Analyzing algorithms

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Complexity

- Time
- Space

Other important issues: understandability, robustness, etc.

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Execution time, assuming a speed of 1 million instructions per second

Cost	$n = 10$	$n = 20$	$n = 100$	
$\log n$	0.004 ms	0.005 ms	0.007 ms	
n	0.01 ms	0.02 ms	0.1 ms	
$n \log n$	0.033 ms	0.09 ms	0.66 ms	
n^2	0.1 ms	0.4 ms	10 ms	
n^4	10 ms	160 ms	1 min 40 sec	
2^n	1 ms	1.05 sec	2.7×10^6 UA	
$n!$	3.6 sec	76 000 years	2×10^{134} UA	
n^n	2 h 48 min	220 UA	2×10^{176} UA	

UA = age of the universe (15 thousand millions years)

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Execution time, assuming a speed of 1 million instructions per second

Cost	$n = 10$	$n = 20$	$n = 100$	$n = 18000$
$\log n$	0.004 ms	0.005 ms	0.007 ms	0.018 ms
n	0.01 ms	0.02 ms	0.1 ms	18 ms
$n \log n$	0.033 ms	0.09 ms	0.66 ms	254 ms
n^2	0.1 ms	0.4 ms	10 ms	5 min 24 sec
n^4	10 ms	160 ms	1 min 40 sec	3 328 years
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Size of the problem that can be solved in 1 hour

Cost	Current size	100 times faster	1000 times faster
n	N	$100N$	$1000N$
n^2	N	$10N$	$31.6N$
n^3	N	$4.64N$	$10N$
2^n	N	$N + 6.64$	$N + 9.97$
3^n	N	$N + 4.19$	$N + 6.29$

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Model of computation

The Real RAM Model:

- Each memory unit can allocate one real number, without precision limit
- Access to one memory position has unit cost
- Unit cost operations are:
  - Comparisons ($<, \leq, = \neq, >, \geq$)
  - Arithmetic operations ($+, -, *, :$)

Analytic functions (such as $\sqrt[k]{\ }, \log, \exp, \cos, \sin, \ldots$) do not have unit cost. Neither do functions floor and ceiling.

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Asymptotic analysis studies the cost of an algorithm (i.e., the number of unit cost operations performed by the algorithm) in terms of the size $n \in \mathbb{N}$ of the input of the problem.

Notation

Given $f, g : \mathbb{N} \longrightarrow \mathbb{R}^+$ increasing functions,

$$g \in O(f) \Leftrightarrow \exists n_0 \in \mathbb{N}\ \exists c \in \mathbb{R}^+\ \forall n \geq n_0\ g(n) \leq cf(n)$$

$$g \in \Omega(f) \Leftrightarrow \exists n_0 \in \mathbb{N}\ \exists c \in \mathbb{R}^+\ \forall n \geq n_0\ g(n) \geq cf(n)$$

$$g \in \Theta(f) \Leftrightarrow g \in O(f) \cap \Omega(f)$$

$$g \in o(f) \Leftrightarrow \lim_{n \to +\infty} \frac{g(n)}{f(n)} = 0$$

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Complexity of an algorithm (in a given computation model)

The worst case running time of an algorithm is $O(f)$ if the number of unit cost operations that it performs for **any** input of size n is $O(f(n))$.

The worst case running time of an algorithm is $\Omega(f)$ if the number of unit cost operations that it performs is $\Omega(f(n))$ for **some** input of size n.

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Complexity of a problem (in a given computation model)

The (time) complexity of a problem is $O(f)$ if **there exists** an algorithm solving it in $O(f)$ running time.

The (time) complexity of a problem is $\Omega(f)$ **all** algorithms solving it run in $\Omega(f(n))$ time.

