Describing and implementing basic geometric objects

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Basic geometric objects

- Points: $p = (x, y) \in \mathbb{R}^2$ and $(x, y, z) \in \mathbb{R}^3$
- Lines and line segments: p_1, p_2
- Halflines: p_1, p_2 or $p, \overrightarrow{v}$
- Polygonal lines and polygons: $p_1, p_2, \ldots, p_n$
- Circles and discs: p_1, p_2, p_3 non collinear
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  - For each vertex v:
    - Its coordinates (x, y, z)
    - An edge $e(v)$ incident to it
  - For each face f:
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  - For each (oriented) edge e:
    - Its starting and ending vertices $v_S(e), v_E(e)$
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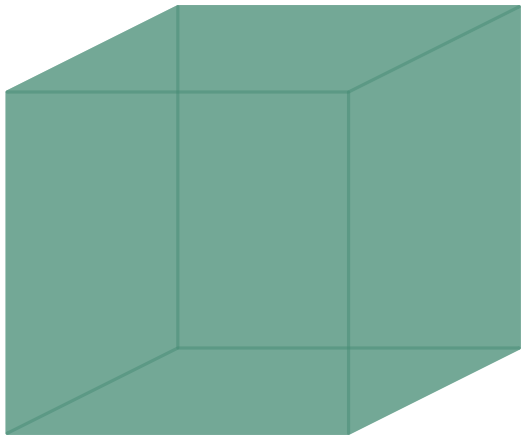

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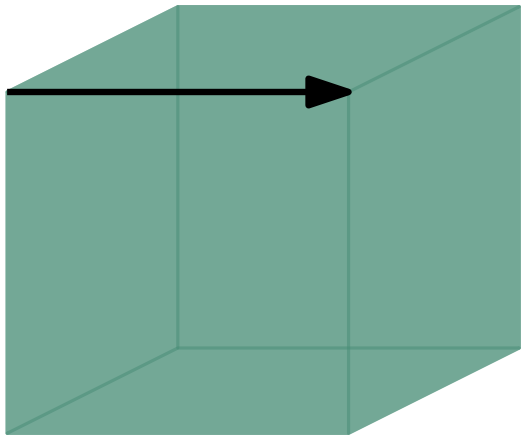

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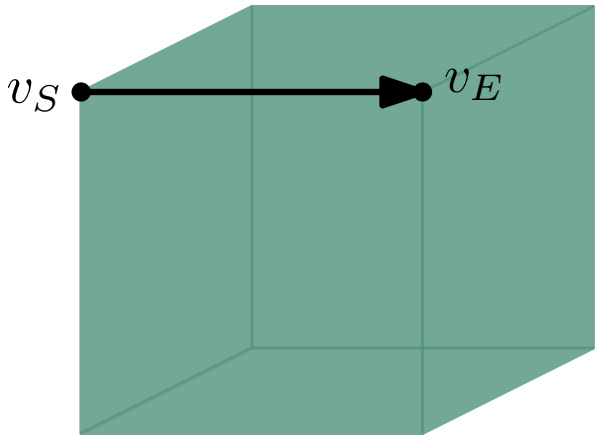

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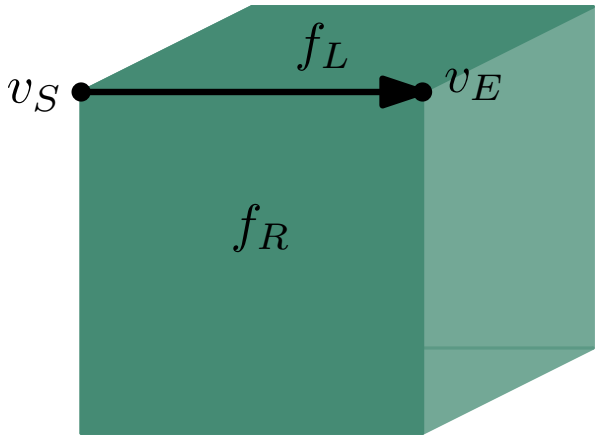

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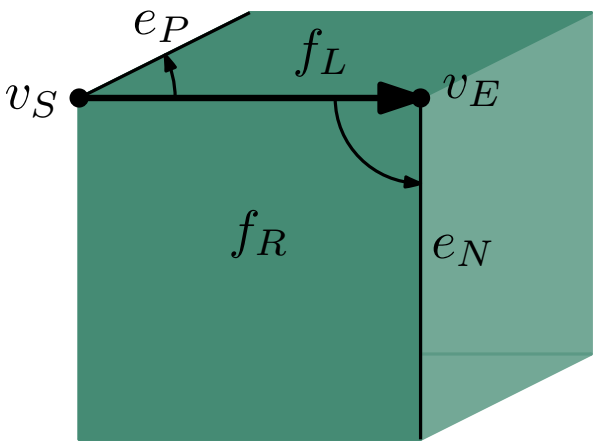
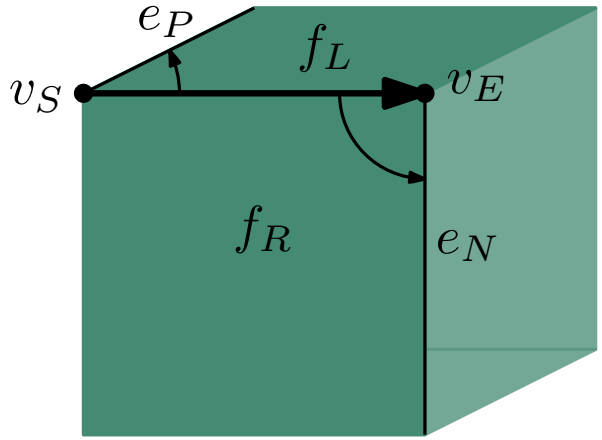


Why do we use a DCEL?



