# INTERSECTING HALF-PLANES and related problems

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# DUALIZING POINTS AND LINES

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In the plane, dualities transform points into lines and lines into points.

One of the most frequently used is:

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1. $D^2 = Id$.
2. $D$ is a bijection between points and non-vertical lines.
3. A point $p$ lies above/on/below a line $\ell$ if and only if the point $\ell^*$ lies above/on/below the line $p^*$.

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   *Proof*: If $p=(a,b)$ and $\ell$ is the line of equation $y=mx+n$, the relative position of $p$ wrt $\ell$ is given by the (in)equation $b \lessgtr ma+n$, which is equivalent to $-n \lessgtr am-b$, and this expresses the relative position of $\ell^*=(m,-n)$ wrt $p^*:\ y=ax-b$.

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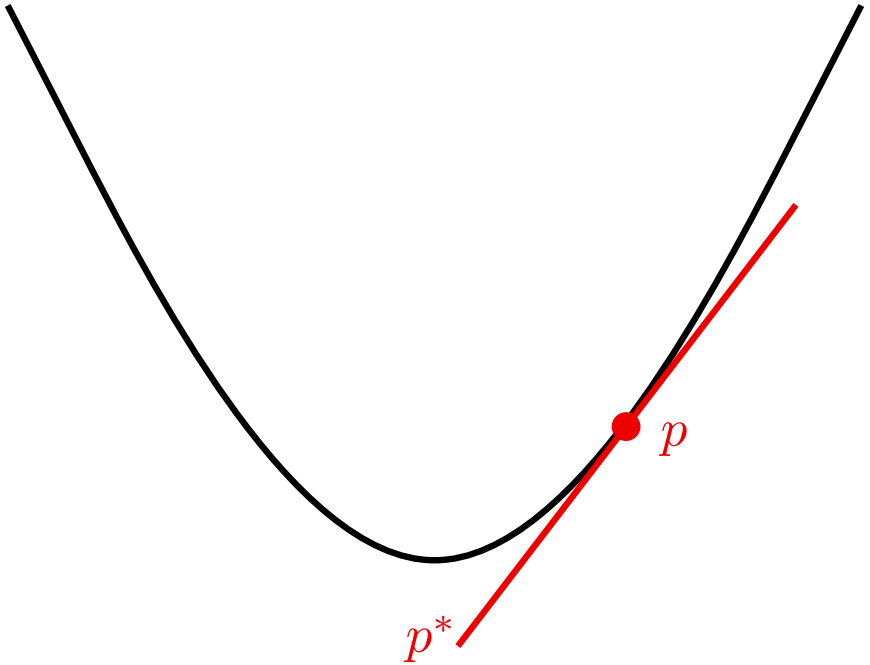

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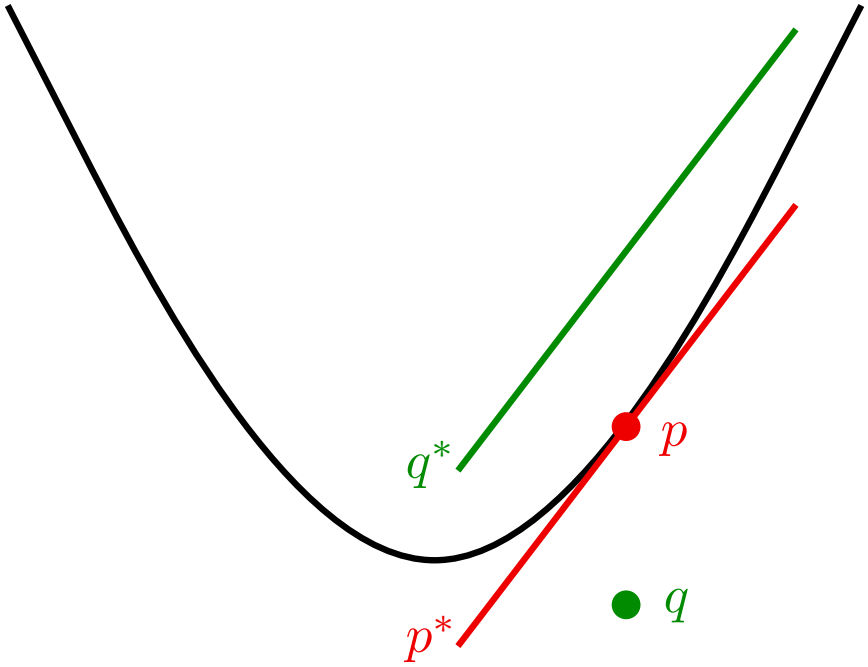

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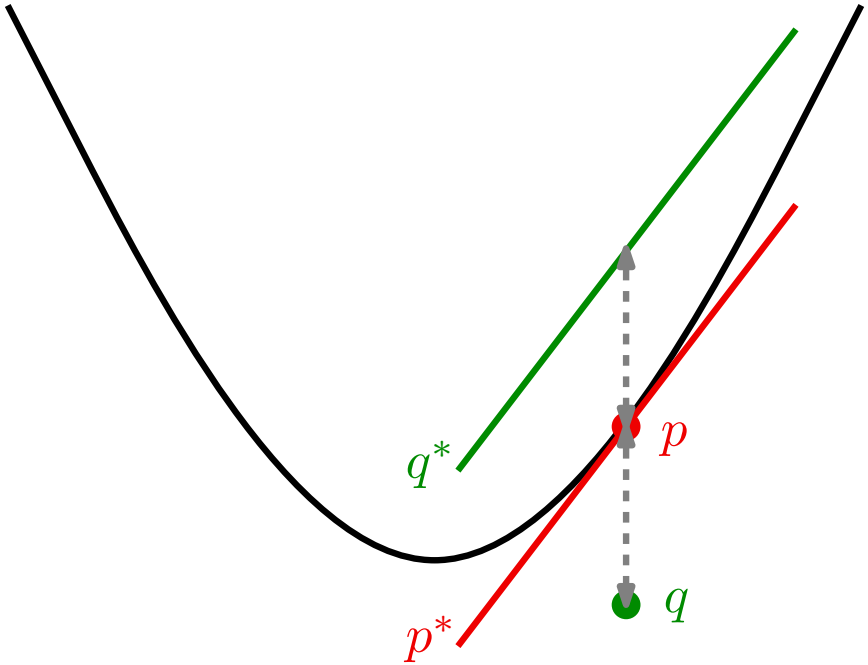

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*Proof*:
Since $y'(x) = x$, the line tangent to the parabola at a point $p = (a, a^2/2)$ has equation $y - a^2/2 = a(x - a)$ i.e., $y = ax - a^2/2$.

For any other point $q = (a, b)$:
$d_{vertical}(p, q) = b - a^2/2$;
$d_{vertical}(p^*, q^*) = -b + a^2/2$.

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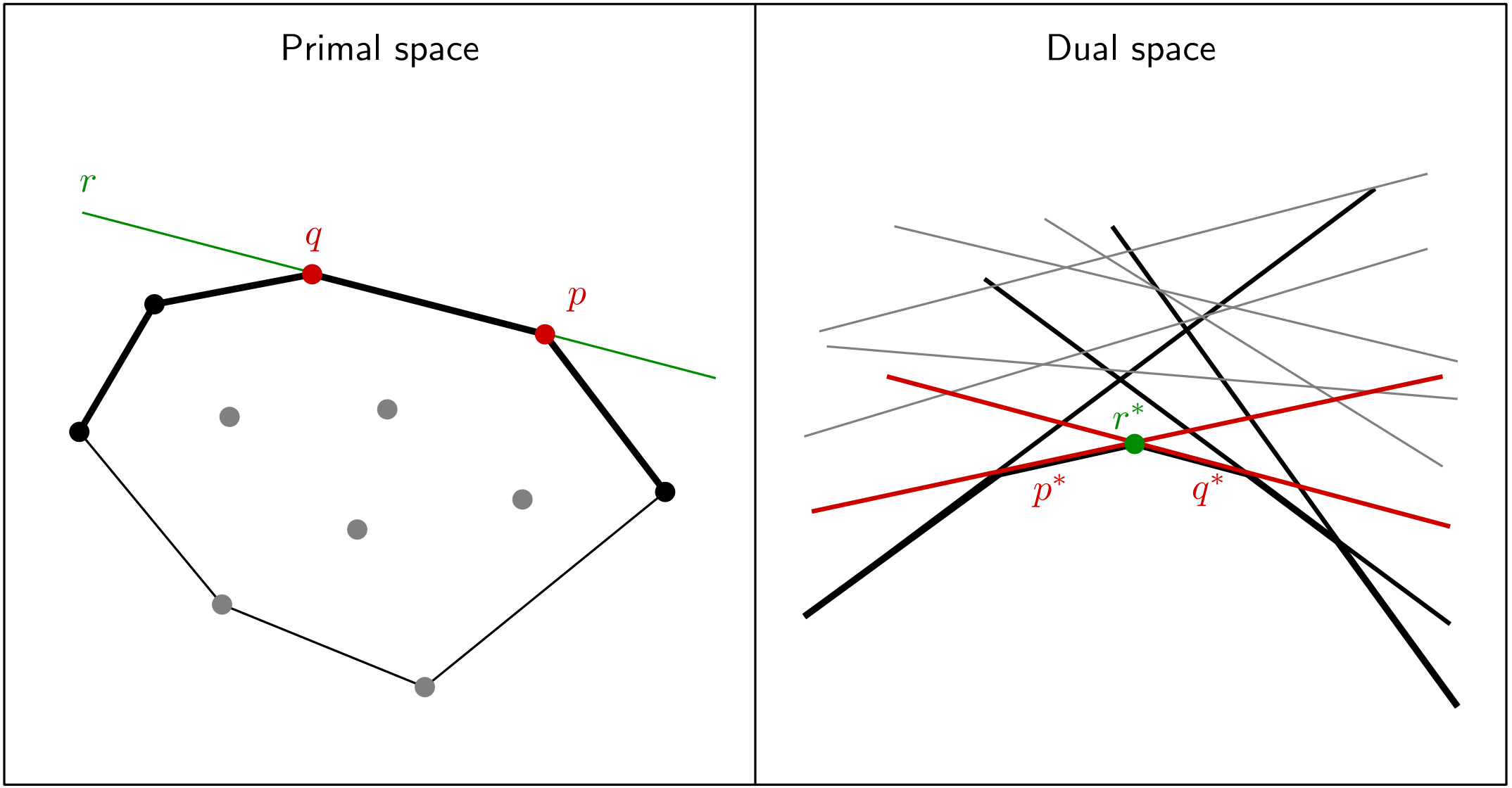

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### Lower bound

Duality reduces computing the convex hull of $n$ points to computing the intersection of $n$ half-planes in $O(n)$ time. Hence, the problem of computing the intersection of $n$ half-planes has an $\Omega(n \log n)$ lower bound.

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## Linear program

Optimize a linear function $ax + by$
subject to $n$ linear restictions $A_i x + B_i y + C_i \leq 0$, where $i = 1, \ldots, n$.

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## Algo 2 (Megiddo, Dyer)

In $O(n)$ time, it is possible to find the vertex of $R$ achieving the optimum, without computing the entire feasible region $R$.

This is done by a prune and search strategy, which at each step eliminates a constant fraction of the restrictions:

- either because they are redundant,
- or because they are not needed to define the solution vertex.

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Let's define $u_1 = \max\{-c_i/a_1 \mid i \in I_0, a_i \leq 0\}$ and $u_2 = \min\{-c_i/a_1 \mid i \in I_0, a_i \geq 0\}$.

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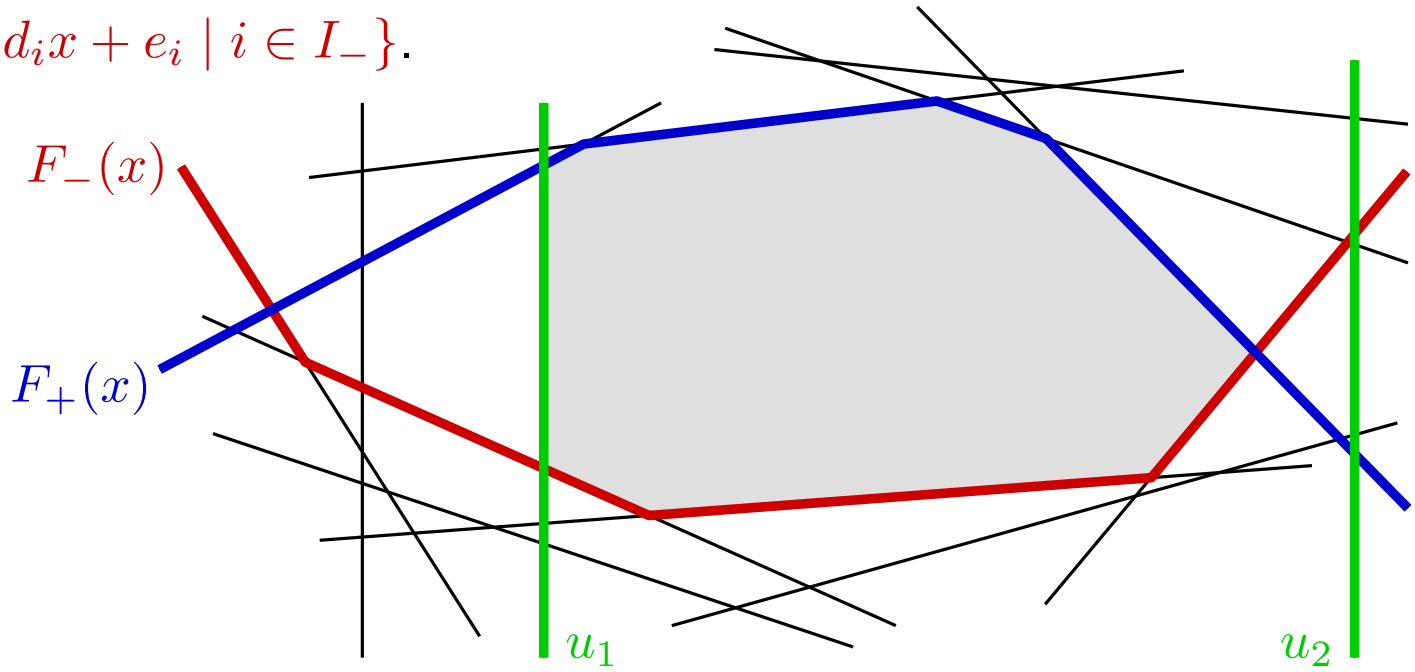

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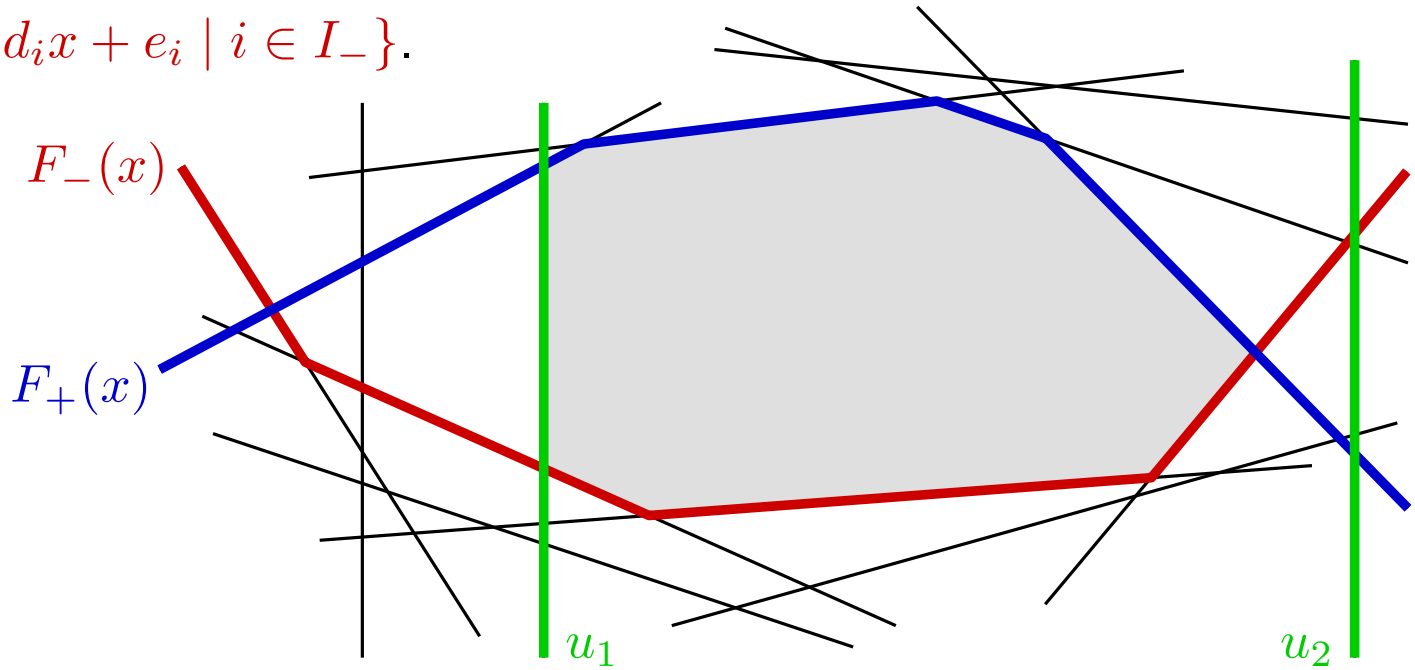

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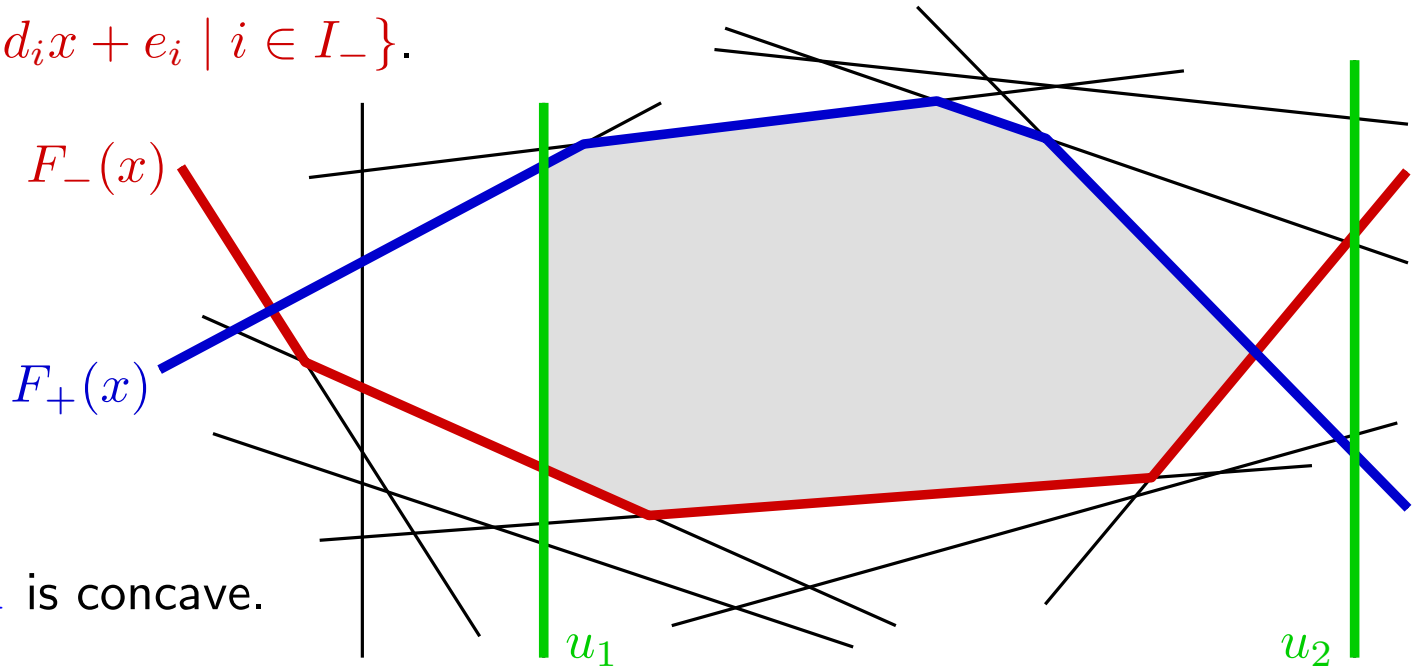
Notice that $F_-$ is convex and $F_+$ is concave.



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This is done by analizing the values of $F_+$, $F_-$ and their slopes $f_+^l, f_+^r, f_-^l, f_-^r$ in $x'$, where the slopes are defined as follows:

- If $F_-(x')$ (resp. $F_+(x')$) is defined by a unique index $i \in I_-$ (resp. $I_+$), then $f_-^l(x') = f_-^r(x') = d_i$ (resp. $f_+^l(x') = f_+^r(x') = d_i$).
- If $F_-(x')$ (resp. $F_+(x')$) is defined by two indexes $i, j \in I_-$ (resp. $I_+$), then $f_-^l(x') = \min(d_i, d_j)$ and $f_-^r(x') = \max(d_i, d_j)$ (resp. $f_+^l(x') = \max(d_i, d_j)$ and $f_+^r(x') = \min(d_i, d_j)$).