Lower bounds

Theorem (Ben-Or): Let X be a semi-algebraic subset of R^d (i.e., X is the set of points in dimensions d satisfying a set of algebraic equations and/or inequations). The membership decision problem associated with X has the following lower bound:

$$\Omega(\log(\max(cc(X), cc(\mathbb{R}^d \setminus X))) - d),$$

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Some known lower bounds. The following problems are $\Omega(n \log n)$ in the Real RAM computation model:

- Sorting n real (integer) numbers.
- Element uniqueness: deciding whether n given real (integer) numbers are all distinct.
- Max-gap: computing the maximum distance between two consecutive numbers from a set of n real (integer) numbers.
- Set disjointness: deciding whether two given sets of n real (integer) numbers are disjoint.
- Set equality: deciding whether two given sets of n real (integer) numbers are equal.

Lower bounds

Reduction

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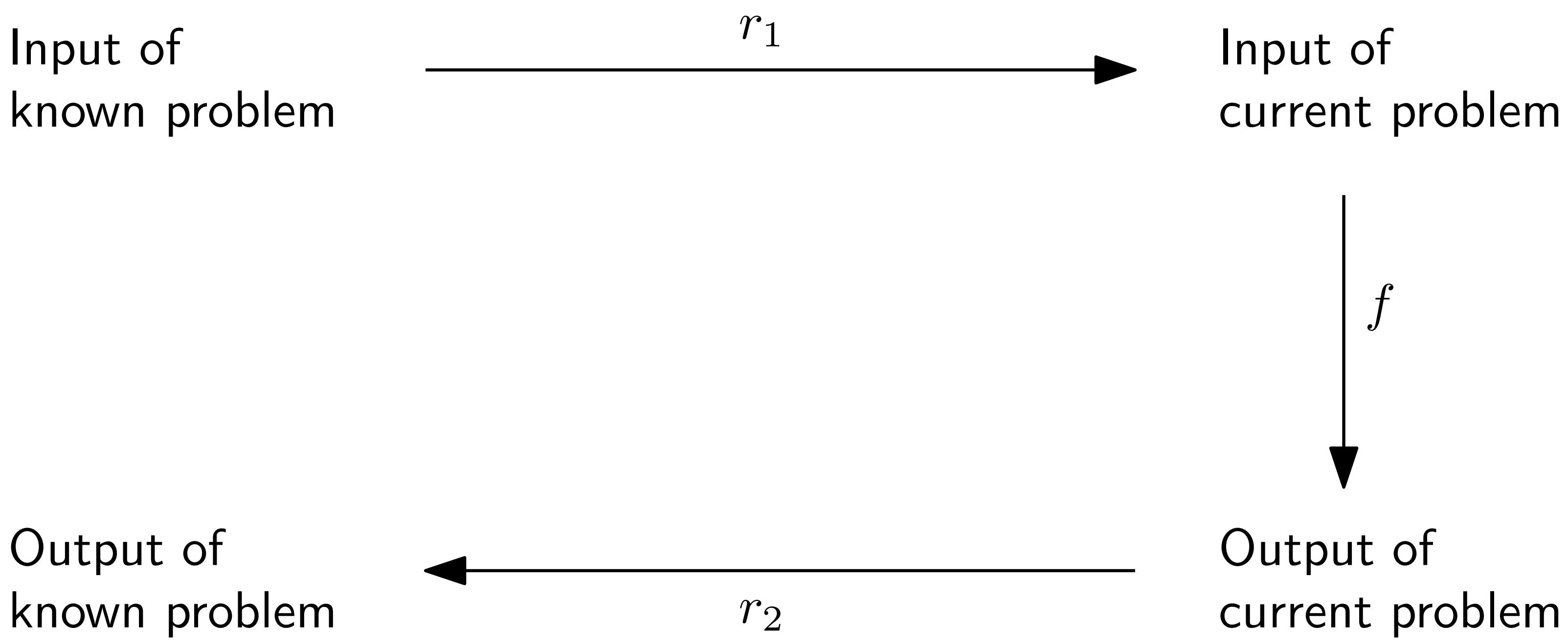

