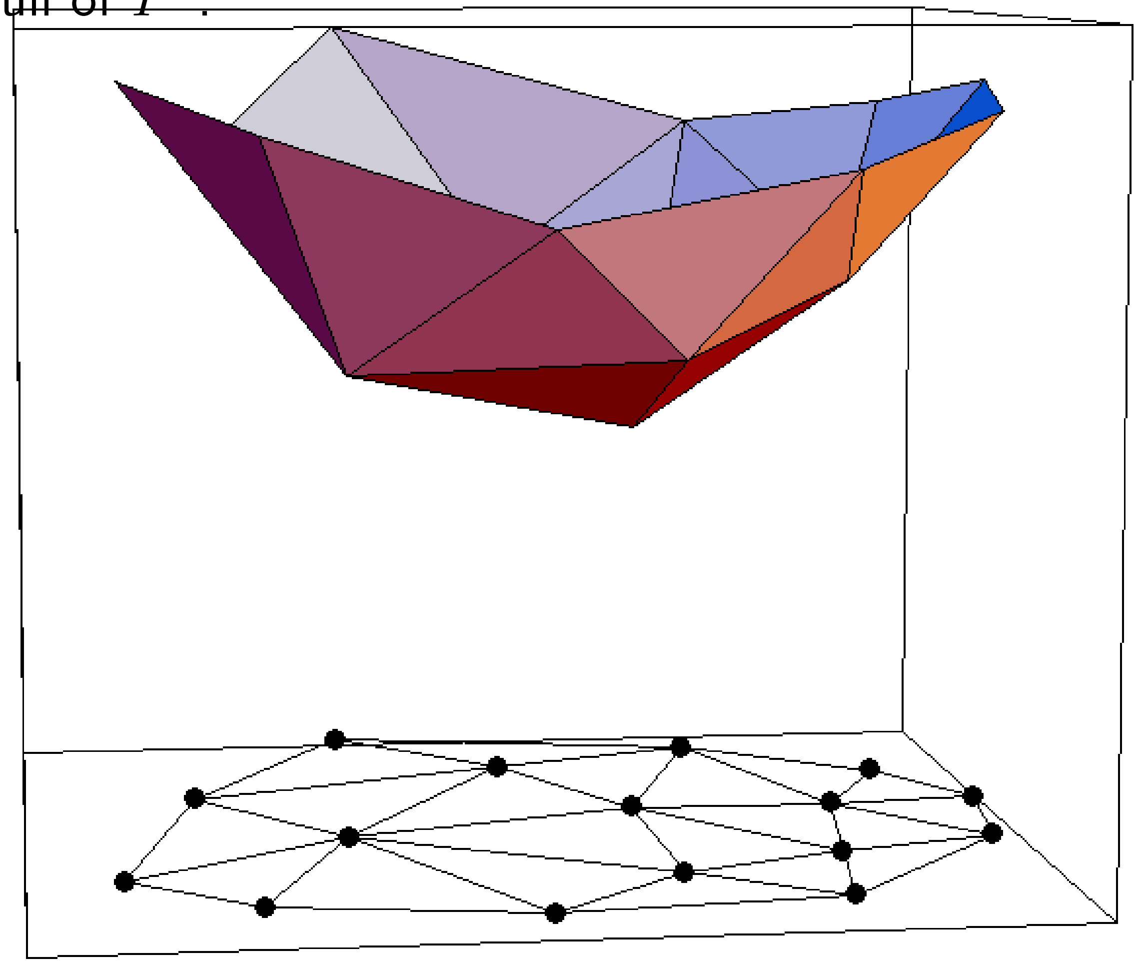
p_i^*, p_j^*, p_k^* form a (triangular) face of the lower convex hull of P^*

$\Updownarrow$

The plane through p_i^*, p_j^*, p_k^* leaves all the remaining points of P^* above it

$\Updownarrow$

The circle through p_i, p_j, p_k leaves all the remaining points of P in its exterior



DELAUNAY TRIANGULATION AND 3D CONVEX HULL

Theorem

Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.

p_i^*, p_j^*, p_k^* form a (triangular) face of the lower convex hull of P^*

$\Updownarrow$

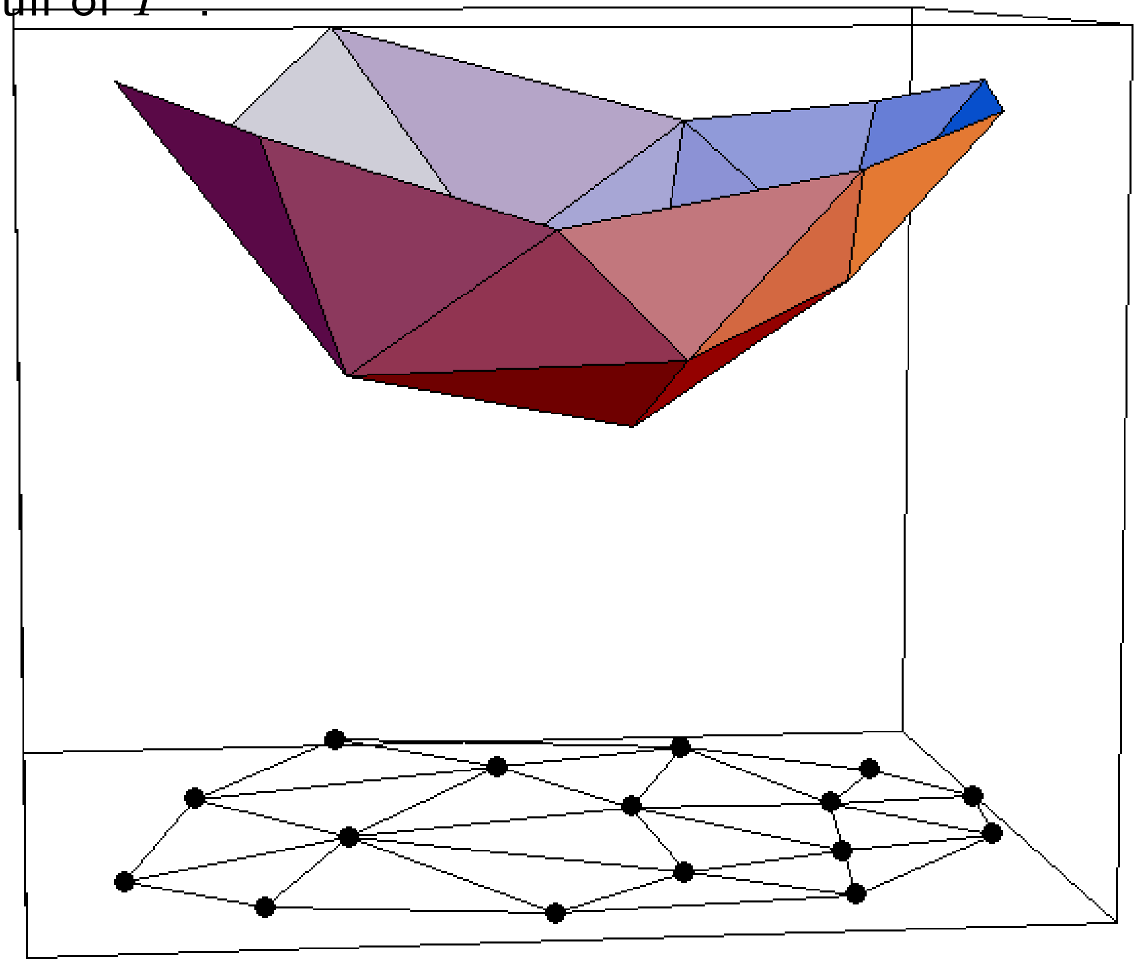
The plane through p_i^*, p_j^*, p_k^* leaves all the remaining points of P^* above it

$\Updownarrow$

The circle through p_i, p_j, p_k leaves all the remaining points of P in its exterior

$\Updownarrow$

p_i, p_j, p_k form a triangle of $Del(P)$



LOCAL CHARACTERIZATION

LOCAL CHARACTERIZATION

Definition

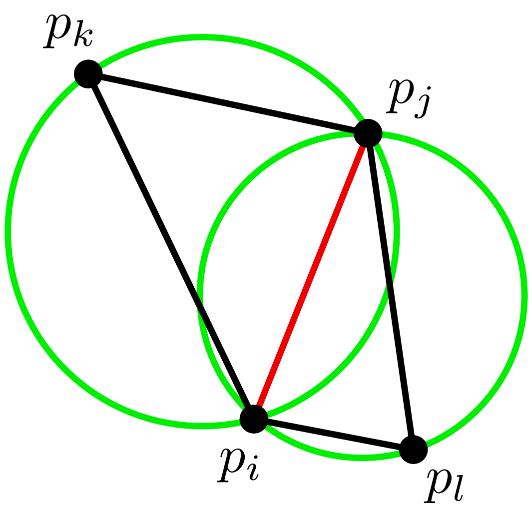
A triangulation $T(P)$ is locally Delaunay if each pair of triangles $p_i p_j p_k$ and $p_i p_j p_l$ sharing an edge $p_i p_j$ satisfies $p_l \notin C_{ijk}$ and $p_k \notin C_{ijl}$.

DELAUNAY TRIANGULATION

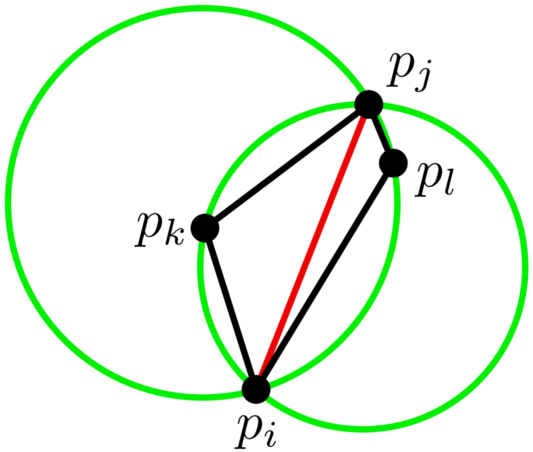
LOCAL CHARACTERIZATION

Definition

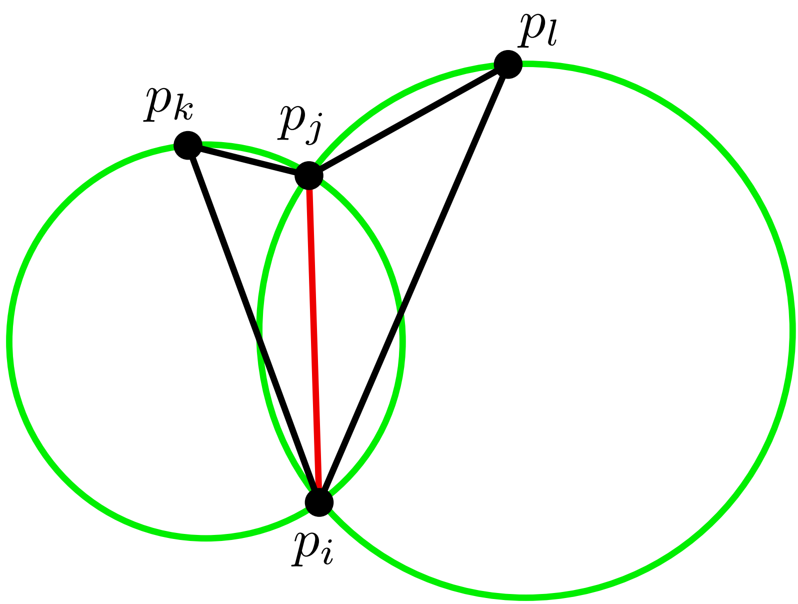
A triangulation $T(P)$ is locally Delaunay if each pair of triangles $p_i p_j p_k$ and $p_i p_j p_l$ sharing an edge $p_i p_j$ satisfies $p_l \notin C_{ijk}$ and $p_k \notin C_{ijl}$.



The edge $p_i p_j$ is locally Delaunay



The edge $p_i p_j$ is not locally Delaunay



The edge $p_i p_j$ is locally Delaunay

In fact, the quadrilateral $p_i p_l p_j p_k$ is not convex

LOCAL CHARACTERIZATION

Theorem

A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

LOCAL CHARACTERIZATION

Theorem

A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

Suppose that $T(P)$ was locally Delaunay without being a Delaunay triangulation.

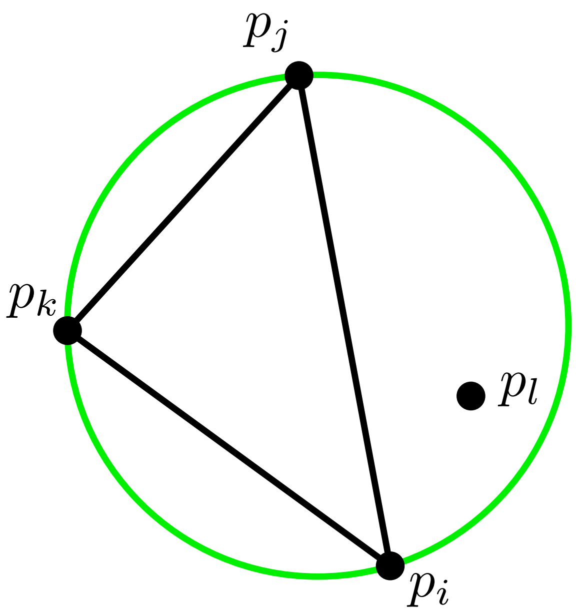
LOCAL CHARACTERIZATION

Theorem

A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

Suppose that $T(P)$ was locally Delaunay without being a Delaunay triangulation.

There would exist a triangle $T_{ijk} = p_i p_j p_k$ and a point p_l such that $p_l \in int(C_{ijk})$.



LOCAL CHARACTERIZATION

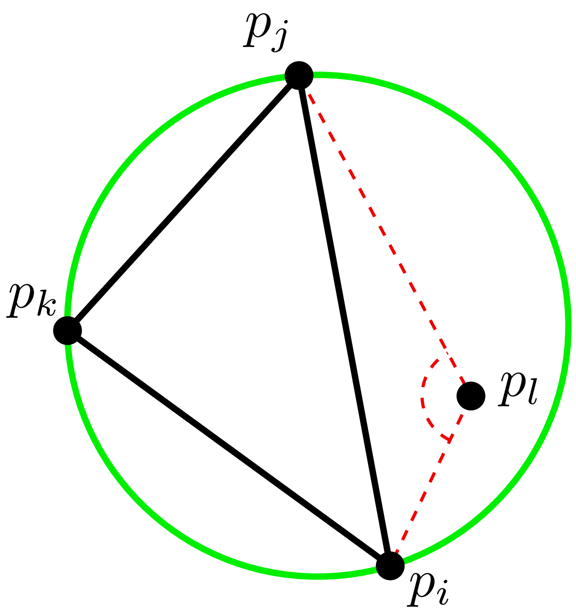
Theorem

A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

Suppose that $T(P)$ was locally Delaunay without being a Delaunay triangulation.

There would exist a triangle $T_{ijk} = p_i p_j p_k$ and a point p_l such that $p_l \in int(C_{ijk})$.