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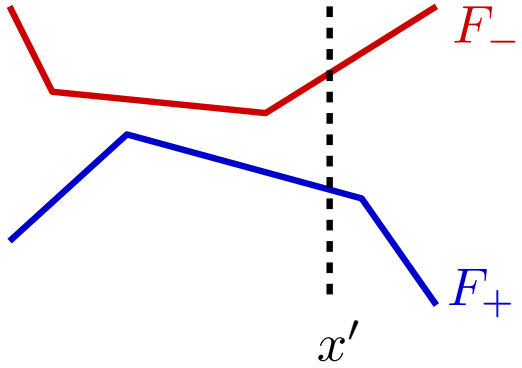


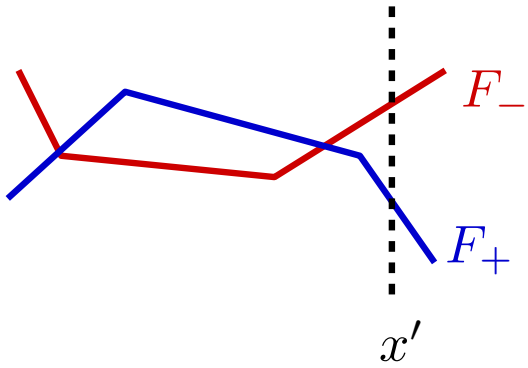

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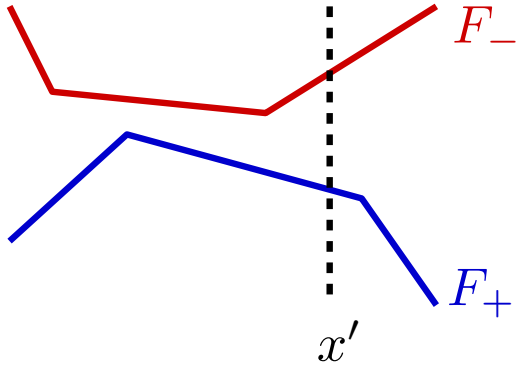


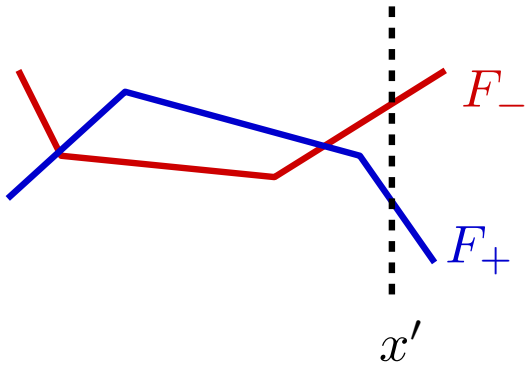


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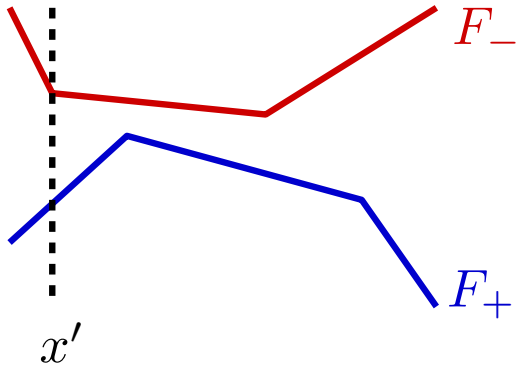


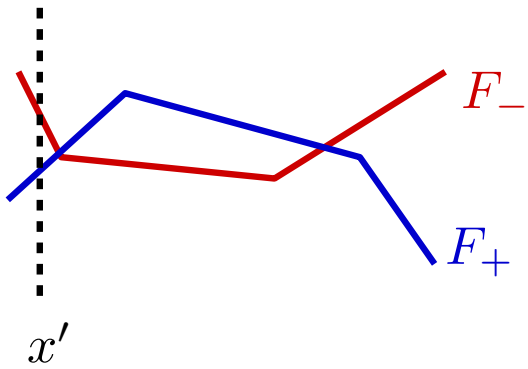

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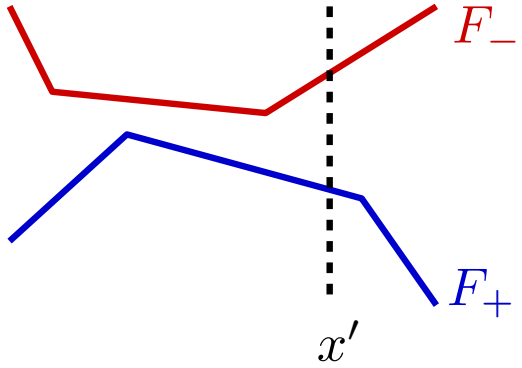


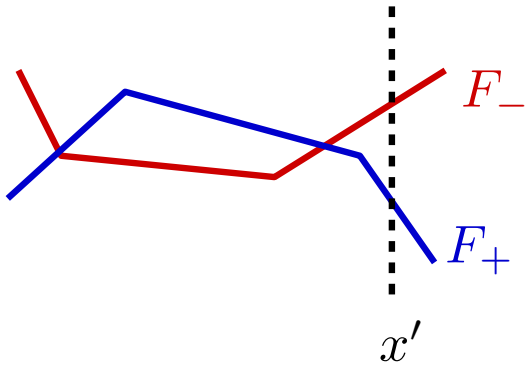


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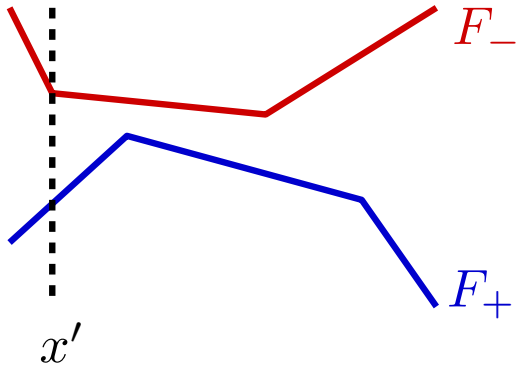
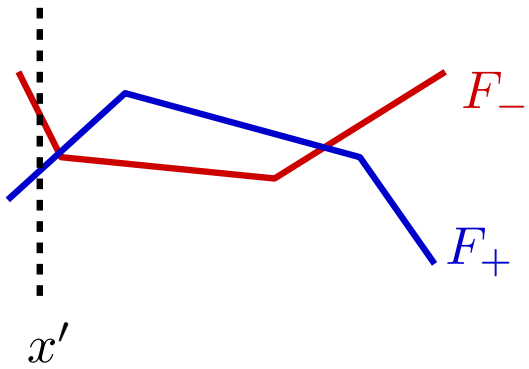
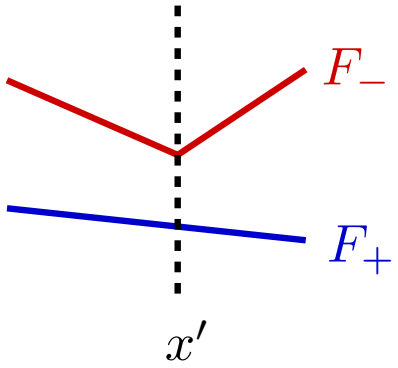
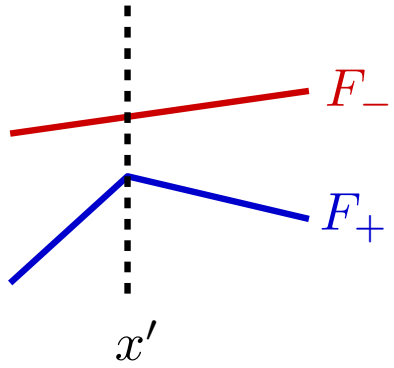



If $f_-^l(x') \leq f_+^l(x')$ and $f_-^r(x') \geq f_+^r(x')$
then there is no solution.



The search

**Case 2.** $F_-(x') \leq F_+(x')$ ($x'$ is feasible)

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If $f_-^l(x') < 0 < f_-^r(x')$,
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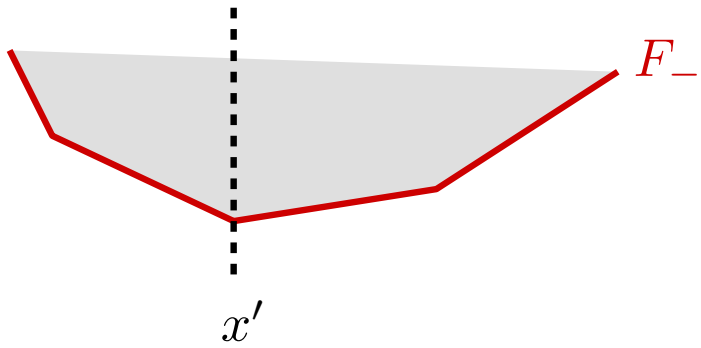

**Case 2.** $F_-(x') \leq F_+(x')$ ($x'$ is feasible)

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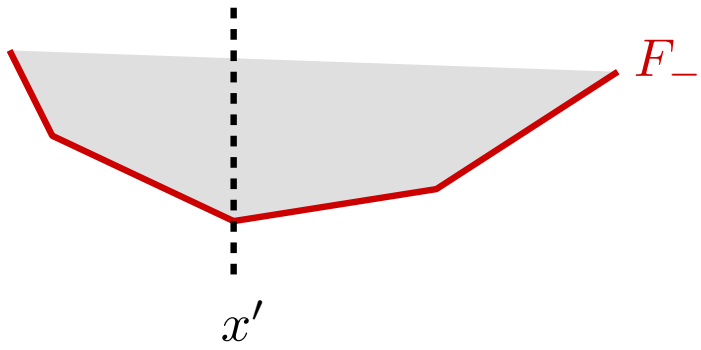
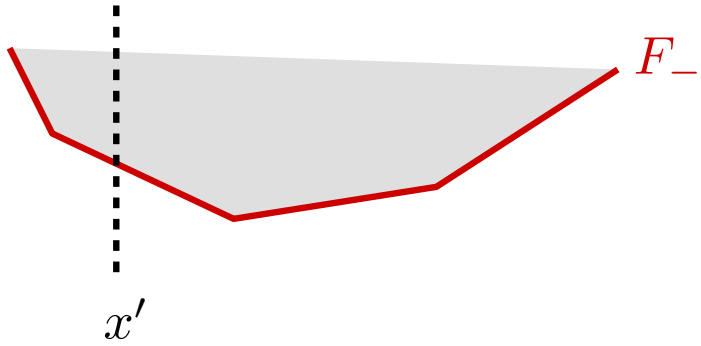



If $f_-^r(x') < 0$,
then search to the right of $x'$.



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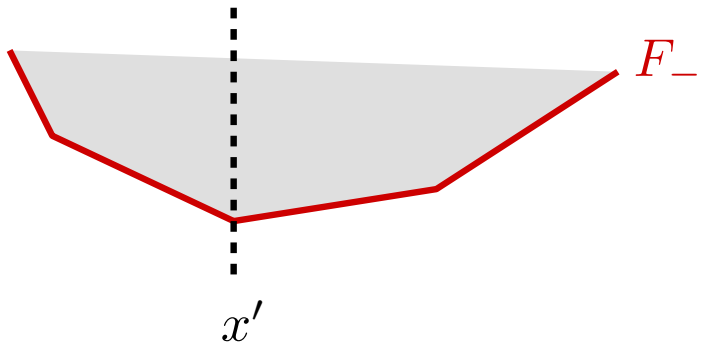


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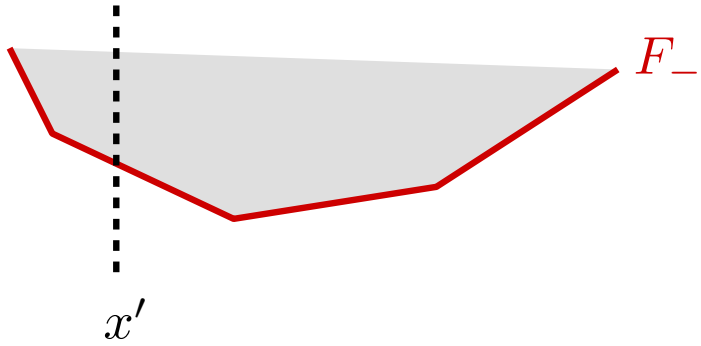


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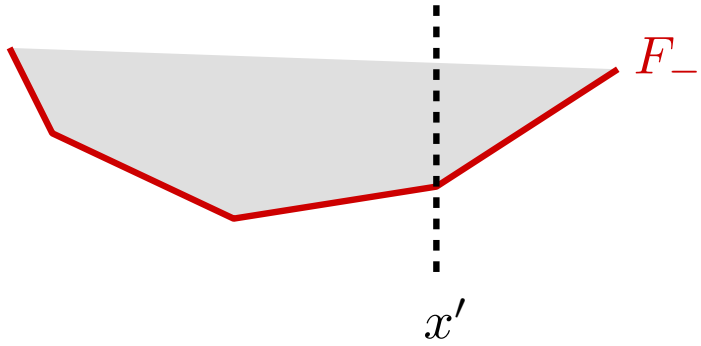

# SOLVING LINEAR PROGRAMS

## The algorithm

## The algorithm

**Initialization** Eliminate all $i \in I_0$ except those corresponding to $u_1$ and $u_2$, if they exist.

# SOLVING LINEAR PROGRAMS

## The algorithm

**Initialization** Eliminate all $i \in I_0$ except those corresponding to $u_1$ and $u_2$, if they exist.

**Advance.** Repeat as many times as necessary:

1. Pair up all restrictions corresponding to $I_-$ (possibly but one). Analogously for $I_+$. Compute the abscissa of the intersection point of each pair of lines, if it exists.

2. For all pairs $i, j \in I_-$ (analogously for $I_+$), do:

   - If $d_i = d_j$, eliminate the line corresponding to $\min(e_i, e_j)$ (resp. $\max(e_i, e_j)$).
   - If $d_i \neq d_j$ and $x_{ij}$ is the abscissa of the intersection point of the two lines, then do:
     - If $x_{ij} < u_1$, eliminate the line corresponding to $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).
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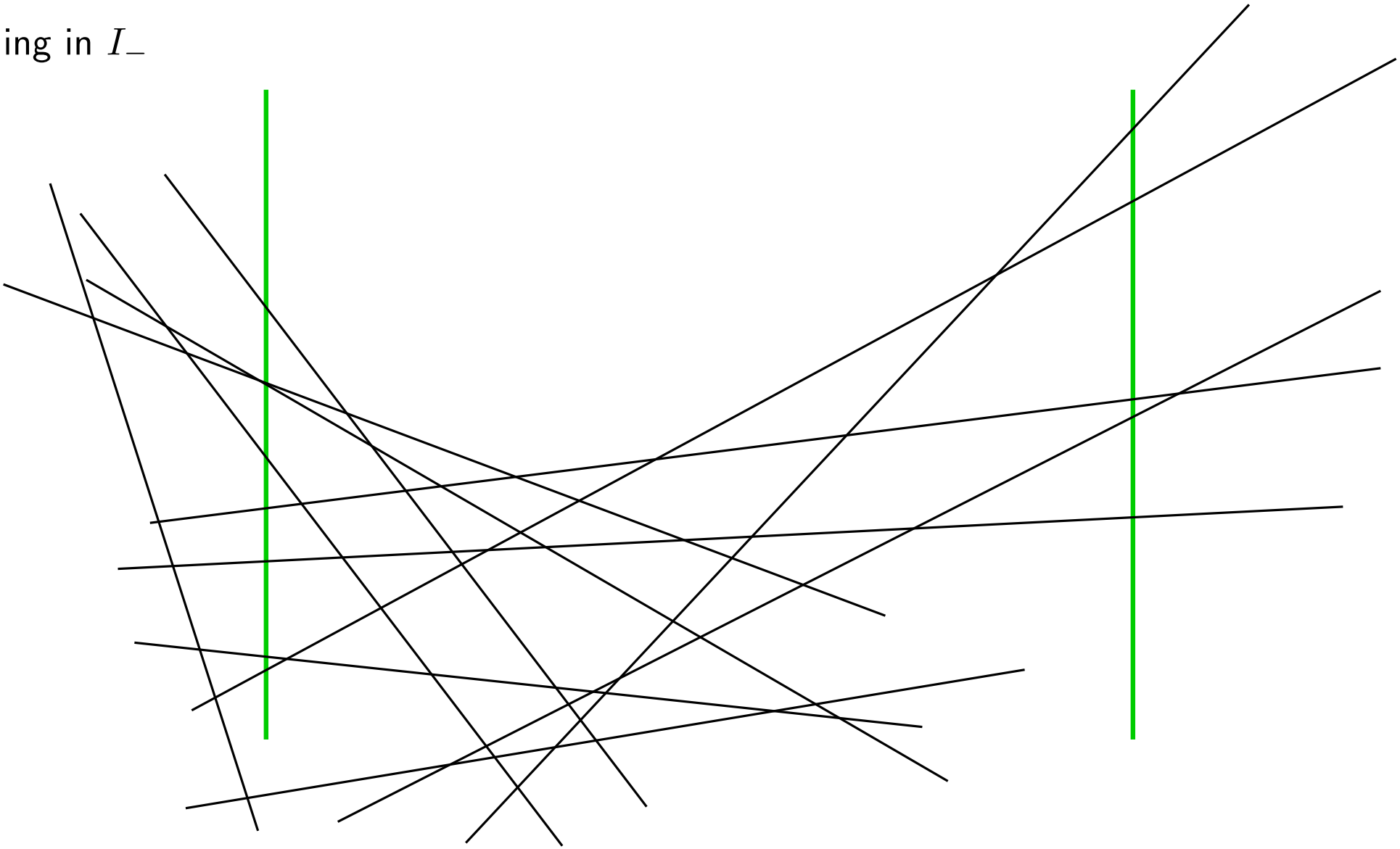
3. Compute the median value $x'$ of the surviving $x_{ij}$.

4. Search: Apply the search procedure to $x'$. If the answer is $x'$, return $x'$.

5. Prune: If the answer is "search to the left" then, for each $x_{ij} > x'$ eliminate the line corresponding to $\max(d_i, d_j)$ (resp. $\min(d_i, d_j)$). If the answer is "search to the right" then, for each $x_{ij} < x'$ eliminate the line corresponding to $\min(d_i, d_j)$ (resp. $\max(d_i, d_j)$).

## The algorithm

Pruning in $I_-$



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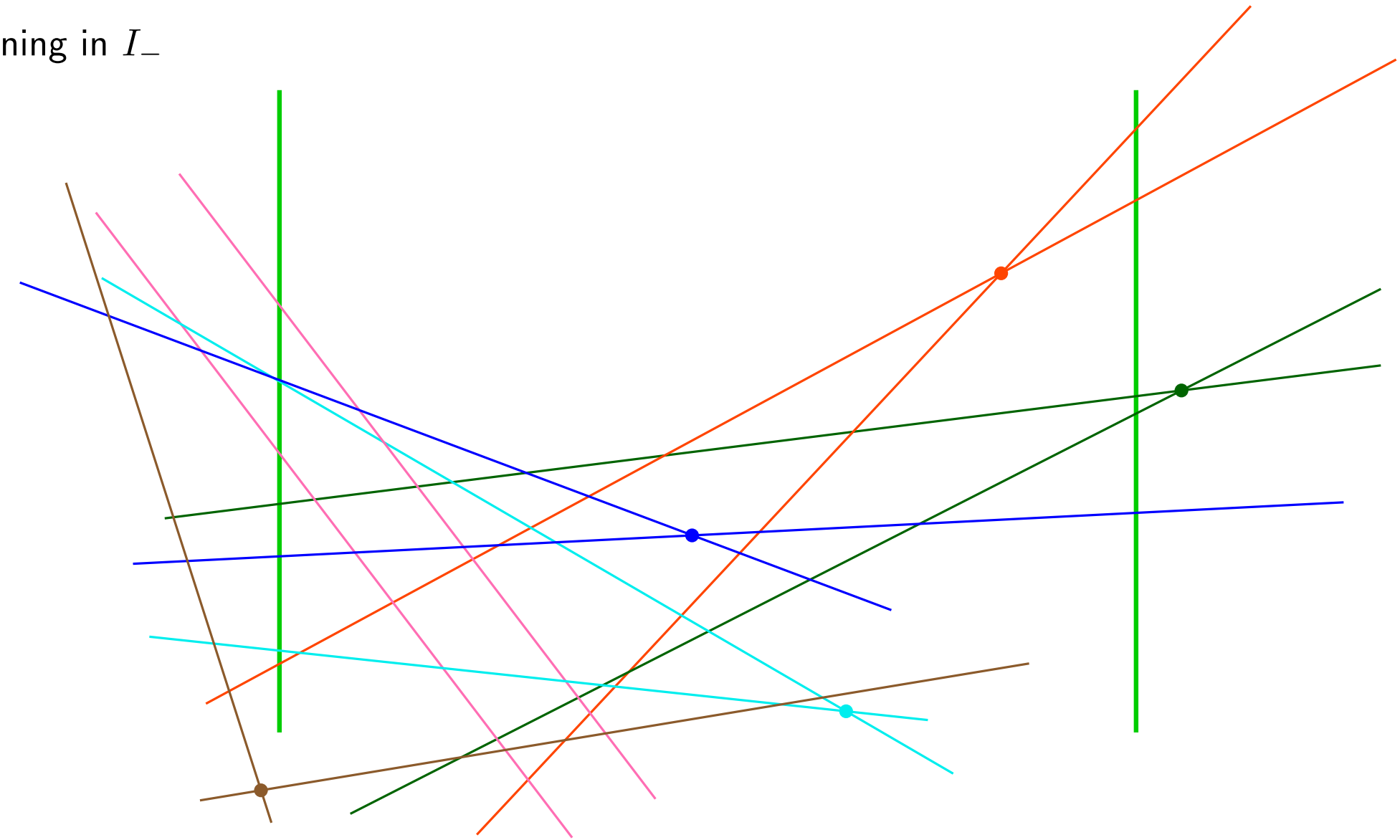

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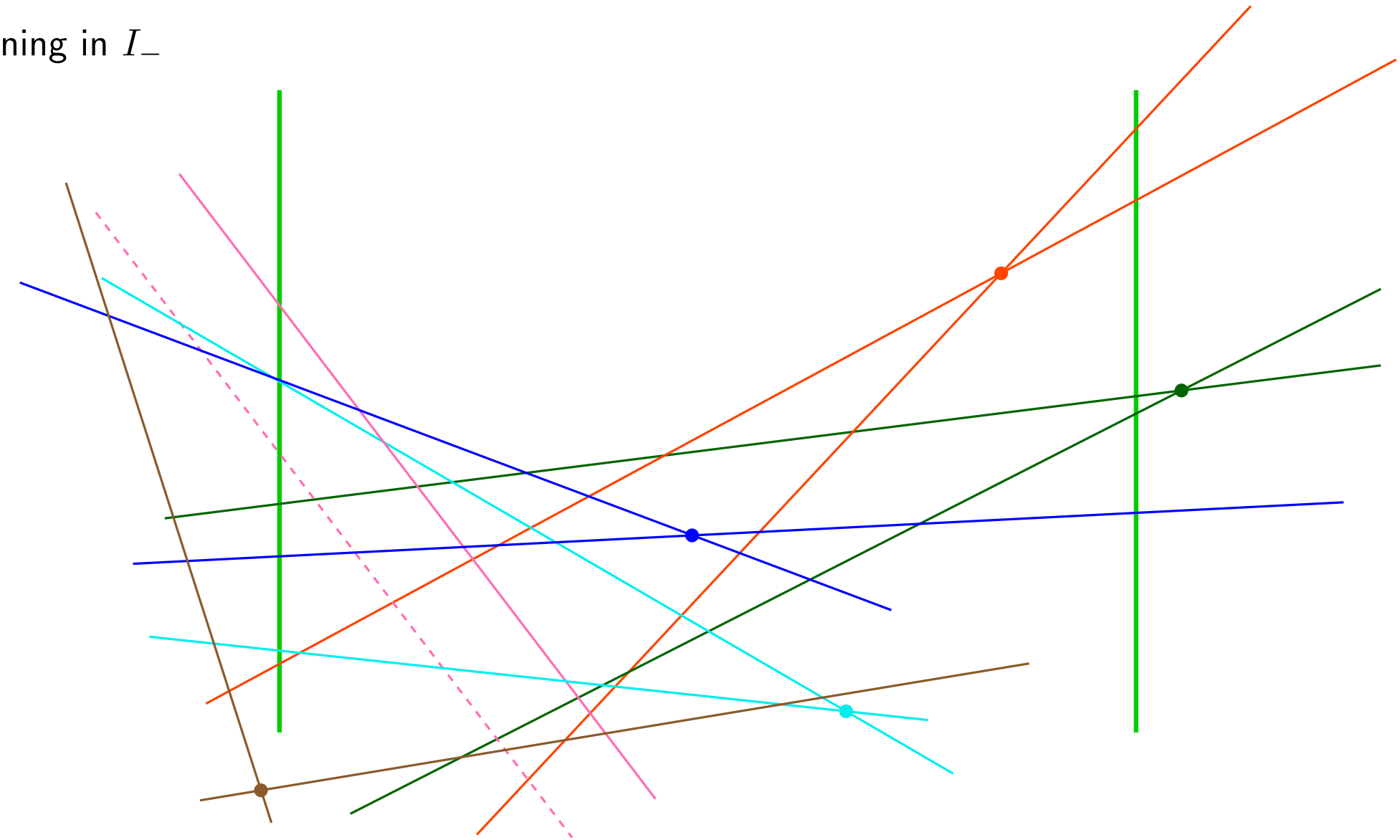

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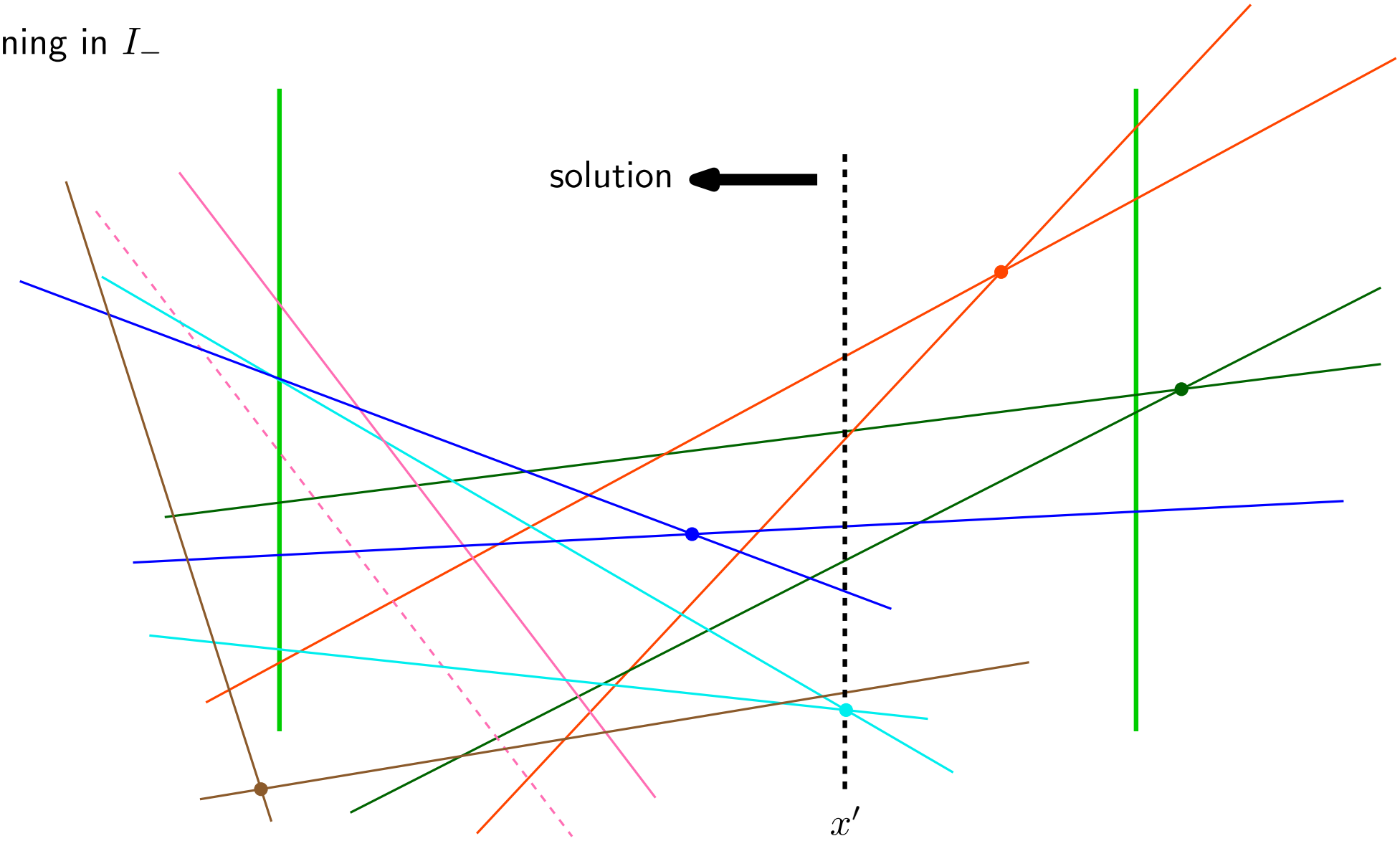