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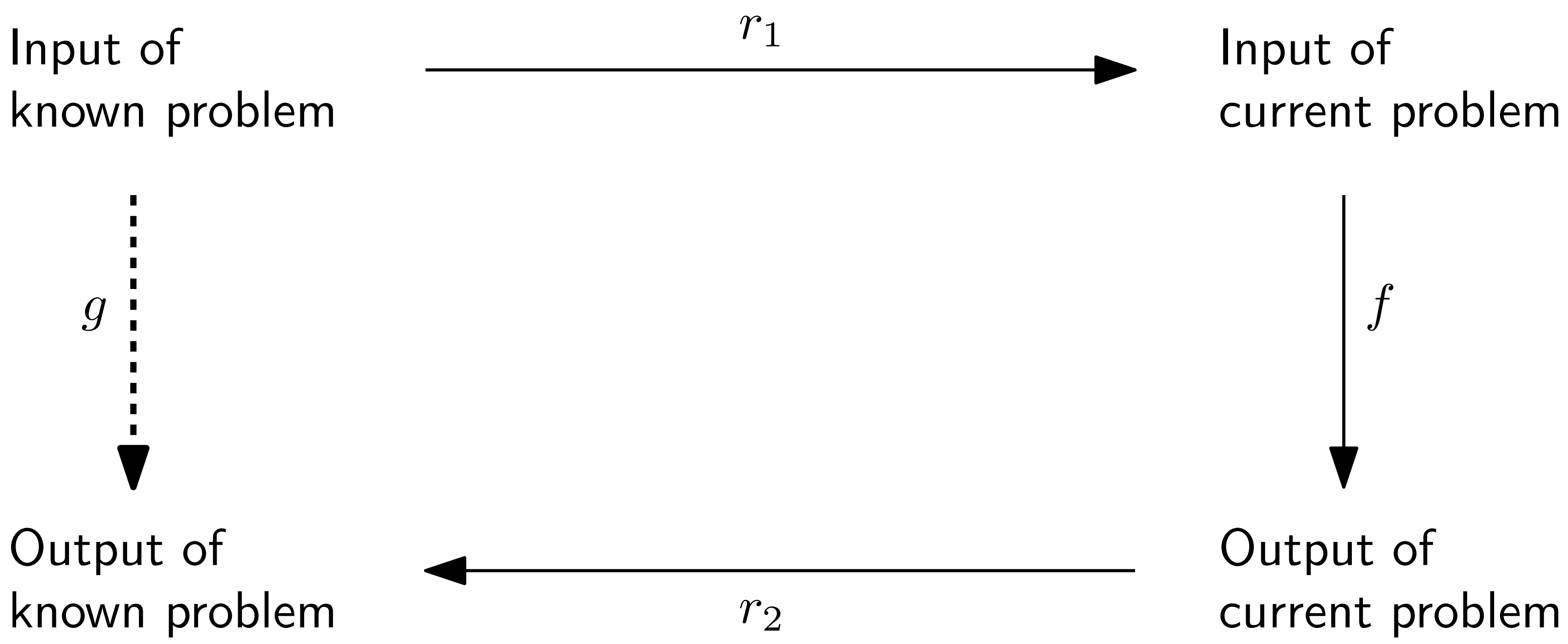

