DELAUNAY TRIANGULATION OF POINT SETS

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DELAUNAY TRIANGULATION

A TOOL FOR INTERPOLATION

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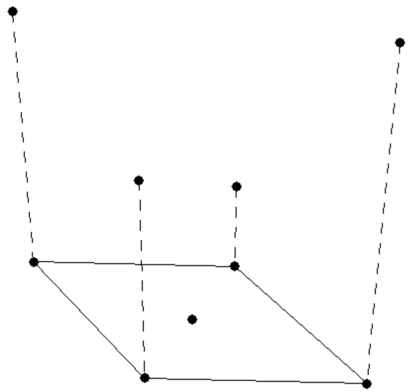

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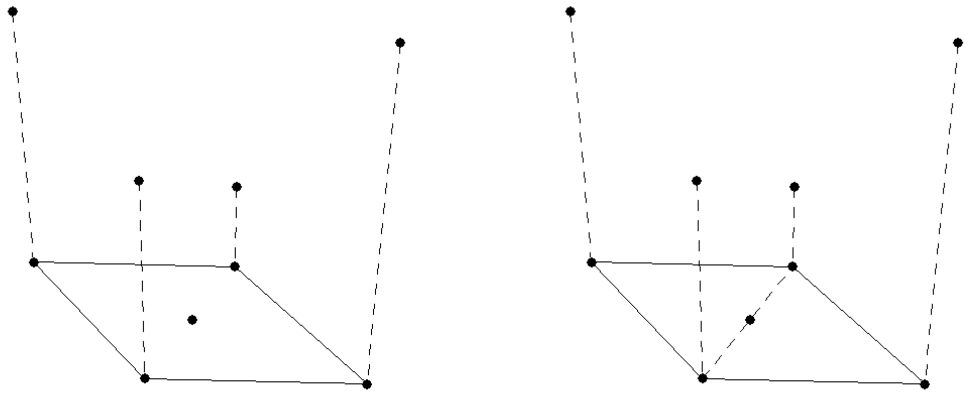

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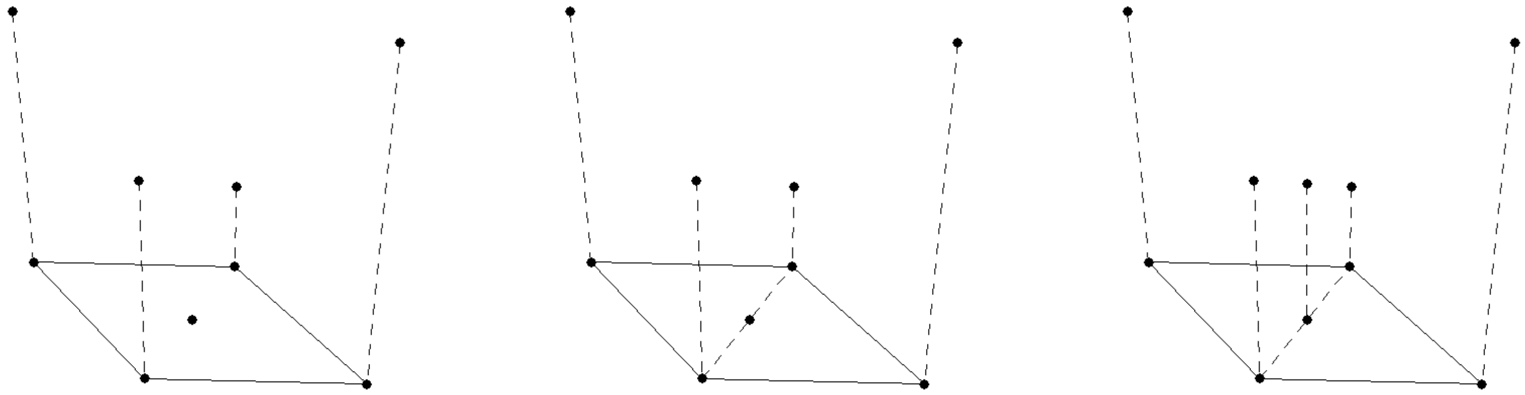

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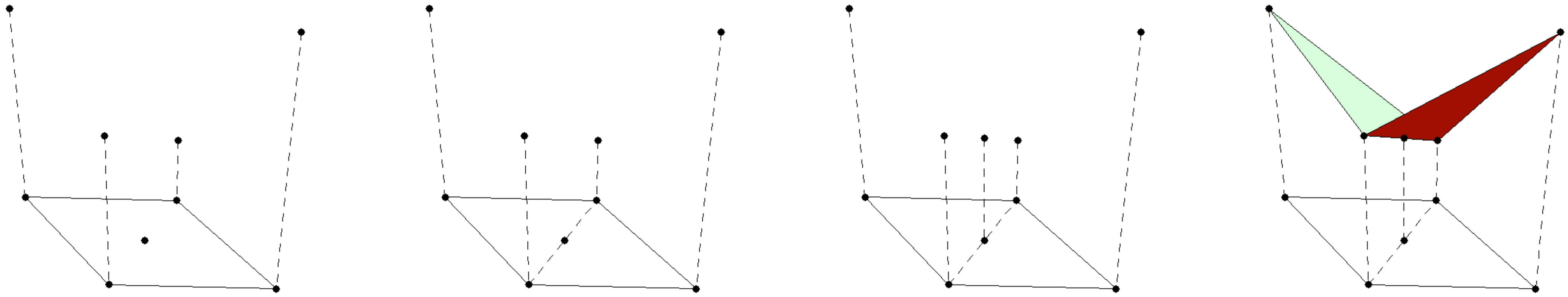

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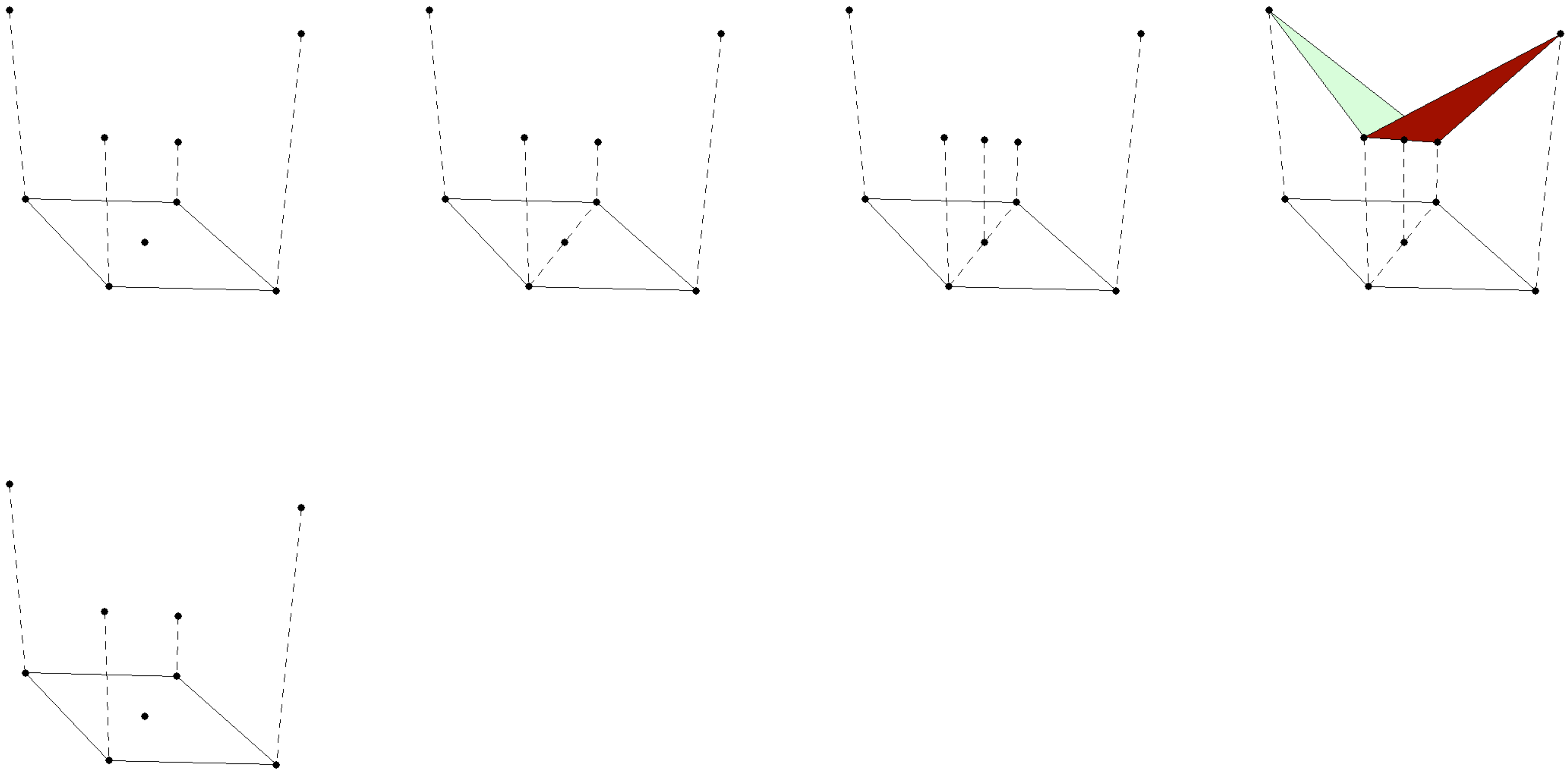

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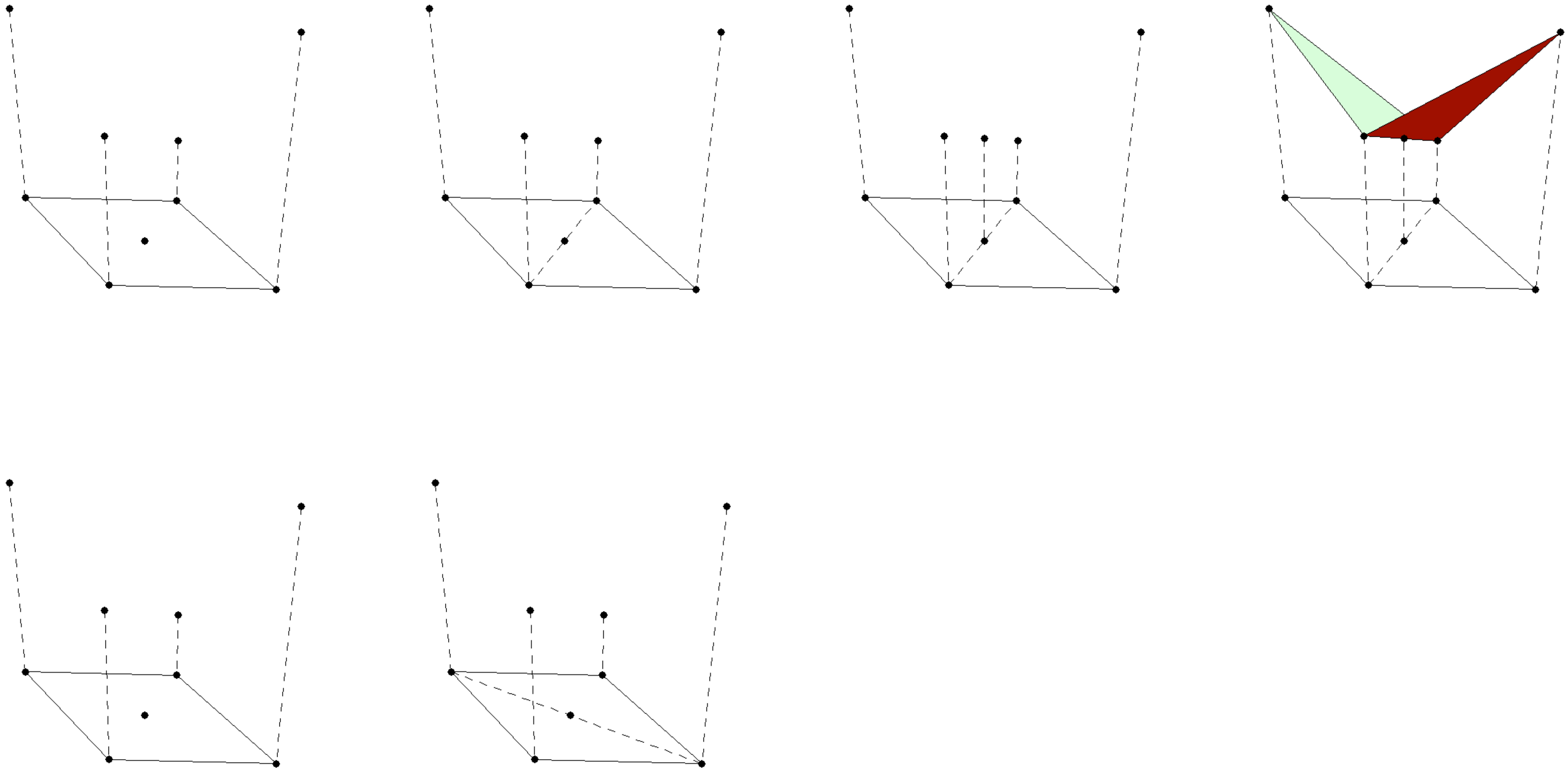

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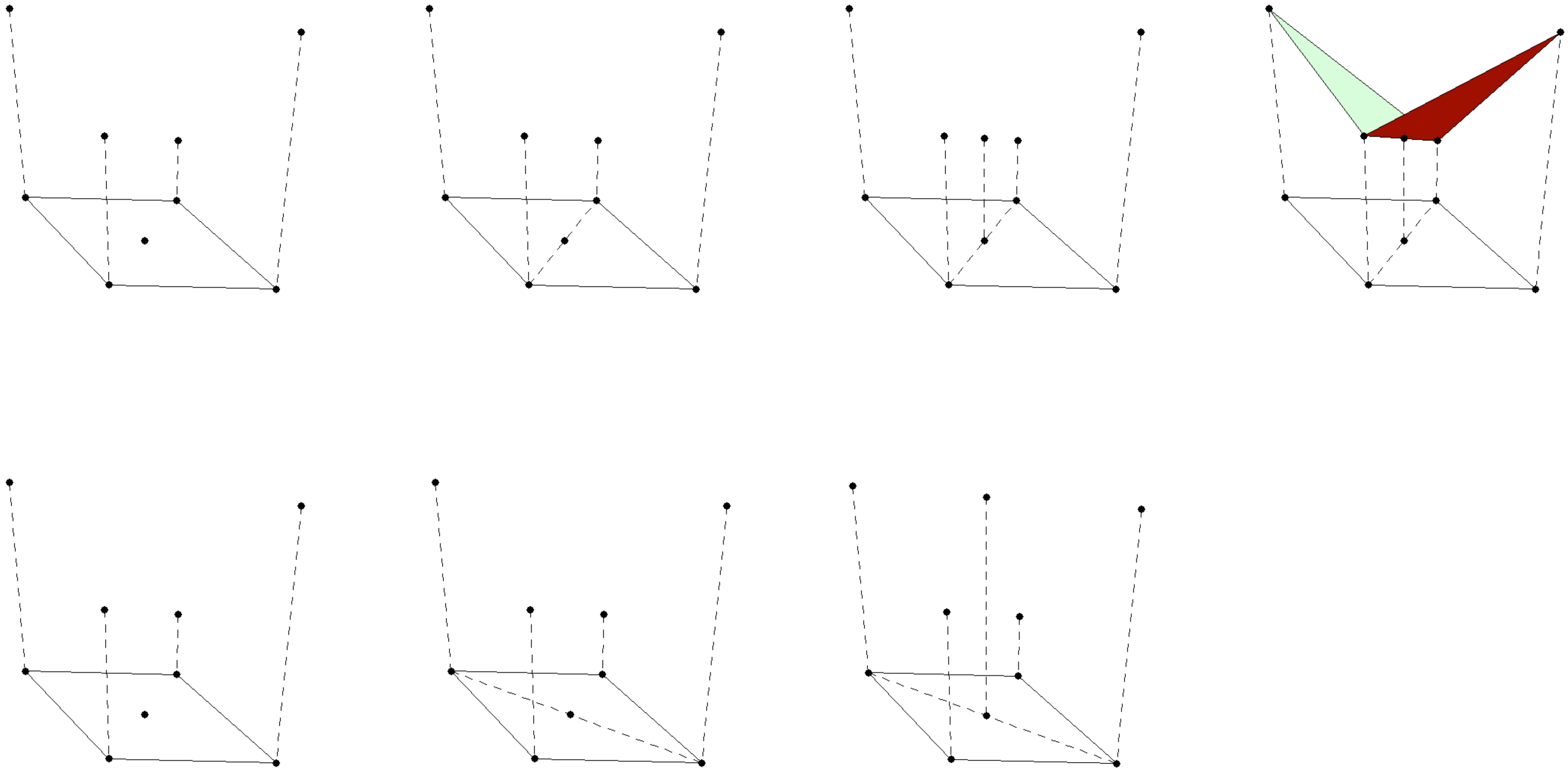

