COMPUTING VORONOI DIAGRAMS

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Naive algorithm

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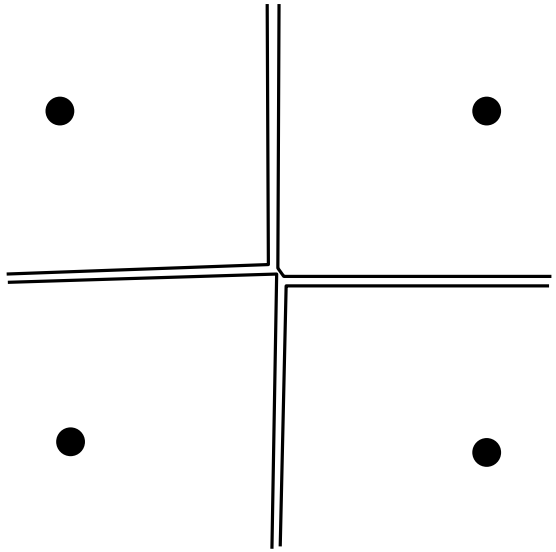

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The fact that each Voronoi region, $Vor(p_i)$, is built in optimal $\Theta(n \log n)$ time does not implie that the construction of the entire diagram, $Vor(P)$, requires $\Omega(n^2 \log n)$ time, as we will see.

incremental algorithm

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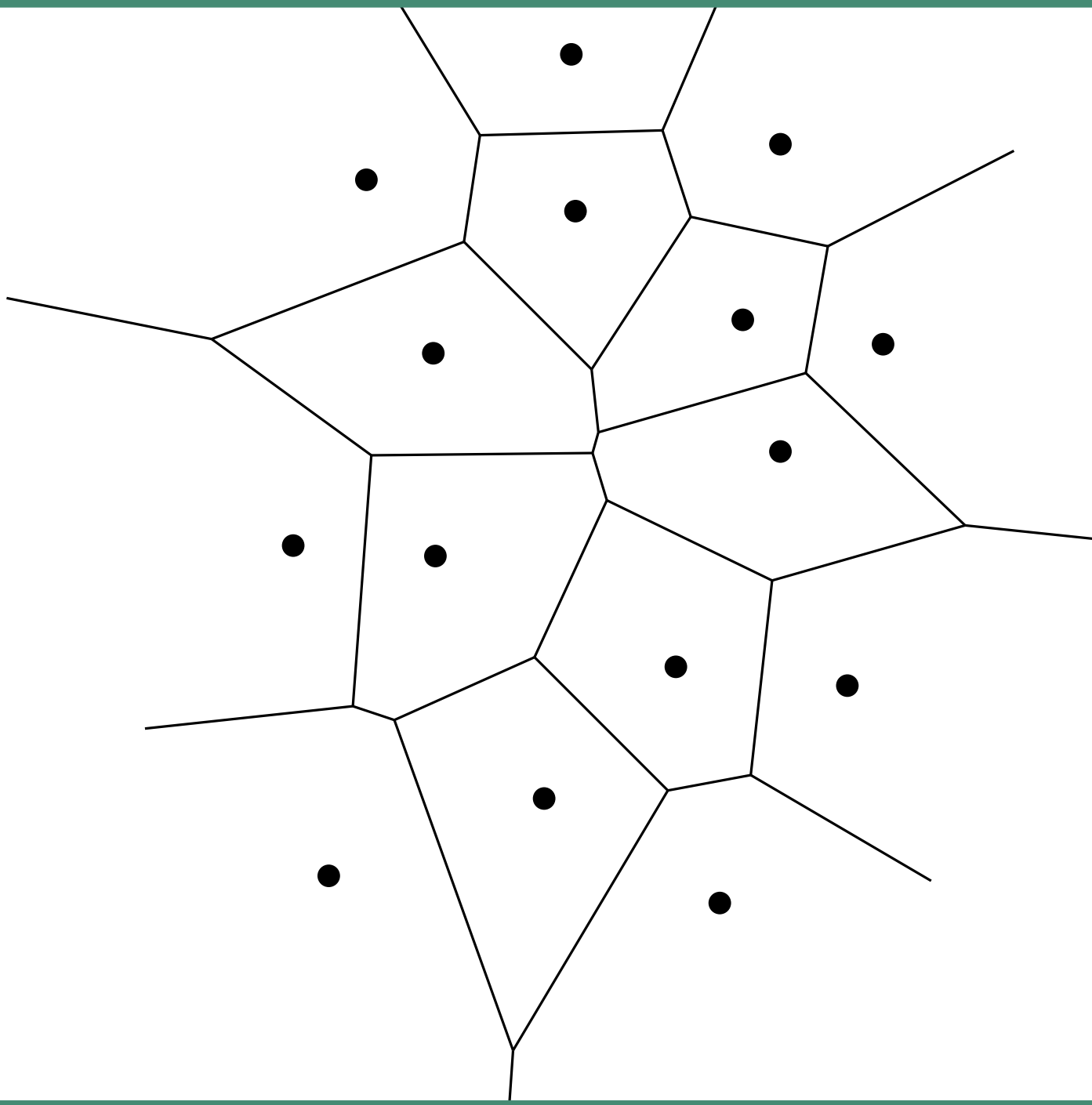

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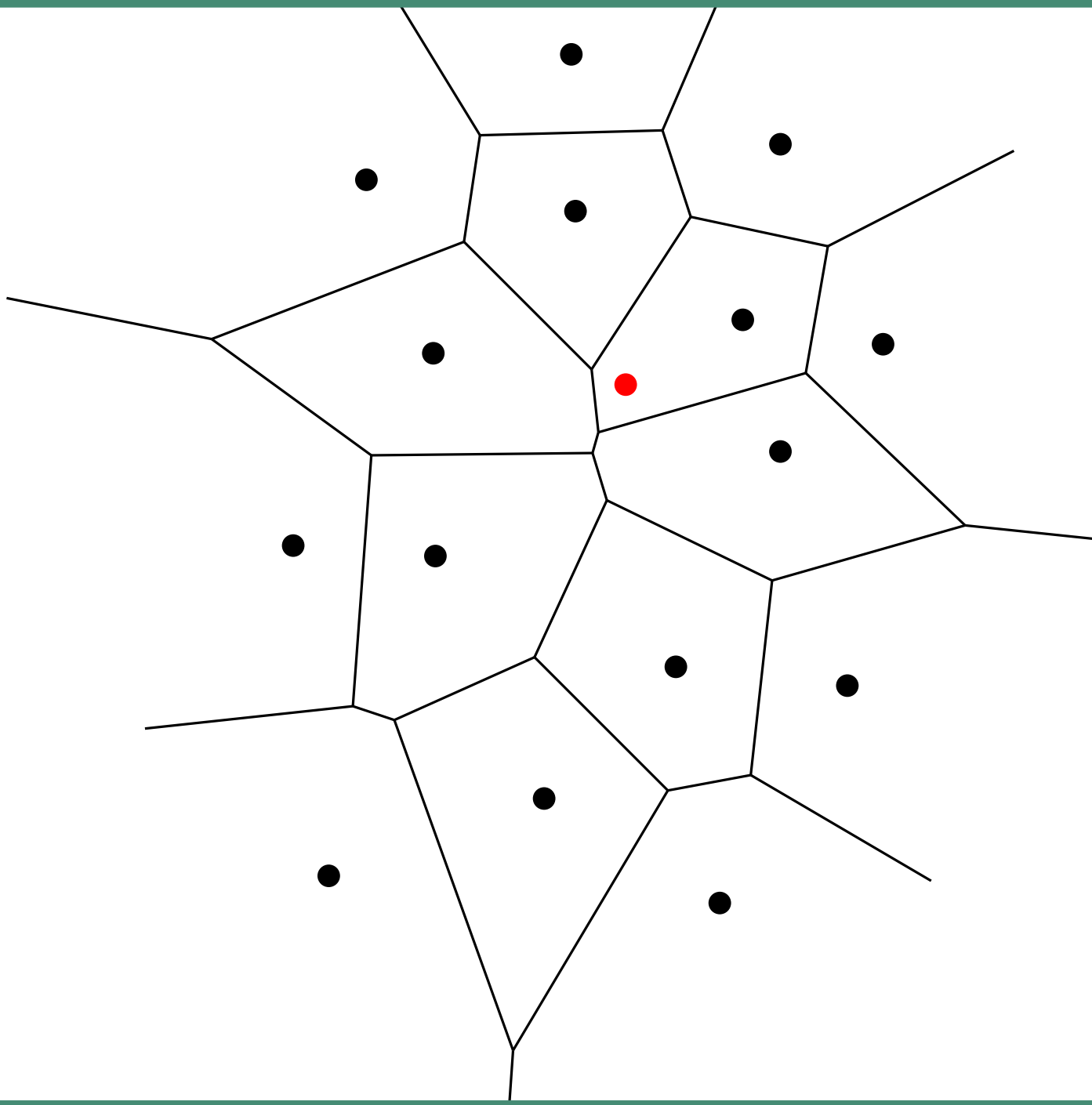

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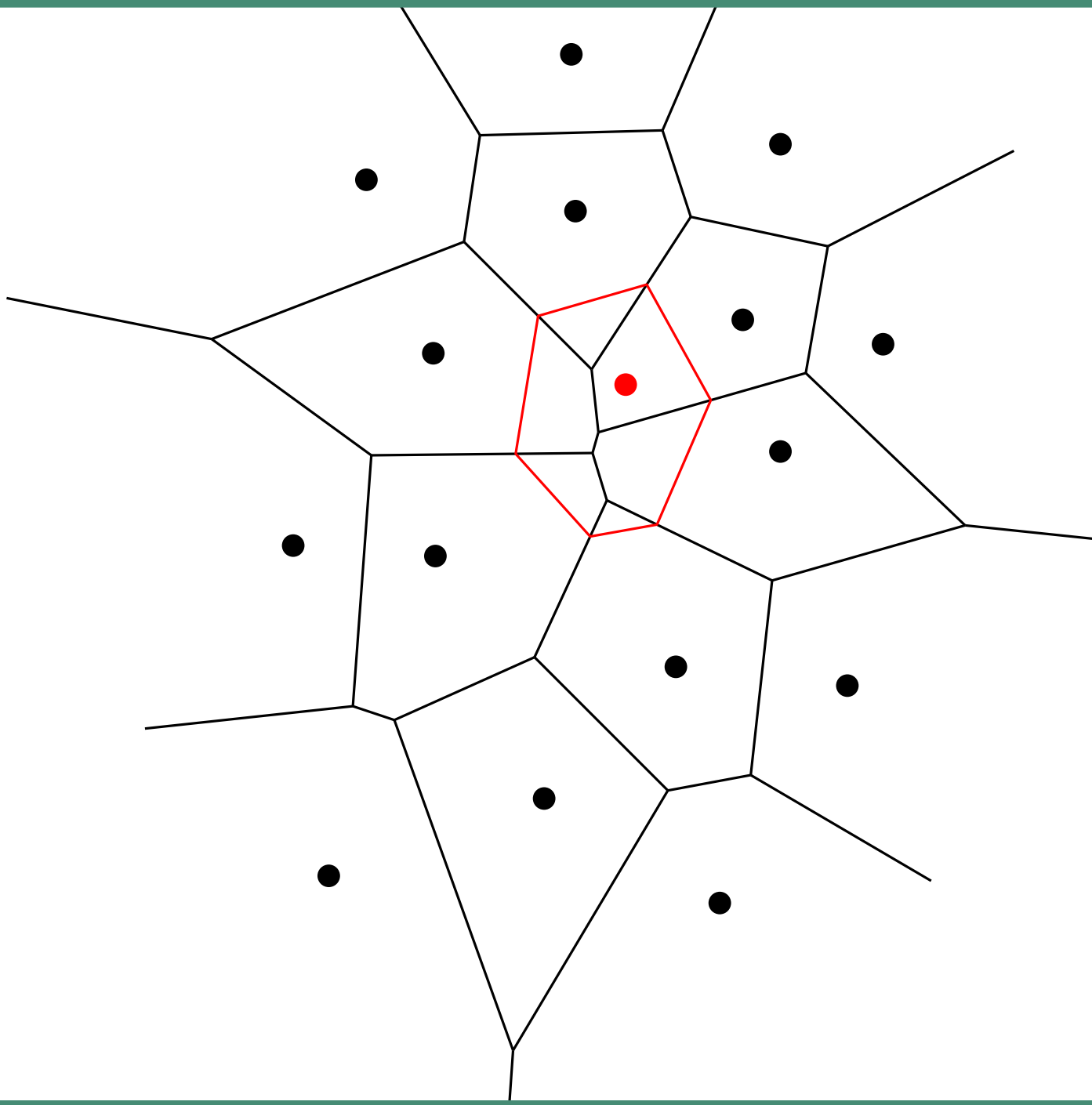

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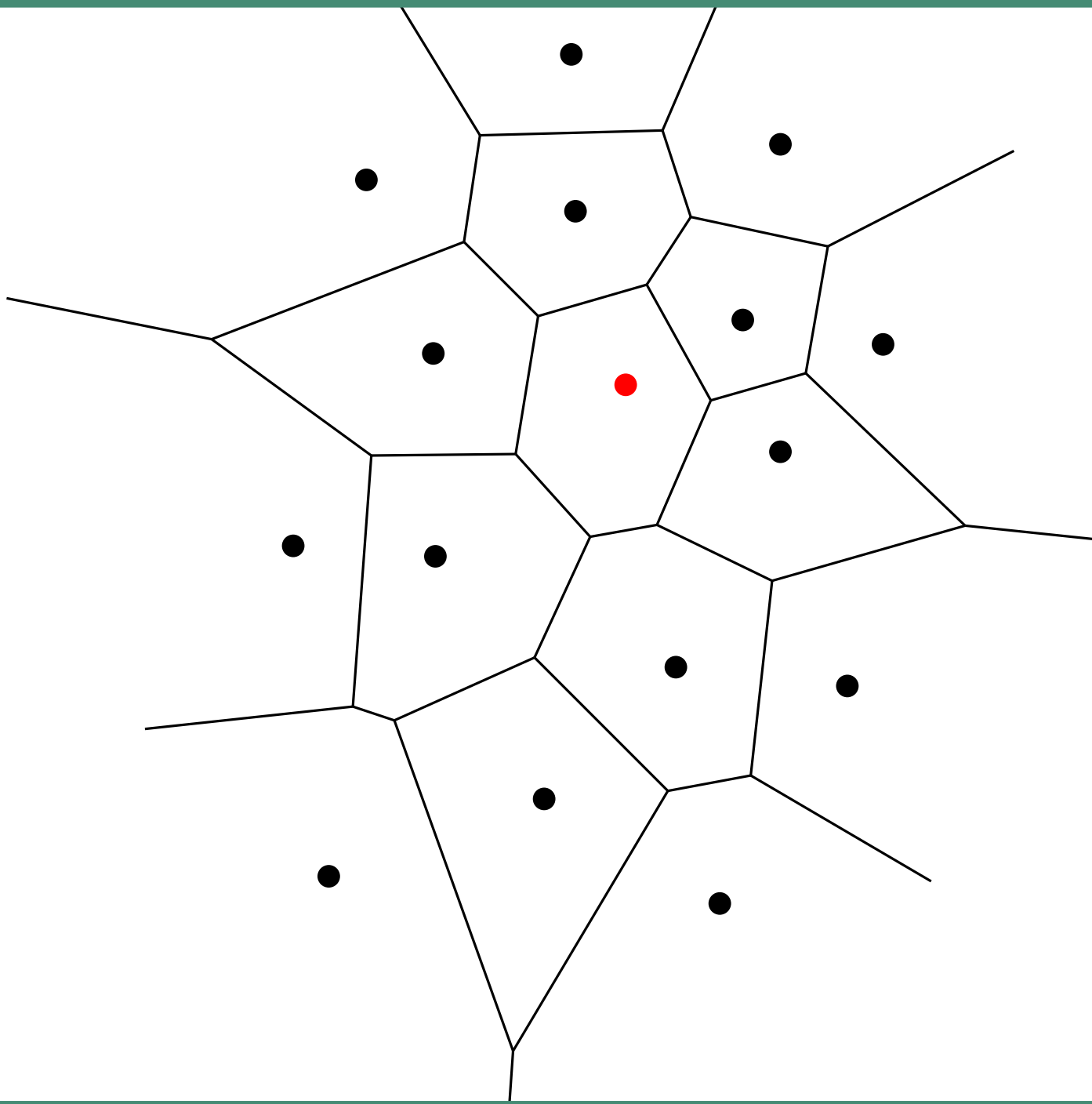

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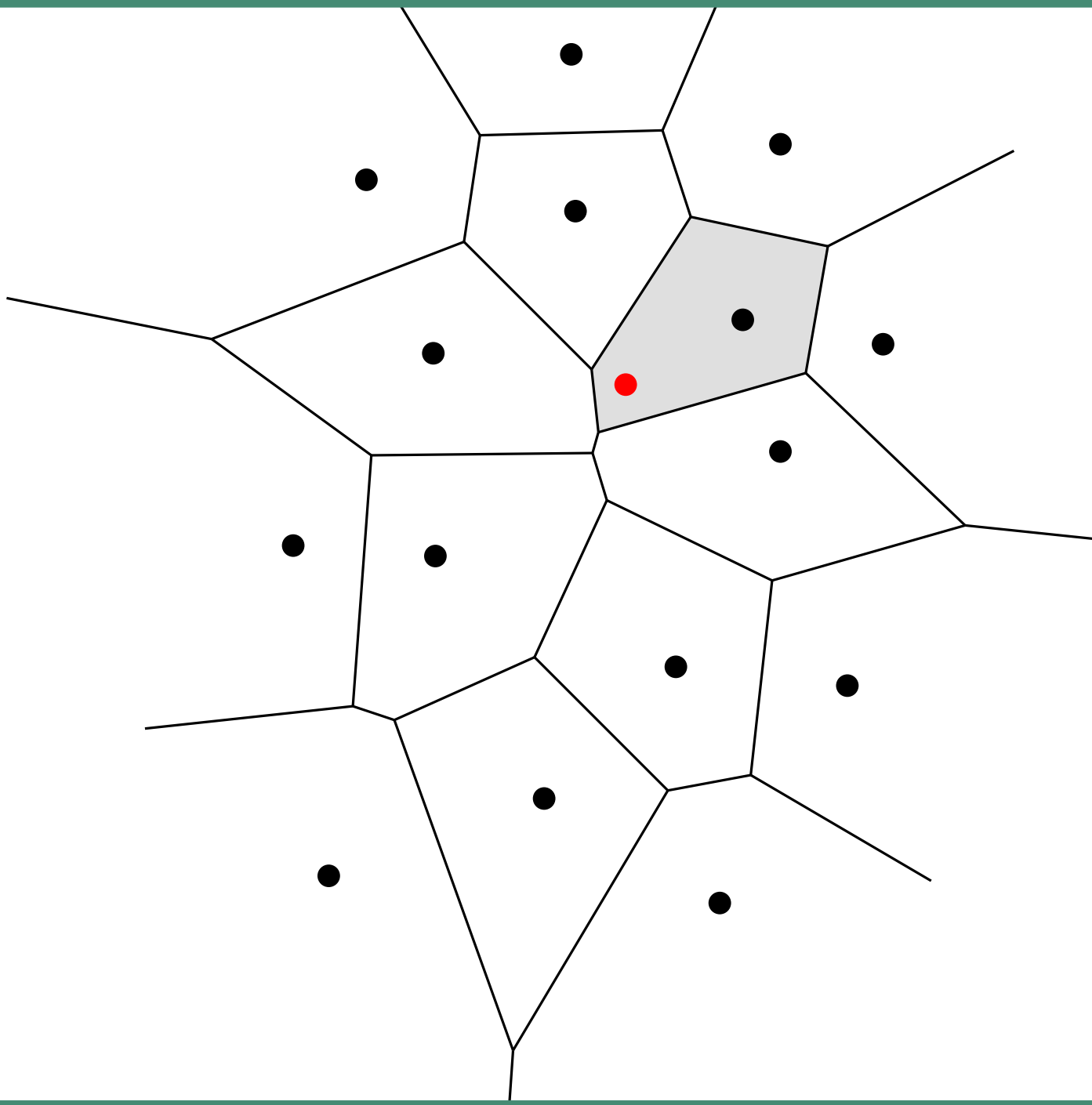

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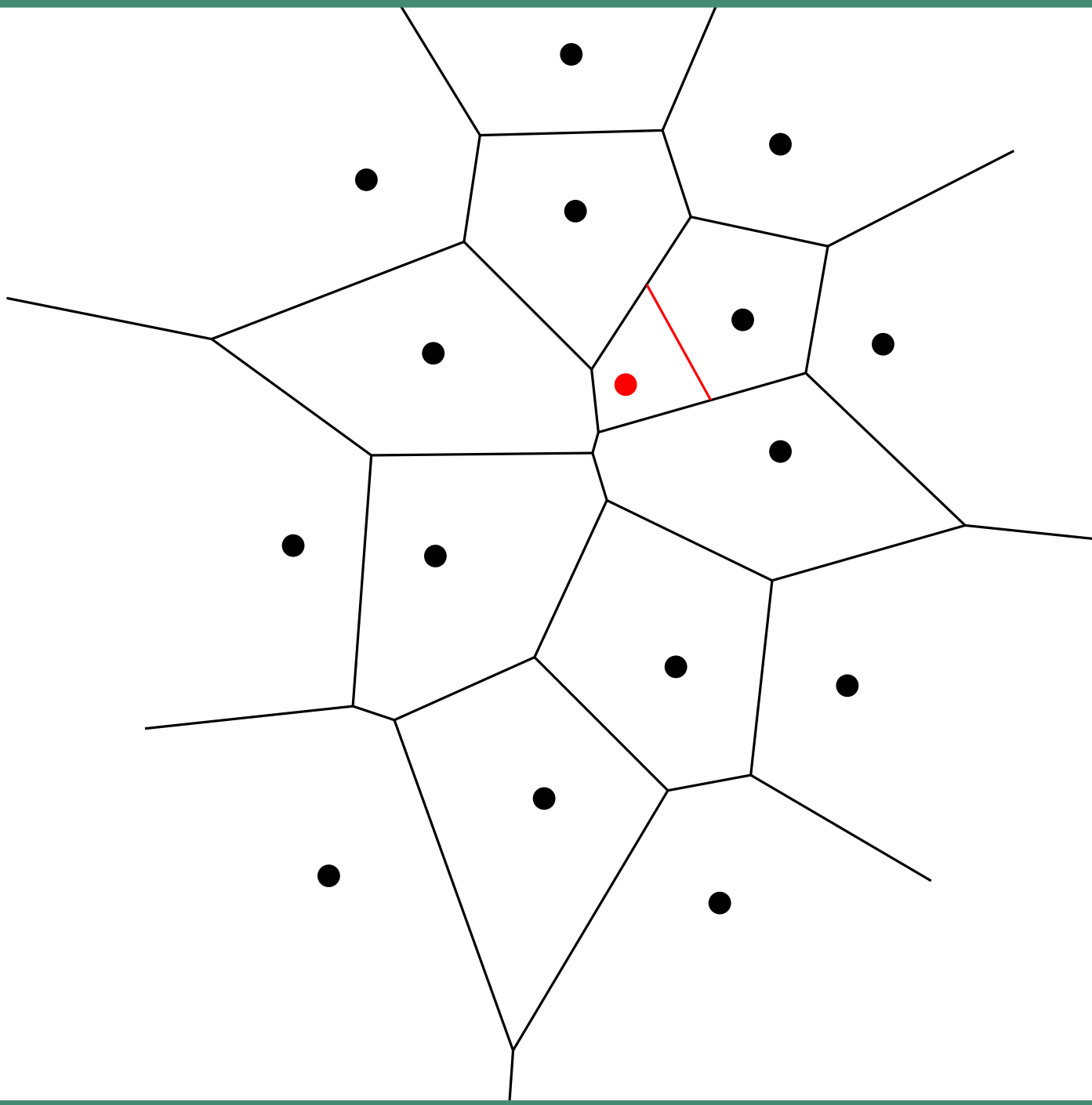

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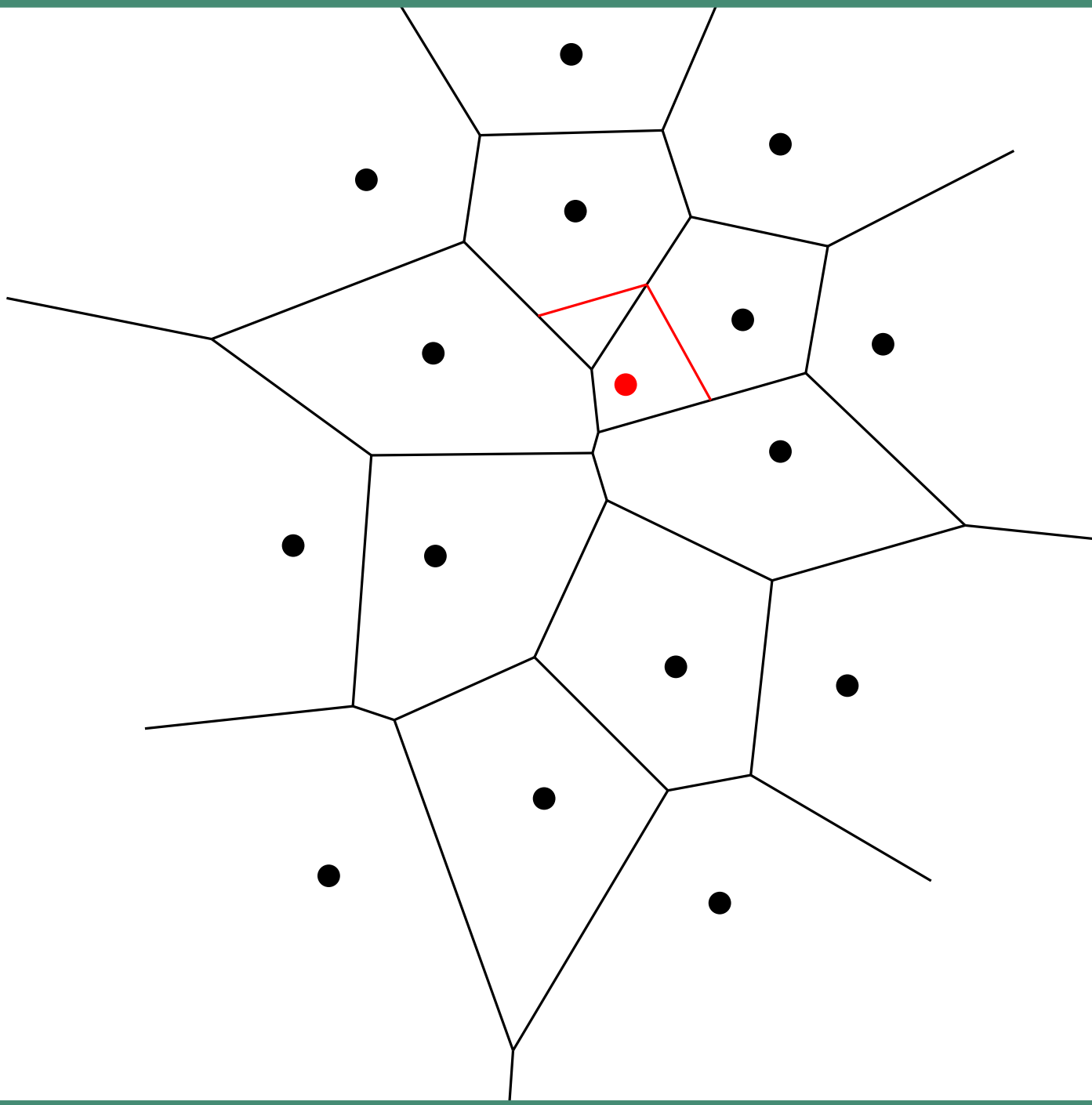

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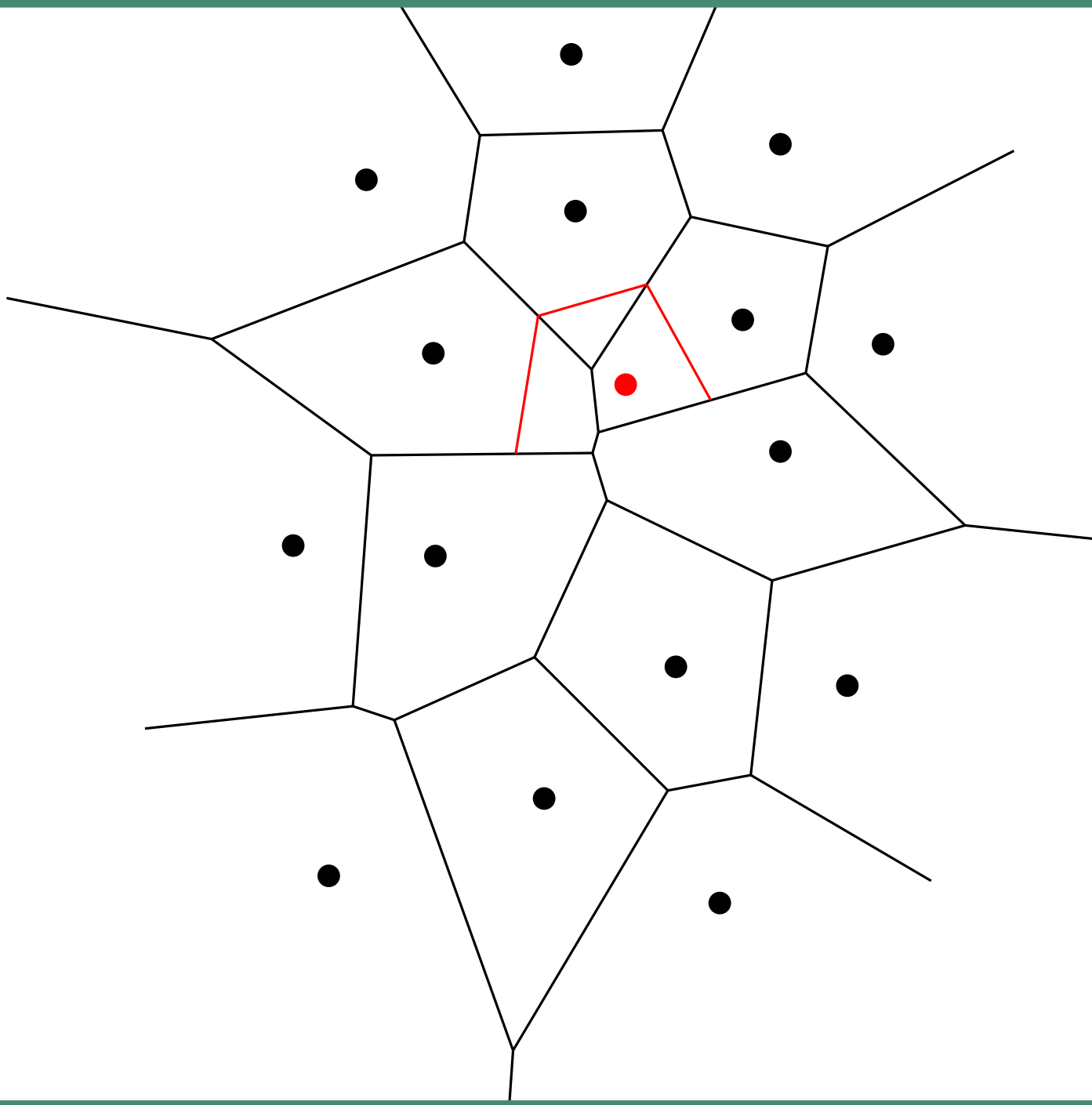

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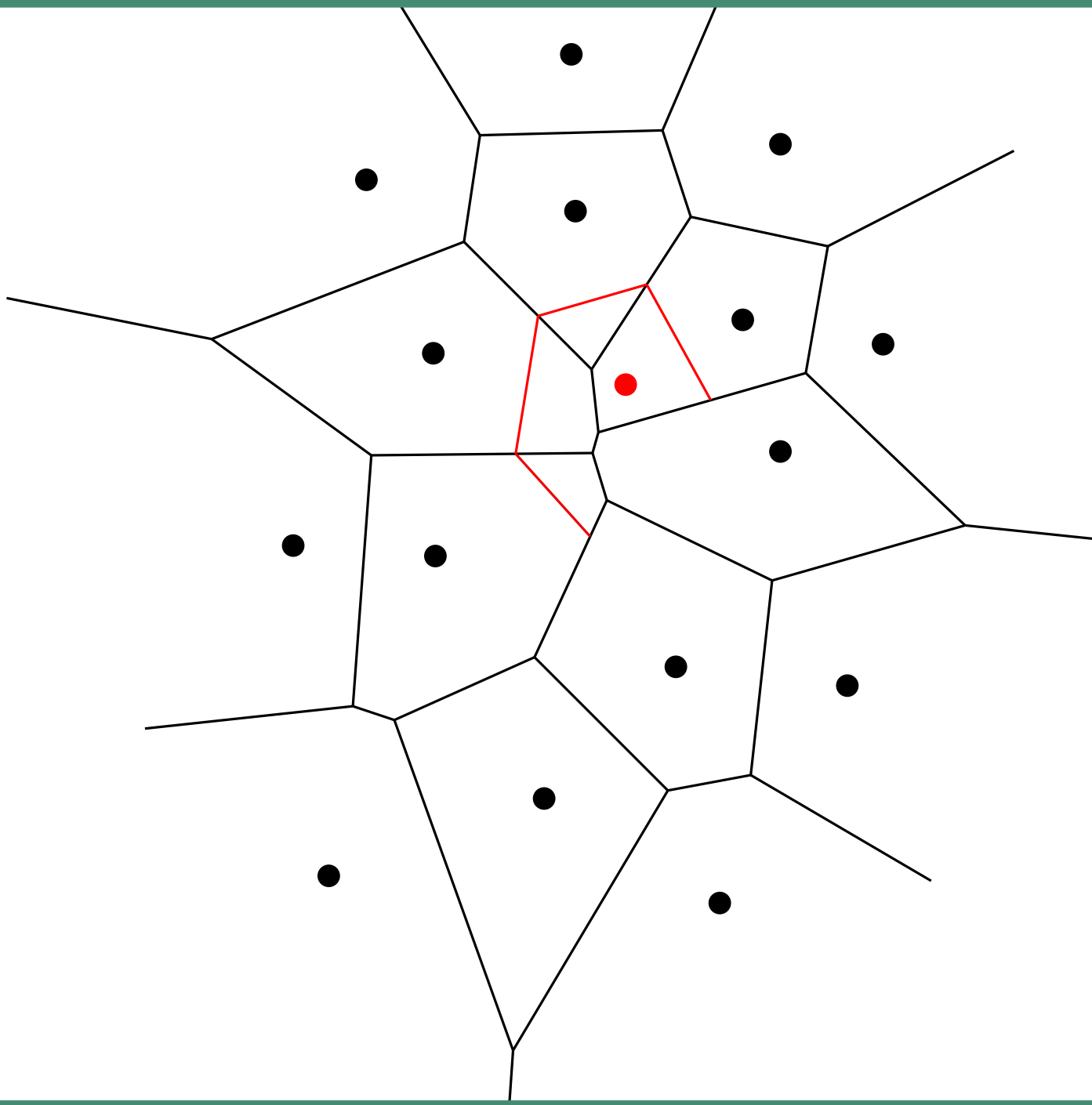

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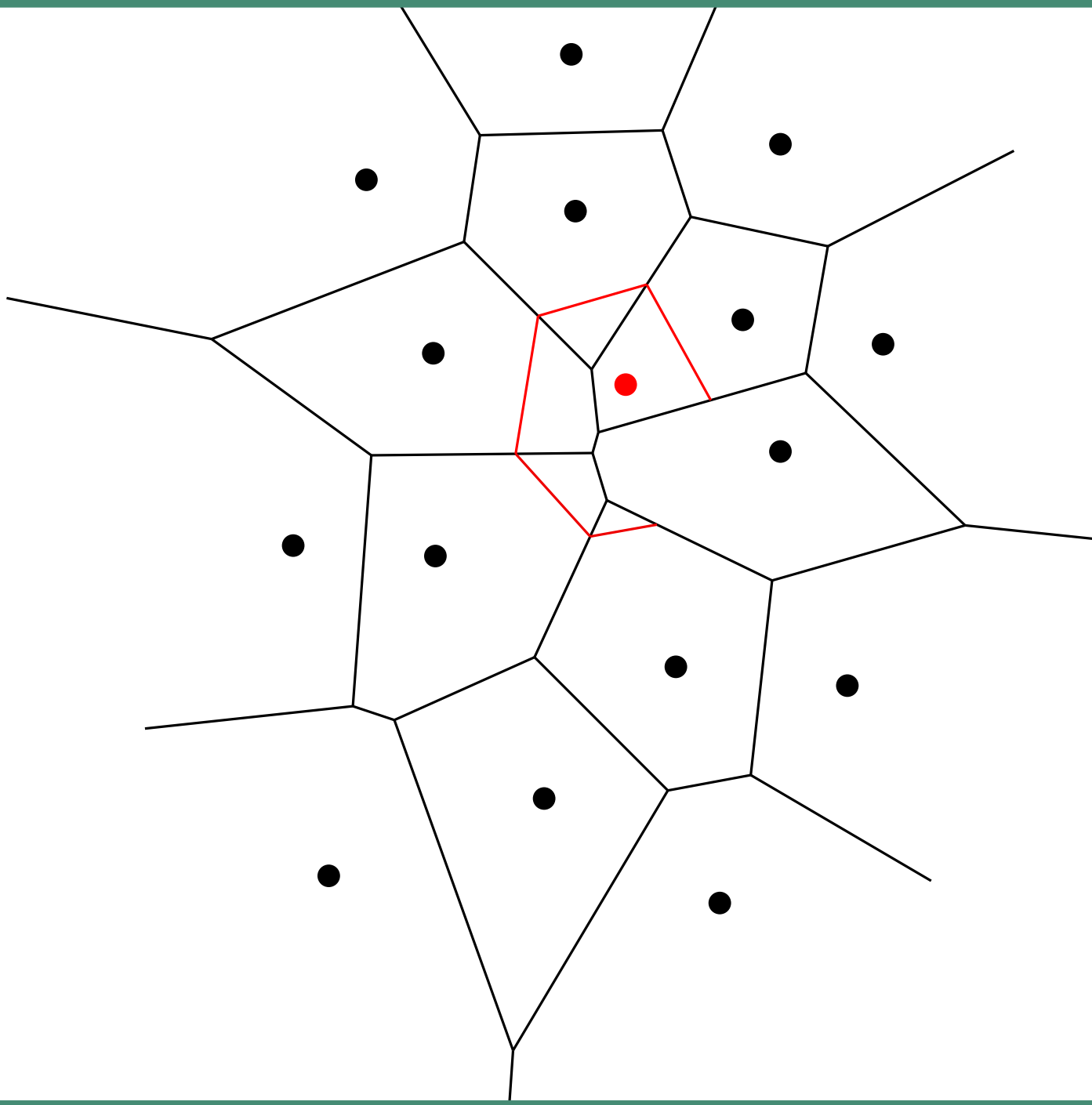

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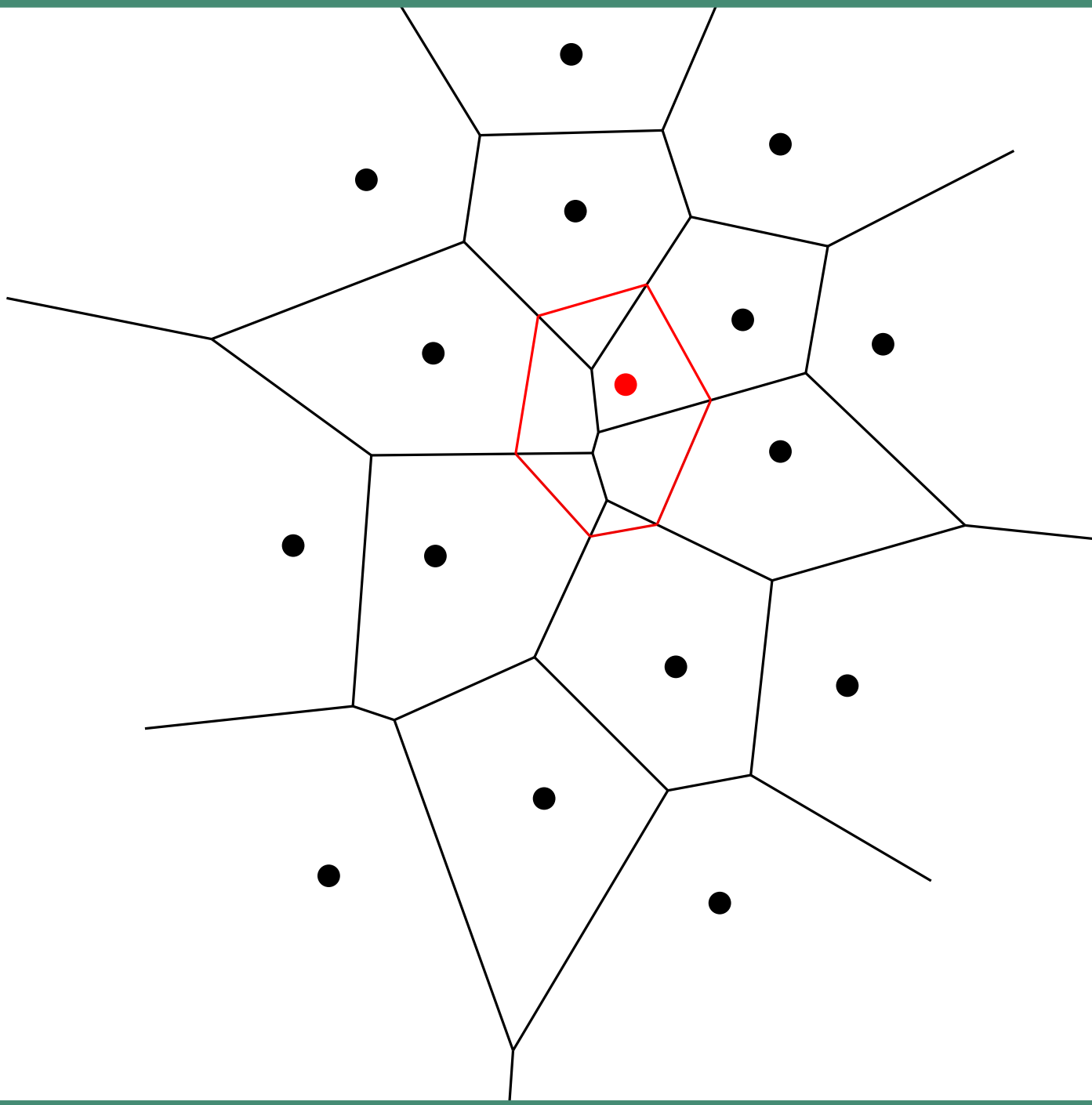

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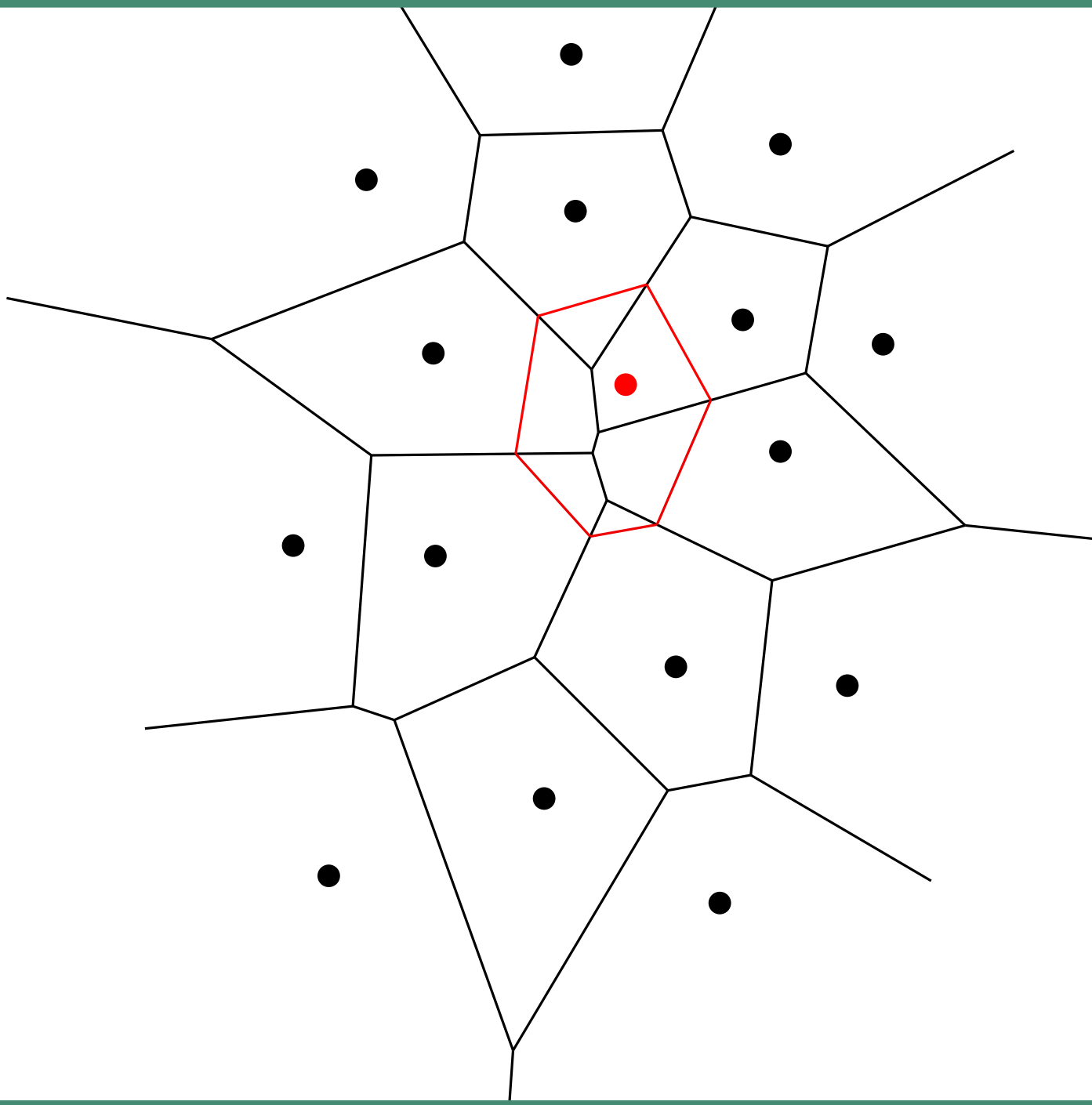

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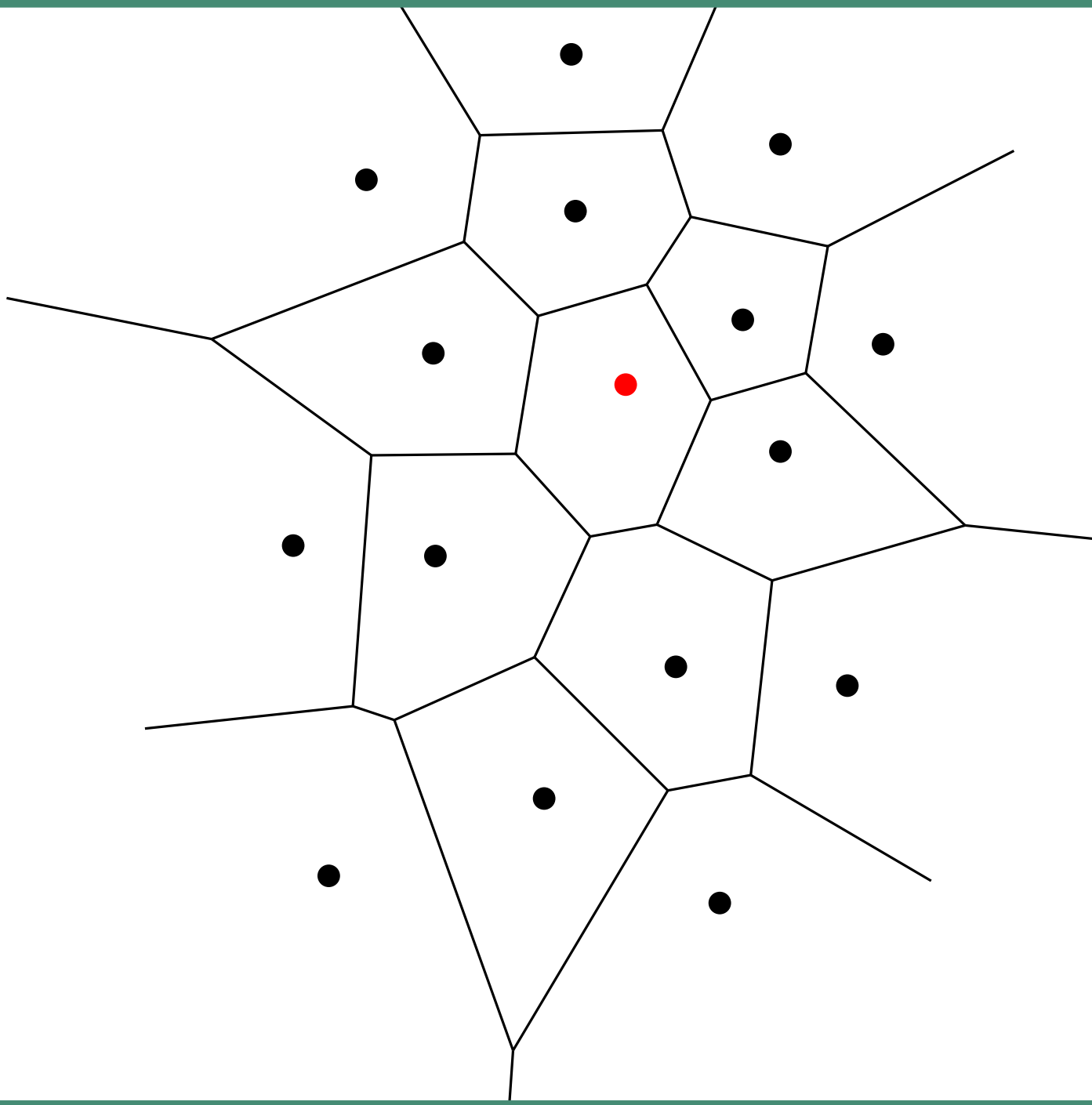

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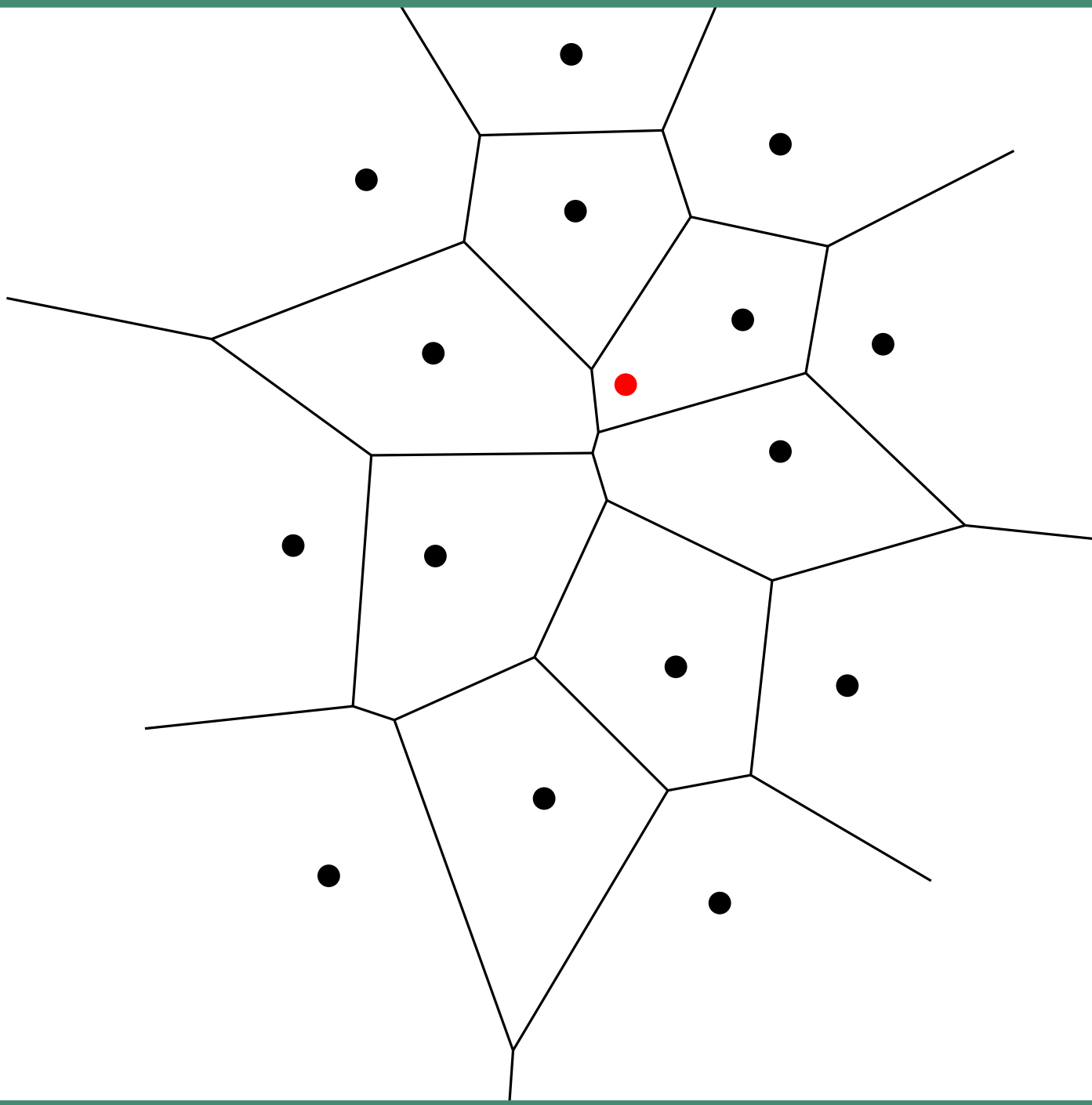

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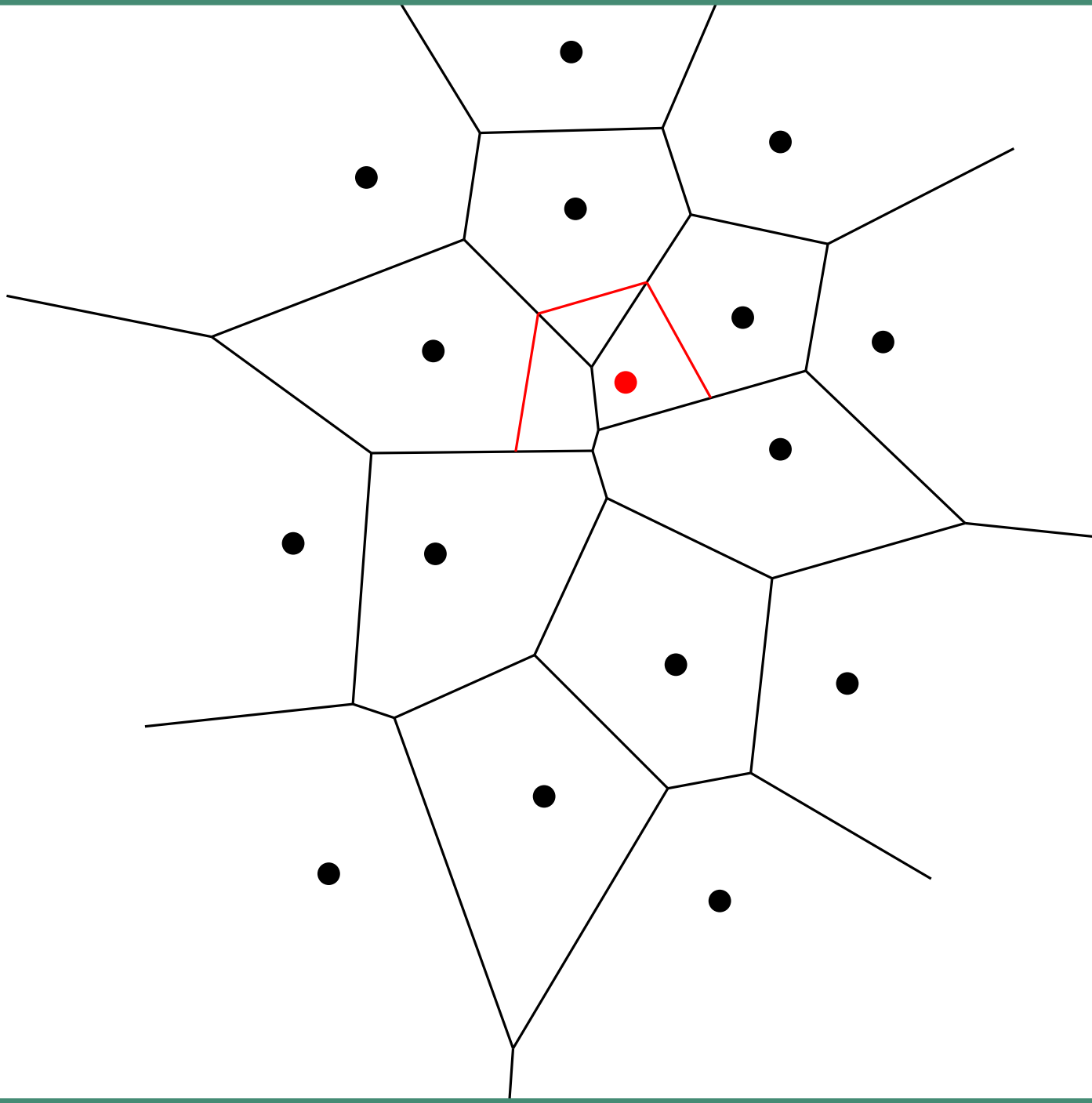