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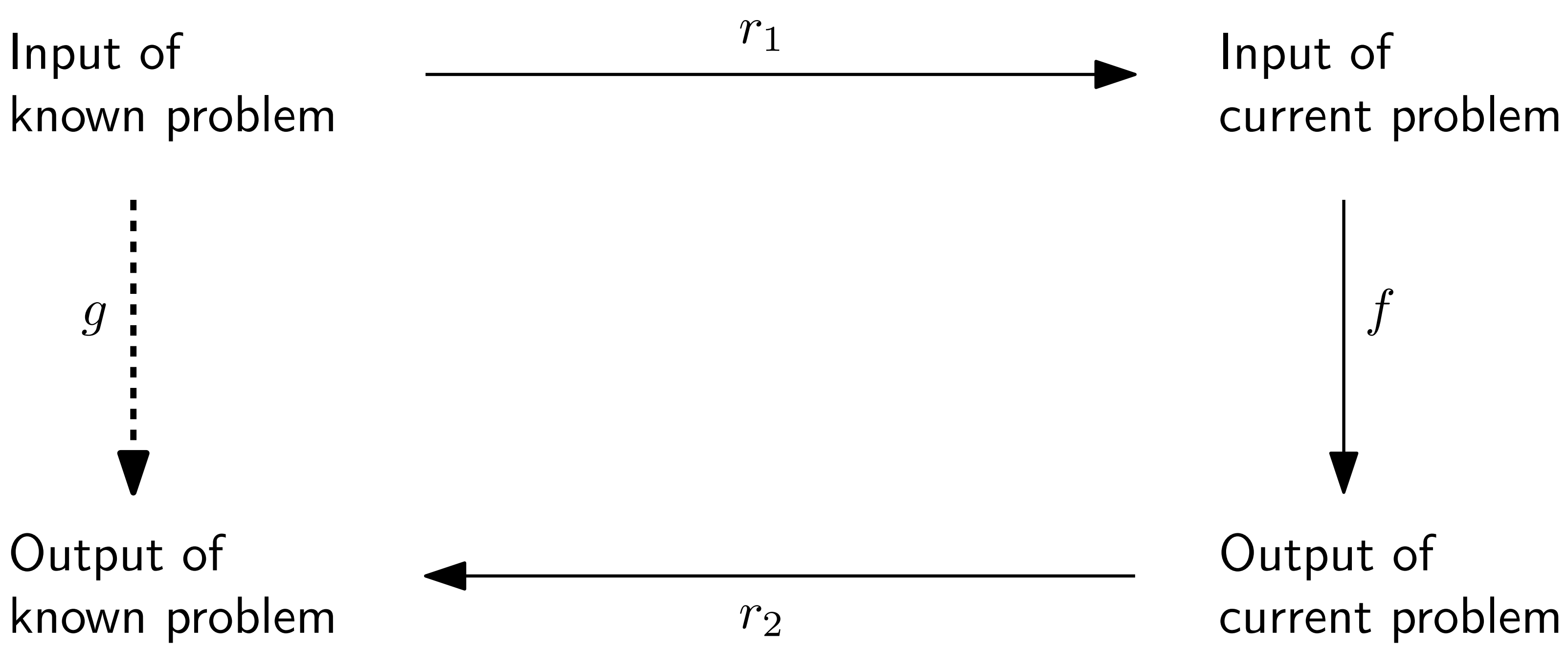

$$\left.\begin{array}{r} g \in \Omega(h(n)) \\ r_1, r_2 \in o(h(n)) \end{array}\right\} \Longrightarrow f \in \Omega(h(n))$$

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Input: $p_1, \ldots, p_n \in \mathbb{R}^2$

Output: $v_1, \ldots, v_k$ the vertices of $ch(\{p_1, \ldots, p_n\})$ in counterclockwise order