Let $\overline{p_i p_j}$ be the edge of T_{ijk} separating p_l from T_{ijk}. Among all 4-tuples in this situation, let $ijkl$ maximize the angle $p_i p_l p_j$.



DELAUNAY TRIANGULATION

LOCAL CHARACTERIZATION

Theorem

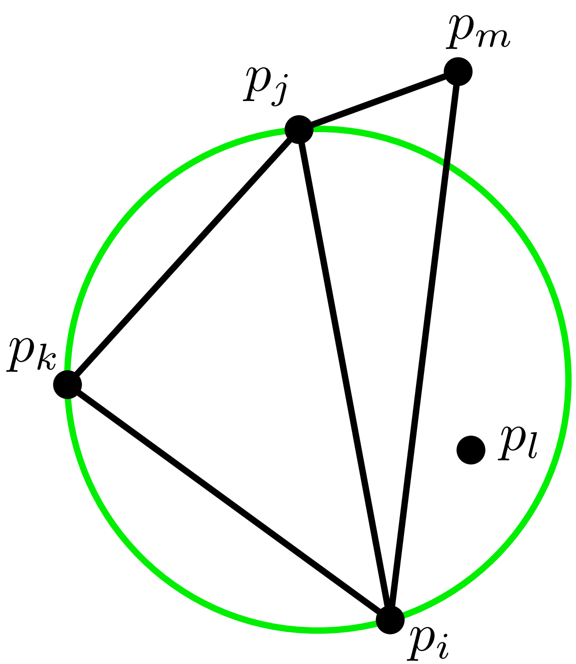
A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

Suppose that $T(P)$ was locally Delaunay without being a Delaunay triangulation.

There would exist a triangle $T_{ijk} = p_i p_j p_k$ and a point p_l such that $p_l \in int(C_{ijk})$.

Let $\overline{p_i p_j}$ be the edge of T_{ijk} separating p_l from T_{ijk}. Among all 4-tuples in this situation, let $ijkl$ maximize the angle $p_i p_l p_j$.

Let T_{ijm} be the triangle adjacent to T_{ijk} though the edge $\overline{p_i p_j}$.



DELAUNAY TRIANGULATION

LOCAL CHARACTERIZATION

Theorem

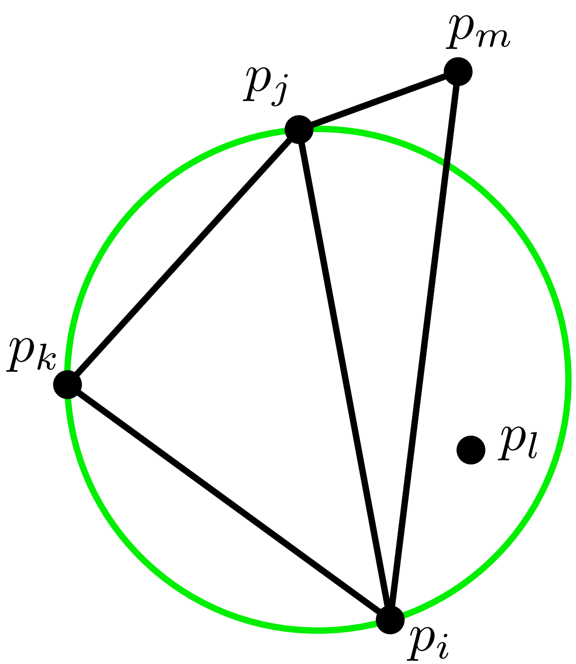
A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

Suppose that $T(P)$ was locally Delaunay without being a Delaunay triangulation.

There would exist a triangle $T_{ijk} = p_i p_j p_k$ and a point p_l such that $p_l \in int(C_{ijk})$.

Let $\overline{p_i p_j}$ be the edge of T_{ijk} separating p_l from T_{ijk}. Among all 4-tuples in this situation, let $ijkl$ maximize the angle $p_i p_l p_j$.

Let T_{ijm} be the triangle adjacent to T_{ijk} though the edge $\overline{p_i p_j}$.



As $T(P)$ is locally Delaunay, $m \neq l$.

LOCAL CHARACTERIZATION

Theorem

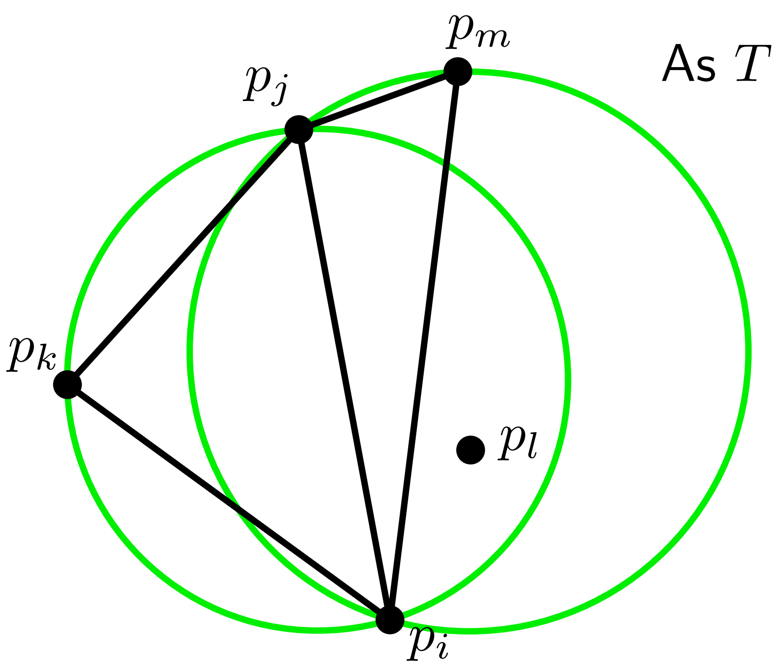
A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

Suppose that $T(P)$ was locally Delaunay without being a Delaunay triangulation.

There would exist a triangle $T_{ijk} = p_i p_j p_k$ and a point p_l such that $p_l \in int(C_{ijk})$.

Let $\overline{p_i p_j}$ be the edge of T_{ijk} separating p_l from T_{ijk}. Among all 4-tuples in this situation, let $ijkl$ maximize the angle $p_i p_l p_j$.

Let T_{ijm} be the triangle adjacent to T_{ijk} though the edge $\overline{p_i p_j}$.



As $T(P)$ is locally Delaunay, $m \neq l$.

Then $p_l \in C_{ijm}$.

LOCAL CHARACTERIZATION

Theorem

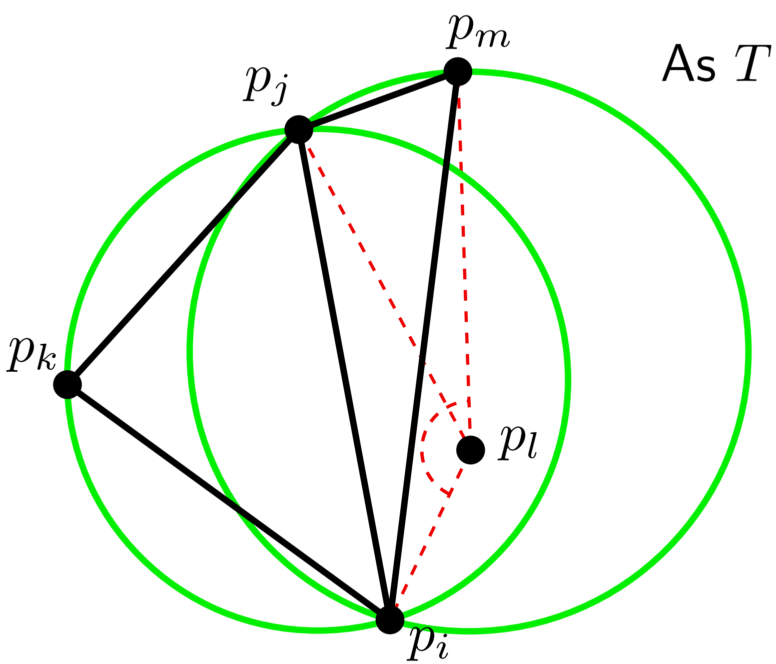
A triangulation $T(P)$ is a Delaunay triangulation if and only if it is locally Delaunay.

Suppose that $T(P)$ was locally Delaunay without being a Delaunay triangulation.

There would exist a triangle $T_{ijk} = p_i p_j p_k$ and a point p_l such that $p_l \in int(C_{ijk})$.

Let $\overline{p_i p_j}$ be the edge of T_{ijk} separating p_l from T_{ijk}. Among all 4-tuples in this situation, let $ijkl$ maximize the angle $p_i p_l p_j$.

Let T_{ijm} be the triangle adjacent to T_{ijk} though the edge $\overline{p_i p_j}$.



As $T(P)$ is locally Delaunay, $m \neq l$.

Then $p_l \in C_{ijm}$.

Hence, one of the angles $p_i p_l p_m$ or $p_j p_l p_m$ would be greater than $p_i p_l p_j$.

DELAUNAY FLIPS

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

DELAUNAY FLIPS

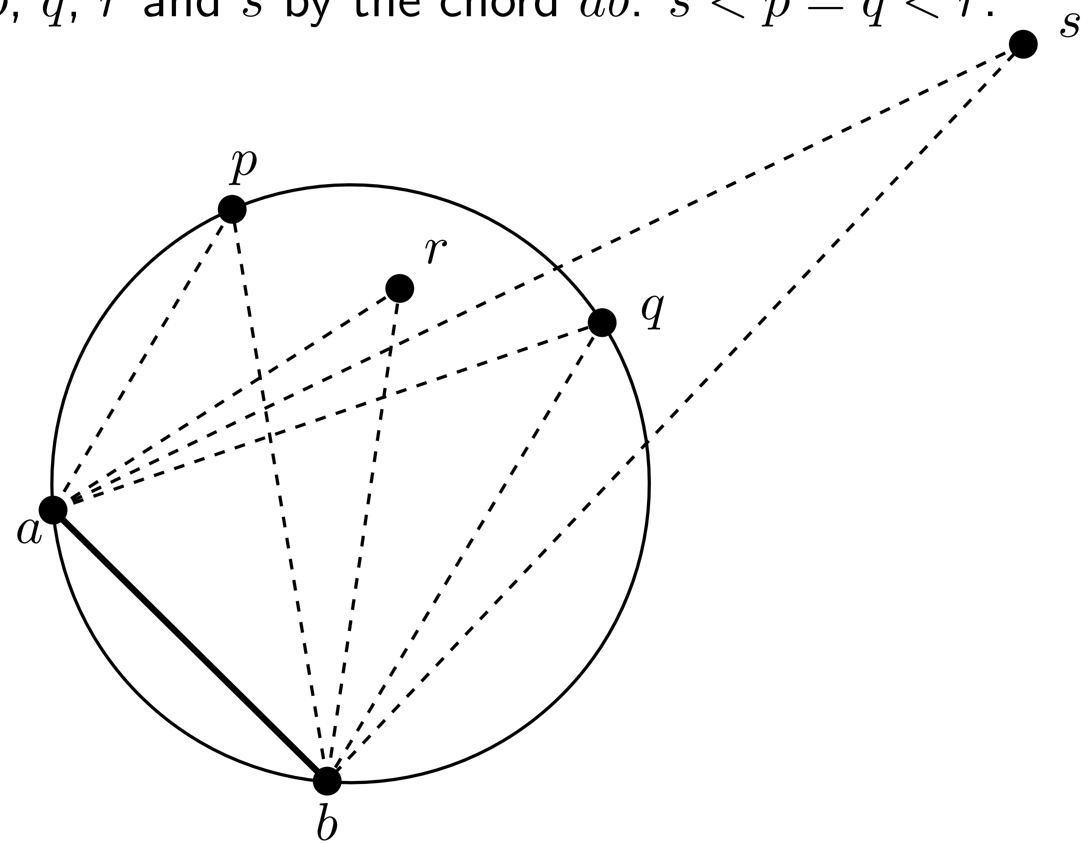
We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 1. Let C be a circle, $\overline{ab}$ a chord of C, and p, q, r and s four points lying to the same side of the line $\overline{ab}$. If r is internal to C, p and q lie in C, and s is external to C, then the following relations hold between the angles formed at p, q, r and s by the chord $\overline{ab}$: $\widehat{s} < \widehat{p} = \widehat{q} < \widehat{r}$.

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 1. Let C be a circle, $\overline{ab}$ a chord of C, and p, q, r and s four points lying to the same side of the line $\overline{ab}$. If r is internal to C, p and q lie in C, and s is external to C, then the following relations hold between the angles formed at p, q, r and s by the chord $\overline{ab}$: $\widehat{s} < \widehat{p} = \widehat{q} < \widehat{r}$.

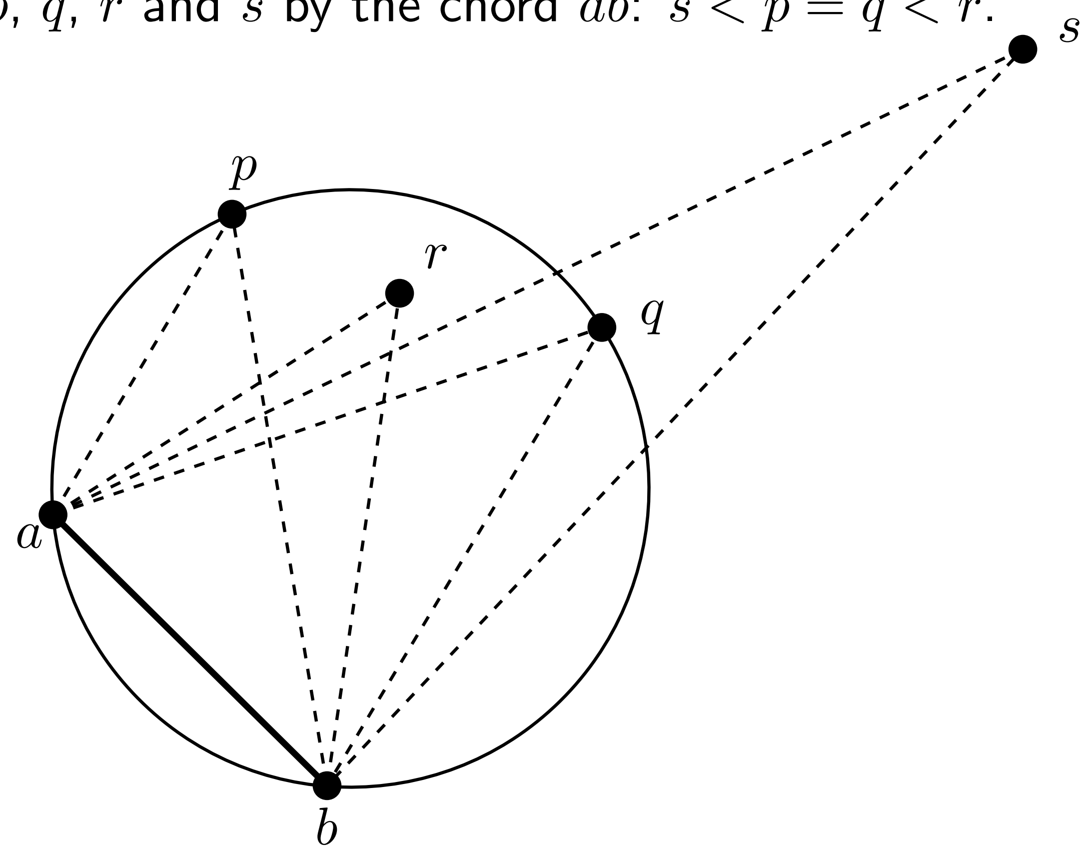


DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 1. Let C be a circle, $\overline{ab}$ a chord of C, and p, q, r and s four points lying to the same side of the line $\overline{ab}$. If r is internal to C, p and q lie in C, and s is external to C, then the following relations hold between the angles formed at p, q, r and s by the chord $\overline{ab}$: $\widehat{s} < \widehat{p} = \widehat{q} < \widehat{r}$.

