**Discrete and Algorithmic Geometry**
**Problems list 4**
**Year 2016-2017 Q1**

1. Find an $O(n)$-time algorithm to triangulate a star-shaped $n$-gon, assuming that a point in its kernel is known.

2. Find an algorithm to decide, given a simple polygon $P$, whether there exists a direction for which $P$ is monotone.

3. The usual definition of convexity reads as follows: A set $S$ is convex iff for all points $p, q \in S$ the line segment $pq$ lies in $S$.

   (a) Prove that the following is an alternative definition: A set $S$ is convex iff for all lines $\ell$, the intersection $S \cap \ell$ is a connected set.

   (b) Obtain as a corollary that a set $S$ is convex iff it is monotone in all directions.

4. An *orthogonal pyramid* is an orthogonal $y$-monotone polygon having one horizontal edge whose length coincides with the sum of the lengths of all the remaining horizontal edges. Find an algorithm to decompose an orthogonal pyramid into convex quadrilaterals, by insertion of internal diagonals. Is is possible to do so in $O(n)$ time?

5. About the number of triangulations of a polygon.

   (a) Which are the polygons that have the least number of triangulations?

   (b) Which are the polygons that have the most triangulations?

   (c) Compute the number $T_n$ of different triangulations of a convex $n$-gon, following these steps:

      i. By counting the number of triangulations that can be built from the initial insertion of one diagonal of the polygon, prove that

      $$T_n = \frac{n(T_3T_{n-1} + T_4T_{n-2} + T_5T_{n-3} + \cdots + T_{n-3}T_5 + T_{n-2}T_4 + T_{n-1}T_3)}{2(n-3)}.$$

      ii. By counting the number of triangulations that can be built from the initial insertion of a triangle whose base is on the edge $p_1p_2$ of the polygon, prove:

      $$T_{n+1} = T_n + T_3T_{n-1} + T_4T_{n-2} + T_5T_{n-3} + \cdots + T_{n-3}T_5 + T_{n-2}T_4 + T_{n-1}T_3 + T_n.$$

      iii. Infer the recursive formula $T_{n+1} = \frac{4n-6}{n}T_n$.

      iv. Finally, prove the expression $T_{n+2} = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$. These numbers are known as Catalan numbers, in honor to the Belgian mathematician Charles Catalan (1814-1894).

6. A *polyomino* is a plane geometric figure formed by connecting one or more equal squares edge to edge. It may be regarded as a finite subset of the regular square tiling with a connected interior or, if you prefer, as a connected set of pixels. For any polyomino $P$, we define two types of guards, namely:

   - Point guards: a point guard $g$ "sees" a point $p \in P$ if the line segment $gp$ lies in $P$.
   - Pixel guards: a pixel guard $g$ "sees" a pixel of $P$ if for all points $p$ in the pixel the line segment $gp$ lies in $P$.

Let $P$ be any polyomino with $m$ pixels. Prove the following statements:

(a) $\lfloor \frac{m+1}{3} \rfloor$ point guards are sometimes necessary to guard $P$.

(b) $\lfloor \frac{m}{2} \rfloor$ pixel guards are always sufficient and sometimes necessary to guard $P$.

7. A *visibility-independent set* in a polygon $P$ is a finite set of points on $P$, $IS(P)$, such that the visibility polygons of its elements are pairwise disjoint, i.e., for all $p, q \in IS(P)$, it holds that $Vis(p, P) \cap Vis(q, P) = \emptyset$. Prove that no single point guard in $P$ is able to see more than one point of $IS(P)$. Conclude that $g(P) \geq is(P)$, where $g(P)$ is the number of point guards in a minimum-cardinality set of point guards of $P$ and $is(P)$ is the number of points in a maximum-cardinality visibility-independent set of $P$. Is the bound tight?

8. A polygon $P$ is *weakly visible* from an edge $a$ if each (internal or boundary) point $p \in P$ can see at least one point $m$ of $a$. Give a linear time algorithm for triangulating such polygons.

9. Prove that any simple polygon with holes can be triangulated. What can you say about the number of triangles of its triangulations?

10. If $P$ is a simple polygon and $a$ is an edge of the convex hull of $P$ but it is not an edge of $P$, then $a$ is called the *lid* of a *pocket* of $P$, limited by $a$ and the edge chain of $P$ connecting the two endpoints of $a$. Let $P$ be a convex $n$-gon and $Q$ a simple $m$-gon. Suppose that we are given a triangulation of the interior of $Q$, as well as of each of its pockets. Prove that it is then possible to compute the intersection of $P$ and $Q$ in $O(n + m)$ time. Suggestion: start enclosing $Q$ in a triangle $T$ big enough to enclose also $P$.

11. Let $P$ be a simple polygon. Propose an algorithm to find the shortest path within $P$ connecting two (internal or boundary) points $s$ and $t$ of $P$. Suggestion: start triangulating the polygon.