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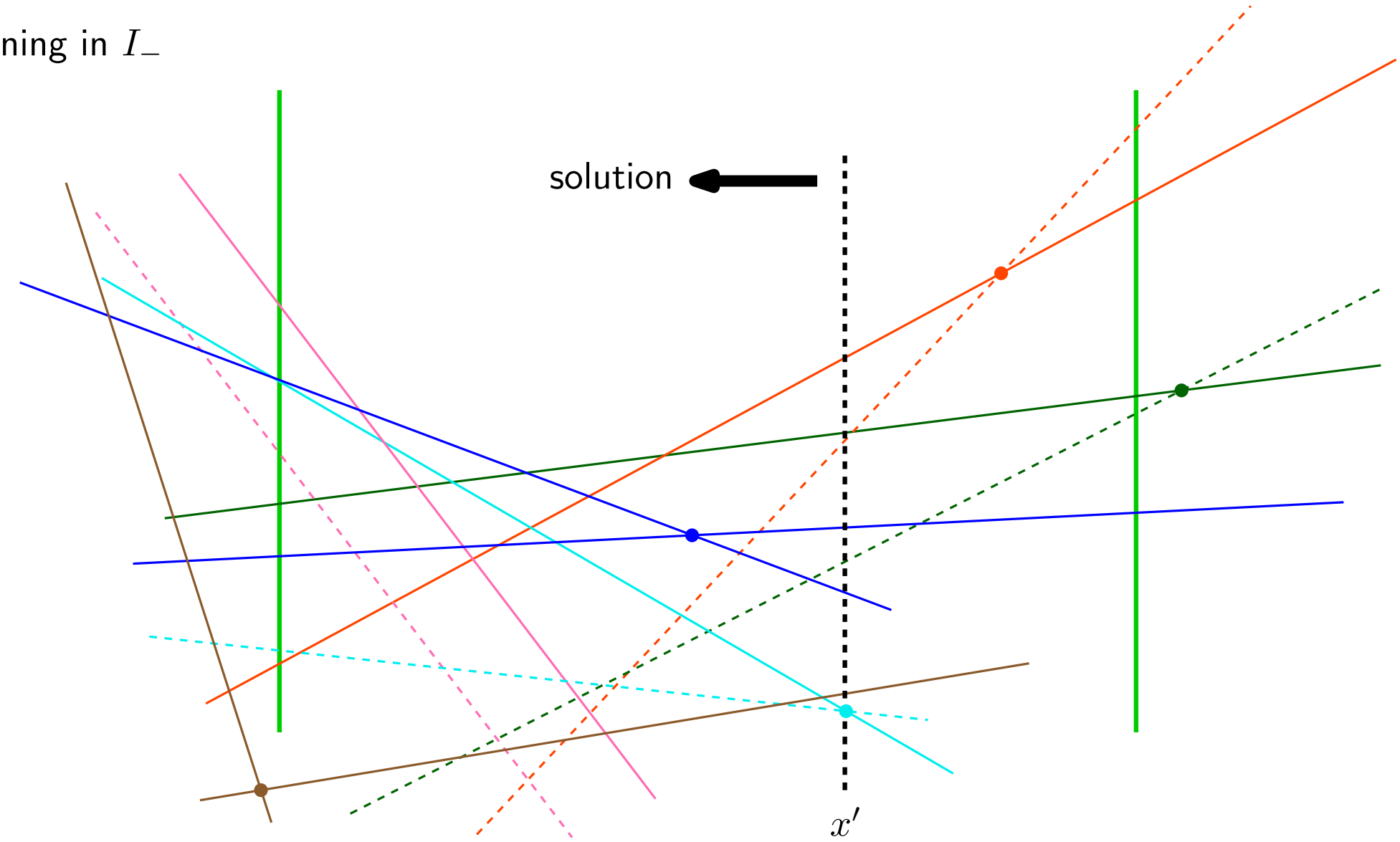

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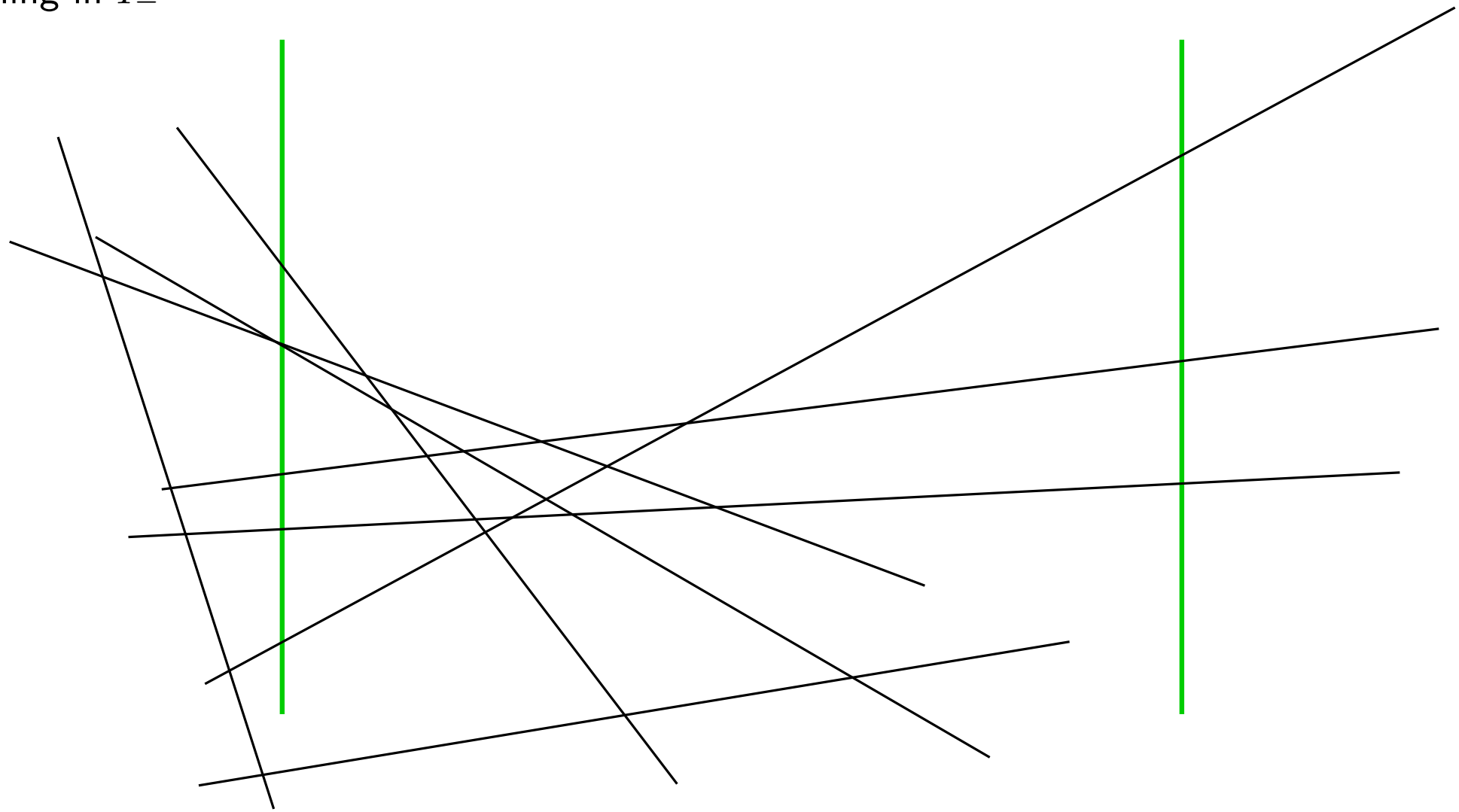

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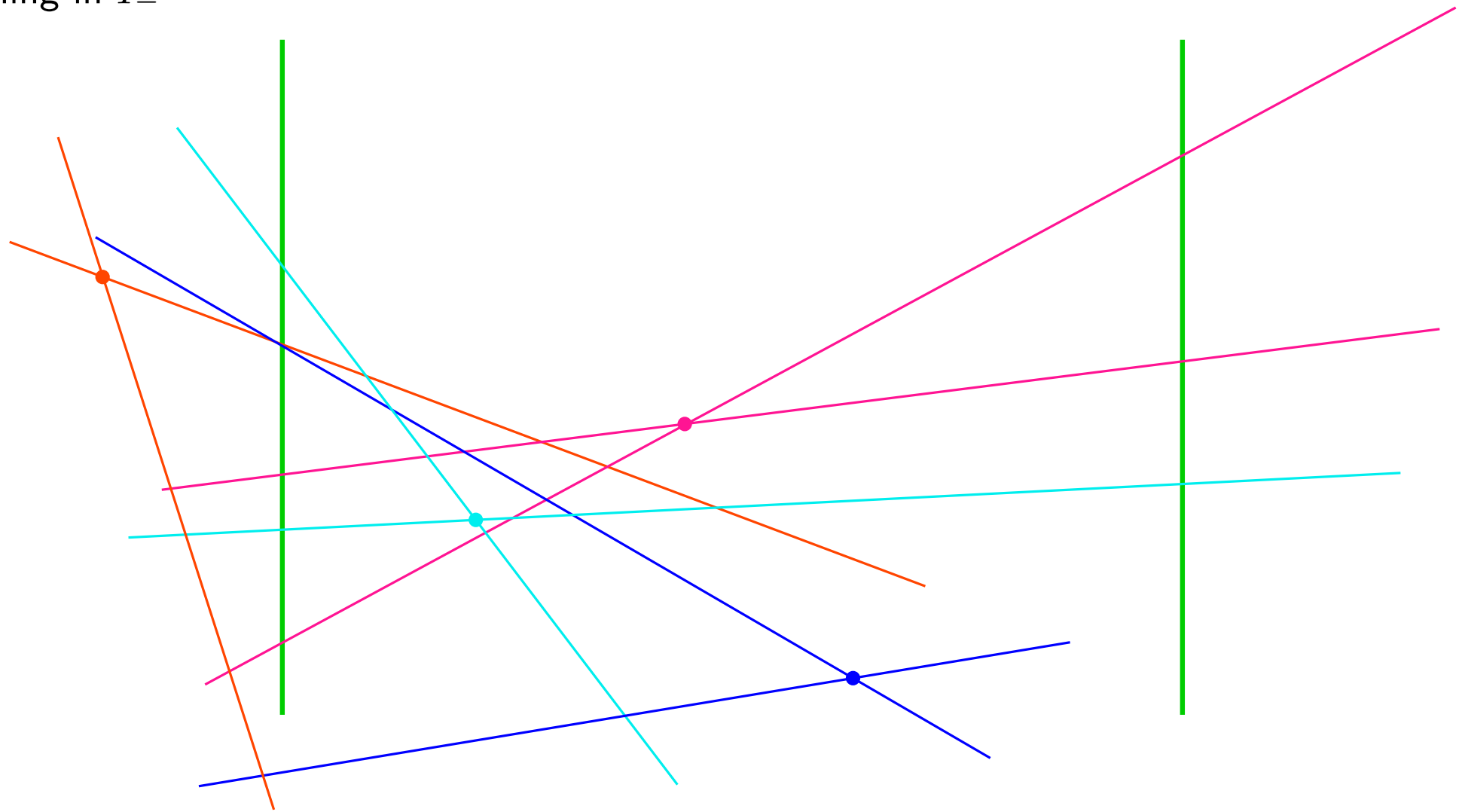

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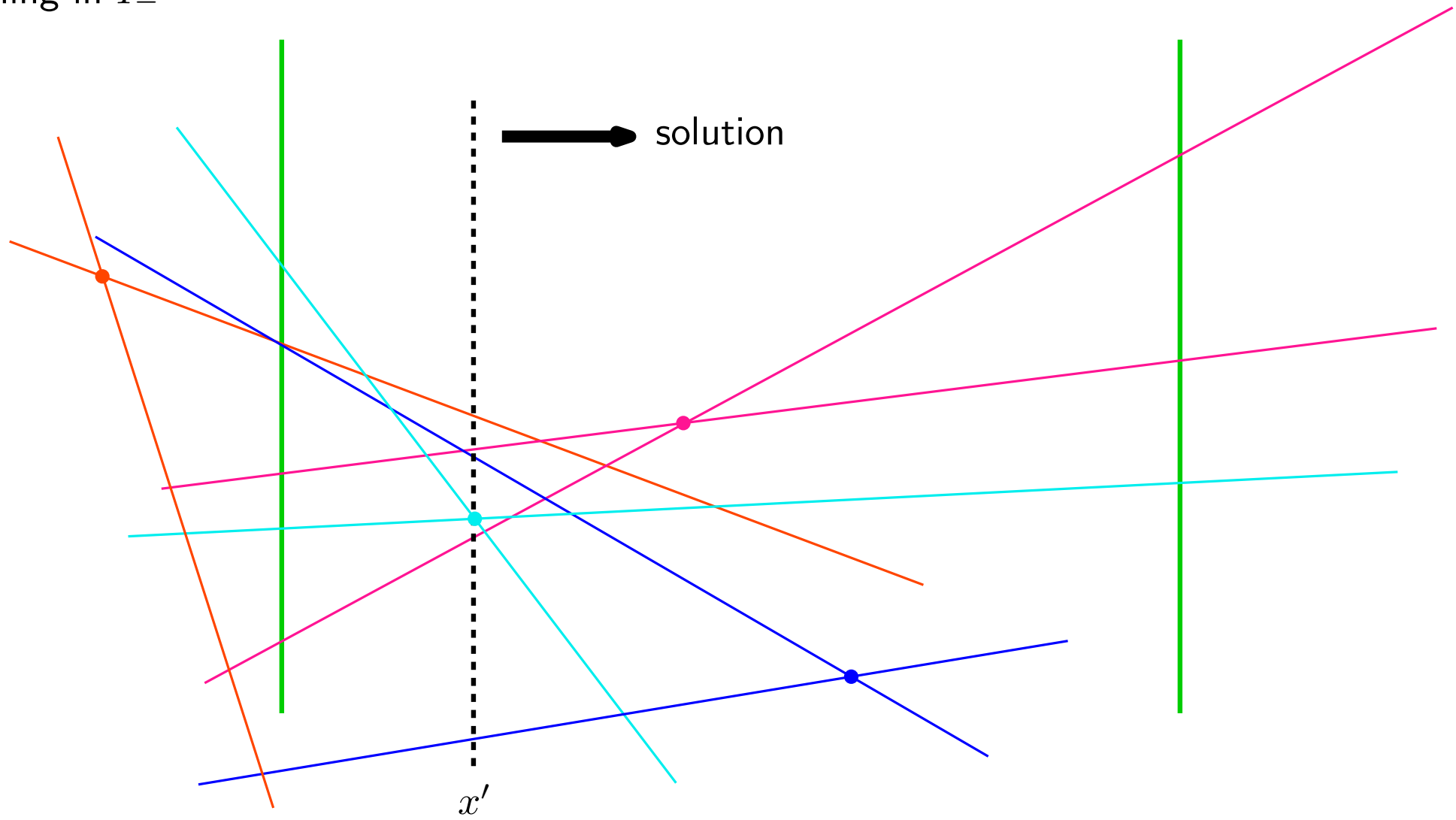

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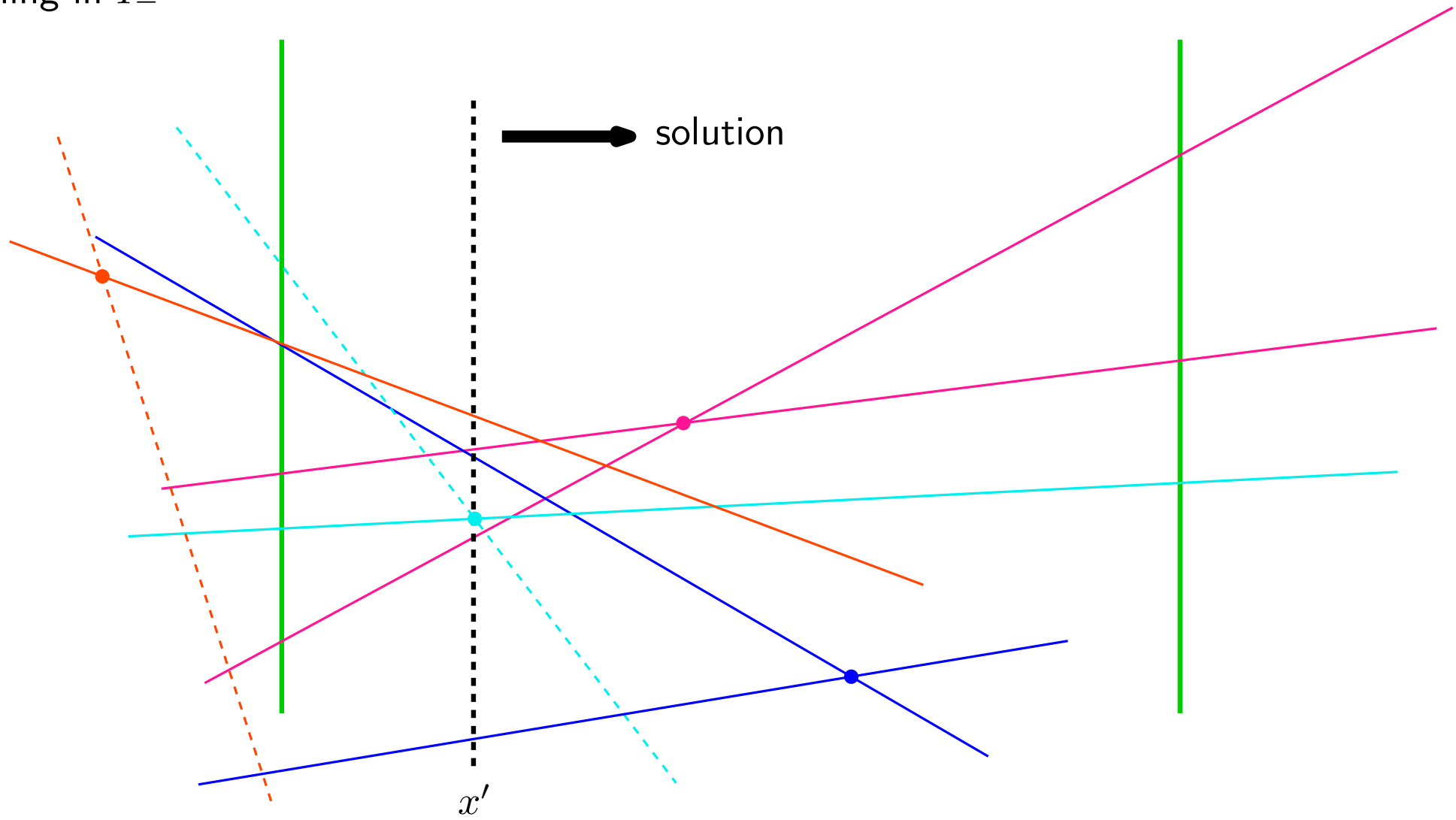

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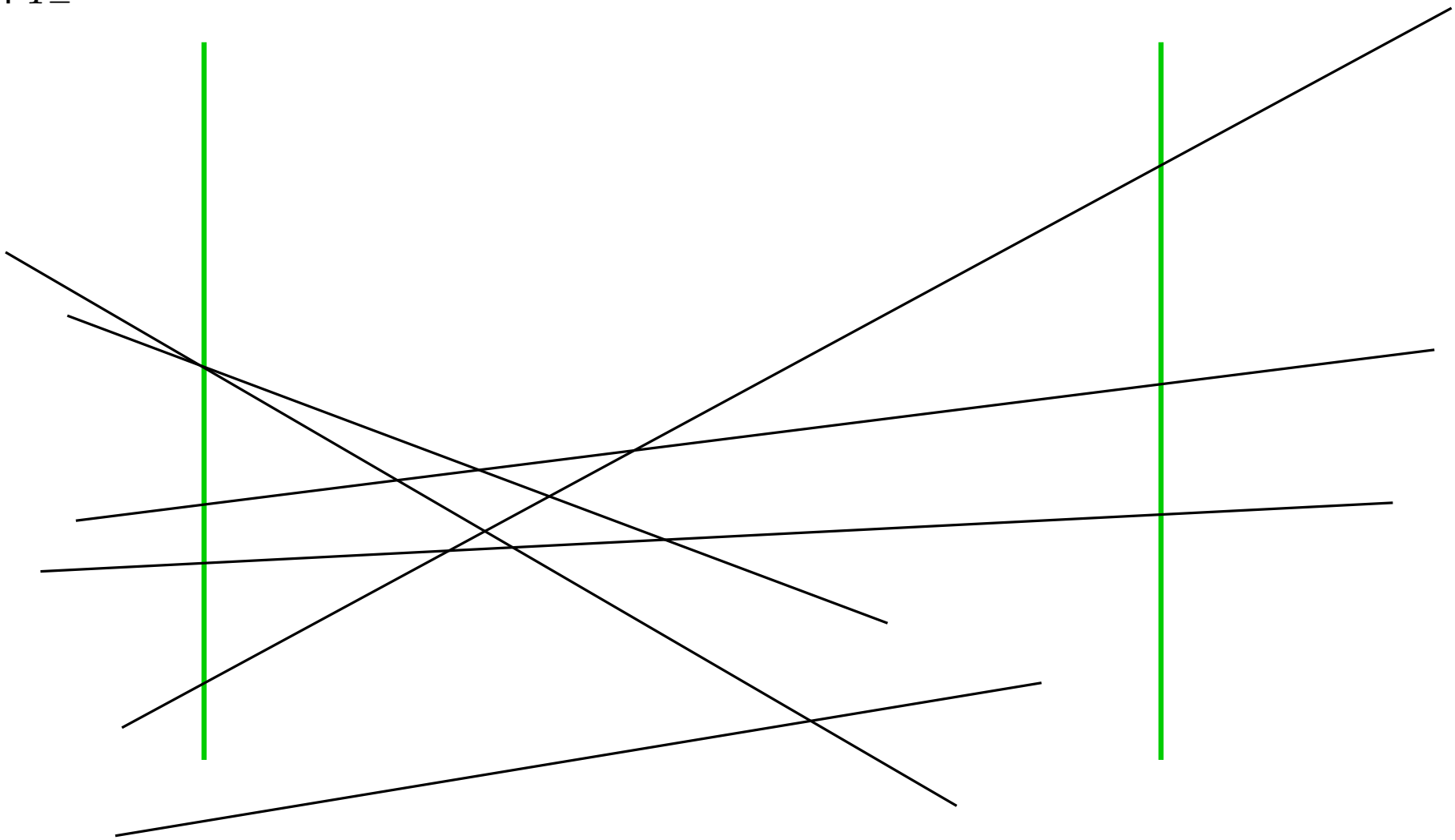