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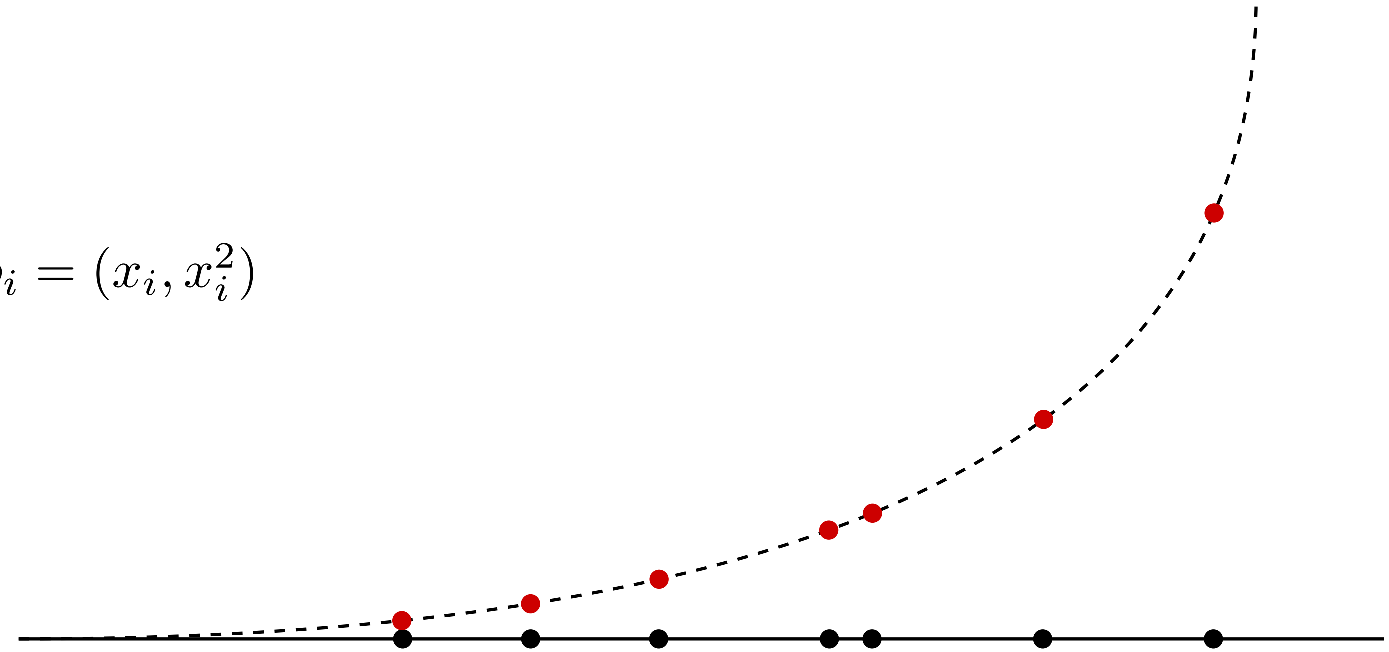

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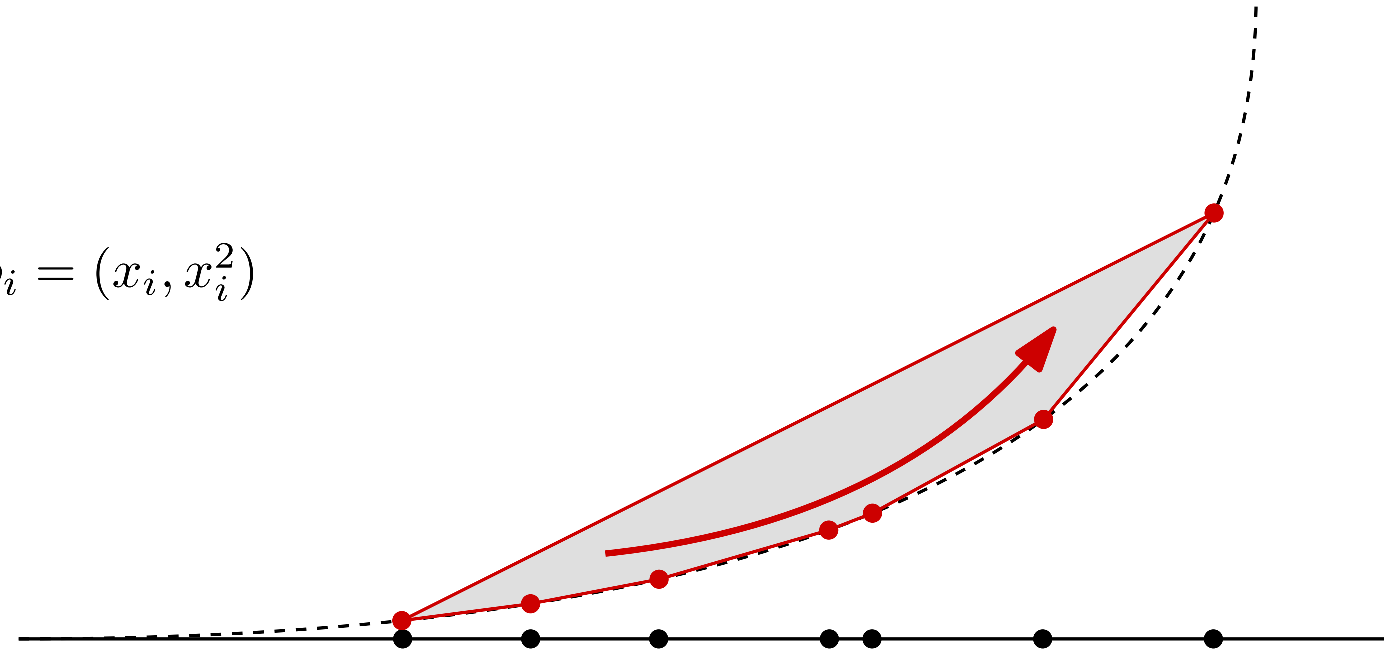