Let us prove that $\widehat{p} = \widehat{q}$:



DELAUNAY FLIPS

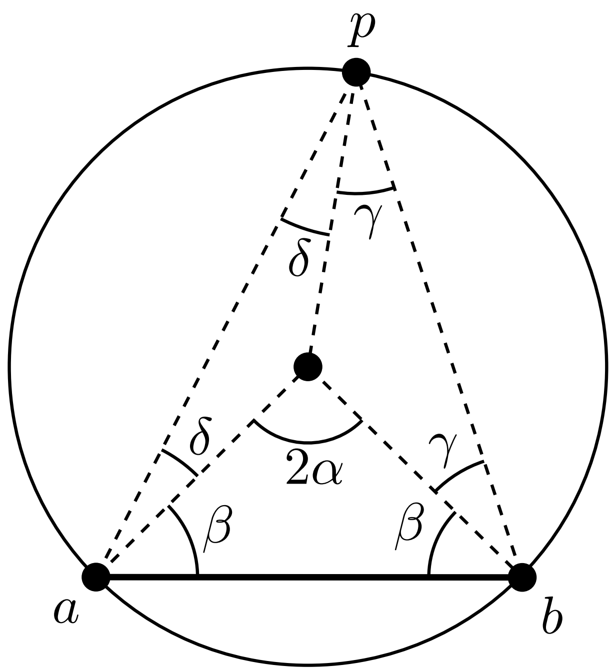
We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 1. Let C be a circle, $\overline{ab}$ a chord of C, and p, q, r and s four points lying to the same side of the line $\overline{ab}$. If r is internal to C, p and q lie in C, and s is external to C, then the following relations hold between the angles formed at p, q, r and s by the chord $\overline{ab}$: $\widehat{s} < \widehat{p} = \widehat{q} < \widehat{r}$.

Let us prove that $\widehat{p} = \widehat{q}$:

First case:

$$\left.\begin{aligned} 2\delta + 2\gamma + 2\beta &= \pi \\ 2\alpha + 2\beta &= \pi \end{aligned}\right\} \Rightarrow 2\alpha = 2\gamma + 2\delta \Rightarrow \alpha = \gamma + \delta \Rightarrow \widehat{p} = \widehat{q} = \alpha$$



DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 1. Let C be a circle, $\overline{ab}$ a chord of C, and p, q, r and s four points lying to the same side of the line $\overline{ab}$. If r is internal to C, p and q lie in C, and s is external to C, then the following relations hold between the angles formed at p, q, r and s by the chord $\overline{ab}$: $\widehat{s} < \widehat{p} = \widehat{q} < \widehat{r}$.

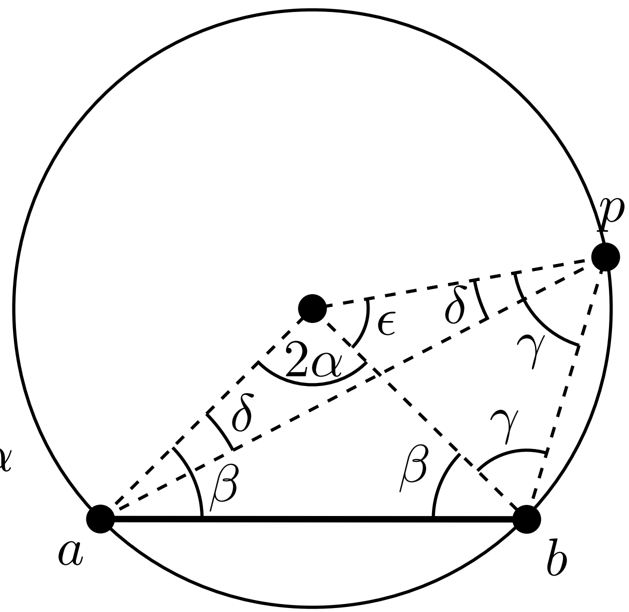
Let us prove that $\widehat{p} = \widehat{q}$:

First case:

$$\left.\begin{aligned} 2\delta + 2\gamma + 2\beta = \pi \\ 2\alpha + 2\beta = \pi \end{aligned}\right\} \Rightarrow 2\alpha = 2\gamma + 2\delta \Rightarrow \alpha = \gamma + \delta \Rightarrow \widehat{p} = \widehat{q} = \alpha$$

Second case:

$$\left.\begin{aligned} 2\alpha + \epsilon + 2\delta = \pi \\ 2\gamma + \epsilon = \pi \end{aligned}\right\} \Rightarrow 2\alpha + 2\delta - 2\gamma = 0 \Rightarrow \alpha = \gamma - \delta \Rightarrow \widehat{p} = \widehat{q} = \alpha$$



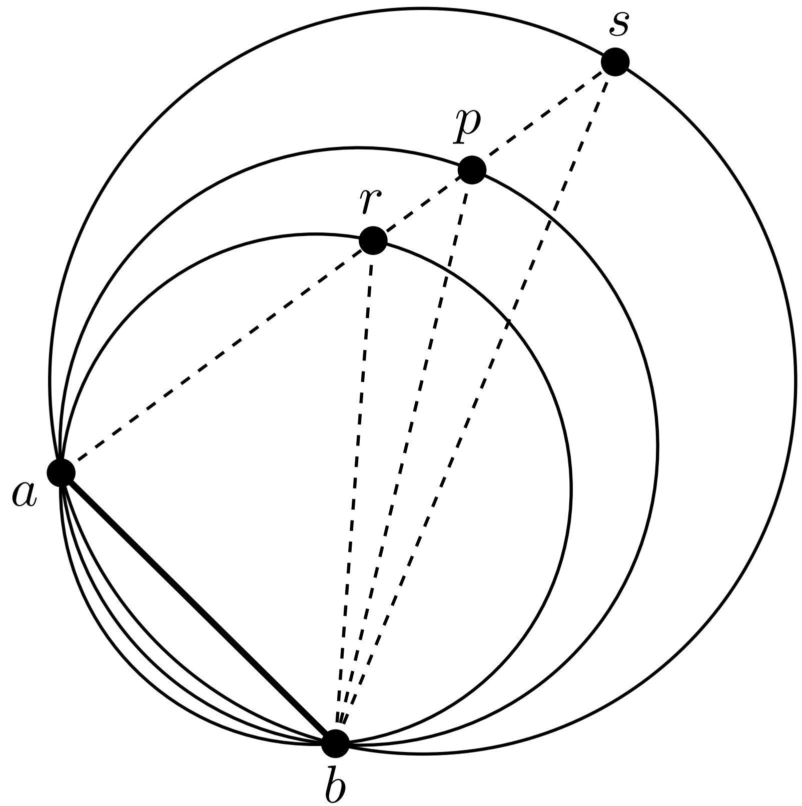
DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 1. Let C be a circle, $\overline{ab}$ a chord of C, and p, q, r and s four points lying to the same side of the line $\overline{ab}$. If r is internal to C, p and q lie in C, and s is external to C, then the following relations hold between the angles formed at p, q, r and s by the chord $\overline{ab}$: $\widehat{s} < \widehat{p} = \widehat{q} < \widehat{r}$.

Let us prove that $\widehat{p} = \widehat{q}$:

The remaining relations follow immediatly:



DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

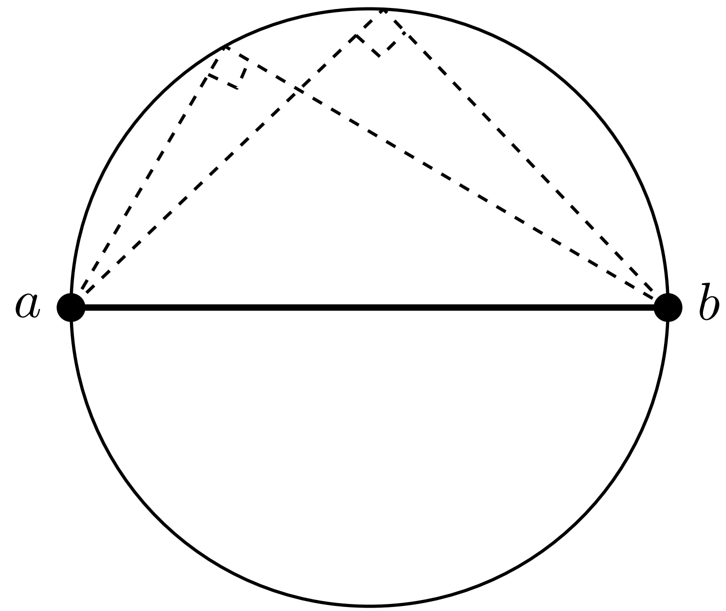
Lemma 2. When the chord $\overline{ab}$ is a diameter of C, the angle $\widehat{p}$ for any $p \in C$ is $\pi/2$.

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 2. When the chord $\overline{ab}$ is a diameter of C, the angle $\widehat{p}$ for any $p \in C$ is $\pi/2$.

Since in this case $2\alpha = \pi$.



DELAUNAY TRIANGULATION

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

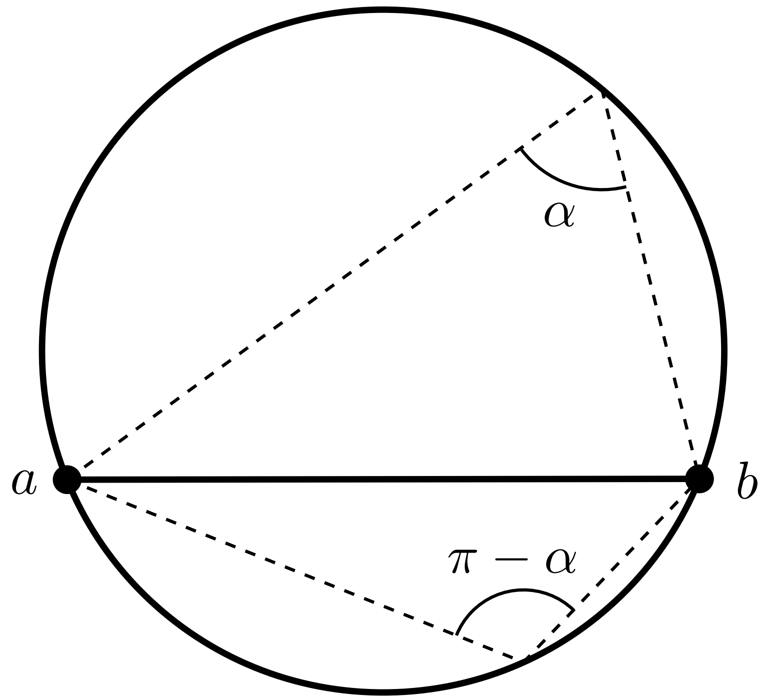
Lemma 3. Given any chord $\overline{ab}$ in a circle C, if one of the arcs corresponds to α, then the other one corresponds to $\pi - \alpha$.

DELAUNAY TRIANGULATION

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

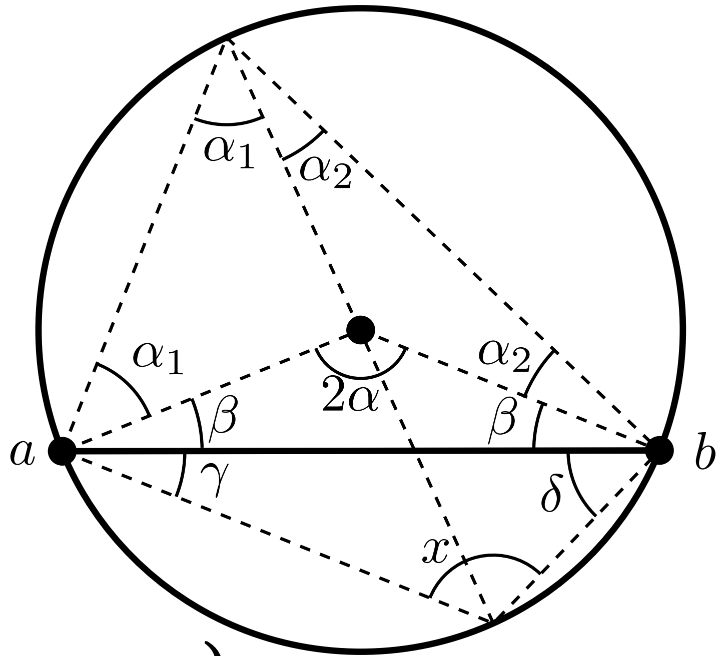
Lemma 3. Given any chord $\overline{ab}$ in a circle C, if one of the arcs corresponds to α, then the other one corresponds to $\pi - \alpha$.



DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 3. Given any chord $\overline{ab}$ in a circle C, if one of the arcs corresponds to α, then the other one corresponds to $\pi - \alpha$.



$$\left.\begin{array}{r} \left.\begin{array}{r} \alpha_1 + \beta + \gamma = \frac{\pi}{2} \\ \alpha_2 + \beta + \delta = \frac{\pi}{2} \end{array}\right\} \Rightarrow \alpha + 2\beta + \gamma + \delta = \pi \\ x + \gamma + \delta = \pi \end{array}\right\} \Rightarrow x = \alpha + 2\beta \\ \left.\begin{array}{r} \\ 2\alpha + 2\beta = \pi \end{array}\right\} \Rightarrow x = \pi - \alpha$$

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 4. Let $\overline{pq}$ be the common edge of the triangles pqa and pqb, forming a convex quatrilateral. Then:

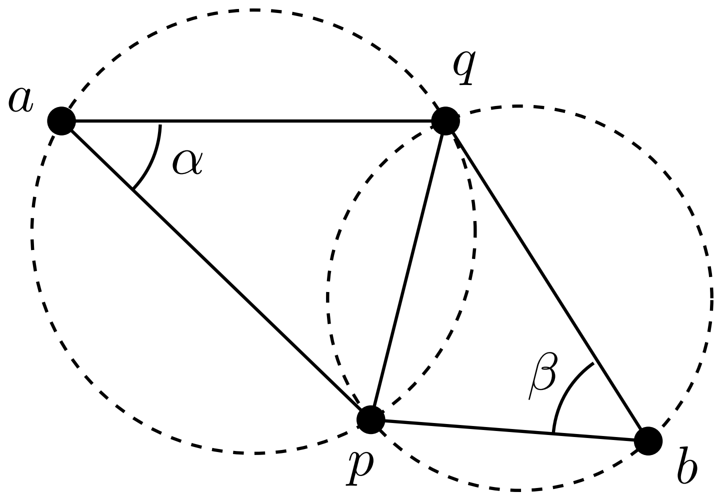
$$a \in ext(C_{pqb}) \Longleftrightarrow b \in ext(C_{pqa})$$

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 4. Let $\overline{pq}$ be the common edge of the triangles pqa and pqb, forming a convex quatrilateral. Then:

$$a \in ext(C_{pqb}) \iff b \in ext(C_{pqa})$$



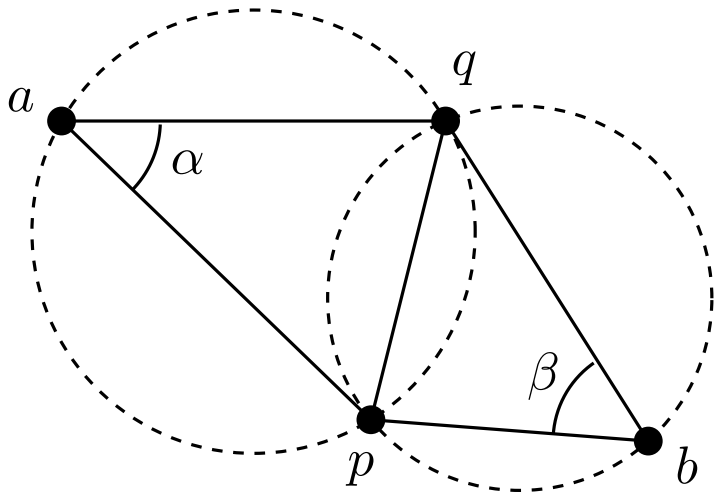
DELAUNAY TRIANGULATION

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 4. Let $\overline{pq}$ be the common edge of the triangles pqa and pqb, forming a convex quatrilateral. Then:

$$a \in ext(C_{pqb}) \iff b \in ext(C_{pqa})$$



$$a \in ext(C_{pqb}) \iff \alpha < \pi - \beta \iff \beta < \pi - \alpha \iff b \in ext(C_{pqa})$$

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 5. Consider a convex quadrilateral with diagonals $\overline{ab}$ and $\overline{pq}$. Then:

$$\overline{ab} \text{ is not locally Delaunay} \iff \overline{pq} \text{ is locally Delaunay}$$

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 5. Consider a convex quadrilateral with diagonals $\overline{ab}$ and $\overline{pq}$. Then:

$$\overline{ab} \text{ is not locally Delaunay} \iff \overline{pq} \text{ is locally Delaunay}$$

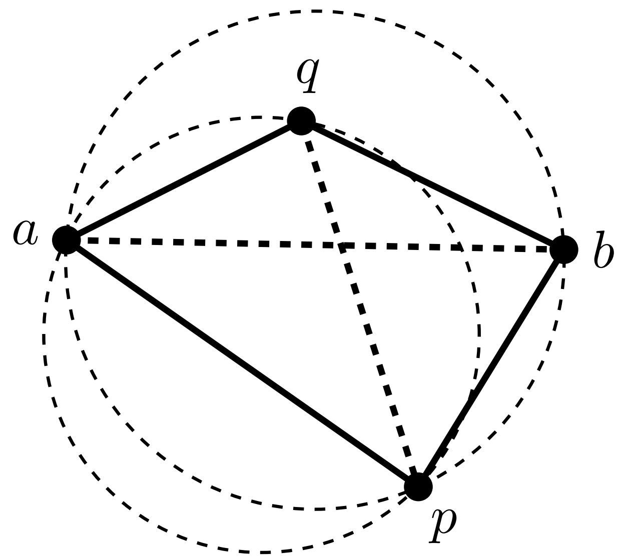
$\overline{ab}$ is not locally Delaunay

$\iff q \in int(C_{abp})$

$\iff \widehat{aqp} > \widehat{abp}$

$\iff b \in ext(C_{apq})$

$\iff \overline{pq}$ is locally Delaunay



DELAUNAY FLIPS

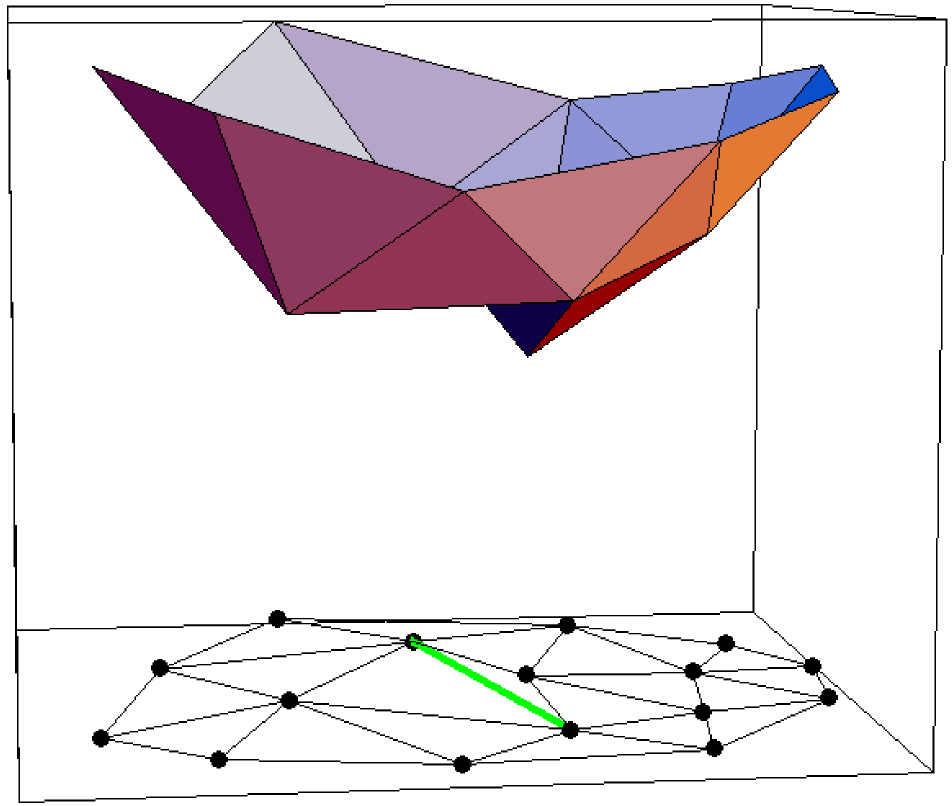
We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 6. Let P be a set of points $p_i = (x_i, y_i, 0)$ in the plane, and let P^* be the set of their vertical projections $p^* = (x_i, y_i, x_i^2 + y_i^2)$ onto the unit paraboloid. Producing a Delaunay flip in a triangulation of P corresponds to "sticking" a tetrahedron from below to the corresponding polyhedrization of P^*.

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 6. Let P be a set of points $p_i = (x_i, y_i, 0)$ in the plane, and let P^* be the set of their vertical projections $p^* = (x_i, y_i, x_i^2 + y_i^2)$ onto the unit paraboloid. Producing a Delaunay flip in a triangulation of P corresponds to "sticking" a tetrahedron from below to the corresponding polyhedrization of P^*.

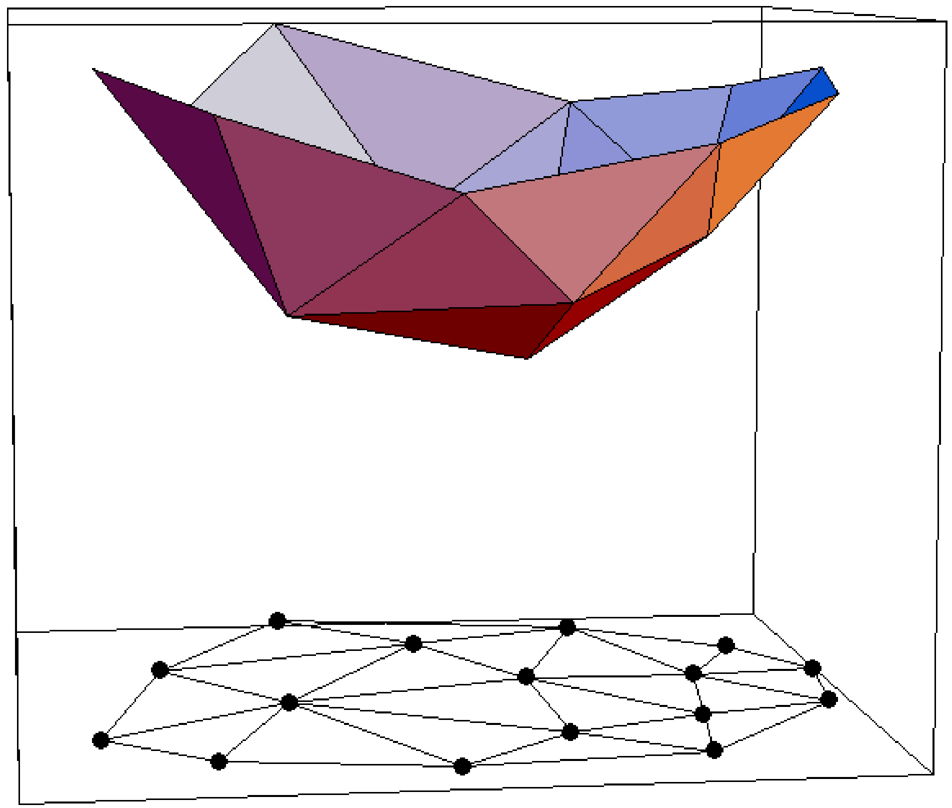


DELAUNAY TRIANGULATION

DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 6. Let P be a set of points $p_i = (x_i, y_i, 0)$ in the plane, and let P^* be the set of their vertical projections $p^* = (x_i, y_i, x_i^2 + y_i^2)$ onto the unit paraboloid. Producing a Delaunay flip in a triangulation of P corresponds to "sticking" a tetrahedron from below to the corresponding polyhedrization of P^*.



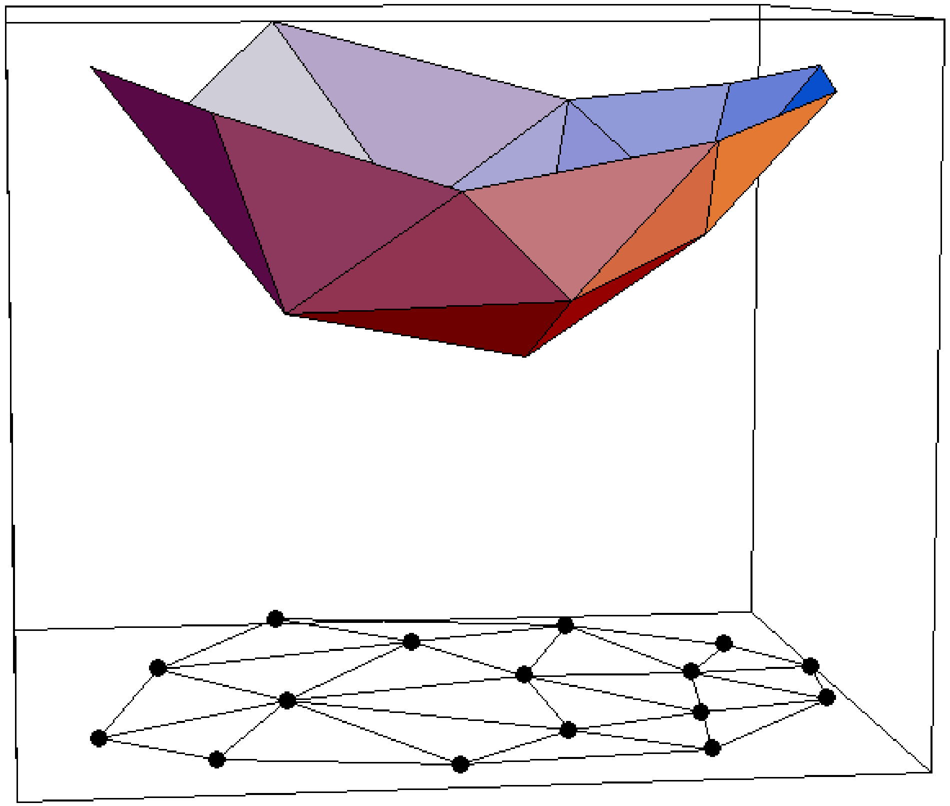
DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Lemma 6. Let P be a set of points $p_i = (x_i, y_i, 0)$ in the plane, and let P^* be the set of their vertical projections $p^* = (x_i, y_i, x_i^2 + y_i^2)$ onto the unit paraboloid. Producing a Delaunay flip in a triangulation of P corresponds to "sticking" a tetrahedron from below to the corresponding polyhedrization of P^*.

Once flipped, the quatrilateral is locally Delaunay: the fourth point lies in the exterior of the circumcircle of the triangle.

In the paraboloid, this means that the fourth point lies above the triangular face of the polyhedrization.



DELAUNAY FLIPS

We intend to prove that $Del(P)$ can be obtained from any triangulation of P by Delaunay flips, which consist in deleting the diagonal of a convex quadrilateral if it is not locally Delaunay, and replacing it by the other diagonal of the quadrilateral.

Corollary. Given any triangulation of P, performing locally Delaunay flips is a procedure converging to $Del(P)$.

ALGORITHMS

ALGORITHMS

1. Compute the Voronoi diagram by any of the known methods and dualize it.

ALGORITHMS

1. Compute the Voronoi diagram by any of the known methods and dualize it.

2. Project the points onto the paraboloid, compute the 3D convex hull by any of the known methods, and project it back onto the plane.

DELAUNAY TRIANGULATION

ALGORITHMS

1. Compute the Voronoi diagram by any of the known methods and dualize it.

2. Project the points onto the paraboloid, compute the 3D convex hull by any of the known methods, and project it back onto the plane.

3. Compute a triangulation, by any of the known methods, and apply Delaunay flips.

DELAUNAY TRIANGULATION

ALGORITHMS

1. Compute the Voronoi diagram by any of the known methods and dualize it.

2. Project the points onto the paraboloid, compute the 3D convex hull by any of the known methods, and project it back onto the plane.

3. Compute a triangulation, by any of the known methods, and apply Delaunay flips.

4. Incremental algorithm

Compute an enclosing triangle for $\{p_1, \ldots, p_n\}$

Compute $Del(p_1, \ldots, p_{i+1})$ from $Del(p_1, \ldots, p_i)$

INCREMENTAL ALGORITHM

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

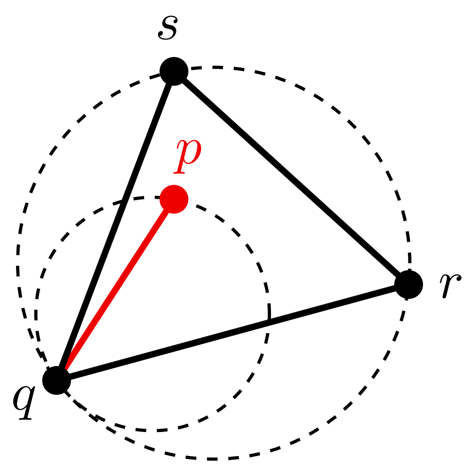
DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

As C_{qrs} is empty, there exist empty circles C_{pq}, such as the circle through p and q tangent to C_{qrs} in q. Similarly for r and s.



INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

DELAUNAY TRIANGULATION

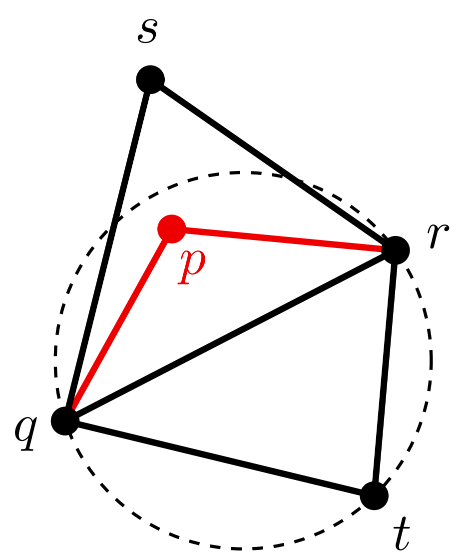
INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

Since p may lie in the interior of C_{qrt}.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Obvious, because the property is local: it affects only quadrilaterals formed by two triangles sharing an edge.

DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.

DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

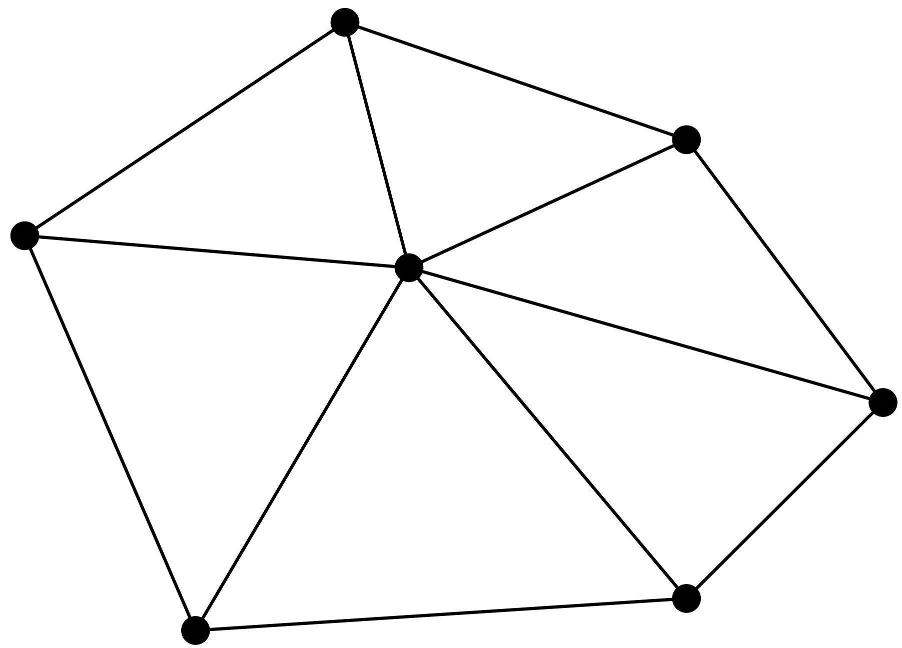
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

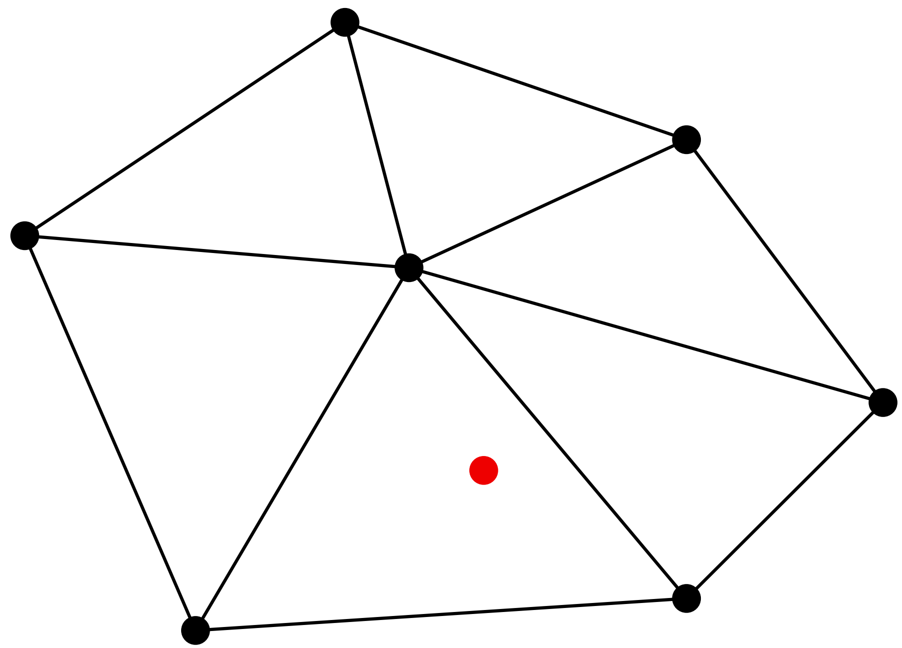
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

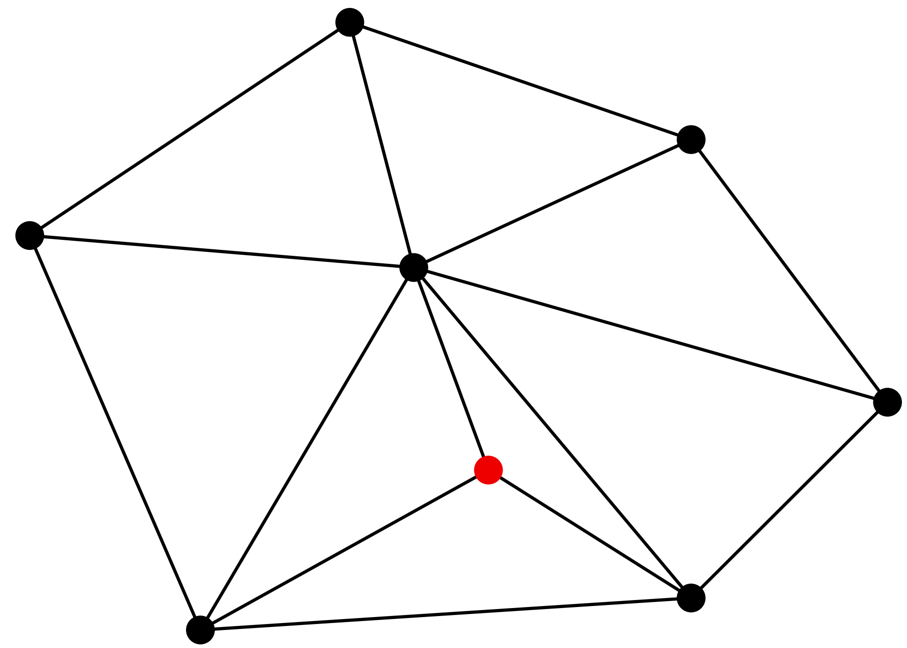
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

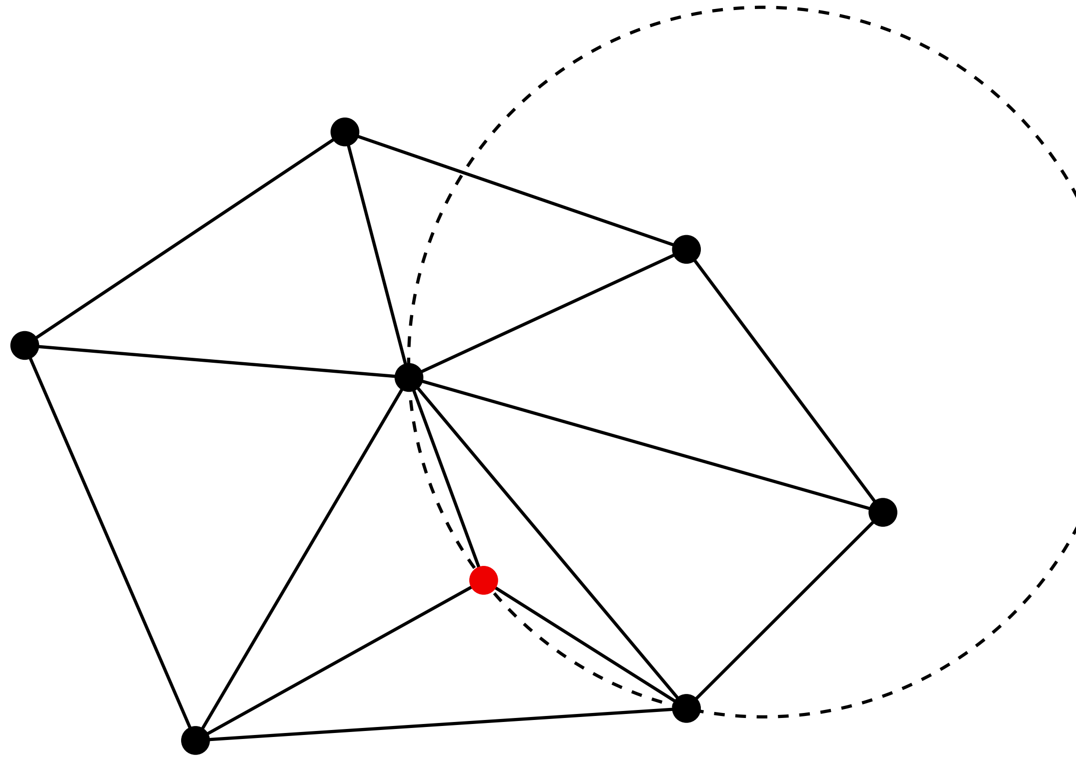
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

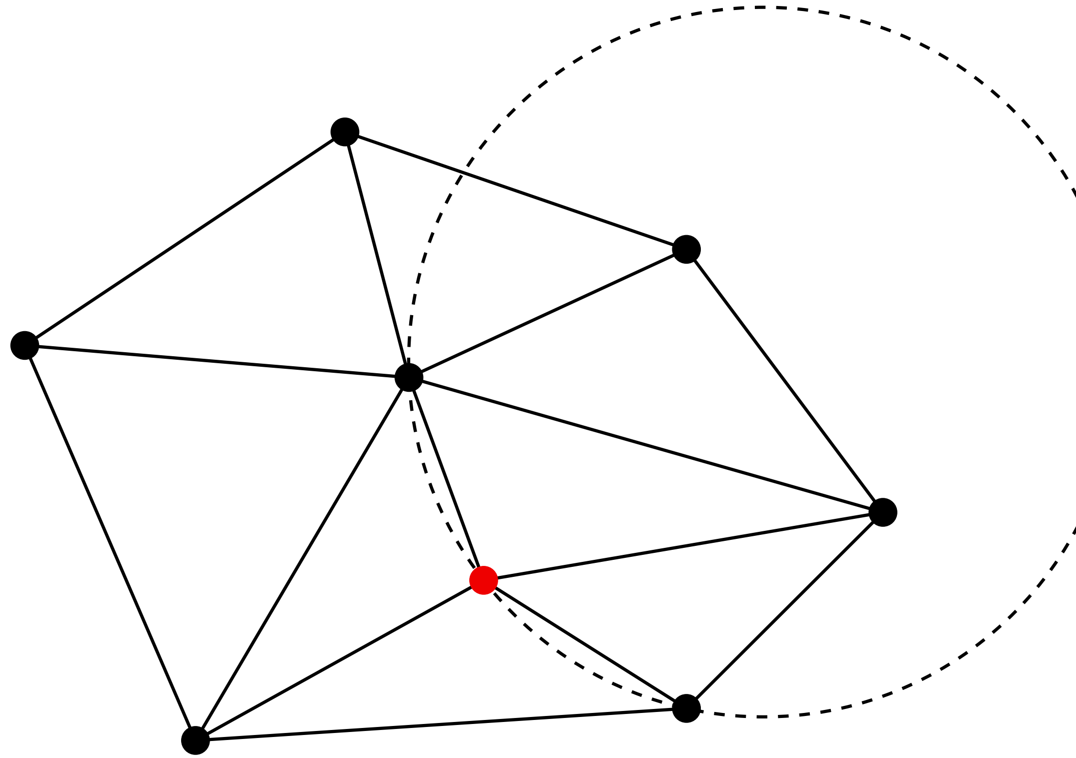
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

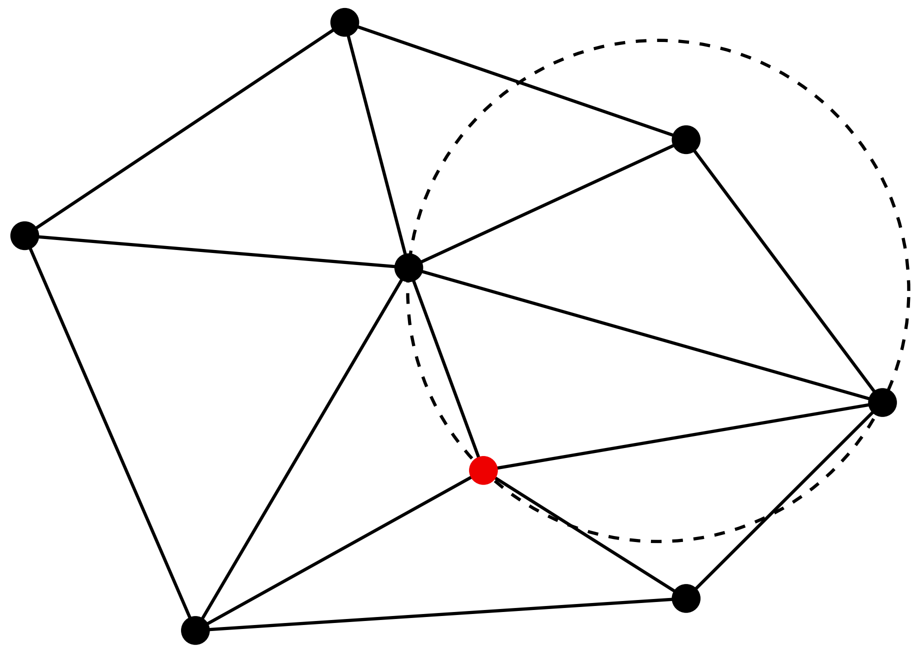
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