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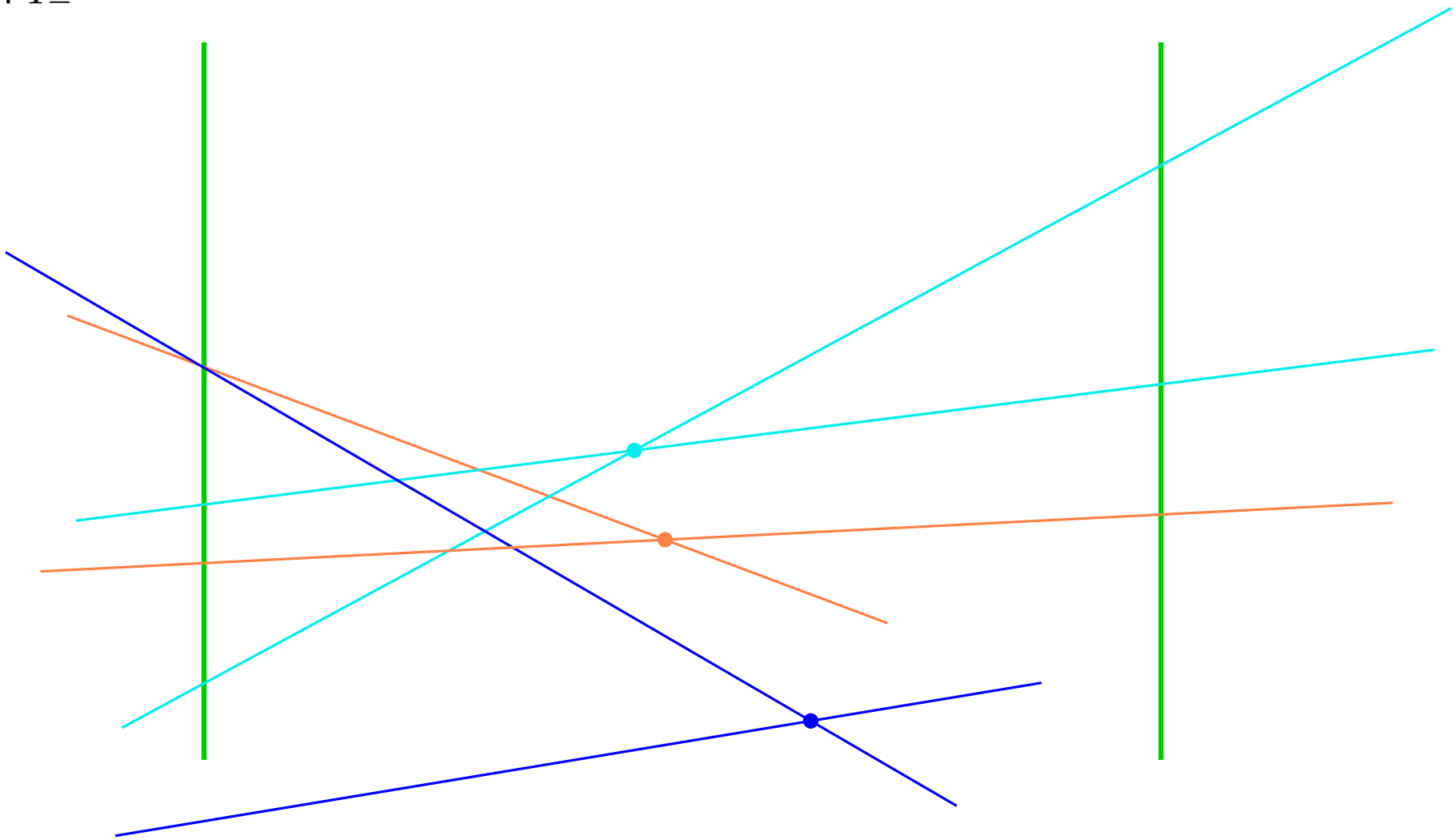

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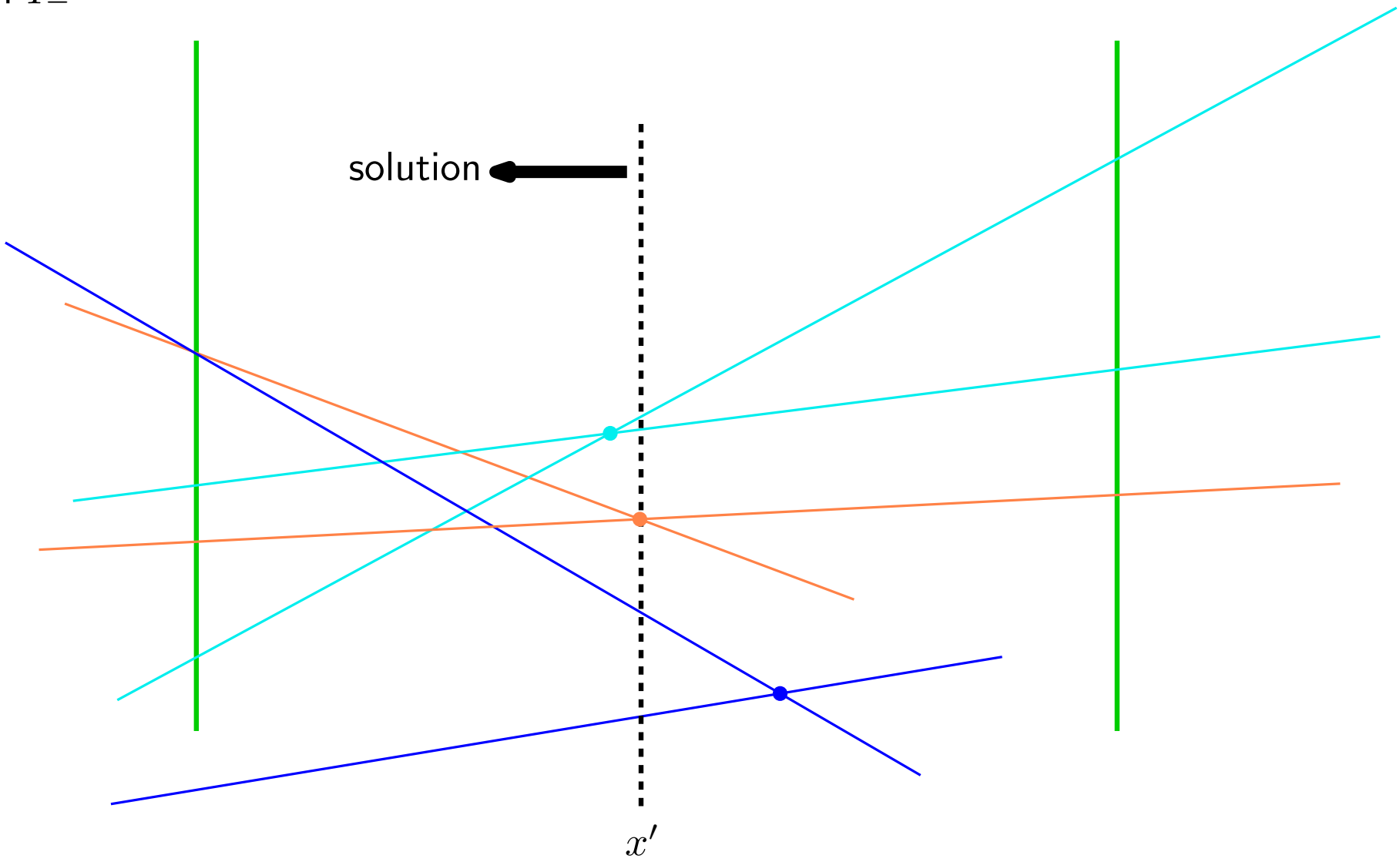

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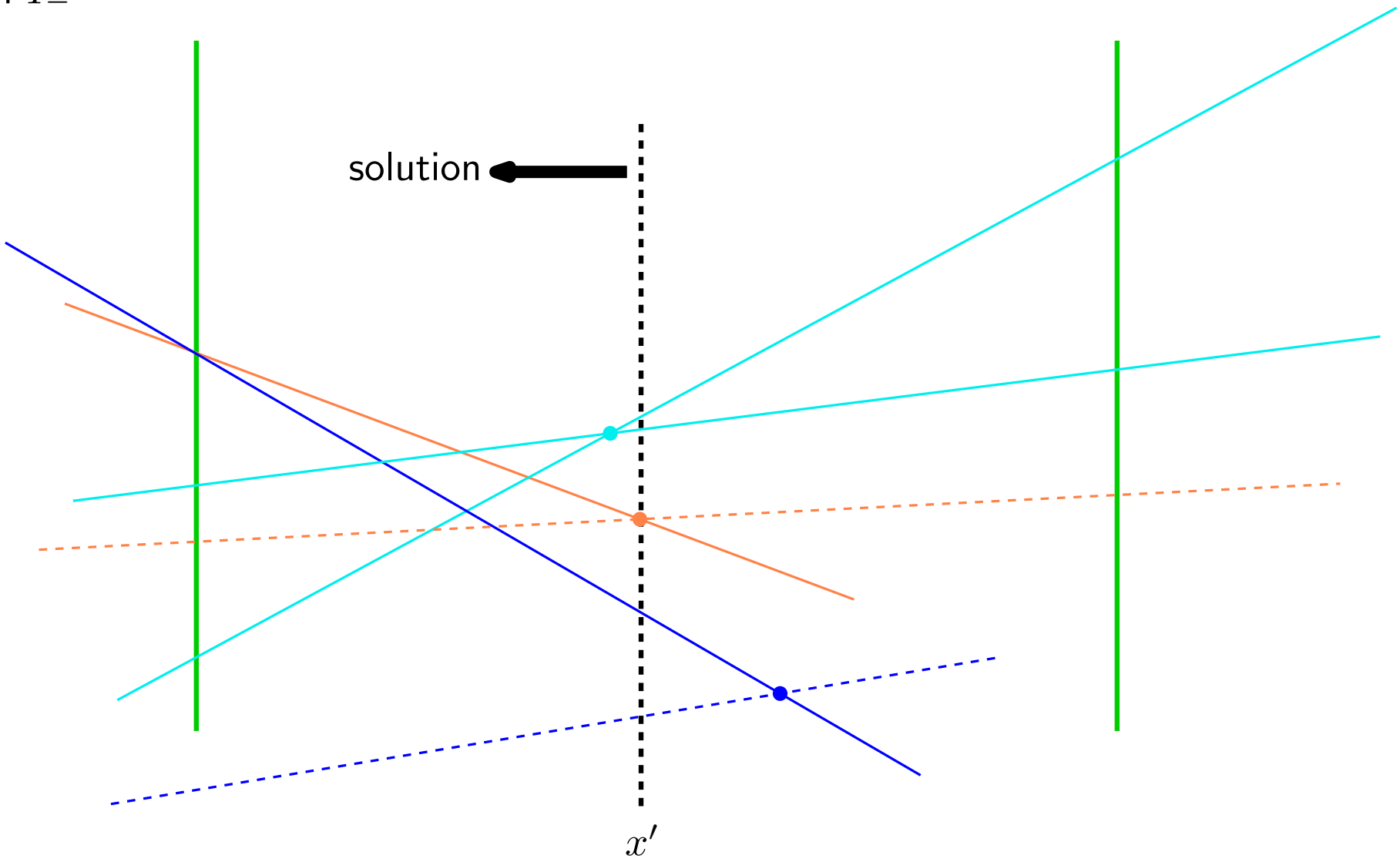

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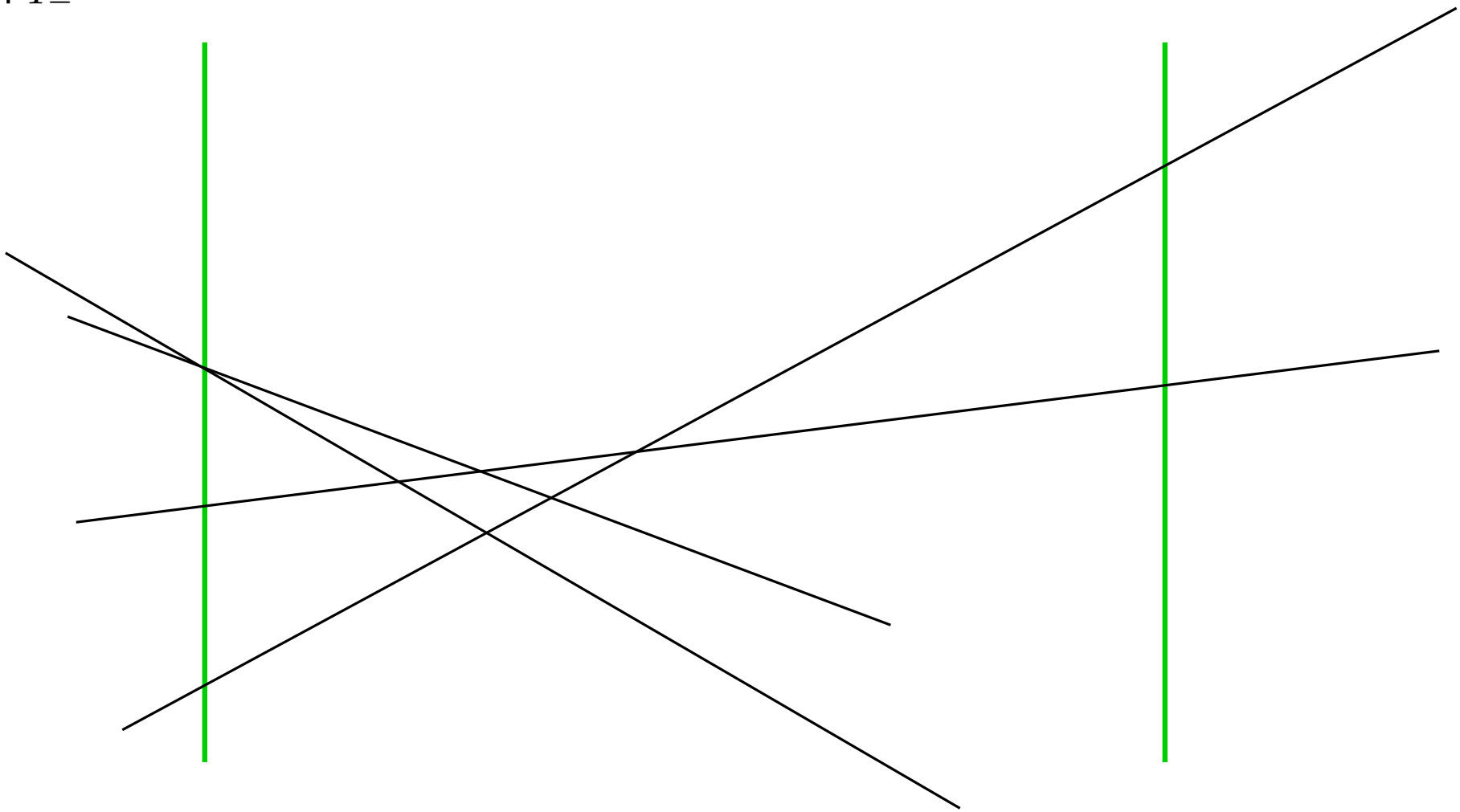

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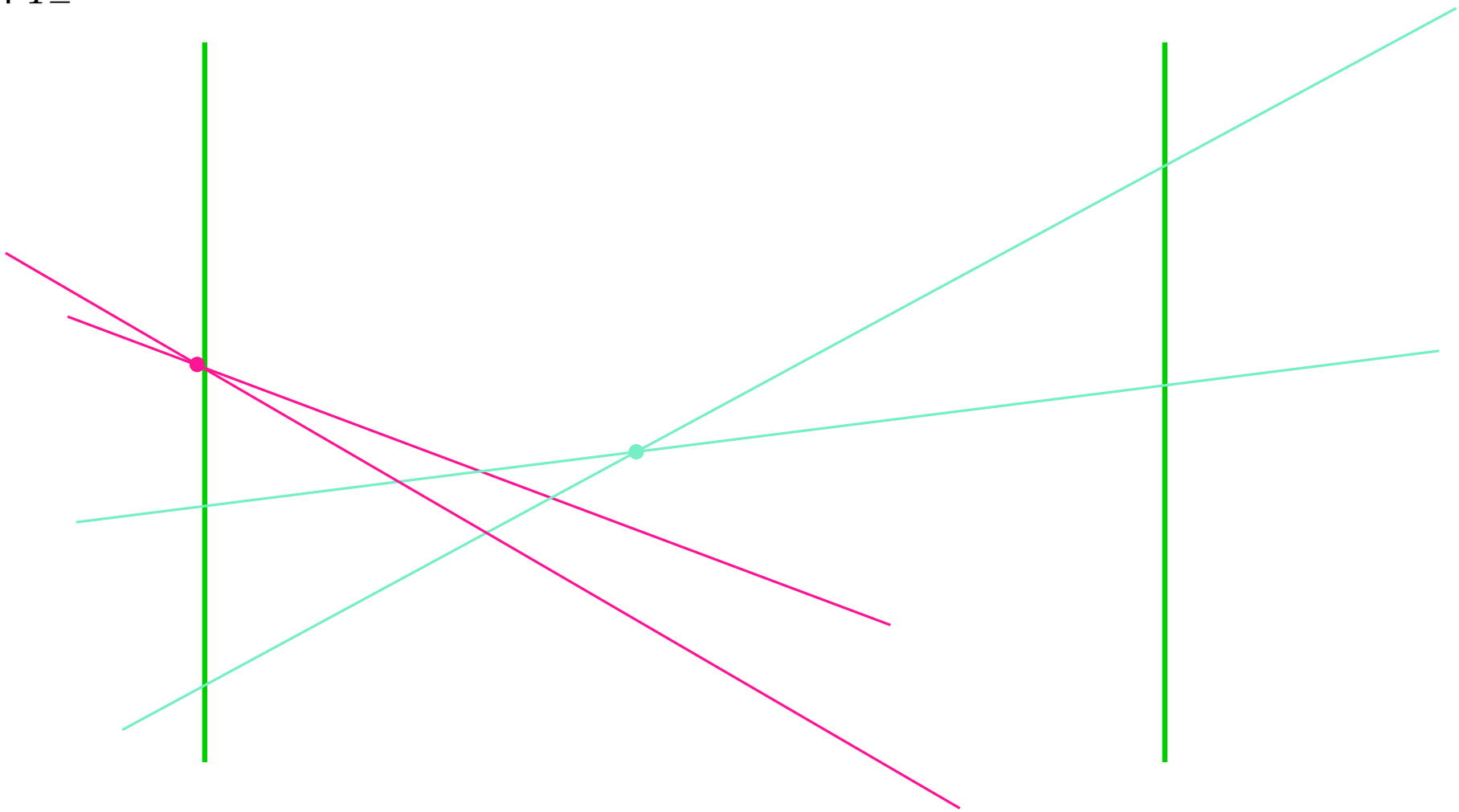

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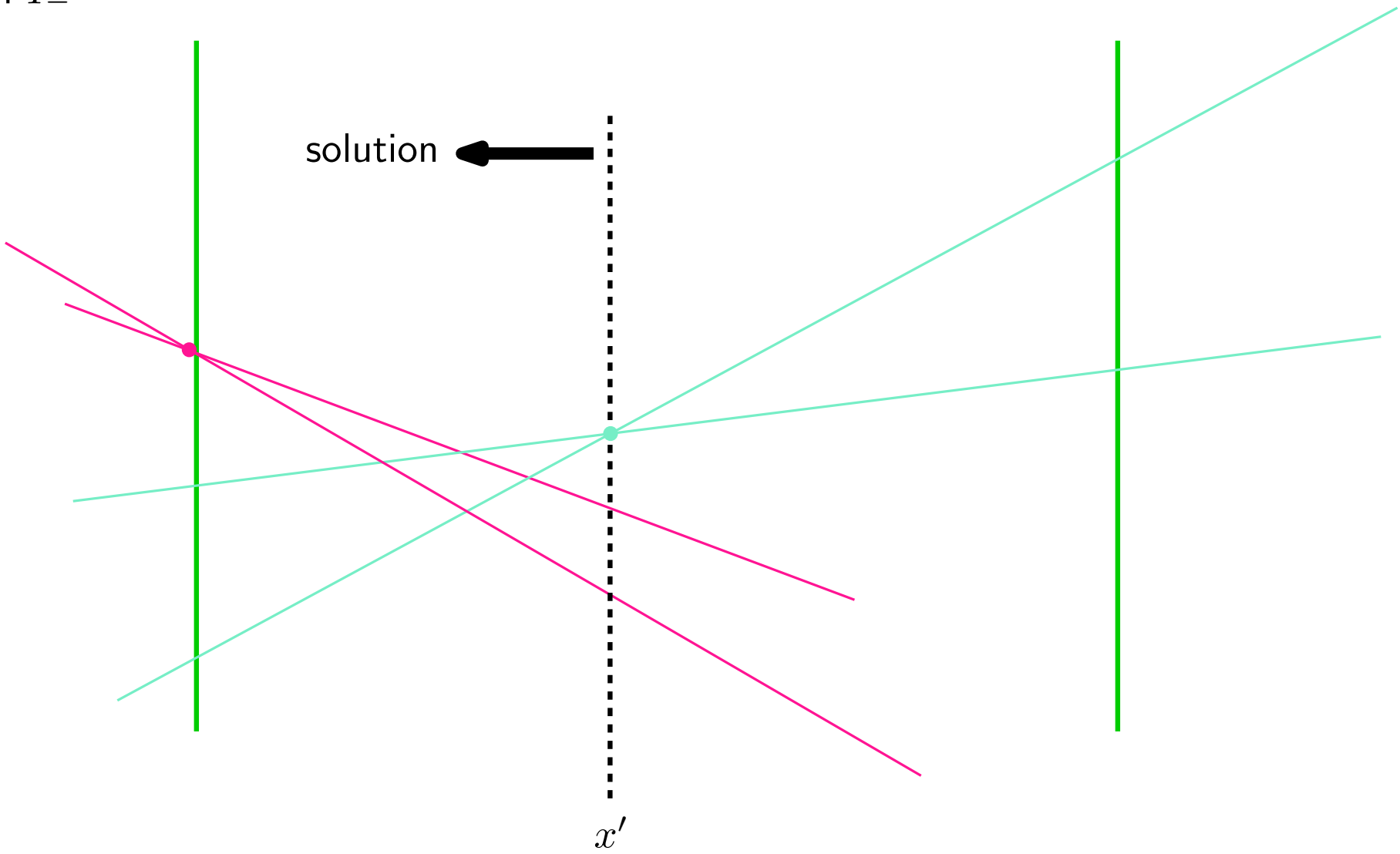

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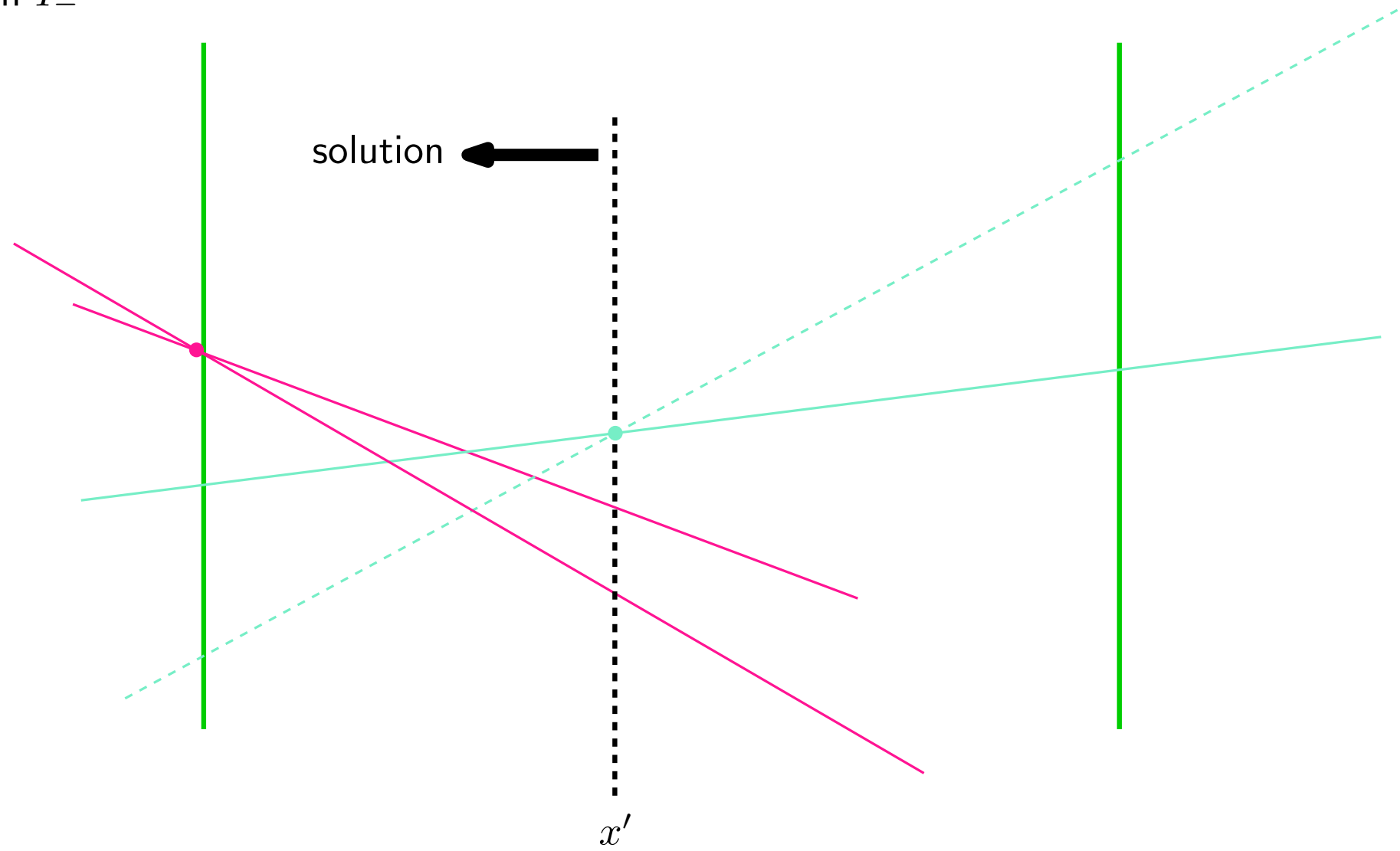