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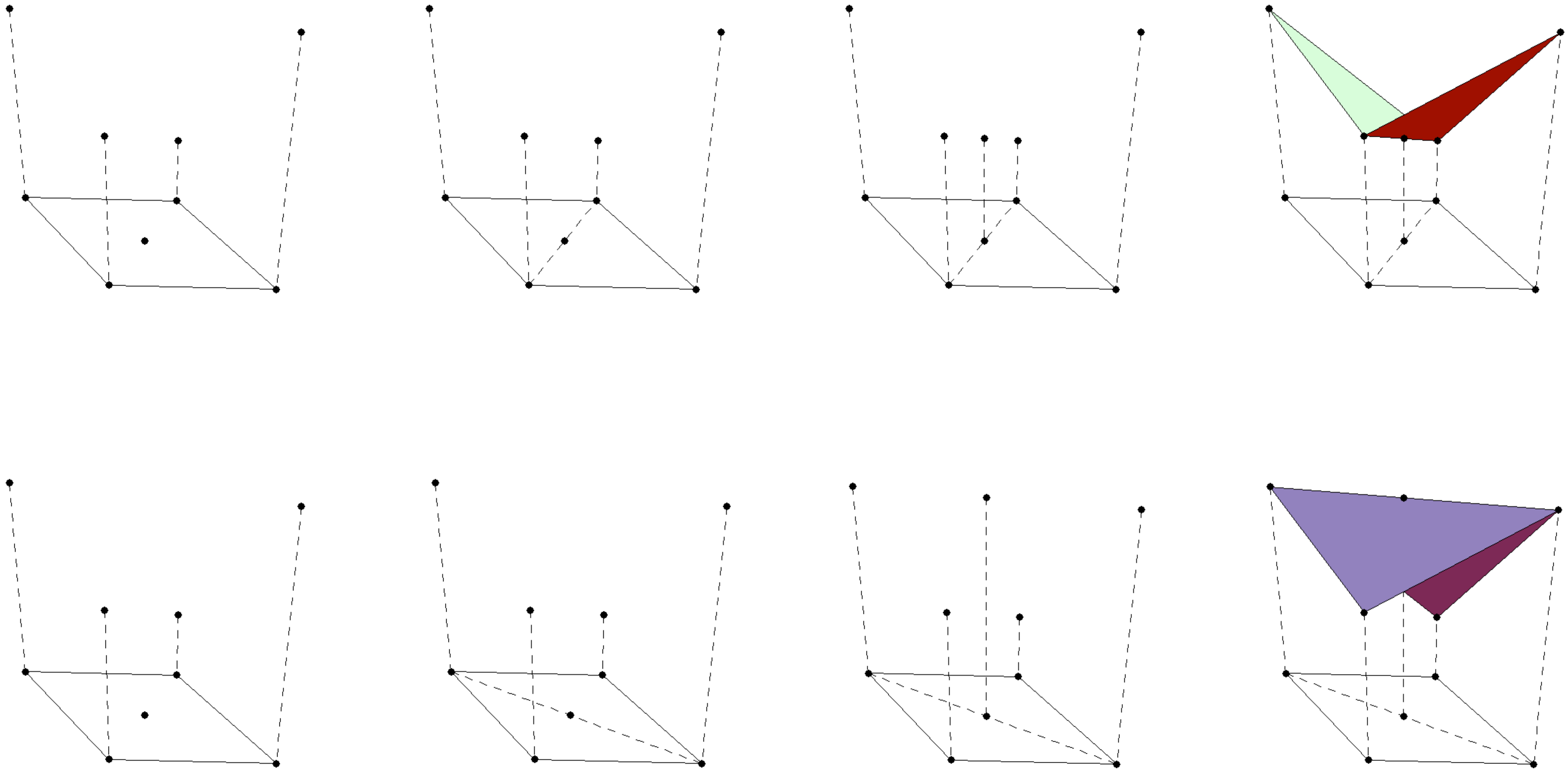

DEFINITION AND PROPERTIES

DELAUNAY TRIANGULATION

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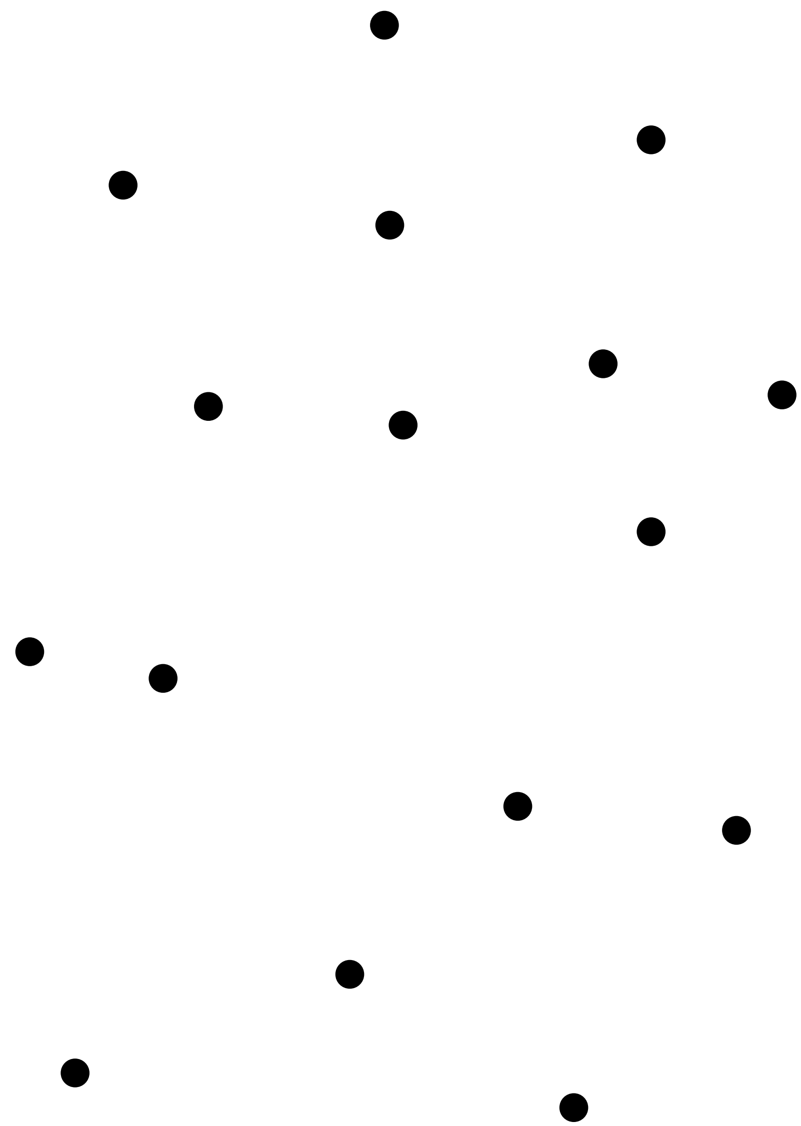

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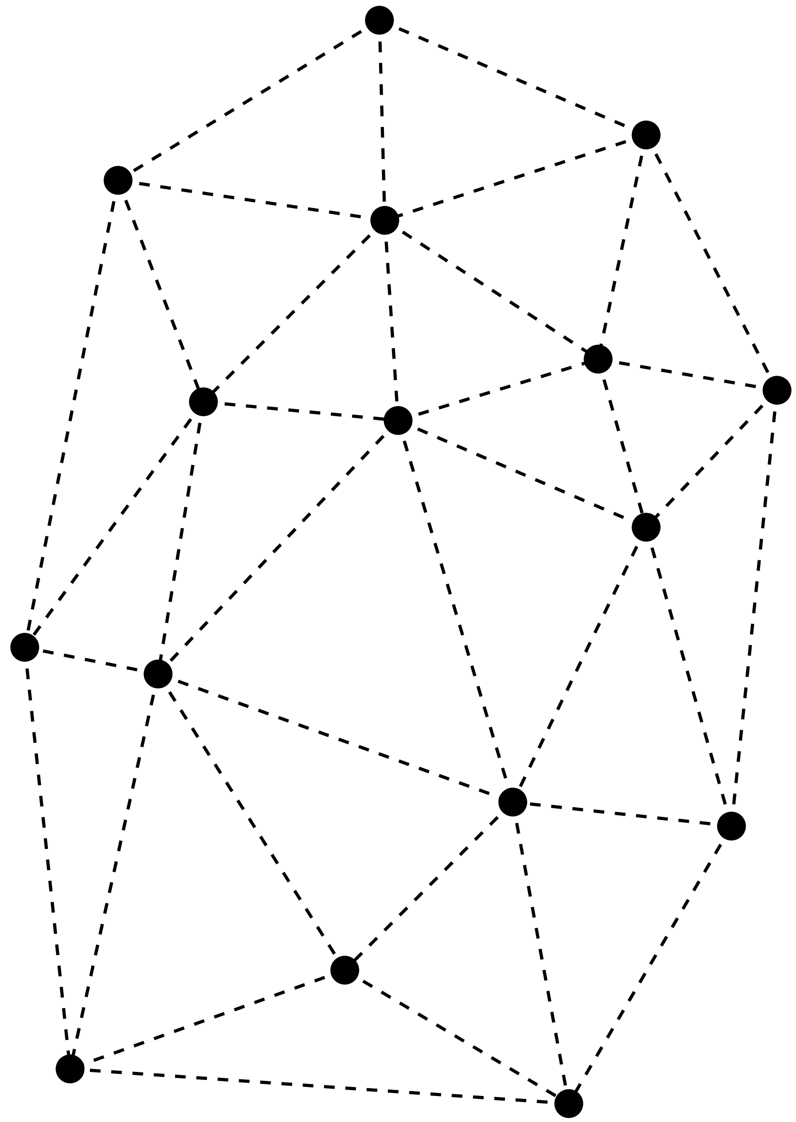

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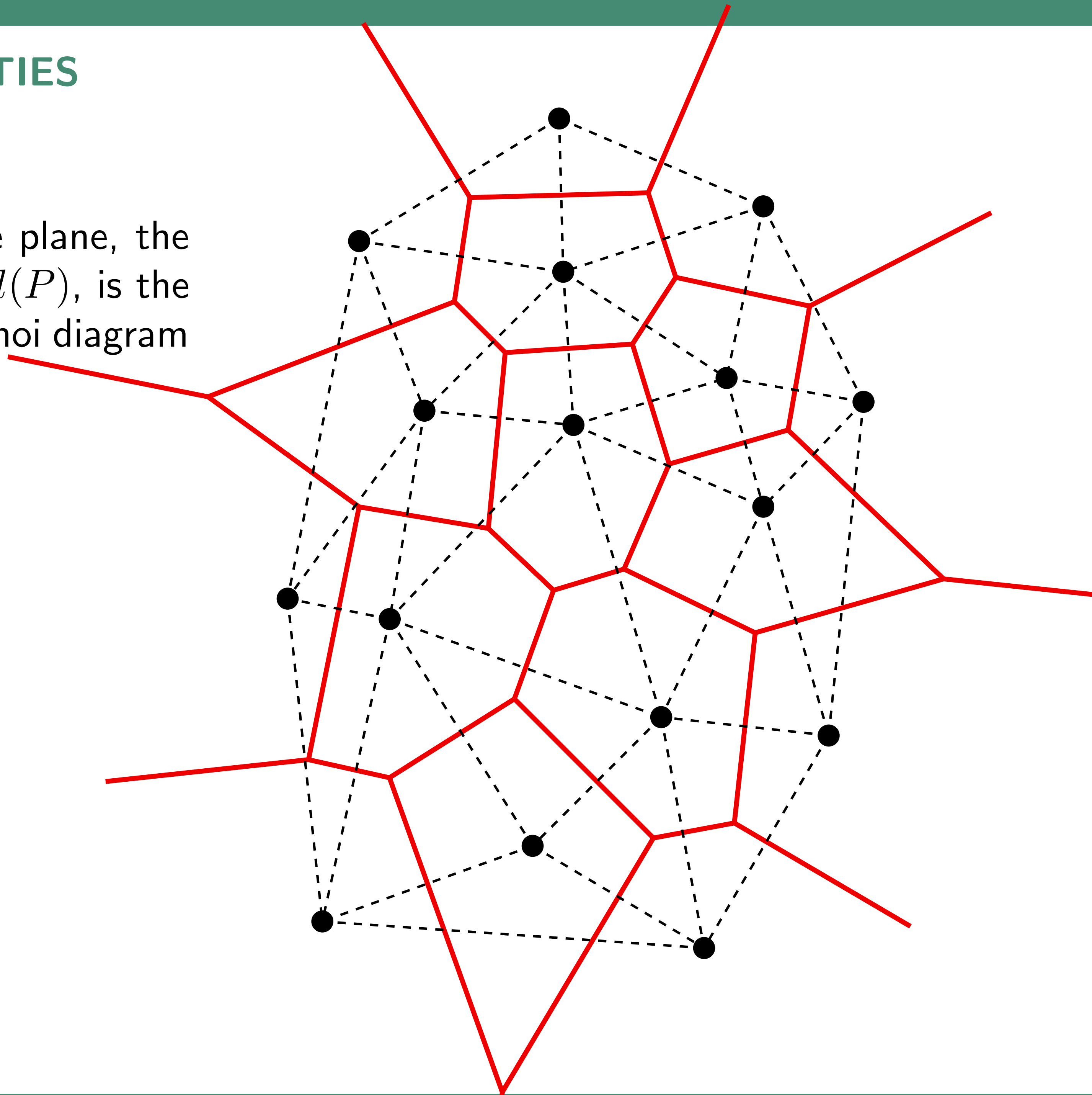

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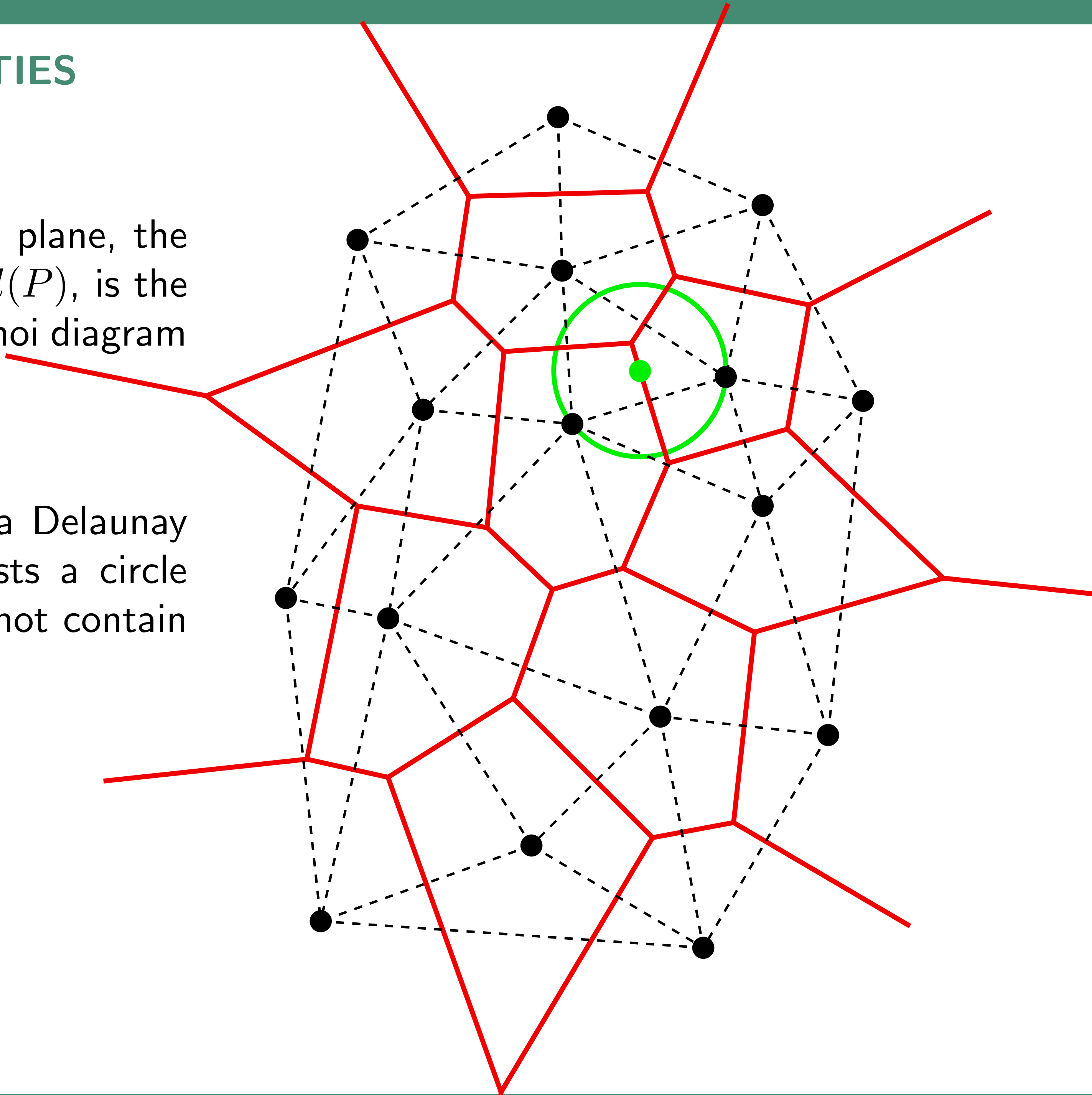
- Two points $p_i, p_j \in P$ form a Delaunay edge if and only if there exists a circle through p_i and p_j wich does not contain any point of P in its interior.