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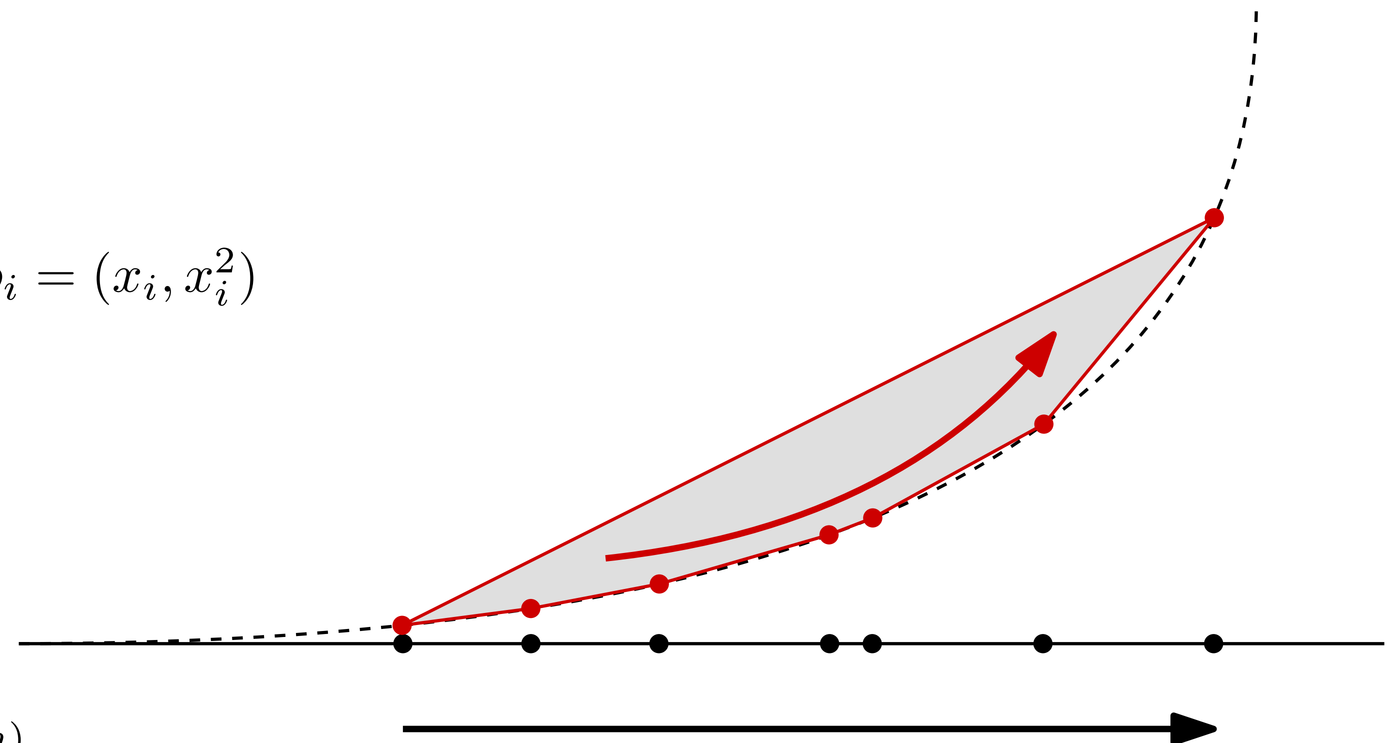