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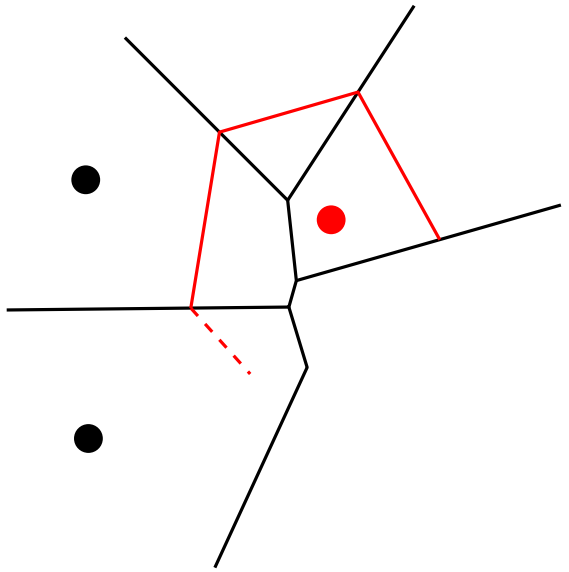

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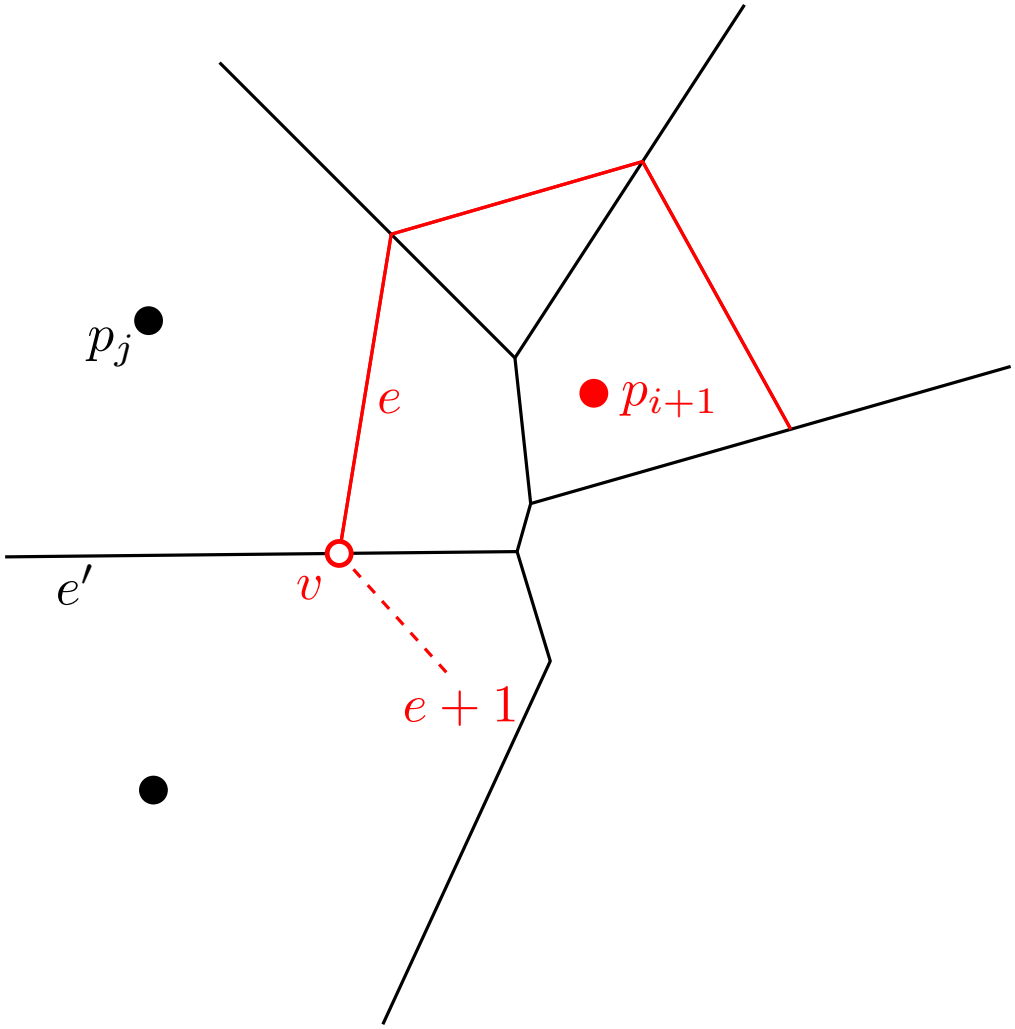
Each time an edge e, generated by p_{i+1} and p_j, intersects a preexistent edge, e', a new vertex v is created and a new edge starts, $e+1$. Then, these are the tasks to perform:

- Create v with $e(v) = e$
- Create $e+1$ and assign $v_B(e+1) = v$, $e_P(e+1) = e$
- Assign $v_E(e) = v$, $e_N(e) = e'$, $f_L(e) = i+1$, $f_R(e) = j$
- Update $v_*(e') = v$ and $e_*(e') = e+1$
- Update $e(p_j) = e$
- Mark for deletion all edges of the region of p_j between $v_B(e)$ and $v_E(e)$ in clockwise order



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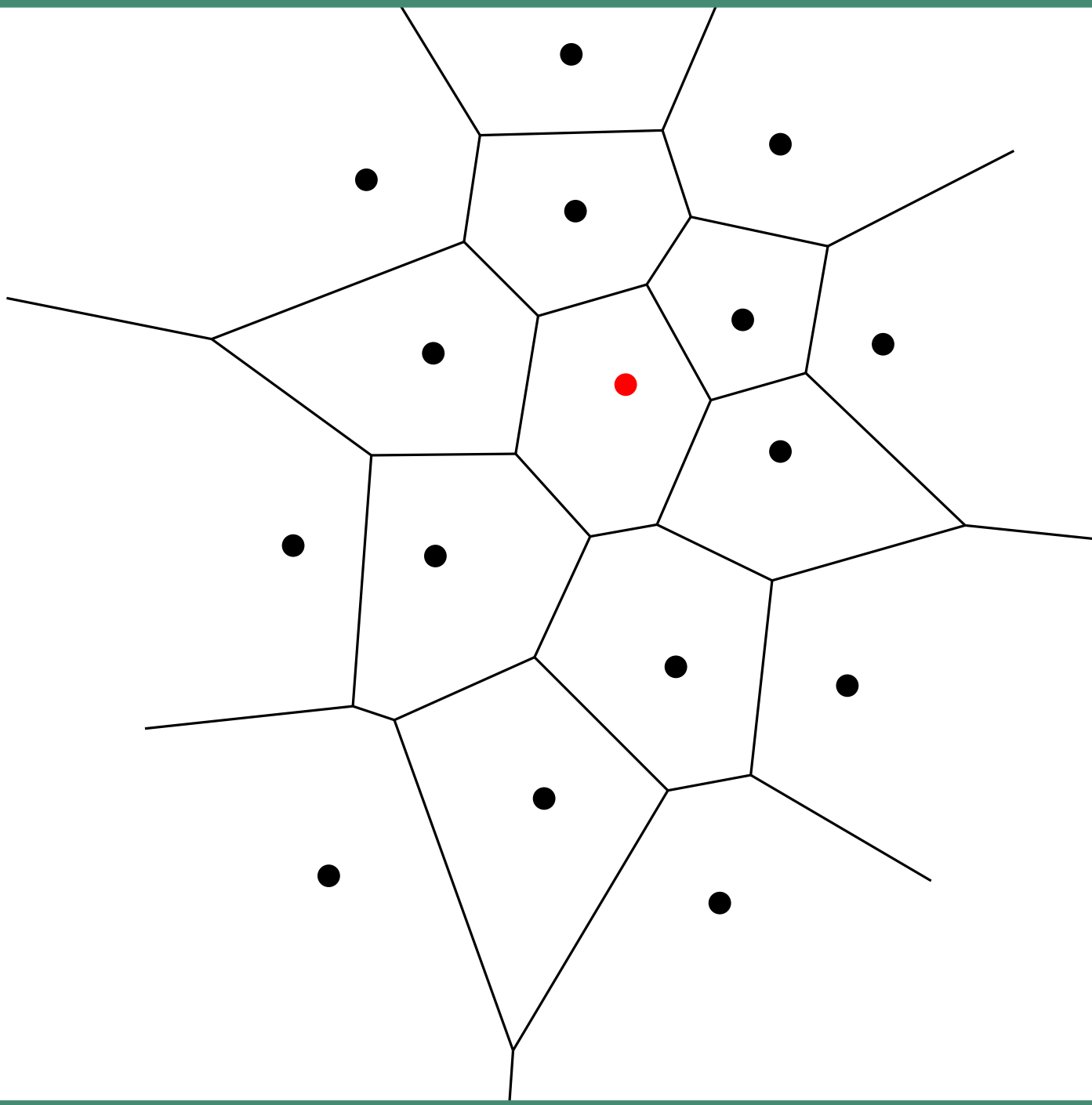

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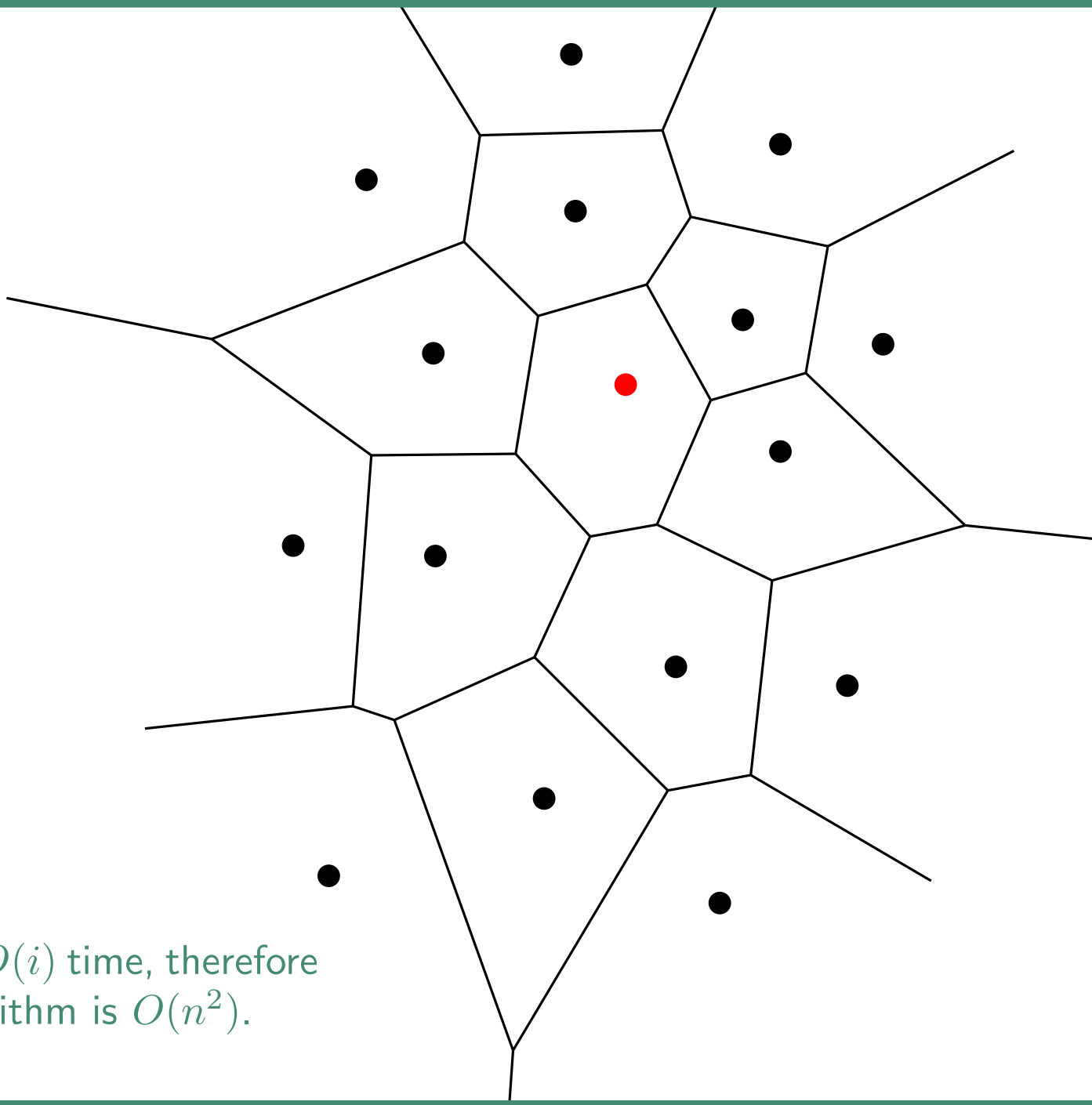

Running time: Each step runs in $O(i)$ time, therefore the total running time of the algorithm is $O(n^2)$.

divide and conquer algorithm

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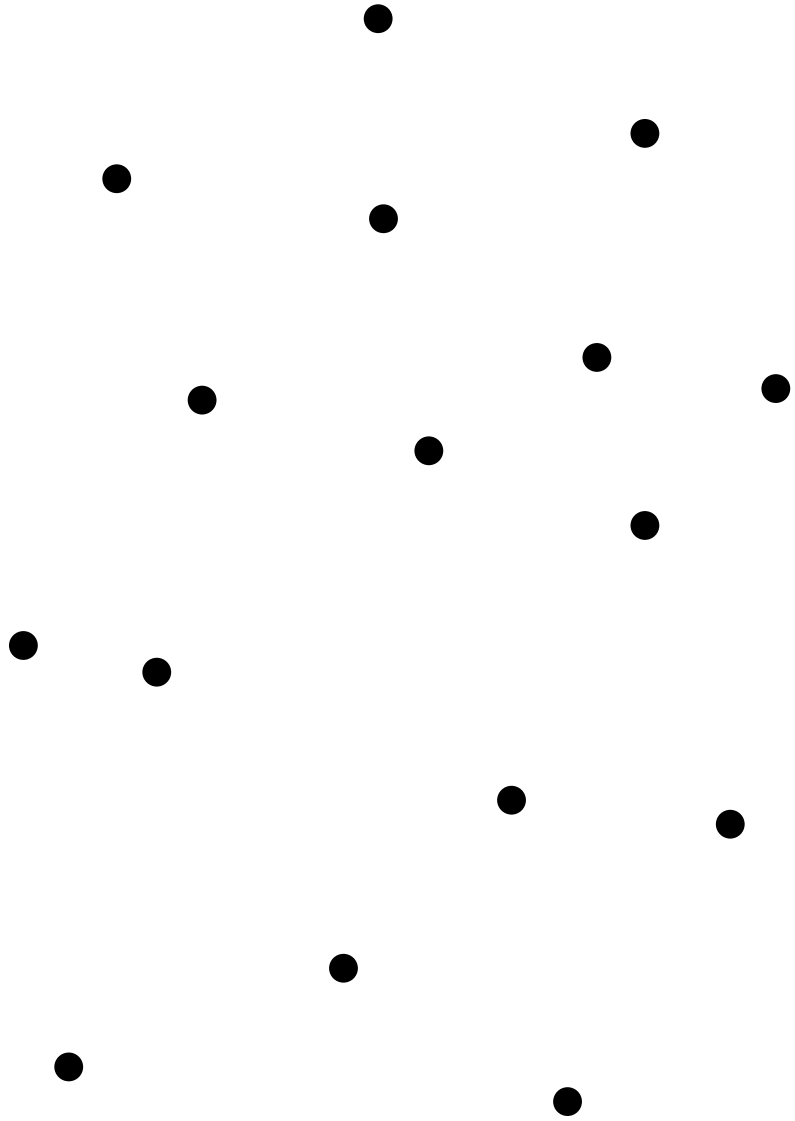

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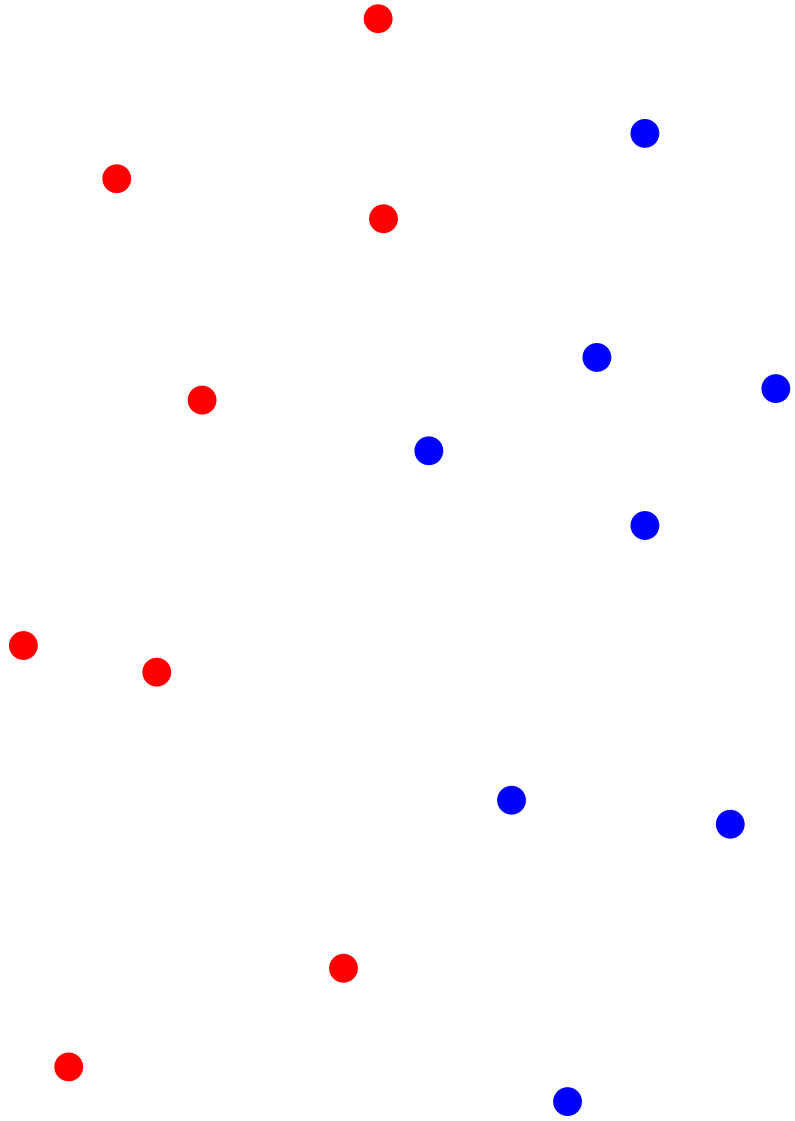

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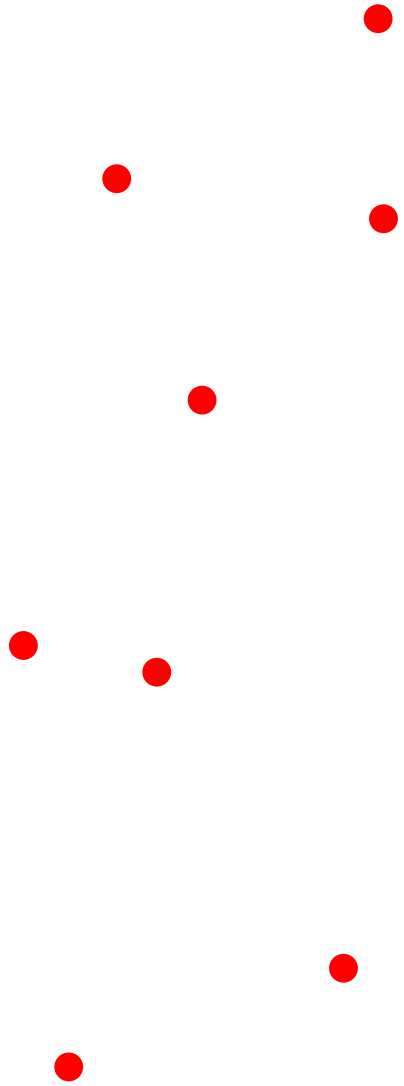

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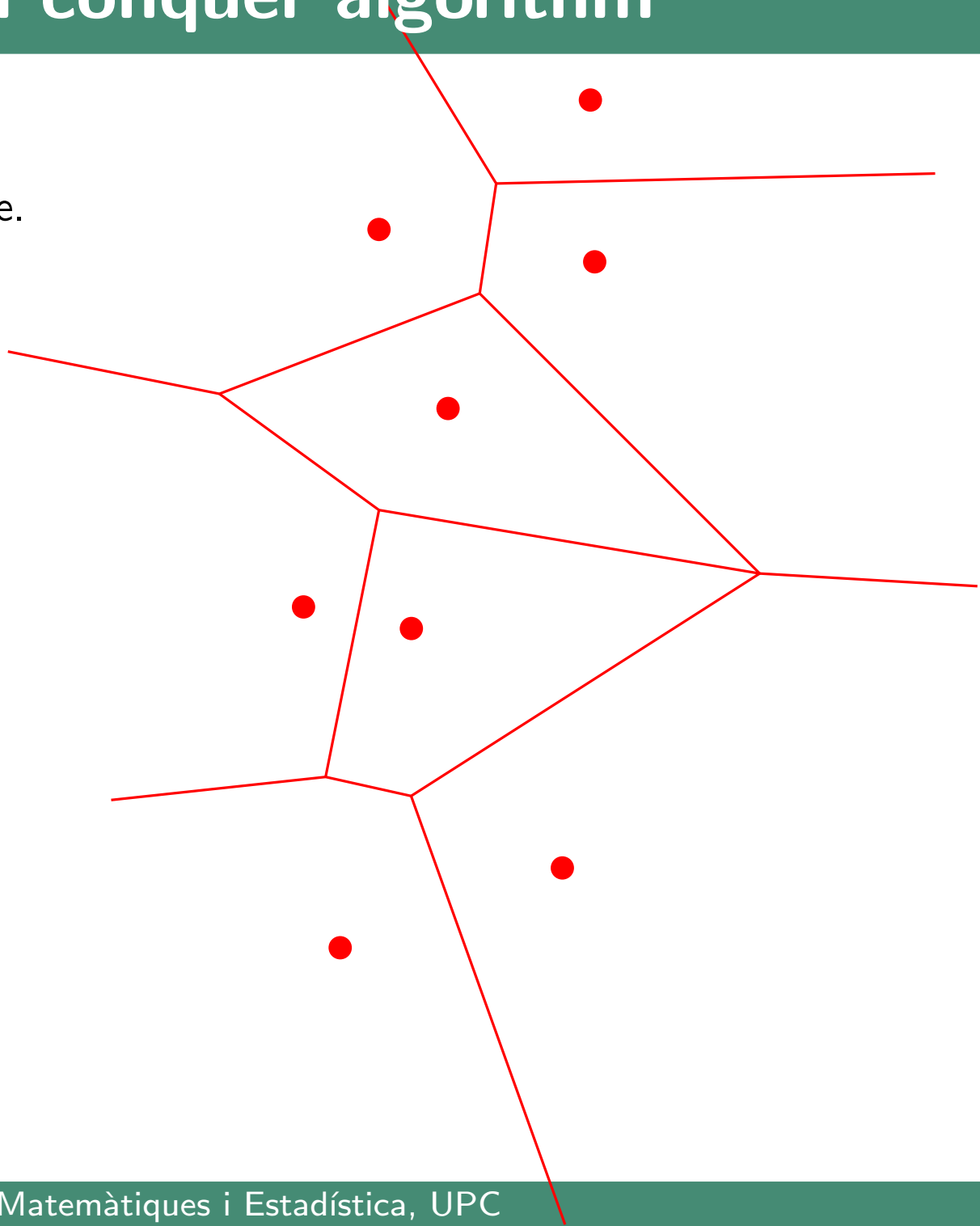

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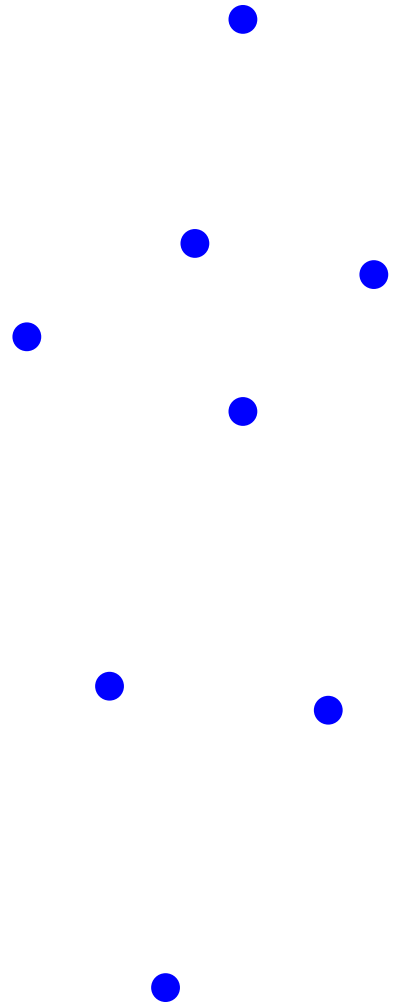

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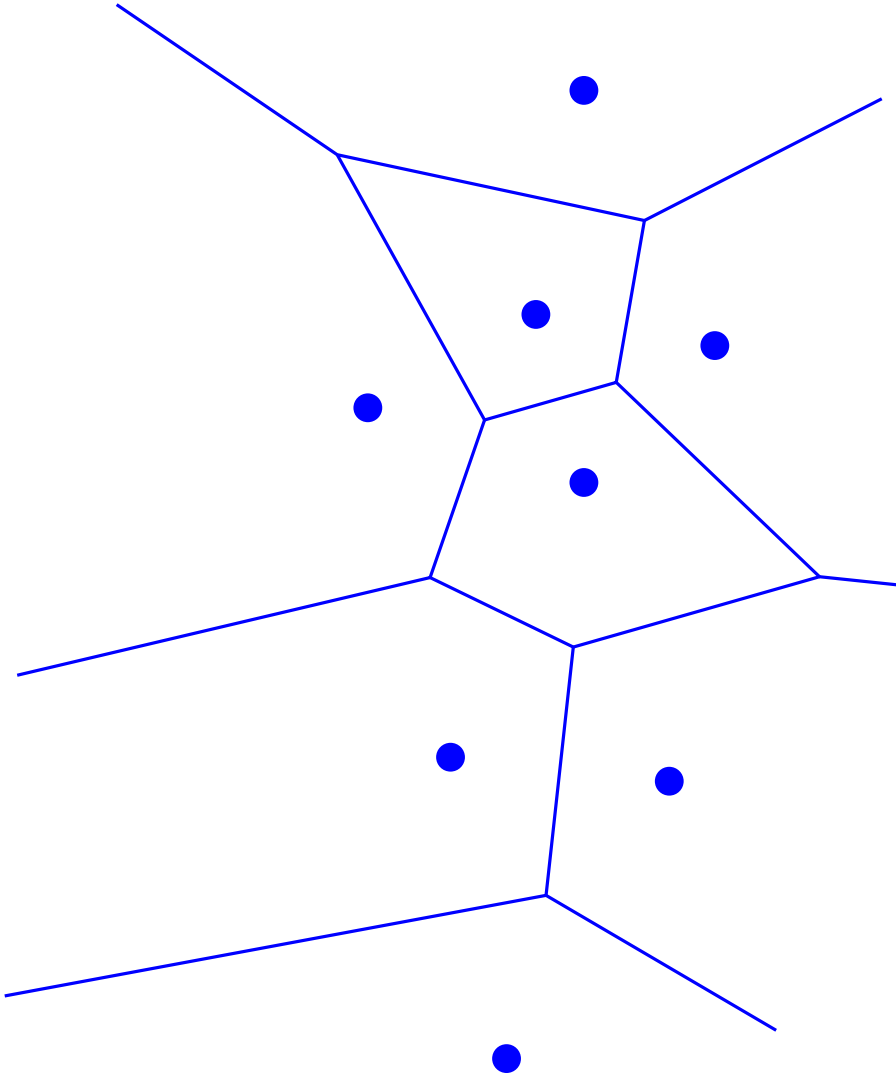

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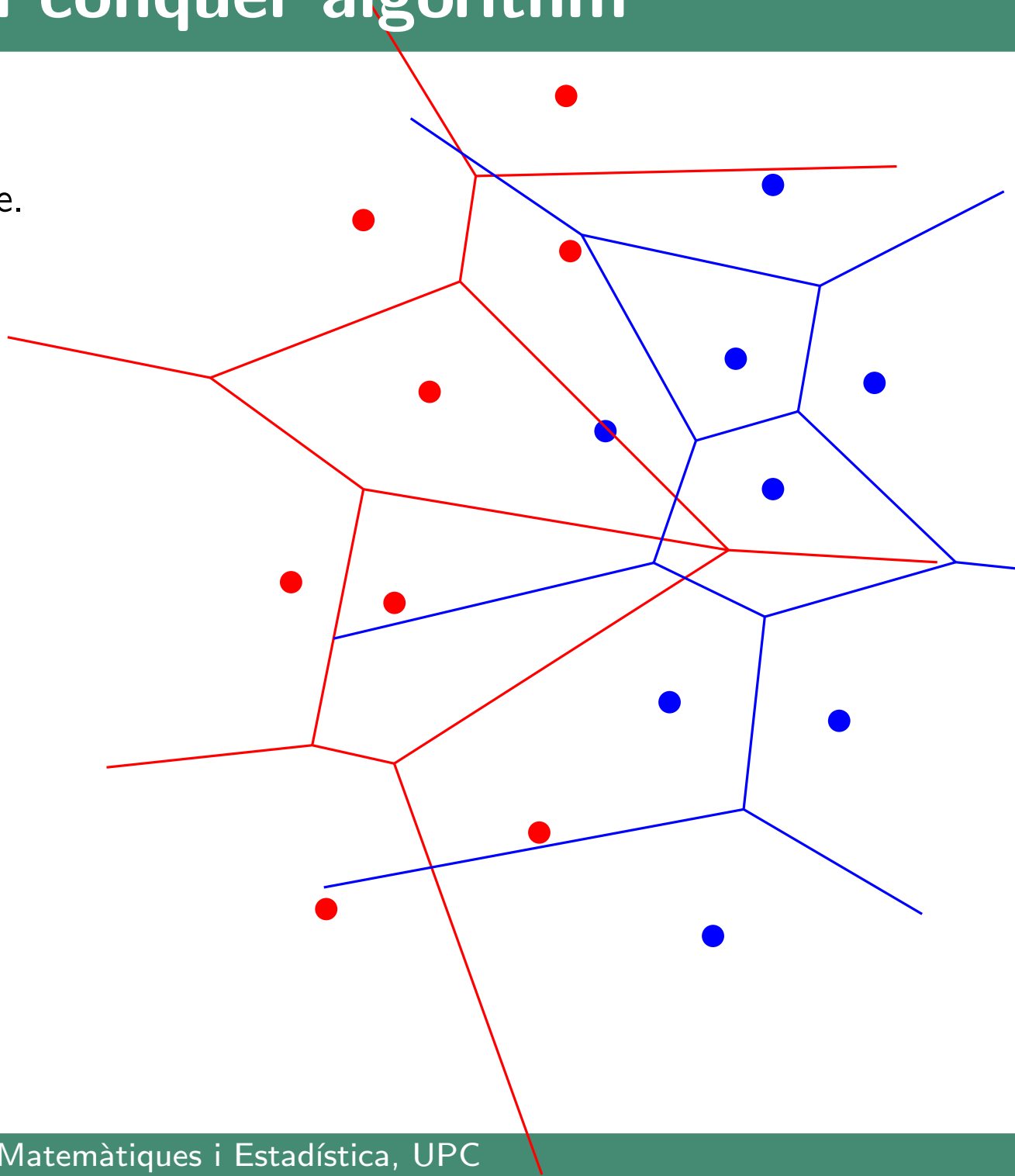

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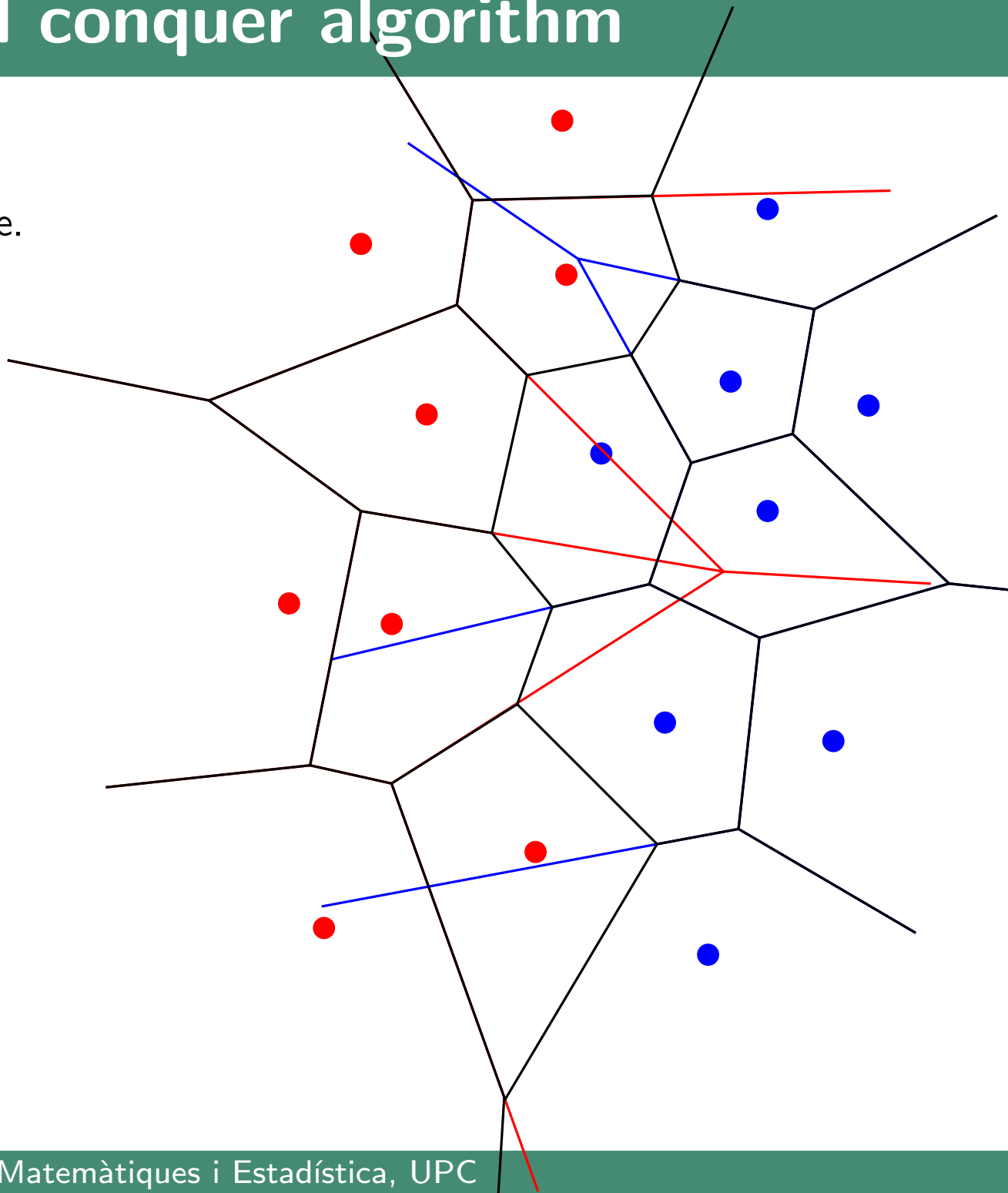

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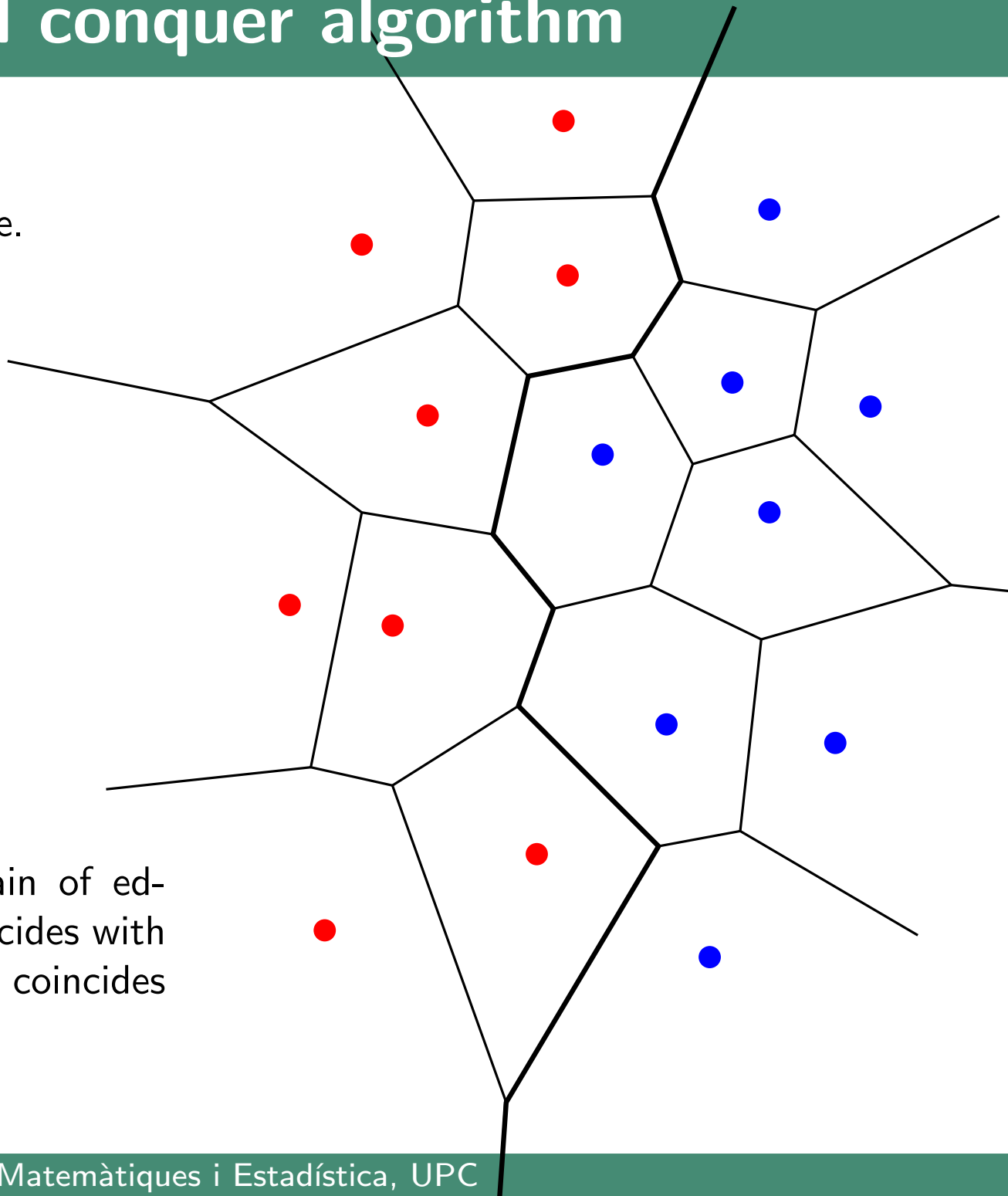

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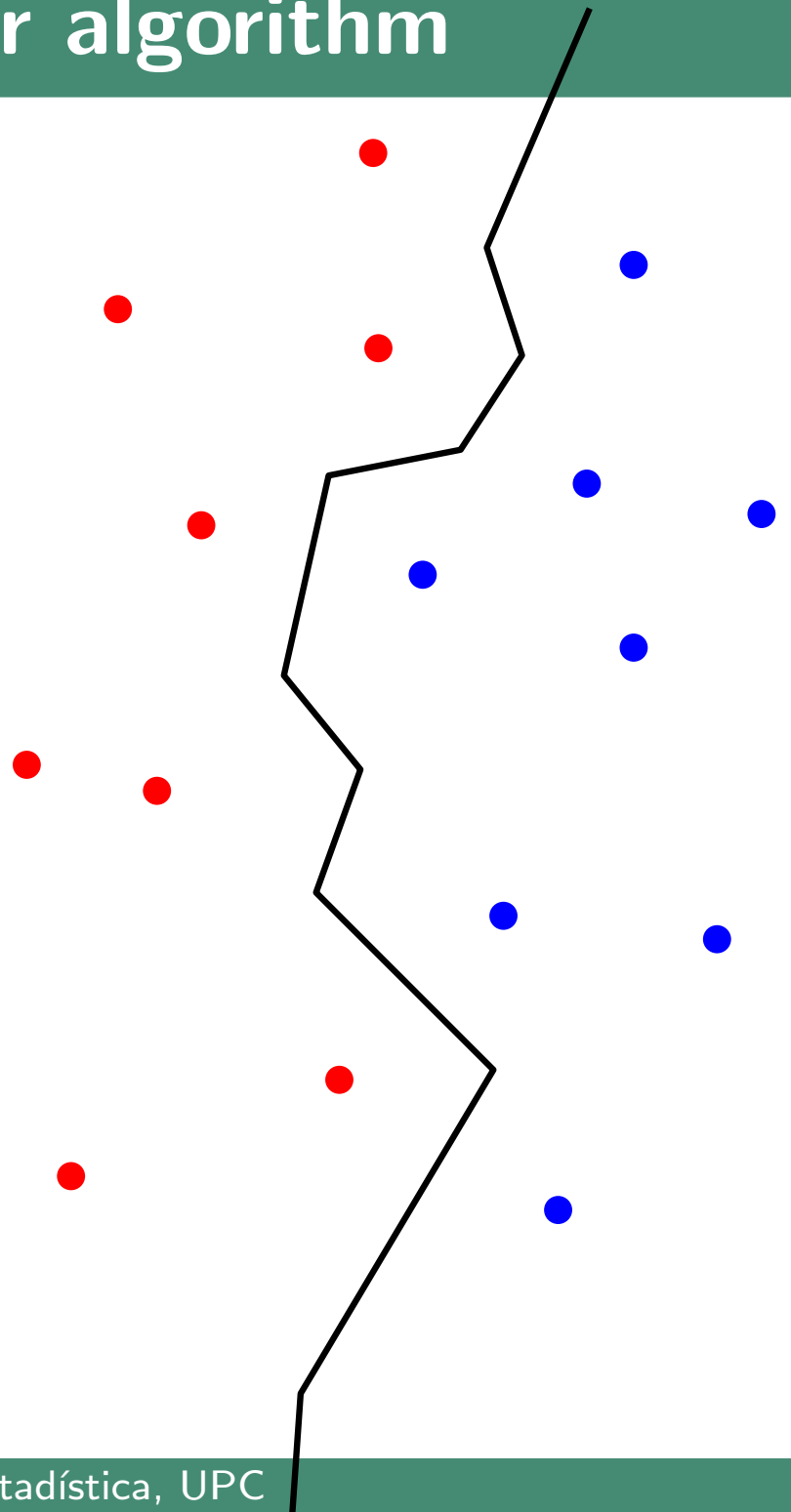

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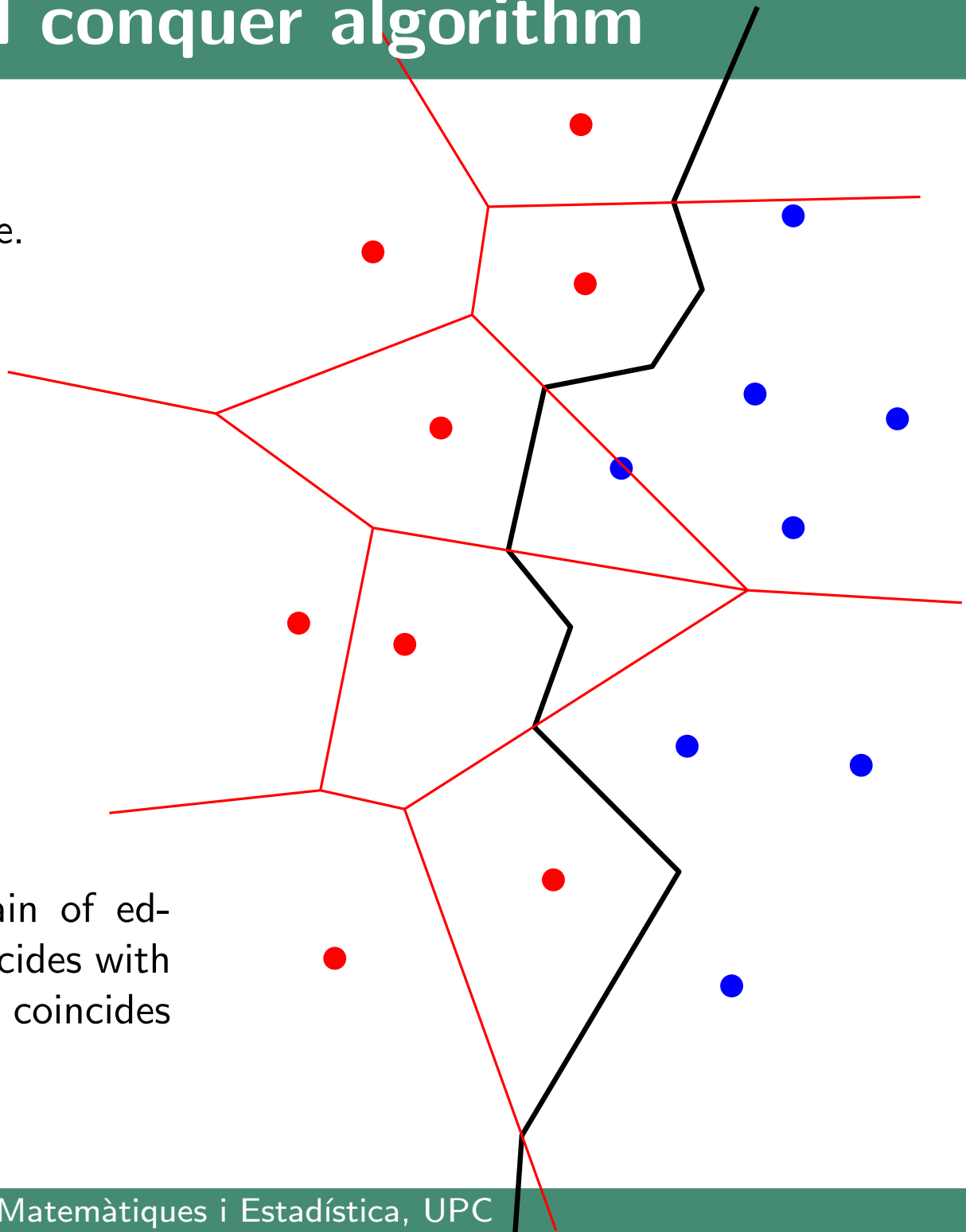

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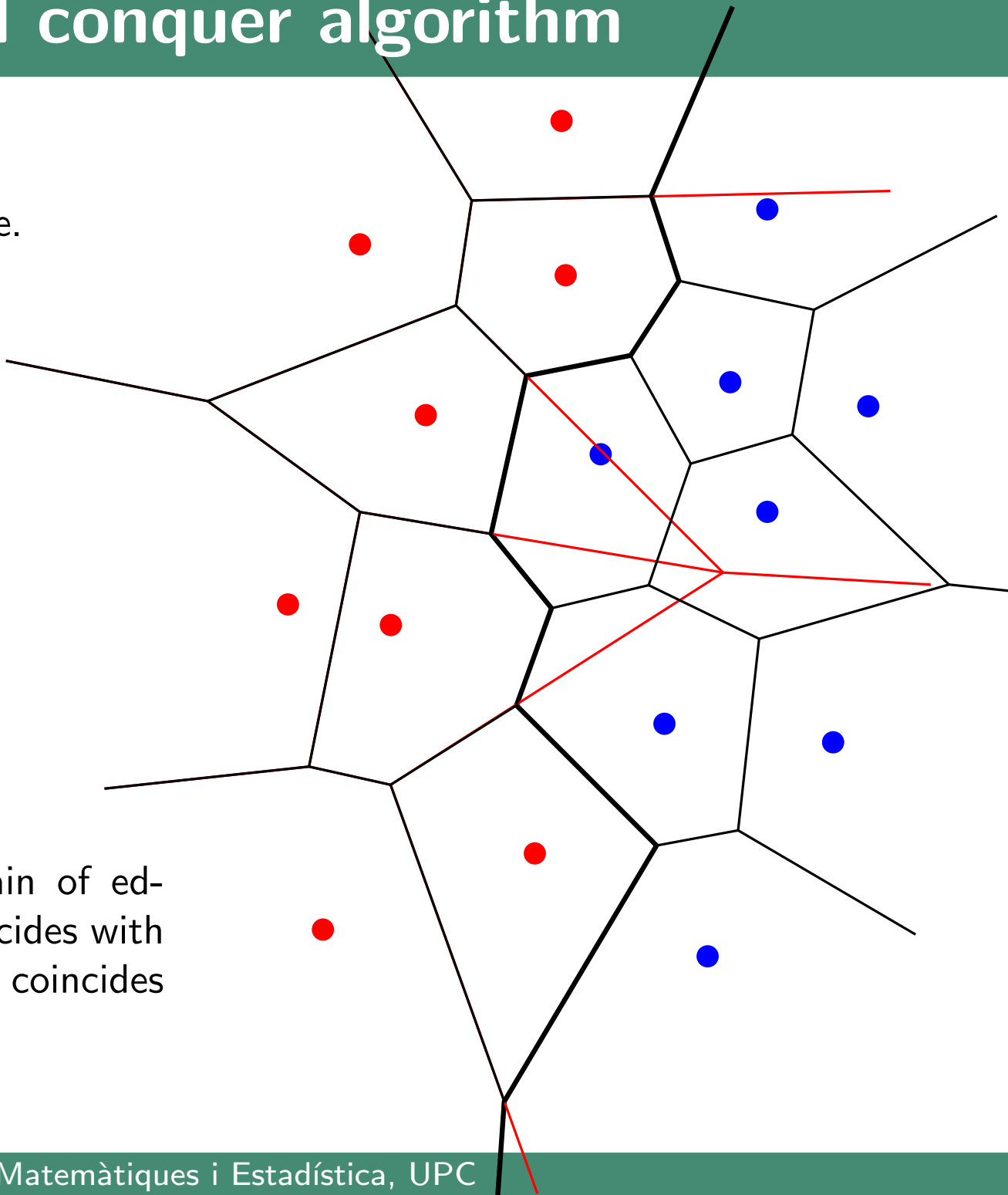

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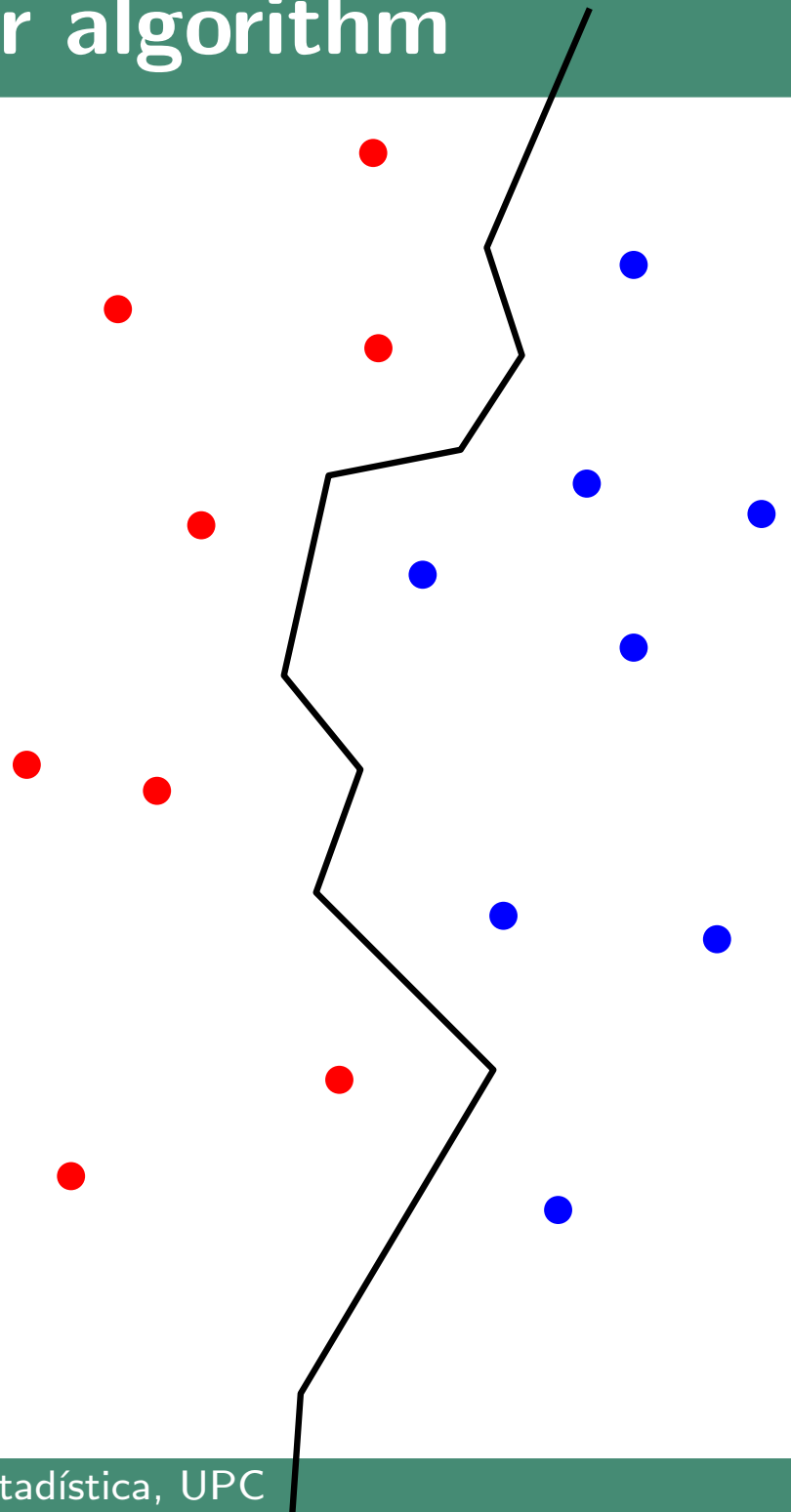

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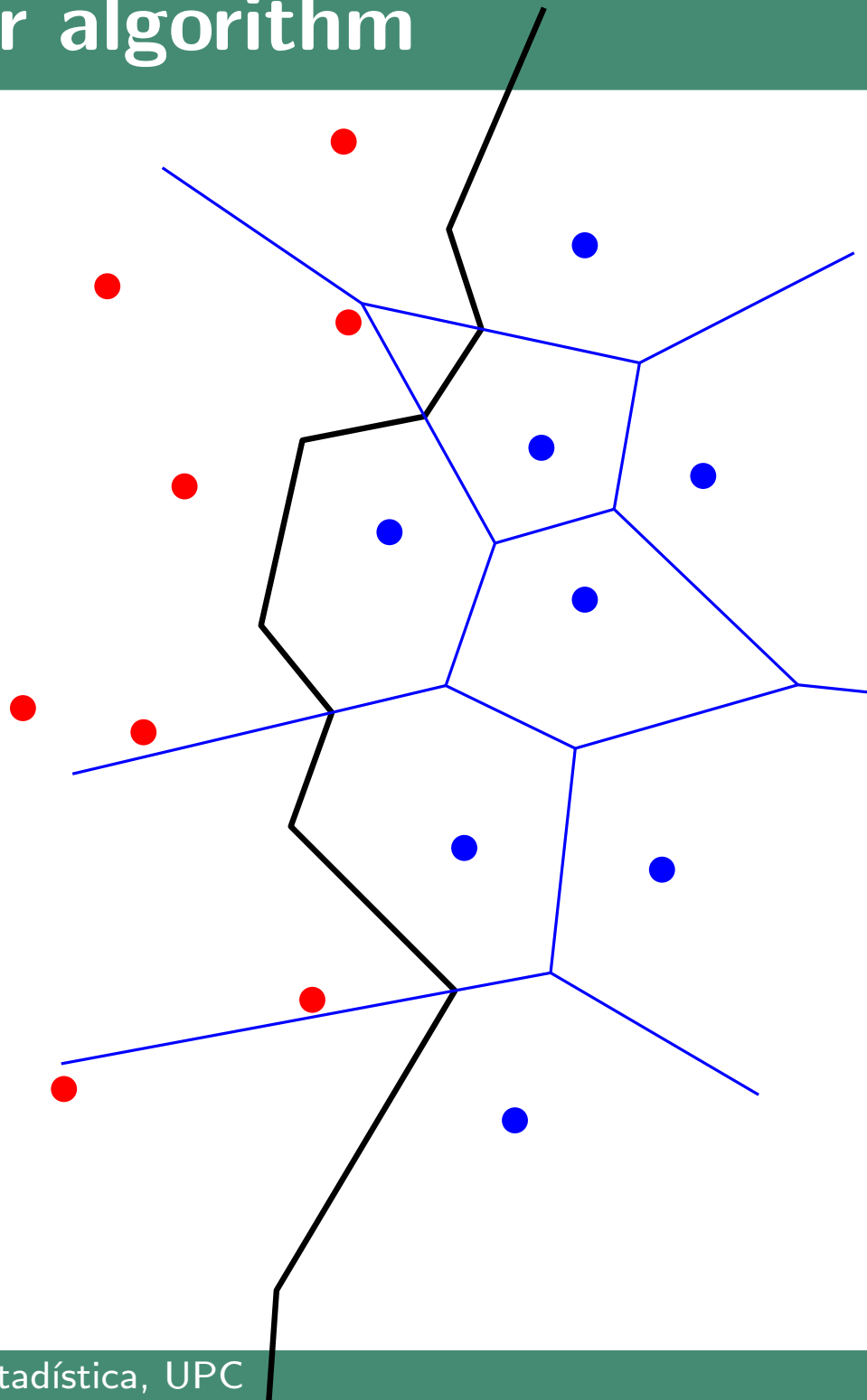

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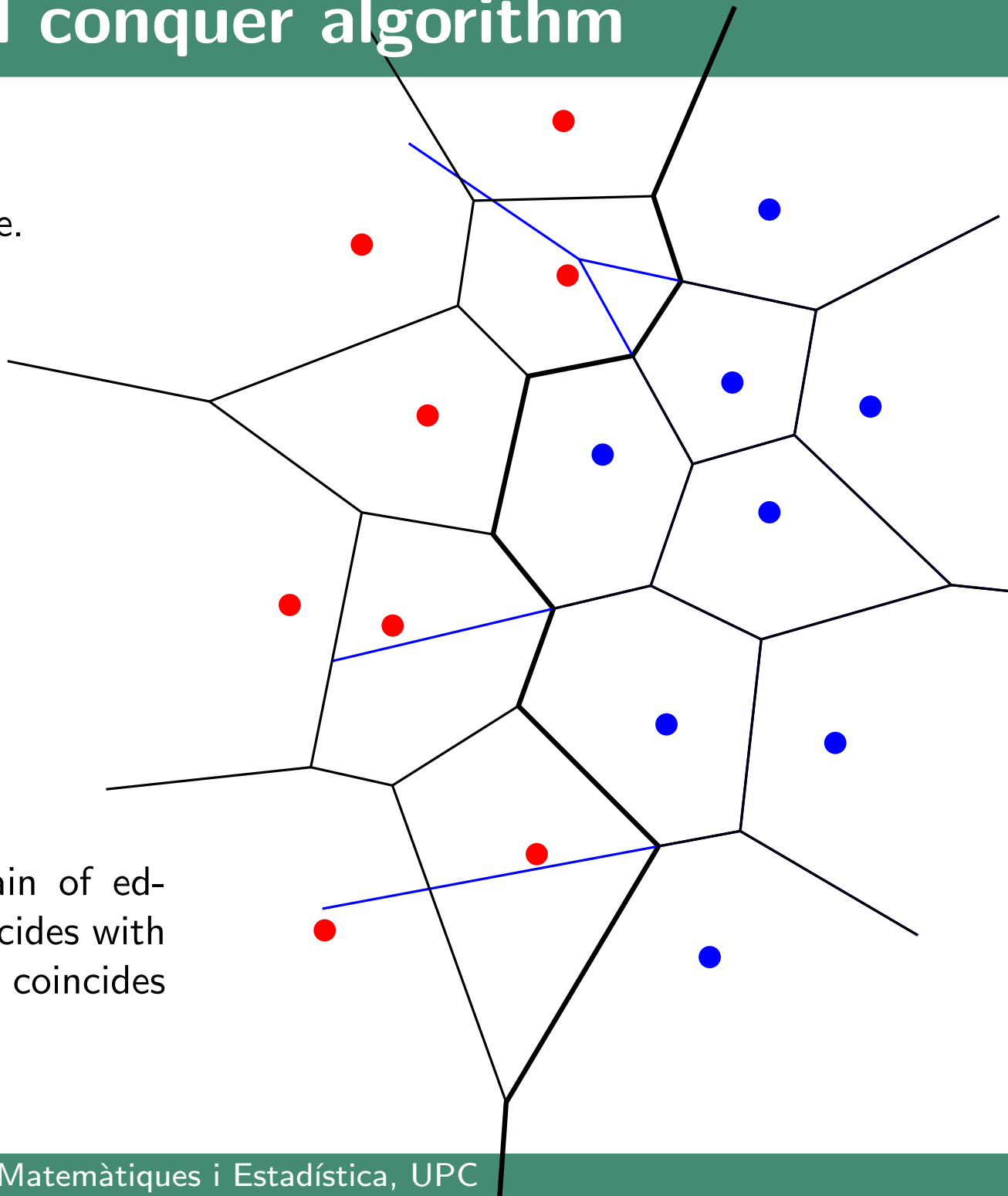

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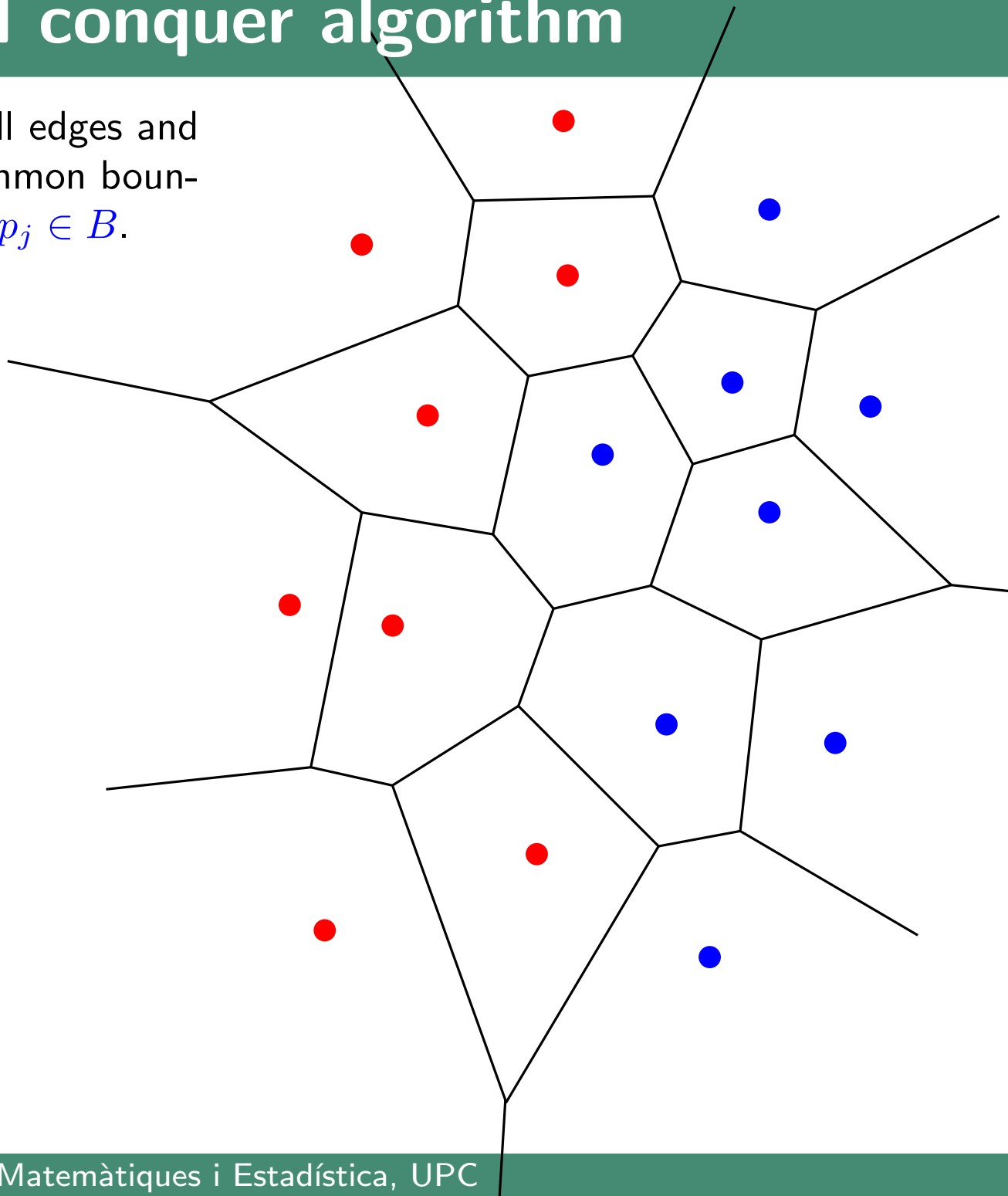
Definition. Let $b(R, B)$ be the set of all edges and vertices of $Vor(P)$ belonging to the common boundary of the regions of some $p_i \in R$ and $p_j \in B$.