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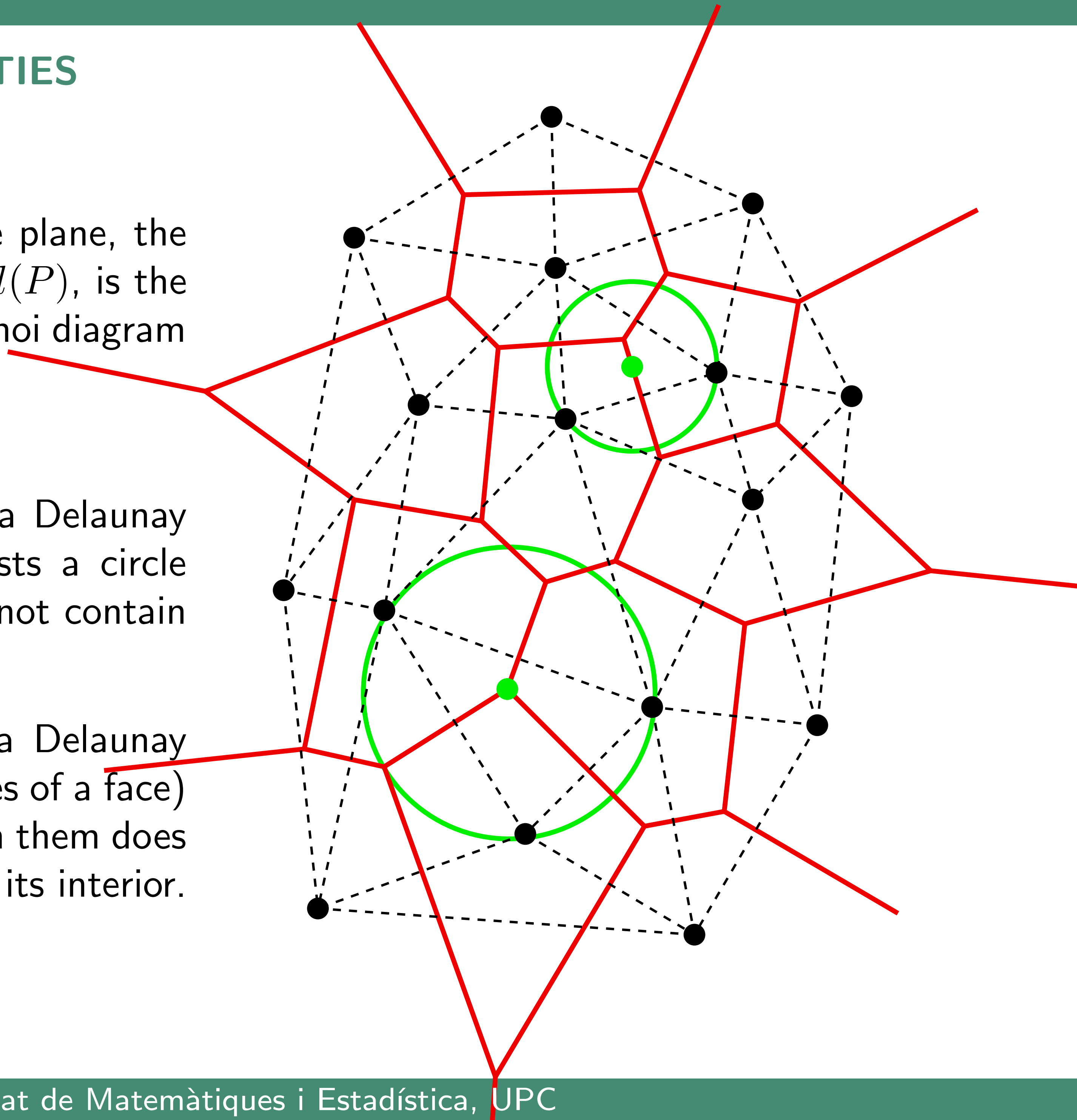

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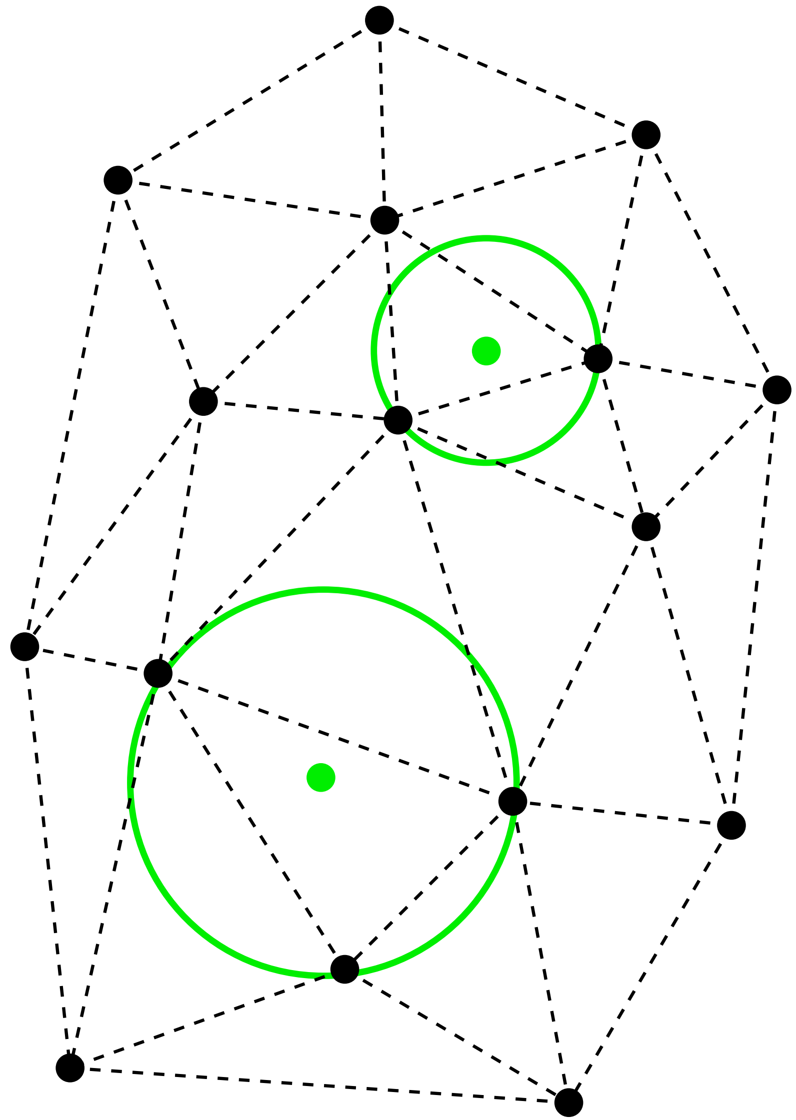

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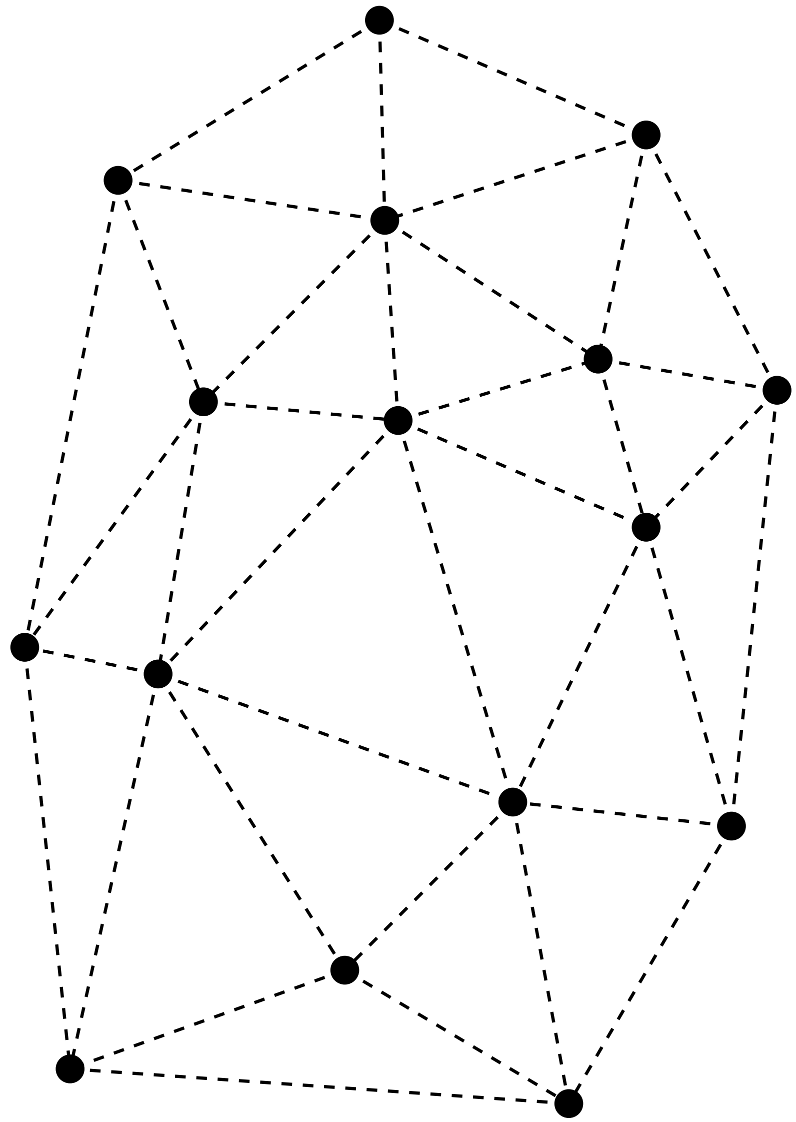

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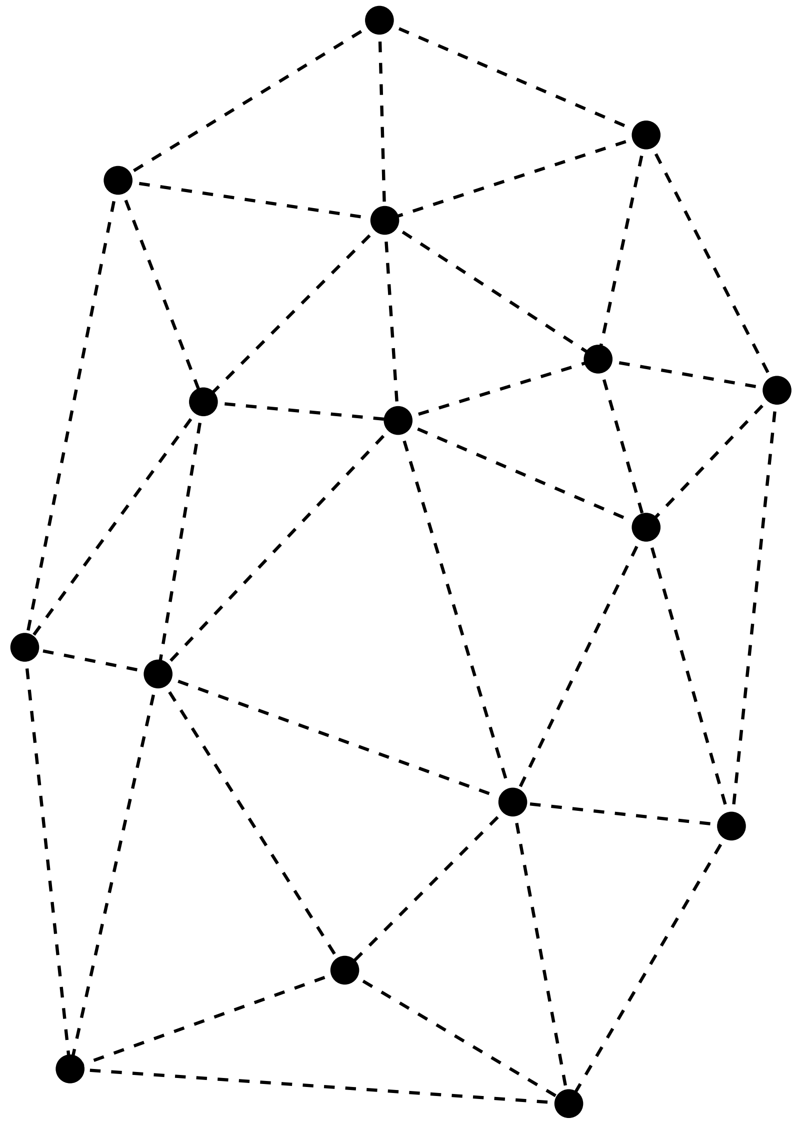

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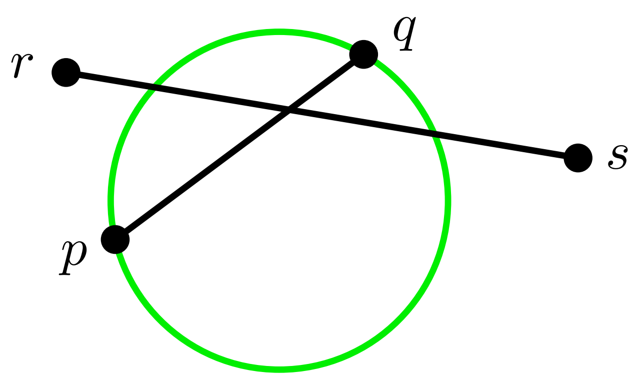
If $\overline{pq}$ is a Delaunay edge, there exists an empty circle through p and q. If a segment $\overline{rs}$ intersects $\overline{pq}$, then every circle through r and s contains at least one of p or q.



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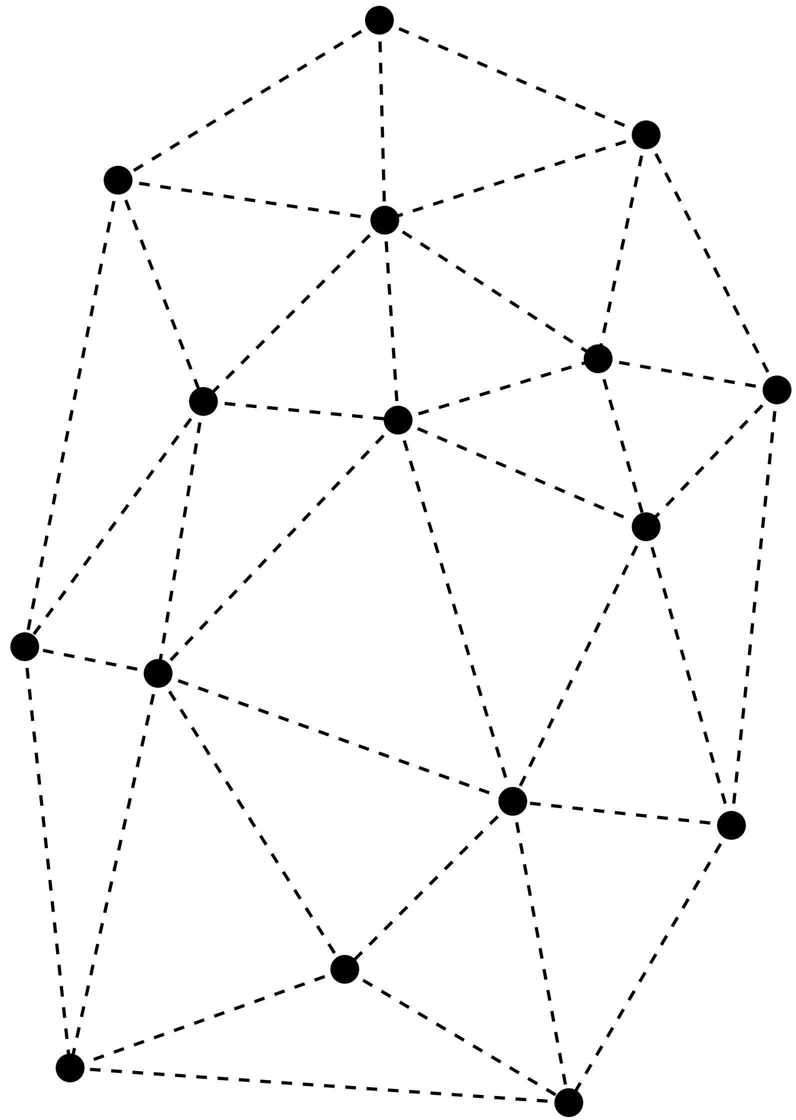

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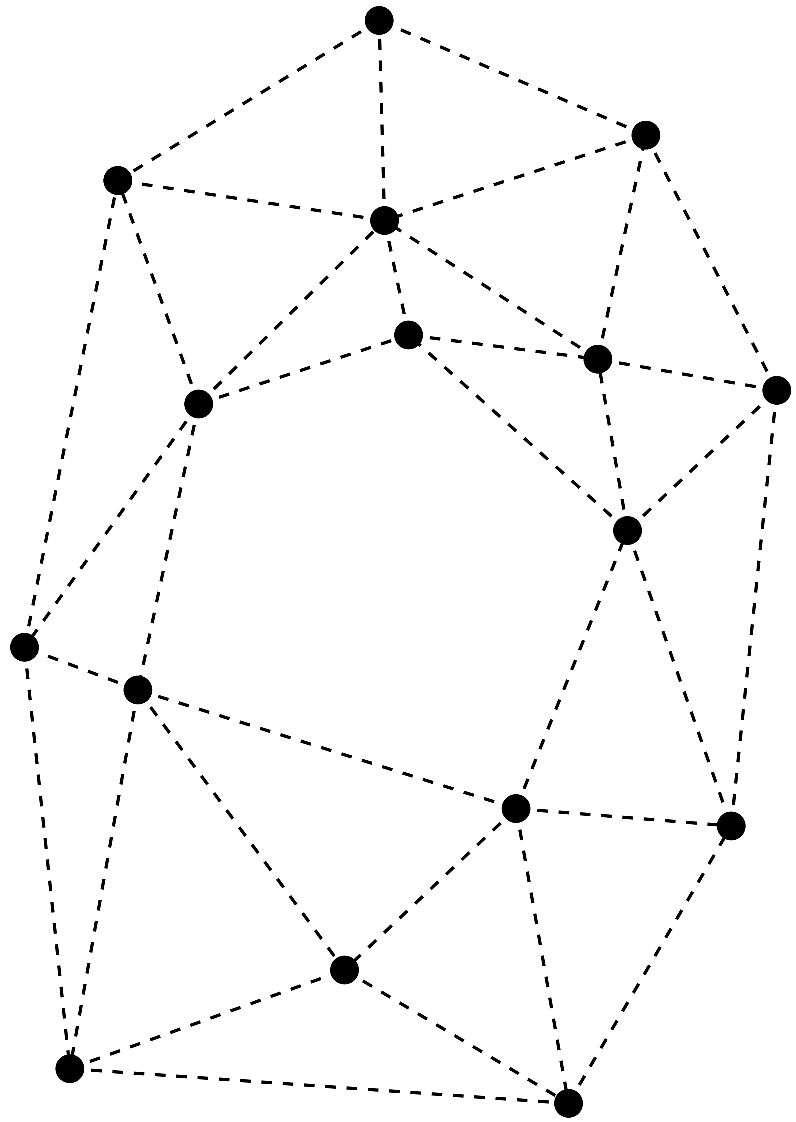

