Discrete and Algorithmic Geometry
Problems list 5
Year 2016-2017 Q1

1. The *Gabriel graph* of a point set P of the plane, $GG(P)$, is defined as follows: its vertices are the points of P; the segment $\overline{pq}$ connecting two points of P is an edge of the graph if and only if the disc that has $\overline{pq}$ as diameter does not enclose any other point of P.
   (a) Prove that $GG(P) \subseteq Del(P)$.
   (b) Prove that two points p and q are adjacent in $GG(P)$ if and only if the Delaunay edge $\overline{pq}$ intersects its dual edge in $Vor(P)$.
   (c) Propose an algorithm to compute $GG(P)$.

2. Two points p and q of a set P in the plane are *relative neighbors* if there is no other point from P simultaneously closer to p and q. In other words, in there is no other point $x \in P$ such that $d(x,p) < d(p,q)$ and $d(x,q) < d(p,q)$. The *relative neighborhood graph* $RNG(P)$ of P is defined as follows: its vertices are the points of P; a segment $\overline{pq}$ connecting two points of P is an edge of the graph if, and only if, p and q are relative neighbors.
   (a) Given two points, p and q, the *lens* $\text{lens}(p,q)$ is the intersection of the discs of radius $d(p,q)$ centered at p and q. Prove that p and q are relative neighbors if, and only if, $\text{lens}(p,q)$ does not contain any other point of P.
   (b) Prove that $RNG(P) \subseteq Del(P)$.
   (c) Propose an algorithm to compute $RNG(P)$.

3. Let $P = \{p_1, \dots, p_n\}$ be a set of n points in the plane. We define $m(p_i, p_j)$ as the minimum number obtained when counting the number of points from P contained in any closed disk containing p_i and p_j; and $m(P) = \max_{i,j=1\dots n} m(p_i, p_j)$.
   (a) Find the value of $m(P)$ for all configurations of P with $n = 3$ and $n = 4$.
   (b) If there exists a disk D containing p_i, p_j and some other k points from P, prove that then there also exists a disk $D' \subseteq D$ having p_i and p_j on its boundary and such that it contains at most k other points from P.
   (c) Use the previous result to obtain an algorithm to compute $m(p_i, p_j)$ for any given pair of points p_i and p_j.
   (d) Give an algorithm for computing $m(P)$.

4. The distance d_∞ between two points in the plane is defined as $d_\infty((a,b),(c,d)) = \max(|c - a|, |d - b|)$.
   (a) Disk centered on p with radius r: if $r > 0$, what is the geometric locus of the points at distance r from a given point p?
   (b) Bisector of p and q: what is the geometric locus of the points that are at the same distance from two given points, p and q? Hint: it may be convenient to distinguish some cases, depending on the relative position of p and q.
   (c) What can you say about the Voronoi region of a point? Is it convex? Is it star-shaped? Is it connected? ...
   (d) Characterize the points whose Voronoi region is unbounded.

5. Let $V_f(P)$ be de farthest-point Voronoi diagram of a 2-dimensional point set $P = \{p_1, \dots, p_n\}$.

(a) Prove that the Voronoi region of a point, $V_f(p_i)$, exists (i.e., is not empty) iff p_i is a vertex of the convex hull of P.

(b) Prove that $V_f(p_i)$ is always a convex polygonal region.

(c) Prove that all the non empty regions $V_f(p_i)$ in $V_f(P)$ are unbounded.

(d) What can you say about the 1-skeleton (i.e., the edge set) of $V_f(P)$?

6. Given a finite point set S, consider two new points $x, y \notin S$, $x \neq y$. Let $Reg(x)$ be the Voronoi region of x in $Vor(S \cup \{x\})$. Let $Reg(y)$ be the Voronoi region of y in $Vor(S \cup \{y\})$. Prove that the boundaries of $Reg(x)$ and $Reg(y)$ can intersect at most in 2 points.