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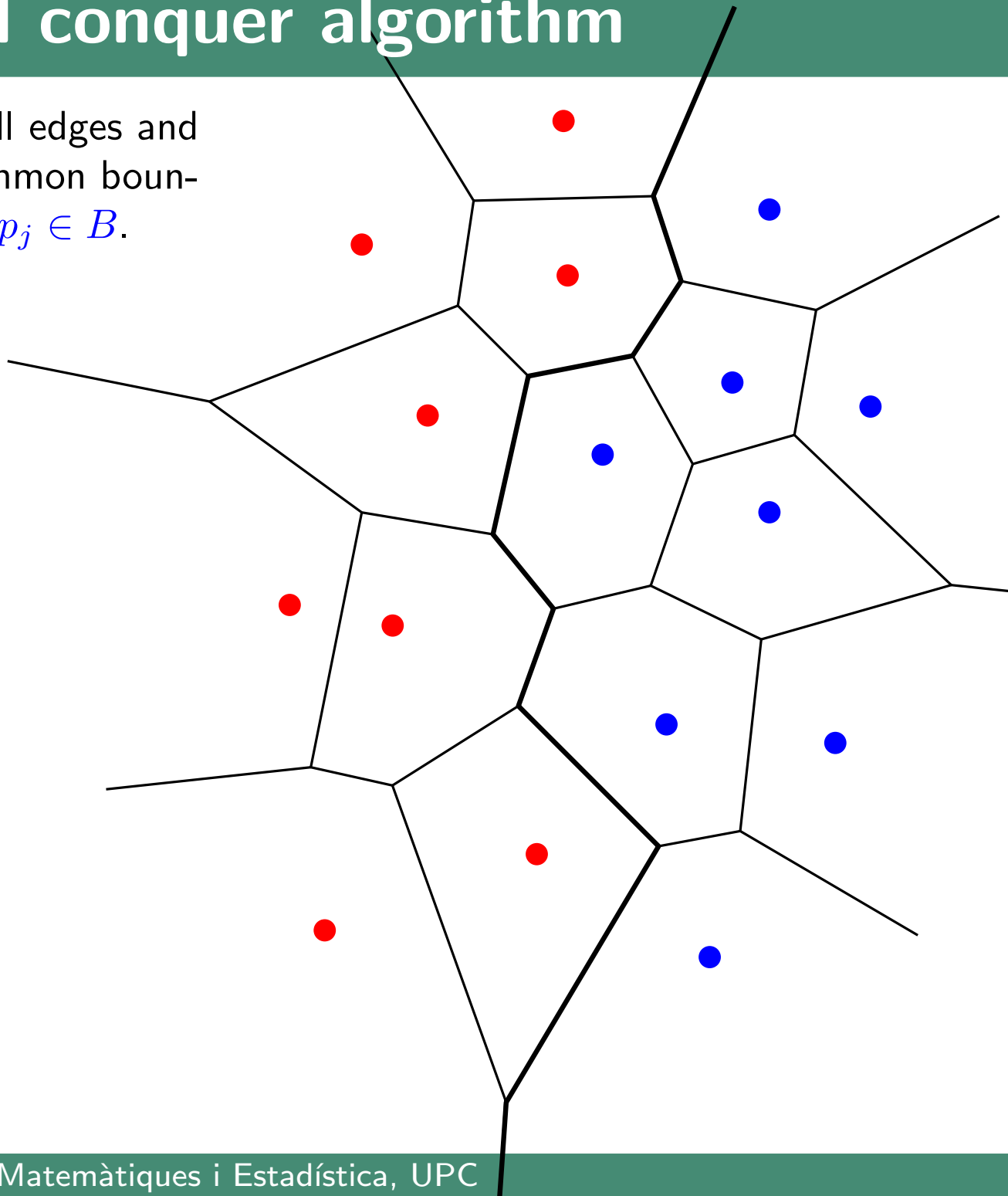

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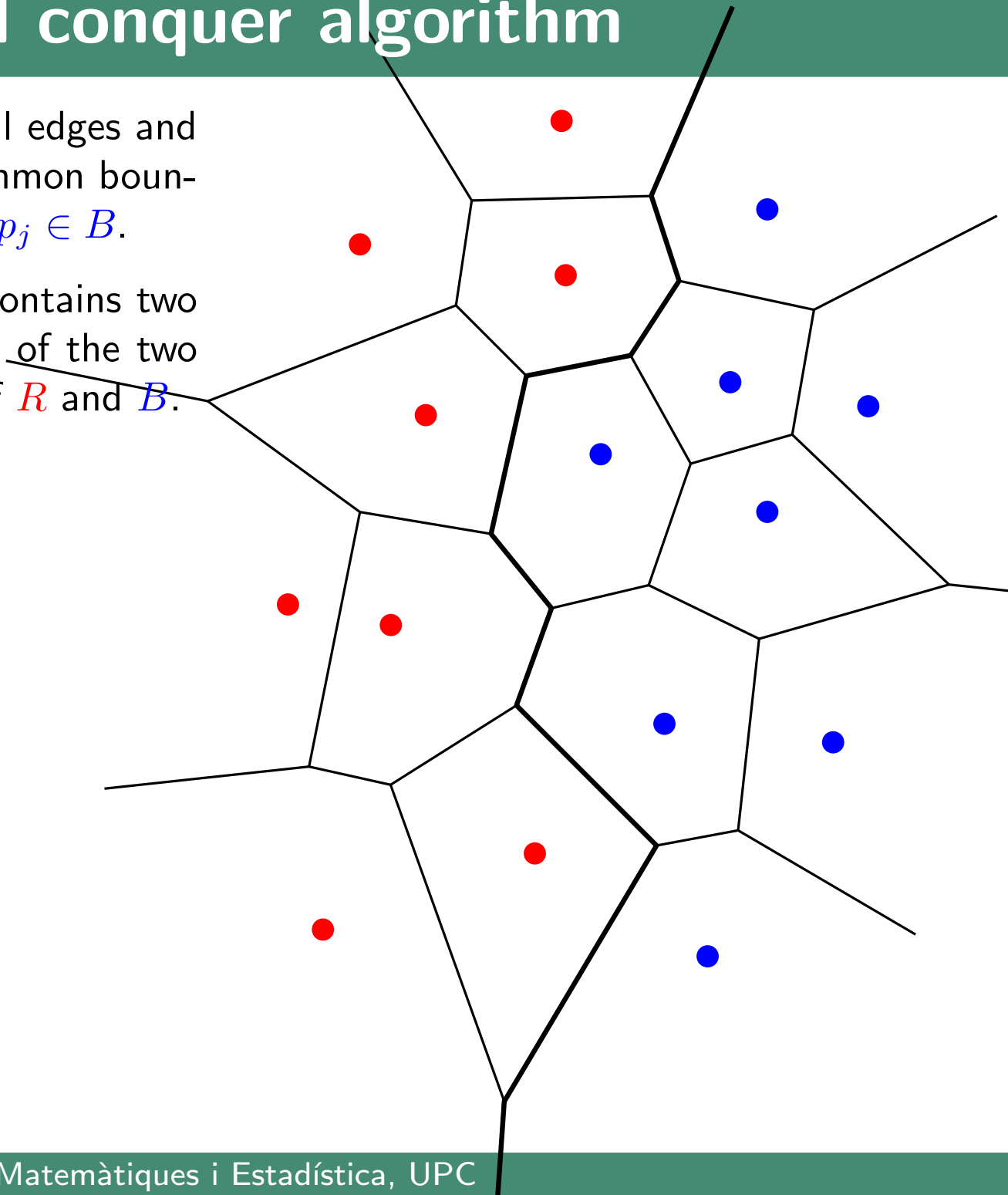

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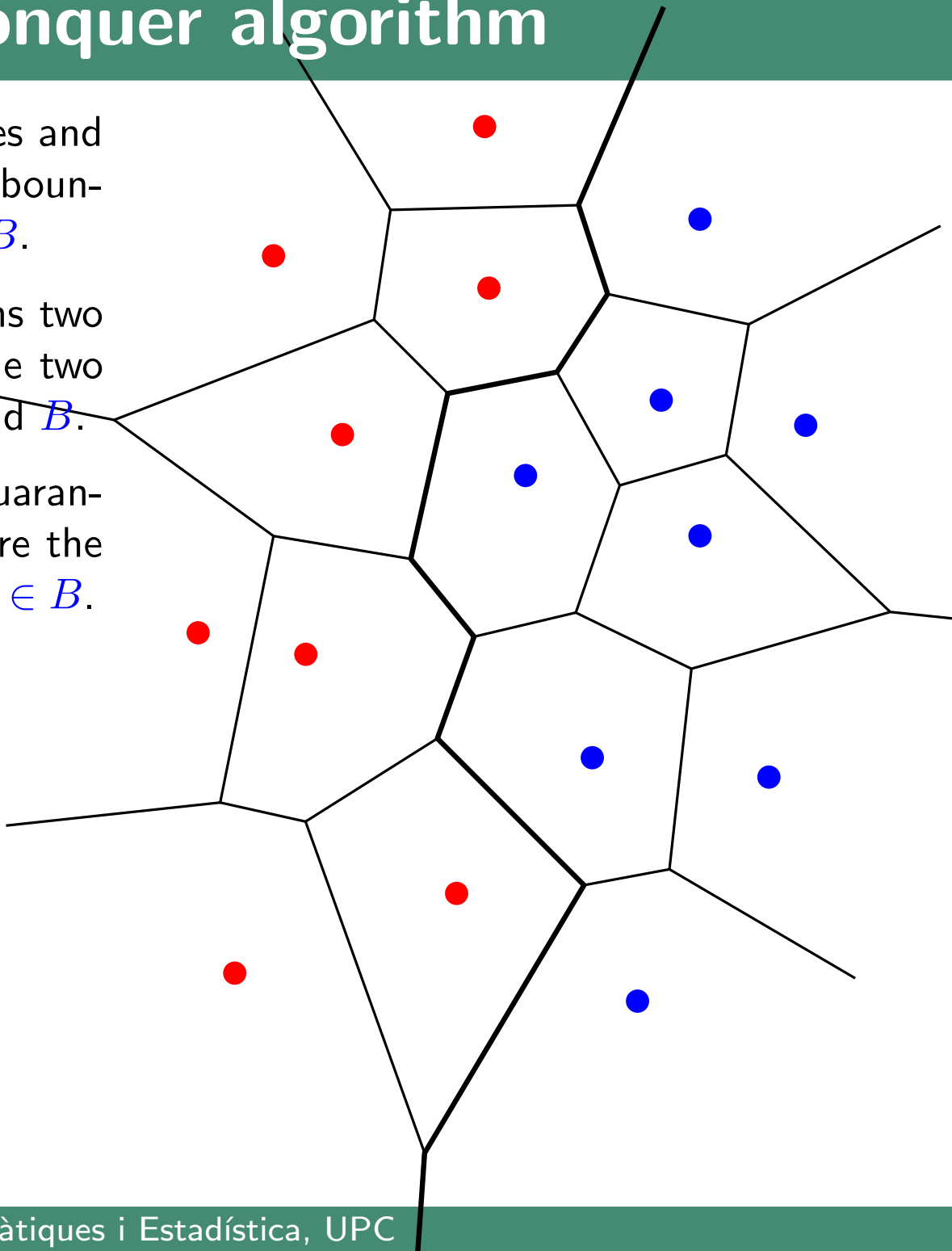
Proof. The vertical separation of R and B guarantees the existence of the "bridges", which are the edges of $ch(P)$ connecting a $p_i \in R$ to a $p_j \in B$.



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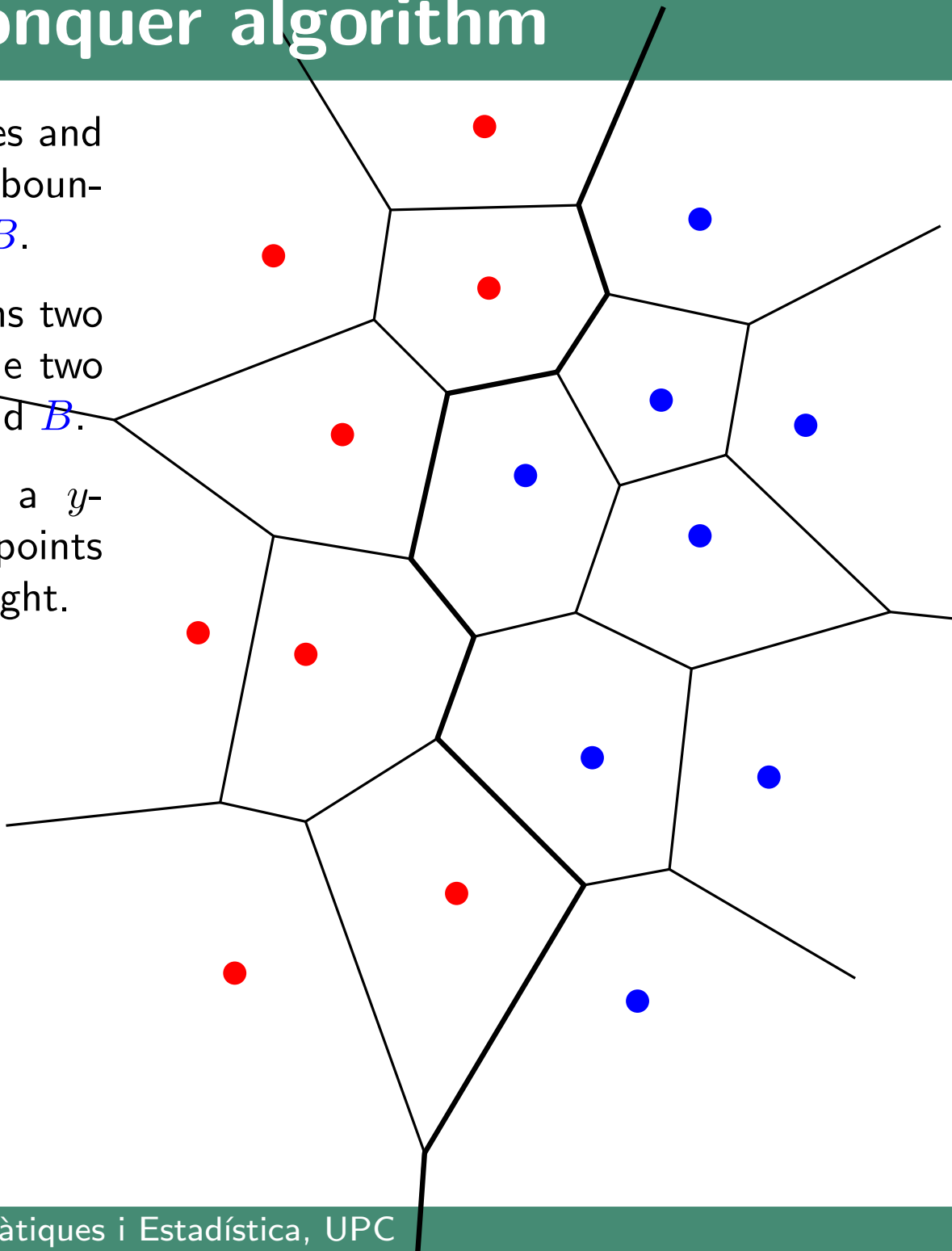

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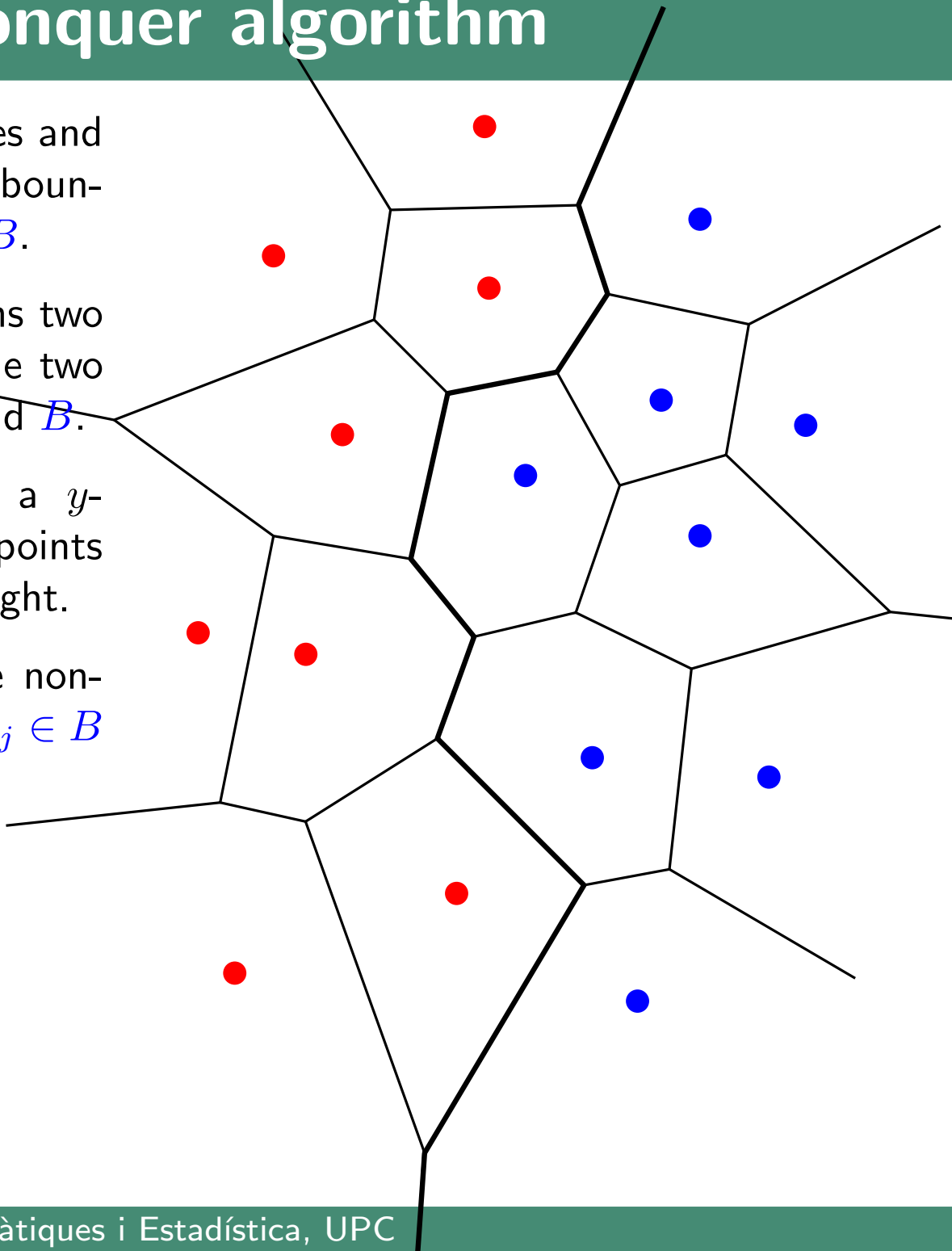
Proof. Every edge e_{ij} of $b(R, B)$ must be non-horizontal, and leave $p_i \in R$ to its left and $p_j \in B$ to its right.



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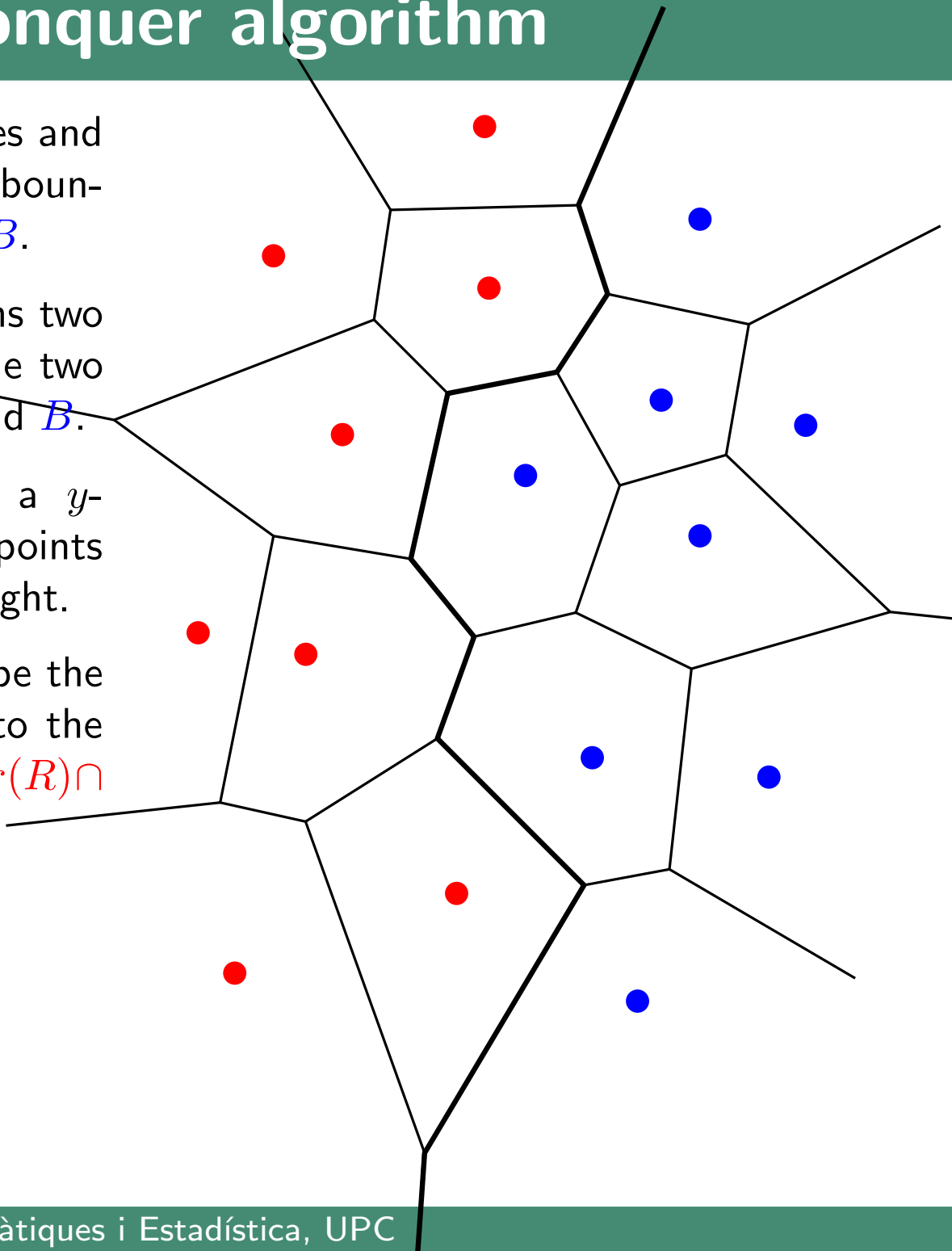
Observation 3. Let π_R and π_B respectively be the regions of the plane located to the left and to the right of $b(R, B)$. Then $Vor(P)$ consists of $Vor(R) \cap \pi_R$, $Vor(B) \cap \pi_B$ and $b(R, B)$.



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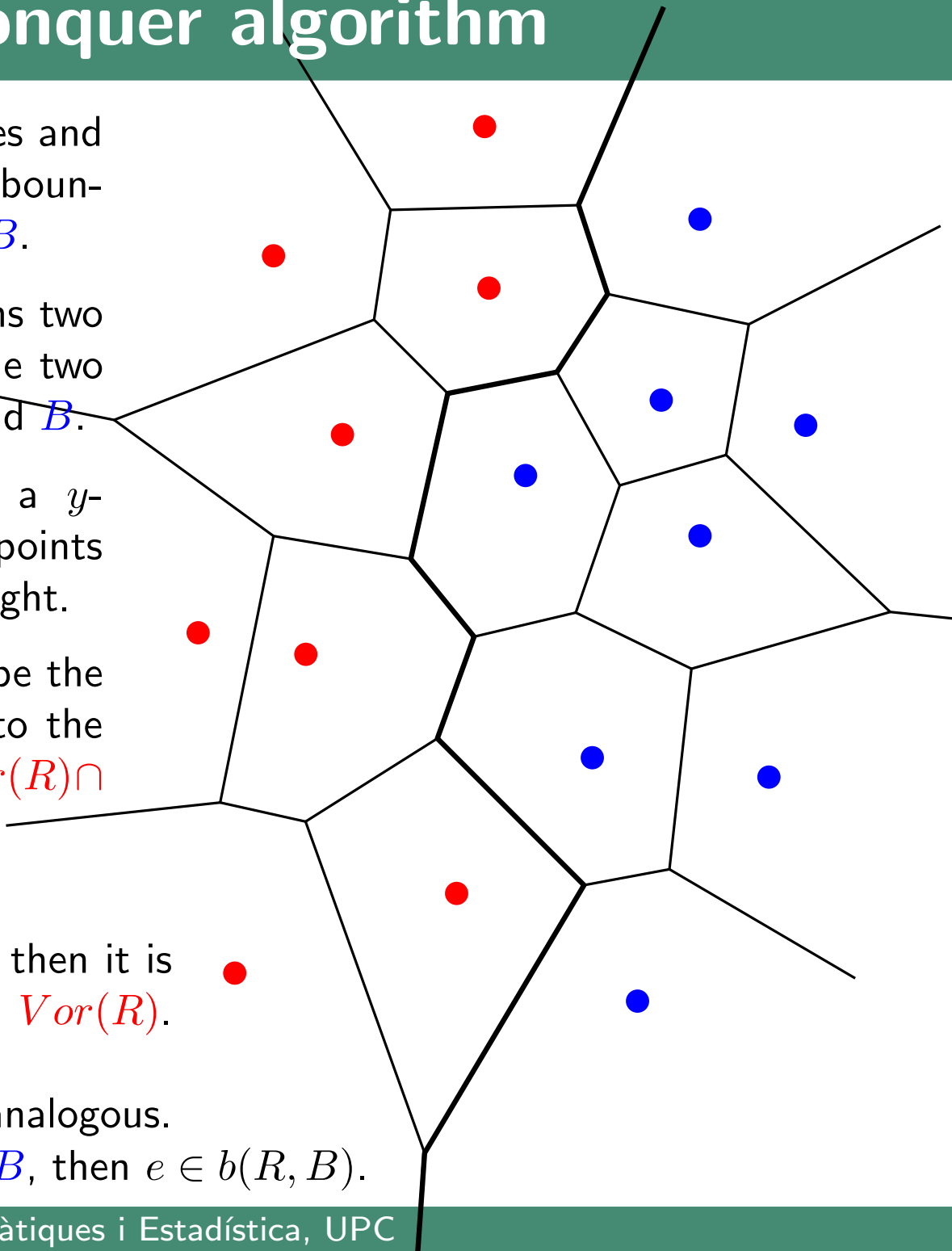

Proof. Let e be an edge of $Vor(P)$:
- If e separates two points of R in $Vor(P)$, then it is (a portion of) the edge separating them in $Vor(R)$. Due to Obs. 2, e cannot belong to π_B.
- If e separates two points of B, the case is analogous.
- If e separates one point of R from one of B, then $e \in b(R, B)$.

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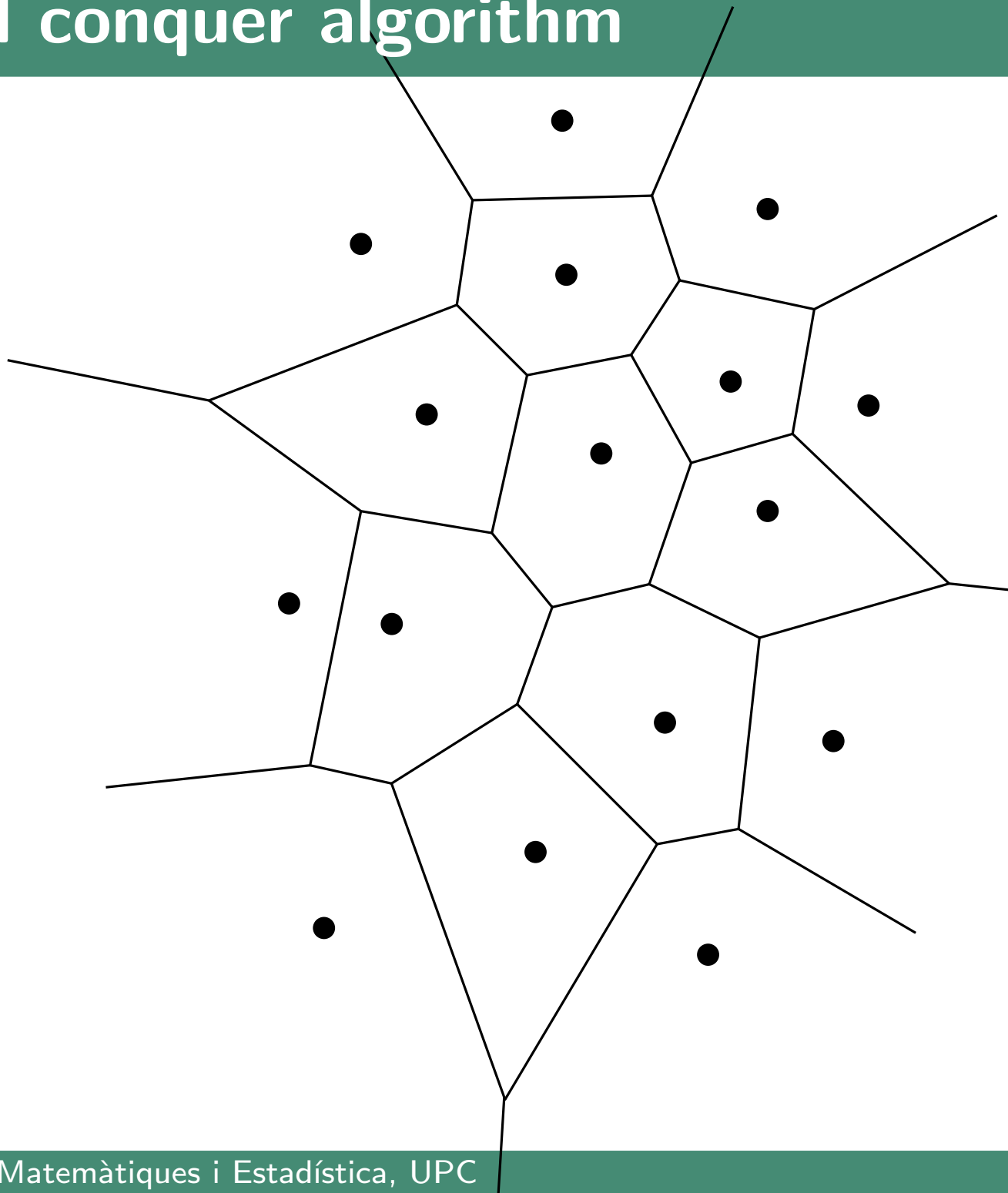

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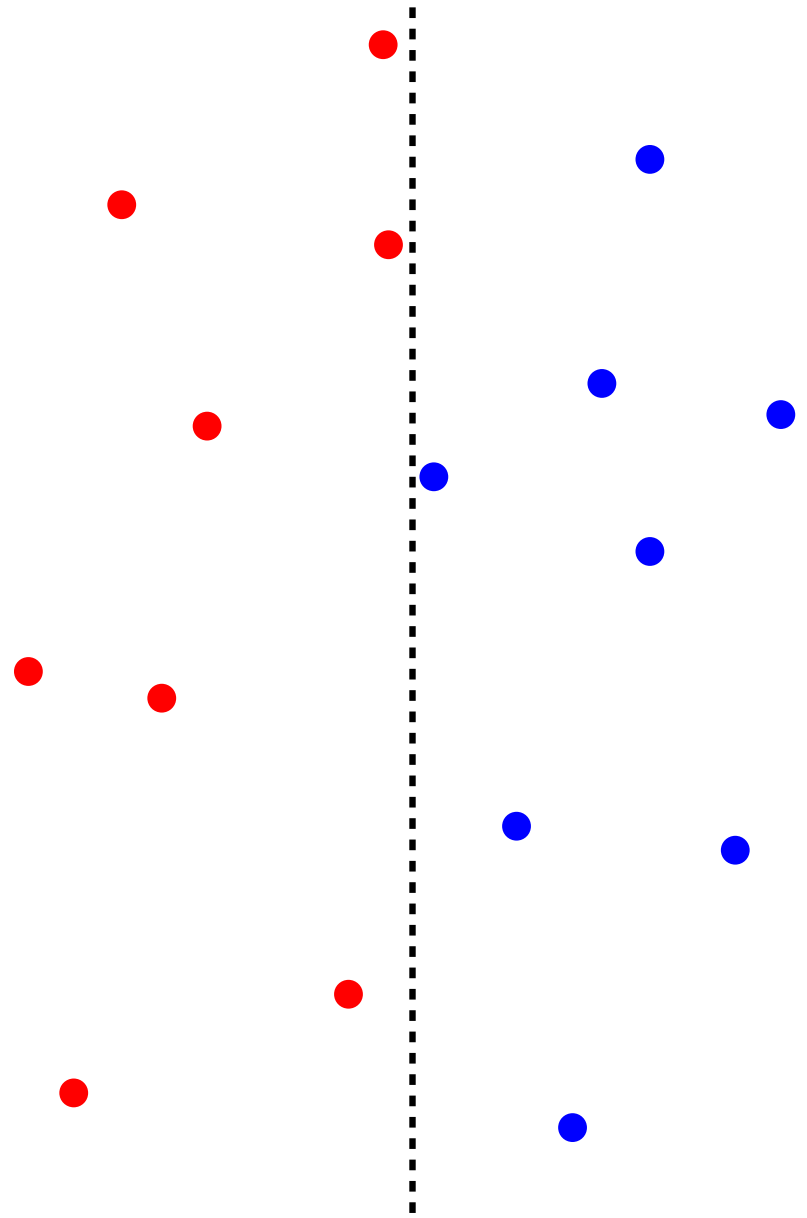

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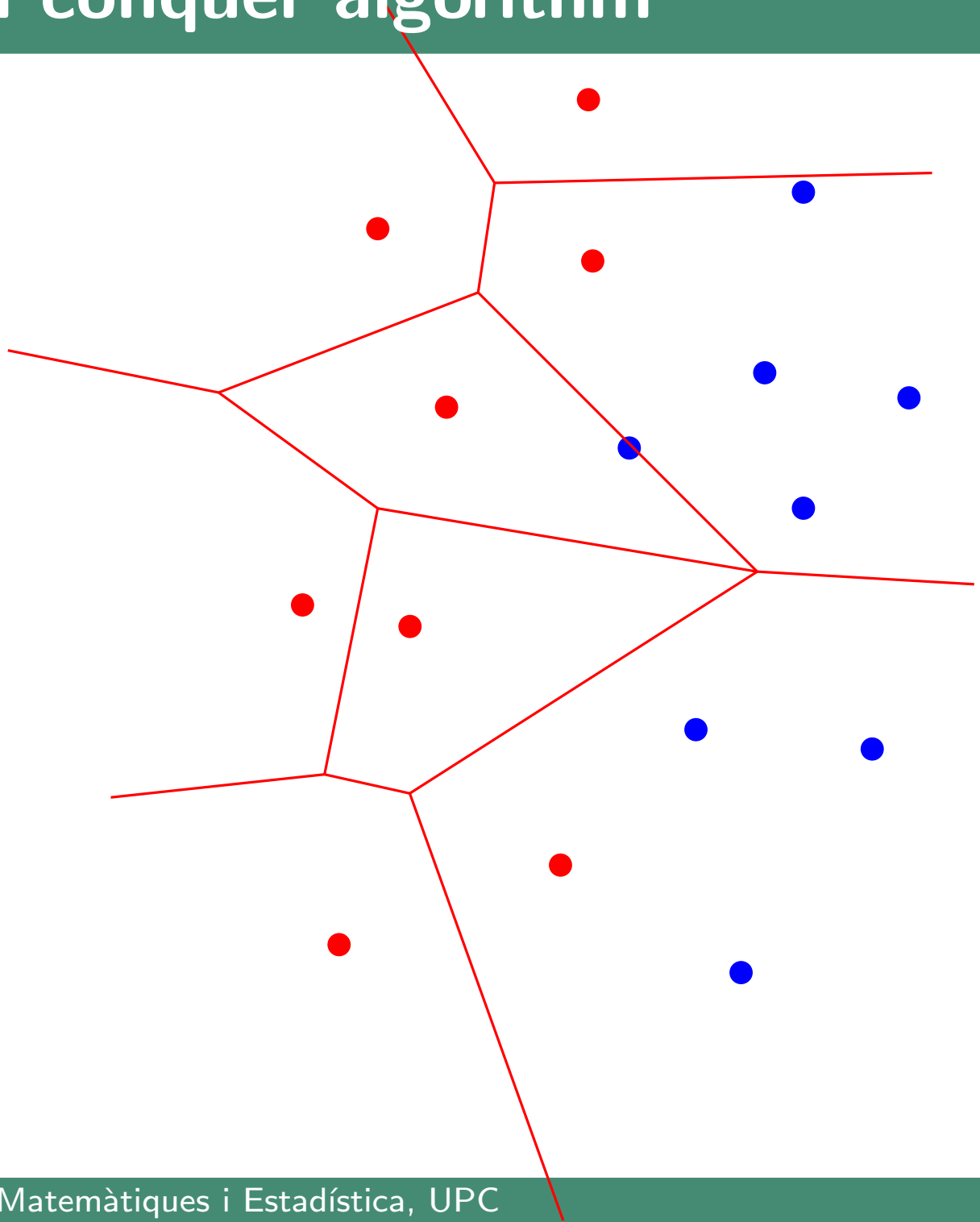

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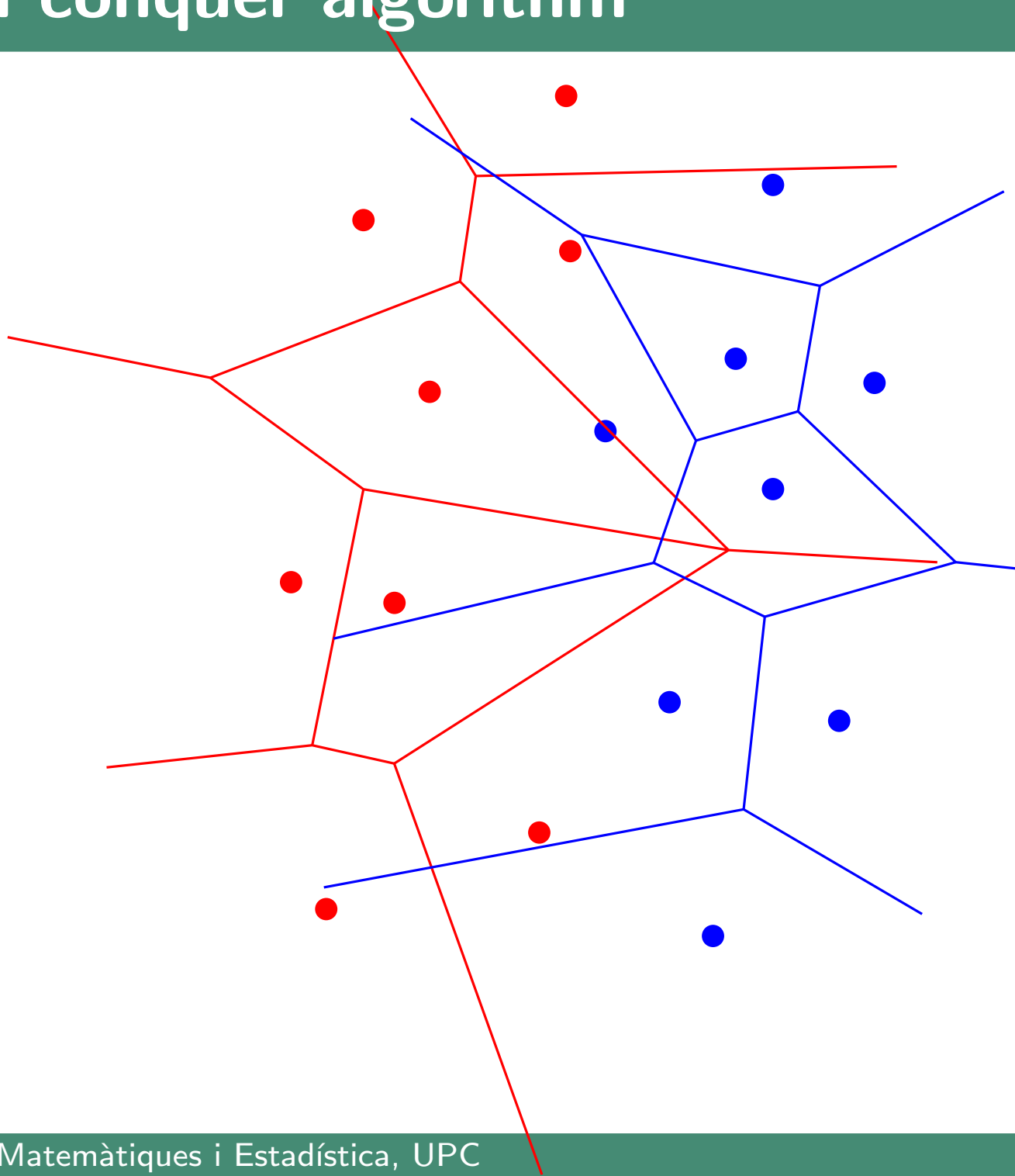

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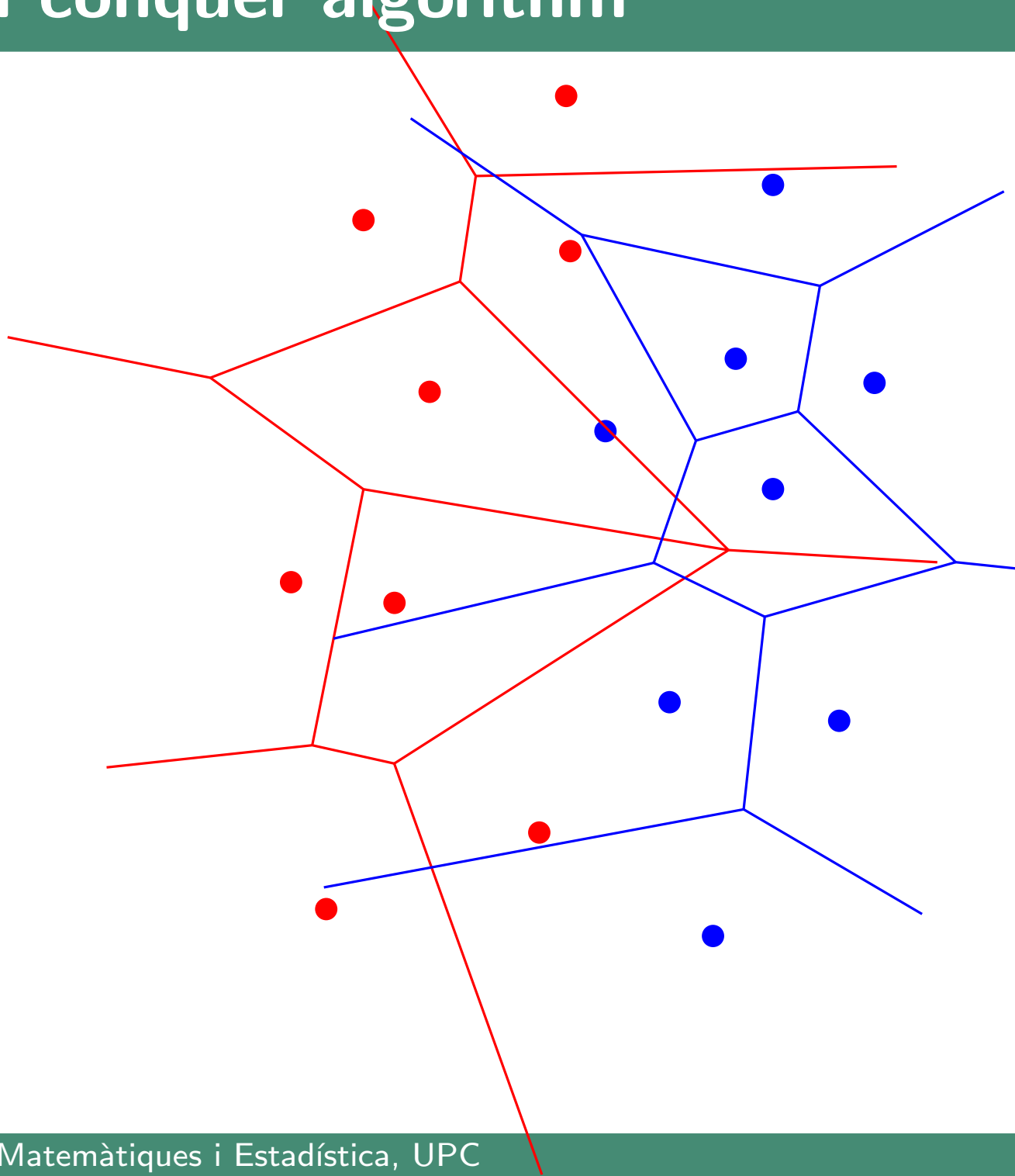

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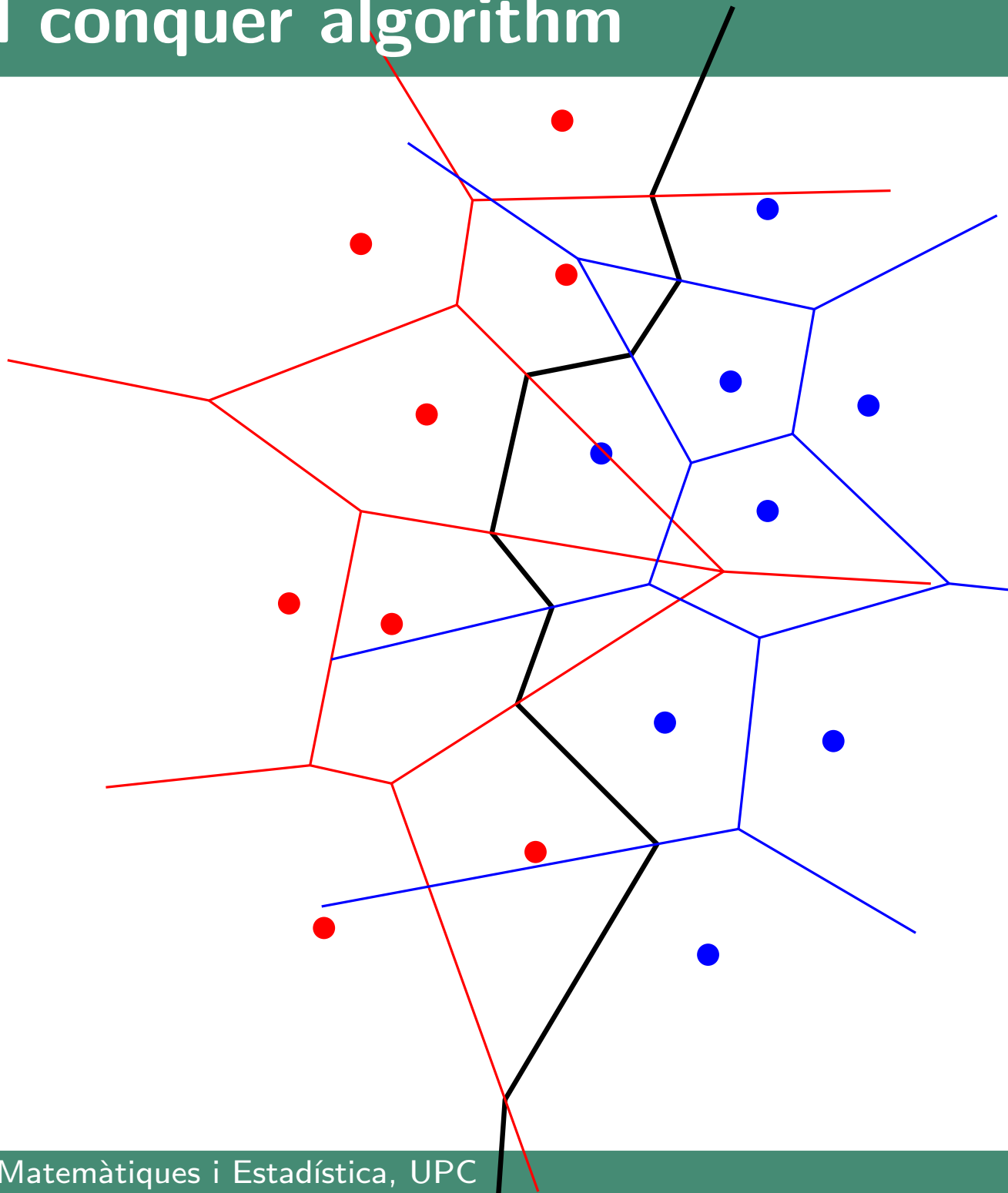
   Prune the portion of $Vor(R)$ lying to the right of the chain and the portion of $Vor(B)$ lying to its left.