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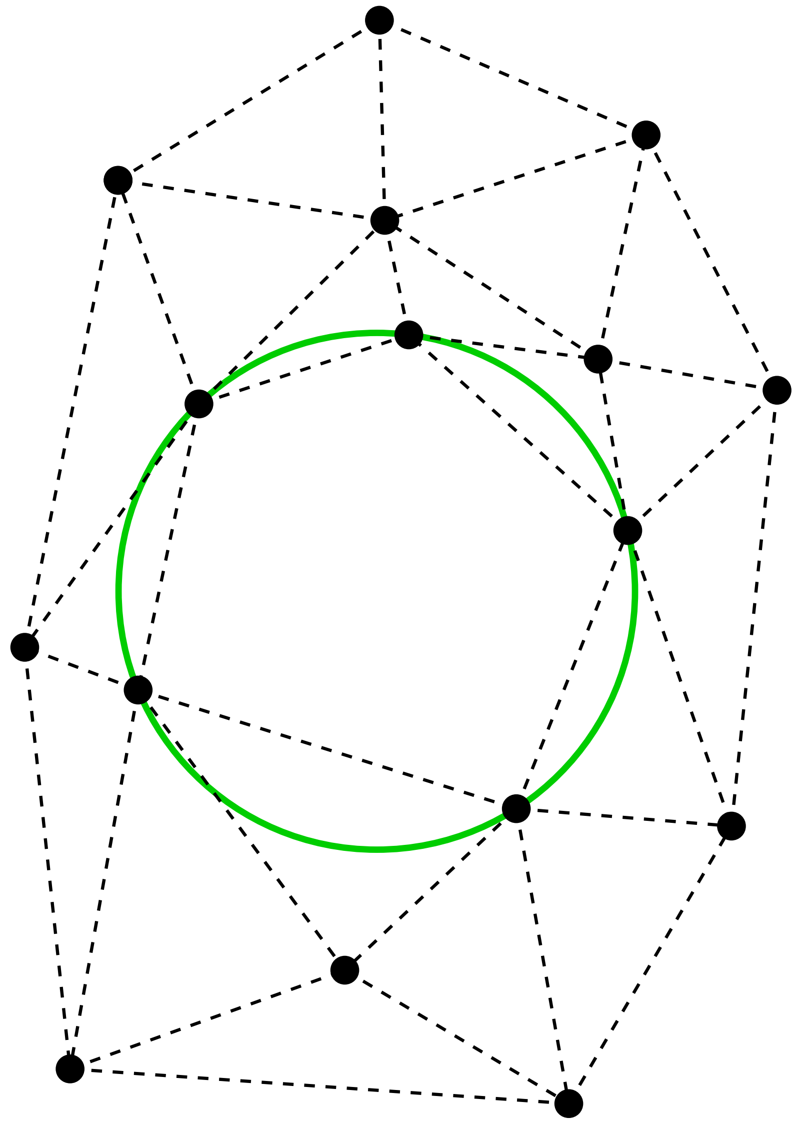

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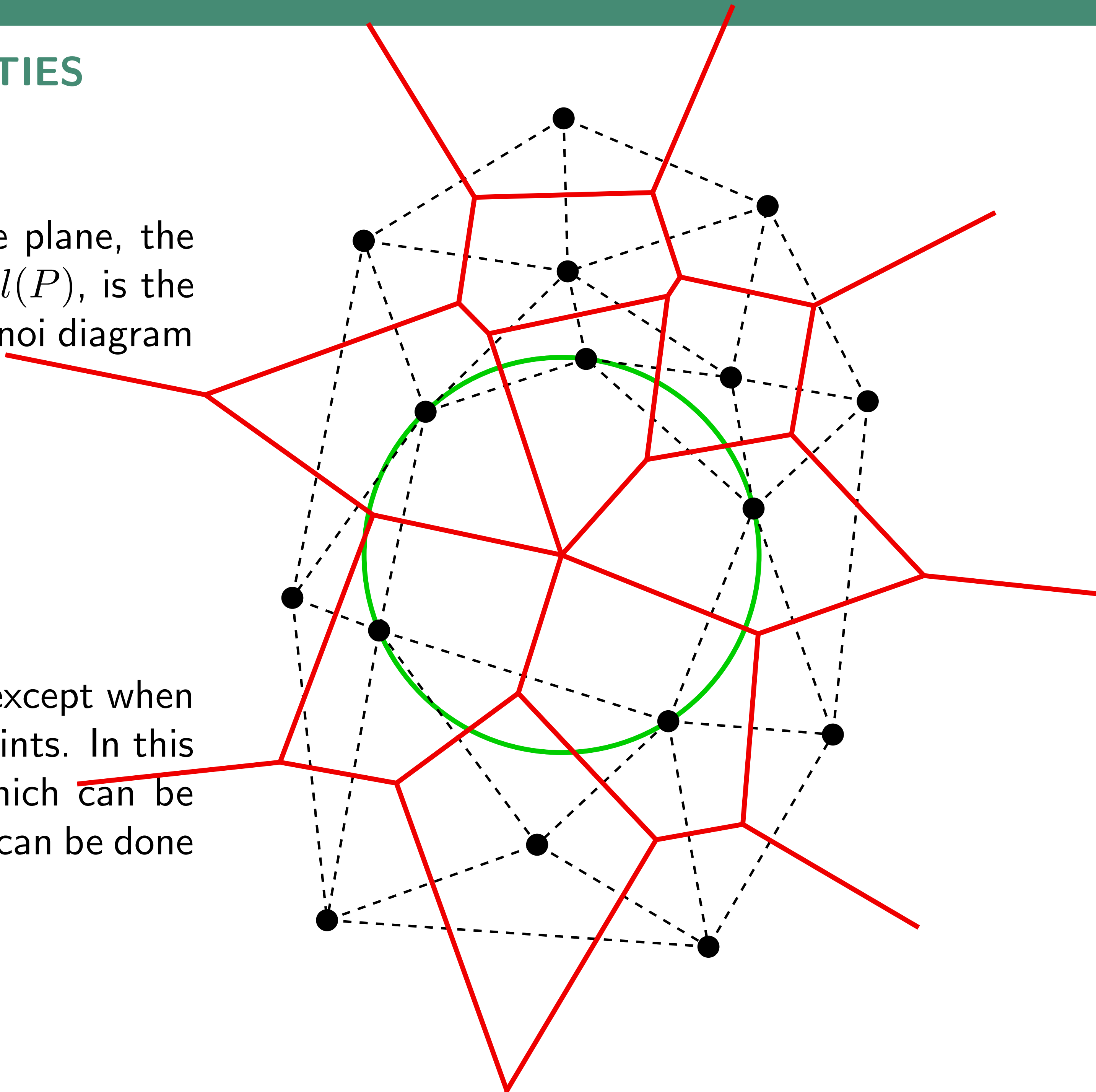

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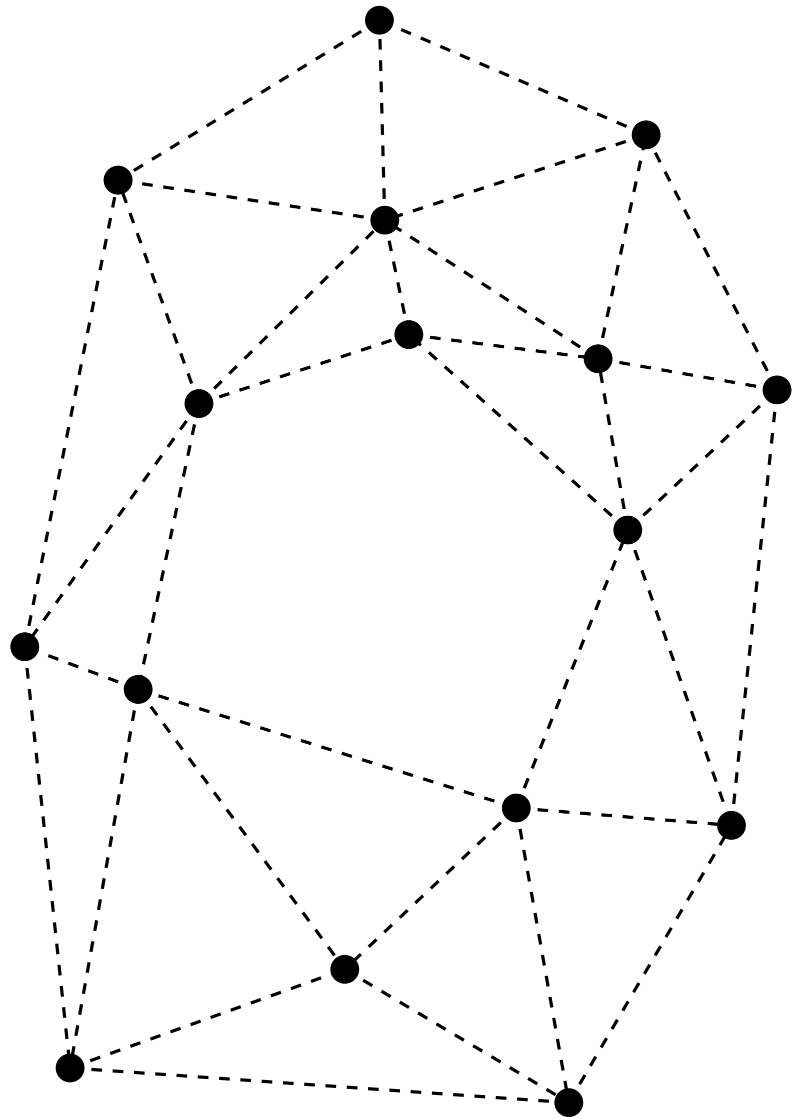

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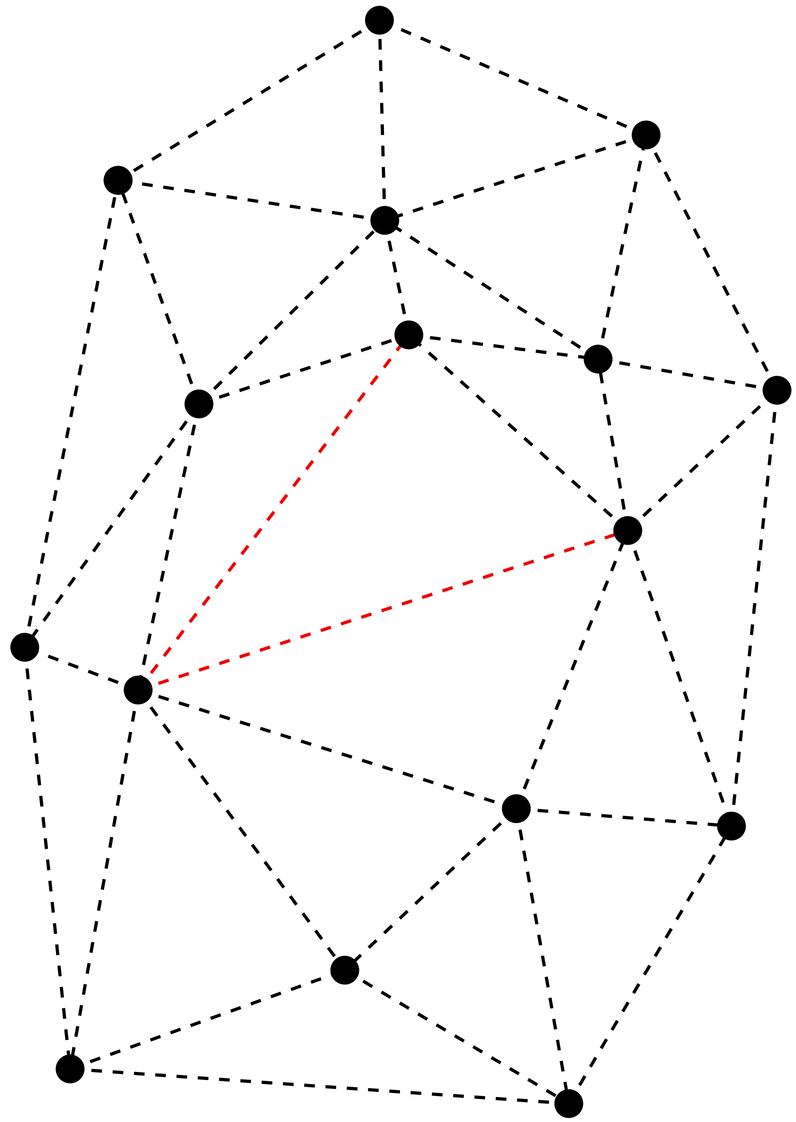

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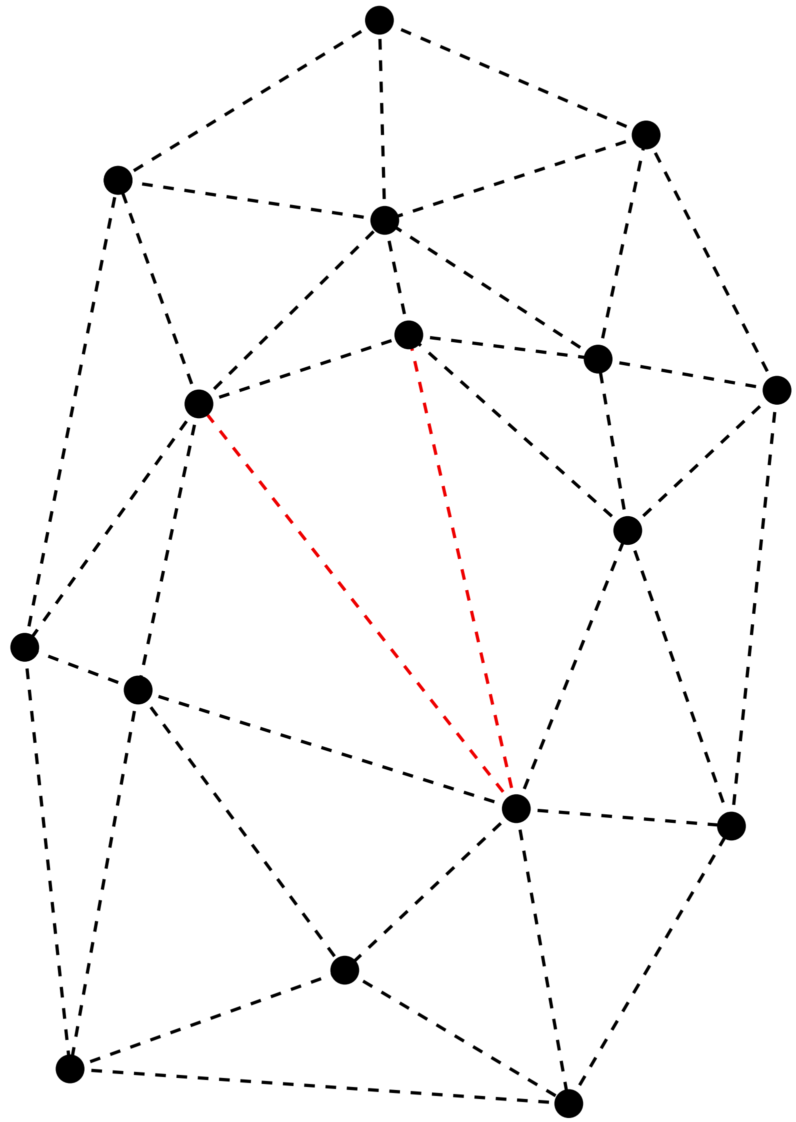

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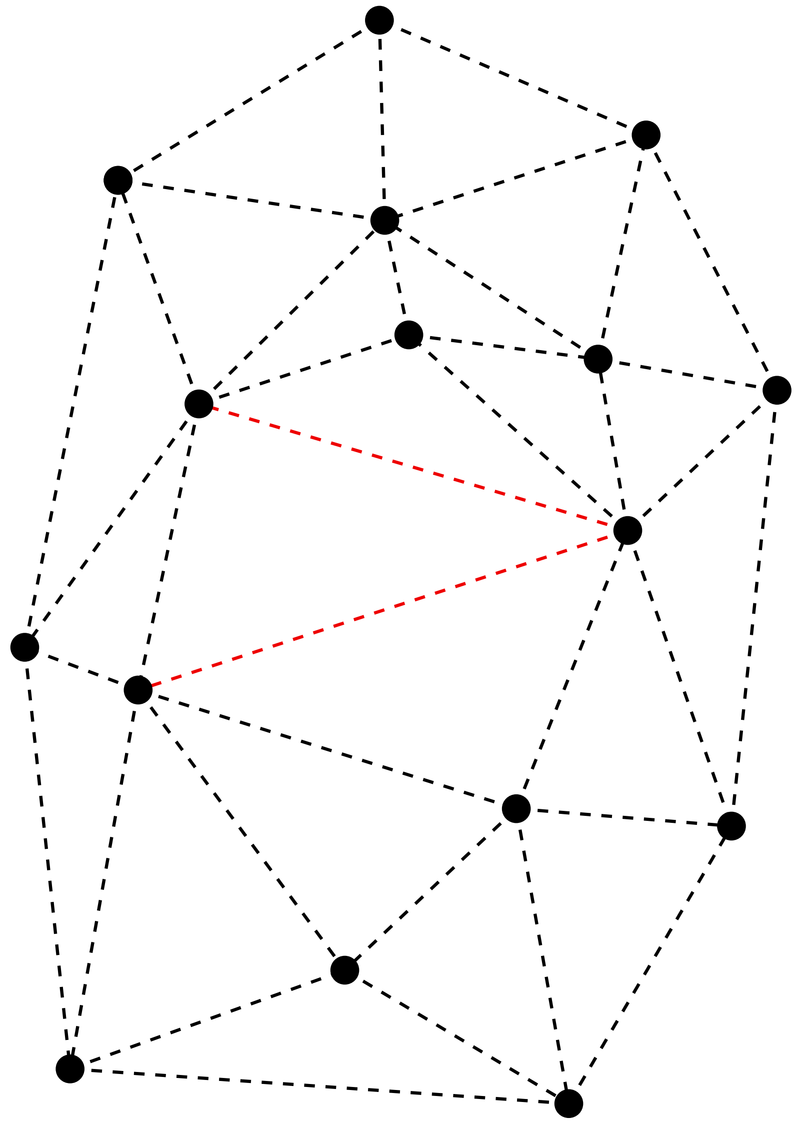