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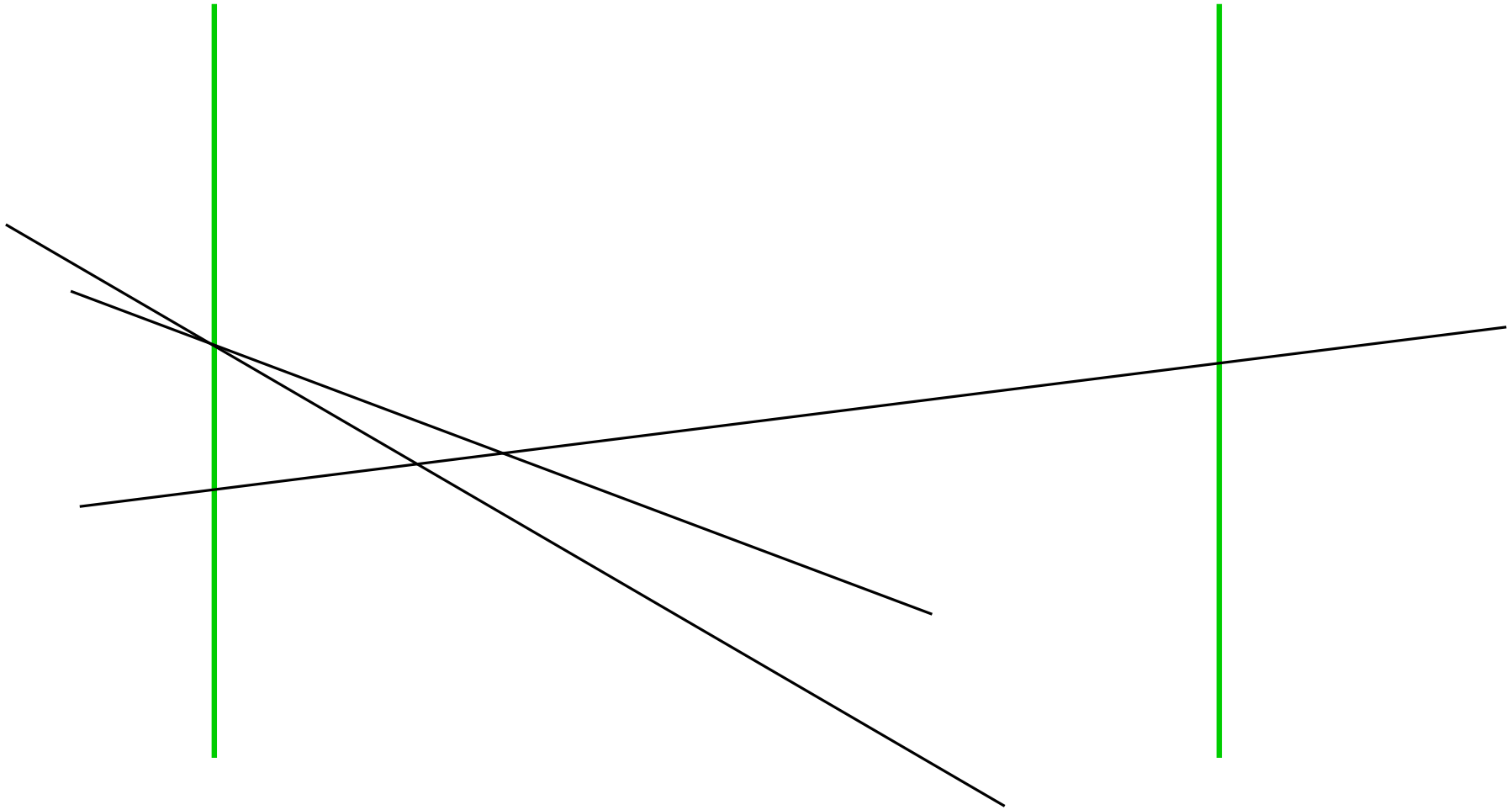

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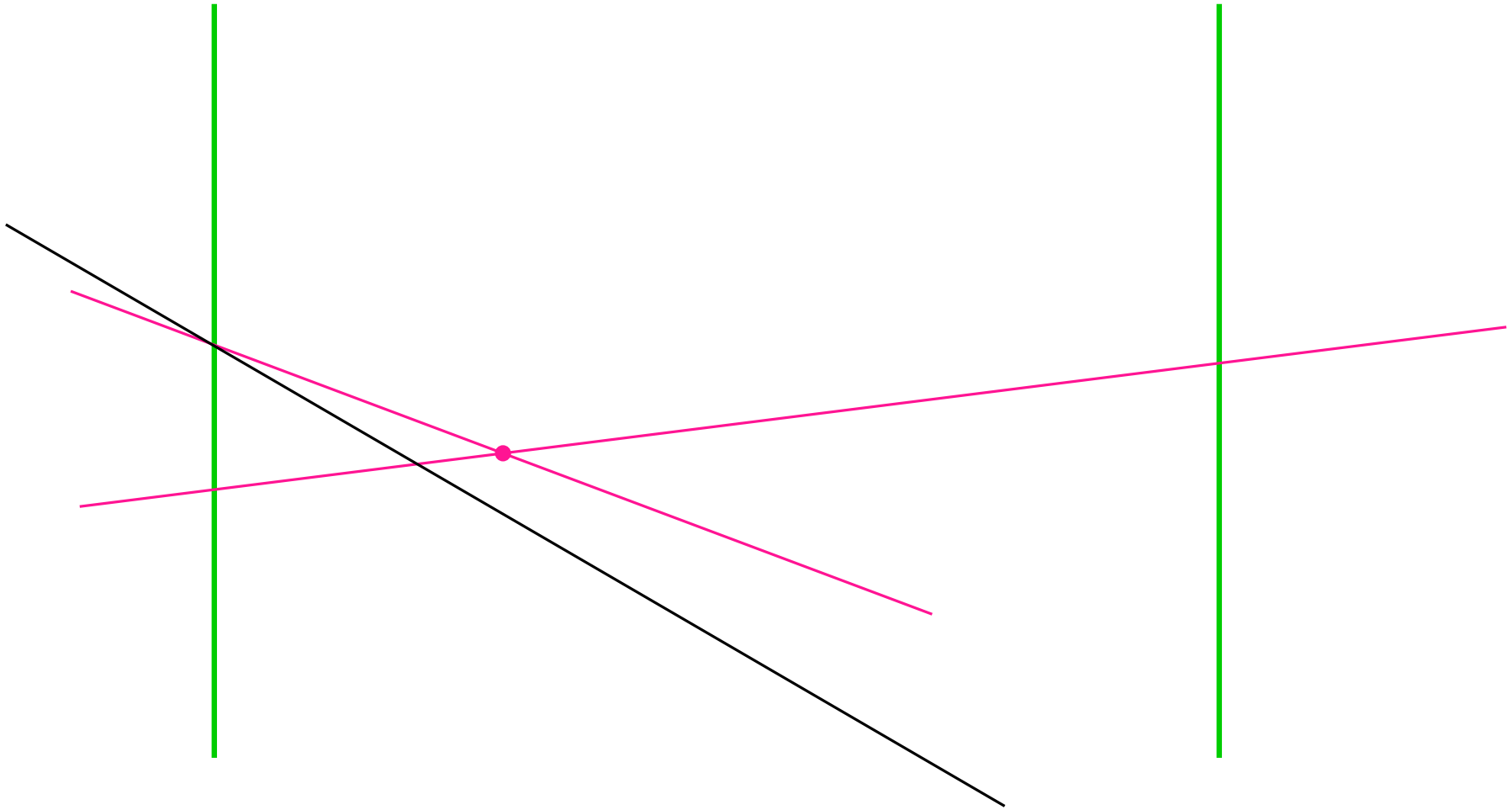

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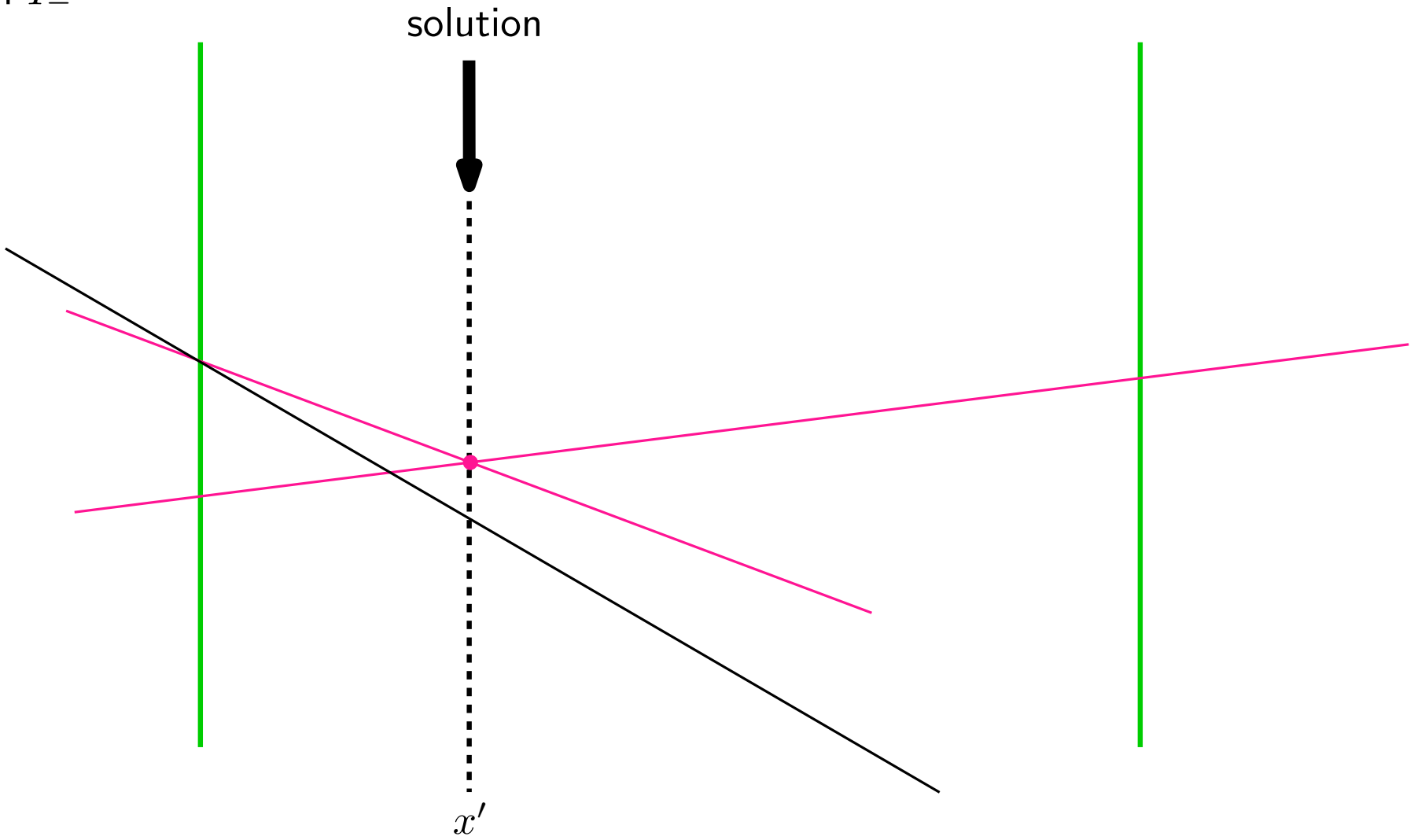

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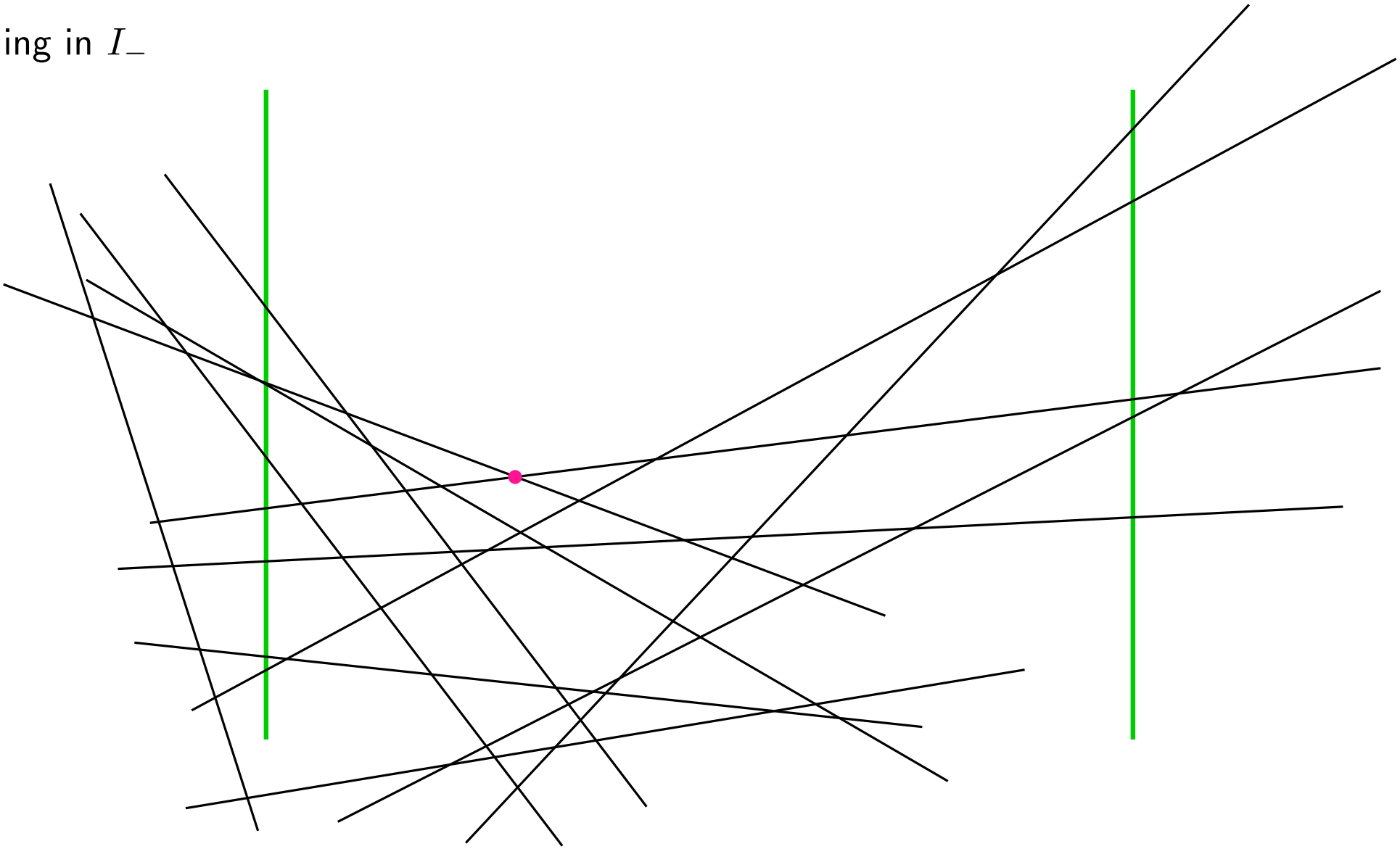

# SOLVING LINEAR PROGRAMS

## The algorithm

**Initialization** Eliminate all $i \in I_0$ except those corresponding to $u_1$ and $u_2$, if they exist. $O(n)$

**Advance.** Repeat as many times as necessary:

1. Pair up restrictions and compute the abscissa of the intersection points. $O(n)$
2. For all pairs, do: $O(n)$
   - If $d_i = d_j$, eliminate one of the two.
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How many restrictions are pruned at each step?

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How many restrictions are pruned at each step? $O(n/4)$

## The algorithm

Exact counting:

- The initial number of vertical restrictions is $k$, where $k \geq 0$.
- The vertical restrictions prune step eliminates all of them but at most 2.
- The initial number of pairs of non vertical lines (i.e., points $x_{ij}$) is $\frac{m}{2}$, $\frac{m-1}{2}$ or $\frac{m}{2} - 1$, depending on the parities of $|I_-|$ and $|I_+|$, where $m + k = n$.
- The prune step eliminates roughly half of them.

Alltogether, the eliminated restrictions are, at least:

$$k + \left\lfloor \frac{\frac{m}{2} - 1}{2} \right\rfloor = \left\lfloor \frac{m}{4} + k - \frac{1}{2} \right\rfloor = \left\lfloor \frac{n}{4} + \frac{3}{4}k - \frac{1}{2} \right\rfloor > \frac{n}{4} - 1.$$

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## Running time

Hence, the running time of the algorithm is

$$cn + c\tfrac{3}{4}n + c\left(\tfrac{3}{4}\right)^2 n + c\left(\tfrac{3}{4}\right)^3 n + \cdots = cn \sum_{k=o}^{\lceil \log_{3/4} n \rceil} \left(\tfrac{3}{4}\right)^k < cn \frac{1}{1 - \frac{3}{4}} = 4cn \in O(n).$$

# COMPUTING THE MINIMUM SPANNING CIRCLE

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## MIN-MAX FACILITY LOCATION

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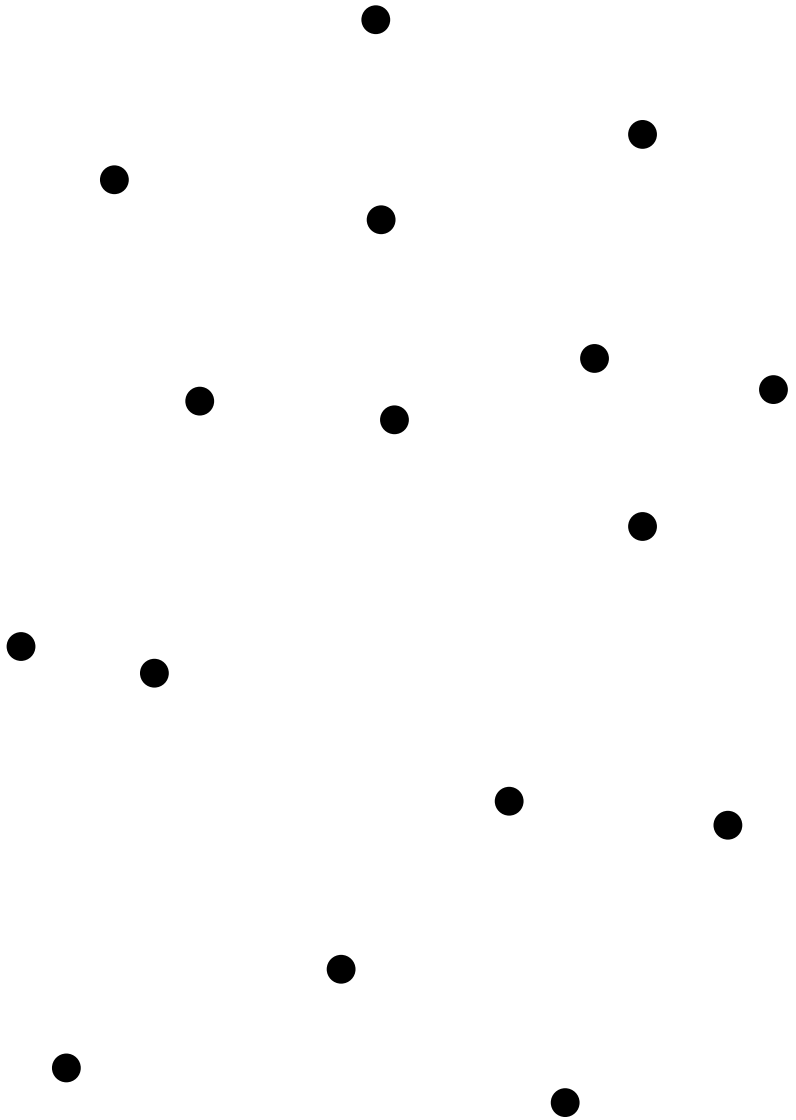
Let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ points in $\mathbb{R}^2$.



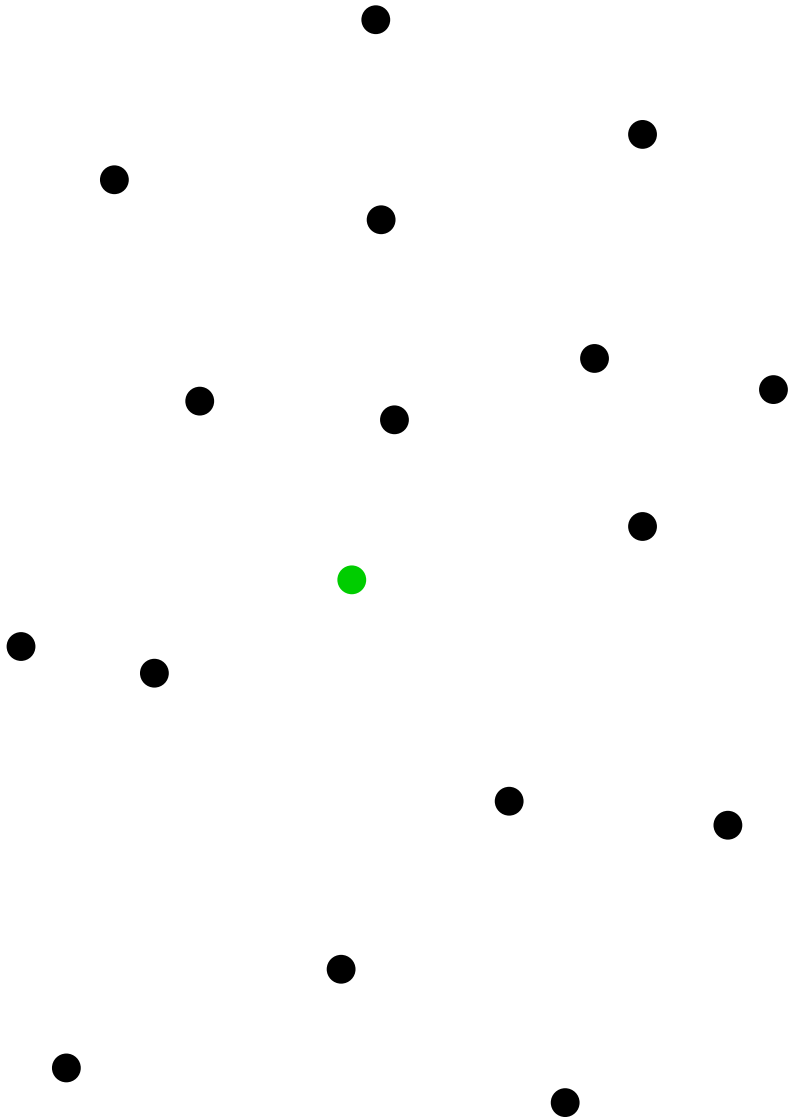
# COMPUTING THE MINIMUM SPANNING CIRCLE

## MIN-MAX FACILITY LOCATION

Let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ points in $\mathbb{R}^2$.