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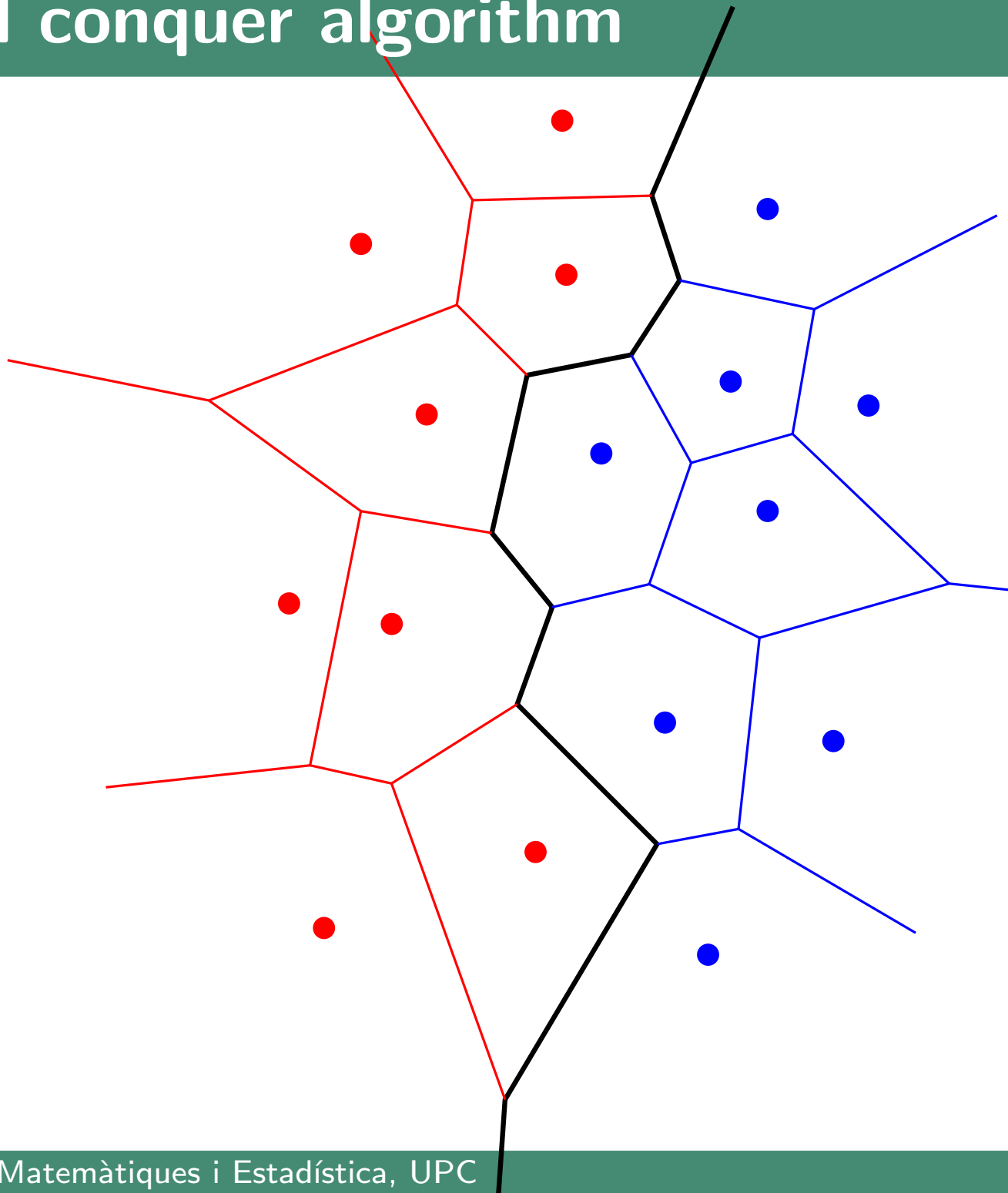

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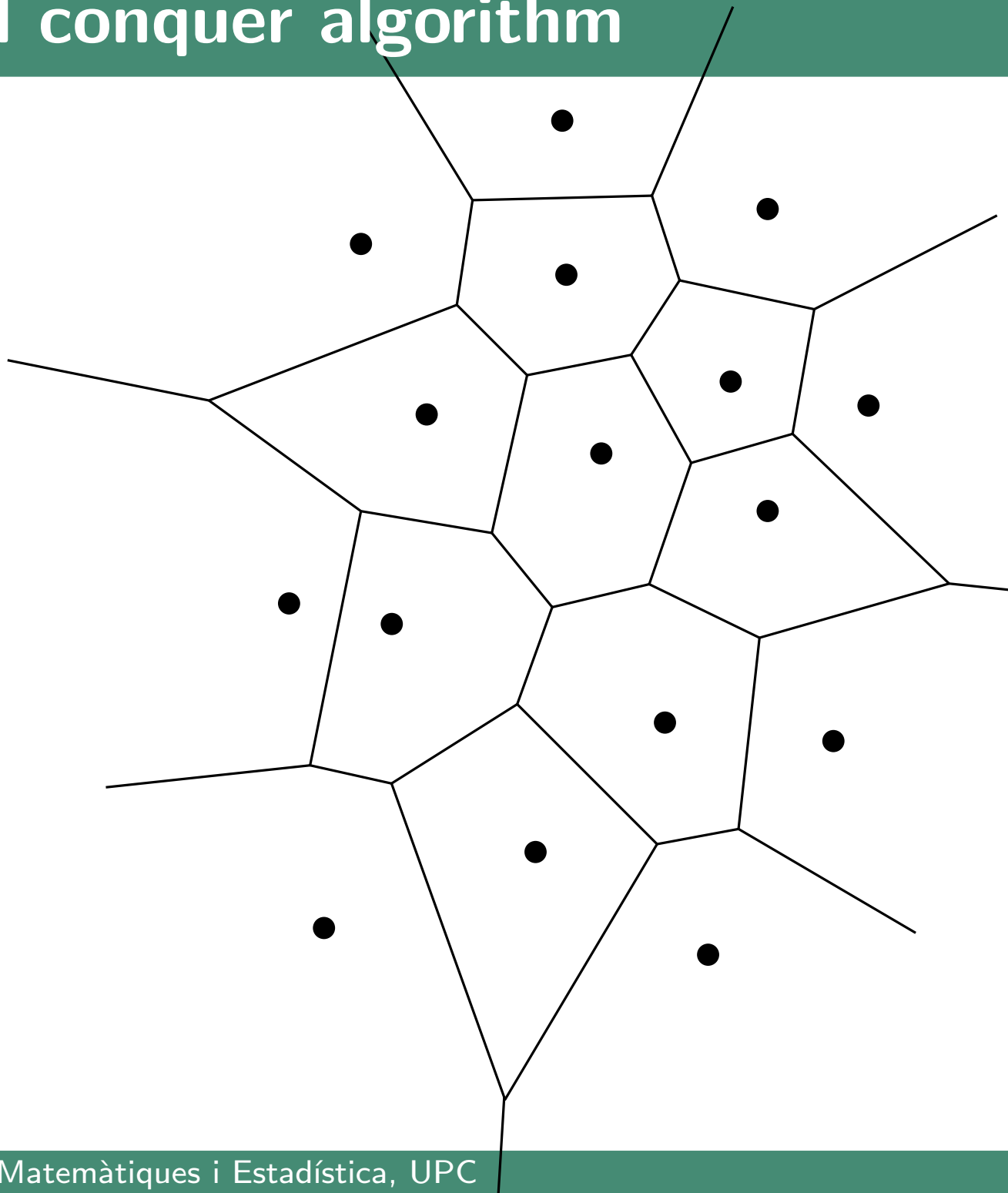

Divide and conquer algorithm

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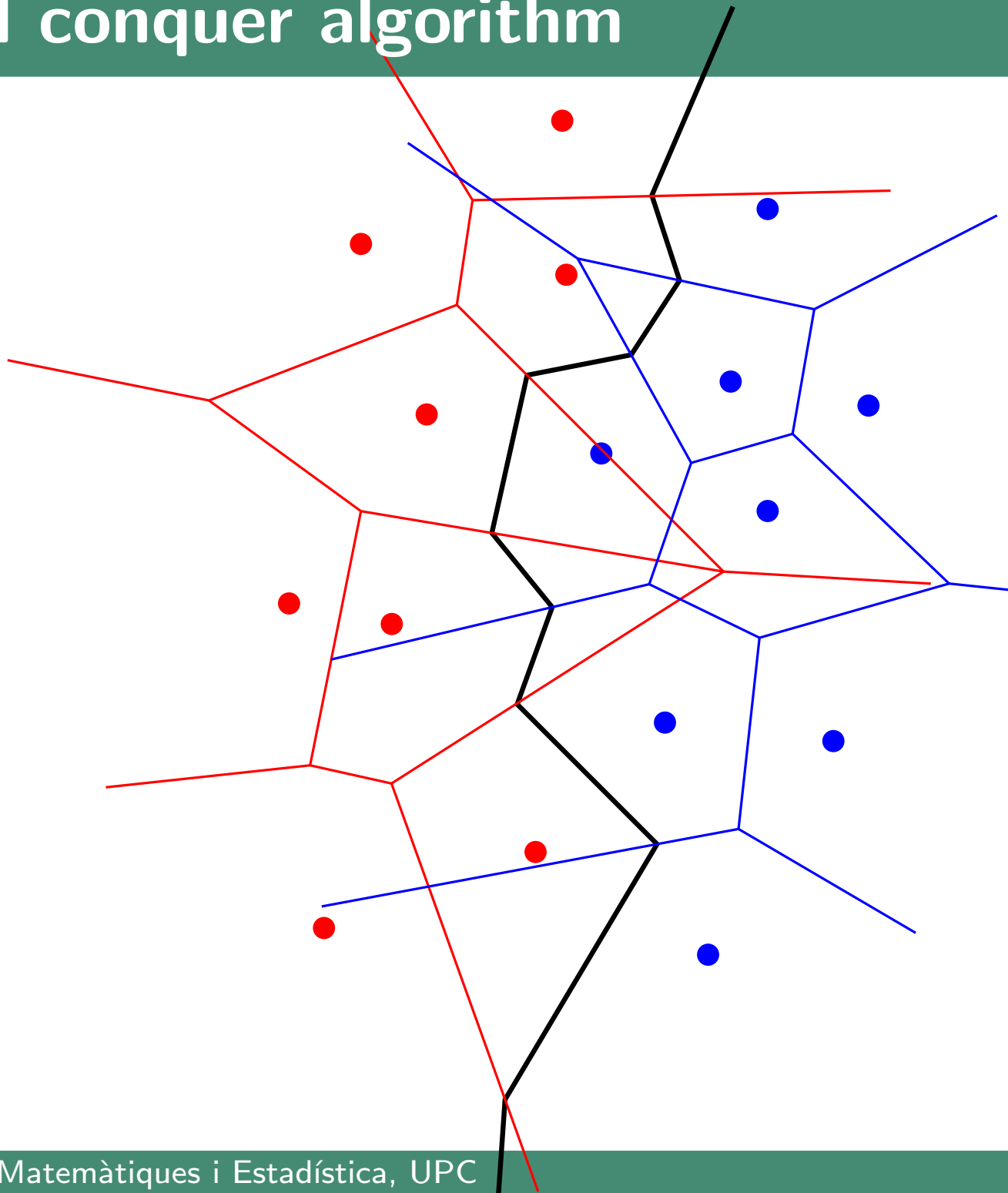
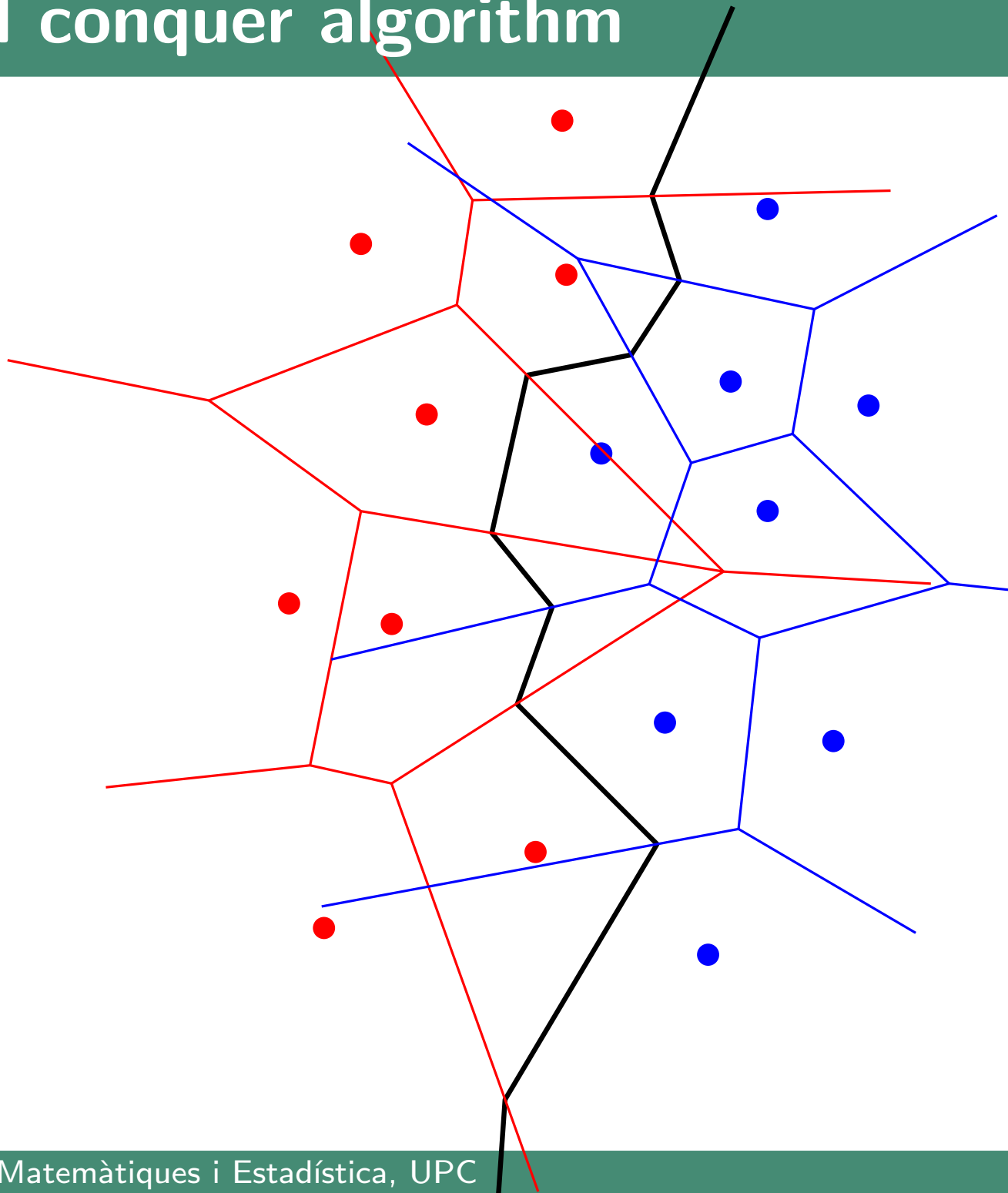


Divide and conquer algorithm

How to compute the chain?

Initialization

Find the two halflines



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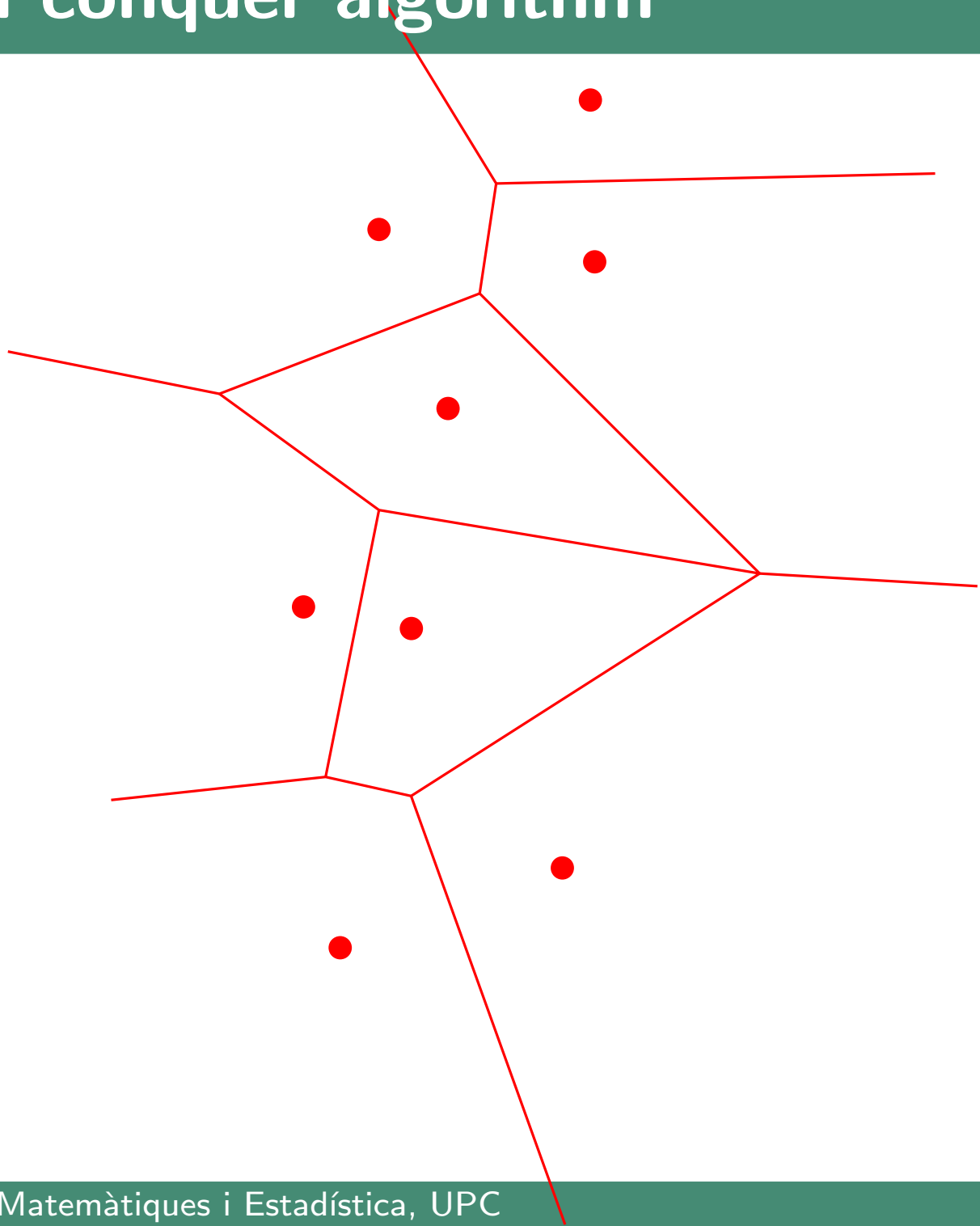

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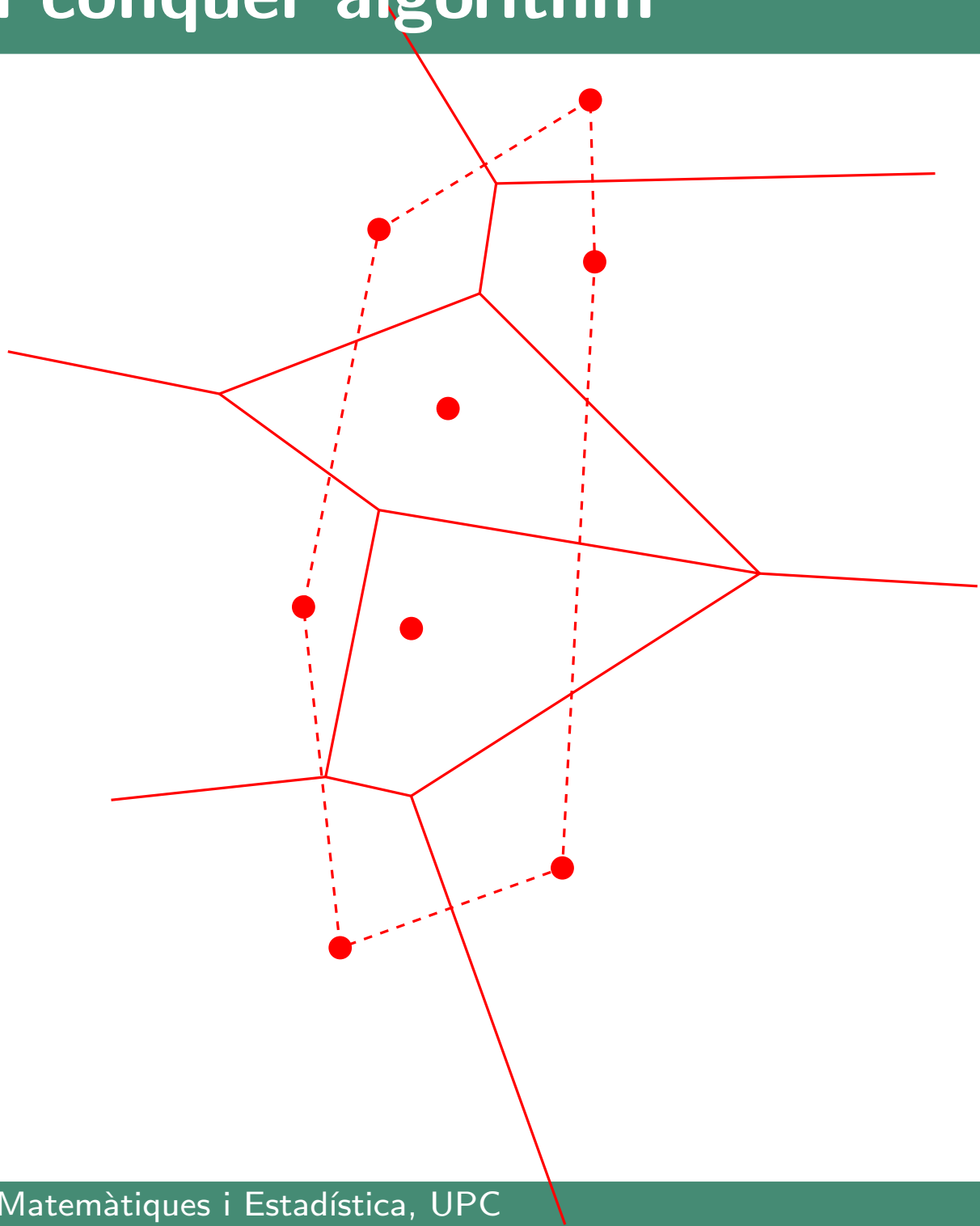

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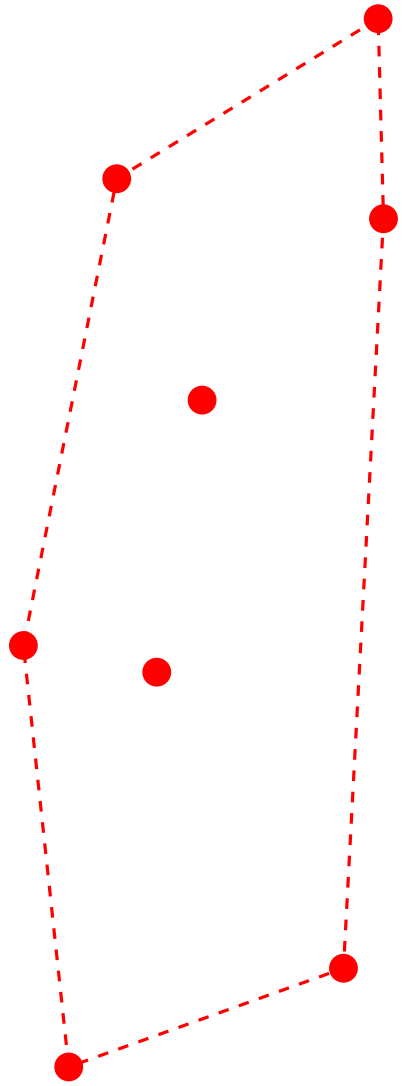

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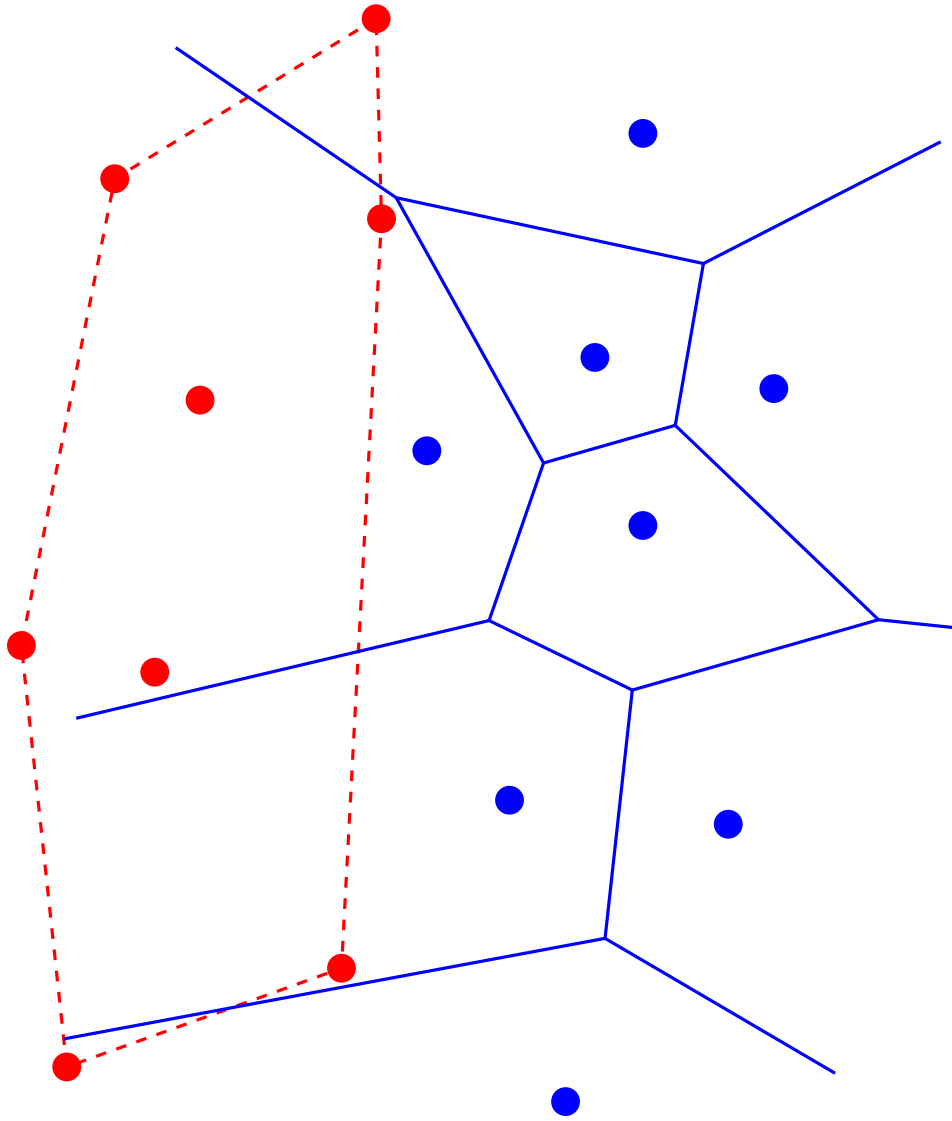

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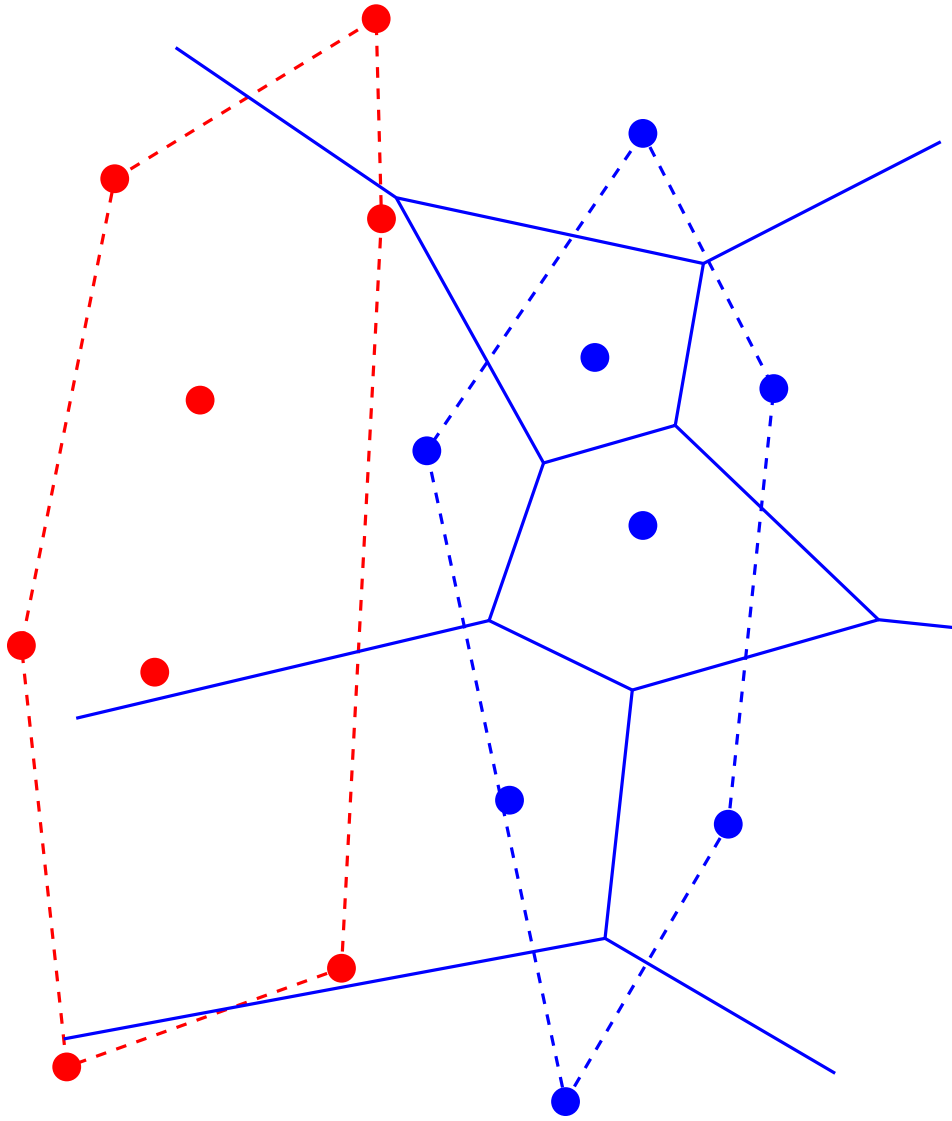

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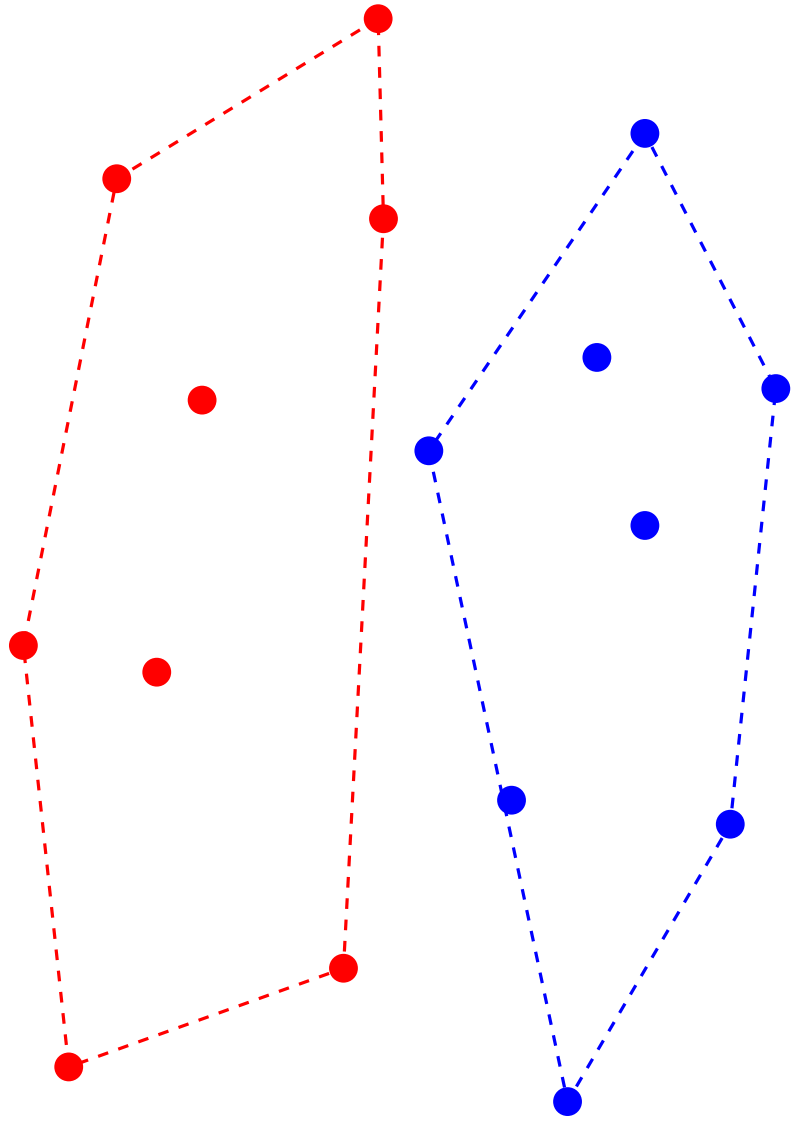

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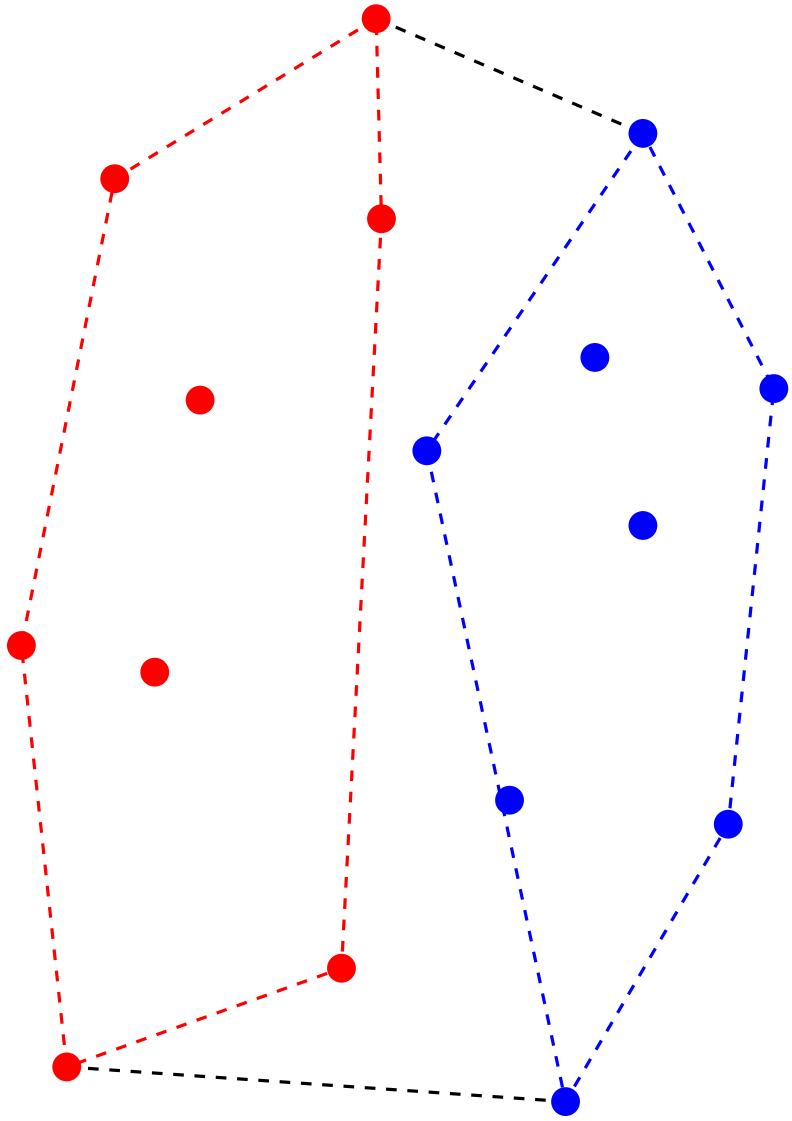

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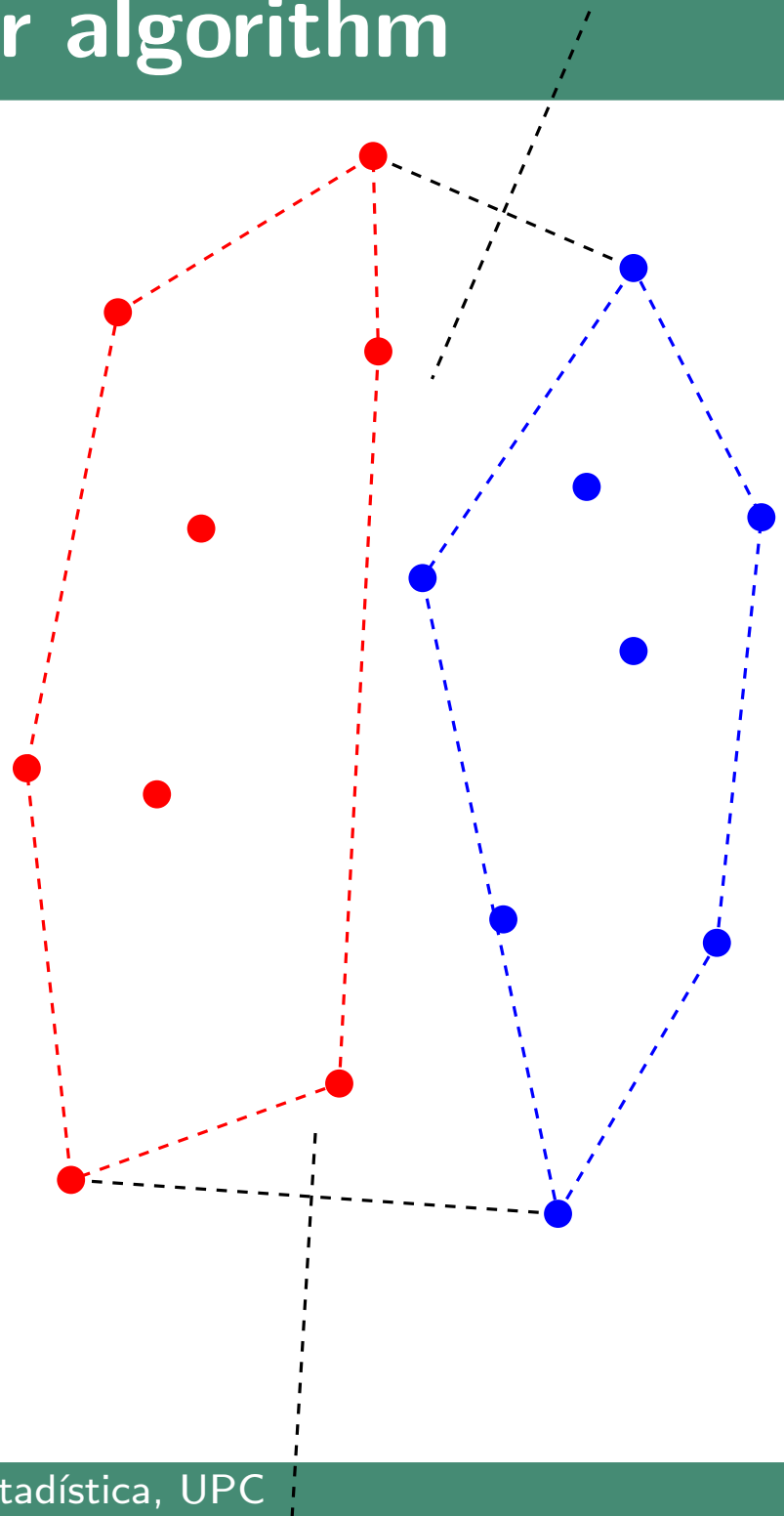

Divide and conquer algorithm

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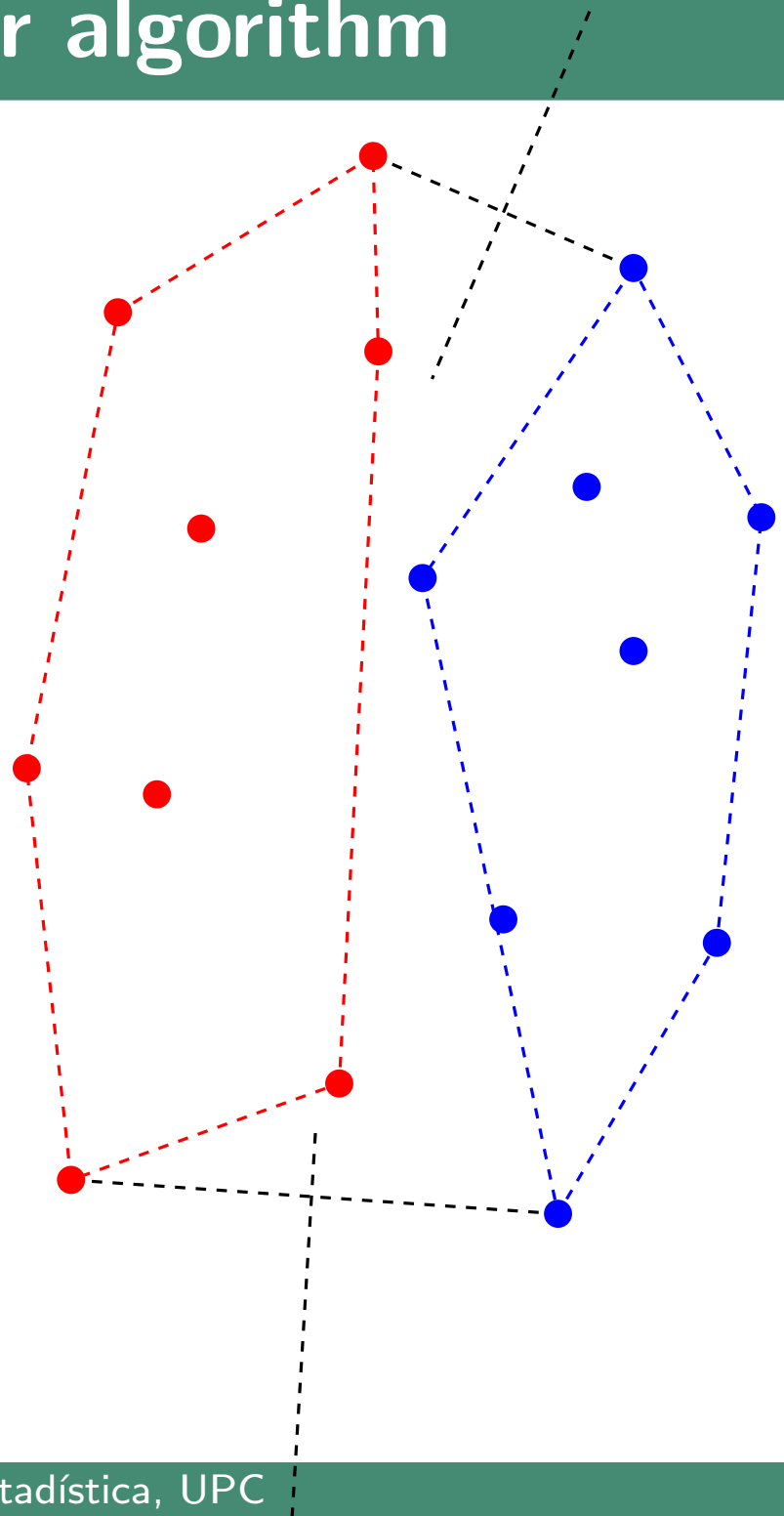

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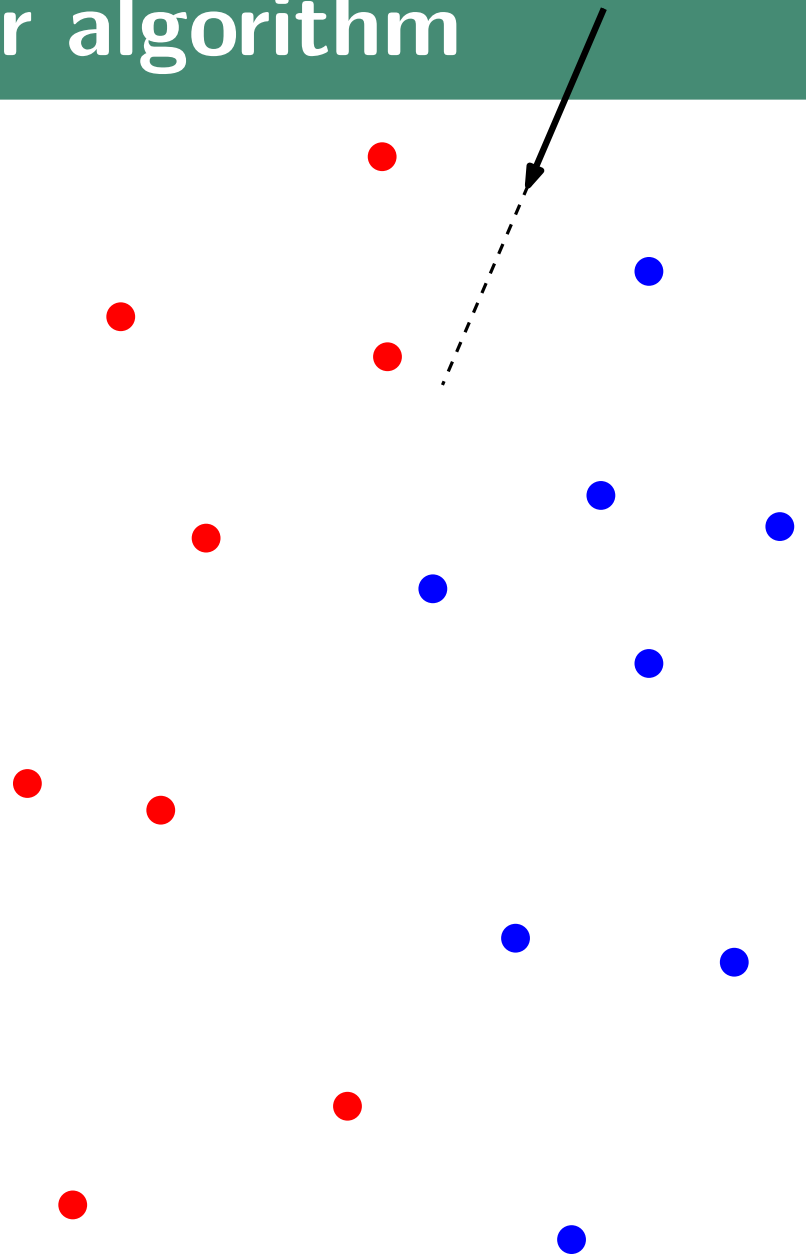

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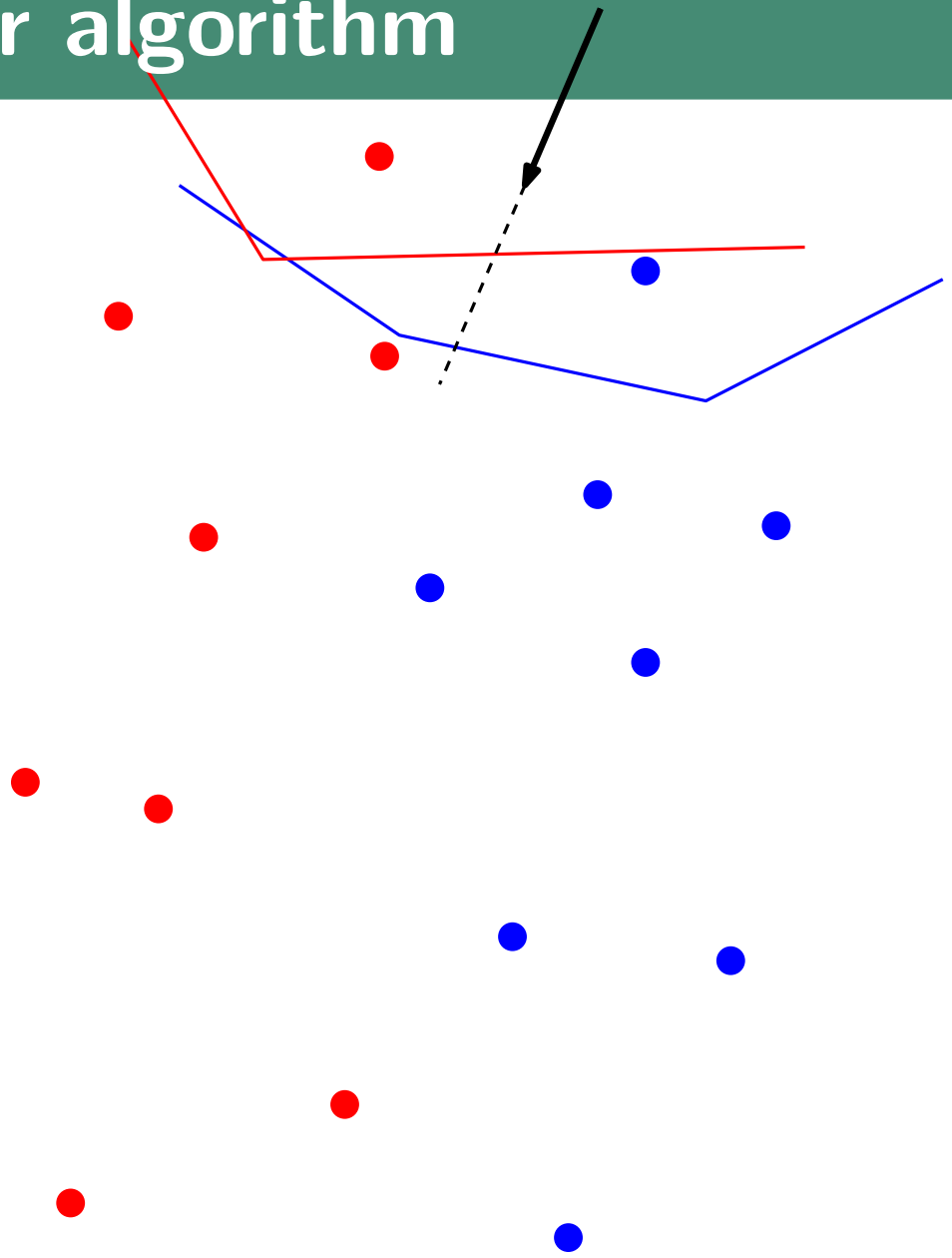

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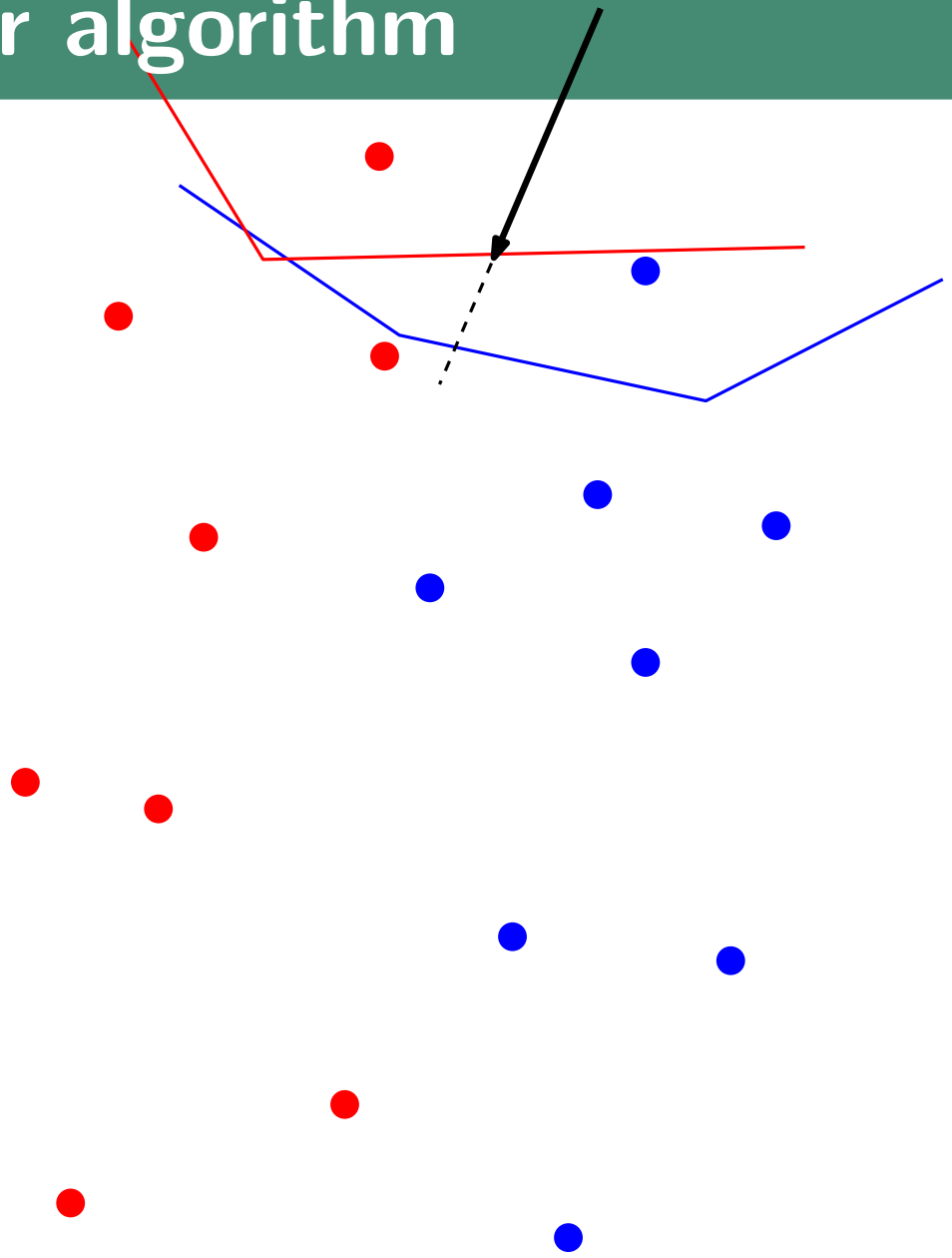

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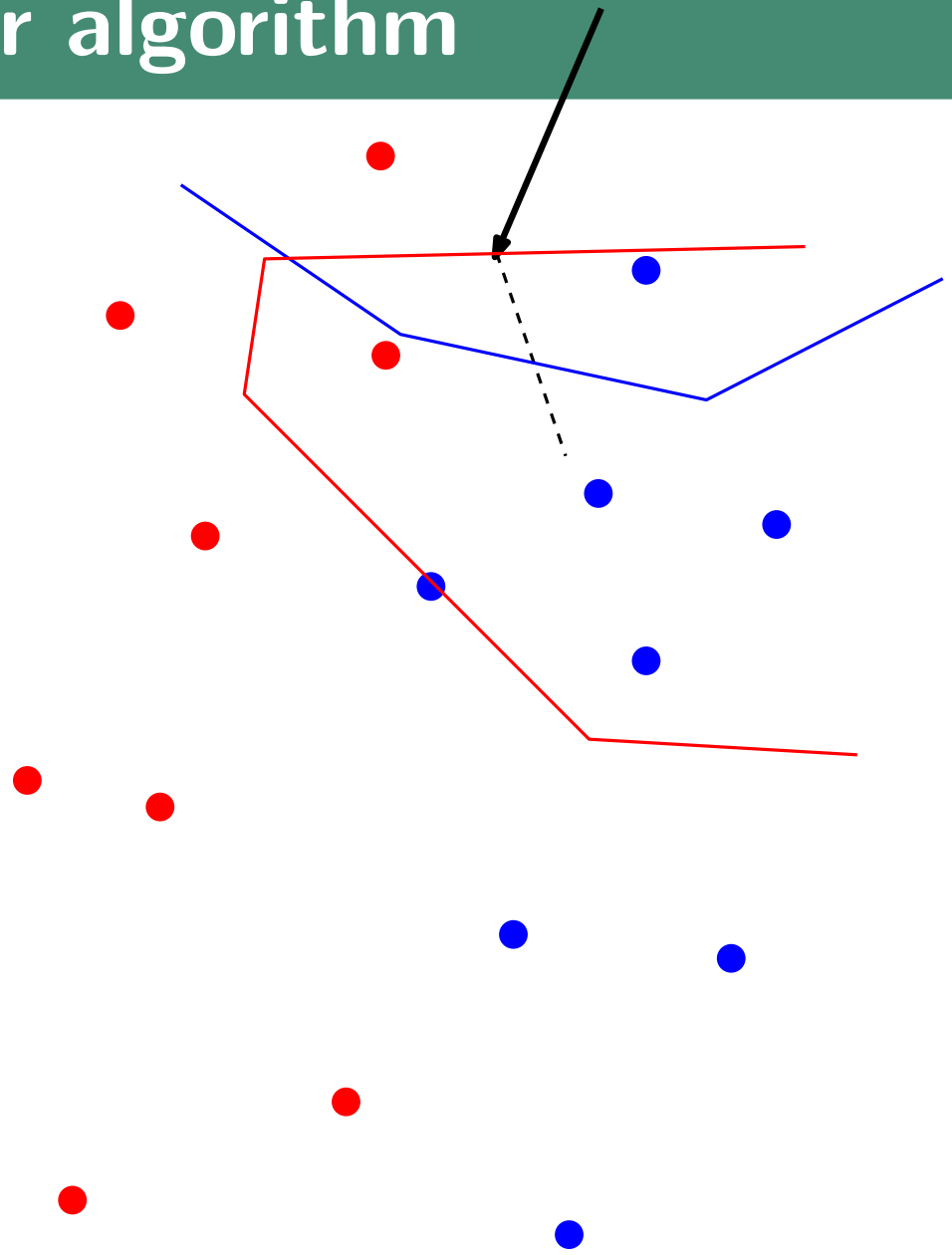