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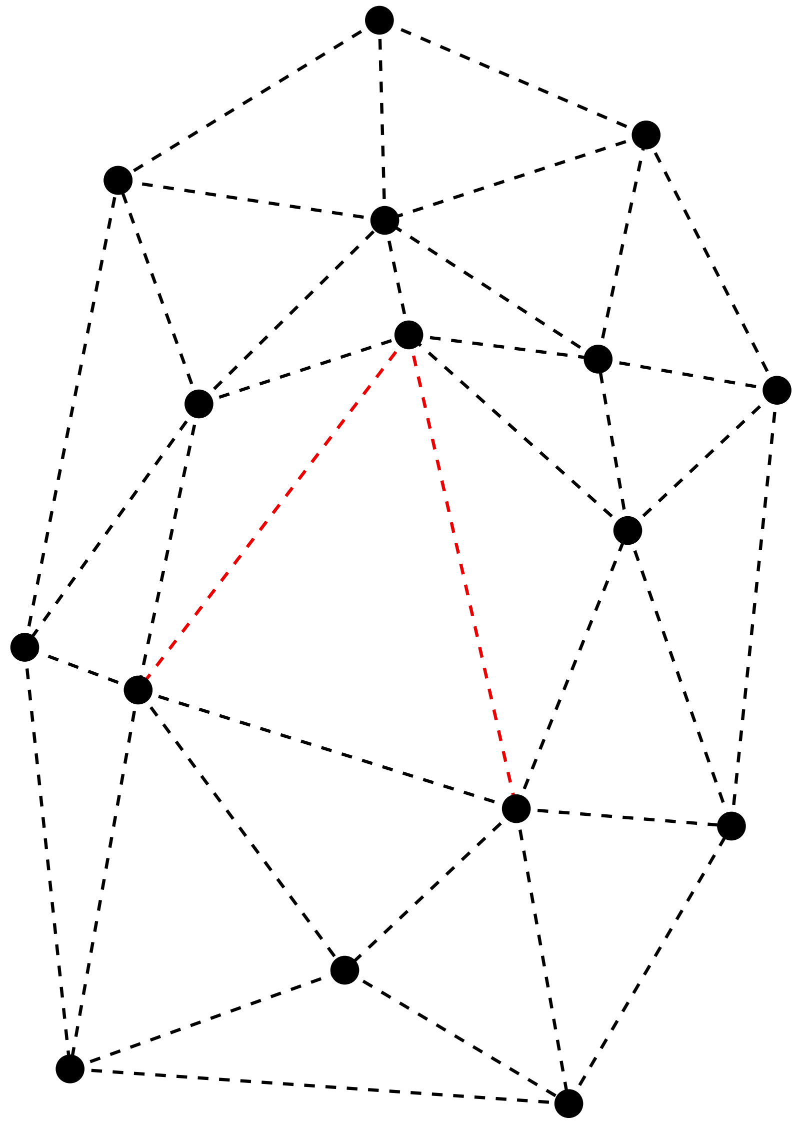

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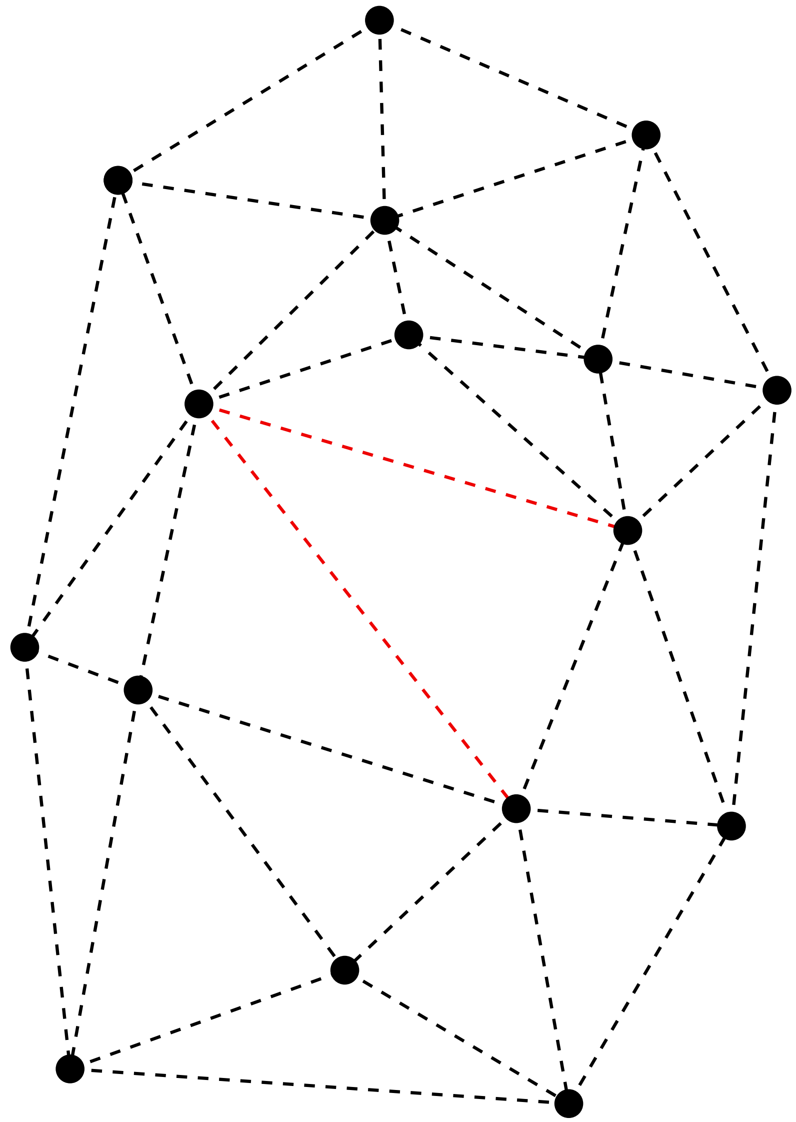

GLOBAL CHARACTERIZATION

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$T(P) = Del(P)$ iff the circumcircles of the triangles of $T(P)$ are empty of points of P.

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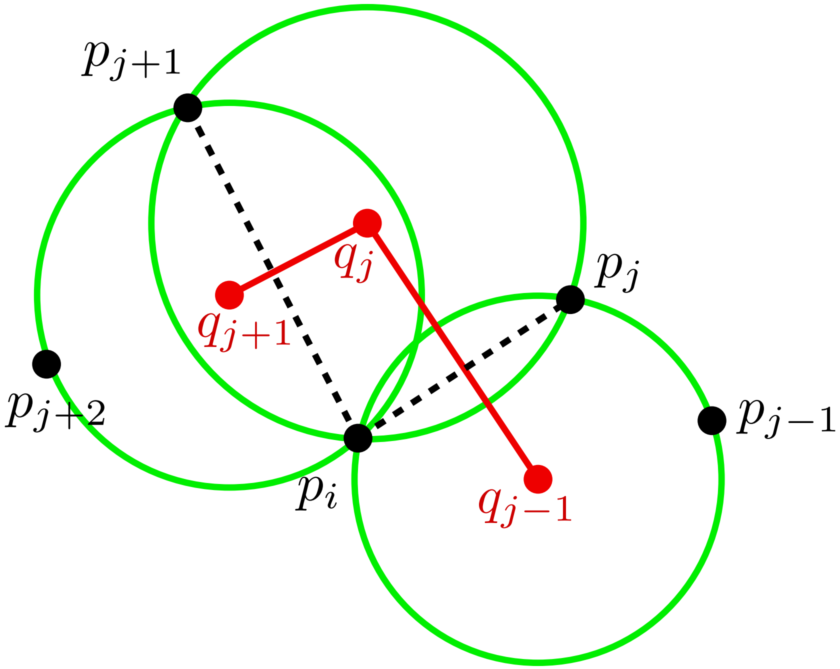
Let $p_i \in P$. Let $p_1, \ldots, p_k$ be the vertices of the triangles of $T(P)$ incident to p_i, sorted in counterclockwise order, $C_1, \ldots, C_k$ be their circumcircles, and $q_1, \ldots, q_k$ their centers (q_j denotes the center of C_j, the circumcircle of p_i, p_j, p_{j+1}). We will prove that the polygon $Q = \{q_1, \ldots, q_k\}$ coincides with $Vor(p_i)$.



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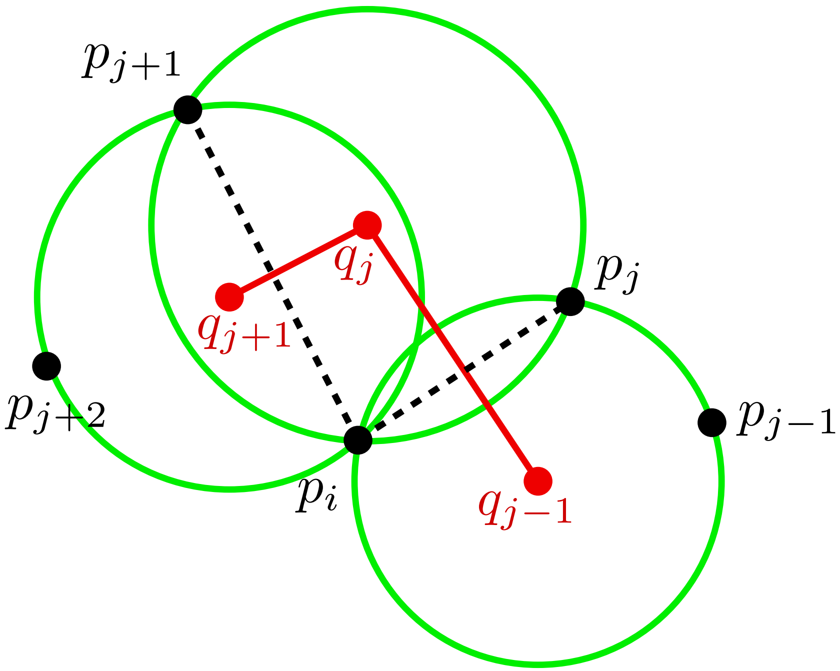
$$\overline{q_{j-1}q_j} \perp \overline{p_i p_j} \Longrightarrow Q = \bigcap_{j=1}^{k} H_{ij}$$



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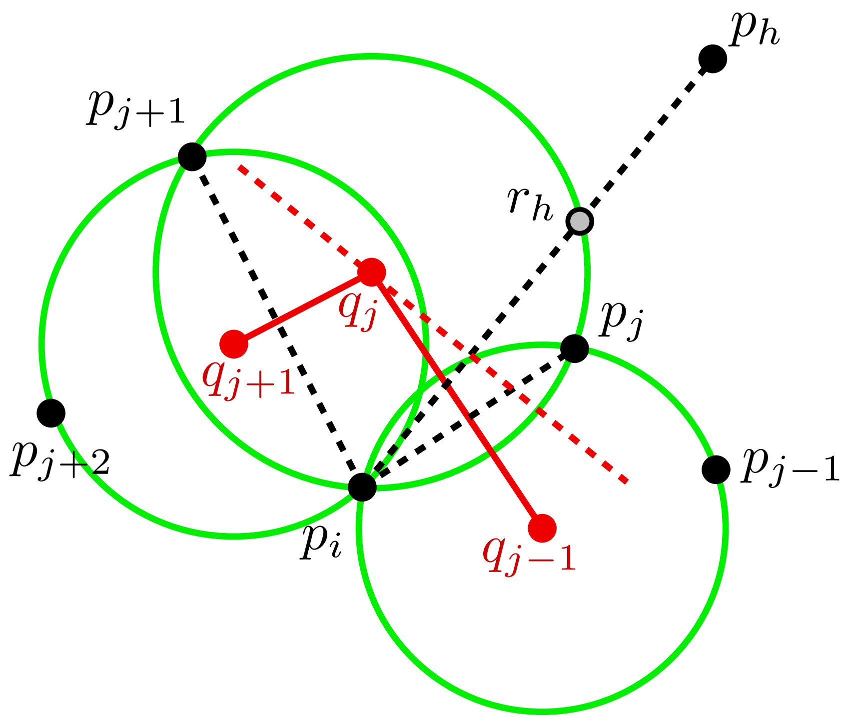

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If $h \neq 1, \ldots, k$ then $q_j \in b(p_i, r_h)$ and, therefore,

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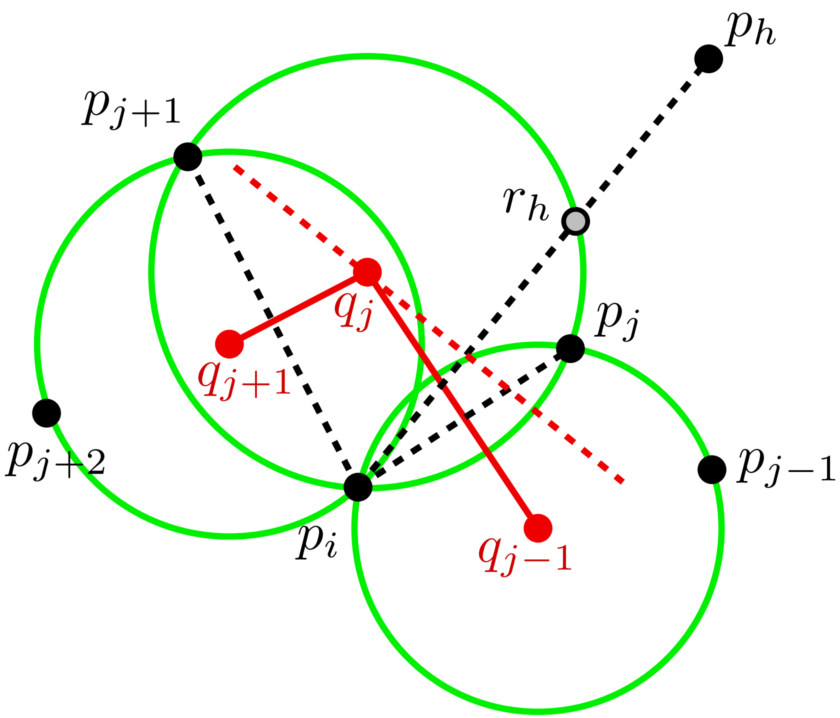

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Hence,

$$Q = \bigcap_{j=1}^{k} H_{ij} = \bigcap_{j \neq i} H_{ij} = Vor(p_i)$$

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Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.

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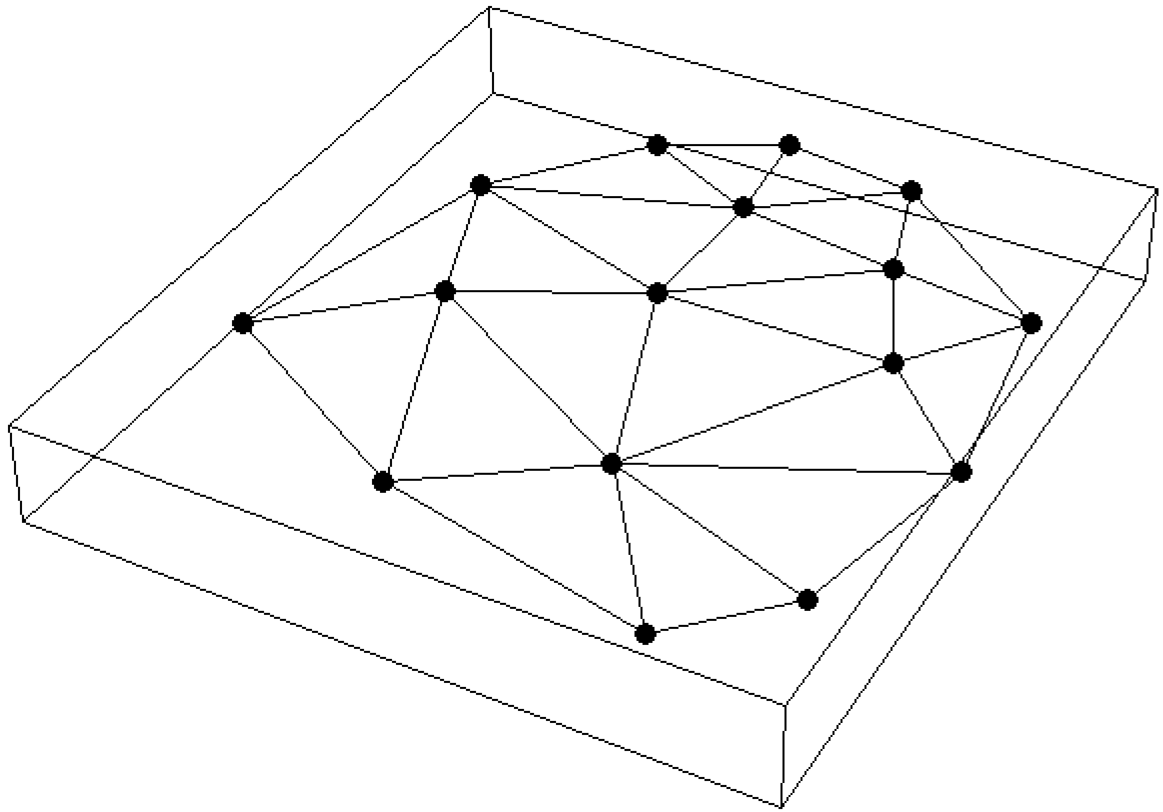