Find the point $x$ on the plane achieving

$$\min_{x \in \mathbb{R}^2} \max_{p_i \in P} d(x, p_i).$$



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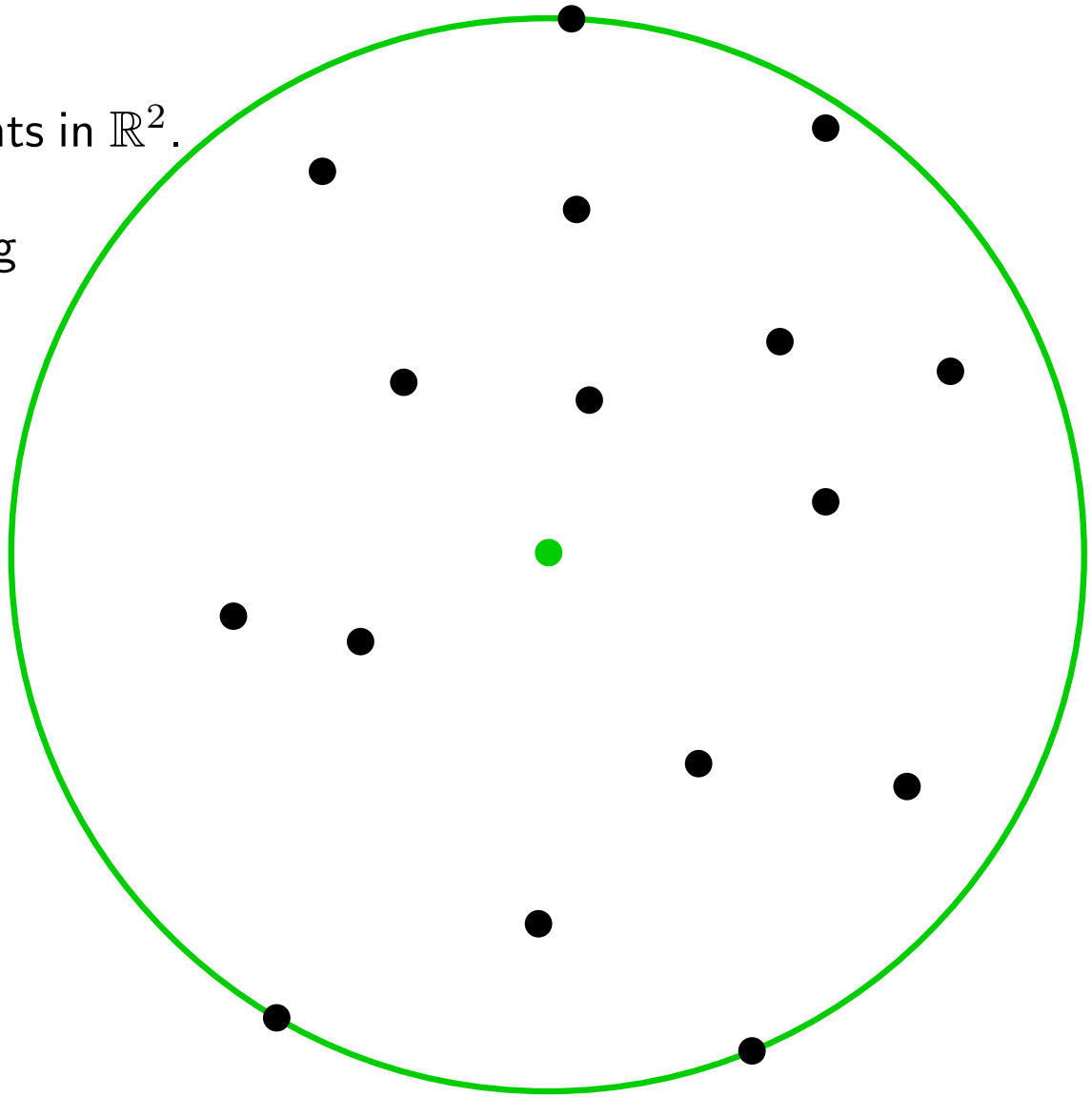
### Geometrically

Find the center of the circle of minimum radius enclosing $P$.



# COMPUTING THE MINIMUM SPANNING CIRCLE

## MSC restricted to a line

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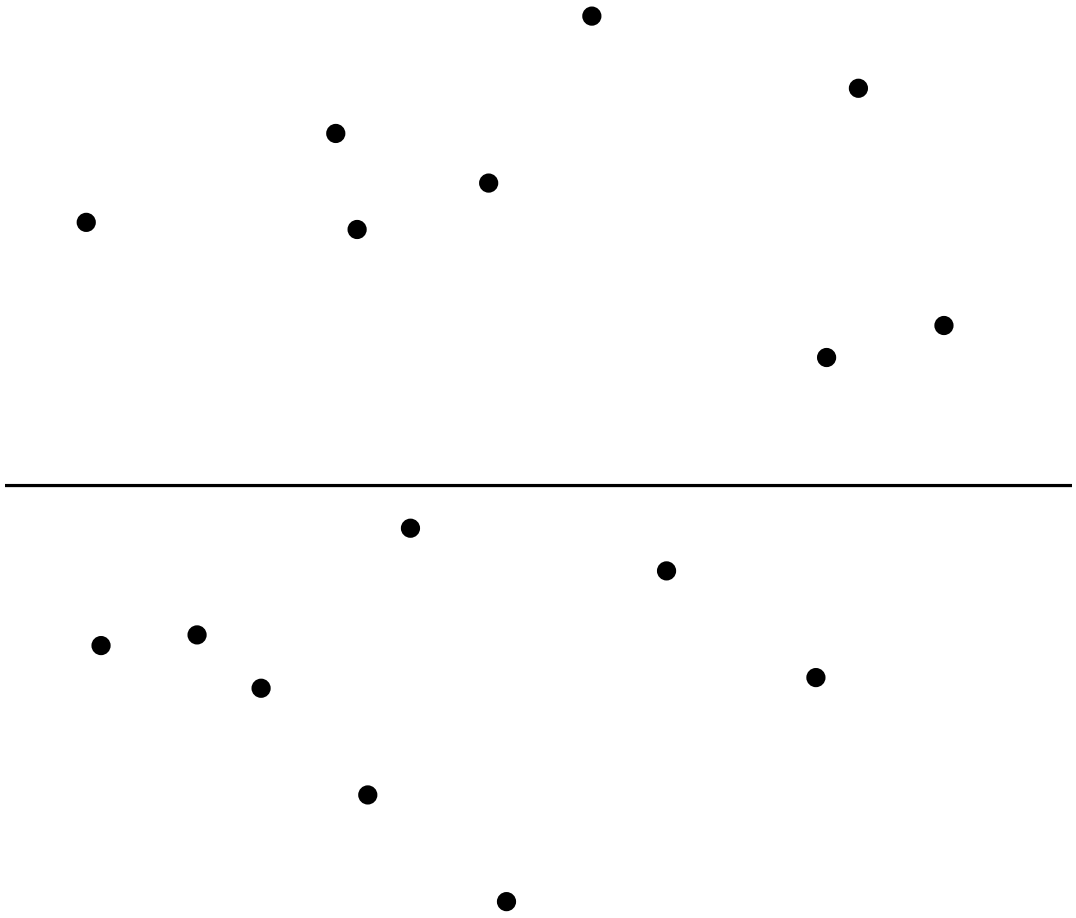
## MSC restricted to a line

**Input:** a set of $n$ points $(a_i, b_i)$, $i = 1, \ldots, n$.
**Output:** $\min_{x \in \mathbb{R}} \max_{i=1 \ldots n} (x - a_i)^2 + b_i^2$.



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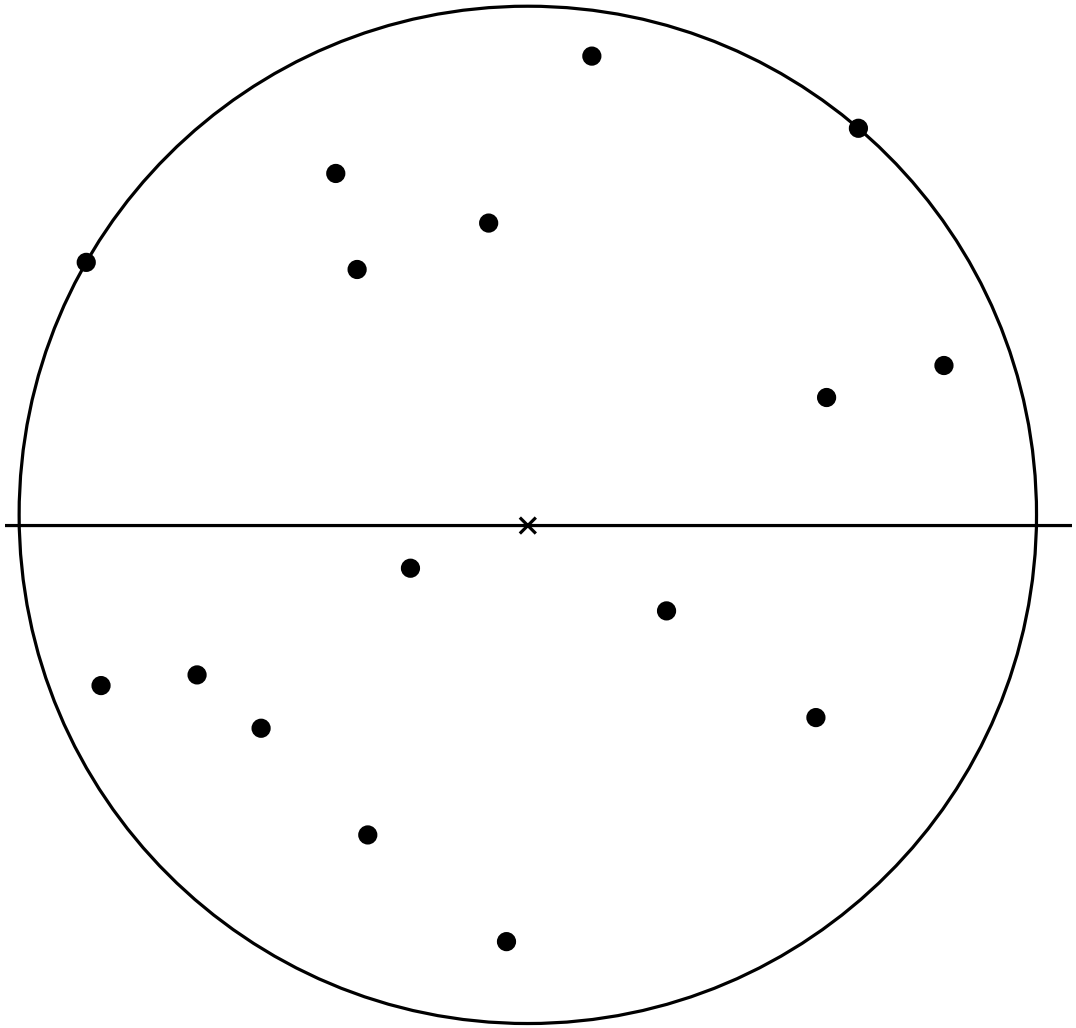

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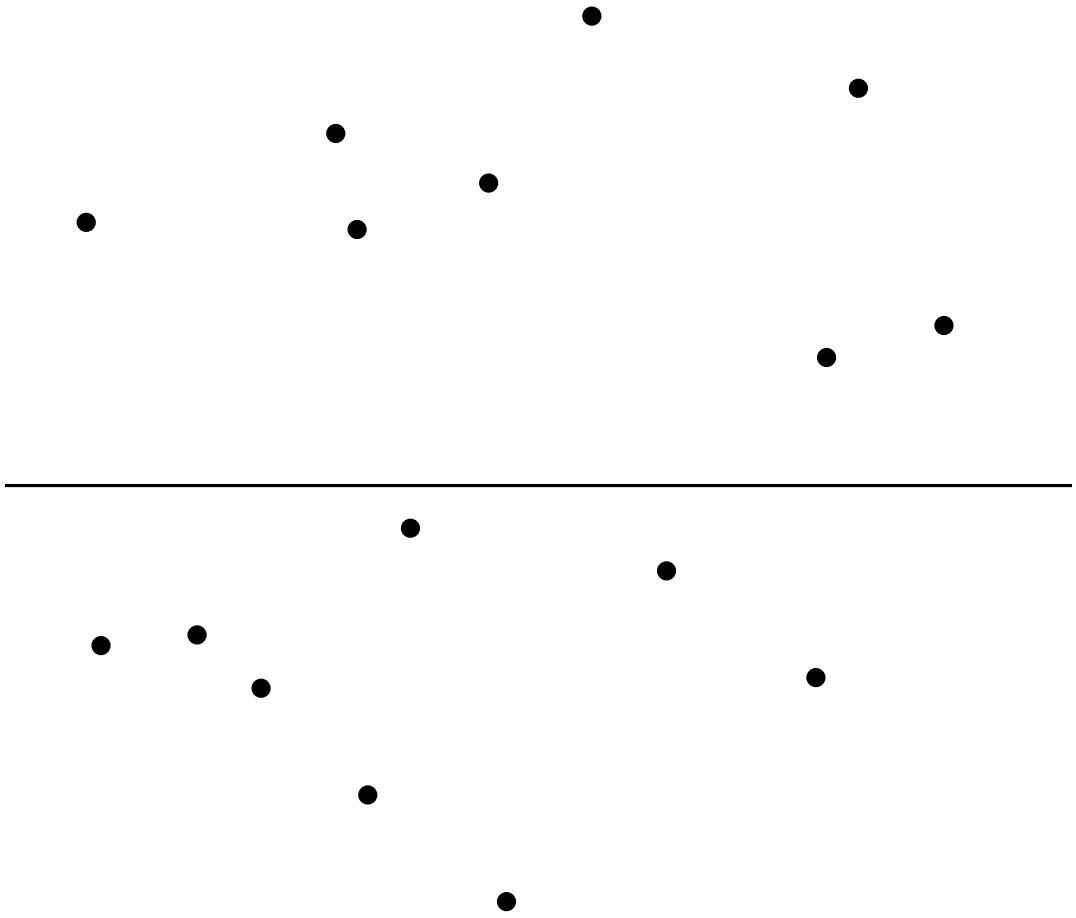
**Output:** $\min_{x \in \mathbb{R}} \max_{i=1 \ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair $p_i, p_j$, compute its perpendicular bisector $b_{ij}$ and find its intersection $x_{ij}$ with the line $y = 0$.



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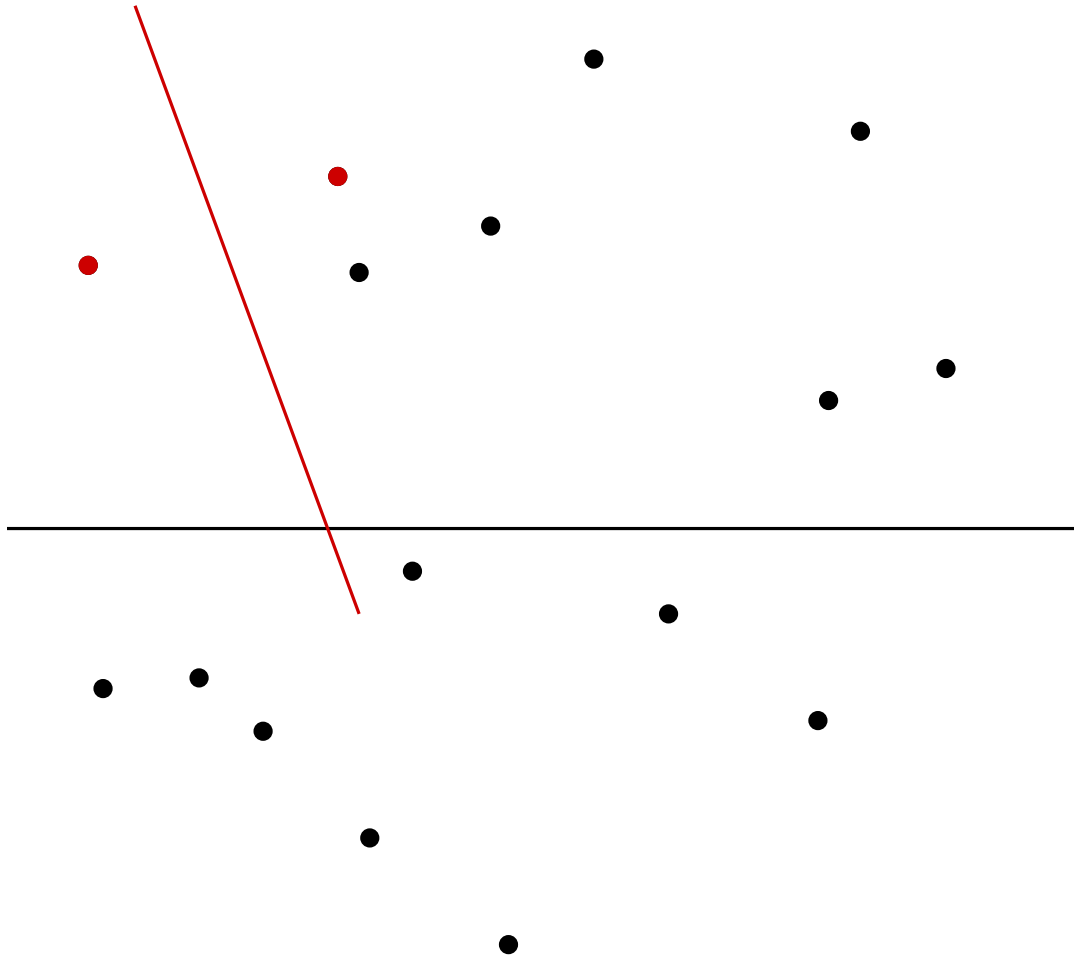
**Output:** $\min_{x \in \mathbb{R}} \max_{i=1 \ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points. For each pair $p_i, p_j$, compute its perpendicular bisector $b_{ij}$ and find its intersection $x_{ij}$ with the line $y = 0$.



# COMPUTING THE MINIMUM SPANNING CIRCLE

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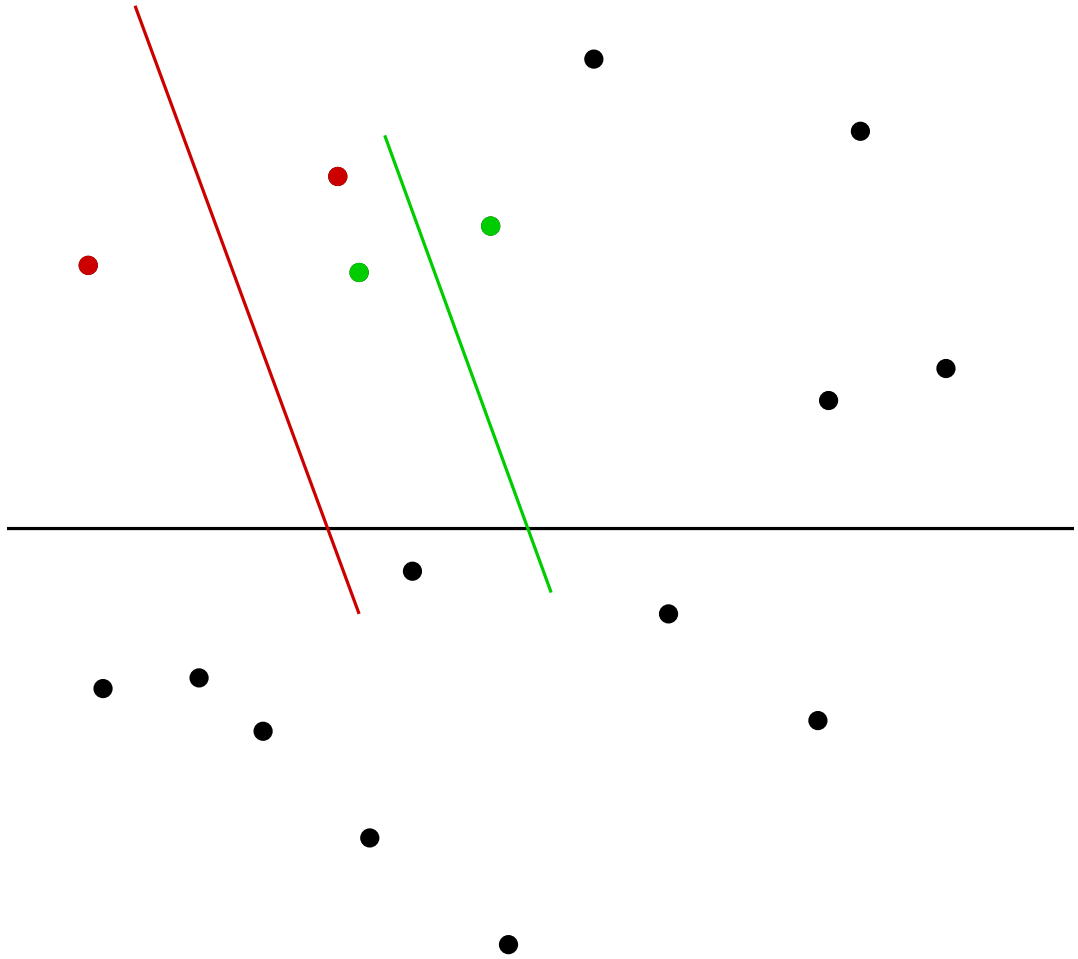

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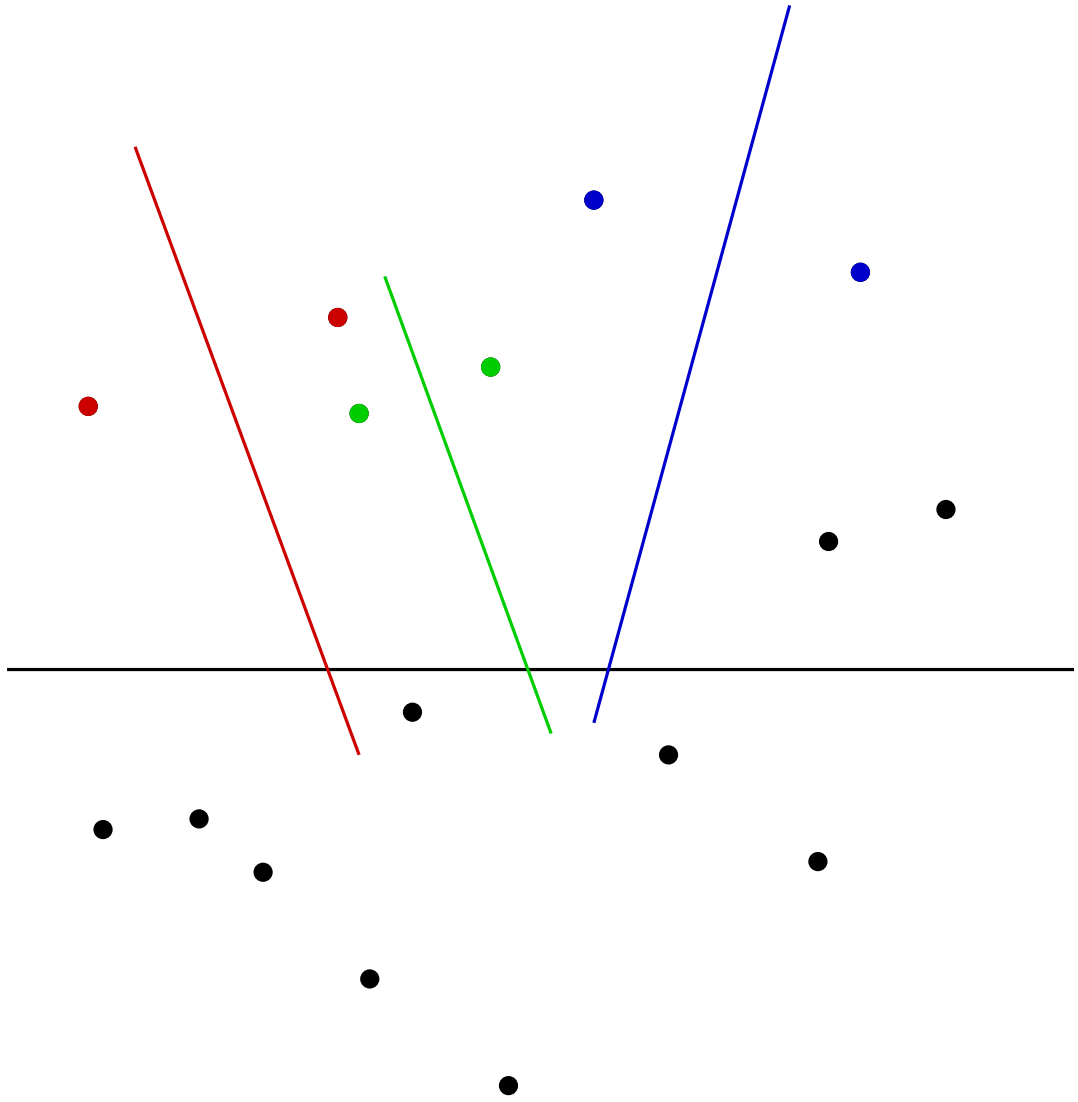