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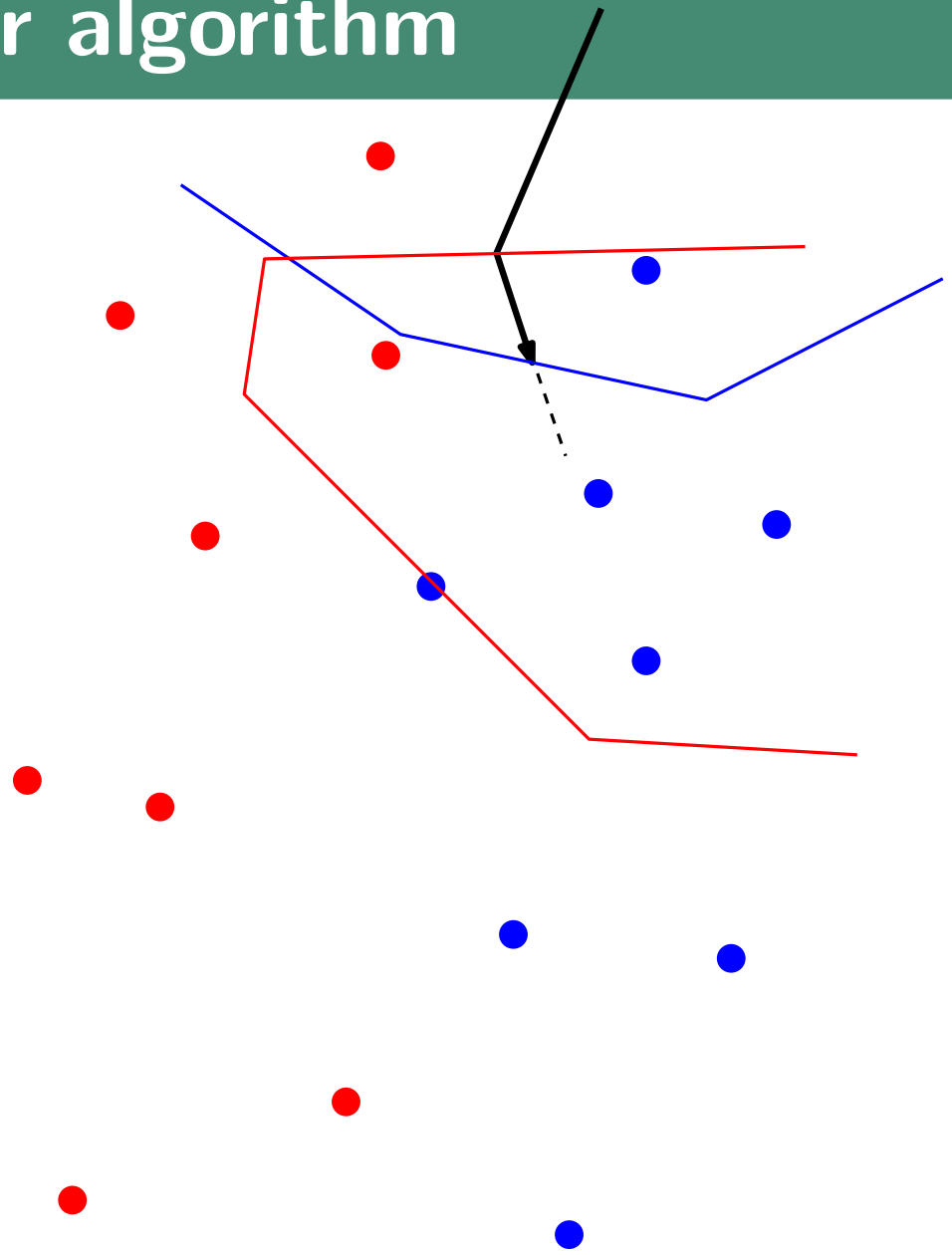

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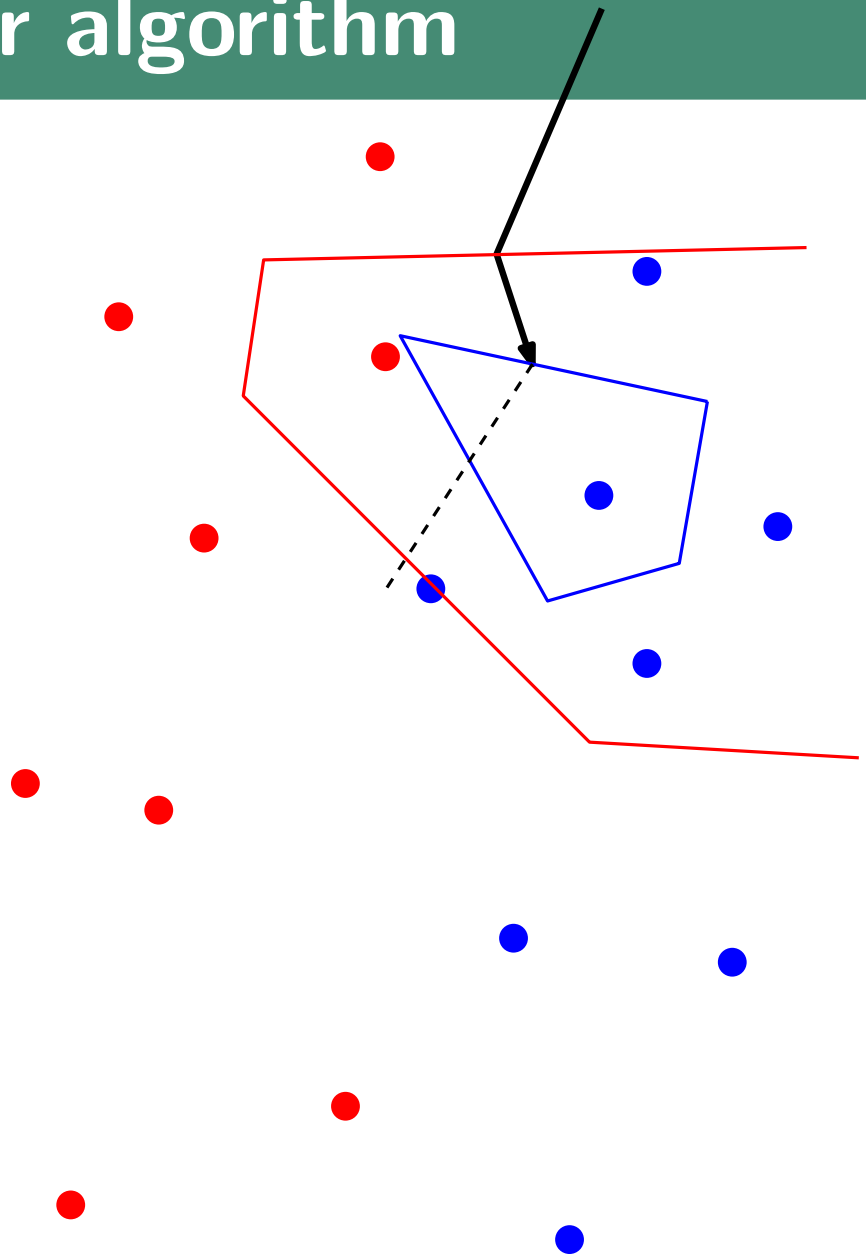

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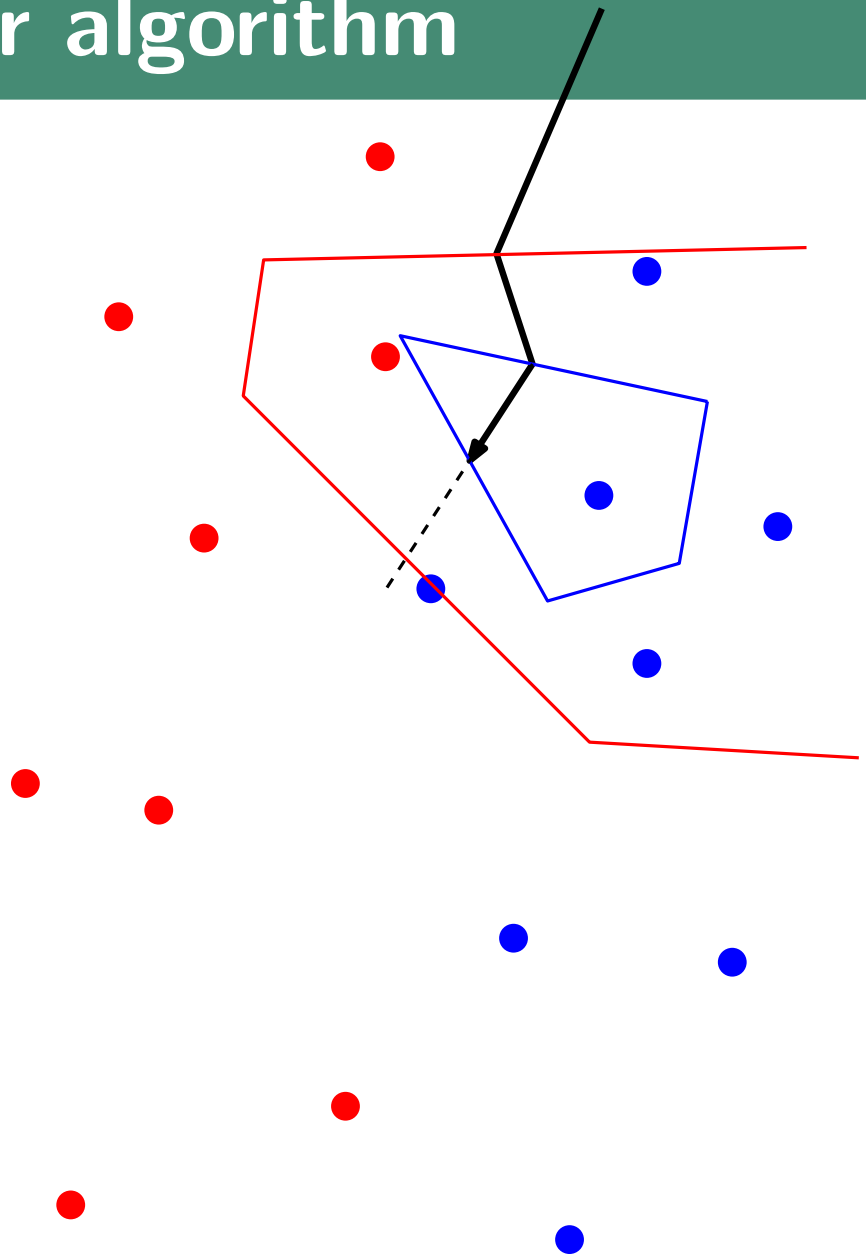

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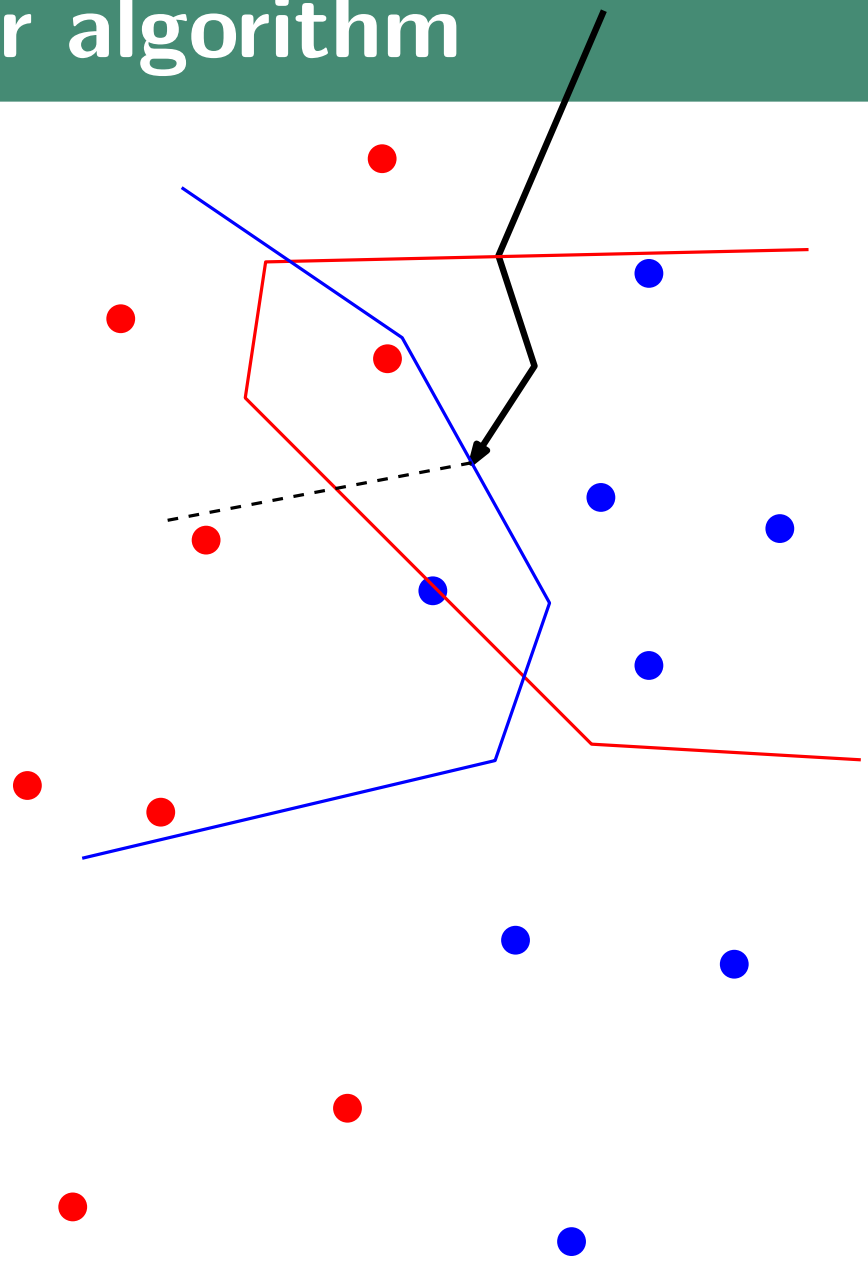

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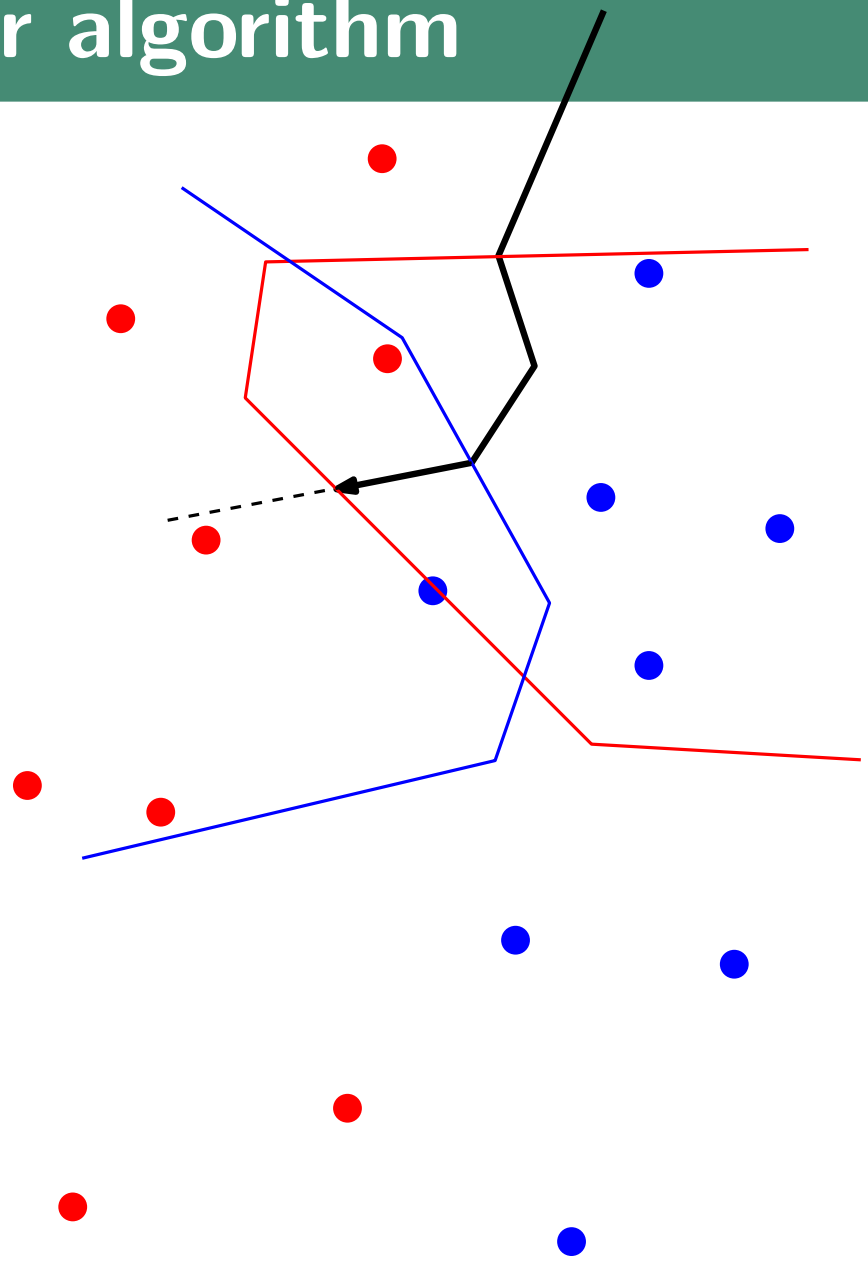

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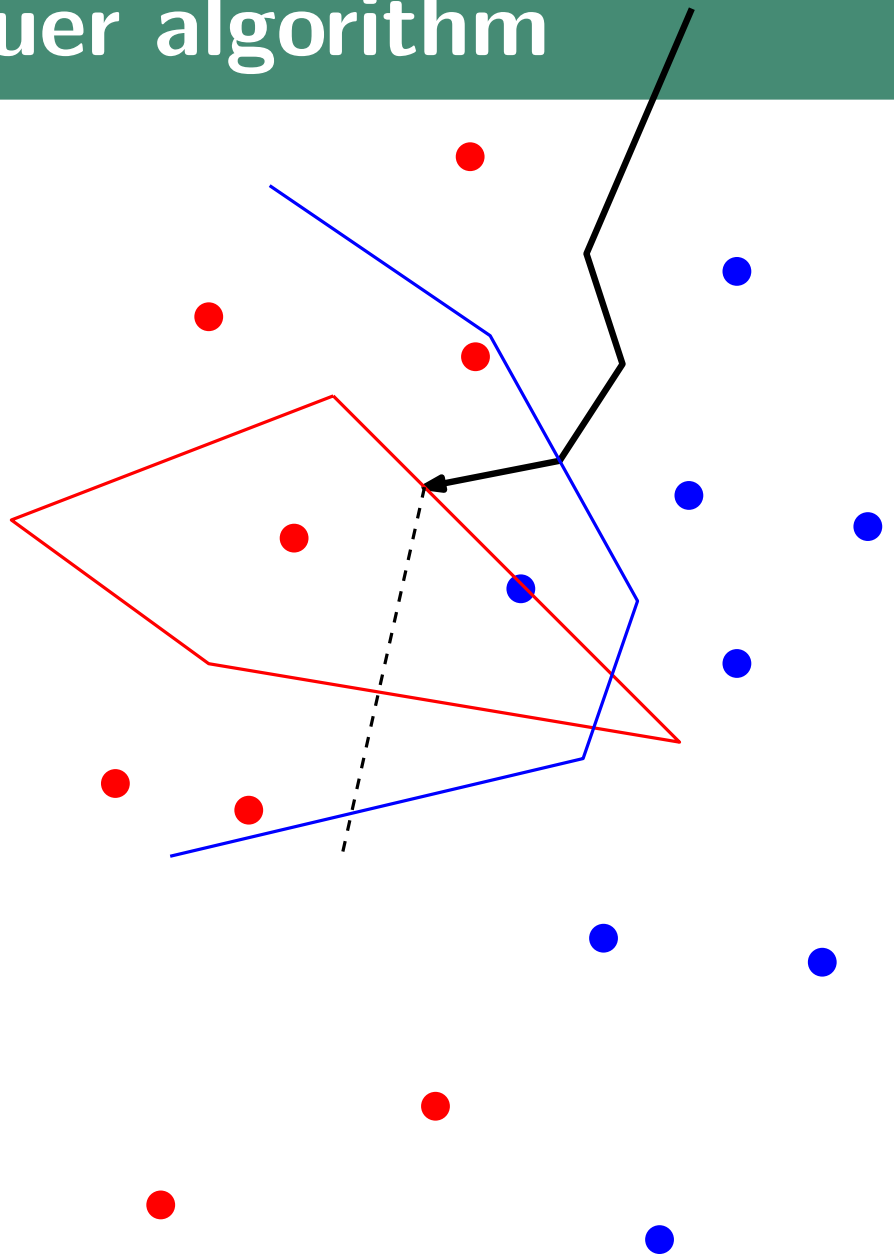

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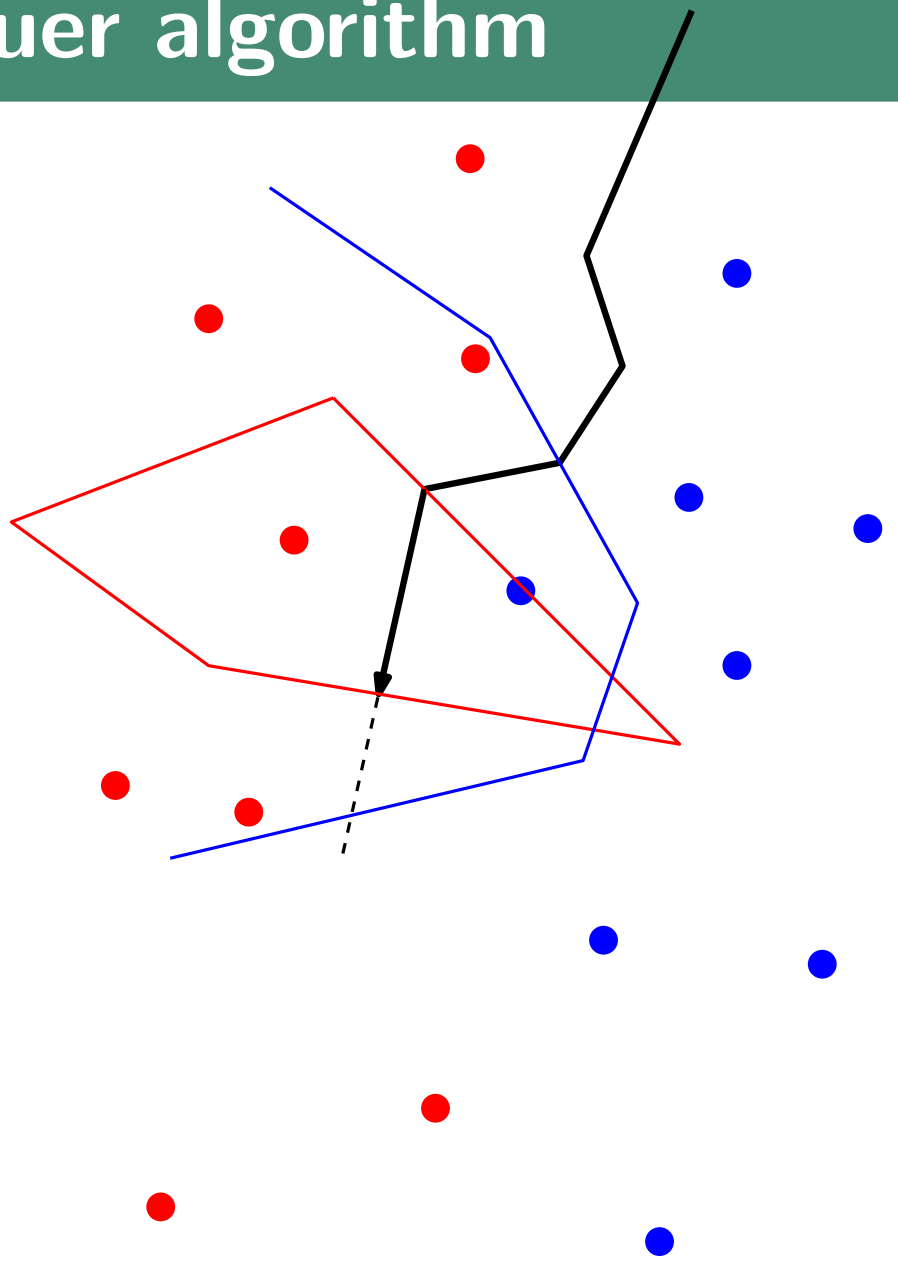

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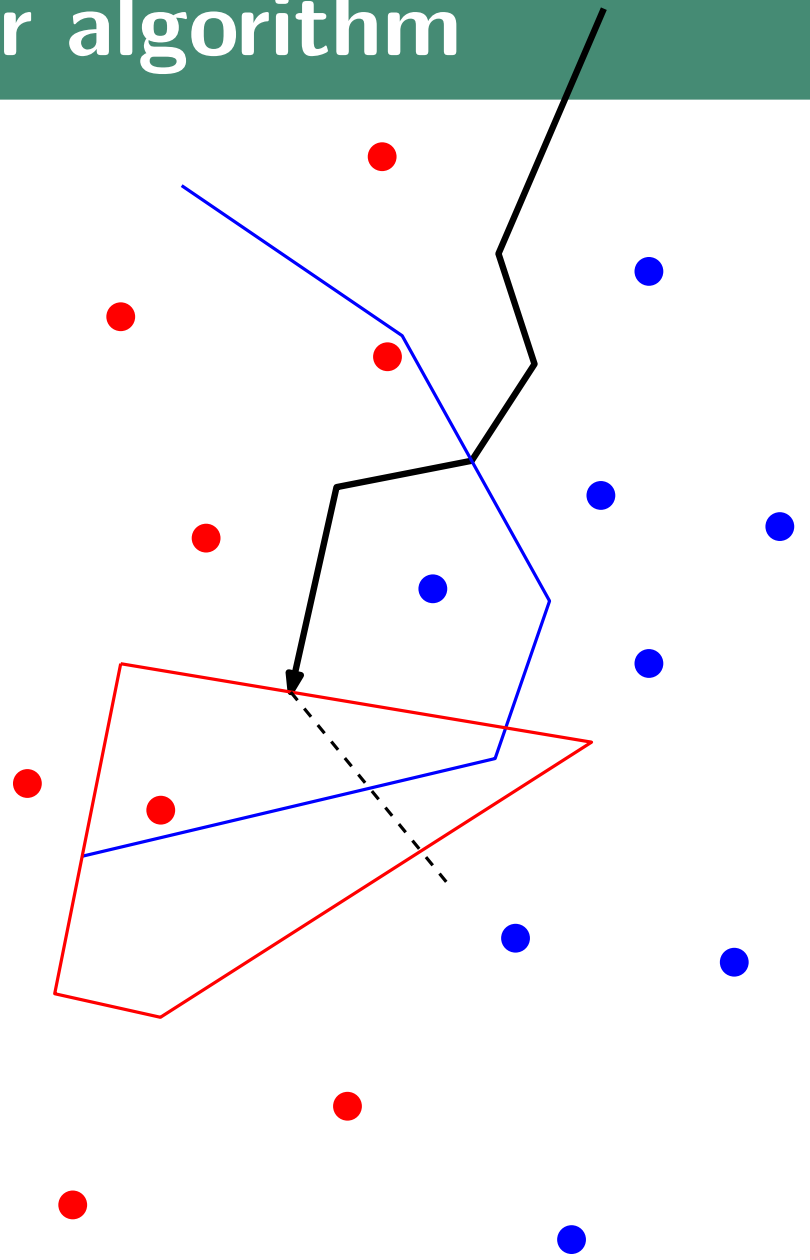

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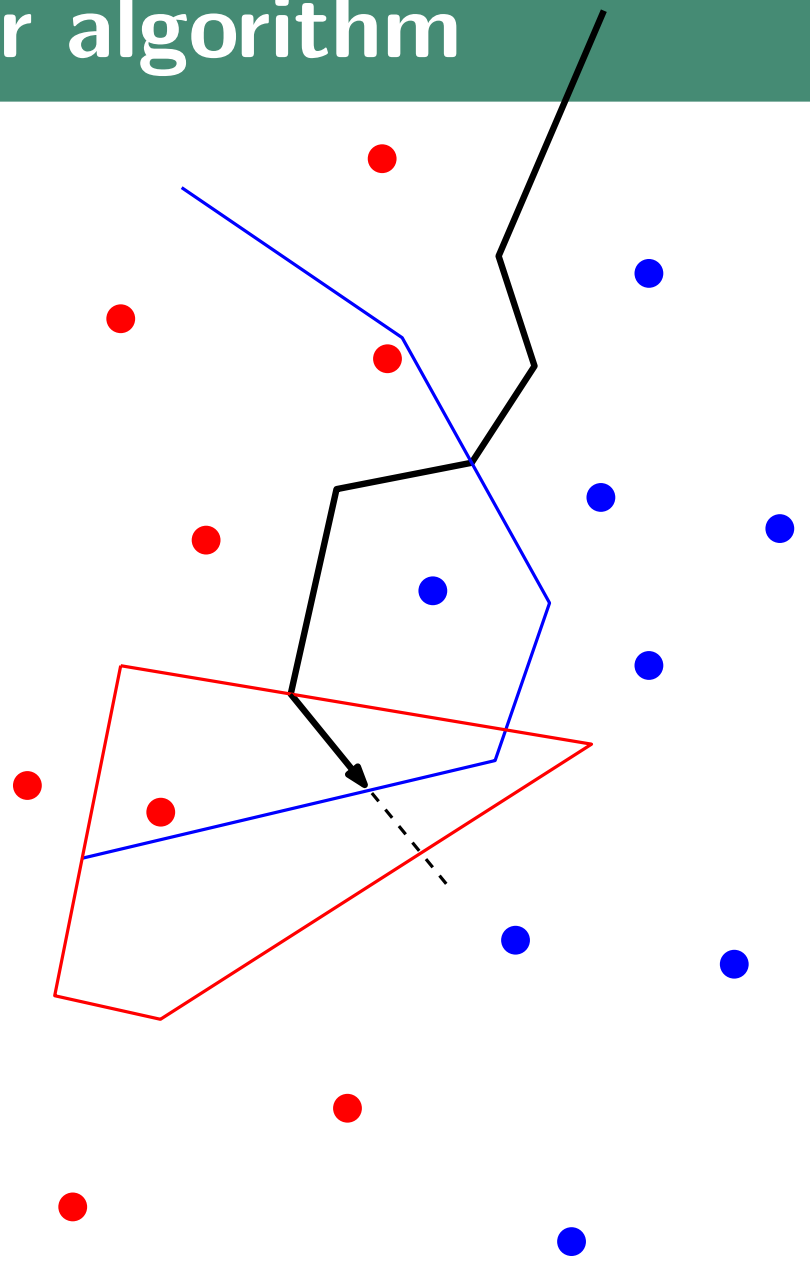

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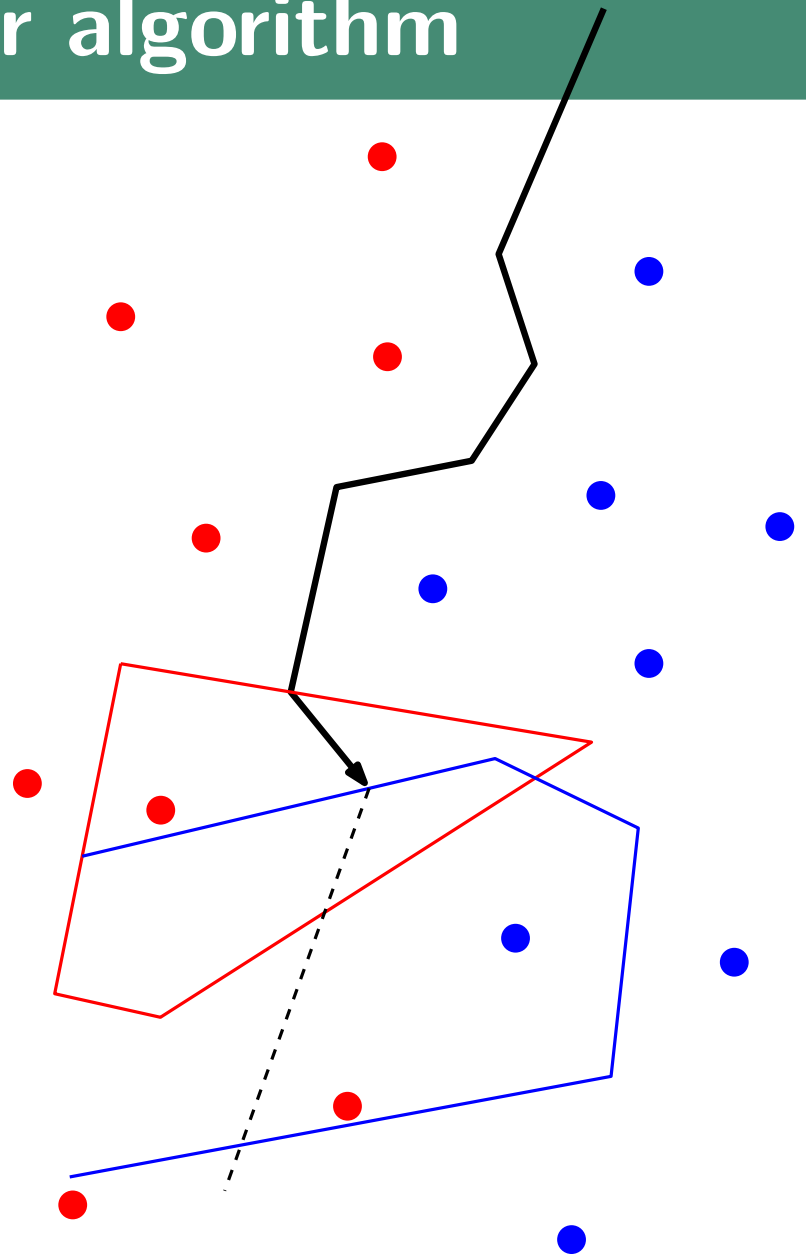

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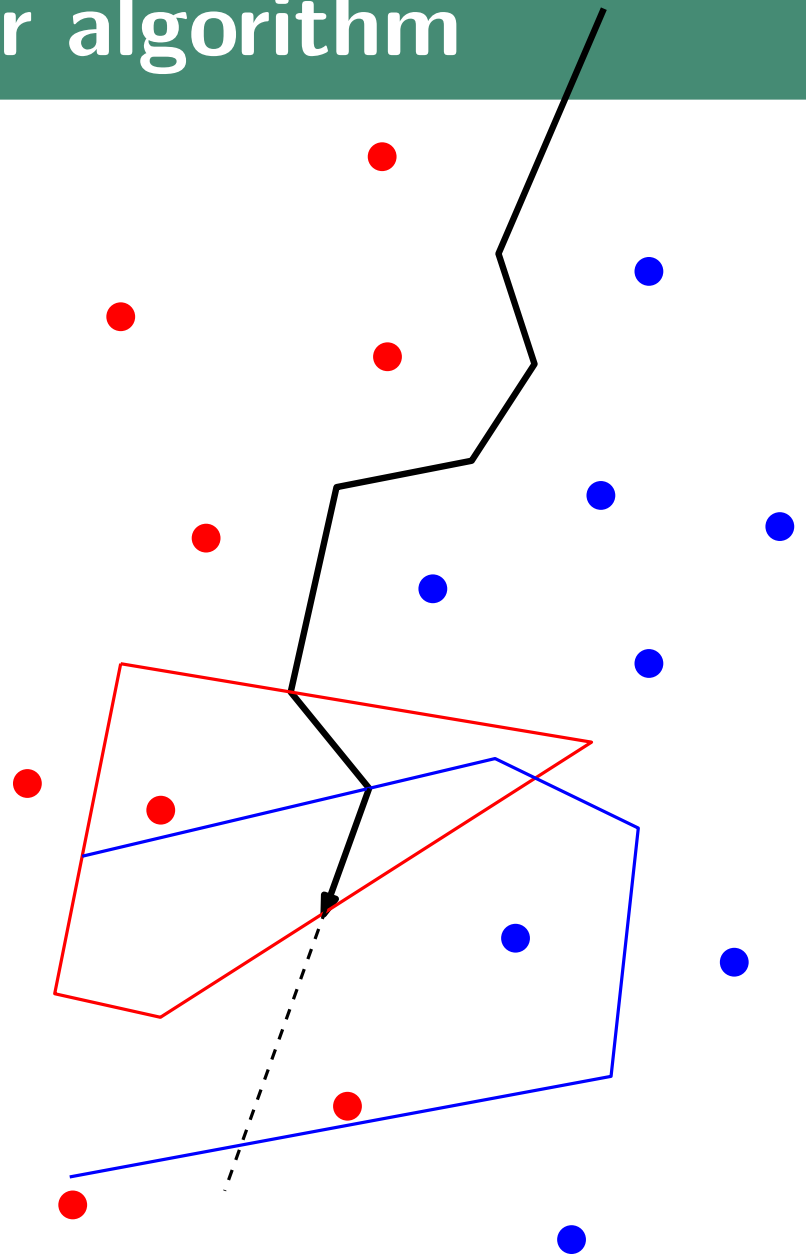

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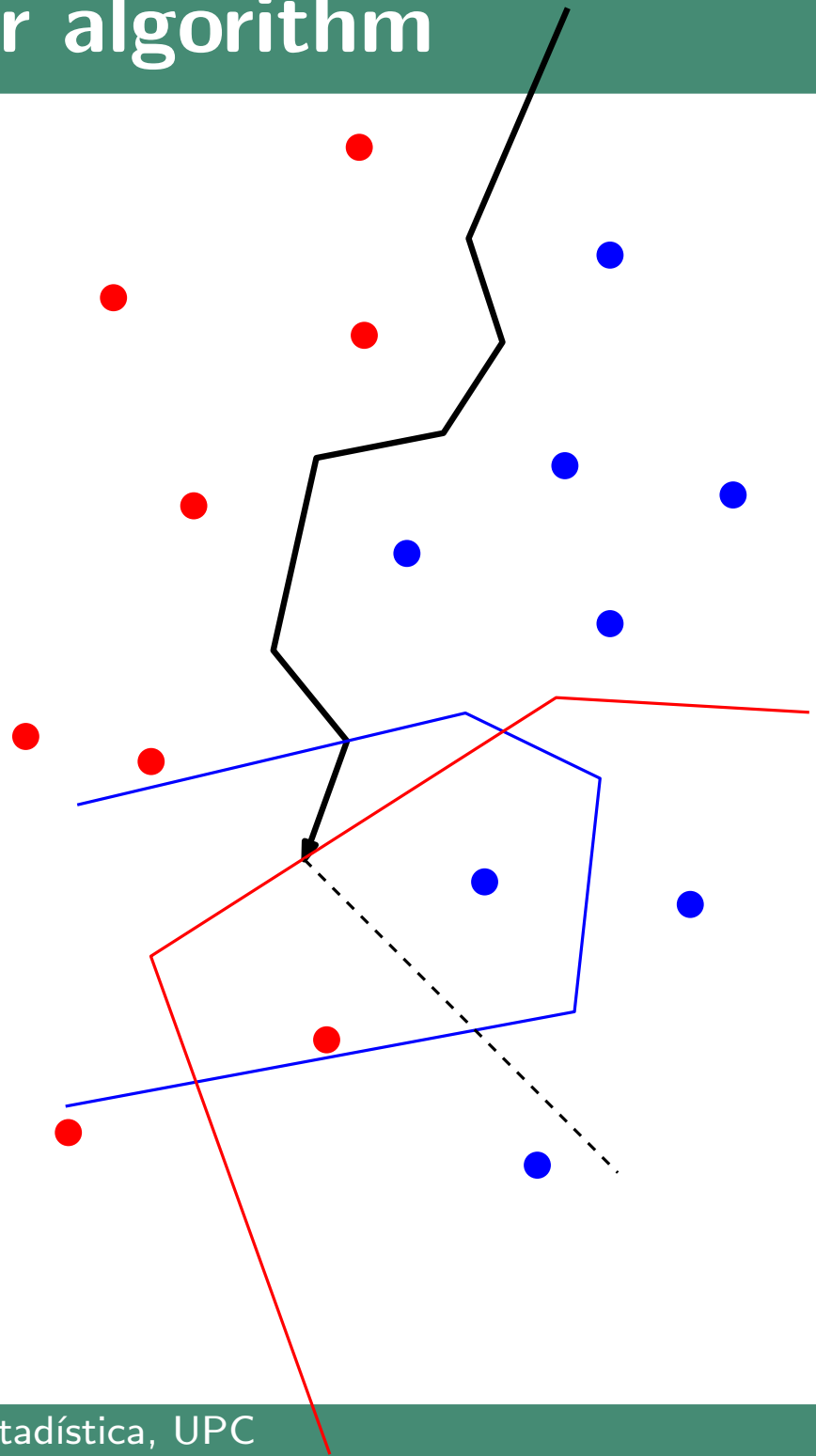

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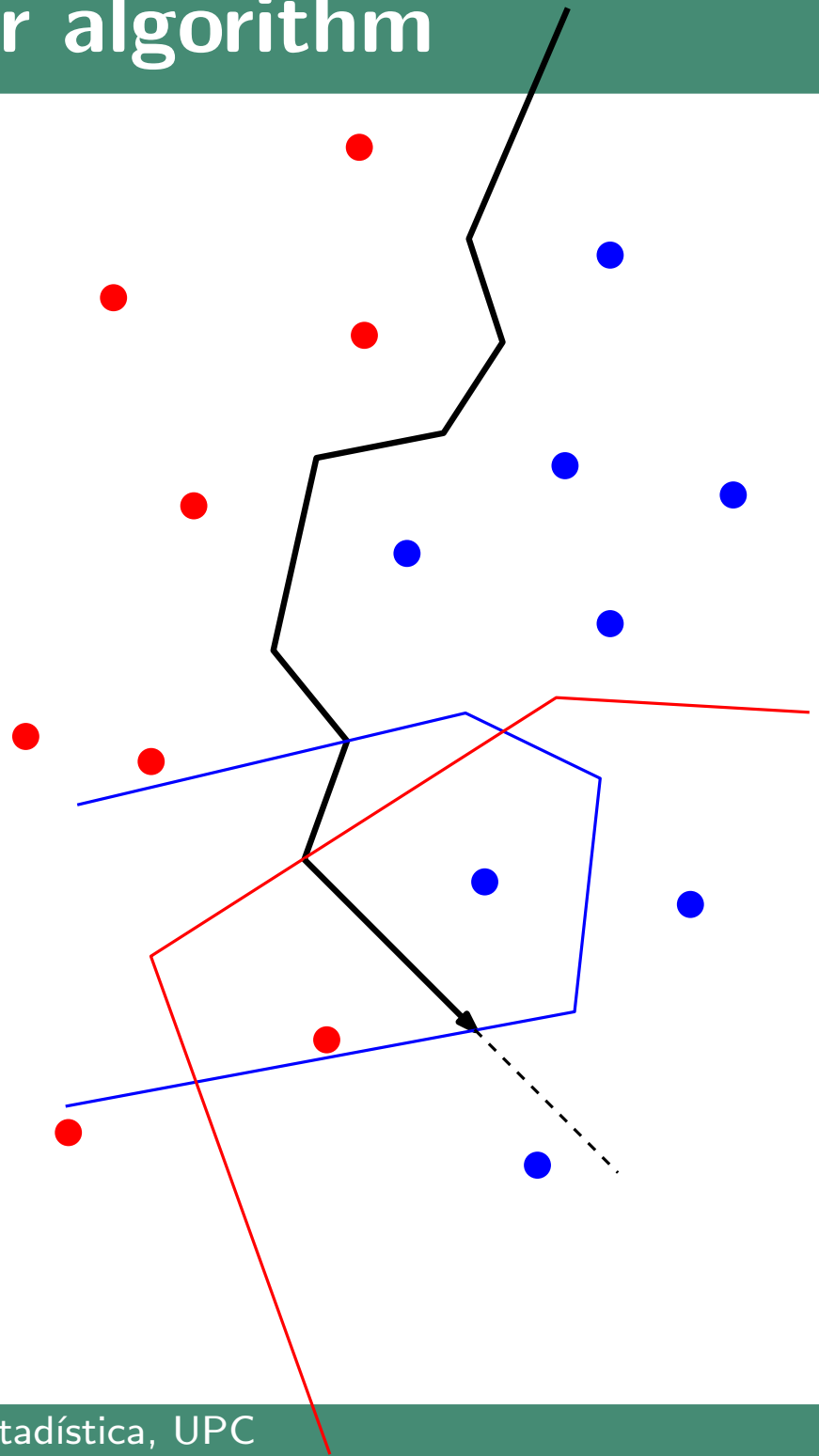

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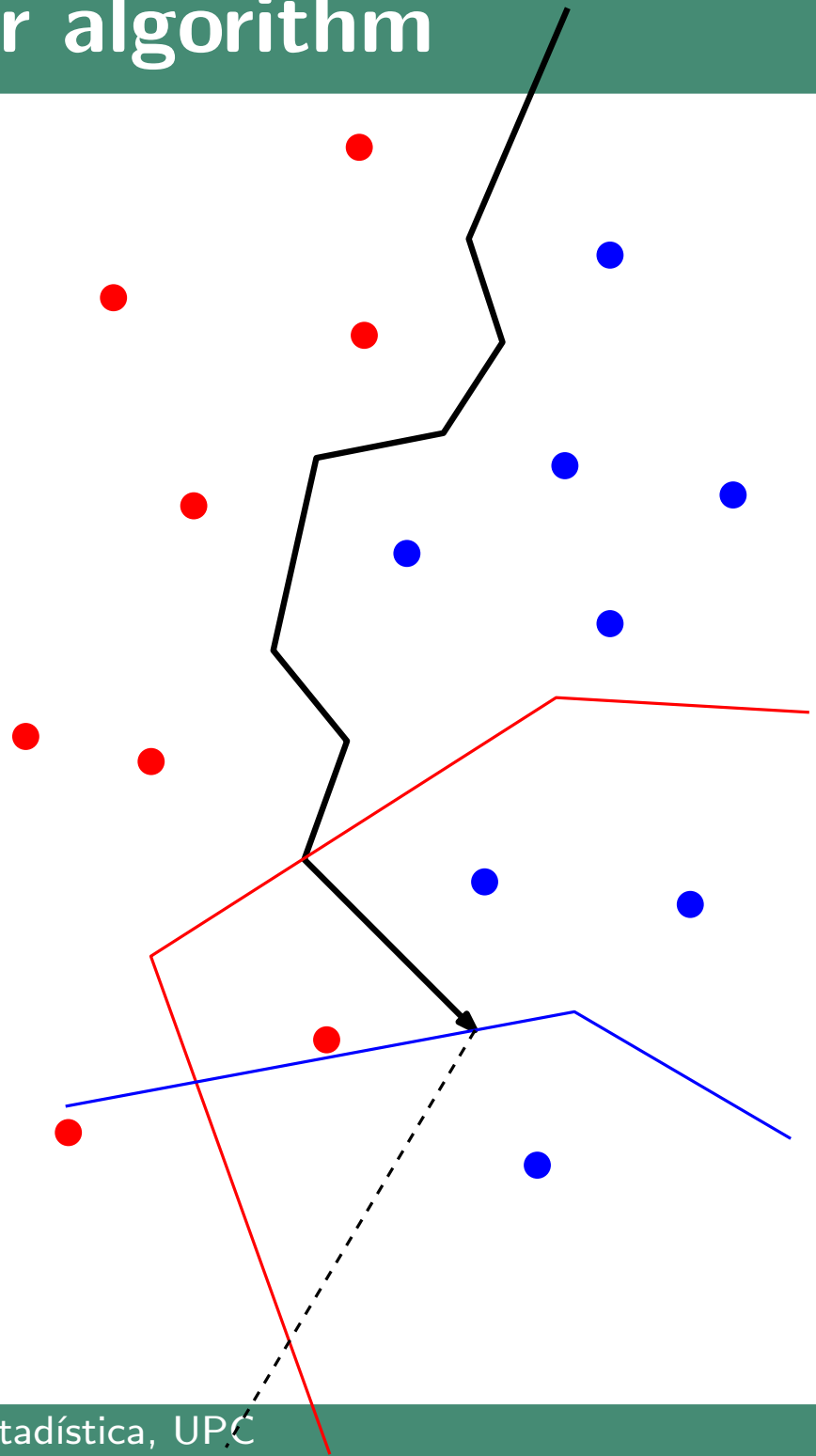

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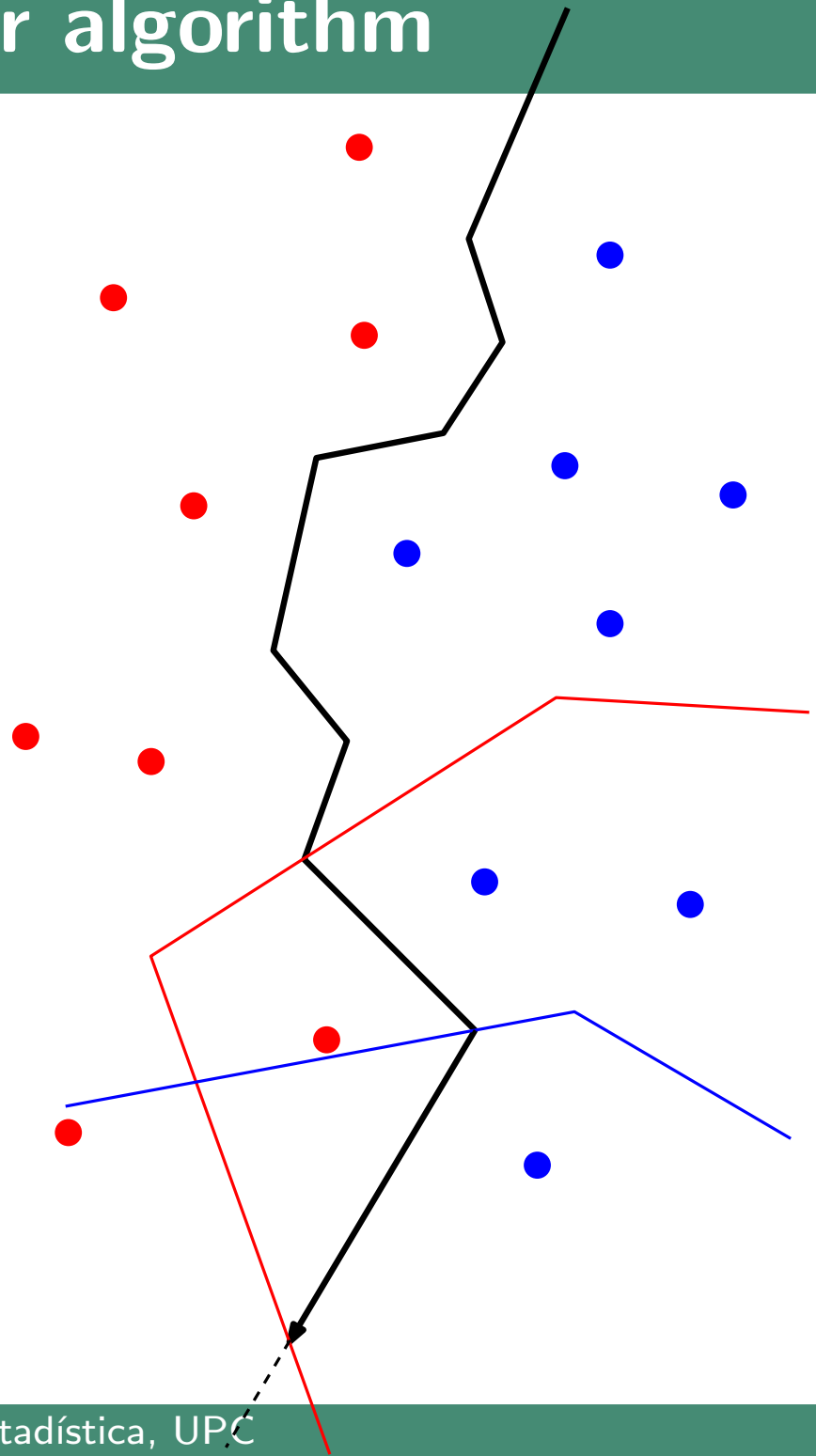

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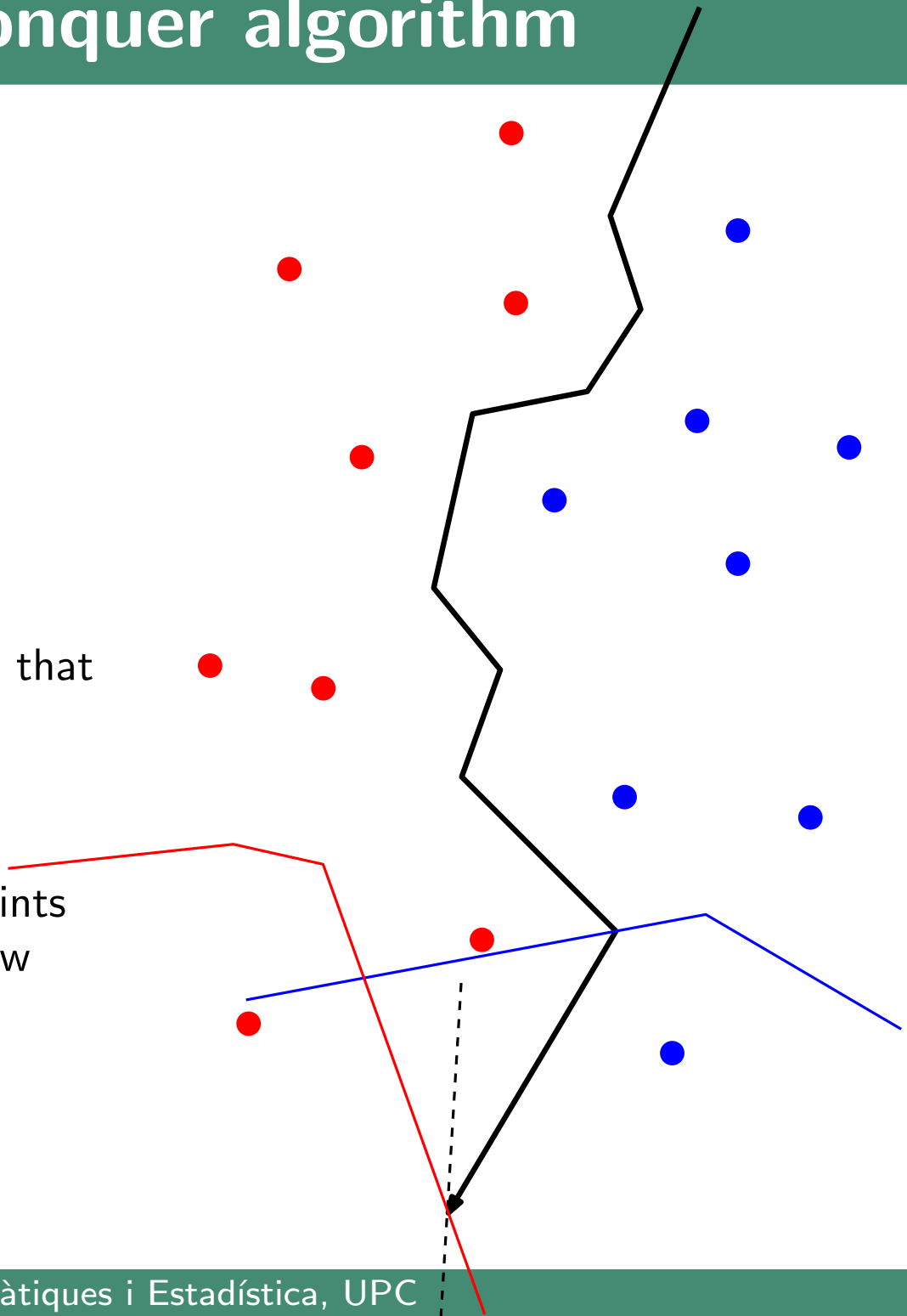

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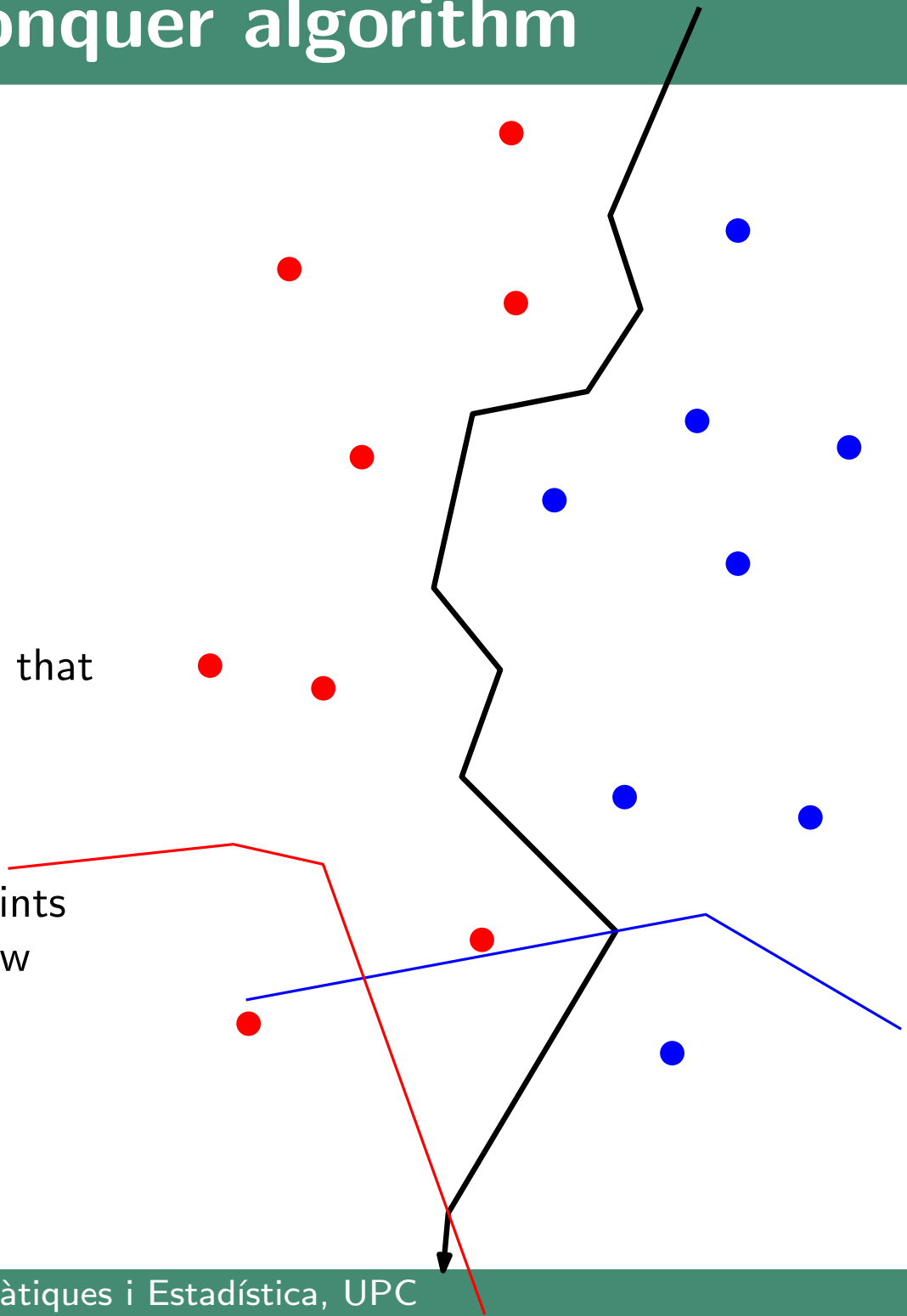