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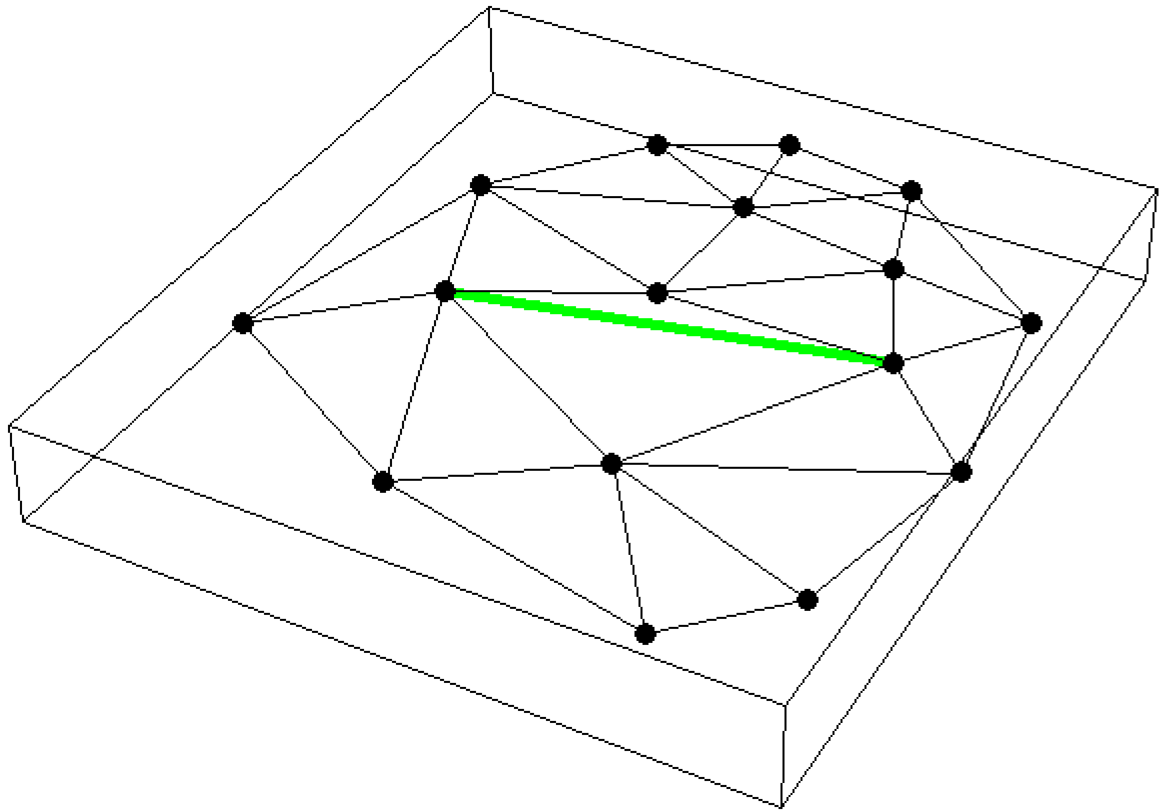

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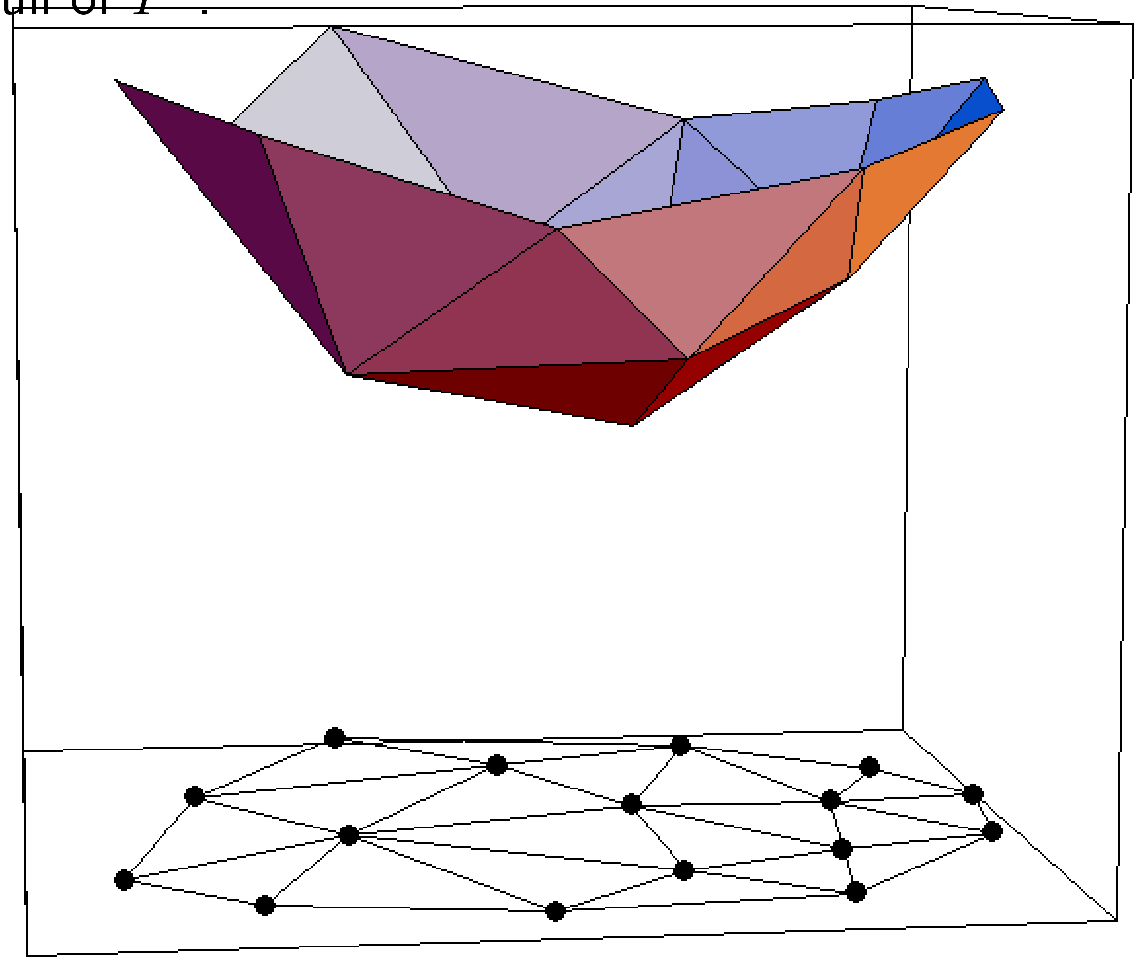

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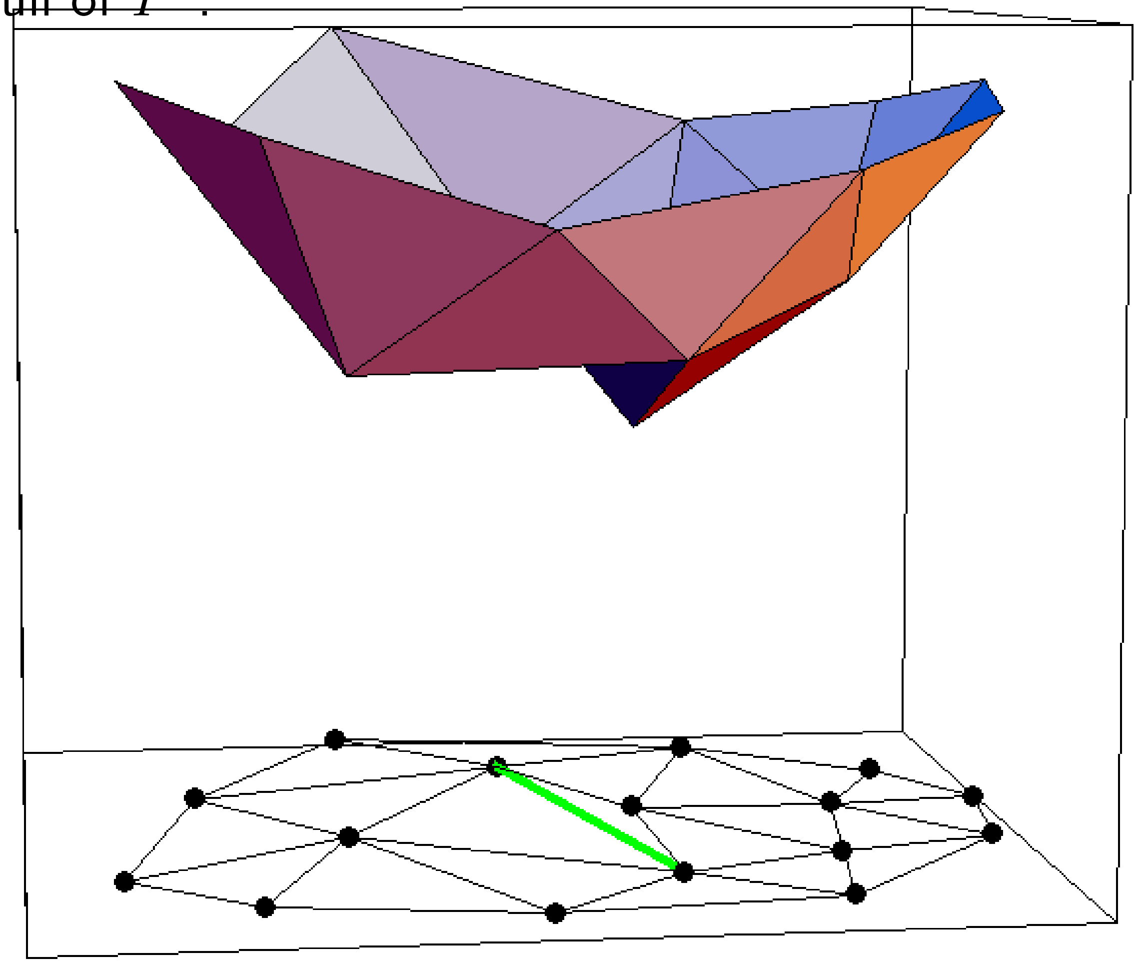

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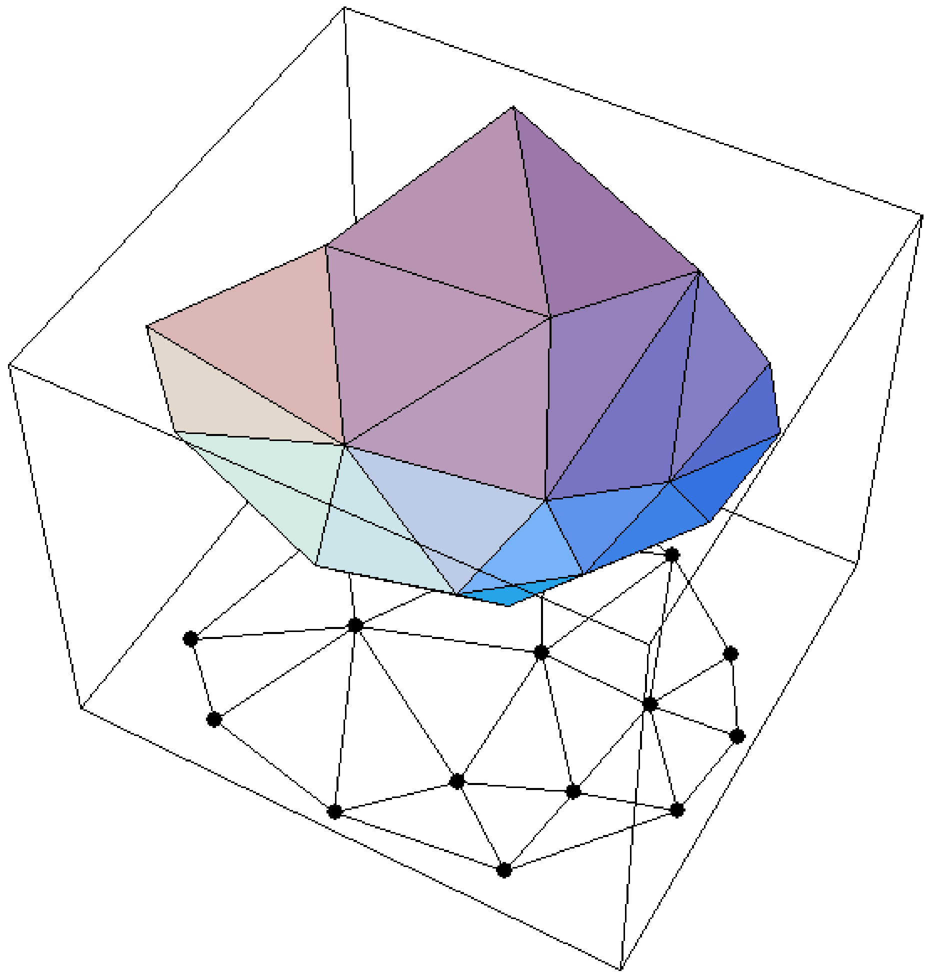

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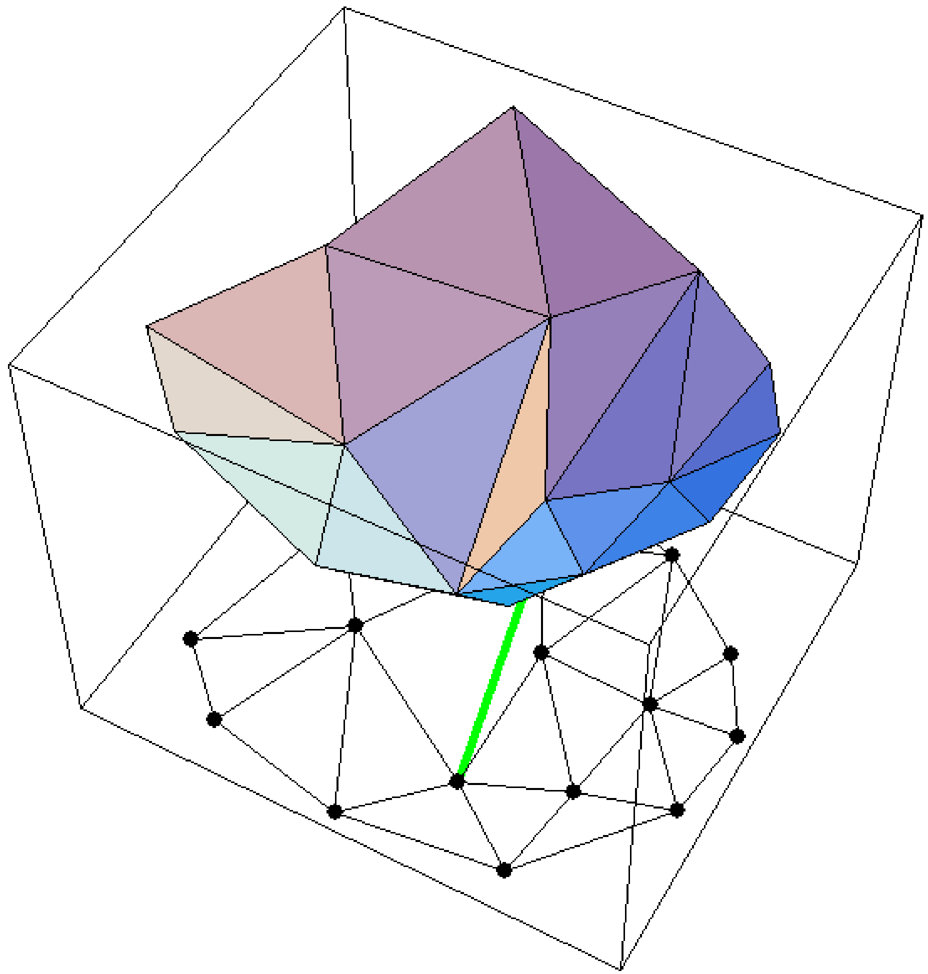
Let $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, b_i, 0)$. Let $p_i^* = (a_i, b_i, a_i^2 + b_i^2)$ be the vertical projection of each point p_i onto the paraboloid $z = x^2 + y^2$. Then $Del(P)$ is the orthogonal projection onto the plane $z = 0$ of the lower convex hull of P^*.