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- Time:
  - It allows to obtain the sorted lists of vertices and edges of any face in time proportional to the size of the face.
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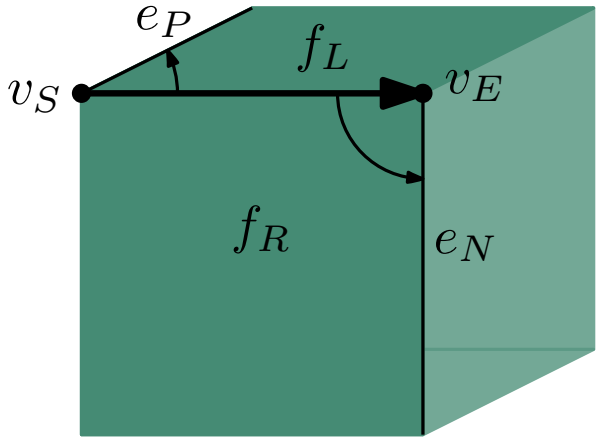

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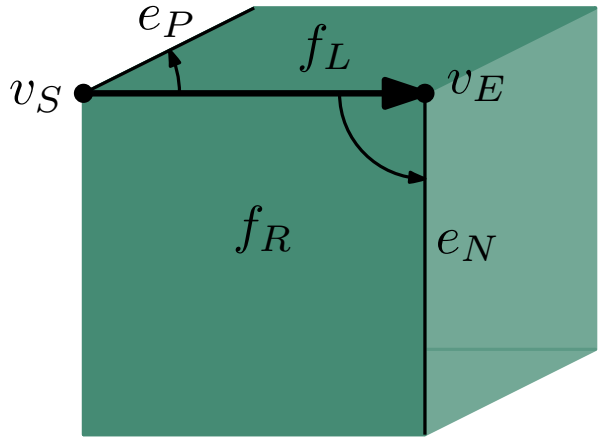

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Proof: Since the graph is planar, $V + F = E + 2$.
Since the faces are n-gons (with $n \geq 3$), $2E \geq 3F$.

Therefore:

$$2V + 2F = 2E + 4 \geq 3F + 4 \Longrightarrow F \leq 2V - 4.$$

$$3E + 6 = 3V + 3F \leq 3V + 2E \Longrightarrow E \leq 3V - 6.$$

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$\mathbb{N}$ and $\mathbb{Z}$ are easy to deal with.

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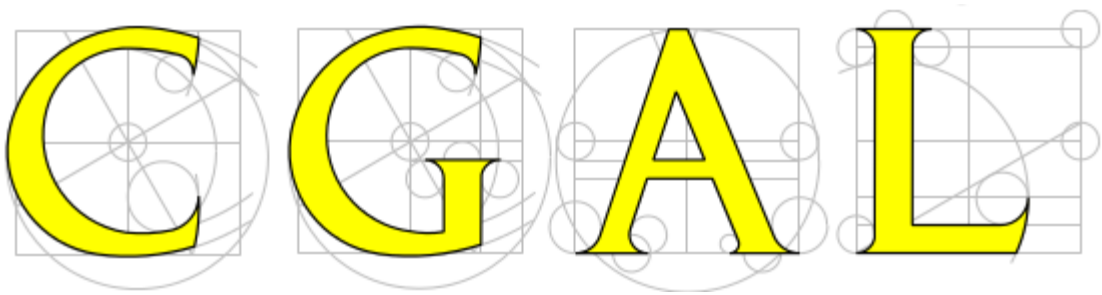
Recommended libraries

The GNU multiple-precision arithmetic library: https://gmplib.org/

The GNU multiple-precision floating-point library: http://www.mpfr.org/

The Computational Geometry Algorithms Library: http://www.cgal.org/



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$$(x, y) \longleftrightarrow (x : y : 1)$$

Lines:

$$ax + by + c = 0 \longleftrightarrow (a : b : c)$$

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$$p = (x : y : 1) \text{ lies in } \ell = (a : b : c) \iff (a, b, c) \cdot (x, y, 1) = 0$$
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Parallel lines

$$\ell_1 \| \ell_2 \Longrightarrow \left\{\begin{array}{l} \ell_1 = (a, b, c) \\ \ell_2 = (\lambda a, \lambda b, d) \end{array}\right\} \Longrightarrow \ell_1 \times \ell_2 = (x, y, 0)$$

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If you implement the cross product, you get a code that is:

- **Correct:** it computes exactly what it should.
- **Reusable:** it is used to solve several (apparently different) problems.
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There is no way you can do the same using affine coordinates!

FURTHER READING

See the documents provided at the end of this course