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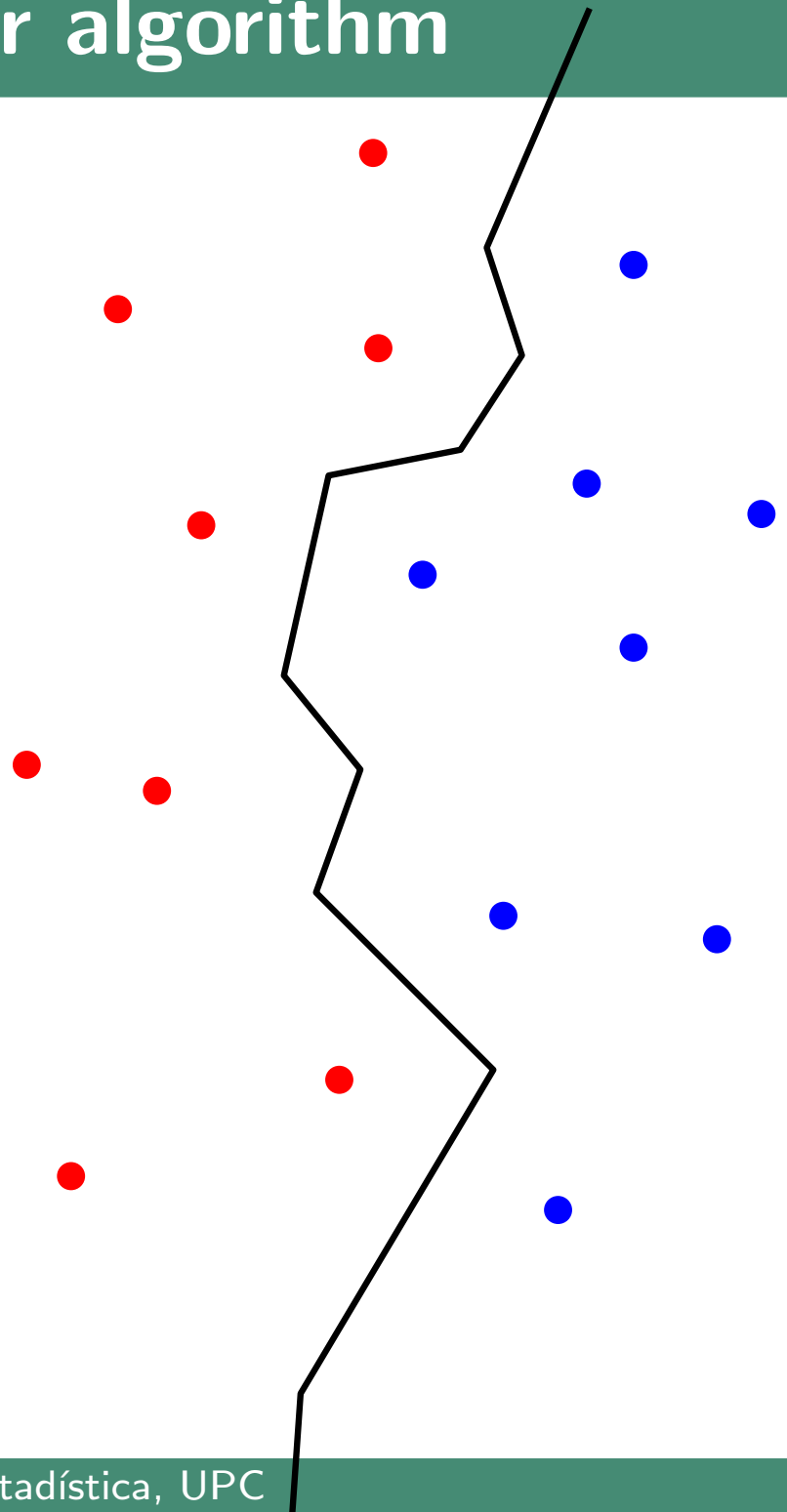
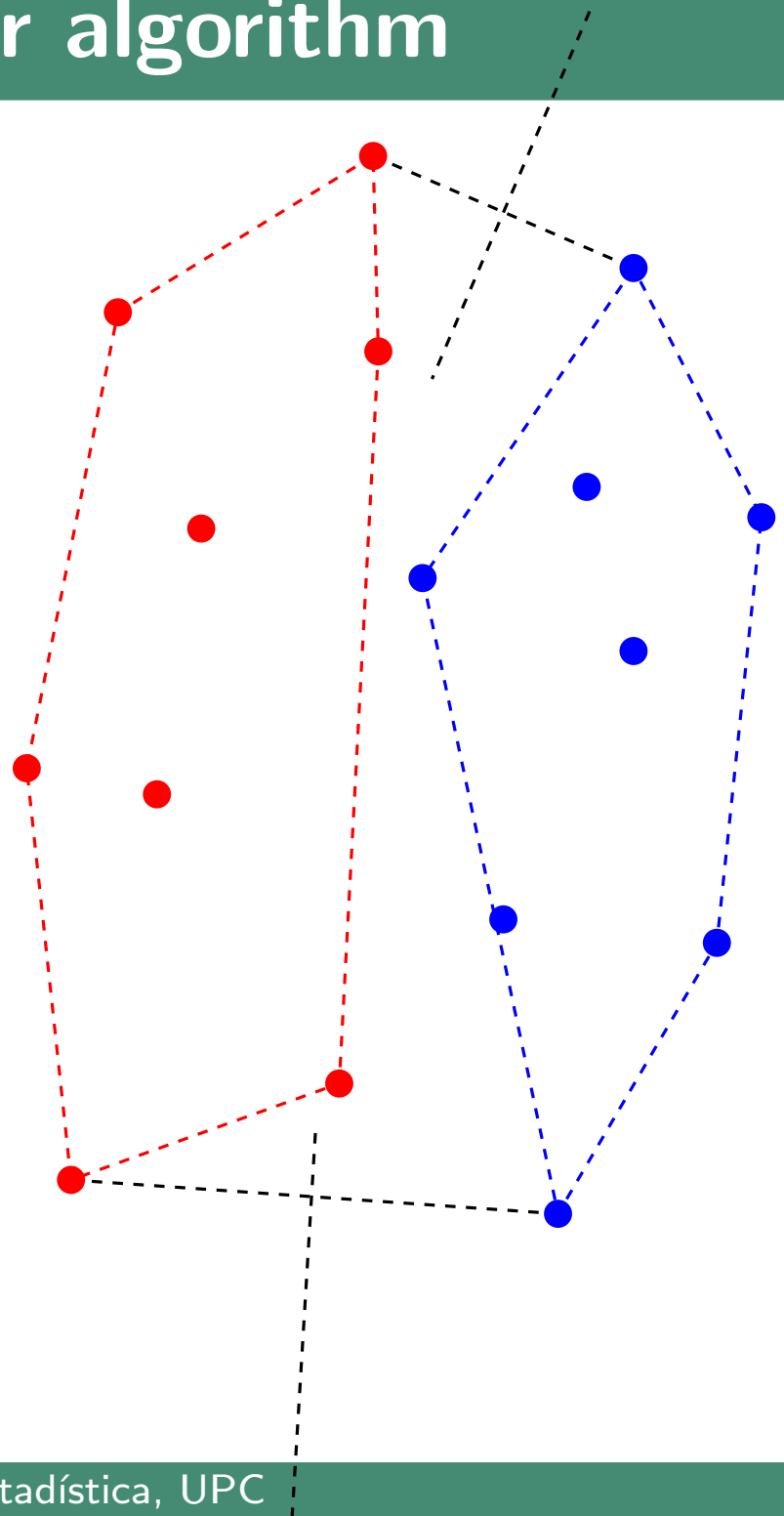


Divide and conquer algorithm

How to compute the chain?

Initialization running time: $O(n)$

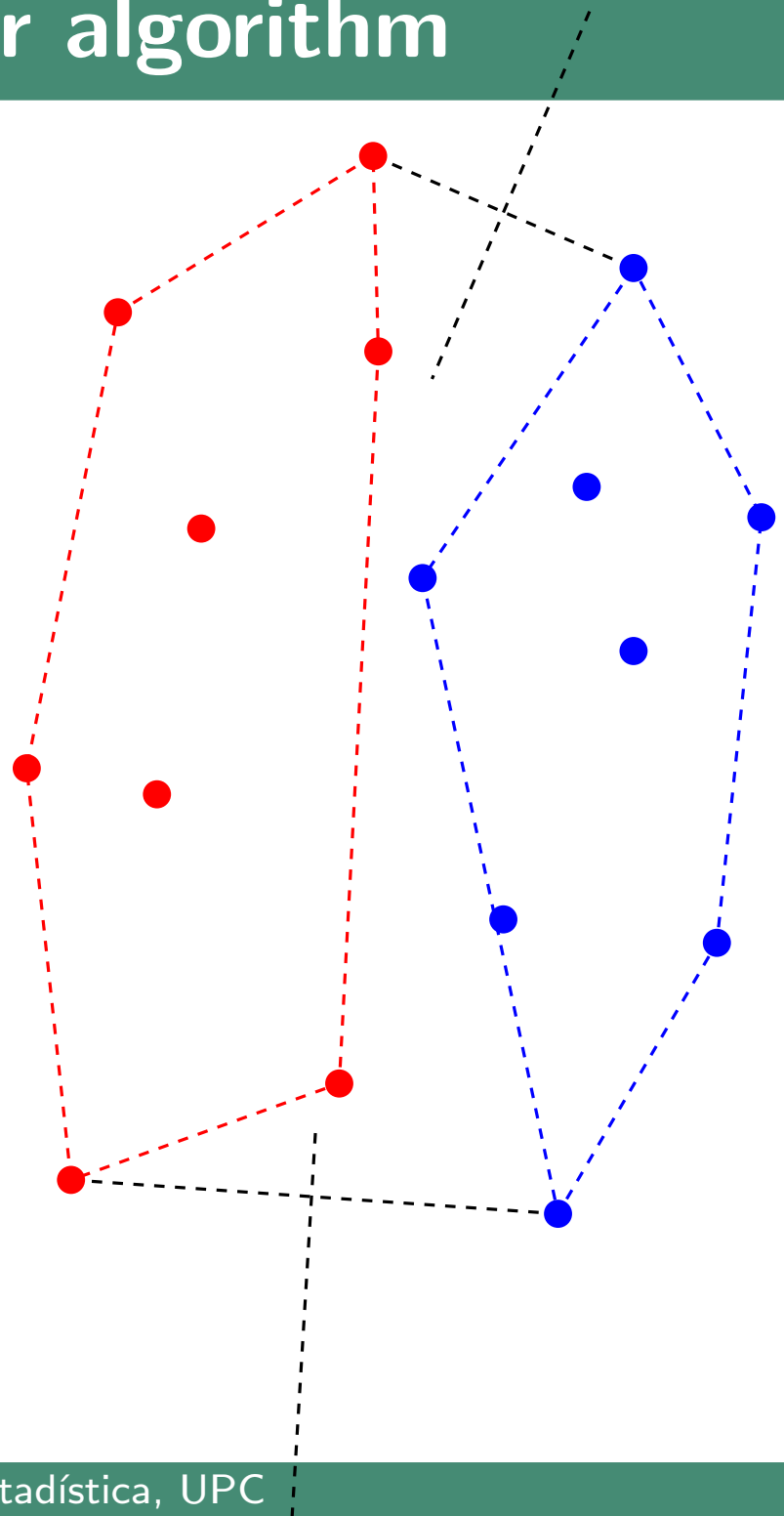


Divide and conquer algorithm

How to compute the chain?

Initialization running time: $O(n)$

From $Vor(R)$ and $Vor(B)$.



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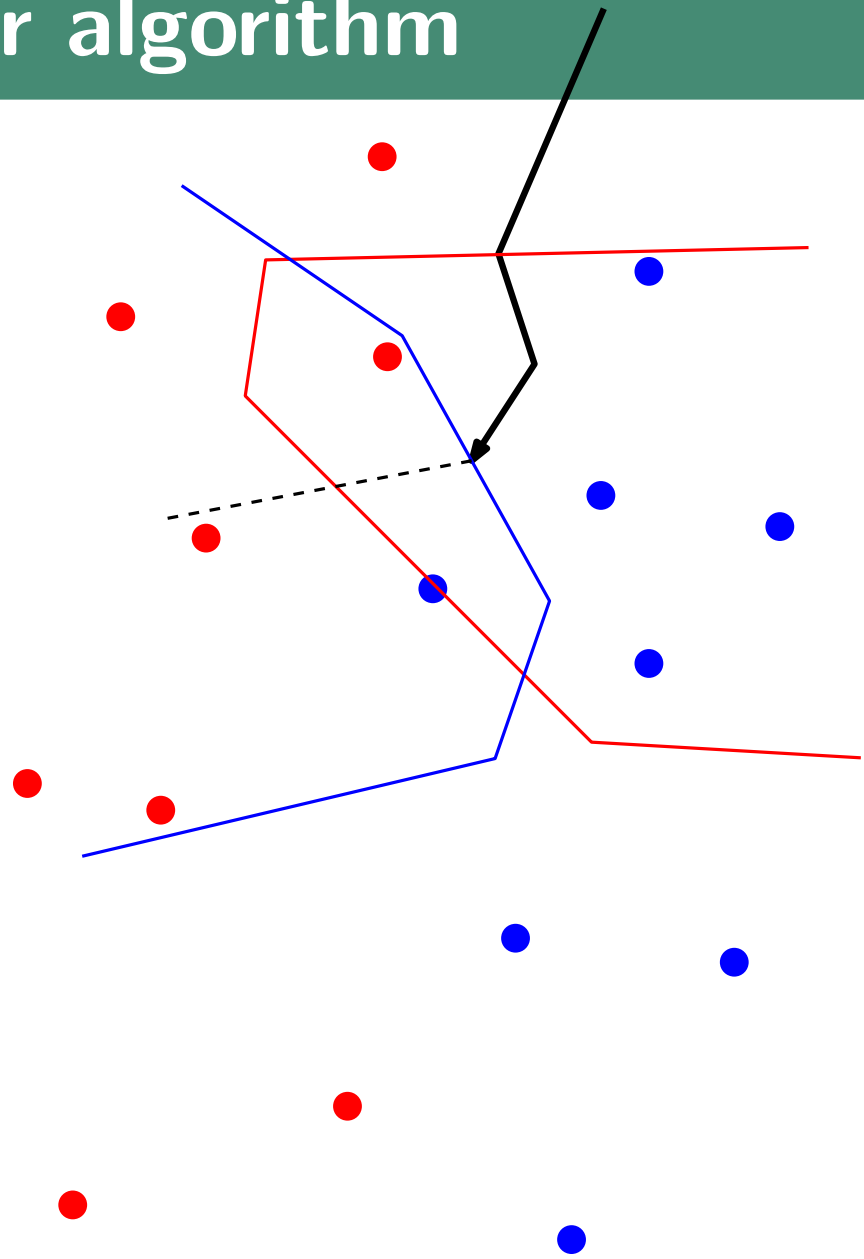

Divide and conquer algorithm

How to compute the chain?

Initialization running time: $O(n)$

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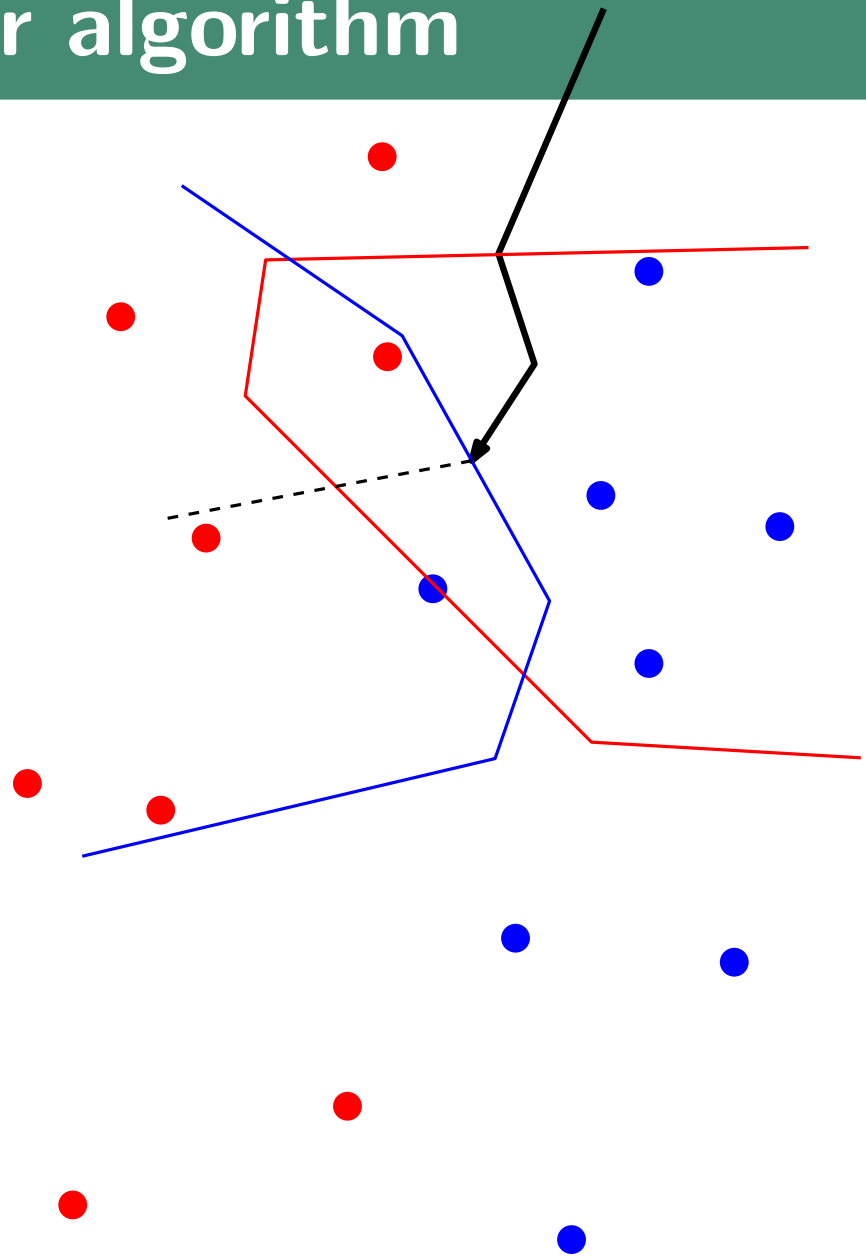
If e is an edge of $b(R, B)$ that entered $Vor_R(p_i)$ through some vertex $v \in Vor(P)$, then the exit point of $b(R, B)$ is found clockwise along the boundary of $Vor_R(p_i)$.



Divide and conquer algorithm

How to do the merging?

Divide and conquer algorithm

How to do the merging?

It consists in updating the DCEL:

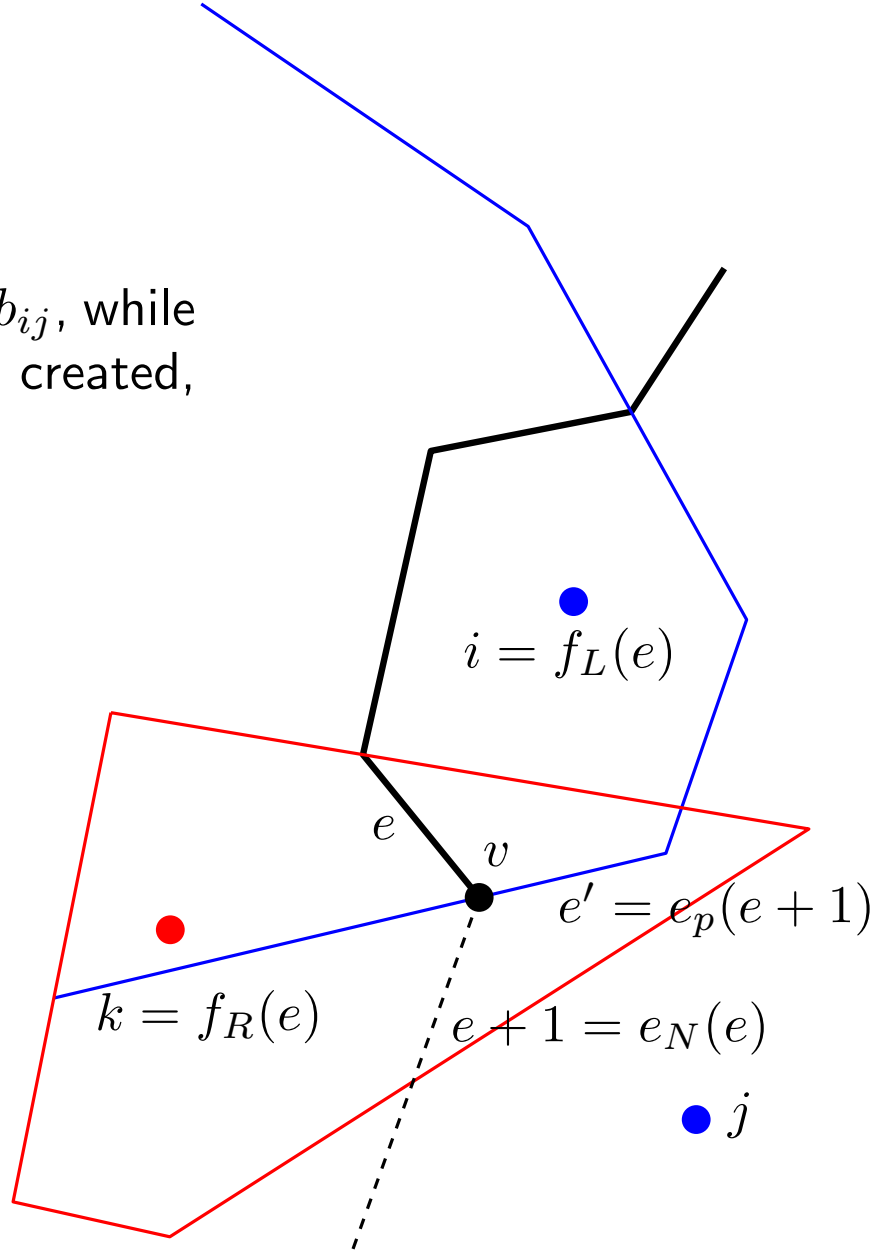
Divide and conquer algorithm

How to do the merging?

It consists in updating the DCEL:

Each time a face $Vor_B(p_i)$ is left through an edge $e' \in b_{ij}$, while staying in the same face $Vor_R(p_k)$, a new vertex v is created, an edge e ends and another edge $e+1$ begins:

- Create the new vertex v and assign $e(v) = e$
- Create $e+1$ and assign to it $v_B = v$ and $e_P = e'$
- Assign to e: $v_E = v$, $e_N = e+1$, $f_L = i$ and $f_R = k$
- Modify for e': $v_* = v$, $e_* = e$
- Update $e(p_i) = e$
- Delete all edges of $Vor_B(p_i)$ found in counter-clockwise order between the entry and exit points



The procedure is analogous when exiting a face $Vor_R(p_i)$.

Divide and conquer algorithm

1. Preprocess: Sort the points of P by abscissa (only once).

2. Division: Vertically partition P into two subsets R and B, of approximately the same size.

3. Recursion: Recursively compute $Vor(R)$ and $Vor(B)$.

4. Merging:

Compute the separating chain.

Prune the portion of $Vor(R)$ lying to the right of the chain and the portion of $Vor(B)$ lying to its left.

The total running time of the algorithm is $O(n \log n)$

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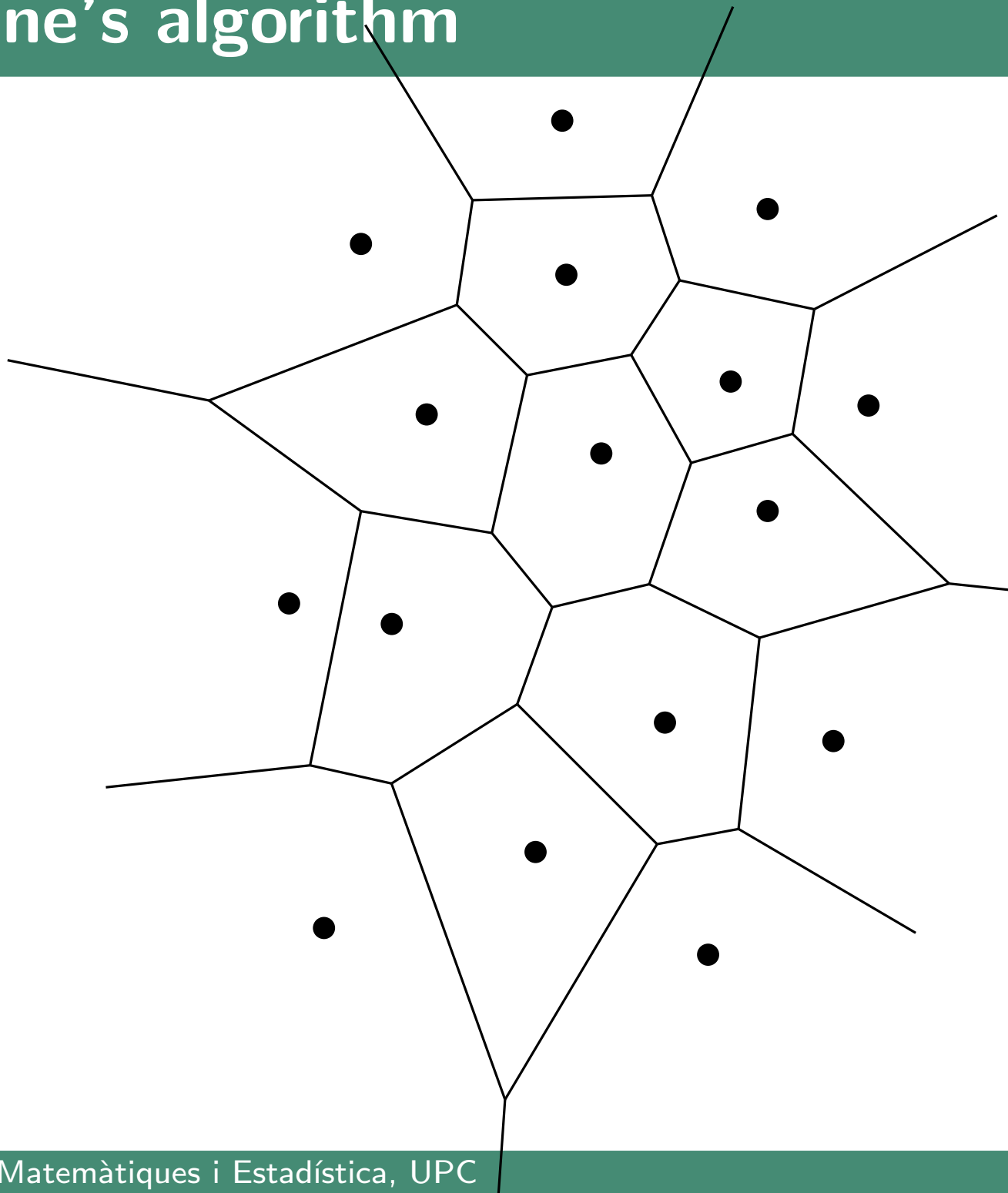
This running time is optimal, because $ch(P)$ can be computed from $Vor(P)$ in $O(n)$ time.

Fortune's algorithm

Fortune's algorithm

How to compute the Voronoi diagram with a sweep line algorithm?



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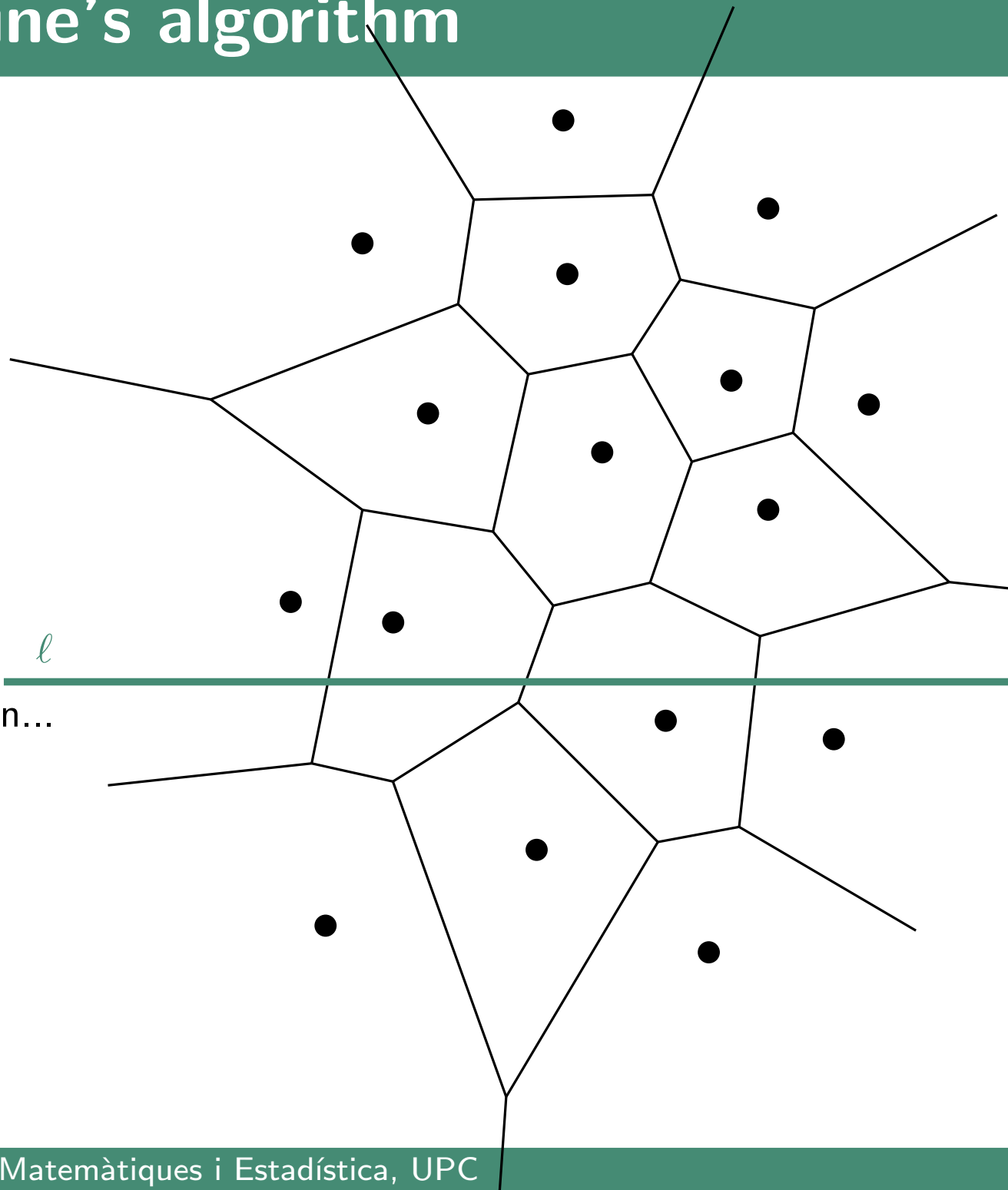

When the sweep line gets to this position...

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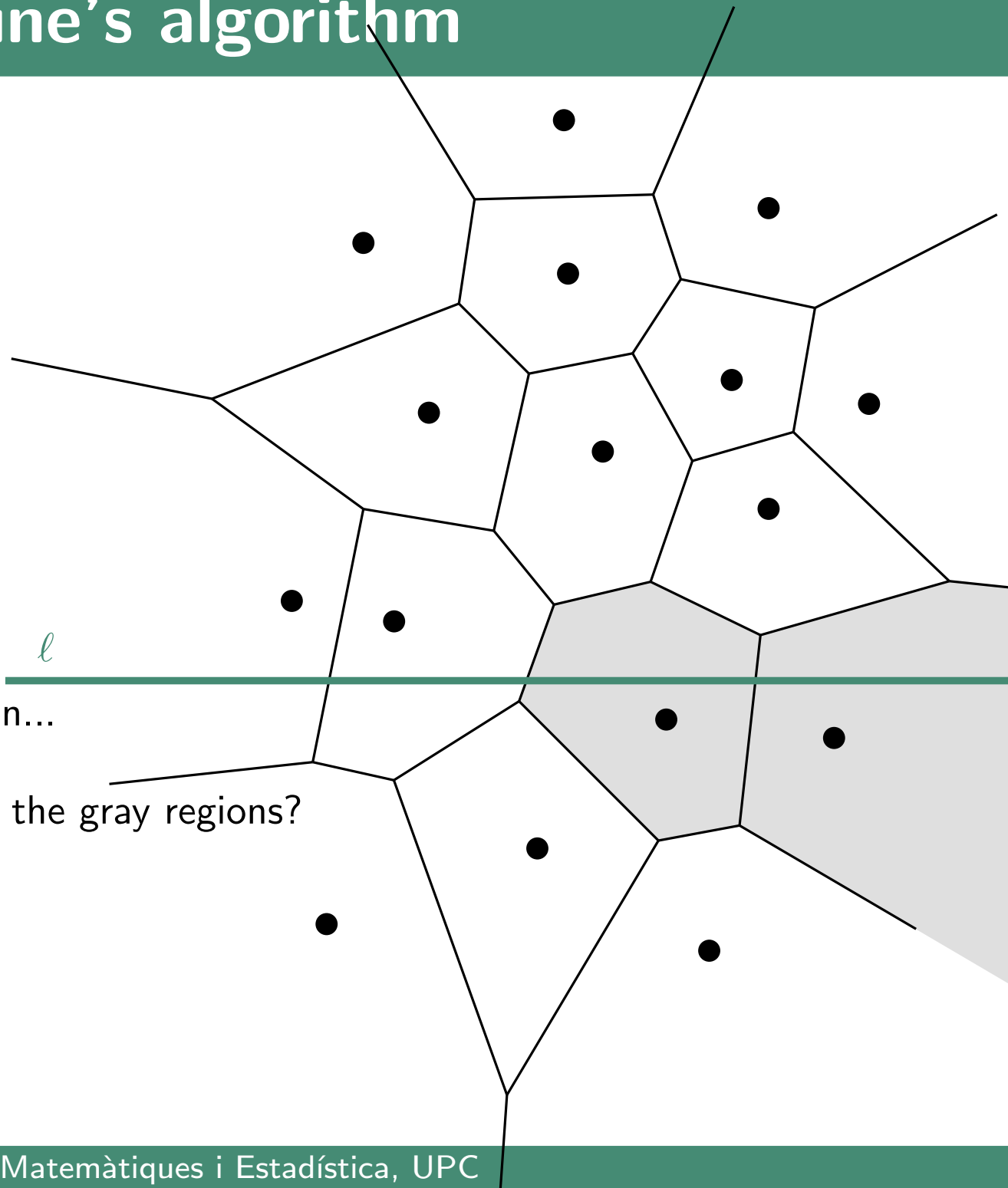

When the sweep line gets to this position...

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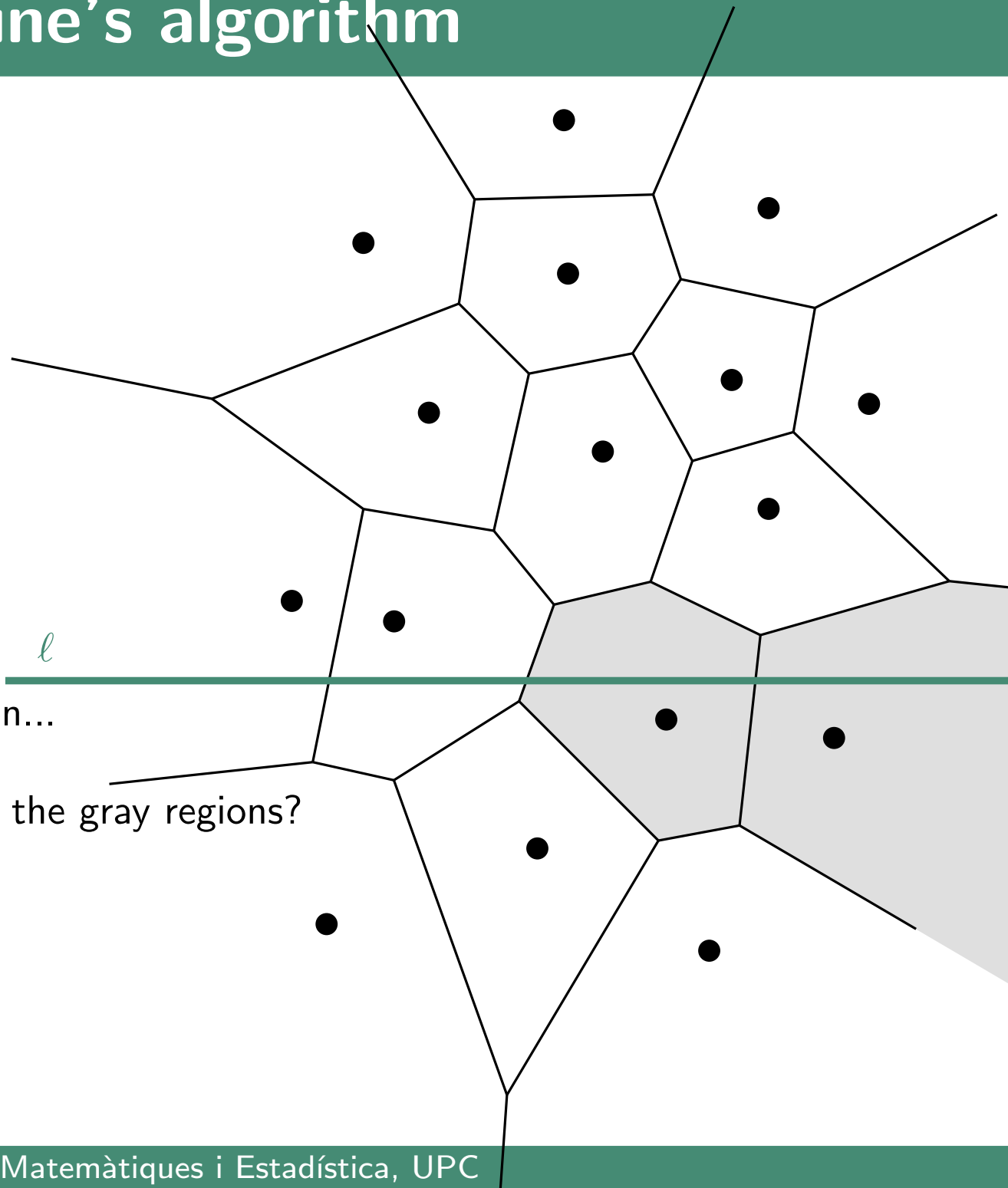

When the sweep line gets to this position...

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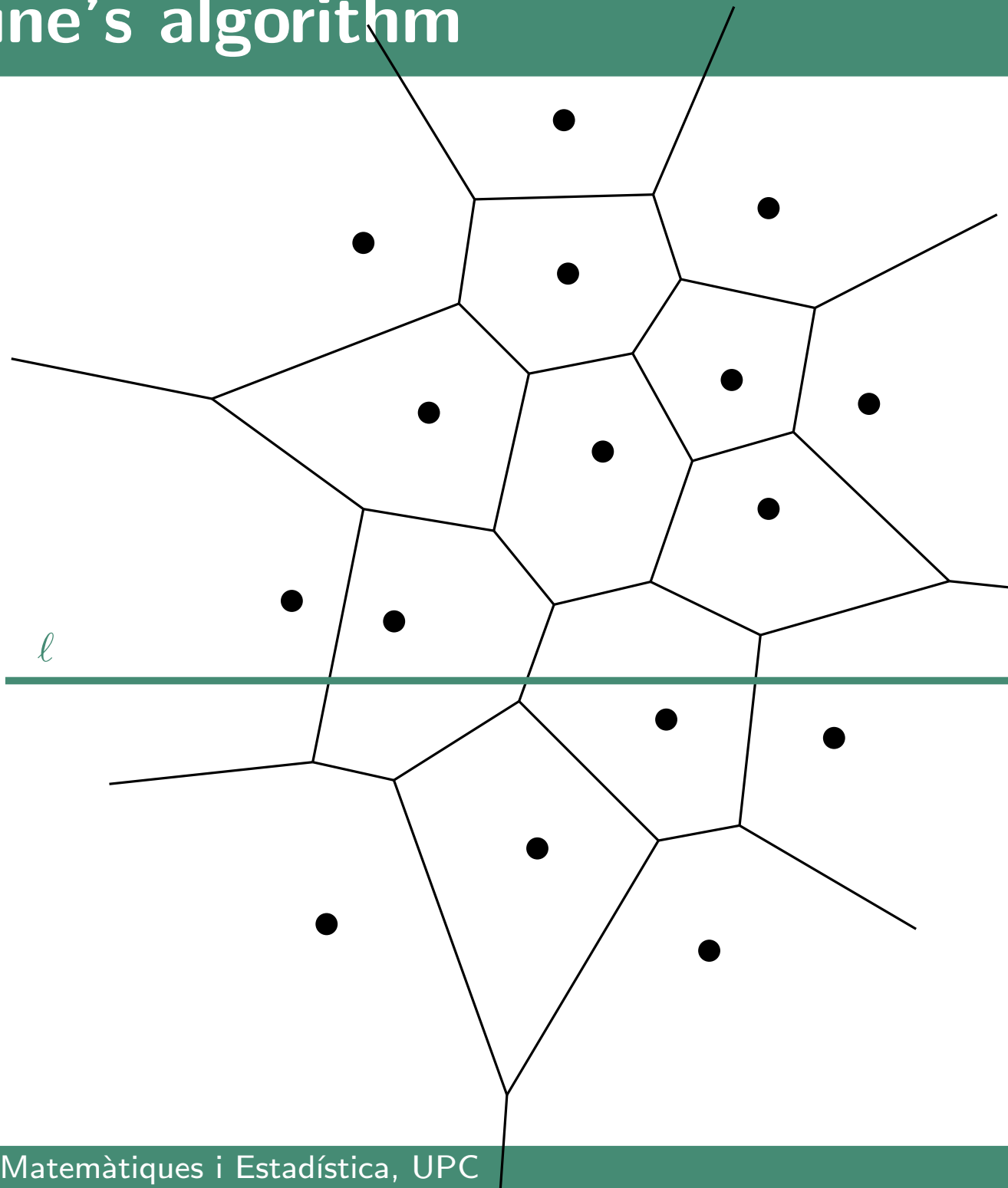
What can we know?



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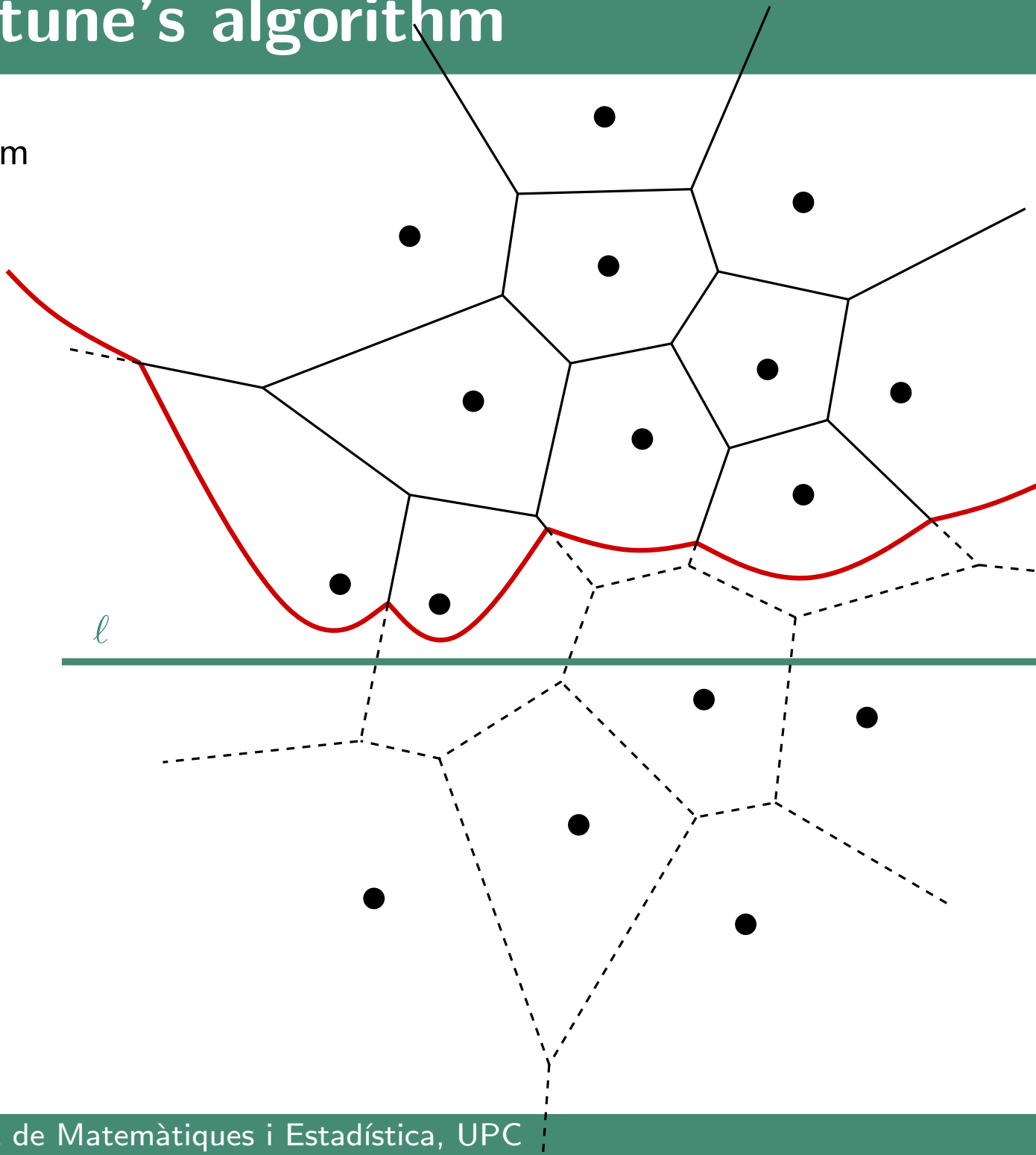

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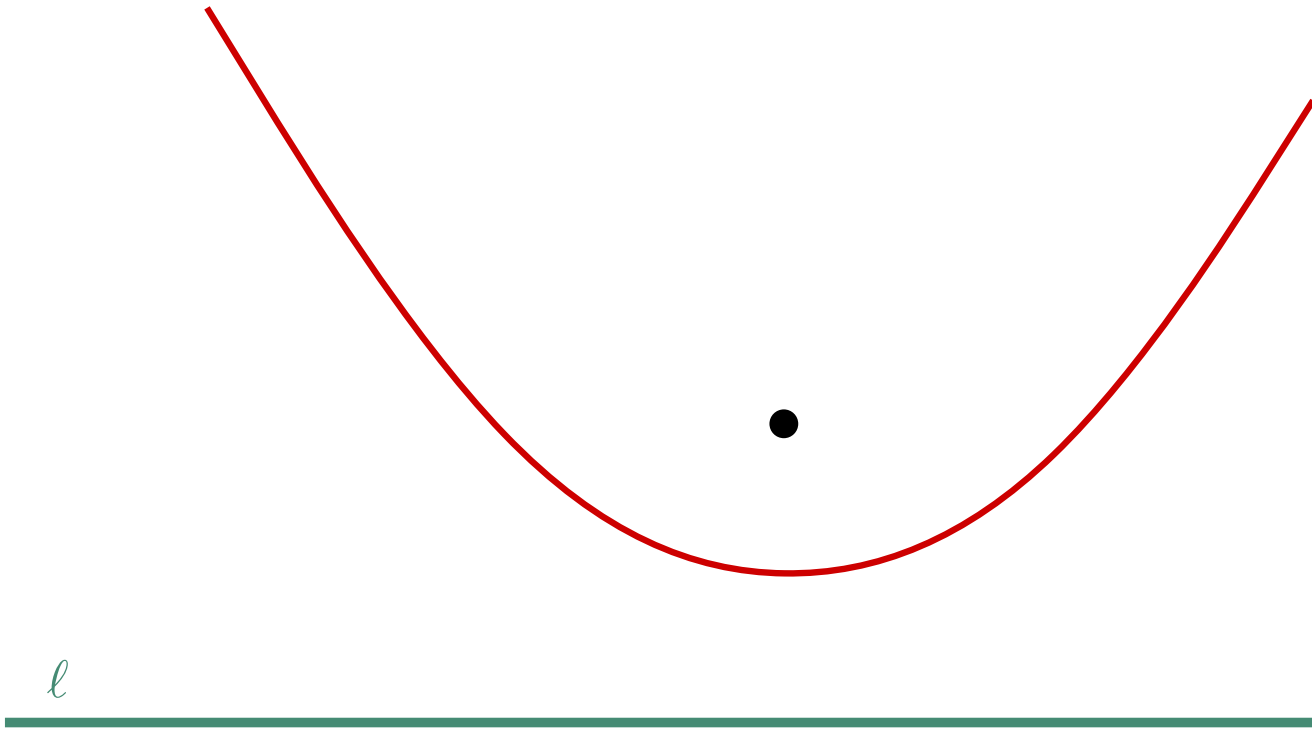

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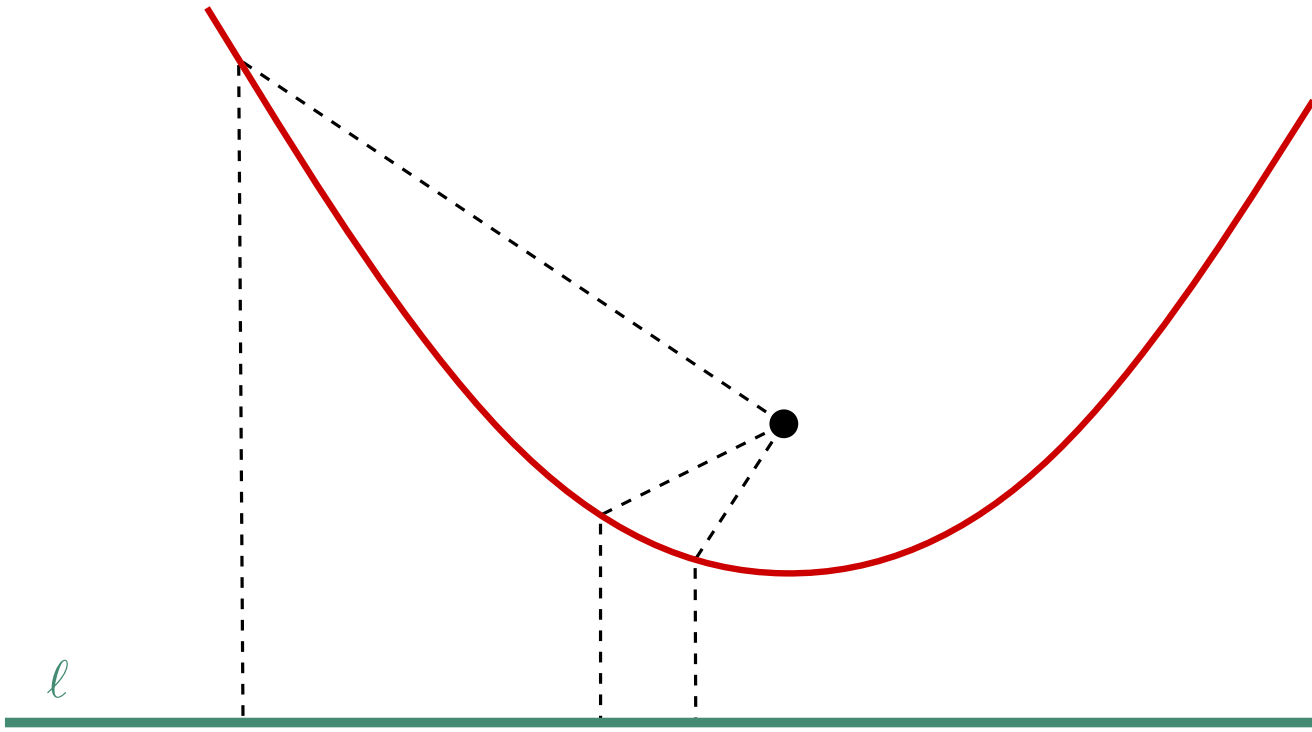

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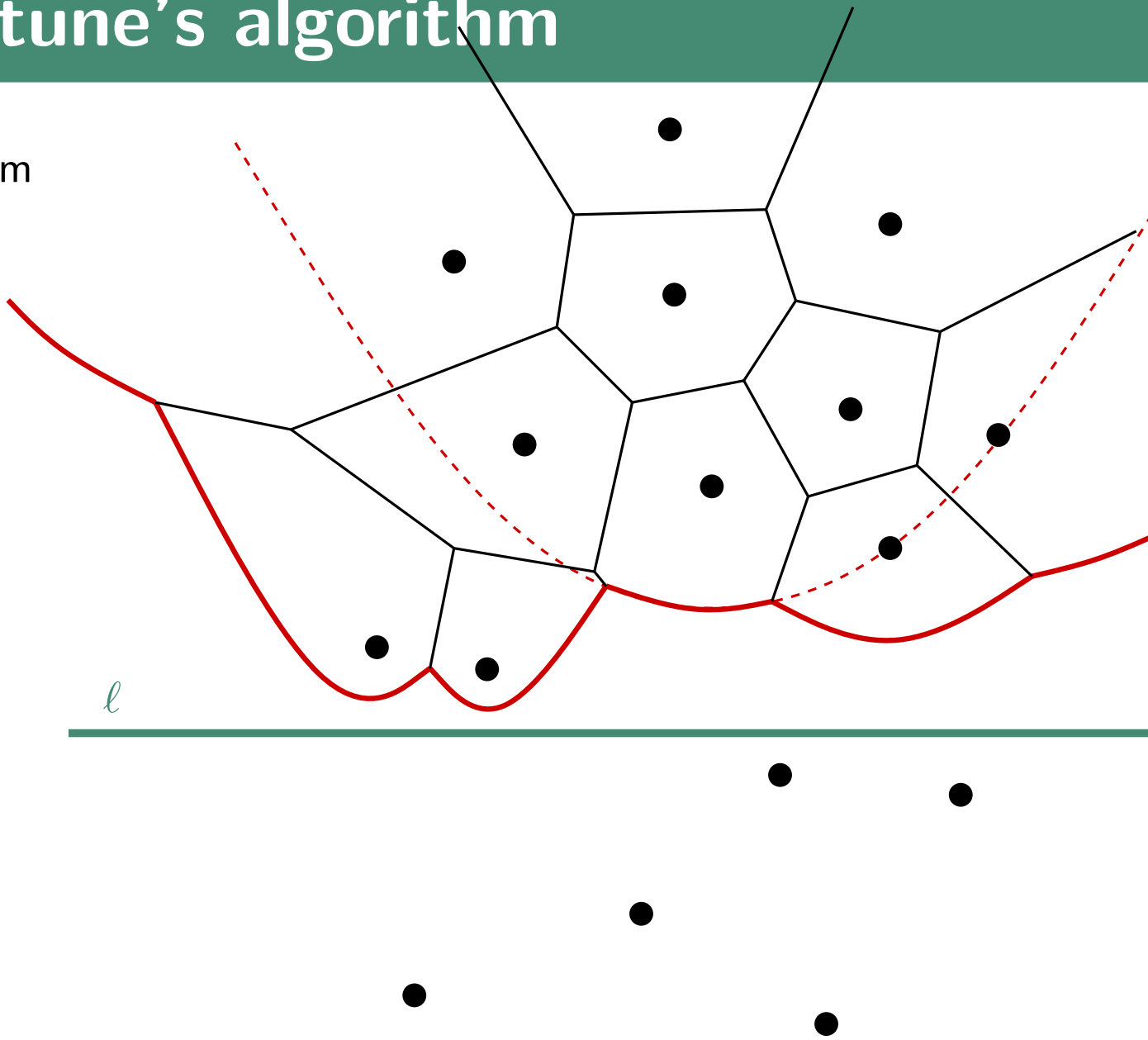

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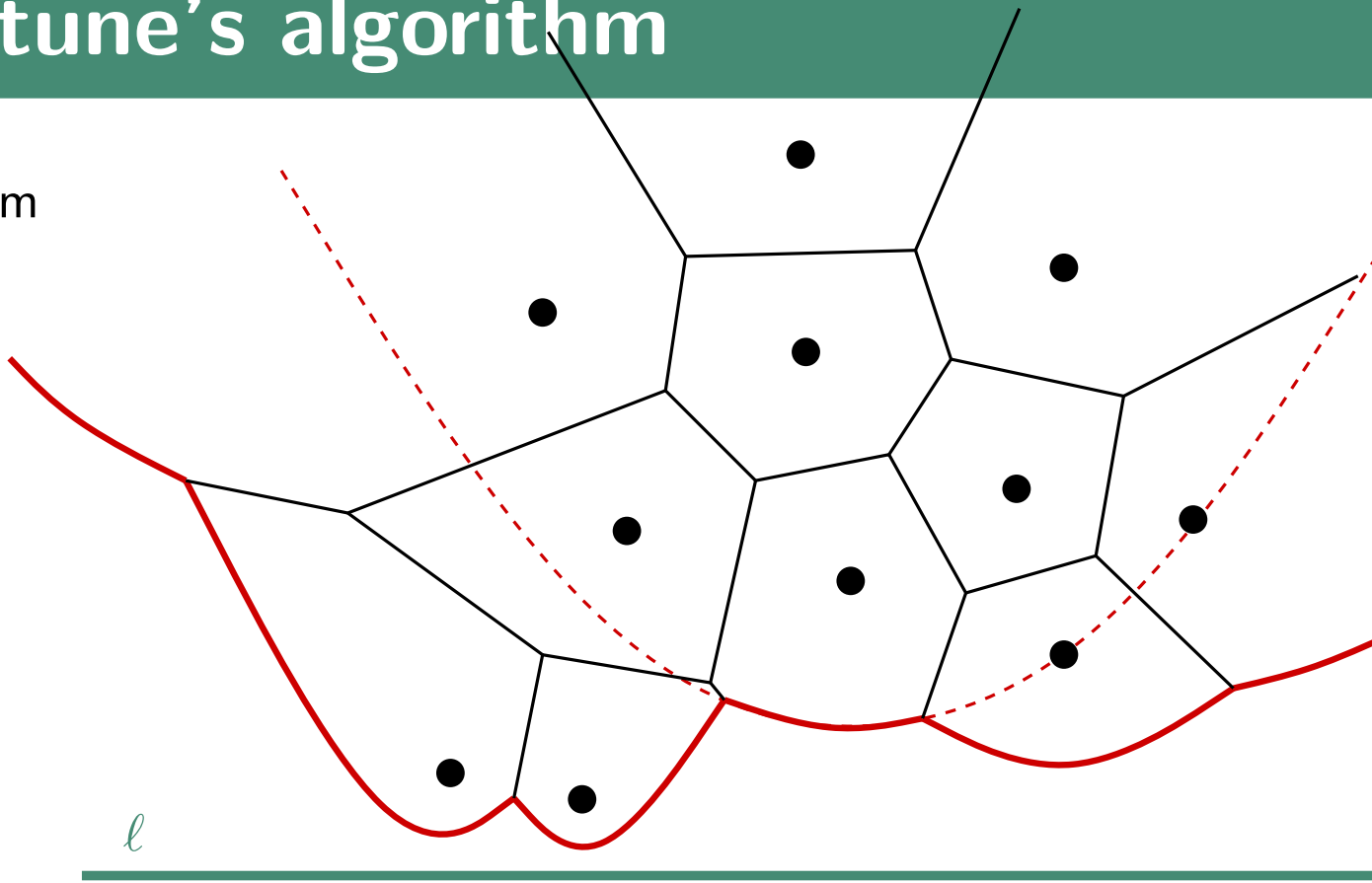
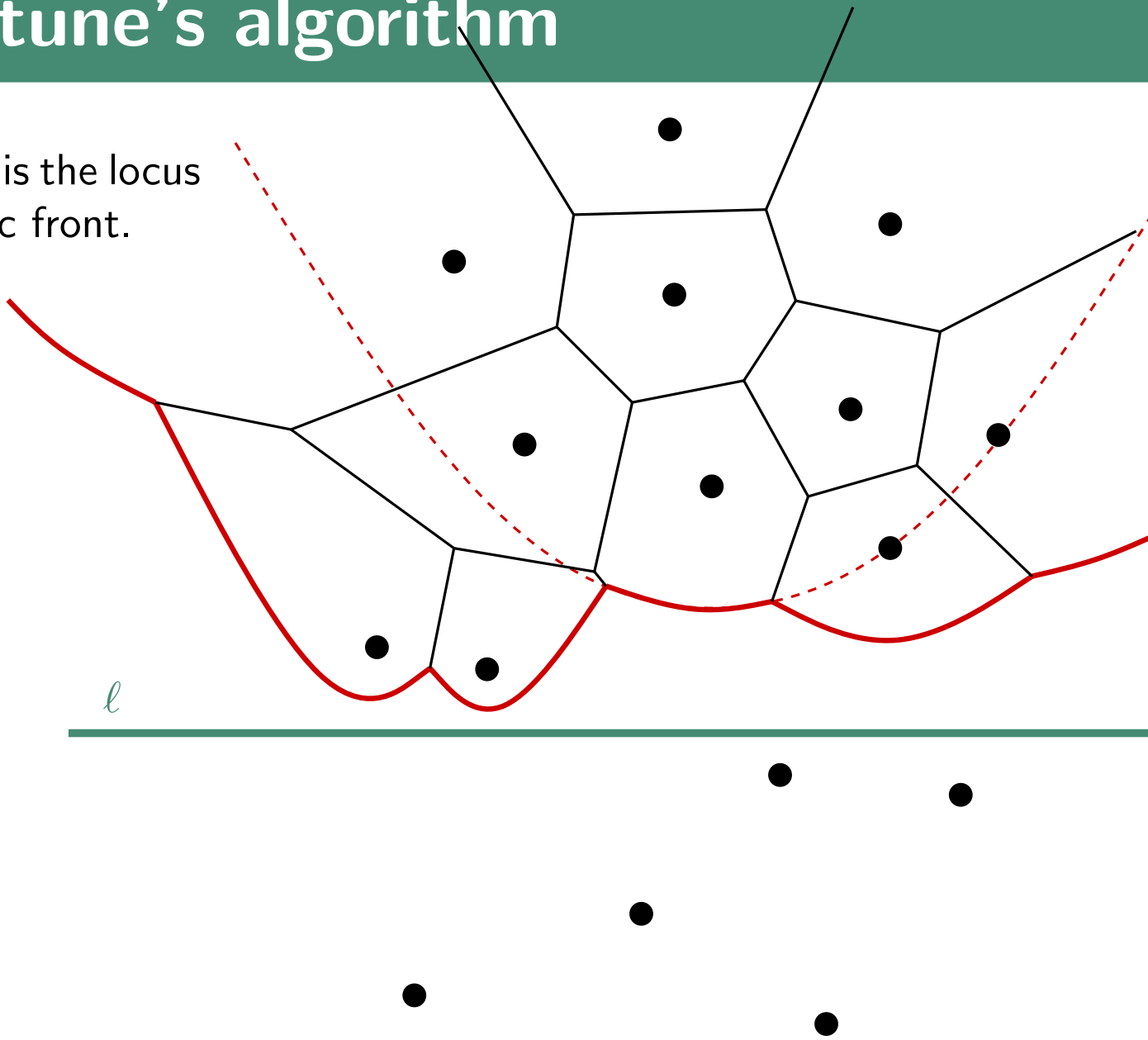


Definition. The *parabolic front* for line ℓ is the lower envelope of the arrangement of the parabolae (p_i, ℓ), for all p_i above ℓ.