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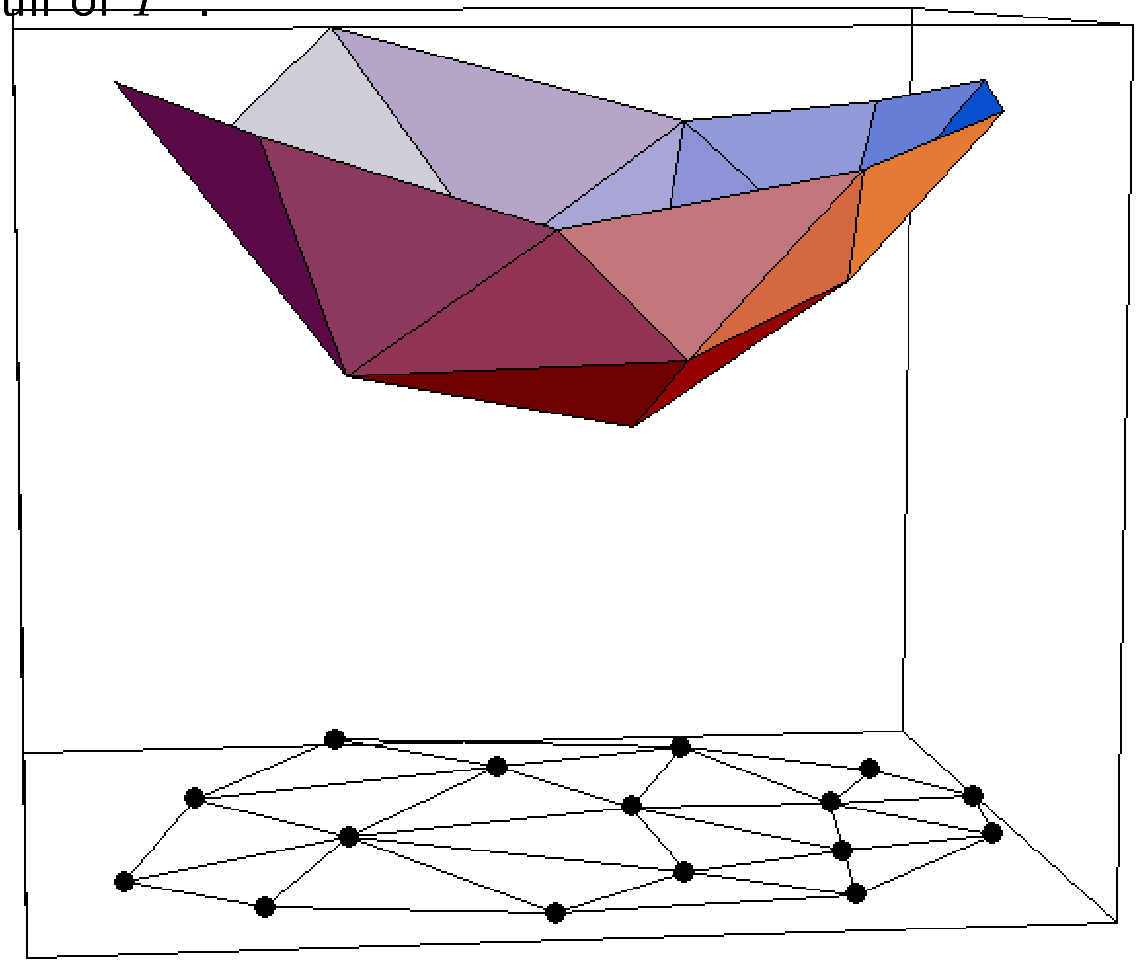

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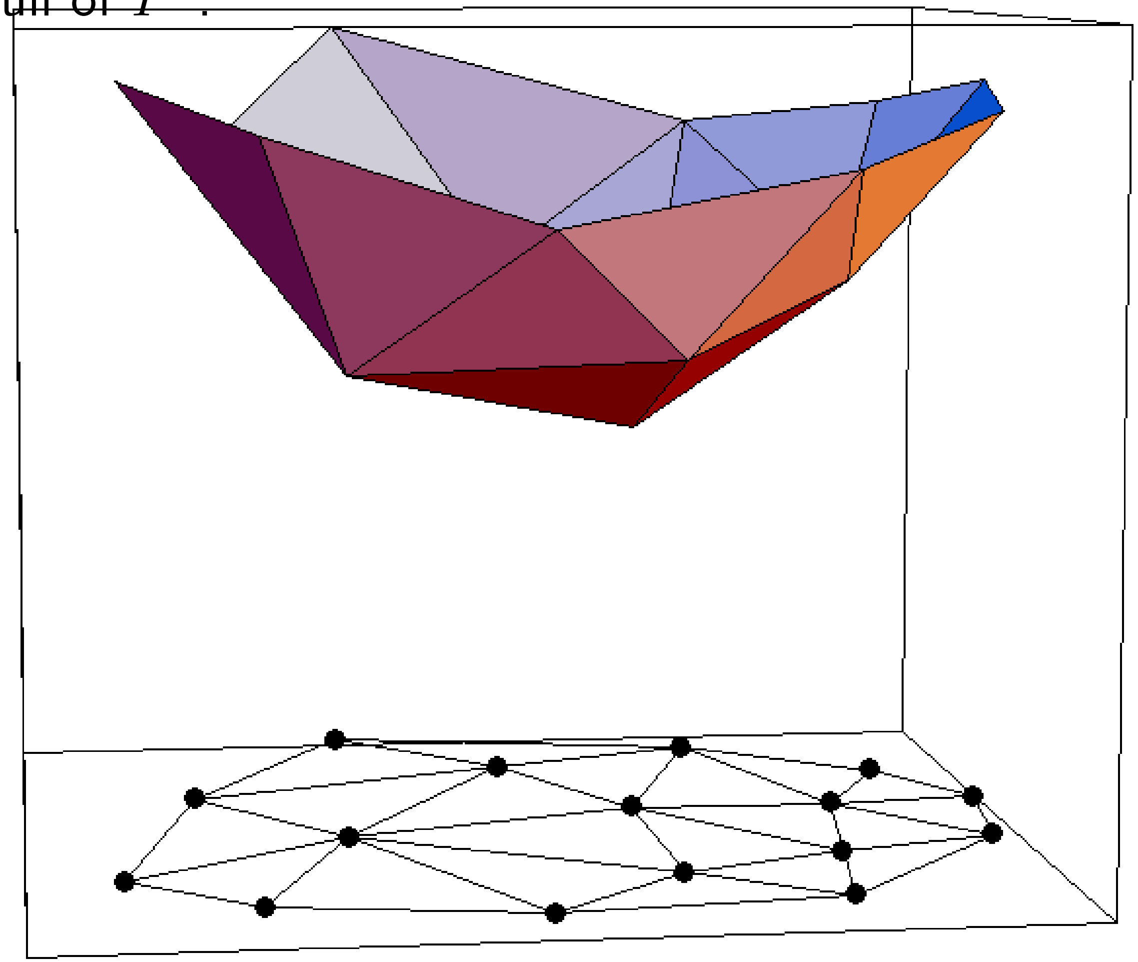

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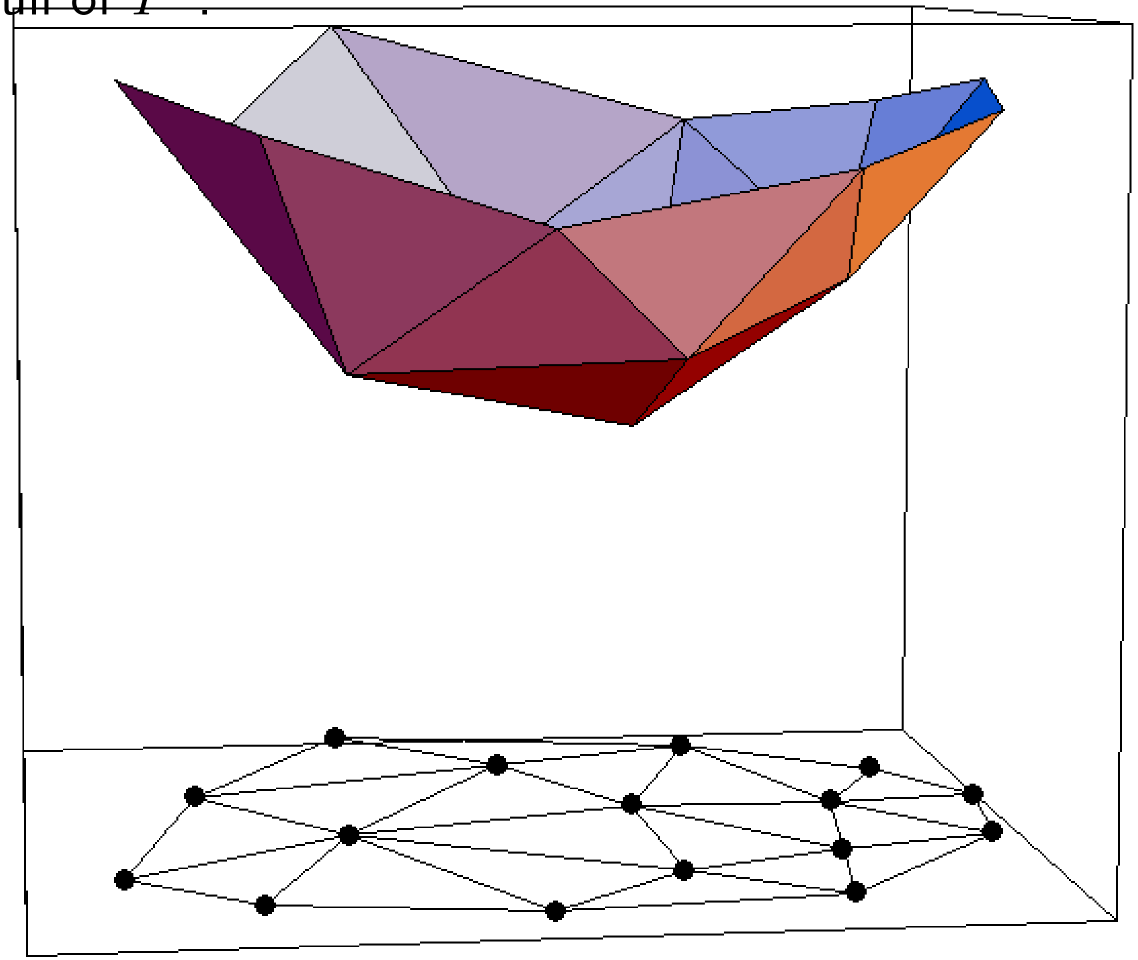
The plane through p_i^*, p_j^*, p_k^* leaves all the remaining points of P^* above it



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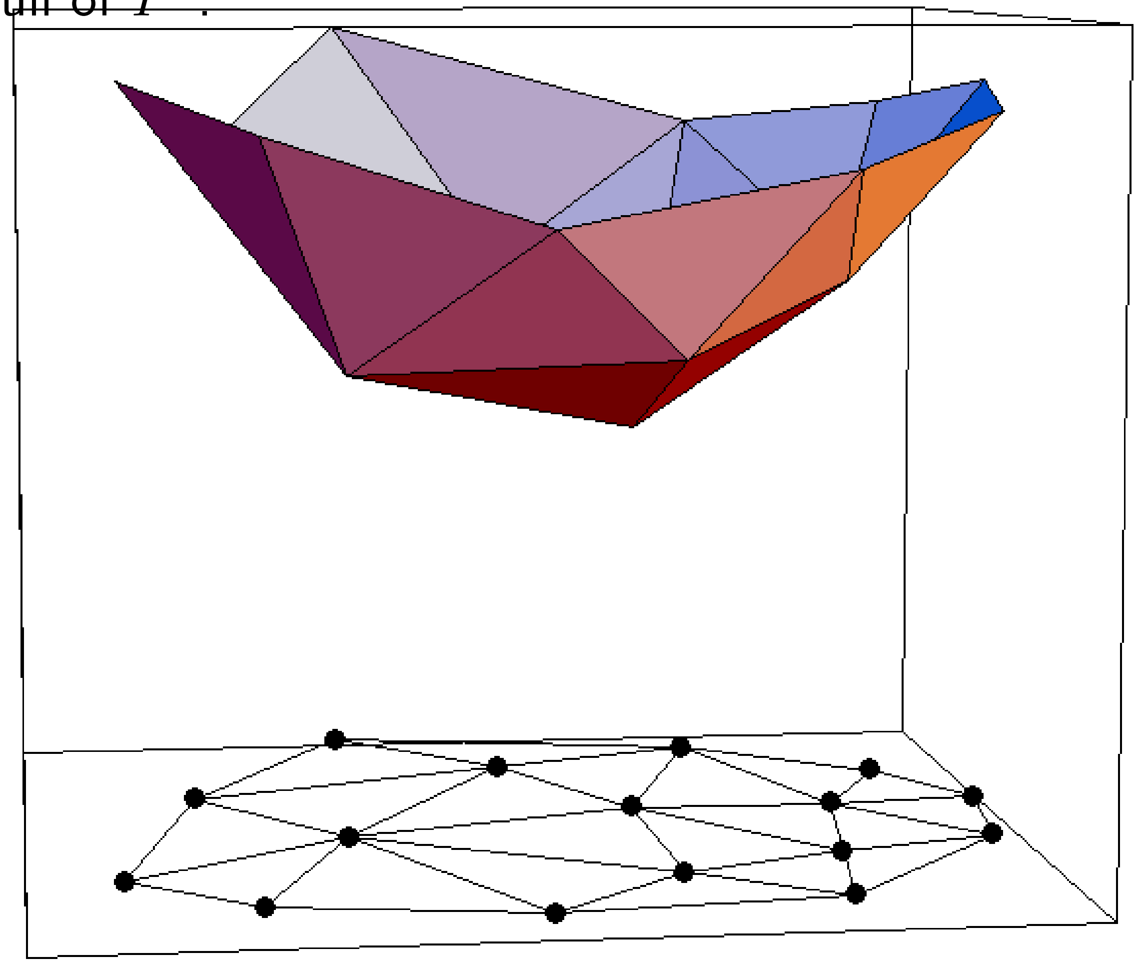
The circle through p_i, p_j, p_k leaves all the remaining points of P in its exterior



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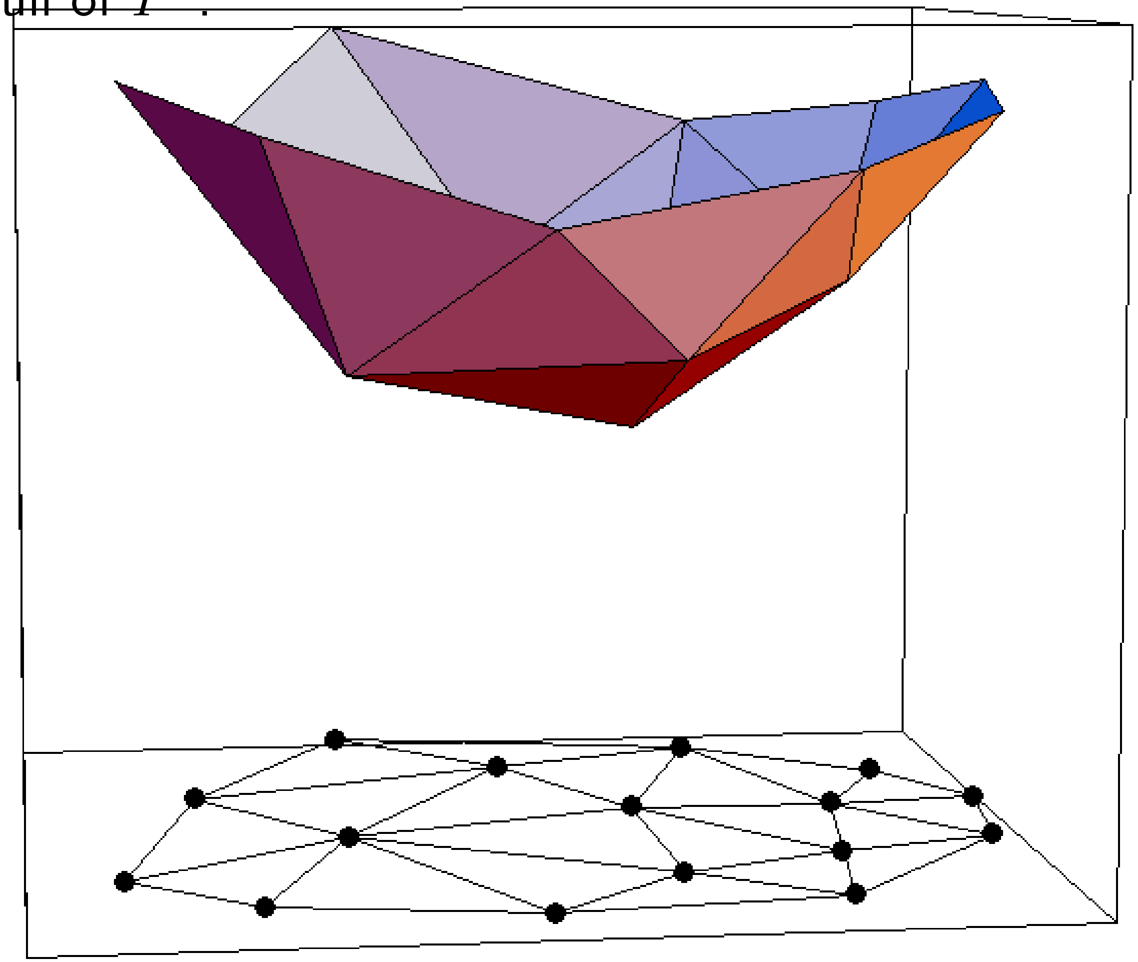
p_i, p_j, p_k form a triangle of $Del(P)$



DELAUNAY TRIANGULATION

LOCAL CHARACTERIZATION

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Definition

A triangulation $T(P)$ is locally Delaunay if each pair of triangles $p_i p_j p_k$ and $p_i p_j p_l$ sharing an edge $p_i p_j$ satisfies $p_l \notin C_{ijk}$ and $p_k \notin C_{ijl}$.