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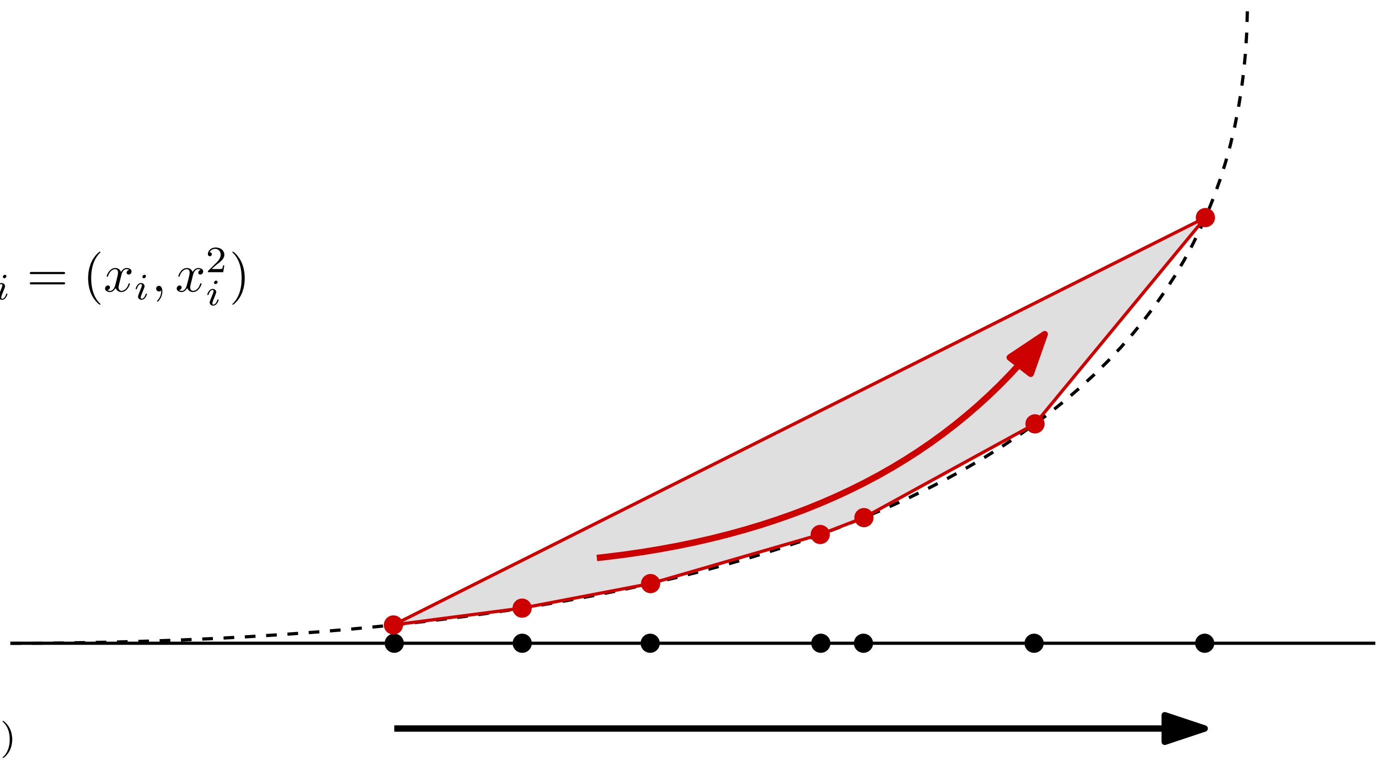

