Discrete and Algorithmic Geometry
Problems list 3
Year 2016-2017 Q1

1. Given n points in the plane, $p_1, \ldots, p_n$, find an algorithm to decide, for any new point q, whether q belongs to the exterior, the interior or the boundary of the convex hull of $p_1, \ldots, p_n$. Is it possible to solve this problem in $o(n \log n)$ time?

2. Let P be a convex n-gon, and t a real number greater than the value of the area of P. For each point q external to P, let $P(q)$ be the polygon resulting from computing the convex hull of $P \cup \{q\}$.

   (a) Prove that the points q such that the area of $P(q)$ equals t form the boundary of a convex polygon $Q(t)$ enclosing P.

   (b) Prove that the number of edges of $Q(t)$ is between n and $2n$, and that, in fact, it is $2n$, except for a finite number of values of t.

   (c) Give an algorithm to compute $Q(t)$.

3. Propose an algorithm to compute the convex hull of a monotone polygon in optimal time.

4. A vertical line leaves to its left n blue points and to its right n red points. If the union of all red and blue points does not contain any three aligned points, prove that it is possible to match the blue and the red points pairwise, such that the resulting red-blue segments do not intersect, and give an algorithm to construct such a matching.

5. Let P be an x-monotone polygonal line with n vertices, all with positive y-coordinate, and let a and b respectively be the minimal and maximal x-coordinates of the points in P. We call *terrain* the portion of the plane enclosed by the x-axis, the vertical lines $x = a$ and $x = b$, and the polygonal line P. We say that P is the profile of the terrain.

   (a) Characterize the set S of all the points lying in the strip $a \leq x \leq b$ and above P which can see all of P. Give an algorithm to find the point of S of minimum y-coordinate.

   (b) Give an algorithm to find the point of the profile allowing to construct the shortest tower guarding the entire profile P.

6. Let P a polytope in $\mathbb{R}^3$ with n vertices. Propose an algorithm to find the longest vertical segment inscribed in P.

7. Find an algorithm to decide, given a polyhedron P in $\mathbb{R}^3$ with n vertices, and a query point $q \in \mathbb{R}^3$, whether q lies in the interior, the boundary or the exterior of P. If the polyhedron P is convex, can you think of a way of preprocessing it so that the queries can be solved in a more efficient way?