7. The second order Voronoi diagram of a finite point set $S = \{p_1, \ldots, p_n\}$ is the decomposition of the plane defined as follows. Each pair of sites $p_i \neq p_j$ defines a region

$$V(p_i, p_j) = \{x \mid \forall k \neq i, j \quad d(x, p_i) \leq d(x, p_k) \wedge d(x, p_j) \leq d(x, p_k)\}.$$

(a) Prove that $V(p_i, p_j)$ is a (possibly empty) convex polygonal region for all i, j.

(b) Characterize the pairs of sites p_i, p_j such that $V(p_i, p_j)$ is bounded.

(c) Propose an algorithm to compute the second order Voronoi diagram of a finite point set S.

8. Suppose we are given a subdivision of the plane into n convex regions. We suspect that this subdivision is a Voronoi diagram, but we do not know its sites. Our goal is to find them.

(a) Prove that this is not always possible, as the sites are not always uniquely defined by the Voronoi diagram.

(b) Give a sufficient condition for the Voronoi diagram to have a uniquely defined set of n sites.

(c) Assume that the given subdivision positively is a Voronoi diagram. Develop an algorithm to find its n sites (assuming they are uniquely defined). Prove the correctness of your algorithm, and analyze its complexity.

9. Number of triangulations of a finite point set in the plane.

(a) Prove that no set of n points can be triangulated in more than $2^{\binom{n}{2}}$ ways.

(b) Prove that there exist sets of n points that can be triangulated in at least $2^{n-2\sqrt{n}}$ ways.

10. Prove that the *flip distance* between two triangulations of a given set of n points can be $\Omega(n^2)$.

11. Prove that the smallest angle of any triangulation of a convex polygon whose vertices lie on a circle is the same. This implies that, when a finite point set P has four or more concyclic points, any completion of its Delaunay pre-triangulation maximizes the minimum angle.

12. The weight of a triangulation is the sum of the lengths of all its edges. Prove or disprove the conjecture that the Delaunay triangulation is a minimum weight triangulation.

13. Delaunay triangulations in three dimensions.

(a) Given a finite point set $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^3$, propose a definition for the Delaunay triangulation of P.

(b) We have shown that the size of the Delaunay triangulation of a set of n points in the plane is $O(n)$. Prove that the size in $\mathbb{R}^3$ can be $O(n^2)$.

14. A triangulation of a finite point set P is called a *Pitteway triangulation* if, for each triangle $T = (a, b, c)$, every point in T has one of a, b or c as its nearest neighbor among the points in P.

    (a) Show that not every Delaunay triangulation is a Pitteway triangulation.

    (b) Characterize those Delaunay triangulations that are Pittaway triangulations.

    (c) Propose an efficient algorithm to check whether a Delaunay triangulation is a Pittaway triangulation.

15. Consider a 2-dimensional robot like Roomba, which can be described as a disk of a given radius r. We want to plan the trajectory of such a robot from a source position s to a target position t in the presence of obstacles that are described as points, $Obs = \{p_1, \dots, p_m\}$. Is this always possible? Propose an algorithm to decide whether it is or not. When the robot can move from s to t, your algorithm should compute a possible path. What is the complexity of your algorithm? Can you tell how long is your path, compared to the shortest one?

16. The *medial axis* is a very useful tool in pattern recognition and computer vision. The medial axis of a polygon P is the locus of all points in P that have more than one closest point among the points of ∂P. Prove the following statements:

    (a) The medial axis of P is a graph whose vertices are centers of circles tangent to ∂P in 3 or more points, and whose edges are the locus of centers of circles tangent to ∂P in 2 or more points.

    (b) The edges of the medial axis of P are line segments if P is convex, and can be curve segments otherwise. Which curves?

    (c) Is there a relationship between the medial axis of P and the Voronoi diagram of the vertices of P?

    (d) Is the medial axis always a tree?

    (e) Propose an algorithm to compute the medial axis of a given polygon P.