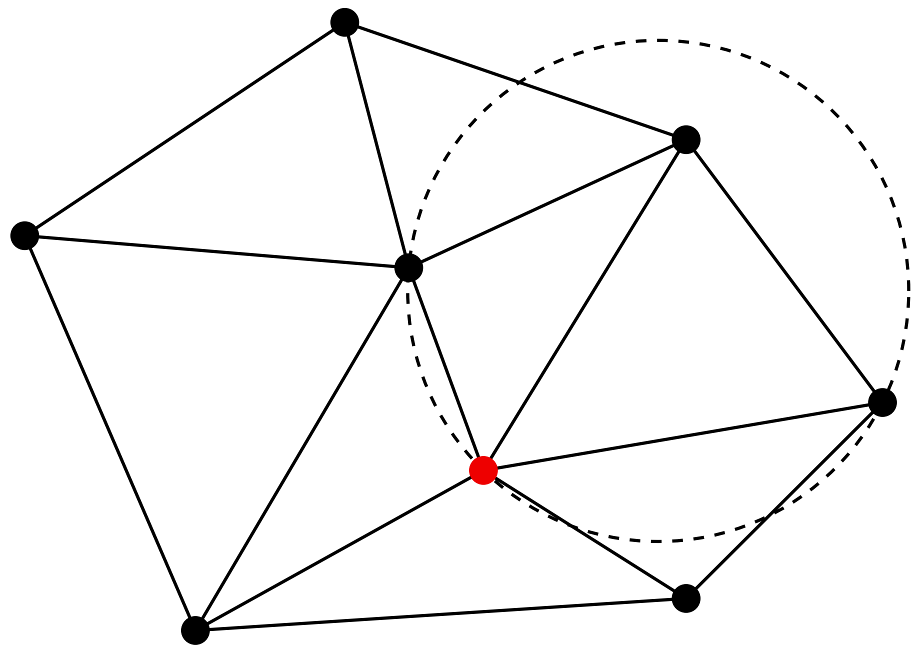
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

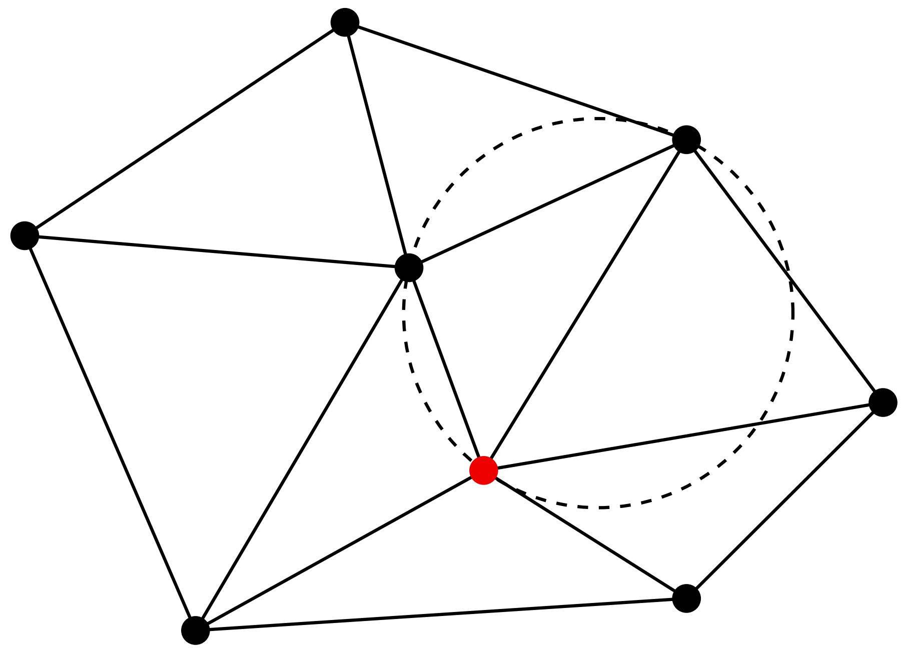
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

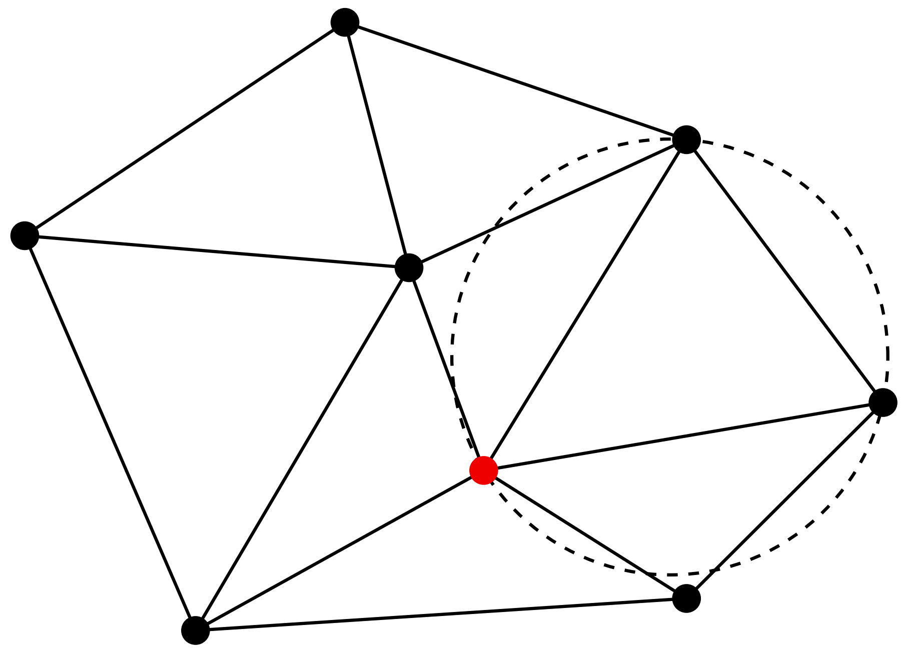
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

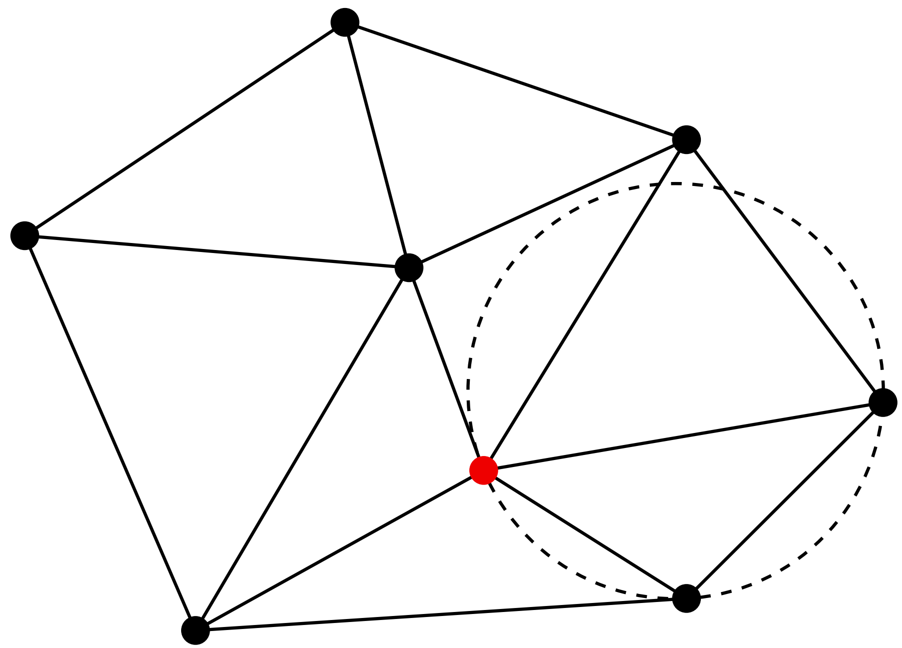
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

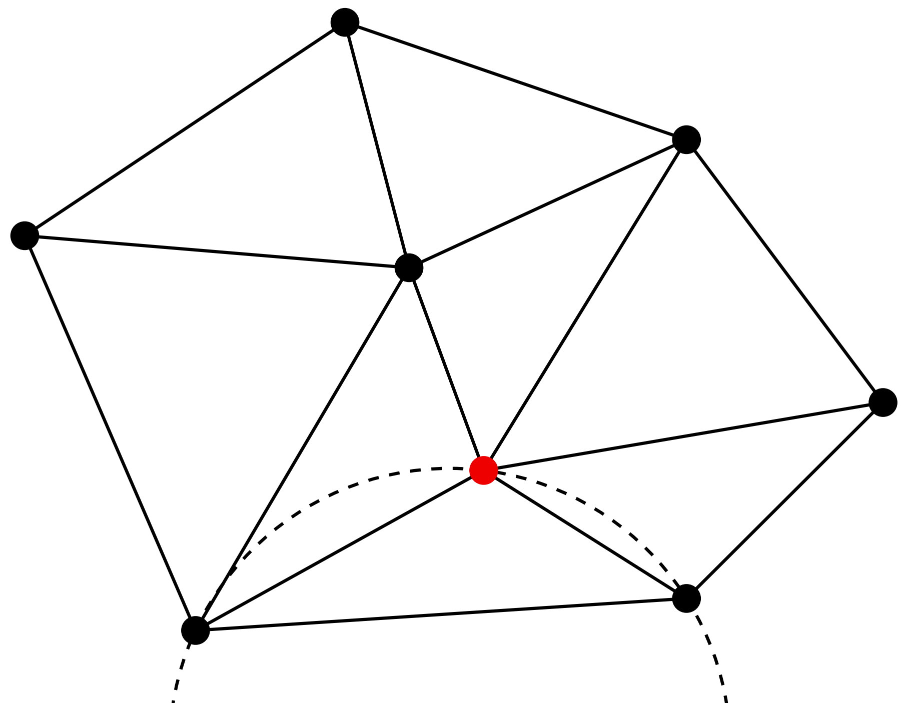
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \dots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

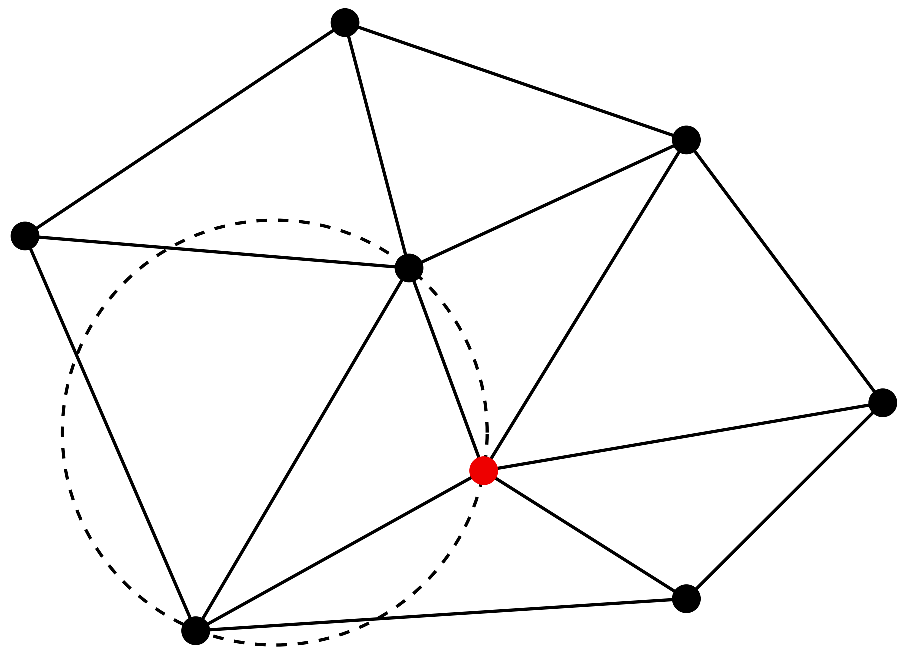
Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.



DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.

Added running time

DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.

Added running time

The added running time of performing the flips when adding p_i is

$$O(\text{degree of } p_i \text{ in } D_i) = O(n).$$

DELAUNAY TRIANGULATION

INCREMENTAL ALGORITHM

Let $D_i = Del(p_1, \ldots, p_i)$ and $p = p_{i+1}$.

Observation 1. If qrs is the triangle of D_i containing p, then $\overline{pq}$, $\overline{pr}$ and $\overline{ps}$ are edges of D_{i+1}.

Observation 2. Let pqr be a triangle incident to p. The edge $\overline{qr}$ may not be a Delaunay edge.

Observation 3. The insertion of the point p can only violate the Delaunay property of the triangles incident to p.

Algorithm

Each time a new point is added to the triangulation, and before adding the next point, the following routine is executed:

Flips

While there are still triangles incident to p non locally Delaunay, flip them.

Added running time

The added running time of performing the flips when adding p_i is

$$O(\text{degree of } p_i \text{ in } D_i) = O(n).$$

As the average order is smaller than 6, the expected added running time is not $O(n^2)$ but simply $O(n)$.

DELAUNAY TRIANGULATION AND EQUIANGULARITY

DELAUNAY TRIANGULATION

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Among all the triangulations of P, the Delaunay triangulation maximizes the minimum angle (the angles of $Del(P)$ are less acute).

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Among all the triangulations of P, the Delaunay triangulation maximizes the minimum angle (the angles of $Del(P)$ are less acute).

Let us be more precise:

If $\mathcal{T} = \{T_1, \ldots, T_t\}$ is a triangulation of P, the "fineness" of $\mathcal{T}$ is the increasingly sorted list of the angles of all the triangles T_i of $\mathcal{T}$: $F(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3t})$.

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Among all the triangulations of P, the Delaunay triangulation maximizes the minimum angle (the angles of $Del(P)$ are less acute).

Let us be more precise:

If $\mathcal{T} = \{T_1, \ldots, T_t\}$ is a triangulation of P, the "fineness" of $\mathcal{T}$ is the increasingly sorted list of the angles of all the triangles T_i of $\mathcal{T}$: $F(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3t})$.

Since every triangulation of P has $t = 2n - h - 2$ triangles, these $3t$-tuples can be compared and lexicographically sorted.

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Among all the triangulations of P, the Delaunay triangulation maximizes the minimum angle (the angles of $Del(P)$ are less acute).

Let us be more precise:

If $\mathcal{T} = \{T_1, \ldots, T_t\}$ is a triangulation of P, the "fineness" of $\mathcal{T}$ is the increasingly sorted list of the angles of all the triangles T_i of $\mathcal{T}$: $F(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3t})$.

Since every triangulation of P has $t = 2n - h - 2$ triangles, these $3t$-tuples can be compared and lexicographically sorted.

The Delaunay triangulation maximizes the "fineness":

$$F(Del(P)) \geq F(\mathcal{T}), \quad \forall \mathcal{T} \text{ triangulation of } P.$$

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Among all the triangulations of P, the Delaunay triangulation maximizes the minimum angle (the angles of $Del(P)$ are less acute).

Let us be more precise:

If $\mathcal{T} = \{T_1, \ldots, T_t\}$ is a triangulation of P, the "fineness" of $\mathcal{T}$ is the increasingly sorted list of the angles of all the triangles T_i of $\mathcal{T}$: $F(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3t})$.

Since every triangulation of P has $t = 2n - h - 2$ triangles, these $3t$-tuples can be compared and lexicographically sorted.

The Delaunay triangulation maximizes the "fineness":

$$F(Del(P)) \geq F(\mathcal{T}), \quad \forall \mathcal{T} \text{ triangulation of } P.$$

The proof of this statement requires a last lemma.

