Discrete and Algorithmic Geometry
Problems list 2
Year 2016-2017 Q1

1. How many connected components can the intersection of two x-monotone polygons have? Give an algorithm to efficiently compute them by exploiting the monotonicity of the polygons.

2. Given n points in the plane, in general position, we want to construct a simple polygon having them as vertices.

   (a) Prove that this problem has complexity $\Omega(n \log n)$.
   (b) Propose an algorithm to solve it.
   (c) Prove that the points admit an x-monotone polygonization, i.e., that there always exists a simple x-monotone polygon having the given points as vertices. Give an algorithm to construct it.

3. Propose an algorithm for decomposing an isothetic polygon (a polygon with only horizontal and vertical edges) into rectangles, using segments aligned with the edges of the polygon. Your algorithm must produce the minimum number of pieces, and the output must give a complete description of the partition.

4. Given a set P with n points in the plane, propose an algorithm to find the position of the isothetic unit square containing the maximum number of points from P.

5. Given a set R with n lines of the plane, find an algorithm to compute the smallest isothetic rectangle containing all the intersection points of the lines in R.

6. Given a planar decomposition S of complexity n, consider a point set P of cardinality m. How can we compute the region of S where each point of P is located?

7. Let S be a set of n pairwise disjoint line segments in the plane. One segment in S can be separated in the $(1, 0)$ direction if it is possible to translate it horizontally towards $+\infty$ without it colliding with any of the remaining segments.

   (a) Prove that there always exists at least one segment in S that can be separated in the $(1, 0)$ direction.
   (b) Obtain an algorithm that gives an ordering of the segments of S that can be used to separate them, one by one.

8. Let S be a set of n pairwise disjoint line segments, $s_1, \ldots, s_n$, and let C be the set of their endpoints.

   (a) Prove that, for each point q lying in the exterior of the convex hull of C, there exists at least one line segment in S that can be entirely seen from q.
   (b) Obtain an efficient algorithm to find it.
   (c) Give an example where q is interior to the convex hull of C and it cannot see any line segment of S entirely.

9. Let S be a set of n pairwise disjoint line segments in the plane. Let p be a point not lying in any of the segments of S. We say that p *weakly sees* a line segment $s \in S$ if there exists a point $x \in s$ such that the interior of the line segment px does not intersect S. Give an algorithm to decide which of the segments of S can be weakly seen from p in $O(n \log n)$ time.

10. Given n line segments in the plane (without assuming them to be disjoint) and two points p and q, find an algorithm to decide whether there exists any point x in the plane that can simultaneously see p and q, i.e., such that the line segments $\overline{xp}$ and $\overline{xq}$ do not intersect any of the given n segments.

11. Find an algorithm to compute the area of the union of n isothetic rectangles. If it suits you, you can start supposing that their edges are not aligned.

12. Let R be a set of n rectangles, $R_1, \ldots, R_n$, all them having their base on the x-axis. Each rectangle R_i is defined by its lower left vertex, $(x_i, 0)$, its width, a_i, and its height, h_i. The *profile* of R is the upper envelope of the rectangles and the x-axis.

    (a) It is quite obvious that the profile of R has at most $2n$ vertical segments (the vertical edges of the rectangles). Prove that it has at most $2n + 1$ horizontal segments.

    (b) Consider the profiles corresponding to two sets of rectangles, R' and R'', consisting of a total number of n', respectively n'' horizontal and vertical segments. Prove that the number of intersections of the two profiles is $O(n' + n'')$. Note: you may suppose that there is no overlapping among the segments of the two profiles.

    (c) Give an algorithm to compute the profile of a set R of n rectangles.

13. Let P be a convex polygon, and consider two points a and b external to P. We will say that *a conquers b* (or that *b is conquered by a*) when b belongs to the convex hull of $P \cup \{a\}$. Give an algorithm, as efficient as possible, receiving as input a convex polygon P with n vertices and a set C with k points, external to P, and producing as output the points of C that are not conquered by any other point in C.