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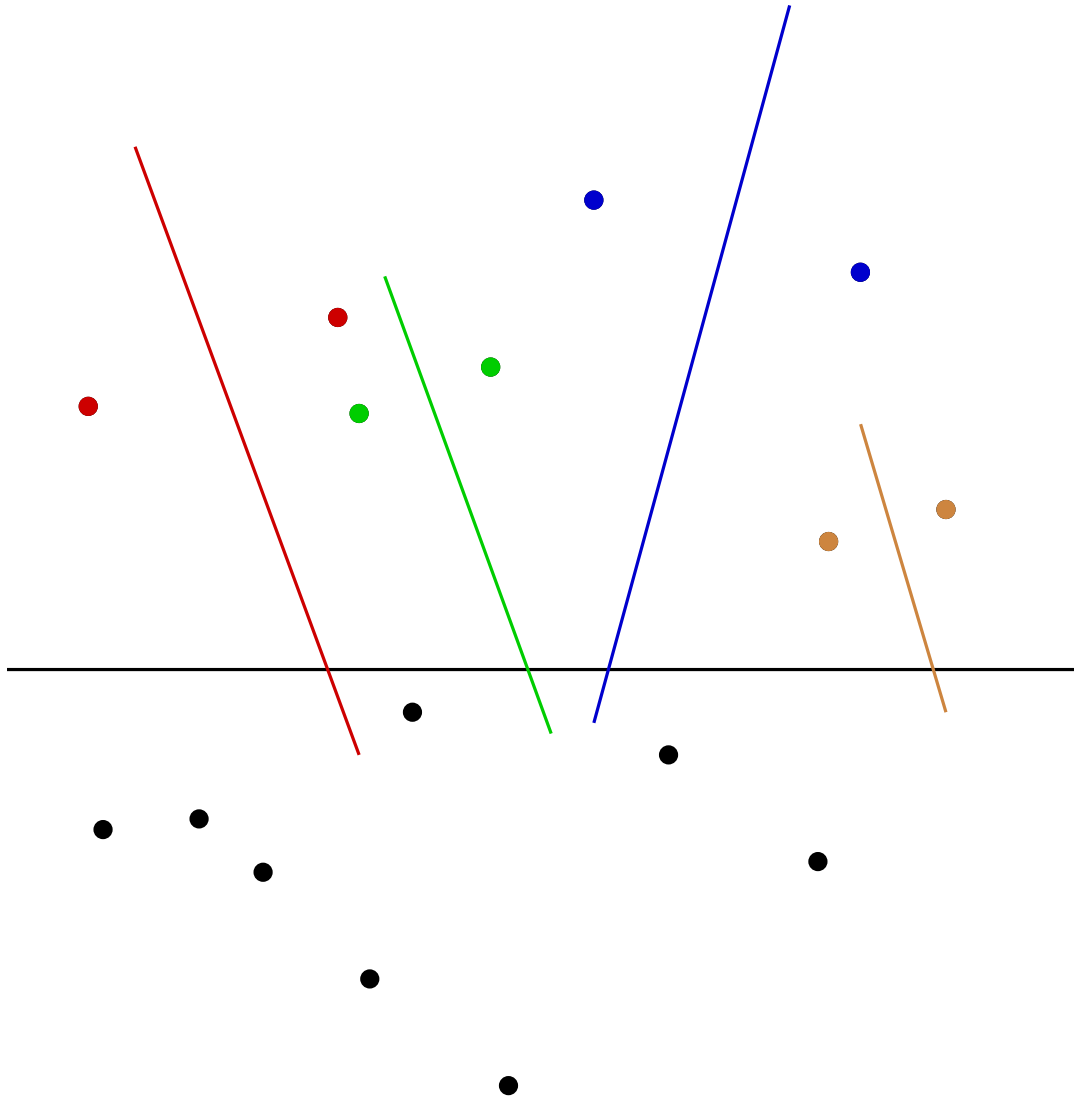

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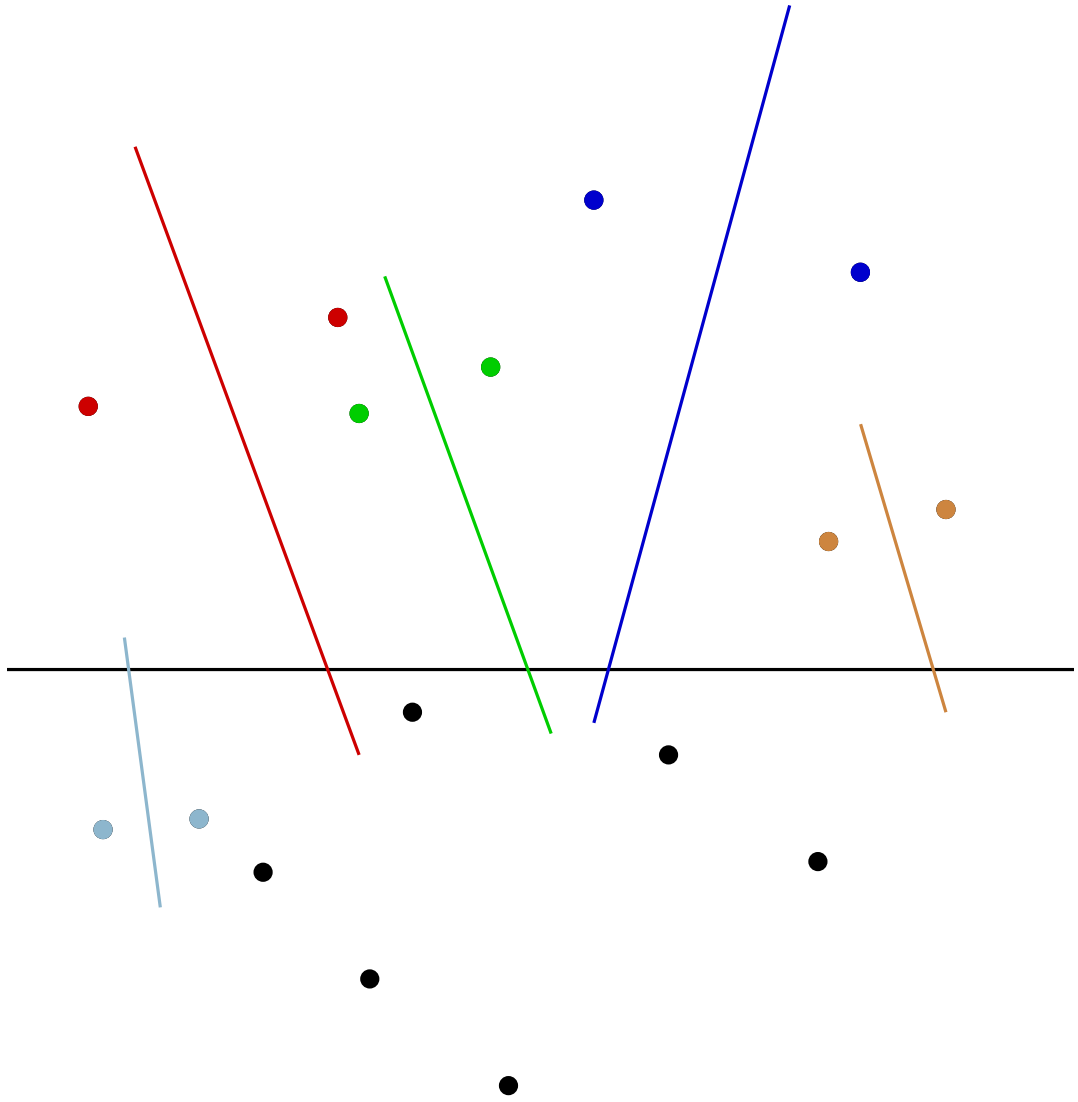

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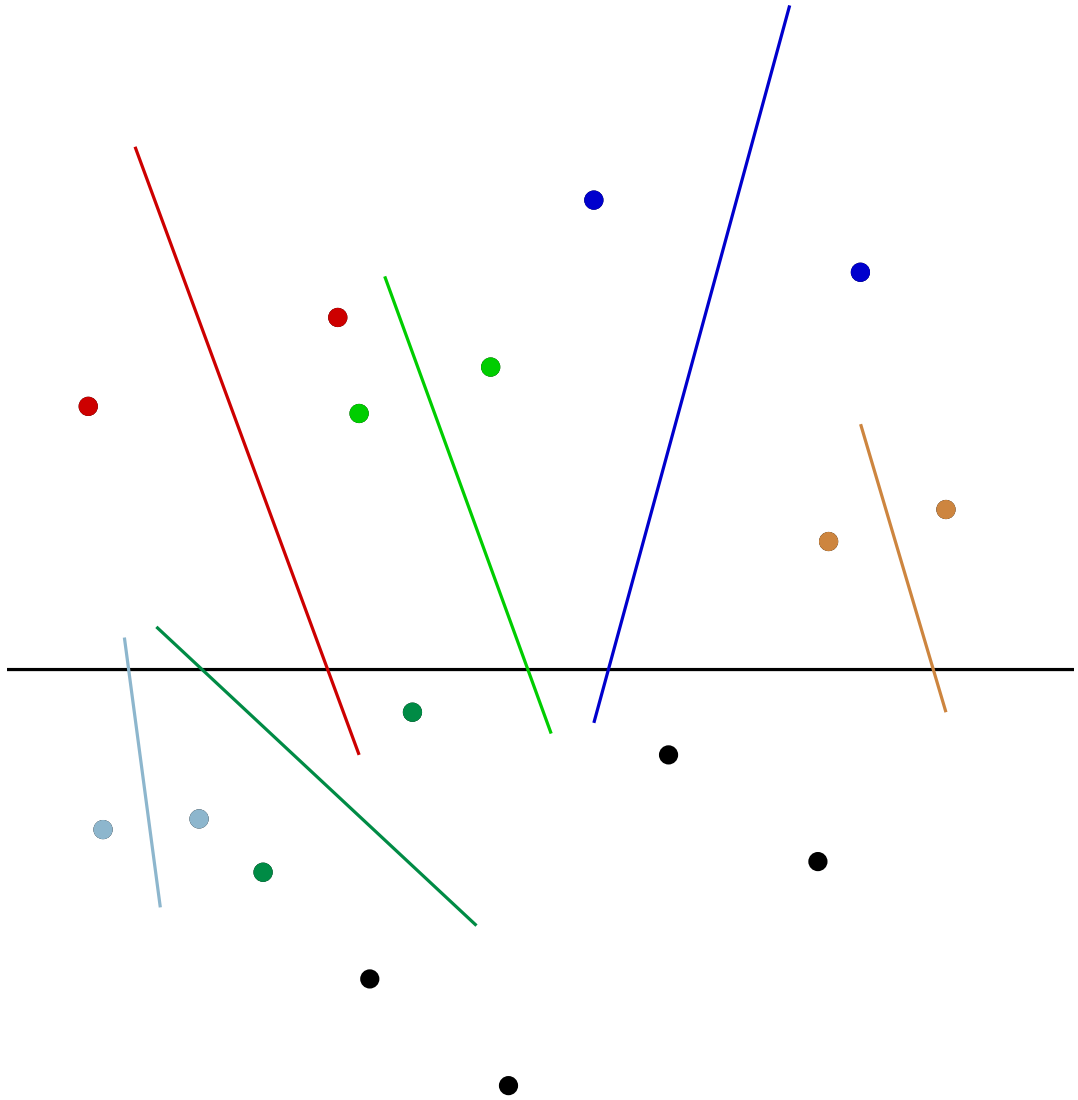

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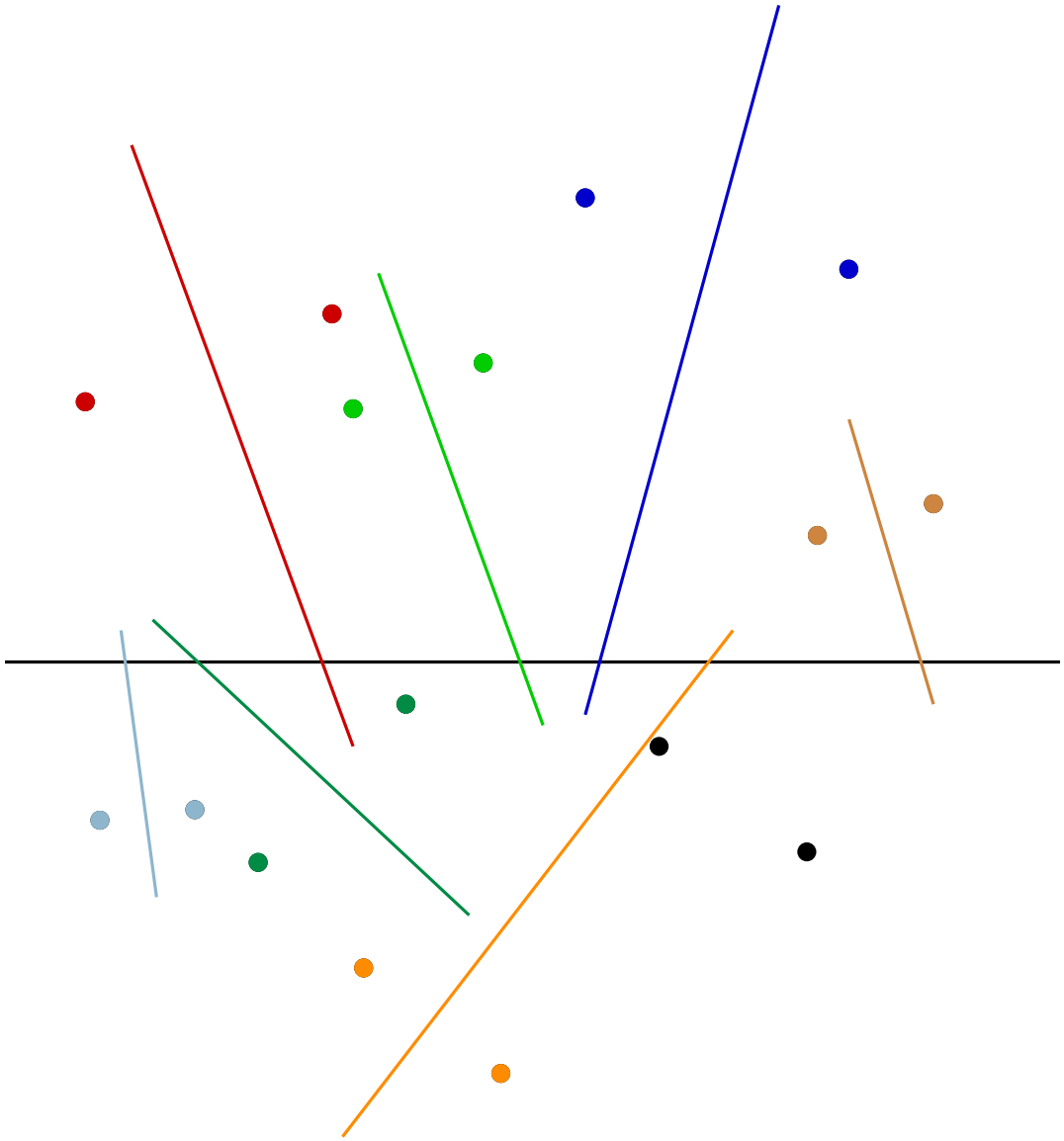

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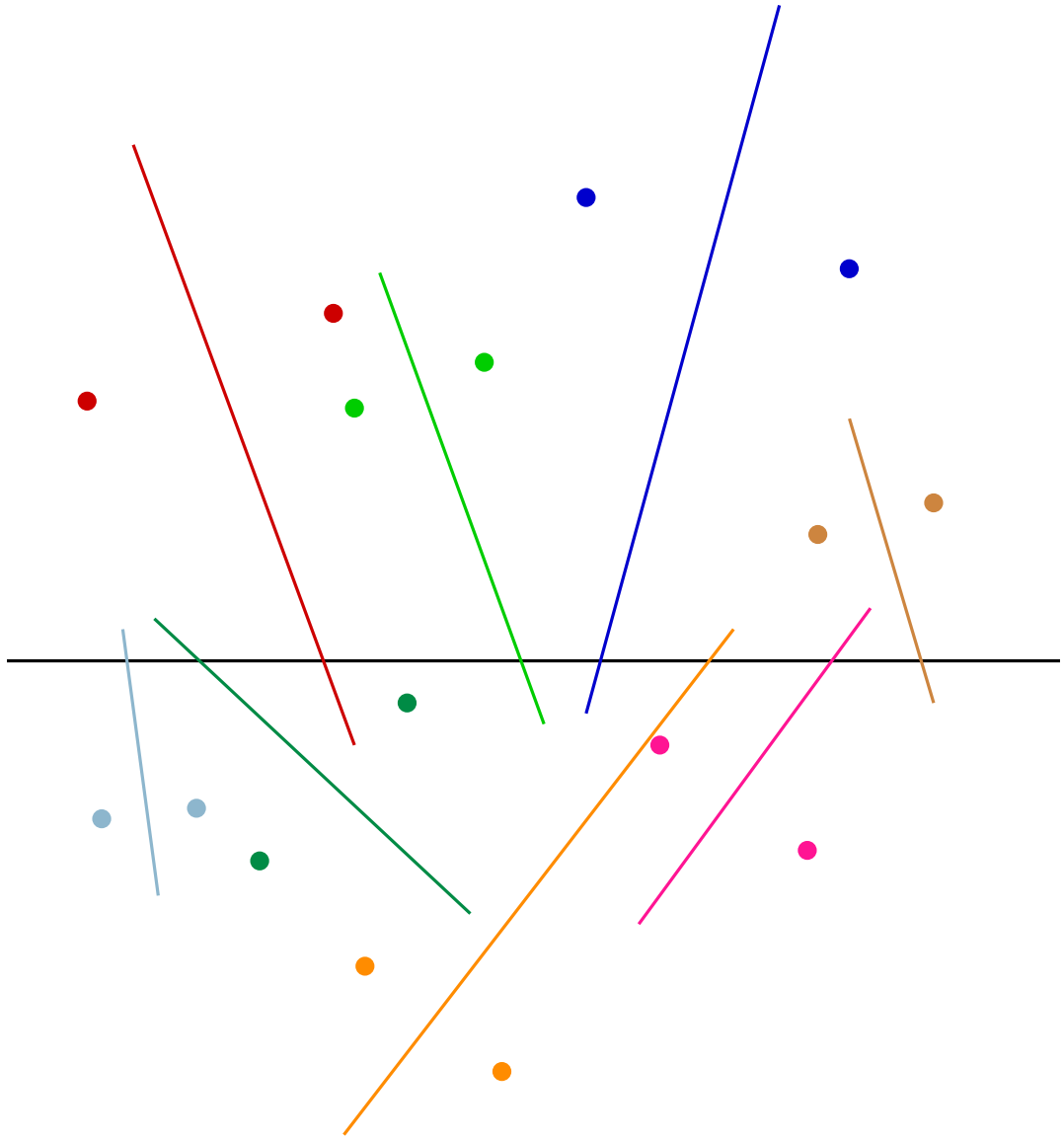

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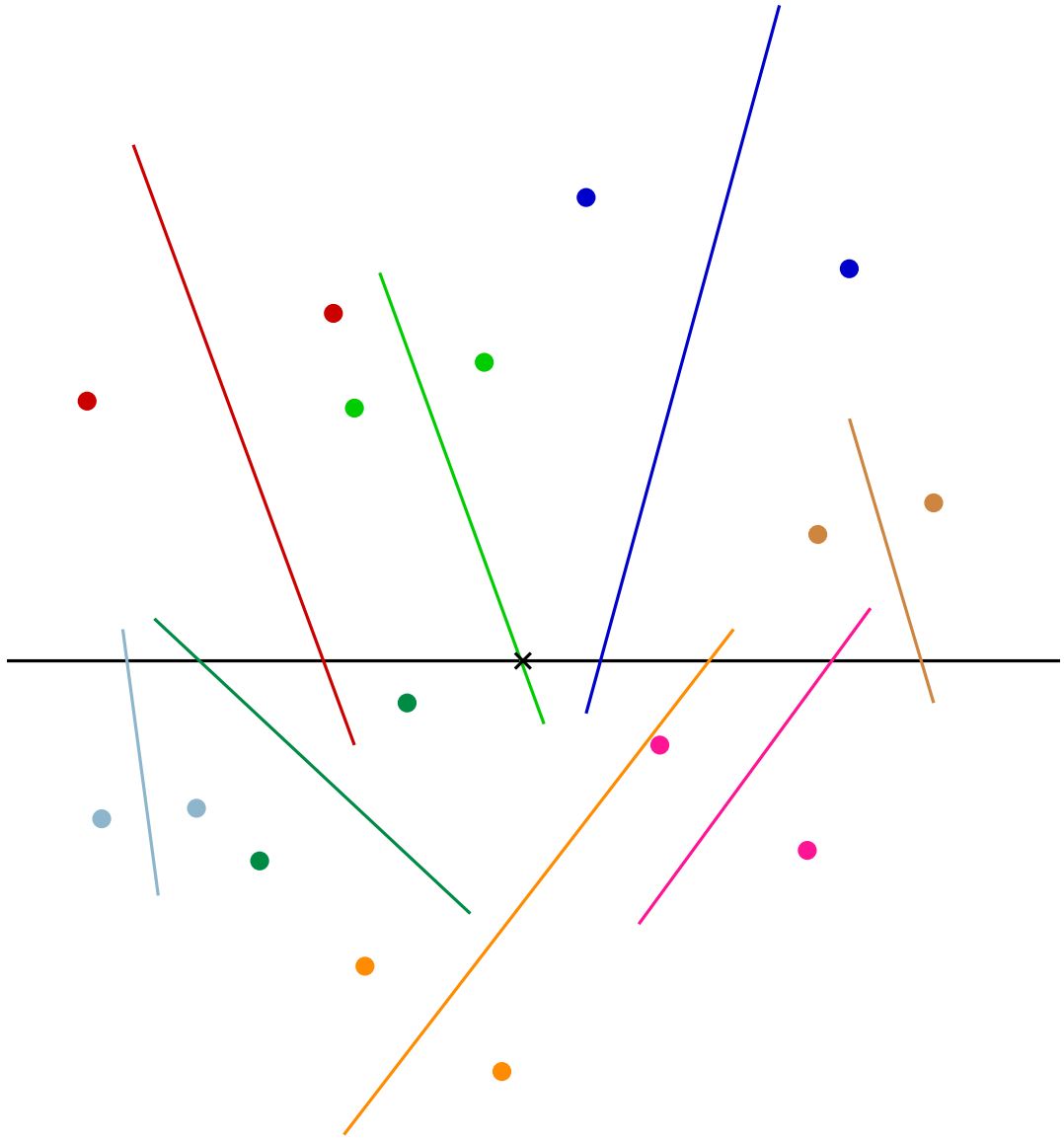
2. Compute $x_m$, the median value of the $x_{ij}$.



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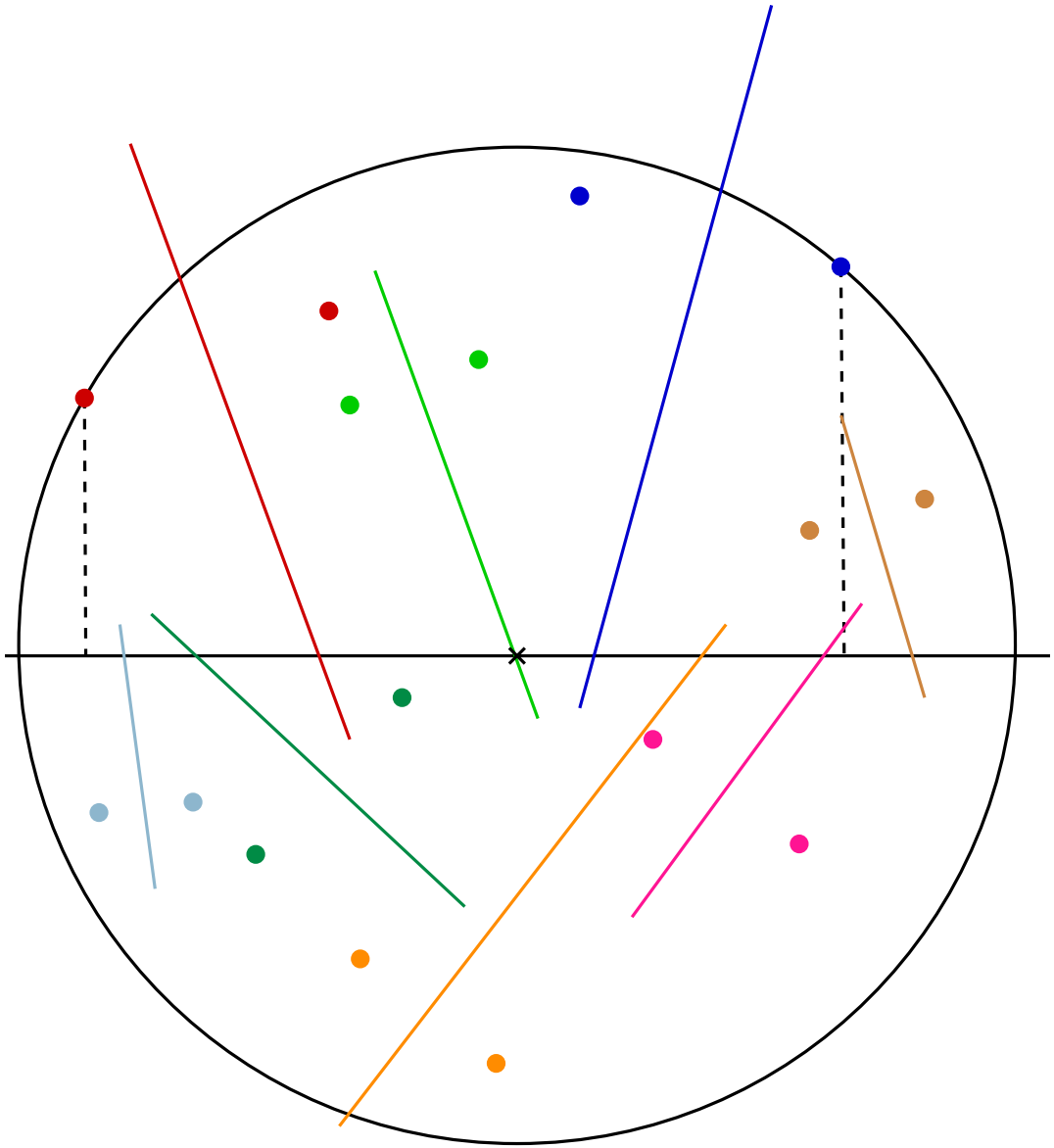
2. Compute $x_m$, the median value of the $x_{ij}$.
3. Search: Compute $C_m$, the MSC centered at $x_m$. If the points $p_i$ lying in the boundary of $C_m$ project ortogonally onto $y = 0$ to different sides of $x_m$, then $x_m$ is the solution. If they all project onto the same side then search on that side.