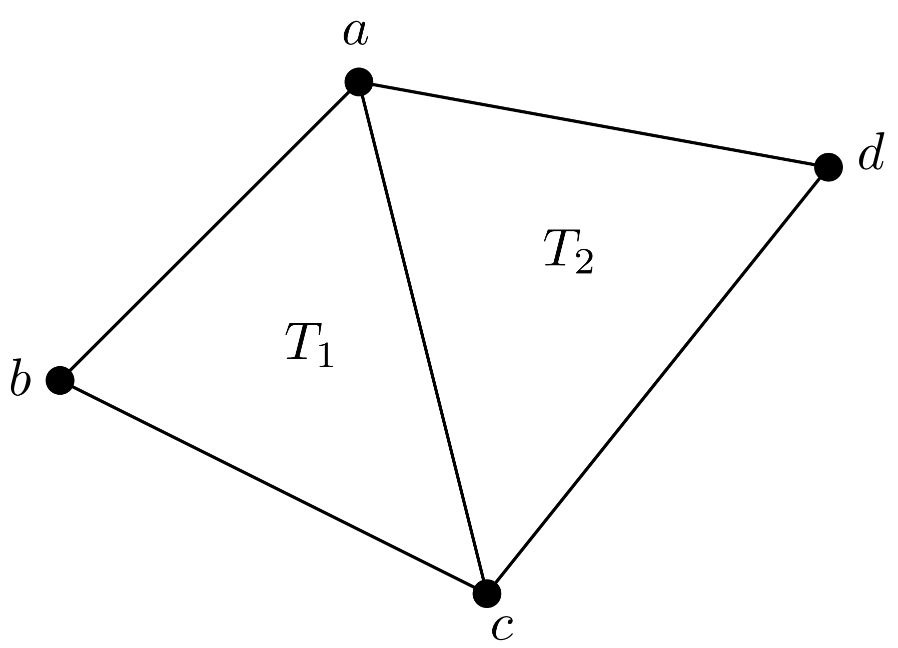
DELAUNAY TRIANGULATION AND EQUIANGULARITY

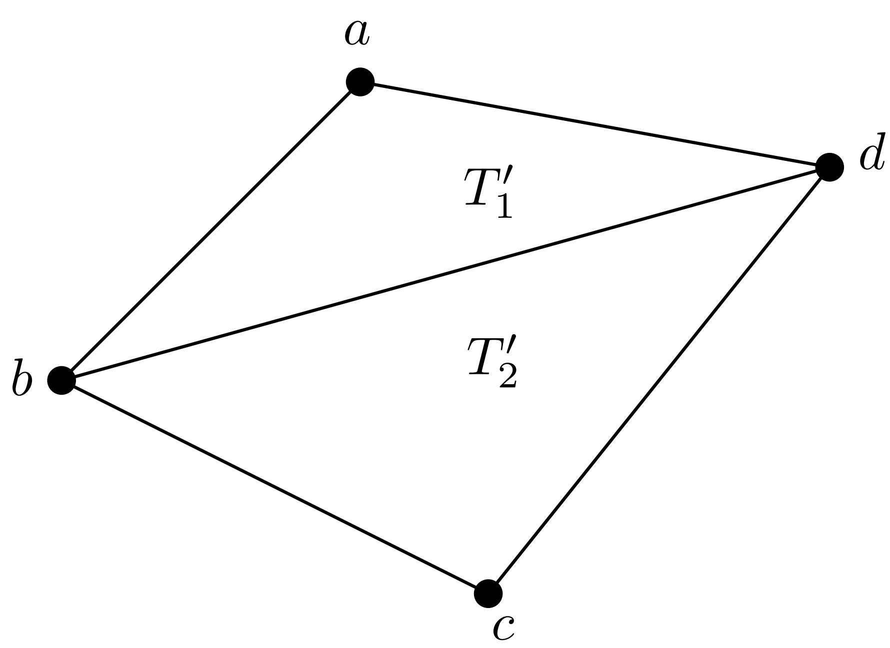
Lemma 7. Let a, b, c and d be four points forming a convex quadrilateral, in counterclockwise order. Let $\mathcal{T}$ and $\mathcal{T}'$ be the two possible triangulations of the quadrilateral: $\mathcal{T}$ uses the diagonal $\overline{ac}$ and $\mathcal{T}'$ uses $\overline{bd}$. Let ϵ and ϵ' respectively be the minimum angles of $\mathcal{T}$ and $\mathcal{T}'$. Then:

$$\epsilon > \epsilon' \iff d \in ext(C_{abc})$$

$$\epsilon = \epsilon' \iff d \in \partial(C_{abc})$$

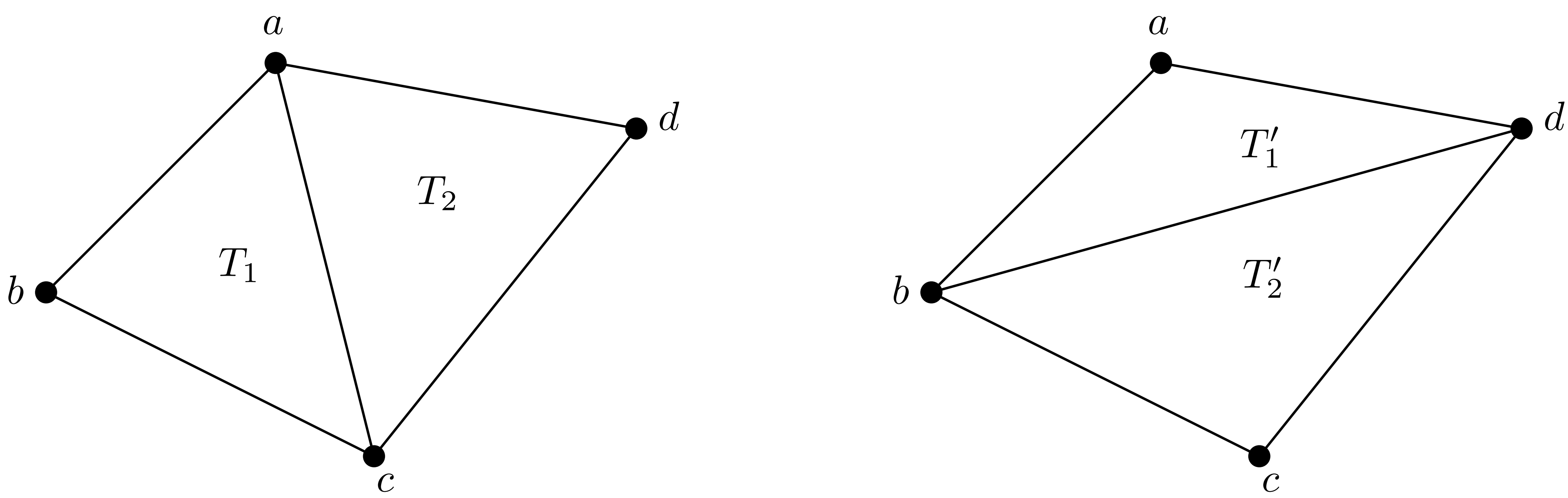
$$\epsilon < \epsilon' \iff d \in int(C_{abc})$$





DELAUNAY TRIANGULATION AND EQUIANGULARITY

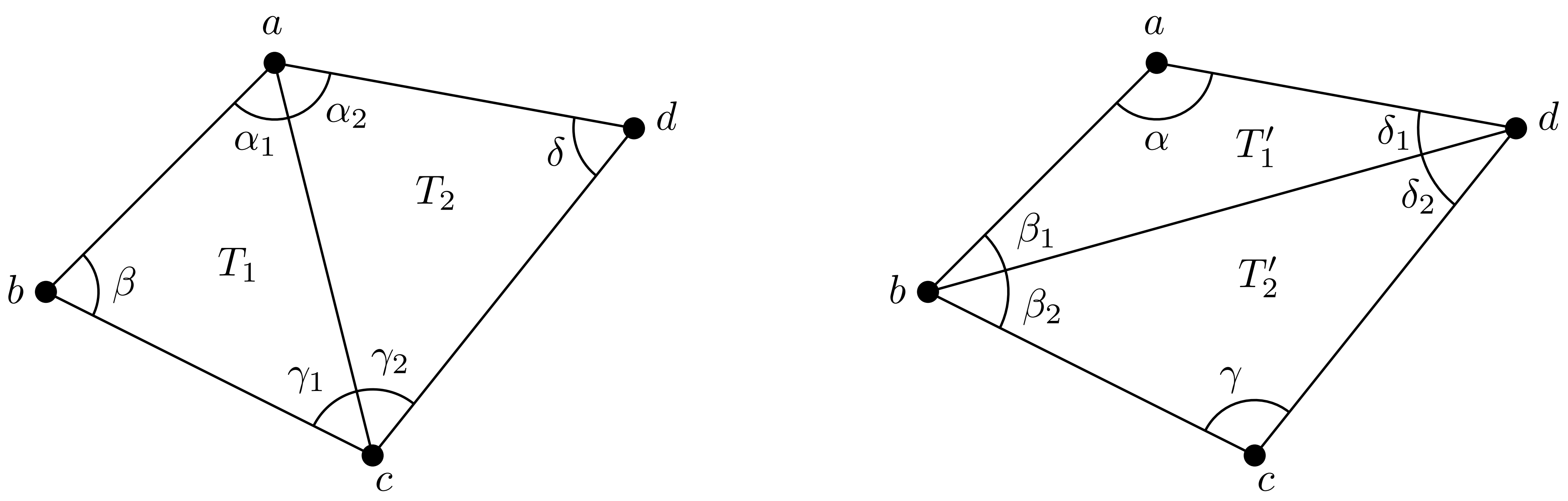
Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.



DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$



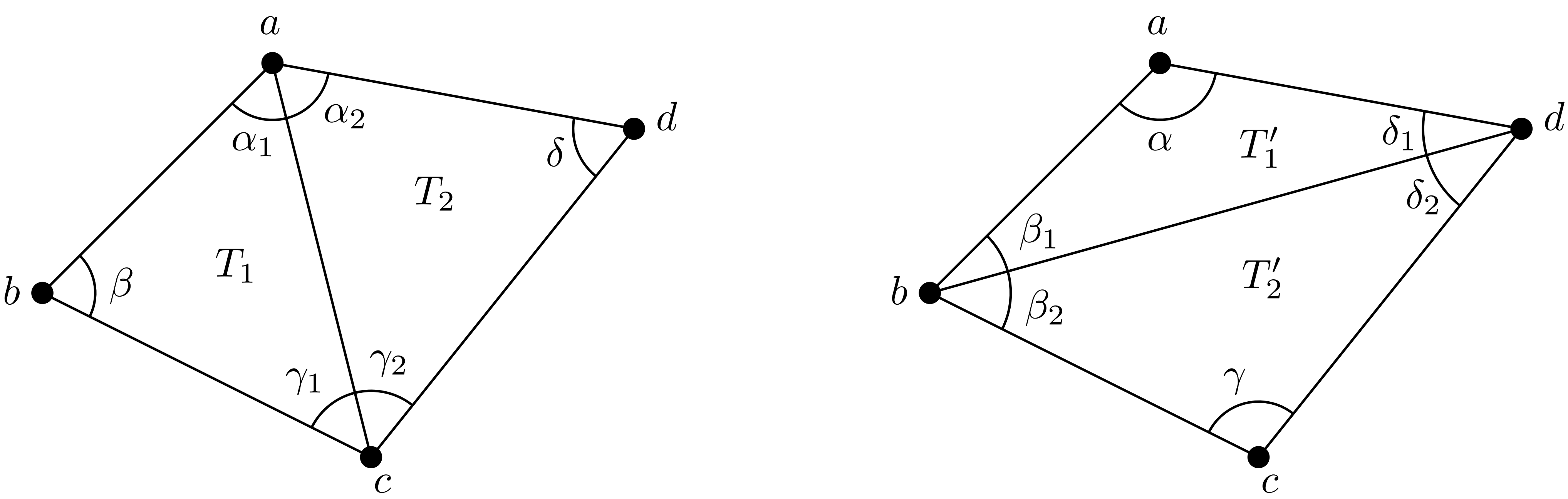
DELAUNAY TRIANGULATION

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

If $\epsilon > \epsilon'$, then ϵ' cannot be α, nor γ.

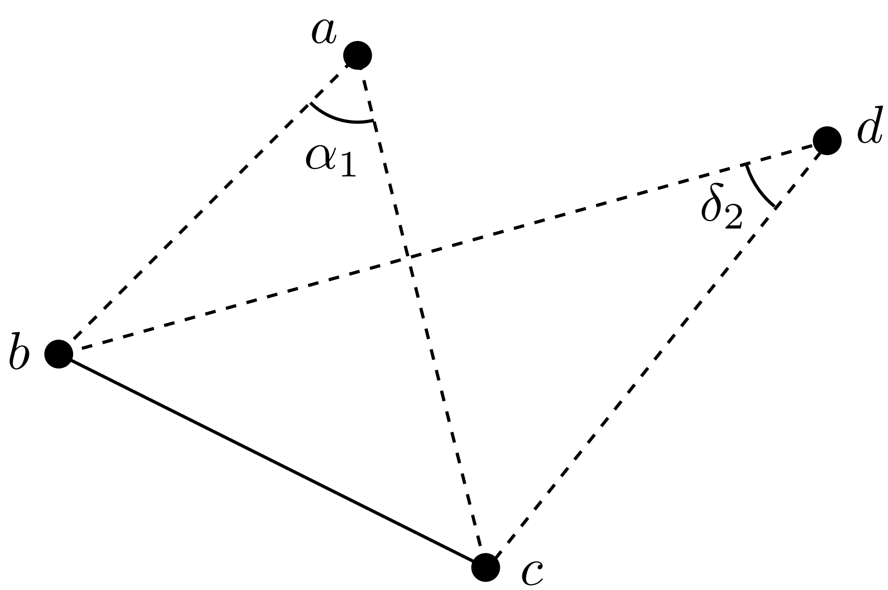


DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

- If $\epsilon > \epsilon'$, then ϵ' cannot be α, nor γ.
- If $\epsilon' = \delta_2$, then $\delta_2 = \epsilon' < \epsilon \leq \alpha_1$ and, therefore, $d \in ext(C_{abc})$.



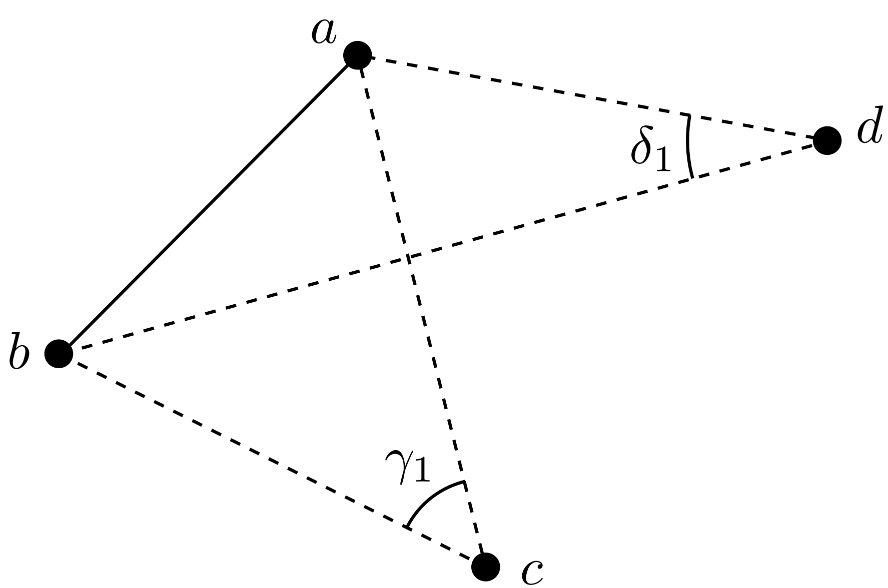
DELAUNAY TRIANGULATION

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

- If $\epsilon > \epsilon'$, then ϵ' cannot be α, nor γ.
- If $\epsilon' = \delta_2$, then $\delta_2 = \epsilon' < \epsilon \leq \alpha_1$ and, therefore, $d \in ext(C_{abc})$.
- If $\epsilon' = \delta_1$, then $\delta_1 = \epsilon' < \epsilon \leq \gamma_1$ and, therefore, $d \in ext(C_{abc})$.