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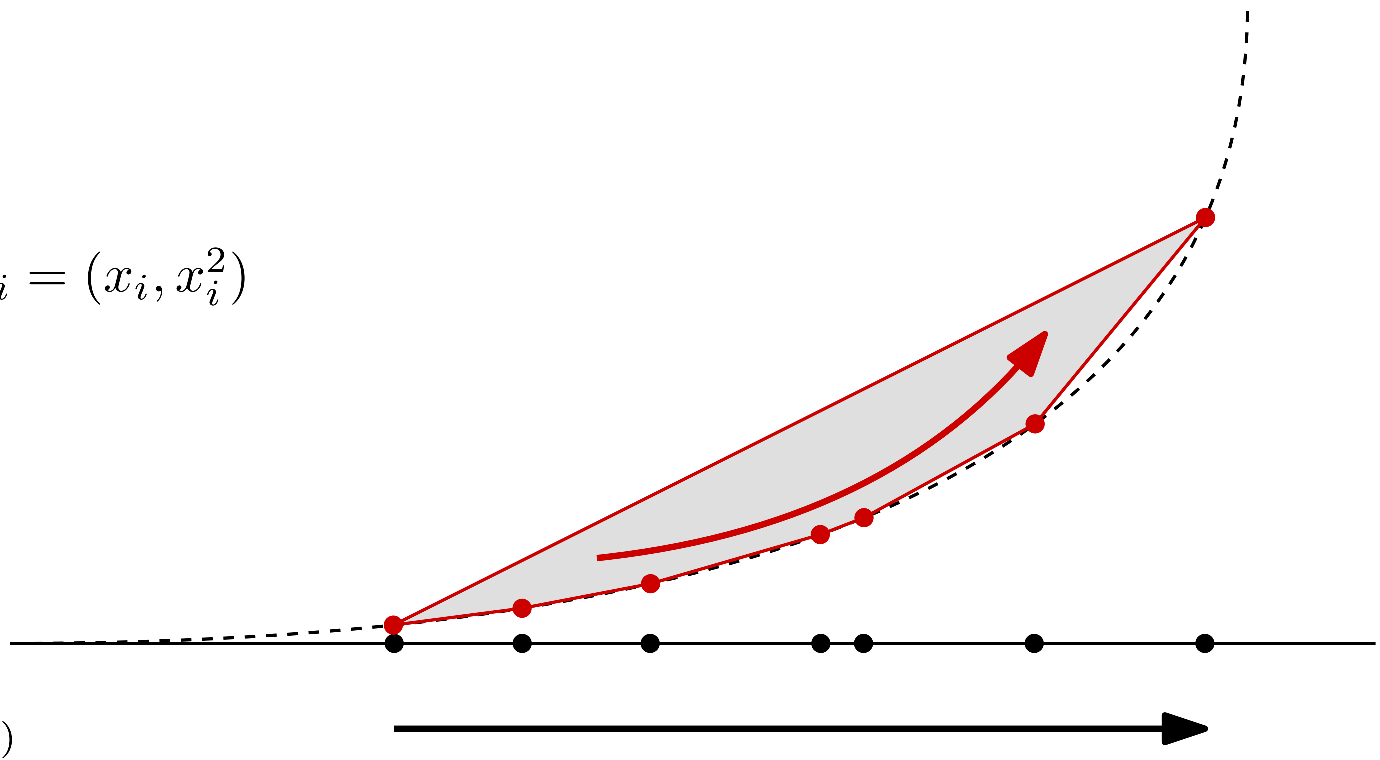

FURTHER READING

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