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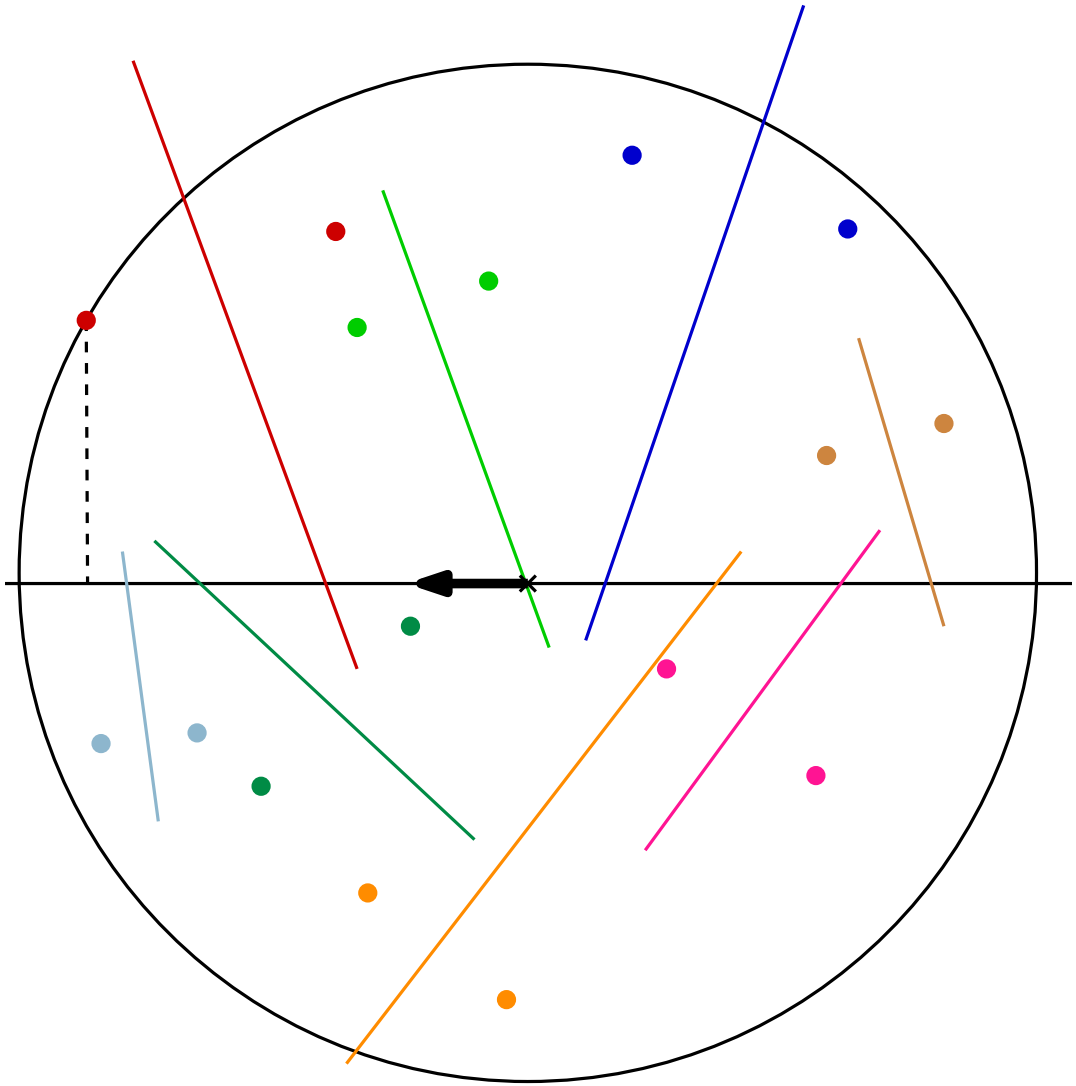

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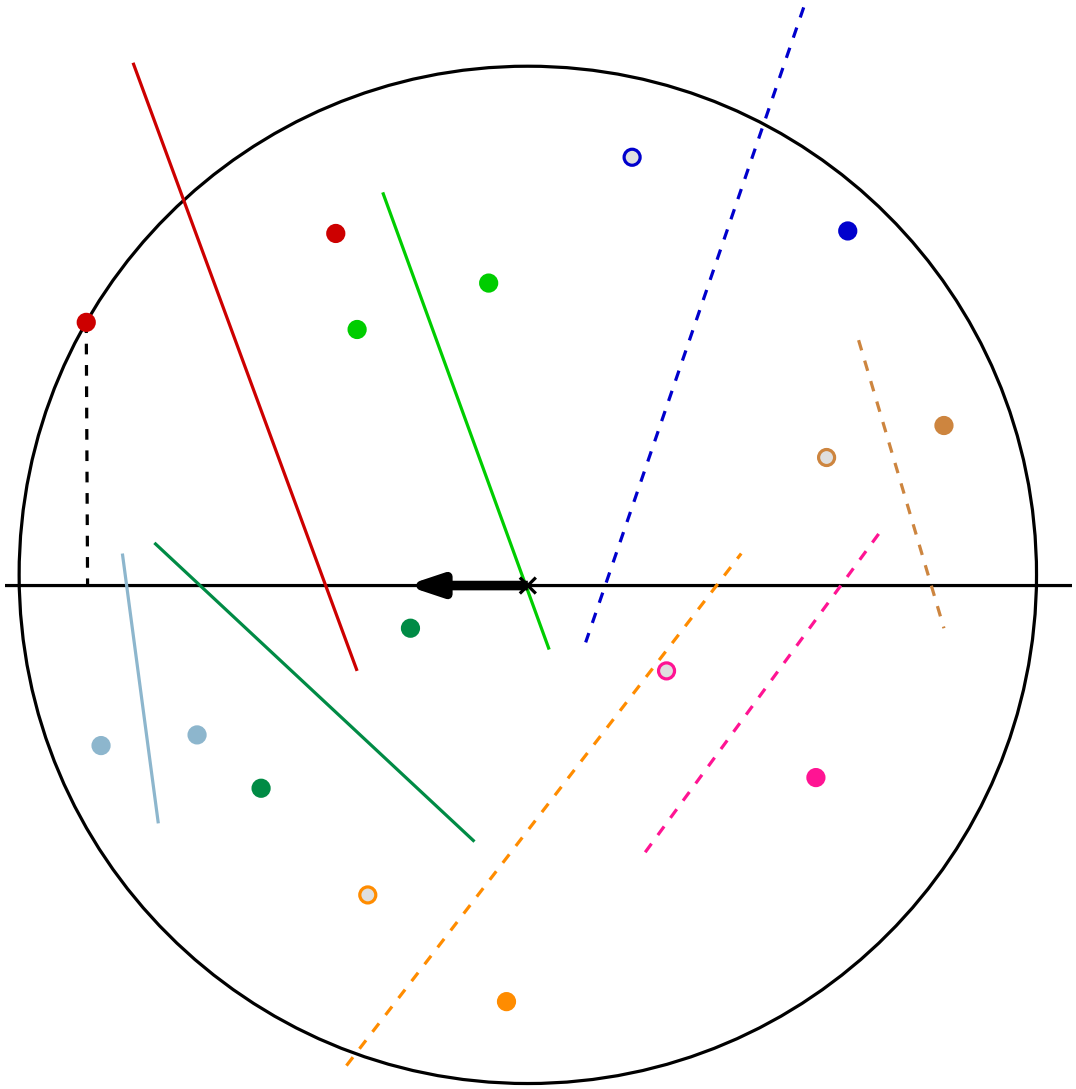
4. Prune: For all $x_{ij}$ located in the side opposite to the solution, eliminate the point $p_i$ or $p_j$ which is closest to $x_m$.



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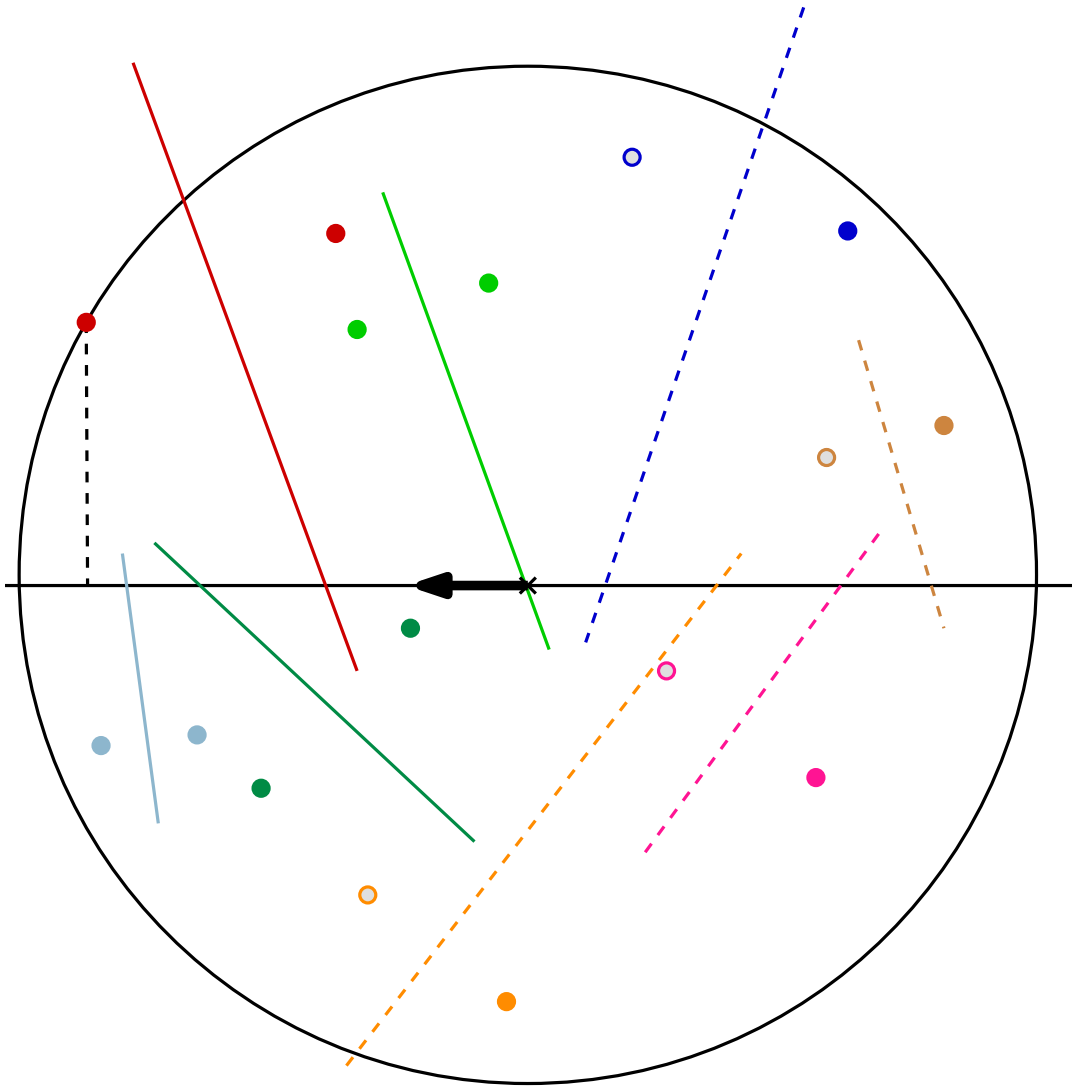
**Output:** $\min_{x \in \mathbb{R}} \max_{i=1 \ldots n} (x - a_i)^2 + b_i^2$.

1. Pair up the points, compute the perpendicular bisector of each pair, and find its intersection with the line $y = 0$. $O(n)$
2. Compute a median value. $O(n)$
3. Search: $O(n)$
4. Prune: In $O(n)$ time, at least $1/4$ of the input is pruned.

The problem is solved in $O(n)$ time.



# COMPUTING THE MINIMUM SPANNING CIRCLE

## MSC in the plane

# COMPUTING THE MINIMUM SPANNING CIRCLE

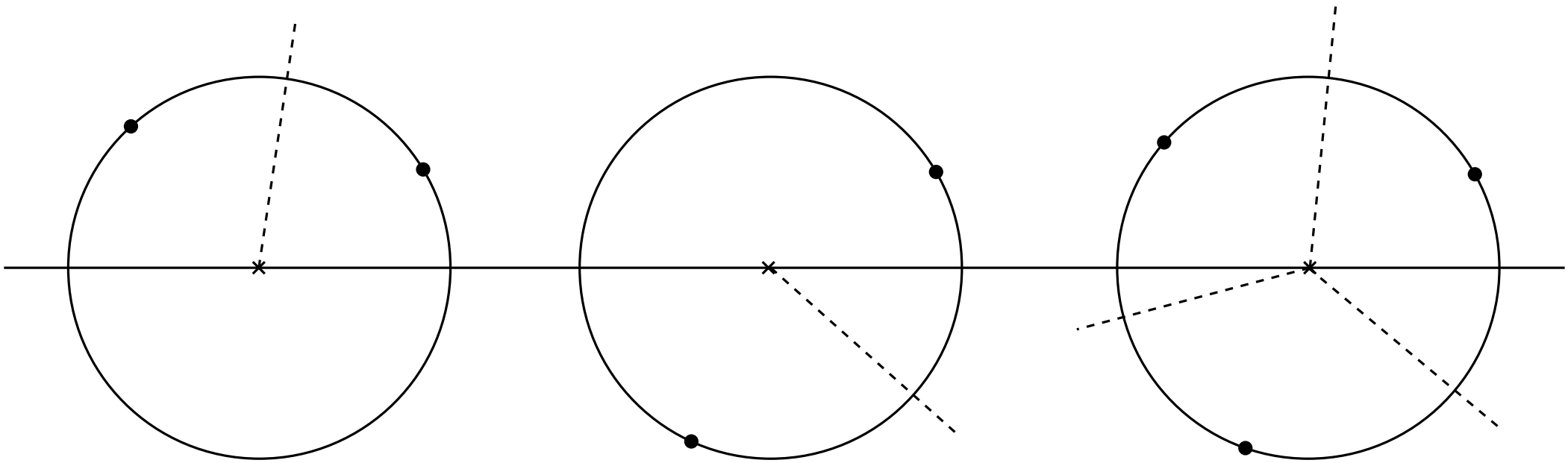
## MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair.
2. Compute the median slope of the non-vertical bisectors, and take it as horizontal.
3. Pair up the non-vertical non-horizontal bisectors, always pairing up one positive slope bisector with one negative slope bisector. Compute the intersection point of each pair.
4. First search: compute the median value $x_m$ of the $x$-coordinate of the intersection points and the vertical bisectors. Solve the MSC problem restricted to the line $x = x_m$. Use the solution to decide whether the unrestricted solution has been found or it lies to the left/right of the vertical line $x = x_m$.
5. Second search: if the solution has not been found, compute the median value $y_m$ of the $y$-coordinate of the intersection points and the horizontal bisectors lying in the opposite halfplane. Solve the MSC problem restricted to the line $y = y_m$. Use the solution to decide whether the unrestricted solution has been found or it lies above/below the horizontal line $y = y_m$.
6. Prune: if the solution has not been found, then each intersection point lying opposite to the solution quadrant is defined by a bisector which does not intersect the solution quadrant. Among the two points defining it, the one closest to the solution quadrant can be eliminated. Analogously, for each vertical bisector in the half-plane opposite to the solution quadrant one point can be eliminated, and the same happens with the horizontal bisectors.

# COMPUTING THE MINIMUM SPANNING CIRCLE

## MSC in the plane

Search step



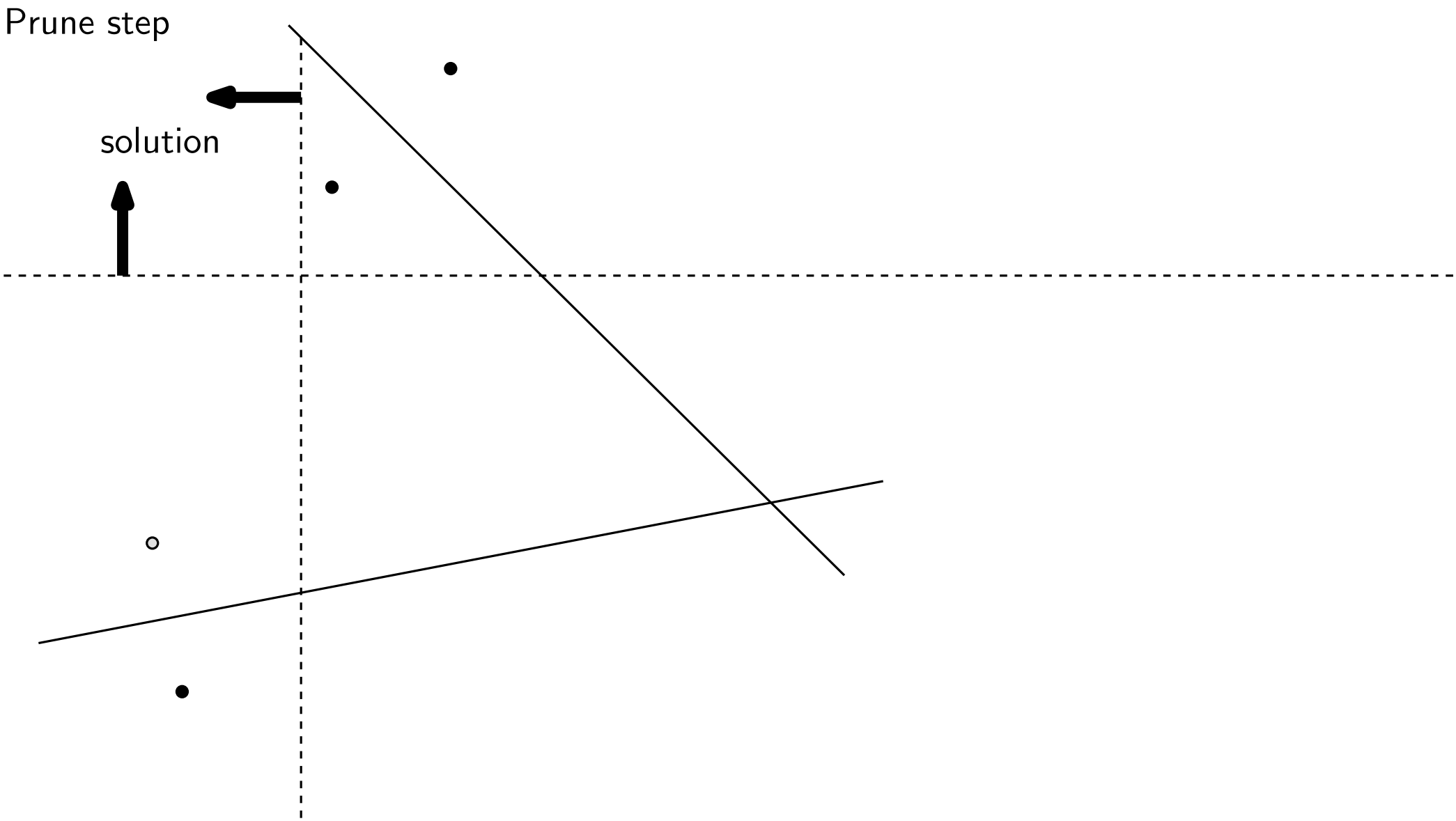
The solution lies above

The solution lies below

The solution has been found

# COMPUTING THE MINIMUM SPANNING CIRCLE

## MSC in the plane



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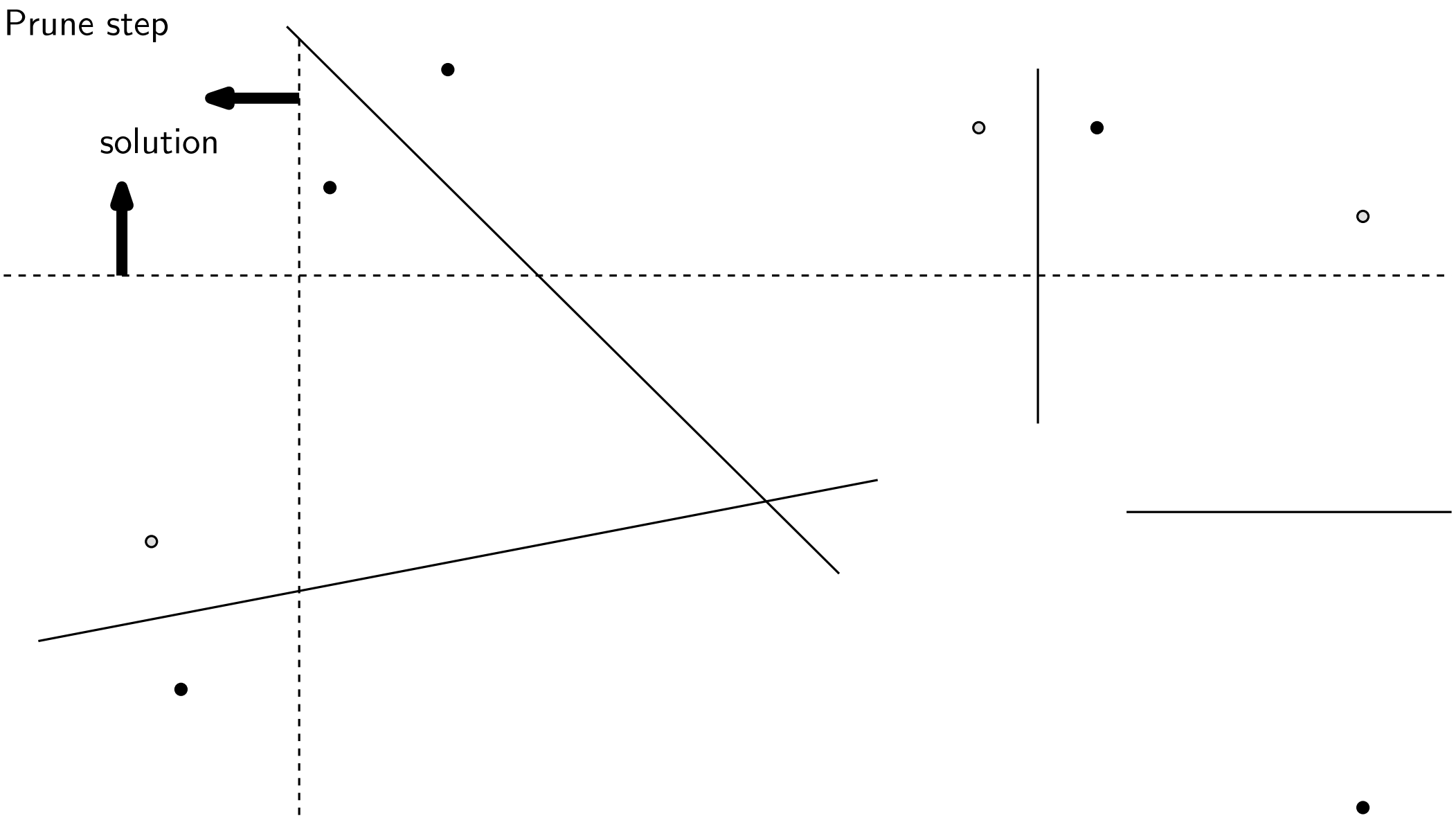

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## MSC in the plane

1. Pair up the points, and compute the perpendicular bisector of each pair. $O(n)$
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3. Pair up the non-vertical non-horizontal bisectors, and compute the intersection point of each pair. $O(n)$
4. Fisrt search: $O(n)$
5. Second search: $O(n)$
6. Prune: In $O(n)$ time, at least $1/16$ of the input is pruned.

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## Conclusion

Given $n$ points in the plane, the min-max facility location problem can be solved in linear time.

# INTERSECTING HALF-PLANES
# and related problems

## FURTHER READING


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- F. Gómez, F. Hurtado, S. Ramaswami, V. Sacristán, G. Toussaint, Implicit Convex Polygons, *Journal of Mathematical Modelling and Algorithms*, Vol. 1, N. 1, pp. 57-85, 2002.