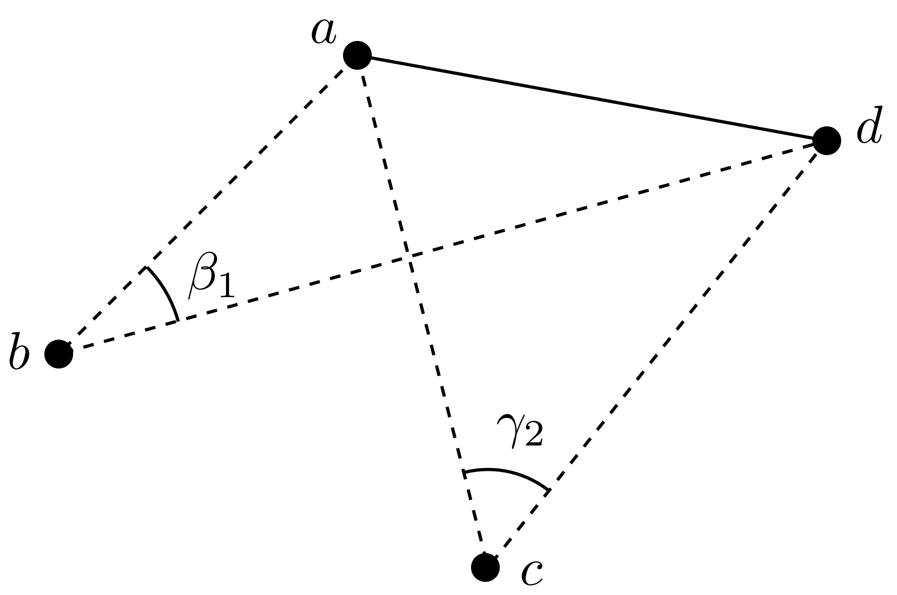


DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

- If $\epsilon > \epsilon'$, then ϵ' cannot be α, nor γ.
- If $\epsilon' = \delta_2$, then $\delta_2 = \epsilon' < \epsilon \leq \alpha_1$ and, therefore, $d \in ext(C_{abc})$.
- If $\epsilon' = \delta_1$, then $\delta_1 = \epsilon' < \epsilon \leq \gamma_1$ and, therefore, $d \in ext(C_{abc})$.
- If $\epsilon' = \beta_1$, then $\beta_1 = \epsilon' < \epsilon \leq \gamma_2$ and, therefore, $d \in ext(C_{abc})$.

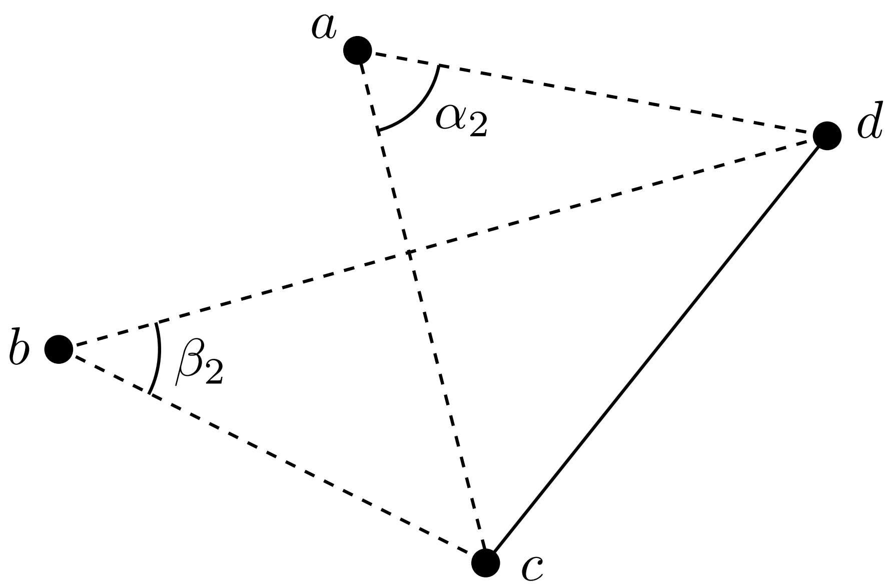


DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

- If $\epsilon > \epsilon'$, then ϵ' cannot be α, nor γ.
- If $\epsilon' = \delta_2$, then $\delta_2 = \epsilon' < \epsilon \leq \alpha_1$ and, therefore, $d \in ext(C_{abc})$.
- If $\epsilon' = \delta_1$, then $\delta_1 = \epsilon' < \epsilon \leq \gamma_1$ and, therefore, $d \in ext(C_{abc})$.
- If $\epsilon' = \beta_1$, then $\beta_1 = \epsilon' < \epsilon \leq \gamma_2$ and, therefore, $d \in ext(C_{abc})$.
- If $\epsilon' = \beta_2$, then $\beta_2 = \epsilon' < \epsilon \leq \alpha_2$ and, therefore, $d \in ext(C_{abc})$.



DELAUNAY TRIANGULATION

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

DELAUNAY TRIANGULATION

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

If $d \in ext(C_{abc})$, then $\epsilon > \epsilon'$

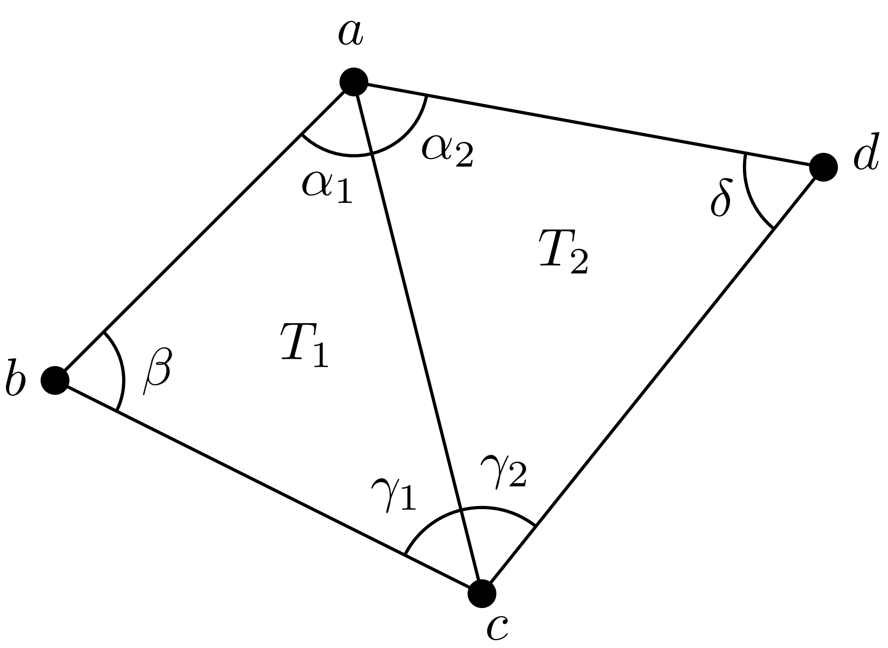
DELAUNAY TRIANGULATION AND EQUIANGULARITY

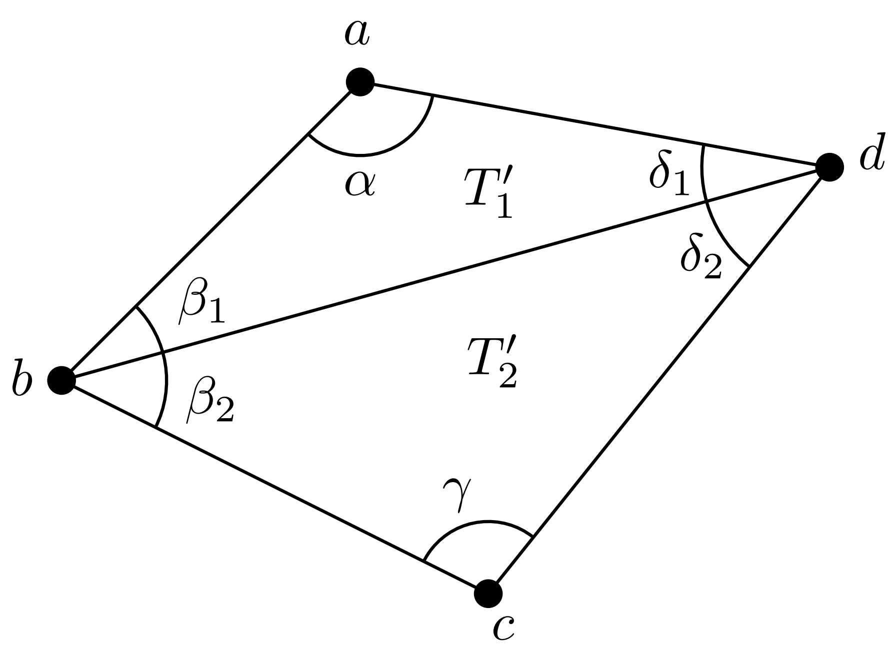
Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

If $d \in ext(C_{abc})$, then $\epsilon > \epsilon'$

- If $\epsilon = \beta$, then $\epsilon = \beta > \beta_1 \geq \epsilon'$
- If $\epsilon = \delta$, then $\epsilon = \delta > \delta_1 \geq \epsilon'$





DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

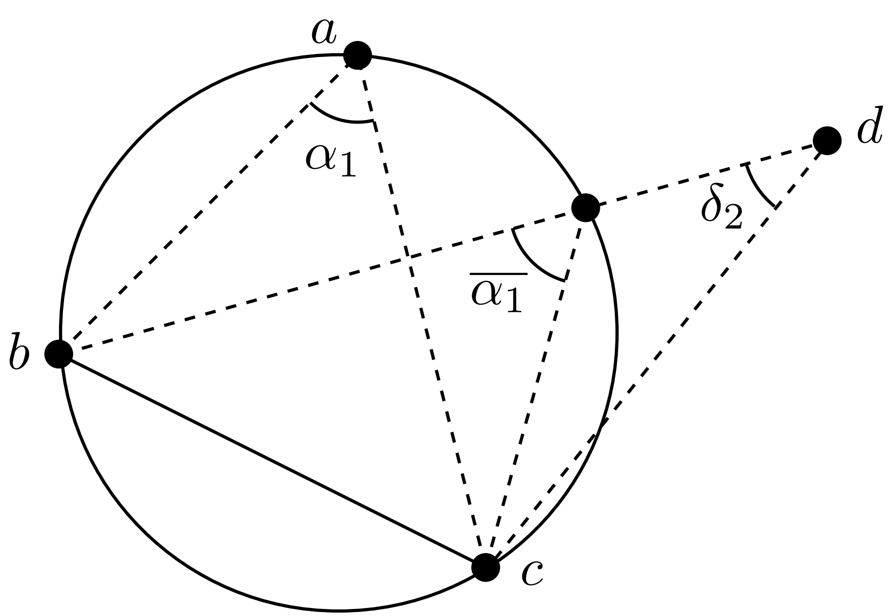
If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

If $d \in ext(C_{abc})$, then $\epsilon > \epsilon'$

If $\epsilon = \beta$, then $\epsilon = \beta > \beta_1 \geq \epsilon'$

If $\epsilon = \delta$, then $\epsilon = \delta > \delta_1 \geq \epsilon'$

If $\epsilon = \alpha_1$, then $\epsilon = \alpha_1 = \overline{\alpha_1} > \delta_2 \geq \epsilon'$



DELAUNAY TRIANGULATION AND EQUIANGULARITY

Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.

If $\epsilon > \epsilon'$, then $d \in ext(C_{abc})$

If $d \in ext(C_{abc})$, then $\epsilon > \epsilon'$

- If $\epsilon = \beta$, then $\epsilon = \beta > \beta_1 \geq \epsilon'$
- If $\epsilon = \delta$, then $\epsilon = \delta > \delta_1 \geq \epsilon'$
- If $\epsilon = \alpha_1$, then $\epsilon = \alpha_1 = \overline{\alpha_1} > \delta_2 \geq \epsilon'$
- If $\epsilon = \alpha_2$, then $\epsilon = \alpha_2 = \overline{\alpha_2} > \beta_2 \geq \epsilon'$
- If $\epsilon = \gamma_1$, then $\epsilon = \gamma_1 = \overline{\gamma_1} > \delta_1 \geq \epsilon'$
- If $\epsilon = \gamma_2$, then $\epsilon = \gamma_2 = \overline{\gamma_2} > \beta_1 \geq \epsilon'$

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Corollary. The Delaunay triangulation is the most equiangular among all triangulations of a given set of points.

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Corollary. The Delaunay triangulation is the most equiangular among all triangulations of a given set of points.

If P does not contain four or more concyclic points, it follows from the previous lemma.

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Corollary. The Delaunay triangulation is the most equiangular among all triangulations of a given set of points.

If P does not contain four or more concyclic points, it follows from the previous lemma.

If P contains four or more concyclic points, $Del(P)$ contains a polygon inscribed in a circle which can be triangulated in several ways. Nevertheless, Lemma 1 (on the geometrical locus of all the points from which a segment is seen under a given angle) guarantees that every triangulation of a polygon inscribed in a circle has the same fineness, since each edge of the polygon belongs to a triangle, and every possible triangle gives rise to the same angle.

DELAUNAY TRIANGULATION AND INTERPOLATION

The Delaunay triangulation is used to interpolate terrains, because it also minimizes the roughness of the terrain, in other words, the integral of the square of the L_2-norm of the terrain's gradient.

It is important to notice that this property is independent from the data, in other words, it is independent from the values of the z-coordinates of the input points.

SOME ADDRESSES TO PLAY WITH DELAUNAY TRIANGULATIONS

http://www.cs.cornell.edu/Info/People/chew/Delaunay.html

http://web.informatik.uni-bonn.de/I/GeomLab/VoroGlide/index.html.en

http://www.dma.fi.upm.es/docencia/segundociclo/geomcomp/voronoi.html

http://www.cs.unc.edu/∼snoeyink/terrain/Demo.html

AND TWO BOOKS WITH MUCH MORE INFORMATION

A. Okabe, B. Boots, K. Sugihara, S. N. Chiu
Spatial Tessellations
2nd ed., J. Wiley & Sons, 2000.

F. Aurenhammer, R. Klein, D.-T. Lee
Voronoi Diagrams and Delaunay Triangulations
World Scientific, 2013.