Proposition. The Voronoi diagram may be known only above the parabolic front.

Fortune's algorithm

Lemma 1. The 1-skeleton of $Vor(P)$ is the locus of all breaking points of the parabolic front.



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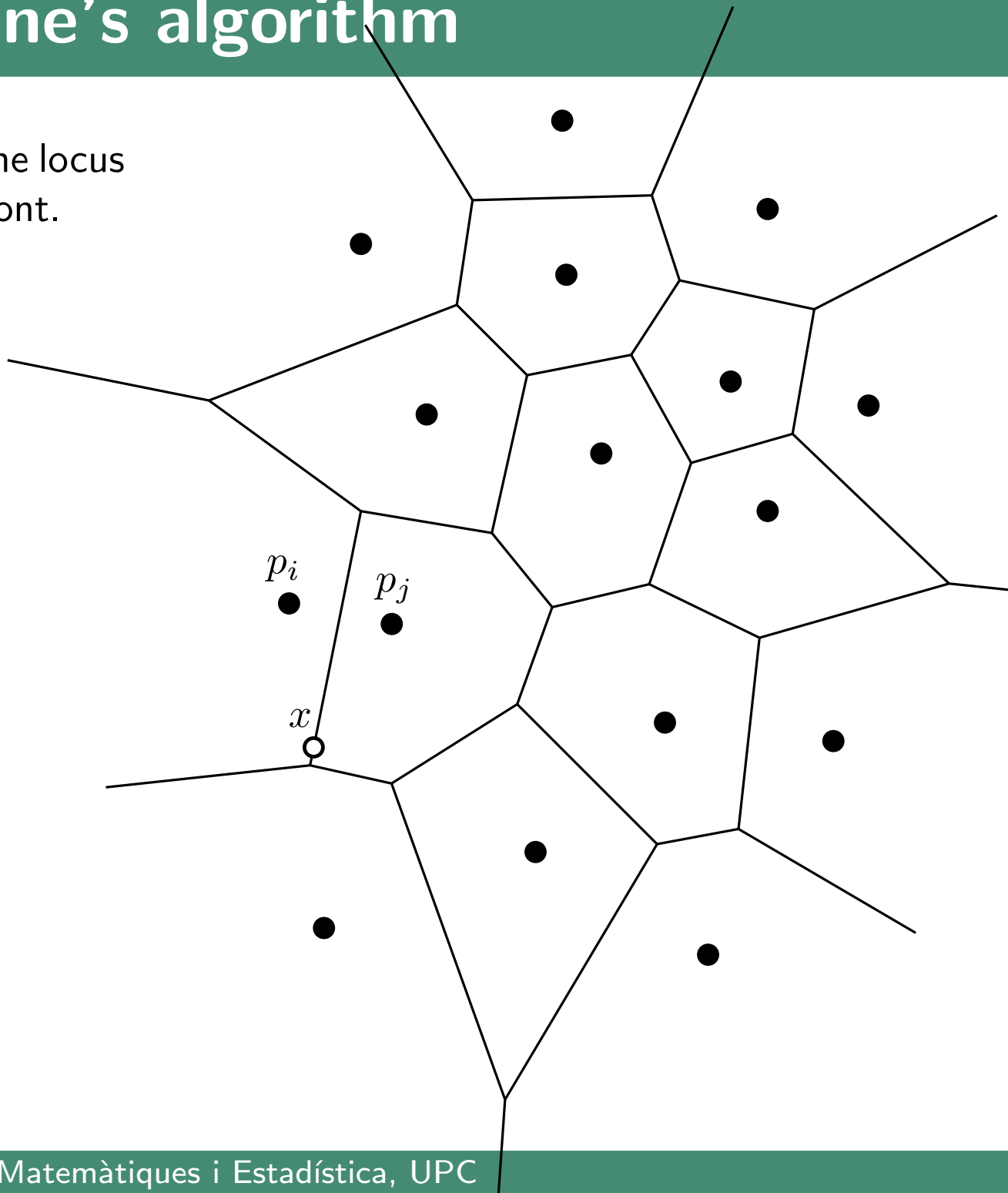

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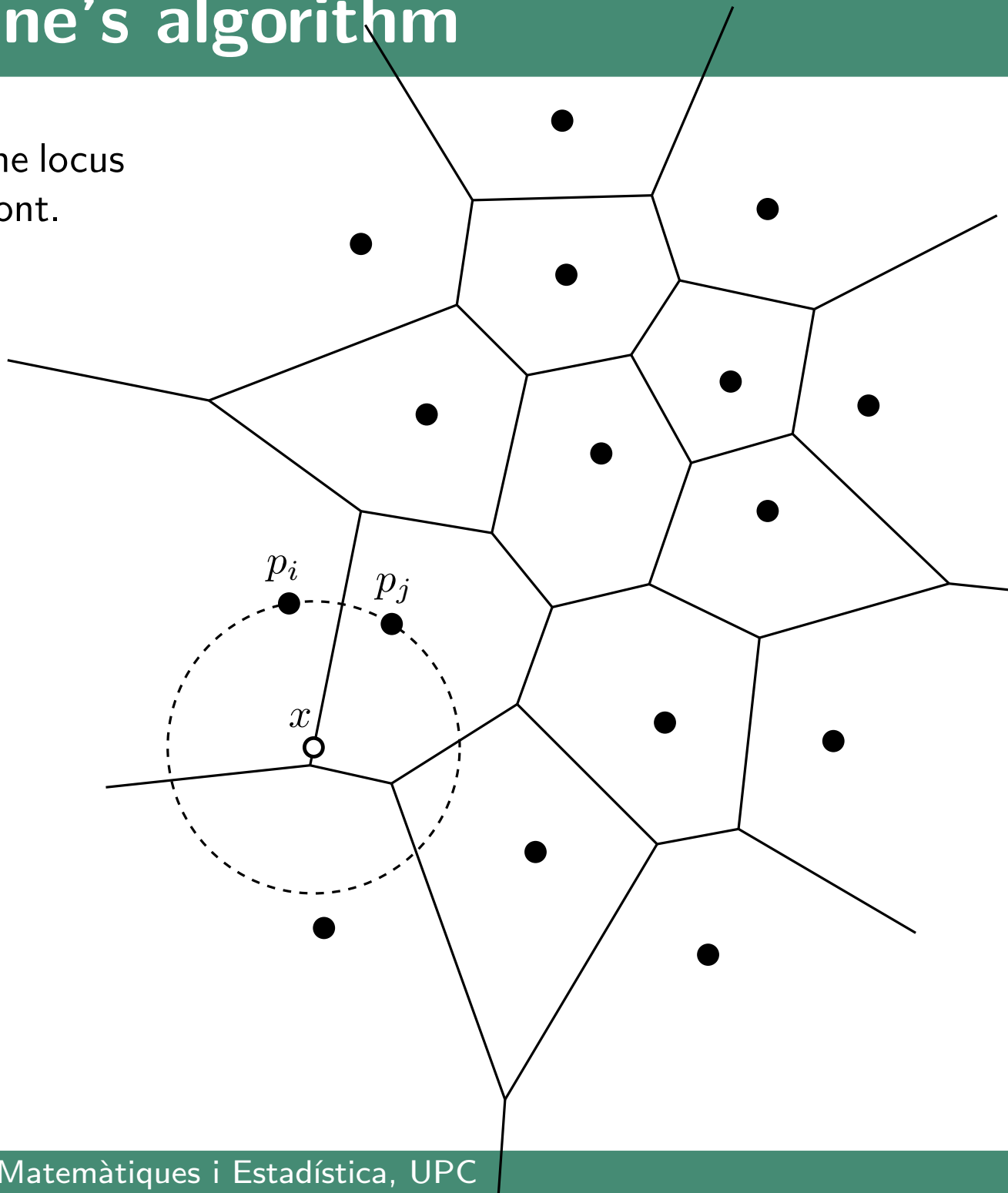

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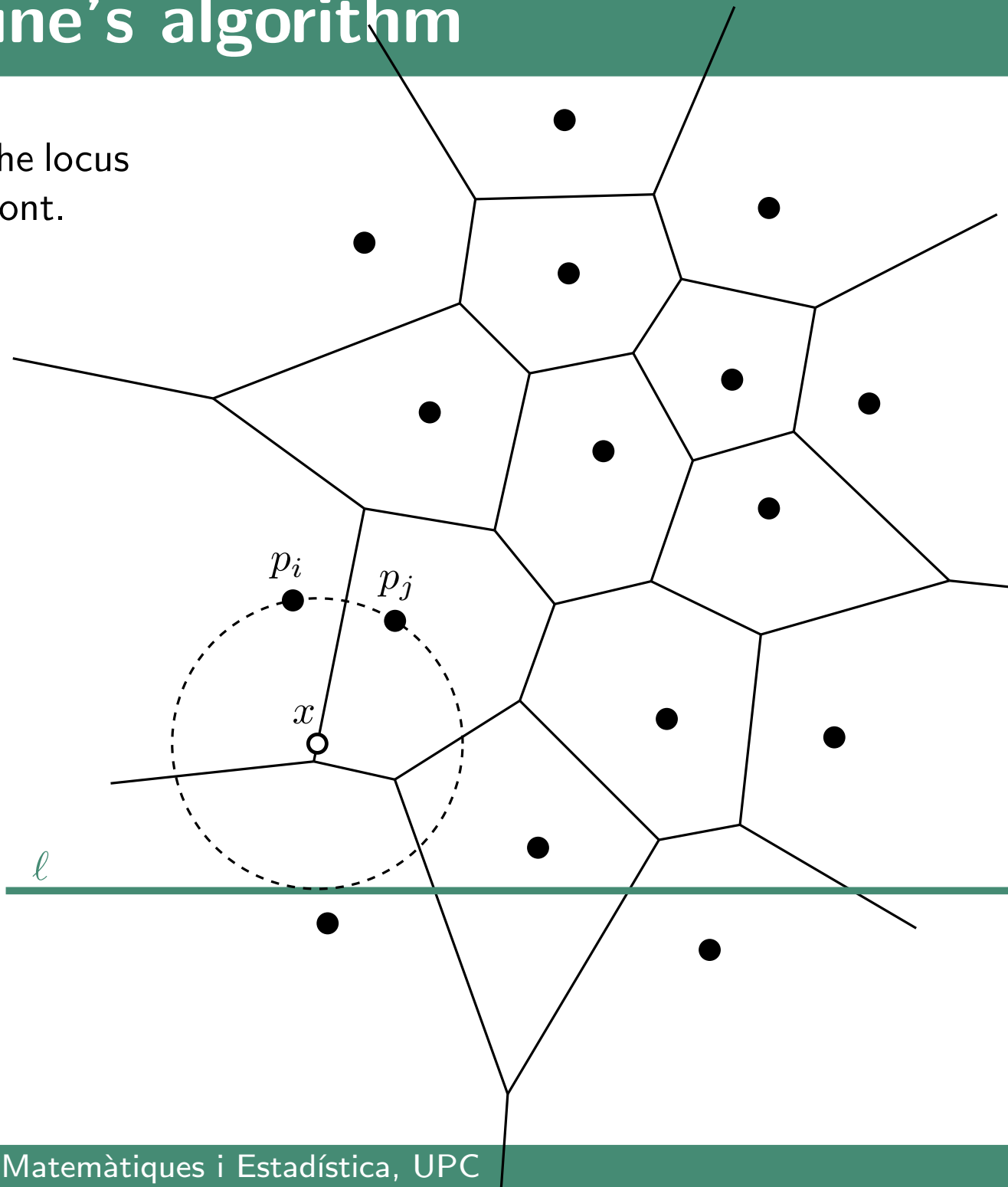

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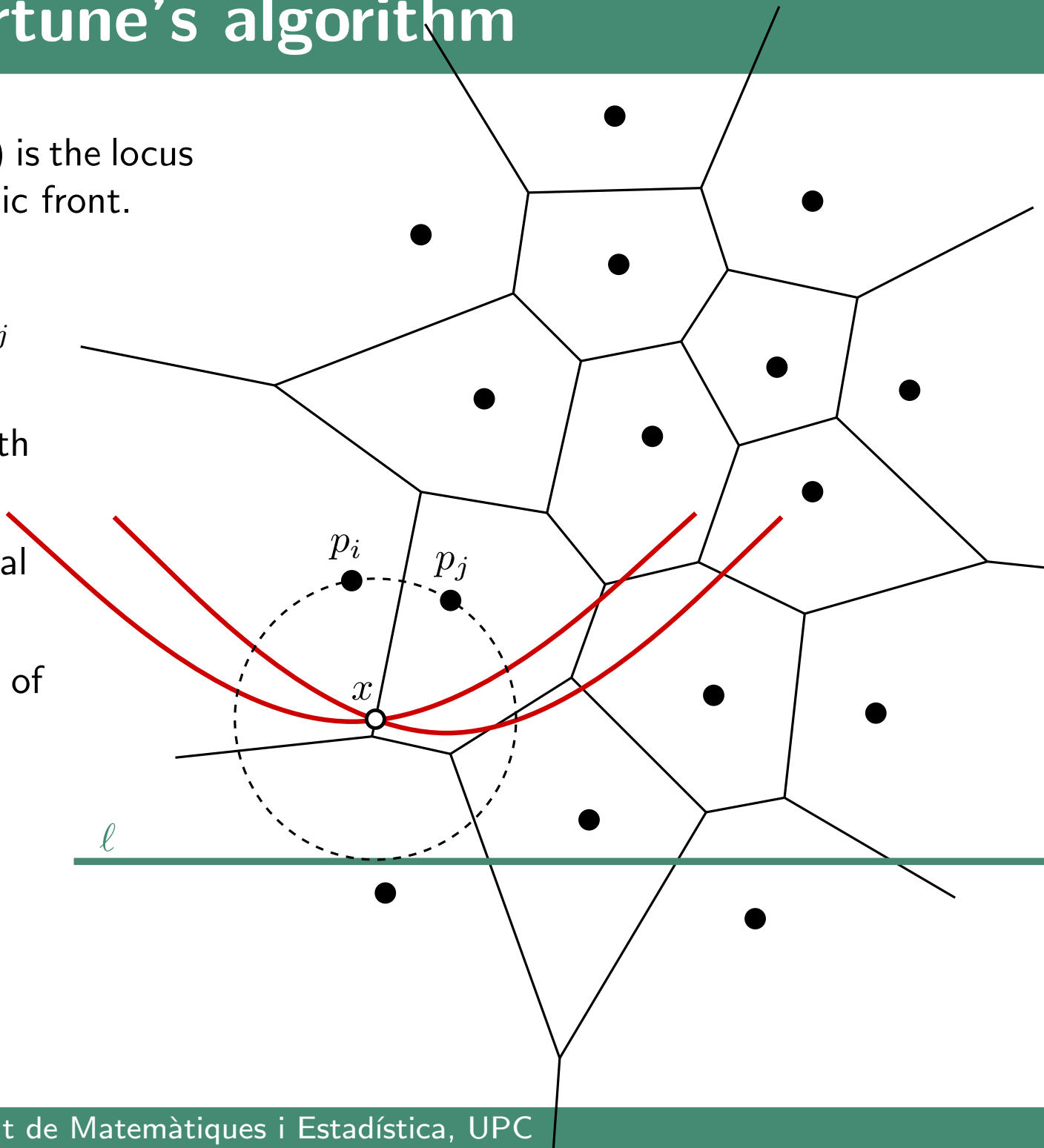

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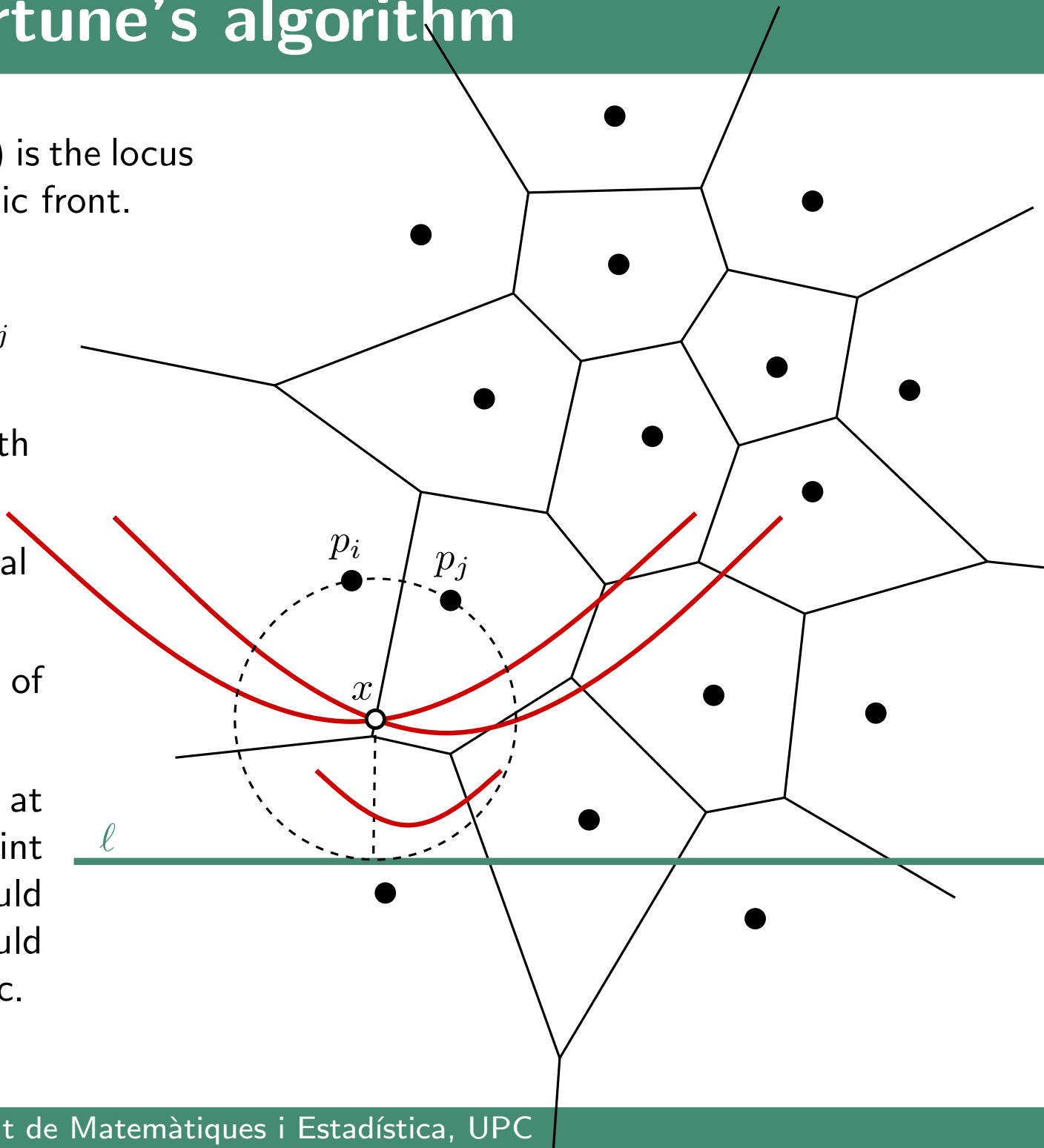

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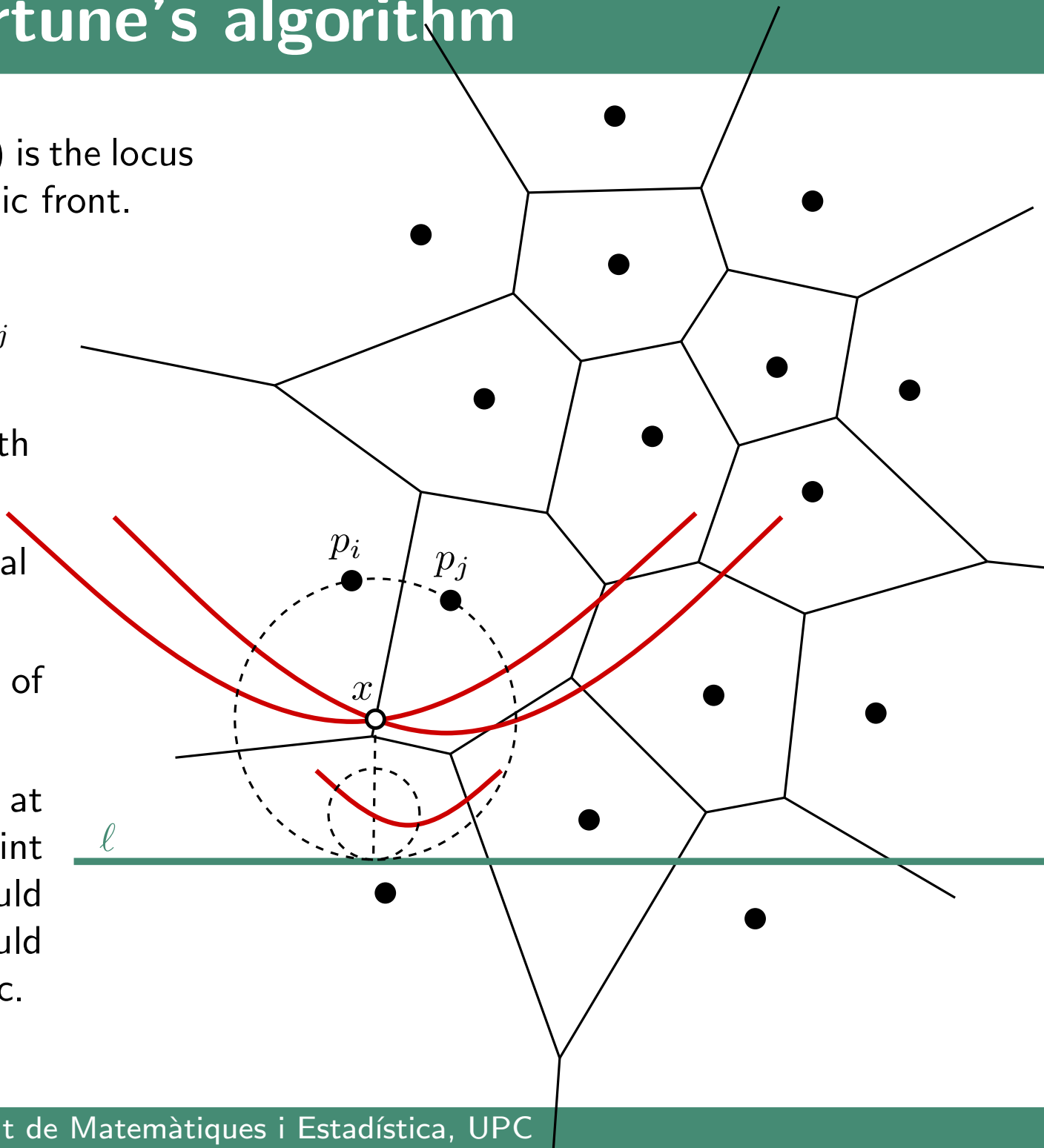

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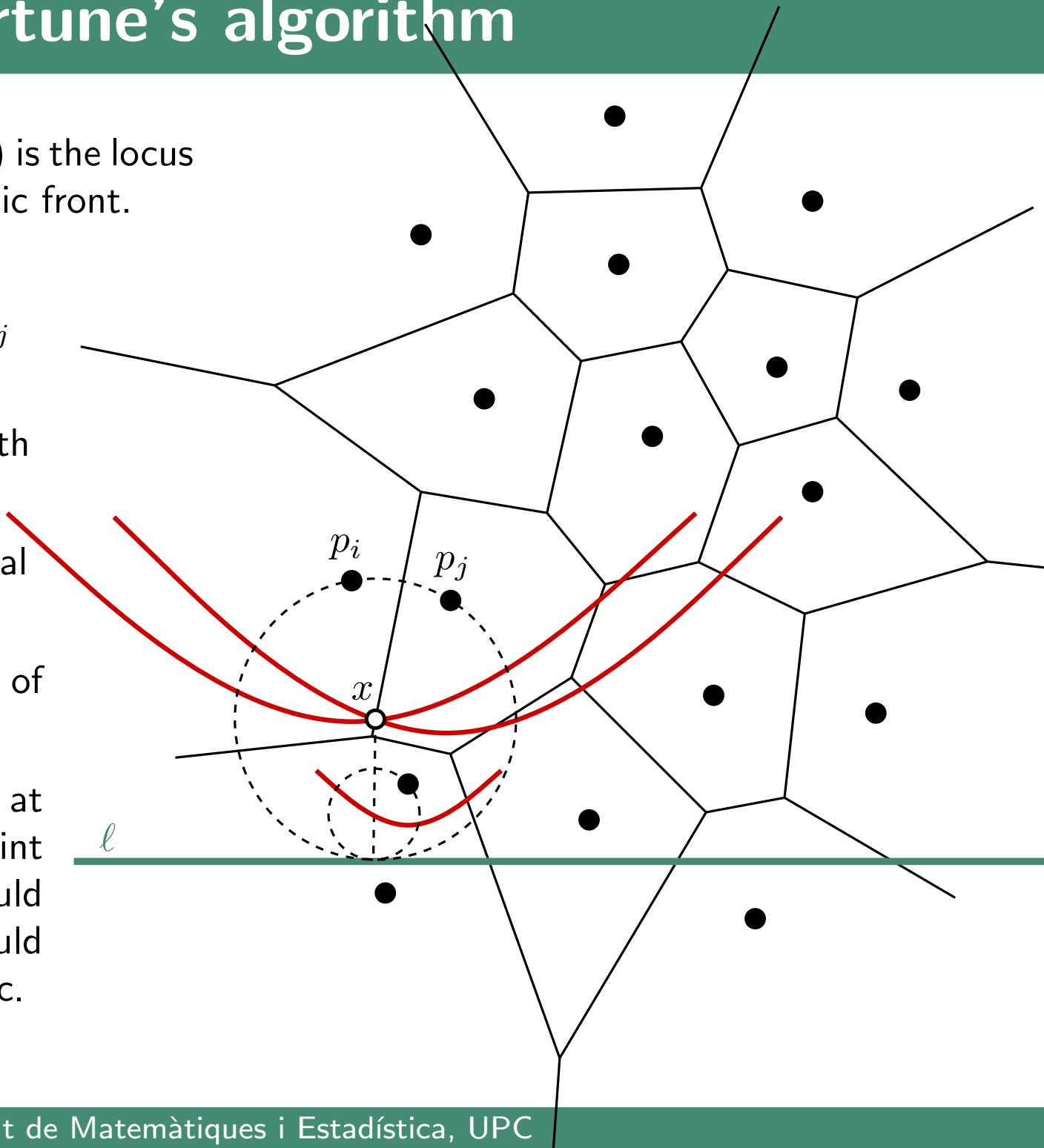

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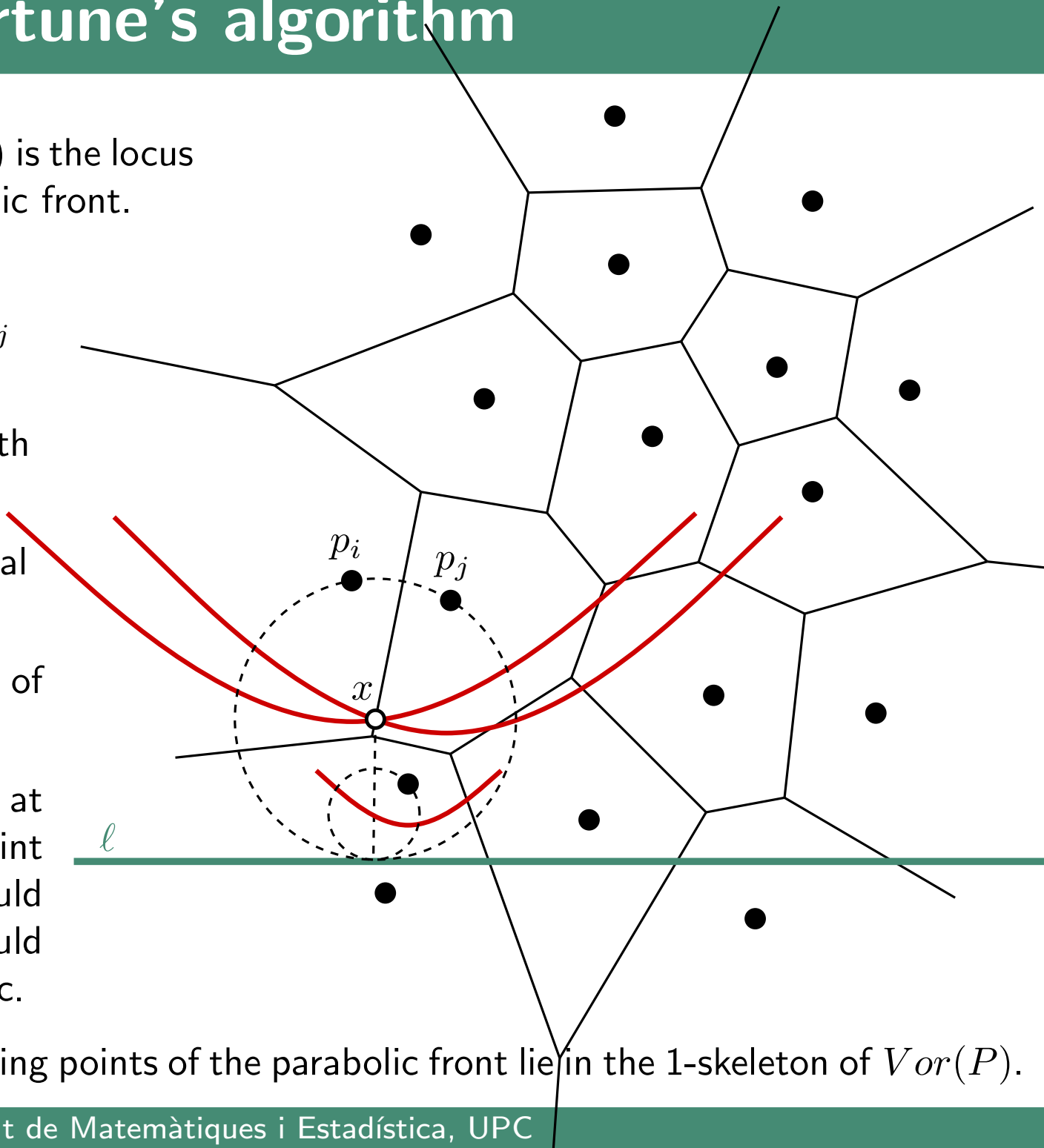
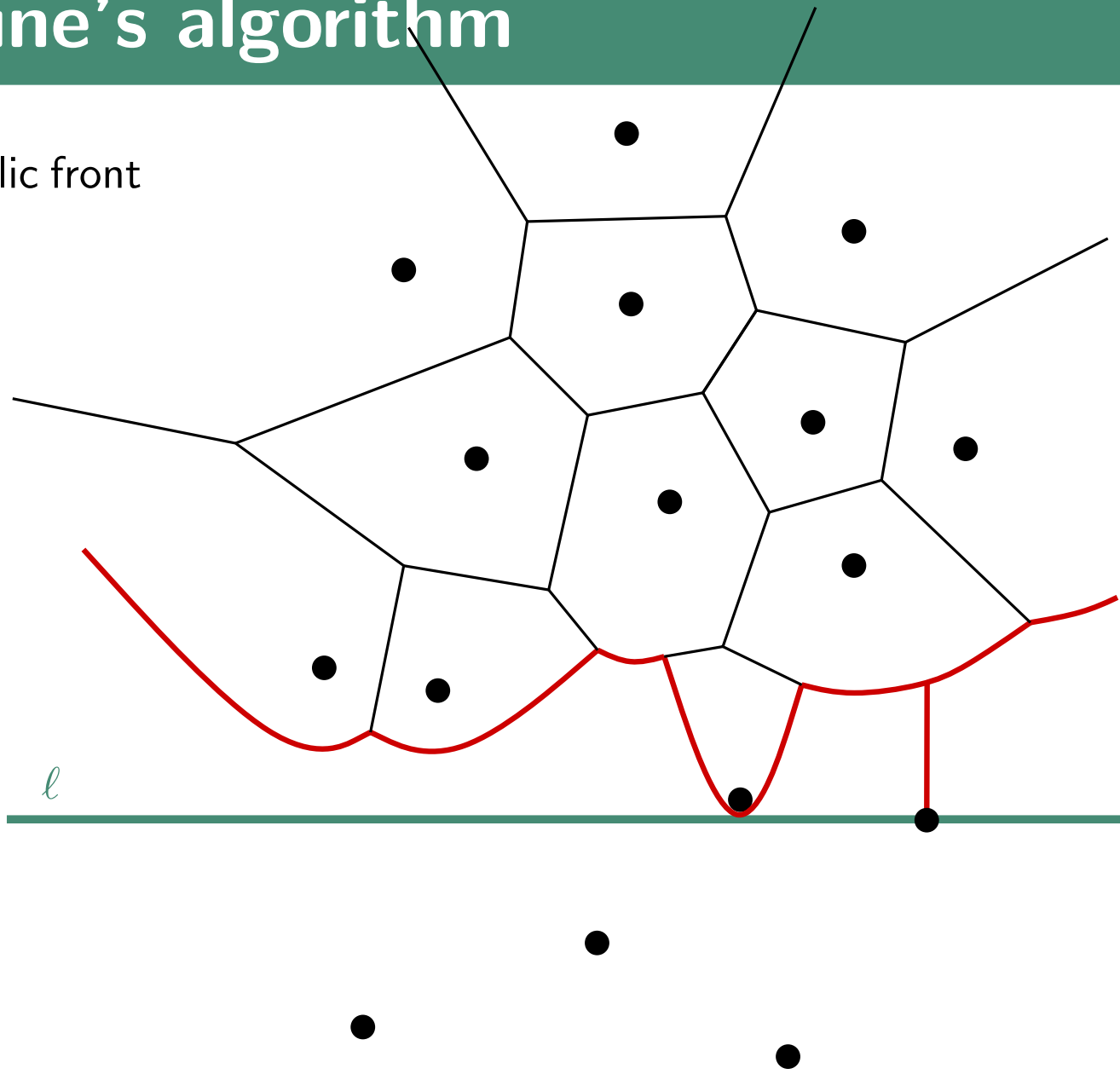


The reverse is trivially true: all breaking points of the parabolic front lie in the 1-skeleton of $Vor(P)$.

Fortune's algorithm

Lemma 2. An arc appears in the parabolic front if and only if ℓ reaches a site.



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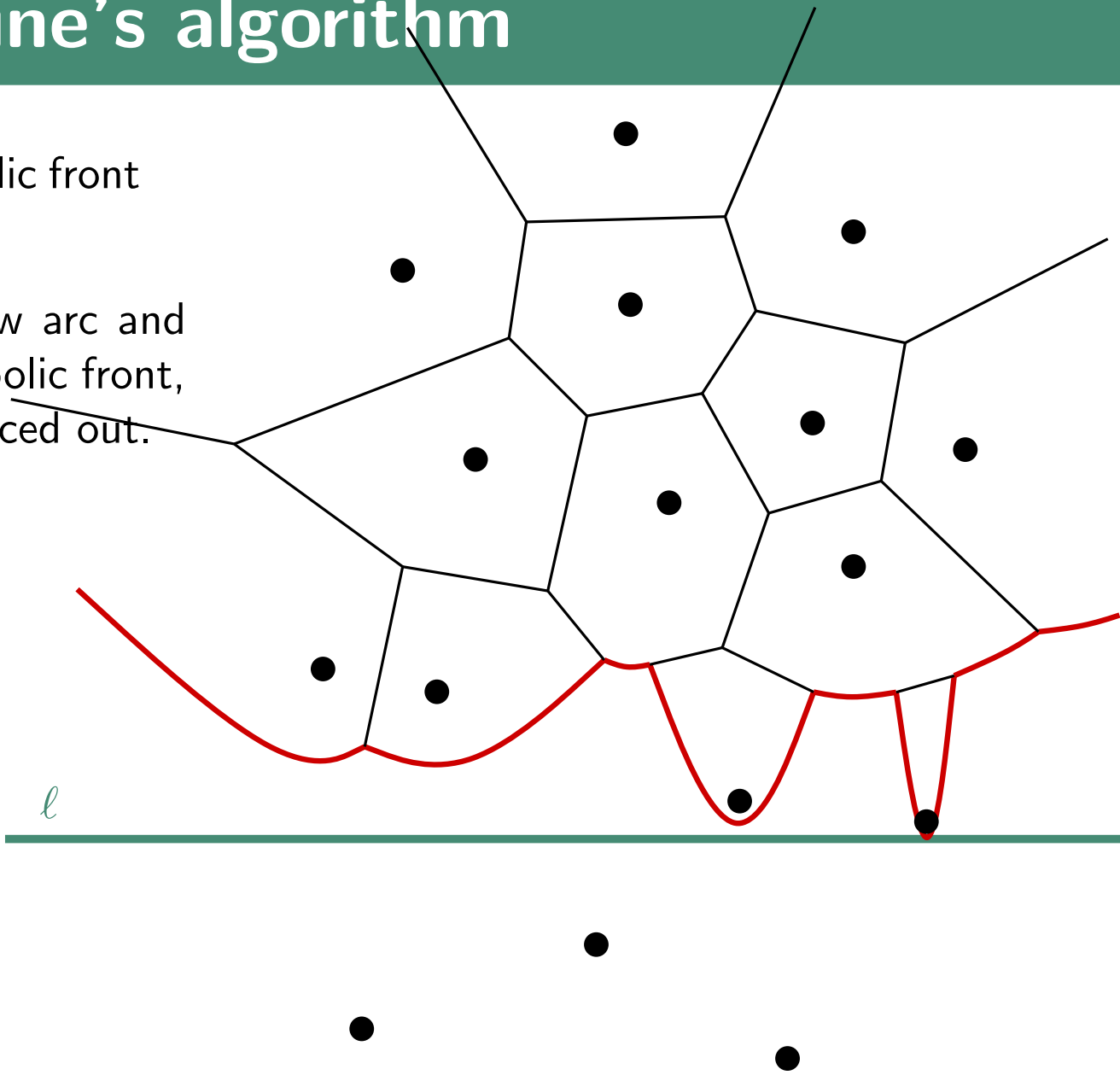

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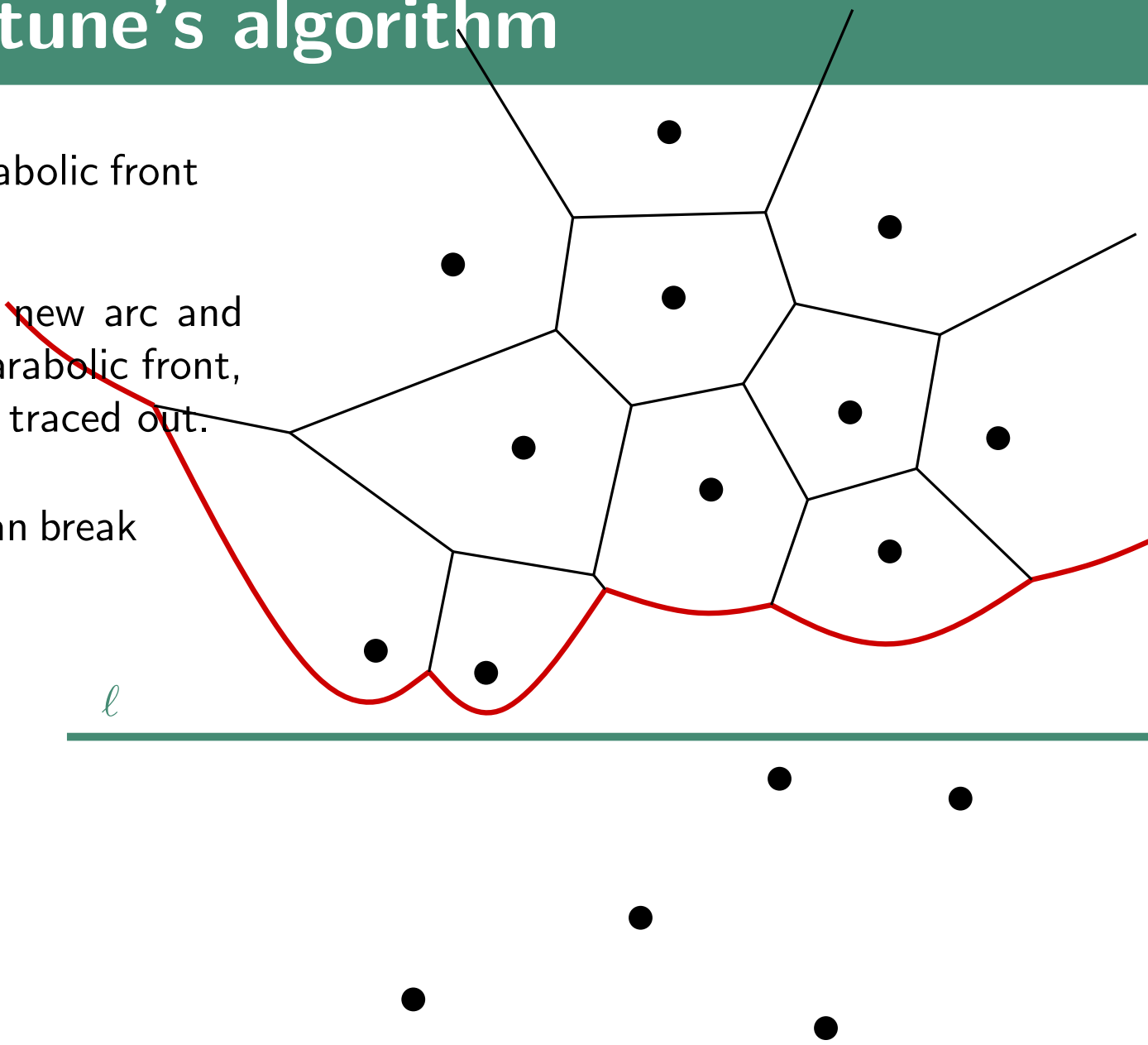

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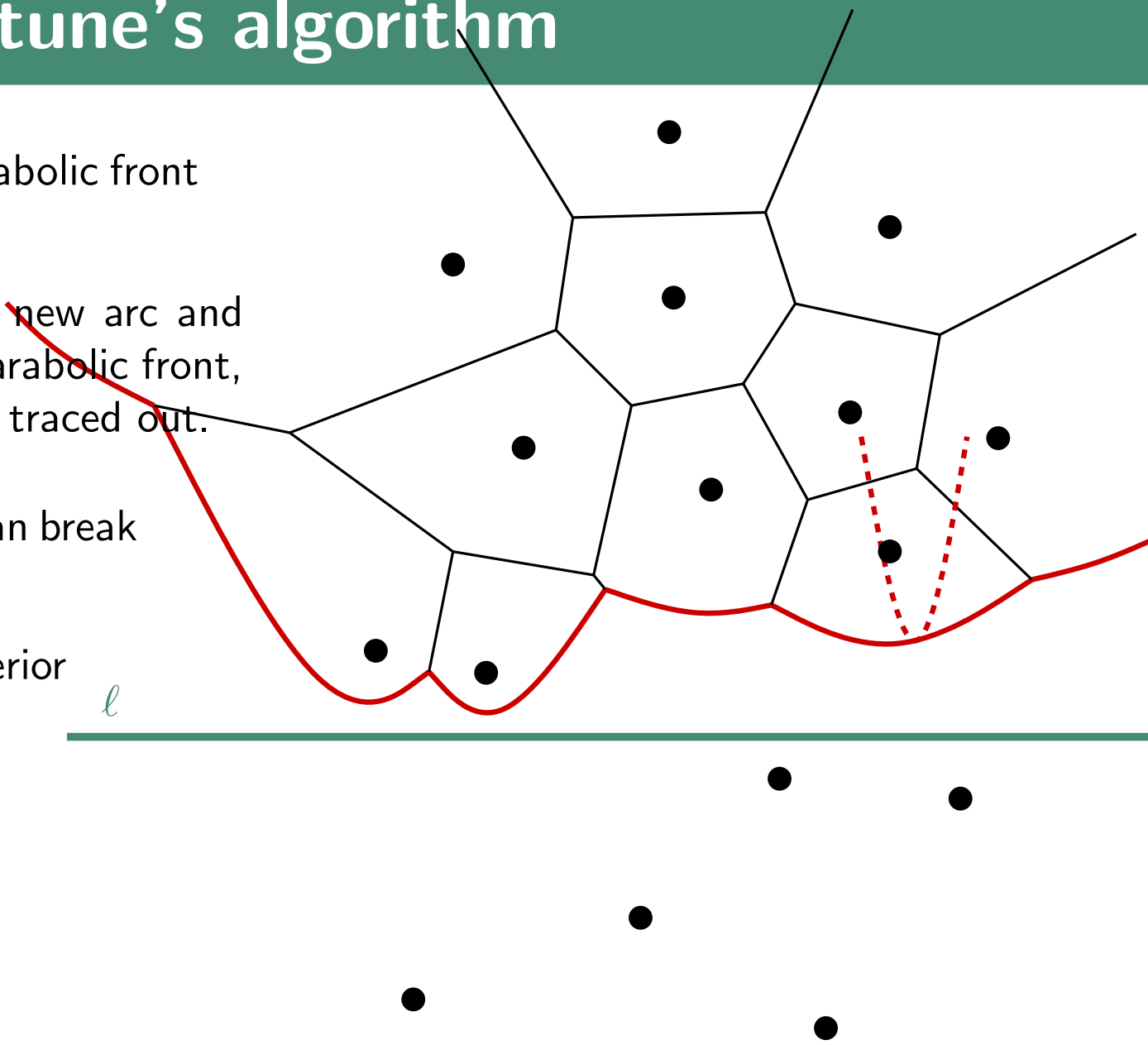

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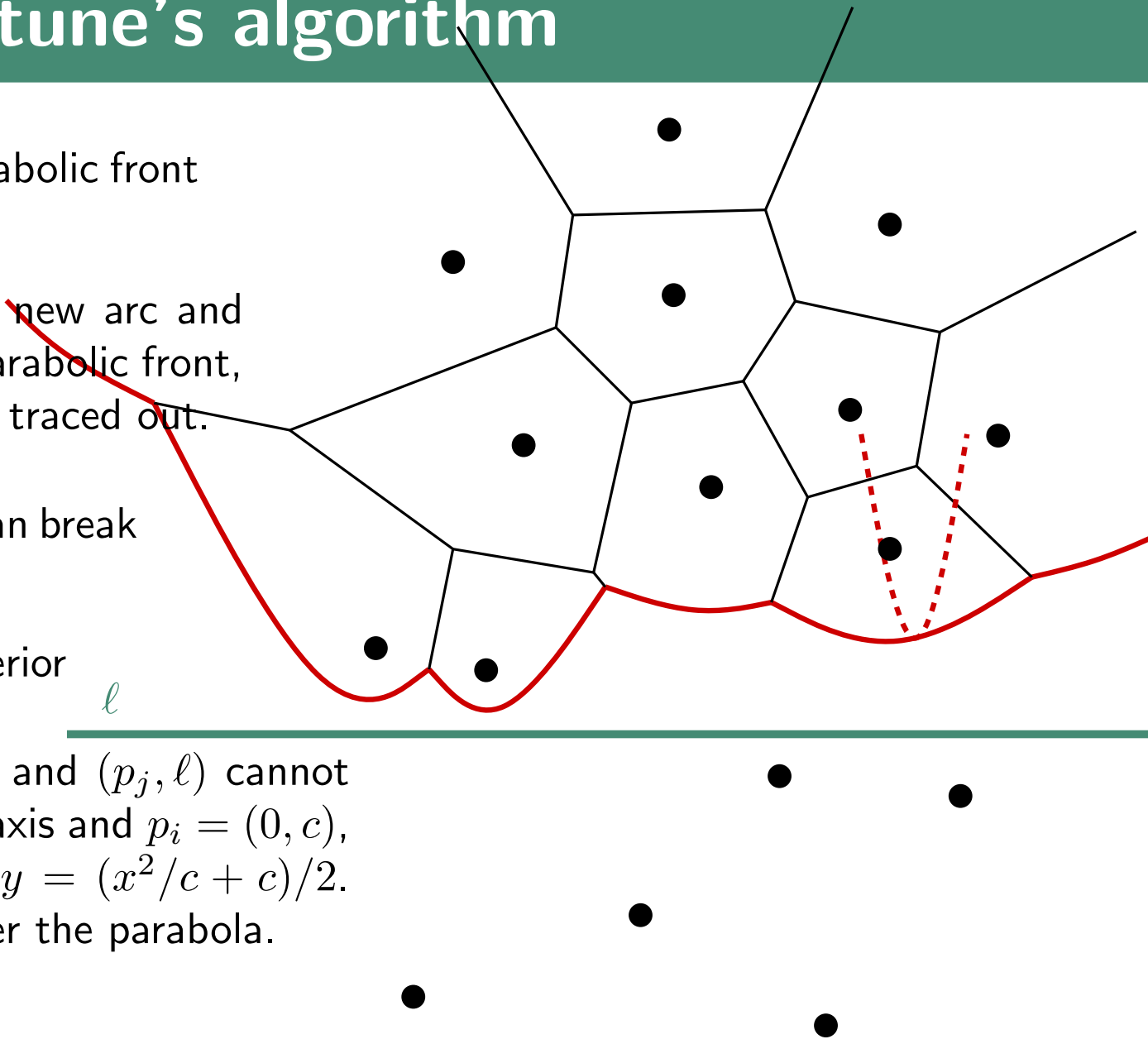
  ... because two parabolae (p_i, ℓ) and (p_j, ℓ) cannot be tangent. In fact, if ℓ is the x-axis and $p_i = (0, c)$, the equation of the parabola is $y = (x^2/c + c)/2$. Therefore, the higher p_i the wider the parabola.



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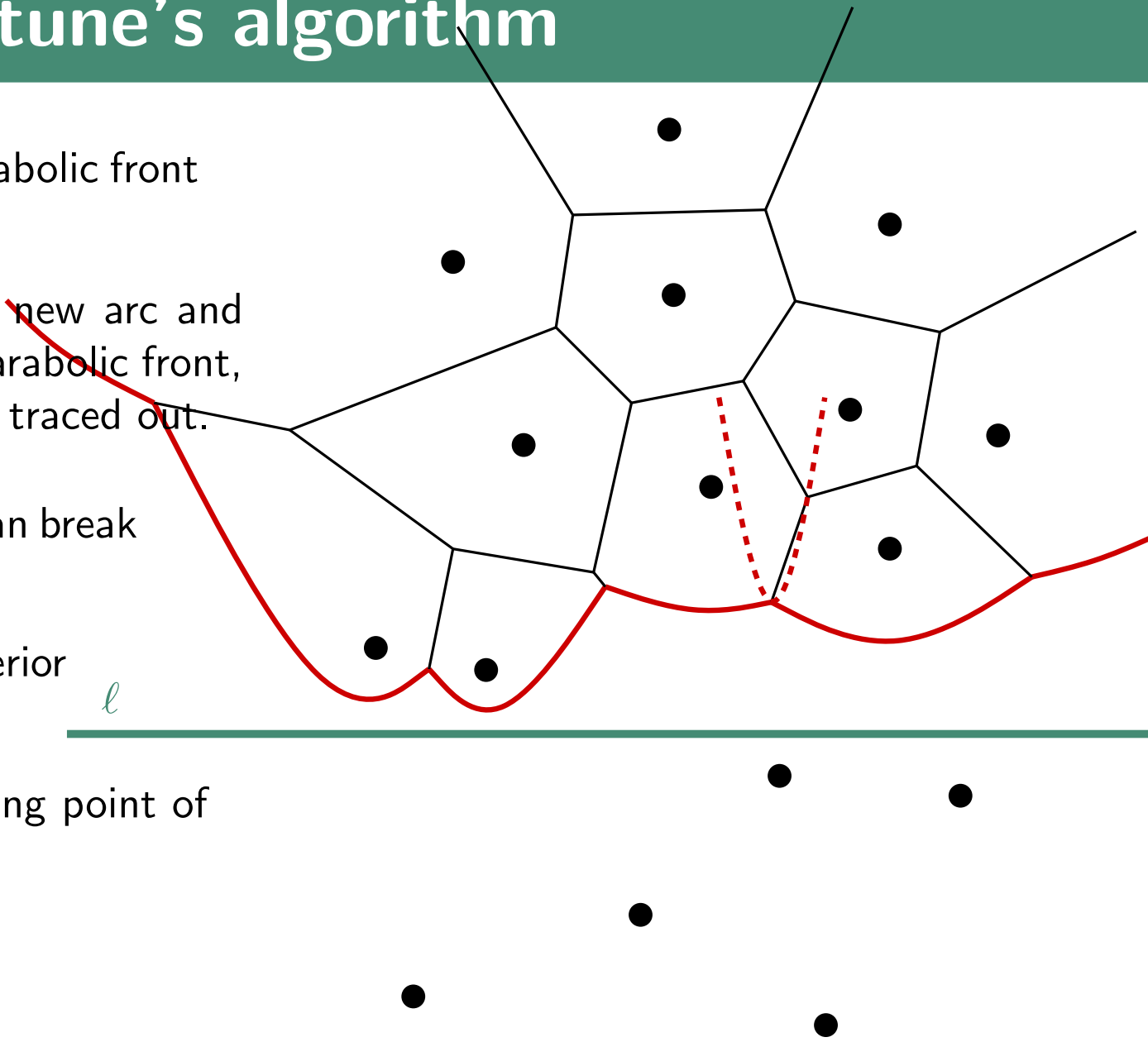

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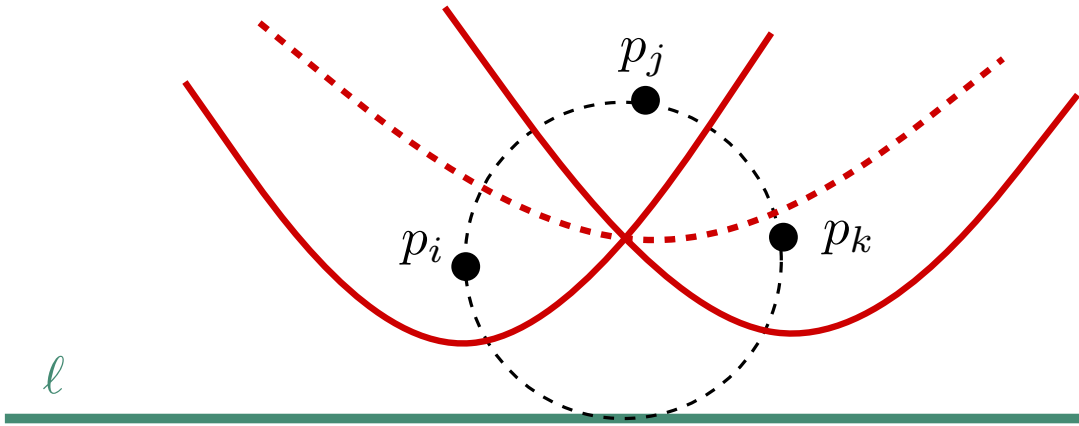

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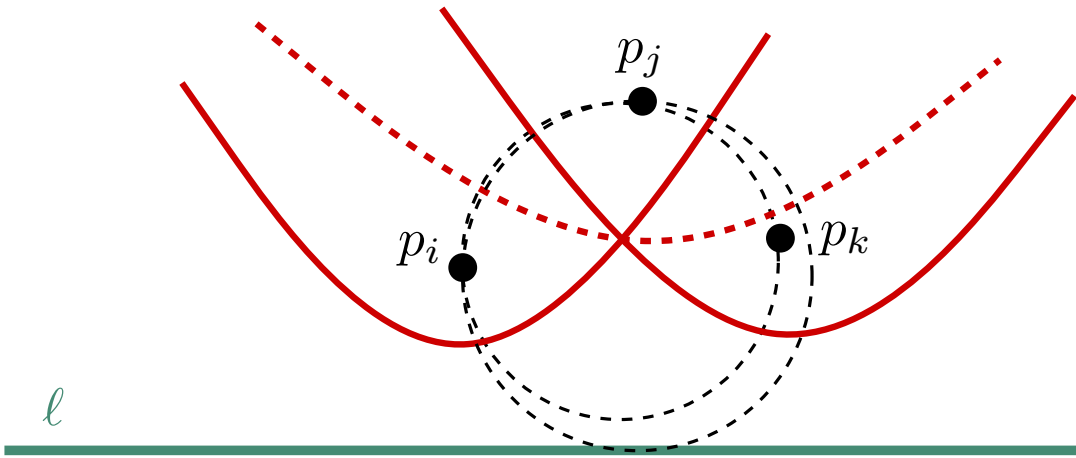

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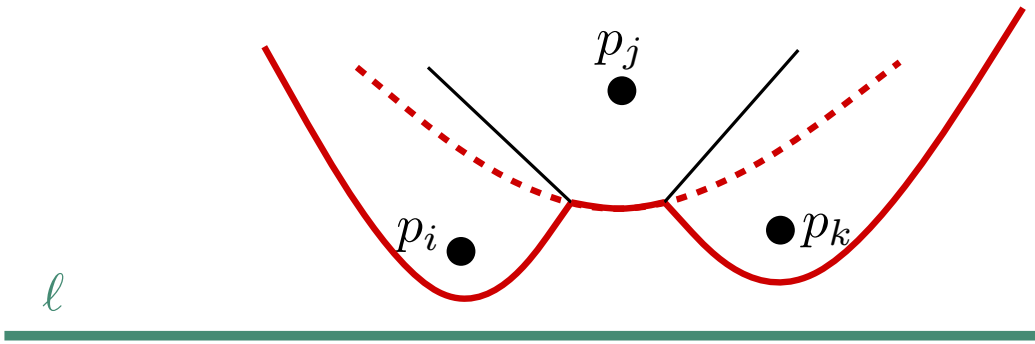

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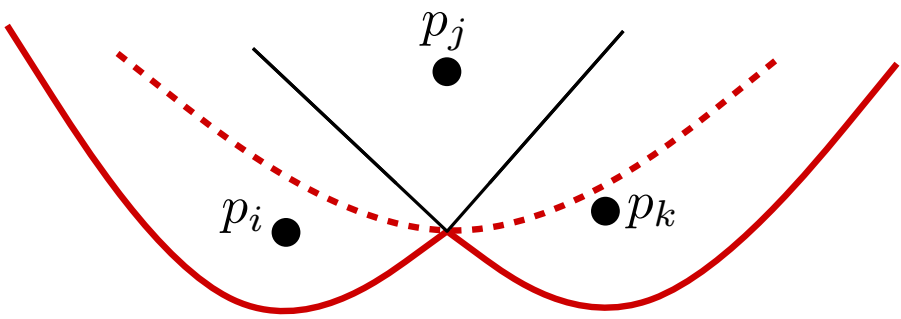
The moment (p_j, ℓ) disappears, the three arcs go through a common point q, equidistant from p_i, p_j, p_k, ℓ.



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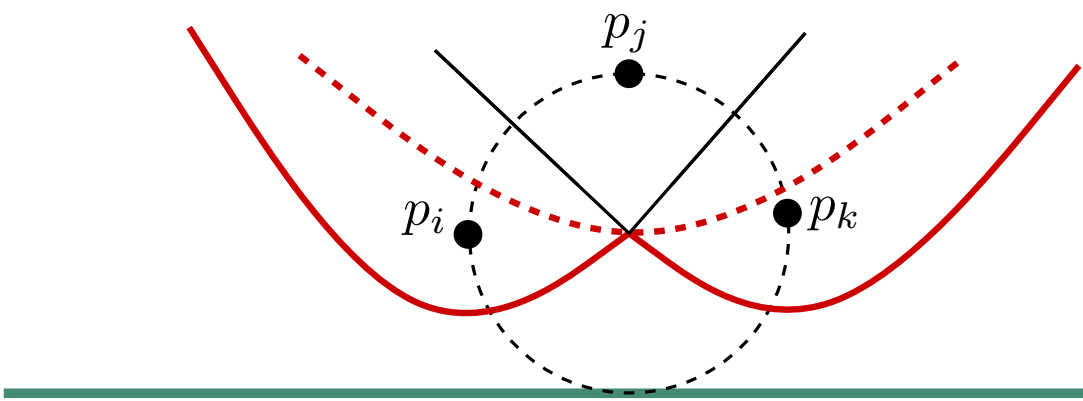
Therefore, the circle though p_i, p_j, p_k is tangent to ℓ.



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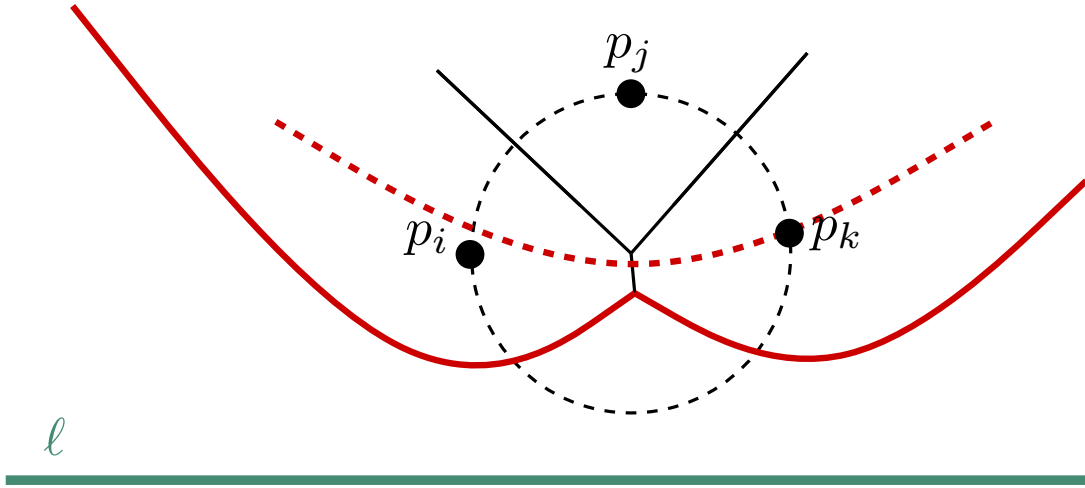
In fact, the circle is empty and a Vornoi vertex has been found.



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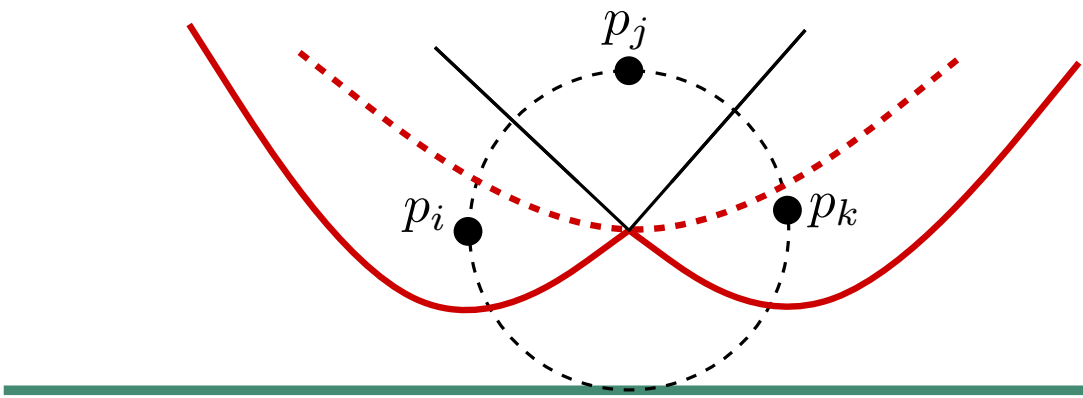
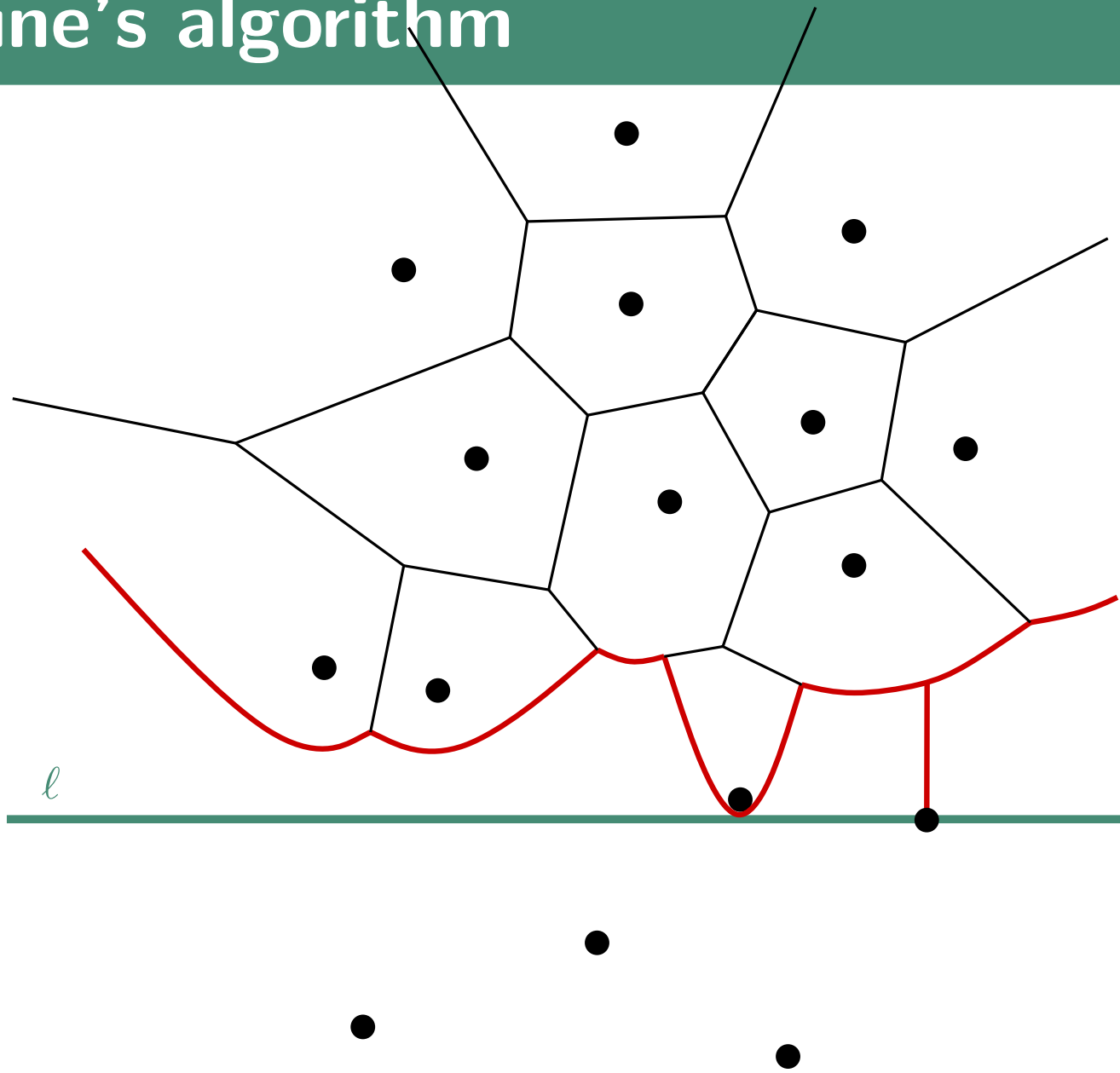
Corollary 1. Every Voronoi vertex is detected in a circle event.



Fortune's algorithm

Corollary 1. Every Voronoi vertex is detected in a circle event.

Corollary 2. The parabolic front consists of at most $2n - 1$ arcs.



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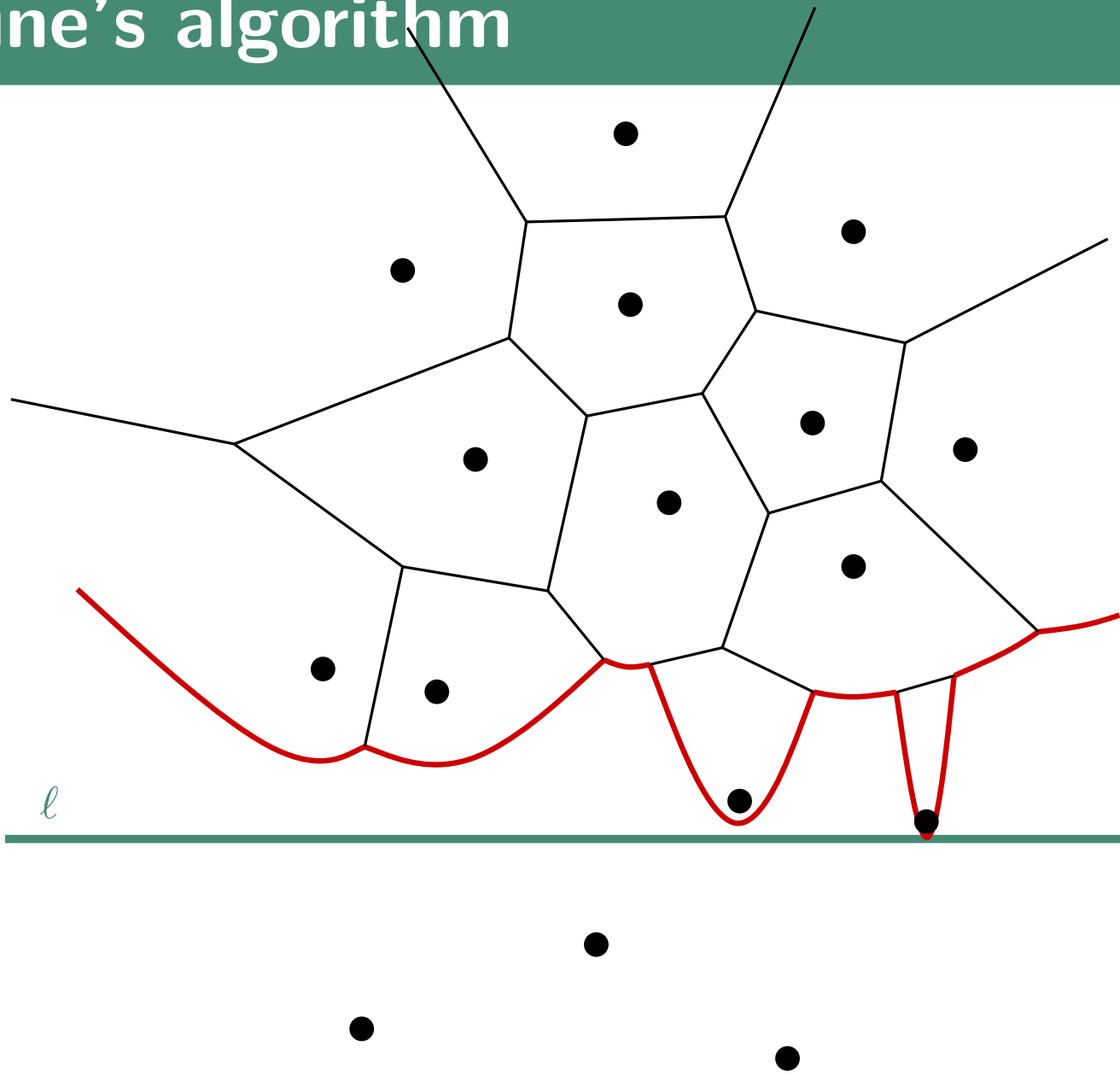

Fortune's algorithm

Output: **Voronoi diagram**

Events queue: **site events and circle events**

Sweep line status: **parabolic front**

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The Voronoi diagram is stored in a DCEL, specifically dealing with halflines.

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Output: **Voronoi diagram**

The Voronoi diagram is stored in a DCEL, specifically dealing with halflines.

Events queue: **site events and circle events**

The events are stored in a priority queue, where the priority of an event is its y-coordinate.

- For every site event $p_i = (x_i, y_i)$, the point itself is stored.
- For every circle event, the lowest point of the circle is stored.

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Sweep line status: **parabolic front**

The sweep line status is stored in a balanced binary search tree in which leaves are arcs and internal nodes are breakpoints of the parabolic front, sorted by x-coordinate.

- Each arc α of a parablola (p_i, ℓ) is represented by the index i of the site p_i, and it has a pointer to the circle event for α.
- Each breaking point is represented by a pair of indices (i, j), and has a pointer to the corresponding Voronoi edge.

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ALGORITHM

Input: A set $P = \{p_1, \ldots, p_n\}$ of n sites in the plane.
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While E is not empty:

- Extract the first event from E.
- Handle the event, depending on whether it is a site or a circle event.

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Complete

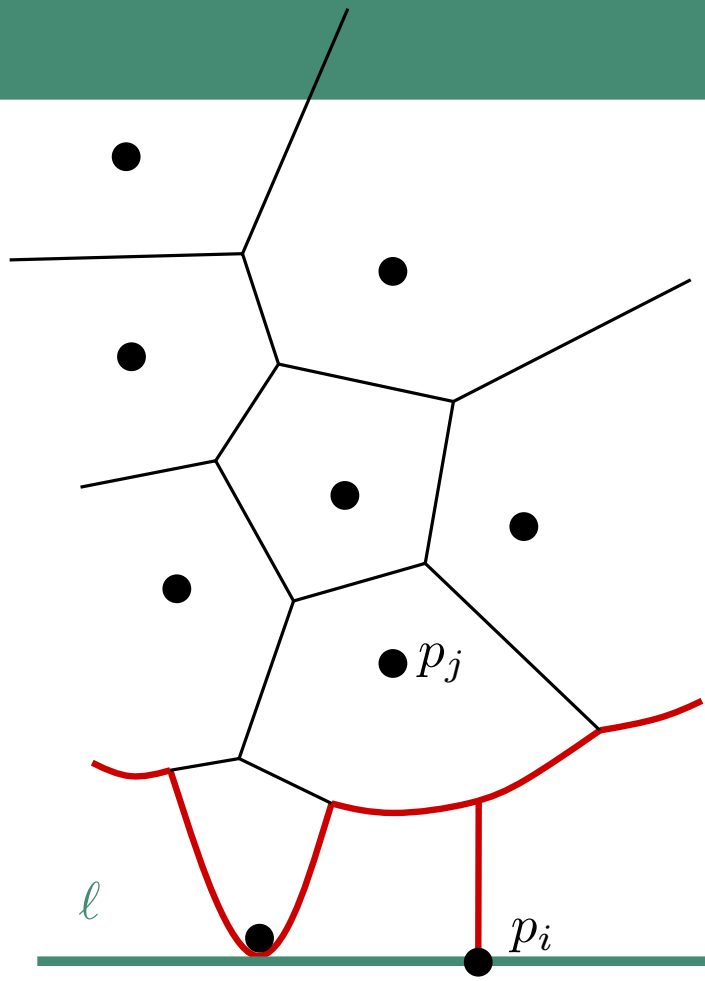
Compute a bounding box of the Voronoi vertices.
Traverse the unbounded faces to update the DCEL appropriately.

Fortune's algorithm

ALGORITHM

Handling a site event p_i

1. Search in the sweep line S for the arc α vertically above p_i. Let p_j be its site.
2. Replace in S the leaf α by a subtree with three leaves: p_j, p_i, p_j and two internal nodes $(p_j, p_i), (p_i, p_j)$. Rebalance S if necessary.
3. Update the events queue:
   - Delete from E all circle events involving α.
   - Insert in E, if necessary, the circle events of all triples involving the three new leaves.
4. Update the Voronoi diagram:
   - Create a new face, with a pointer to the new edge.
   - Create a new edge, with pointers to the incident faces.



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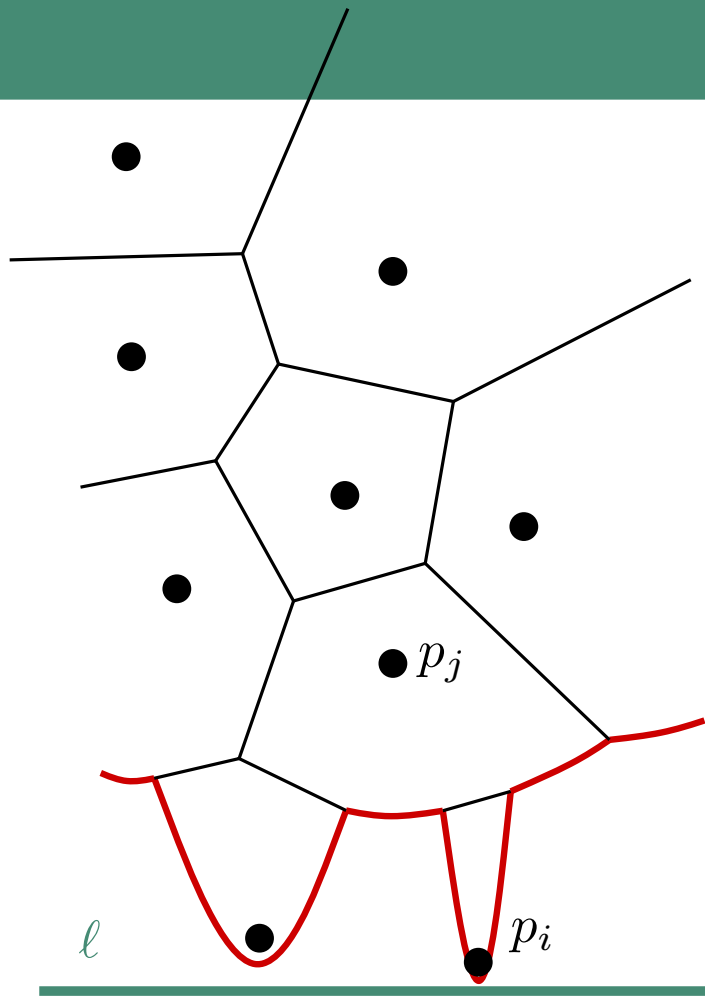

Fortune's algorithm

ALGORITHM

Handling a circle event c_ℓ

1. Search in the sweep line S for the arc α vertically above c_ℓ. Let p_j be its site. Let p_i, p_k respectively be the previous and the next sites in S.

2. Delete in S the leaf α and replace the internal nodes $(p_i, p_j), (p_j, p_k)$ by (p_i, p_k). Rebalance S if necessary.

3. Update the events queue:

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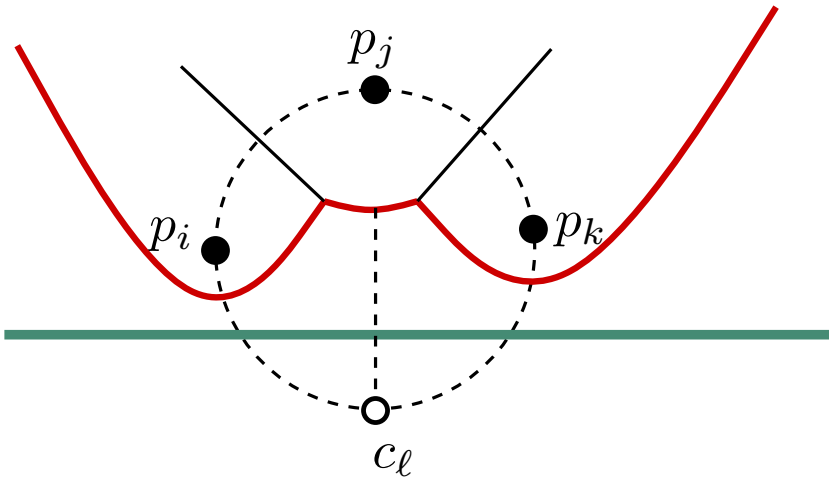
   - Insert in E, if necessary, the circle events of all triples involving the new consecutive arcs.



4. Update the Voronoi diagram:

   - Create a new vertex (the center of the circle), with a pointer to the new edge.
   - Complete the information of the two ending edges: final vertex and next edges.
   - Create a new edge, with pointers to its starting vertex, previous edge, and incident faces.

Fortune's algorithm

ALGORITHM

Handling a circle event c_ℓ

1. Search in the sweep line S for the arc α vertically above c_ℓ. Let p_j be its site. Let p_i, p_k respectively be the previous and the next sites in S.

2. Delete in S the leaf α and replace the internal nodes $(p_i, p_j), (p_j, p_k)$ by (p_i, p_k). Rebalance S if necessary.

3. Update the events queue:

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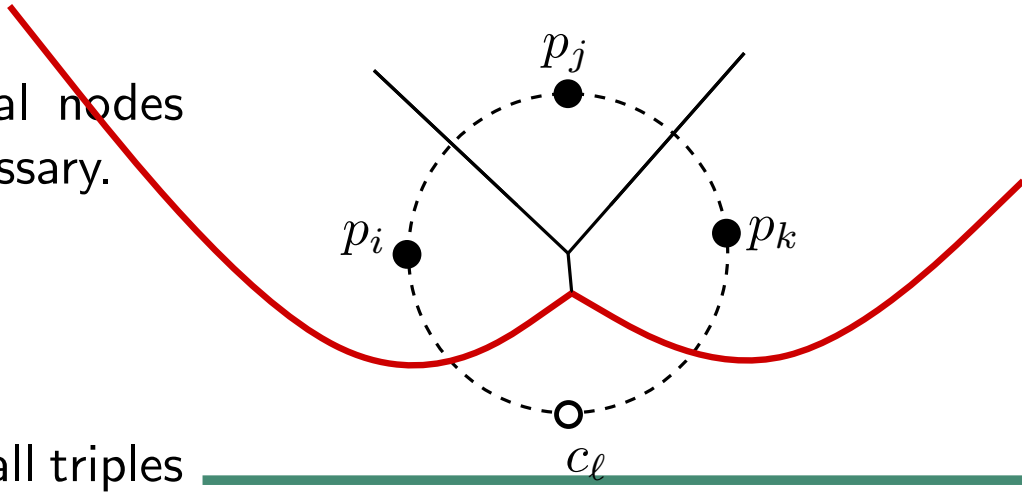

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Fortune's algorithm

COMPLEXITY

The algorithm runs in $O(n \log n)$ time and uses $O(n)$ space.

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Proof. As for running time:

- Each primitive operation on the events queue E and the sweep line statuts S takes $O(\log n)$ time. Each operation on the Voronoi diagram takes $O(1)$ time.
- Handling each event requires a constant number of such primitive oprations.
- The number of site events is n. The number of circle events actually processed is $2n - 3$, each corresponding to a Voronoi vertex (false alarms are not processed, as they are deleted before they are processed).

As for space:

- The sweep line complexity is $\leq 2n - 1$.
- The events queue stores $\leq 3n - 3$ events.
- The DCEL structure storing the Voronoi diagram uses $O(n)$ space.

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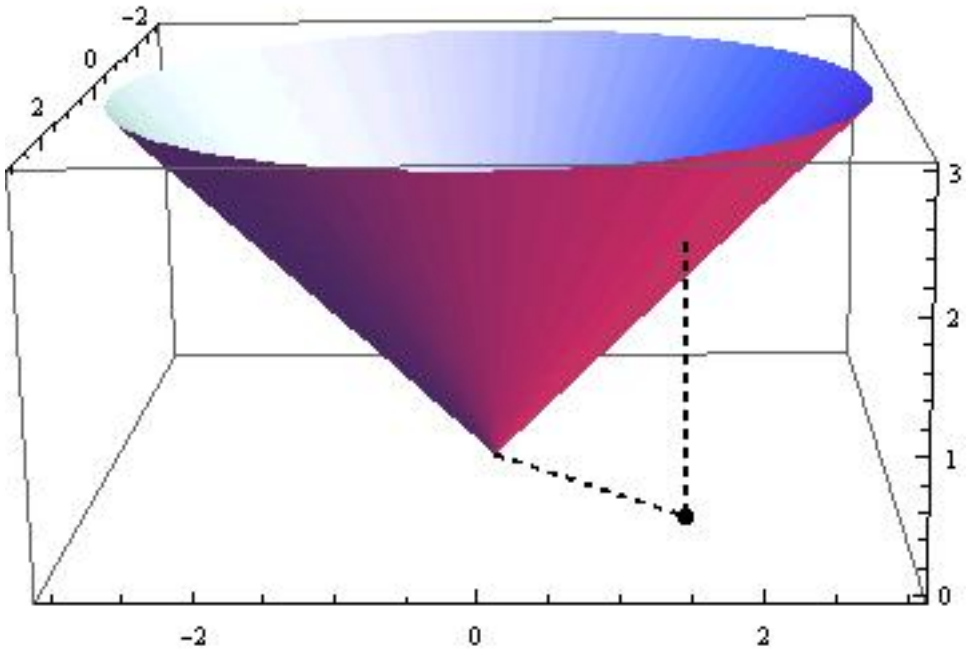
Degeneracies can be handled within the same complexity.

Fortune's algorithm

3-DIMENSIONAL INTERPRETATION

Fortune's algorithm

3-DIMENSIONAL INTERPRETATION



Consider the point set P to be embedded in the plane $z = 0$.

For each site $p_i = (x_i, y_i)$, consider the cone C_i:

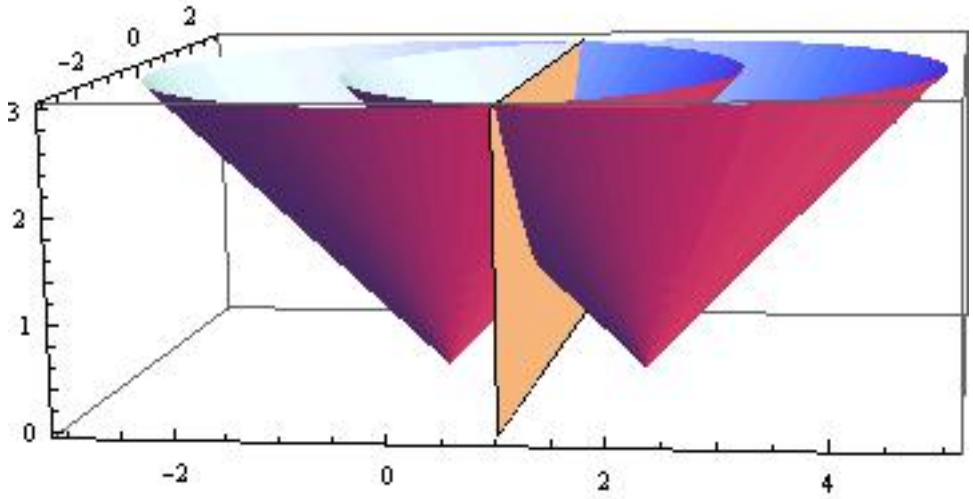
$$z^2 = (x - x_i)^2 + (y - y_i)^2.$$

Notice that, for every point q in the plane,

$$d(q, p_i) = \text{vertical\_distance}(q, C_i).$$

Fortune's algorithm

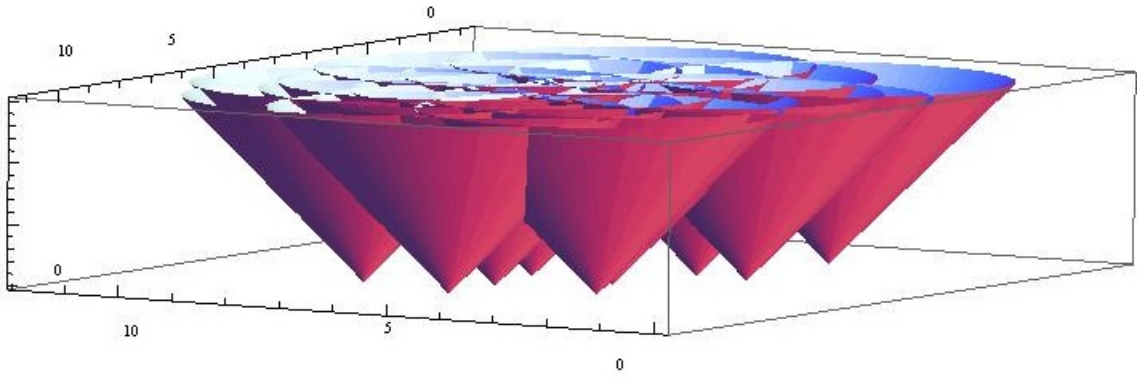
3-DIMENSIONAL INTERPRETATION



The intersection of two cones $C_i \cap C_j$ vertically projects onto the bisector b_{ij} of the sites p_i and p_j.

Fortune's algorithm

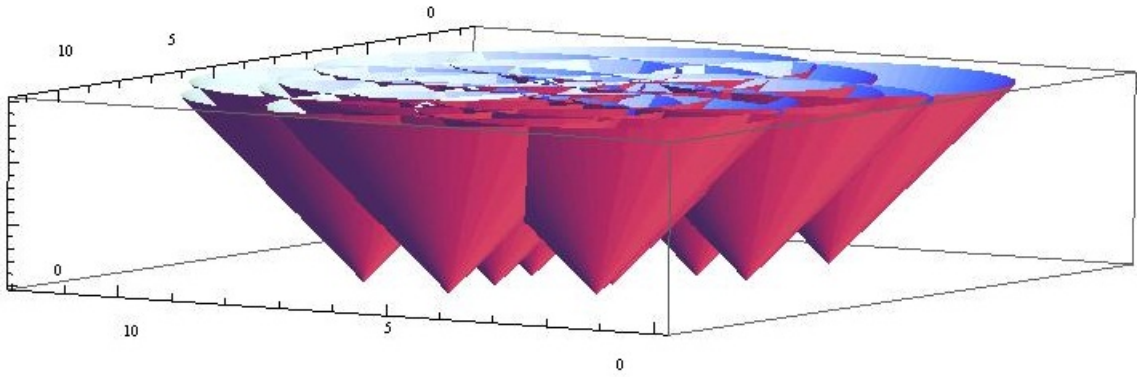
3-DIMENSIONAL INTERPRETATION



Consider the arrangement of all such cones.

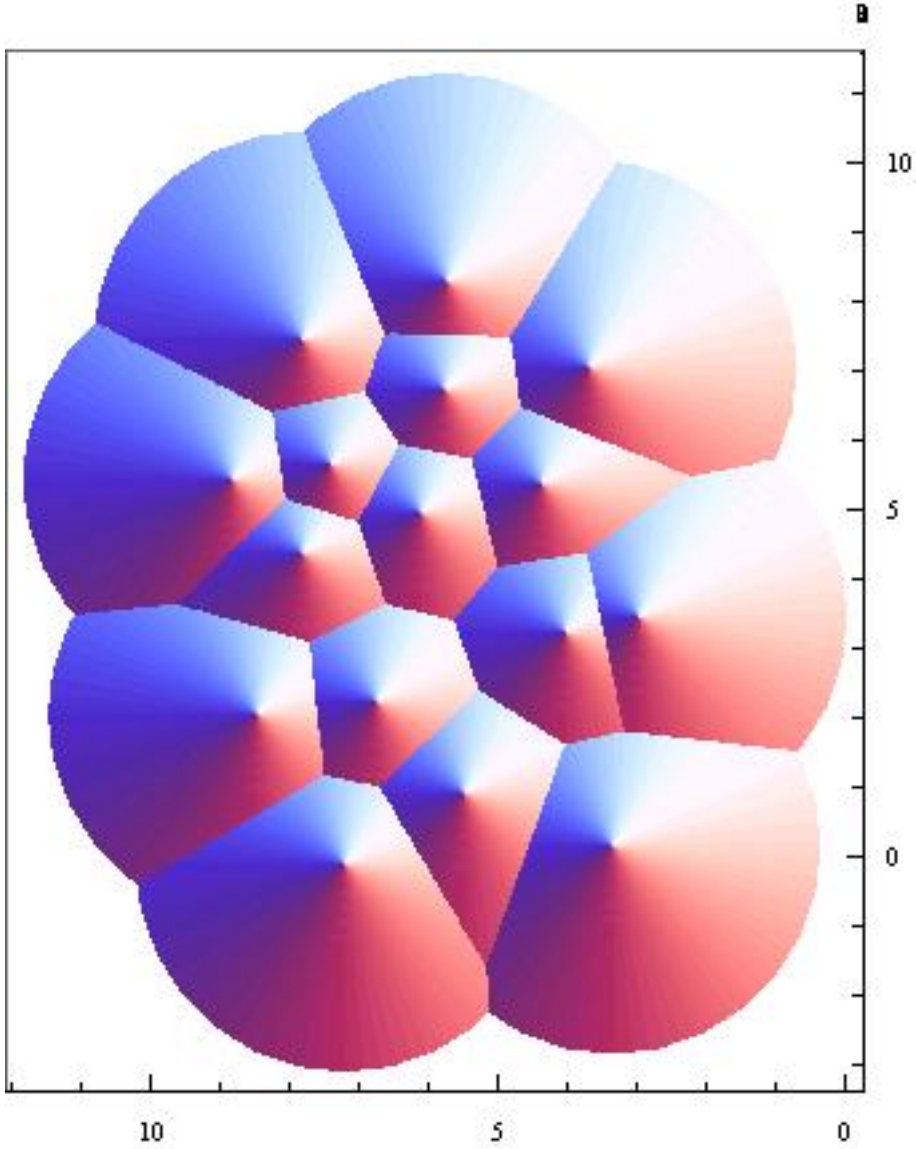
Fortune's algorithm

3-DIMENSIONAL INTERPRETATION



Consider the arrangement of all such cones.

Its lower envelope vertically projects onto $Vor(P)$.



Fortune's algorithm

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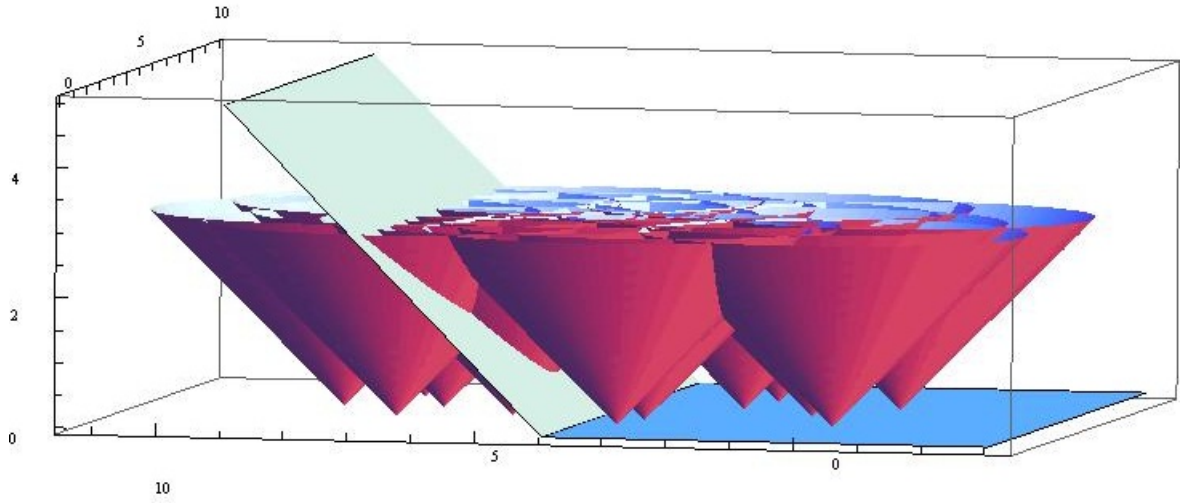

Sweep the arrangement of cones with a plane forming angle $\pi/4$ with the horizontal.

Fortune's algorithm

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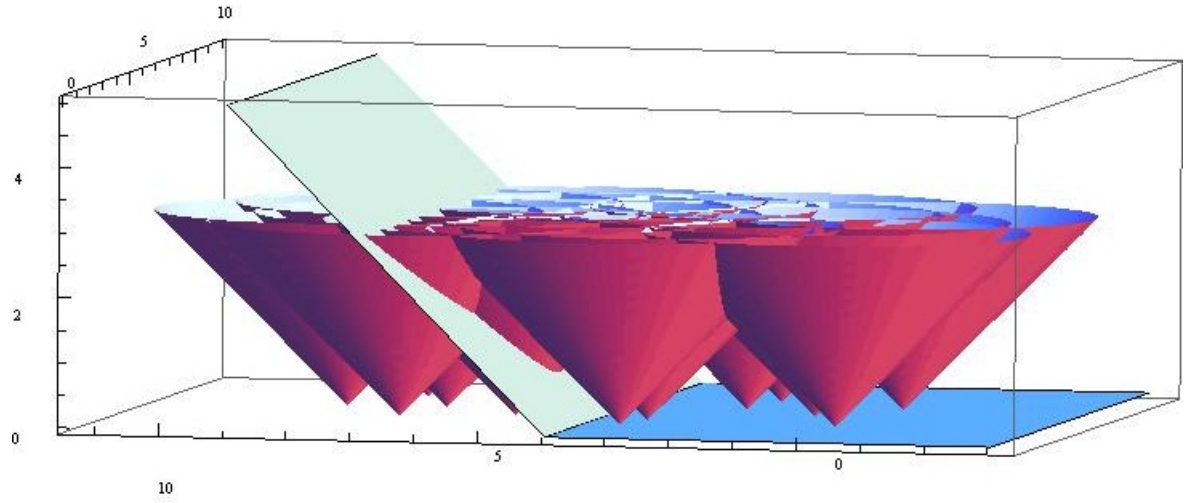
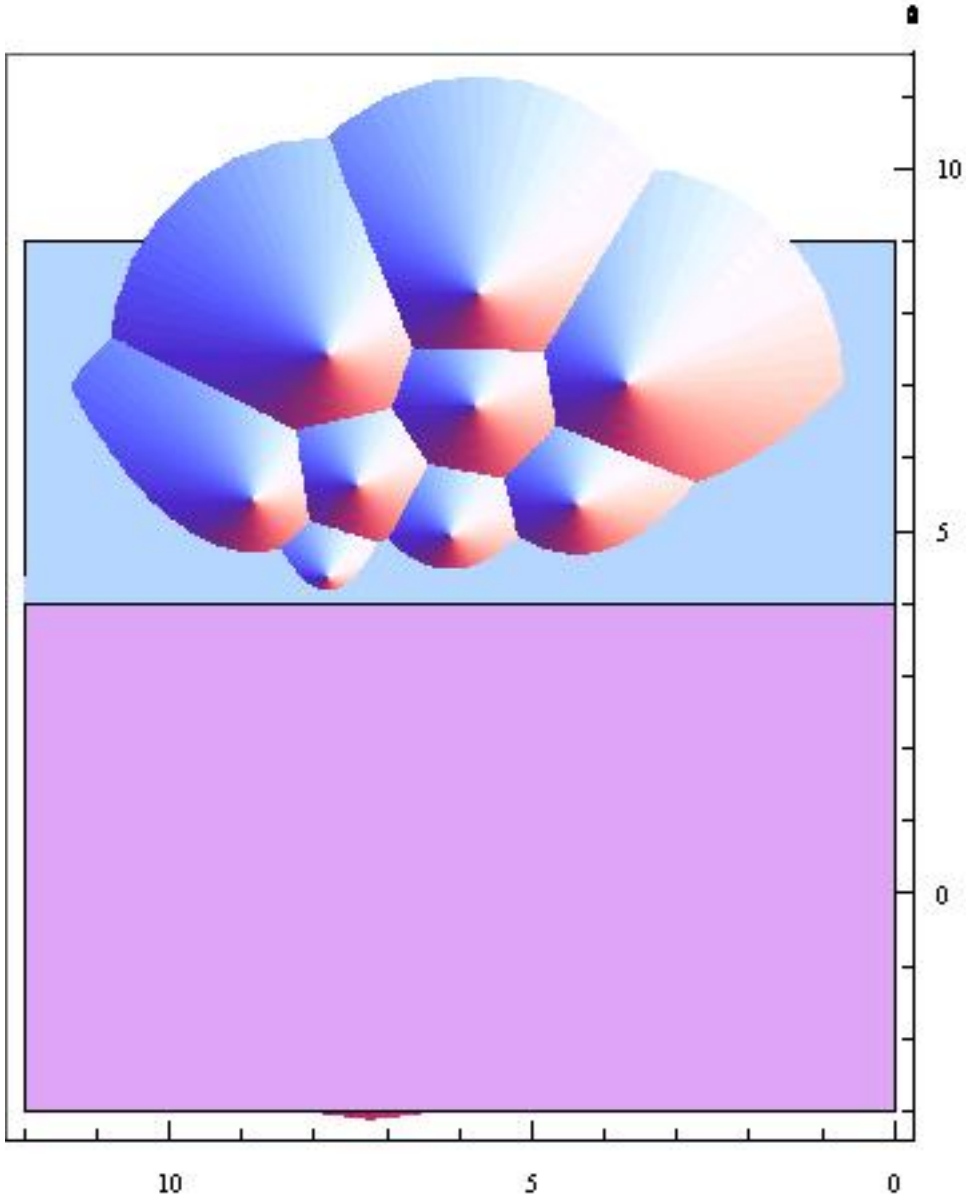




Sweep the arrangement of cones with a plane forming angle $\pi/4$ with the horizontal.

This is the same as Fortune's algorithm!

3D projection algorithm

3D projection algorithm

Consider the point set P as being embedded in the plane $z = 0$.

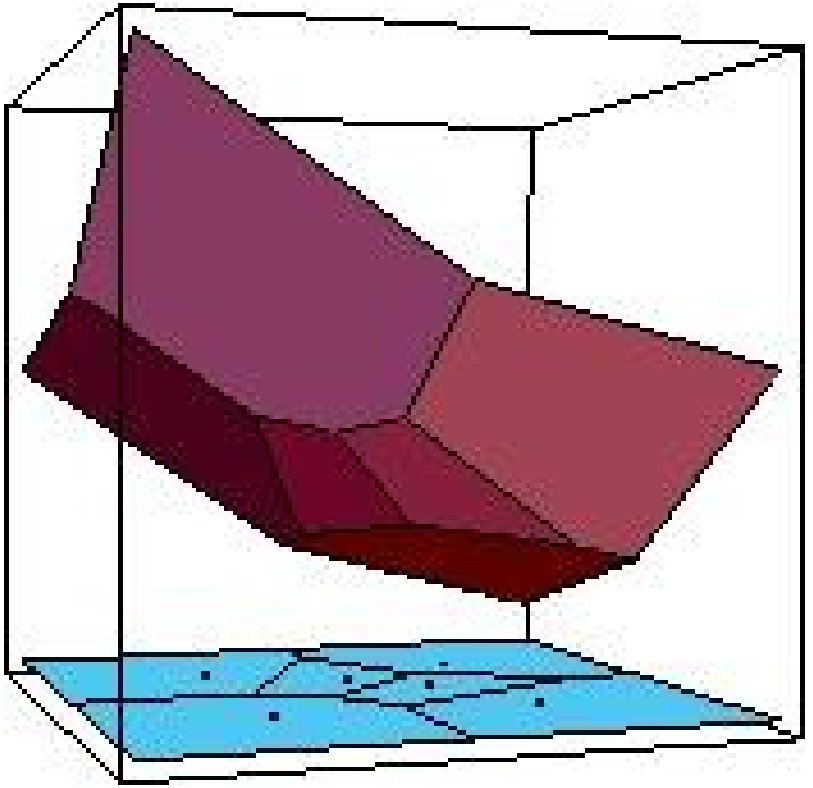
Consider the paraboloid $z = x^2 + y^2$.

For each point $p_i \in P$ let p_i^* be its vertical projection of p_i onto the paraboloid, i.e.:

$$\text{if } p_i = (a_i, b_i, 0), \text{ then } p_i^* = (a_i, b_i, a_i^2 + b_i^2).$$

For each point p_i^* consider the plane which is tangent to the paraboloid at p_i^*.

The Voronoi diagram of P is the orthogonal projection onto the plane $z = 0$ of the polyhedral convex region obtained when intersecting the upper halfspaces defined by these planes.



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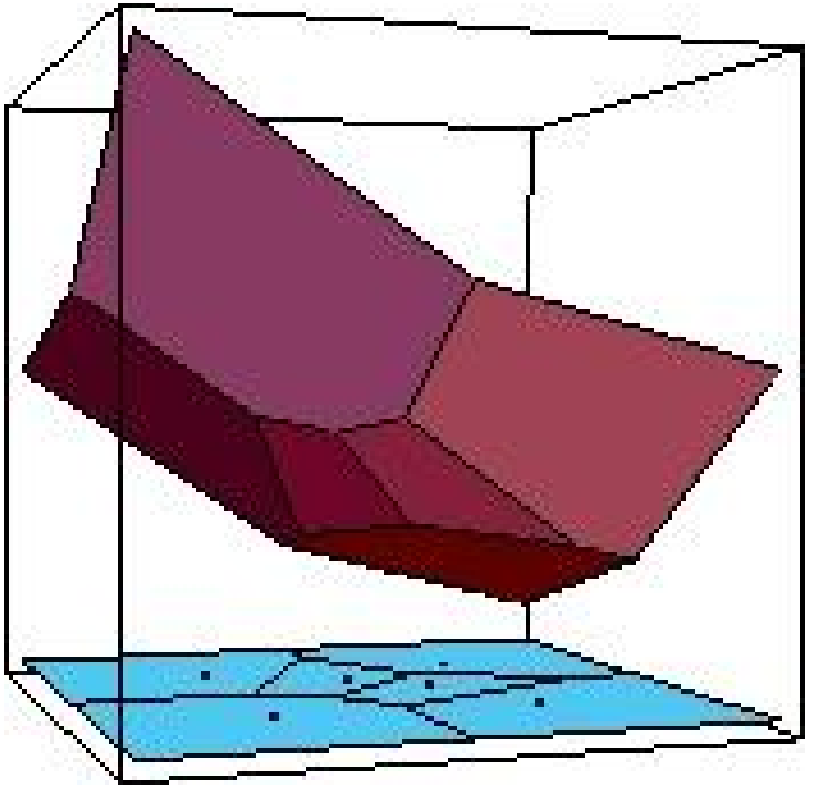

This gives rise to an algorithm that runs in $O(n \log n)$ time and uses $O(n)$ space.

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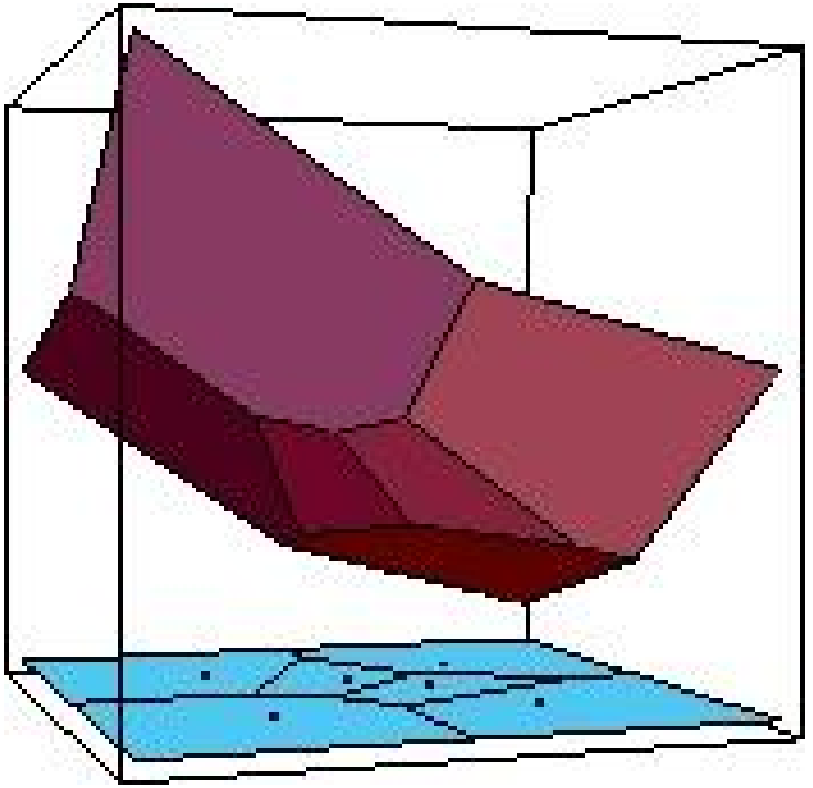

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Notice that the DCEL is the same for the polyhedron and for the Voronoi diagram!