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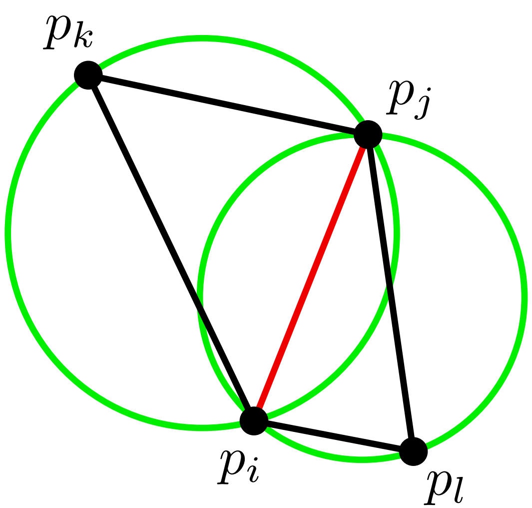
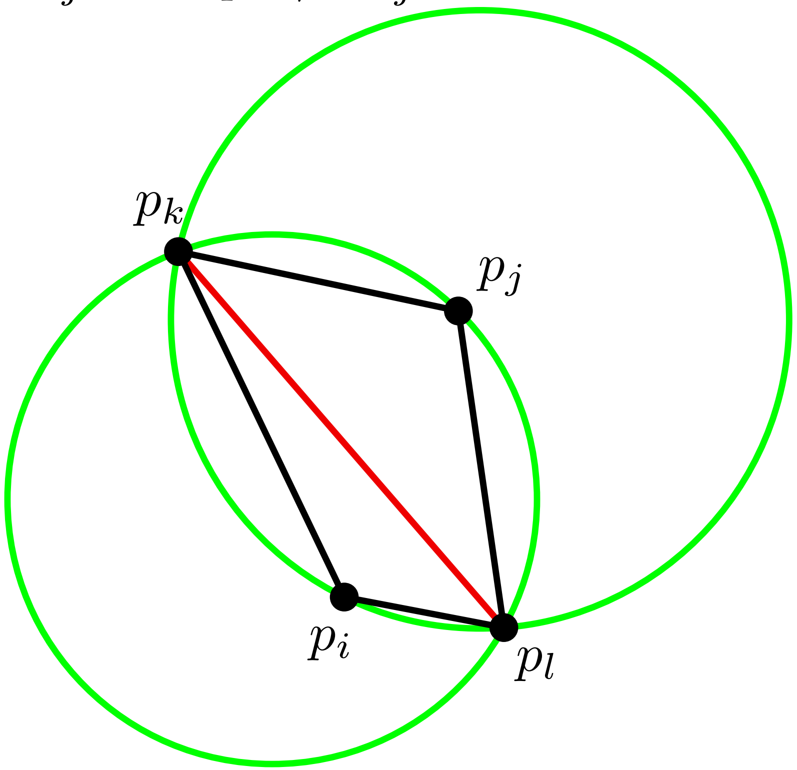
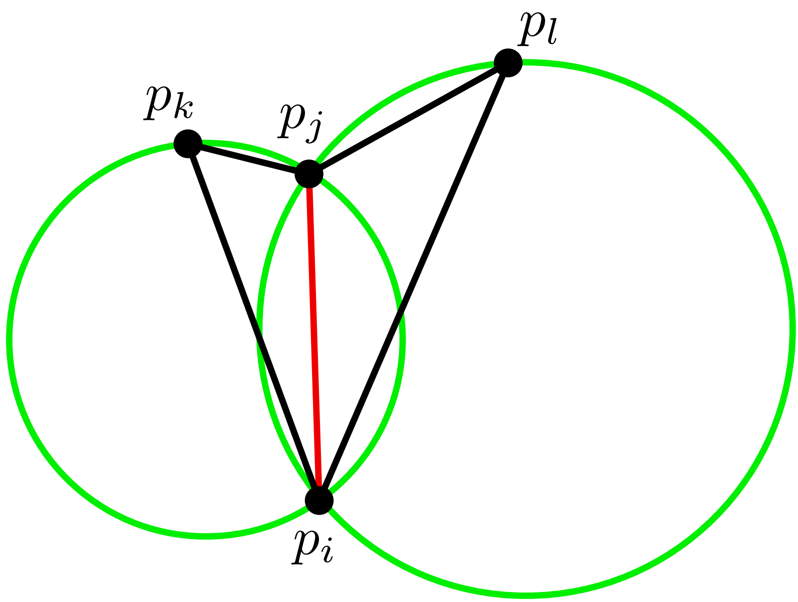


The edge p_ip_j is locally Delaunay



The edge p_kp_l is not locally Delaunay



The edge p_ip_j is locally Delaunay

In fact, the quadrilateral $p_ip_lp_jp_k$ is not convex

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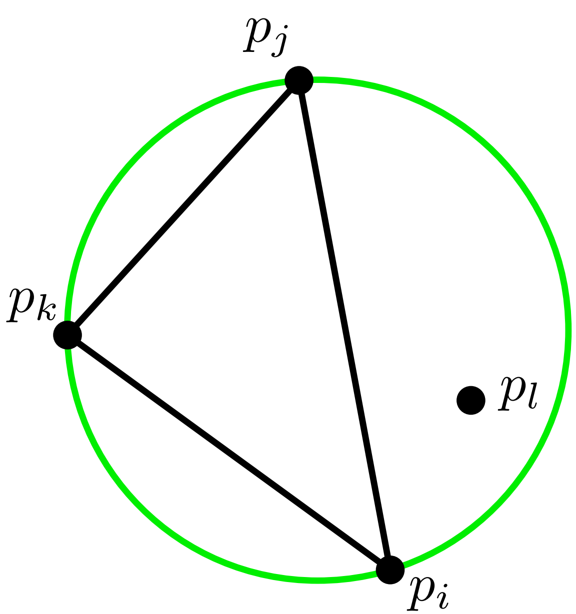

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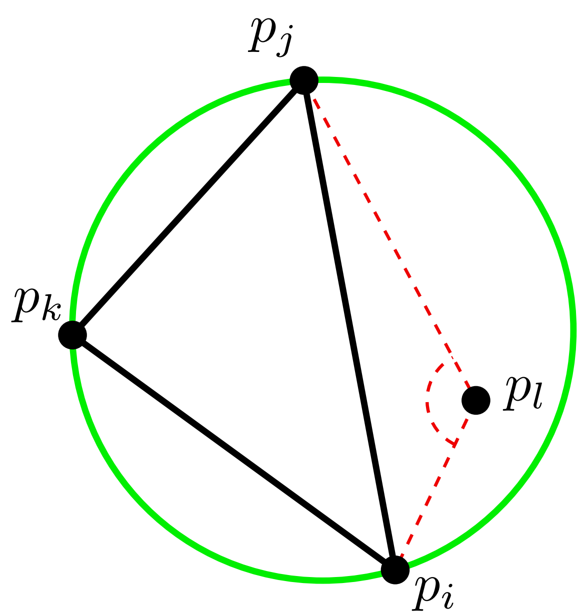

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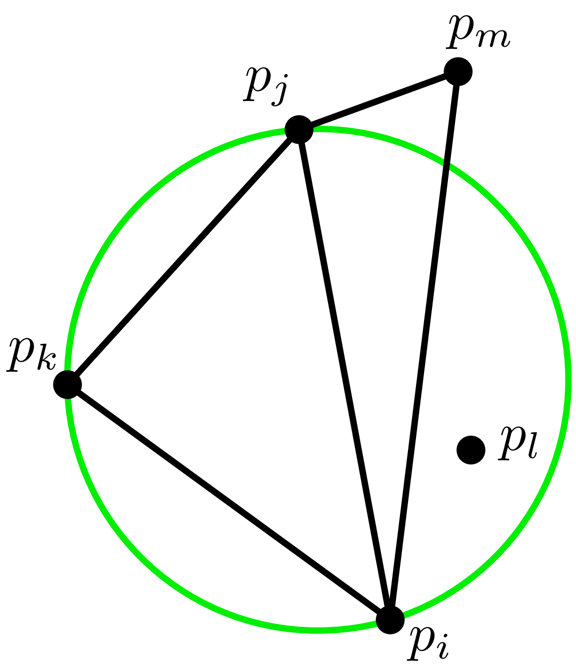

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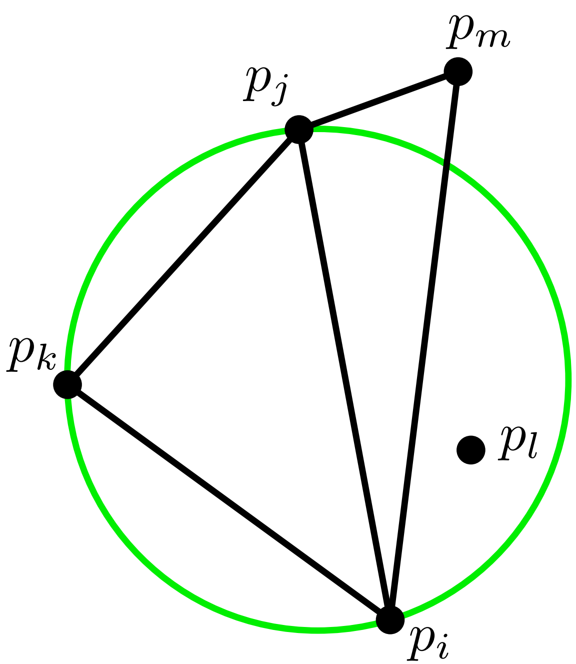

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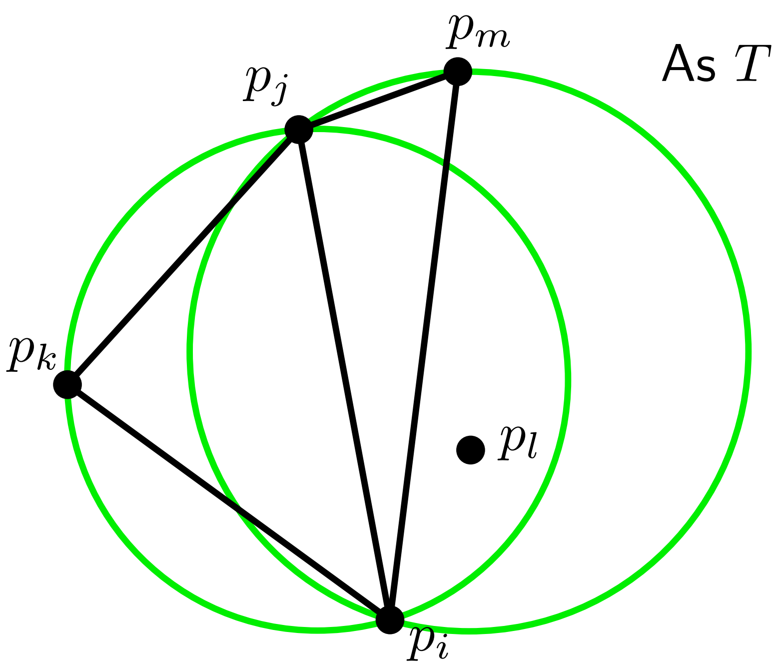

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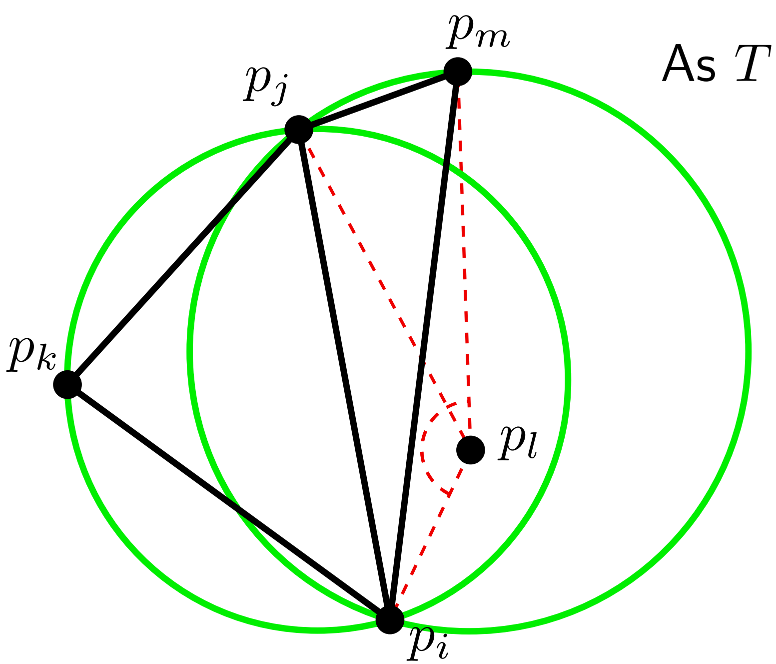

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Then $p_l \in C_{ijm}$.

Hence, one of the angles $p_i p_l p_m$ or $p_j p_l p_m$ would be greater than $p_i p_l p_j$.

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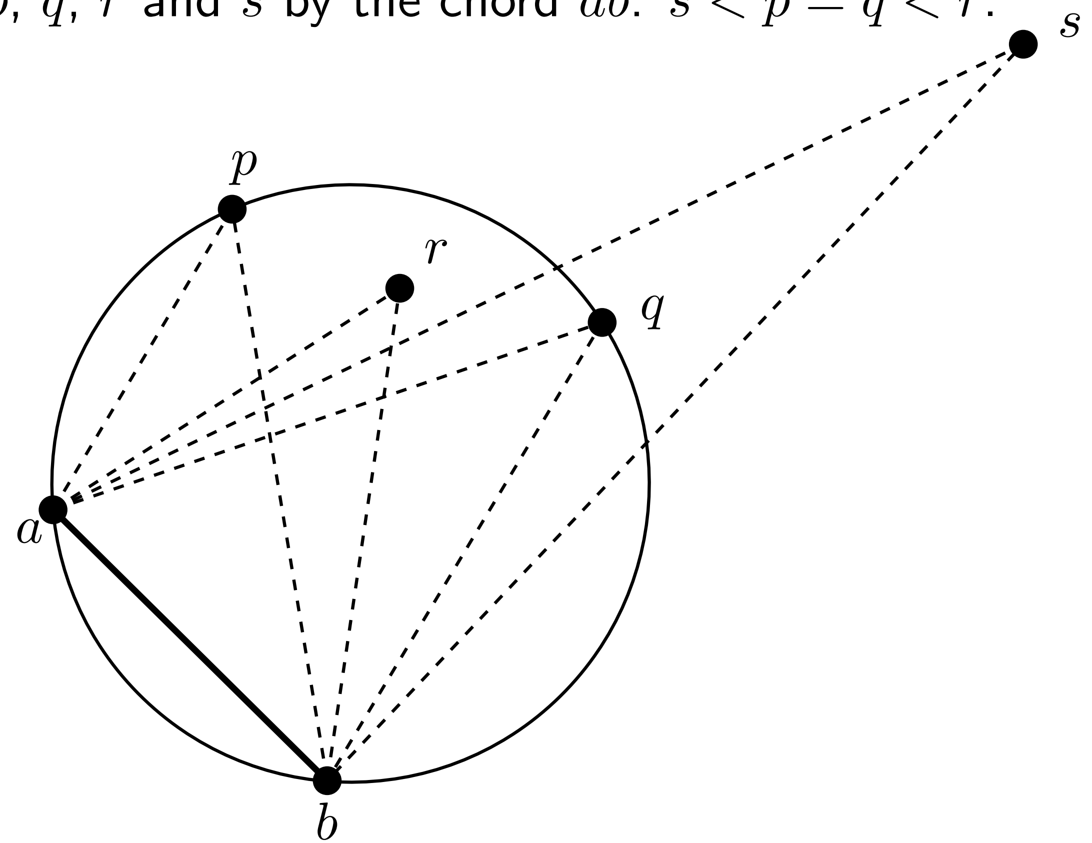

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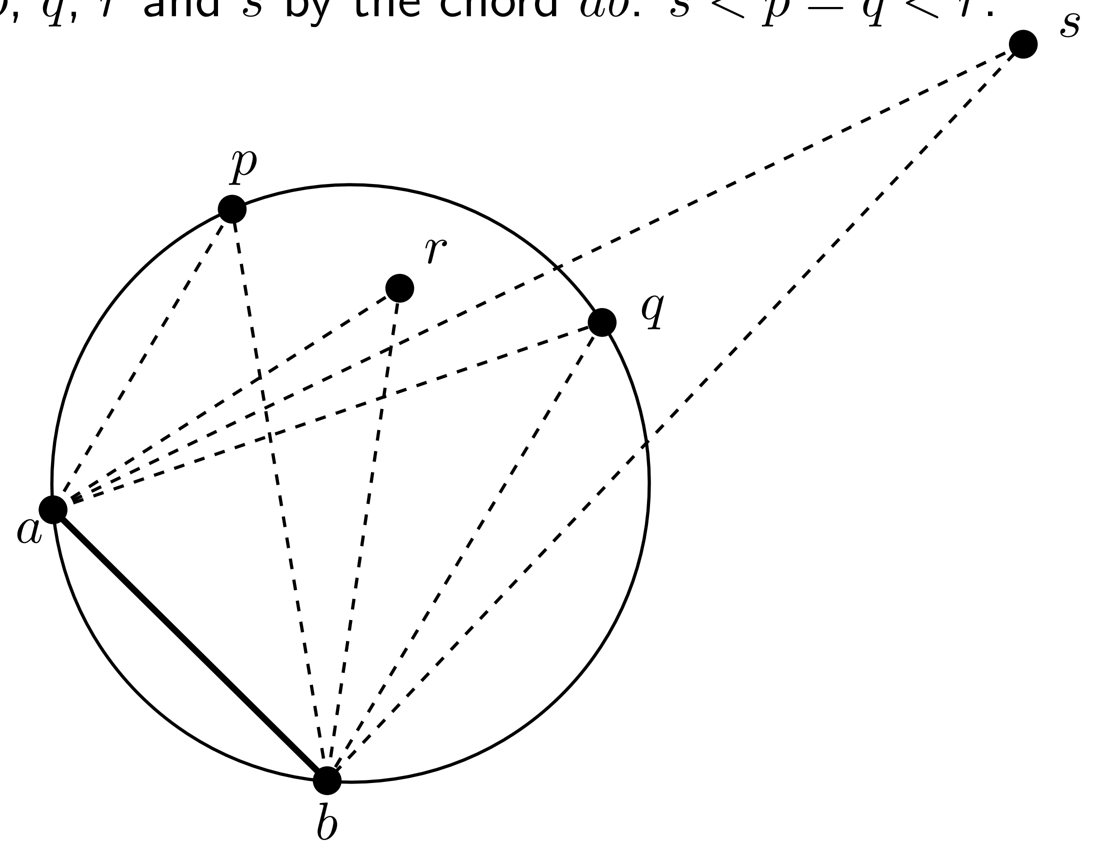

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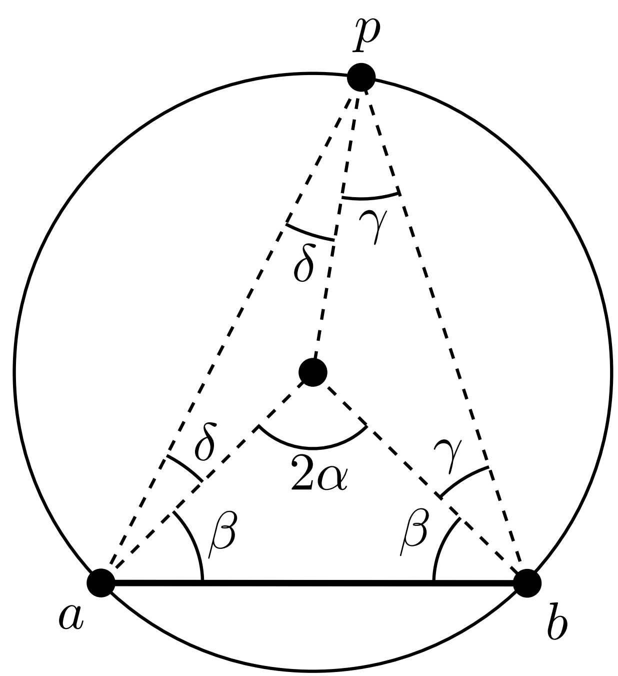
$$\left.\begin{aligned} 2\delta + 2\gamma + 2\beta = \pi \\ 2\alpha + 2\beta = \pi \end{aligned}\right\} \Rightarrow 2\alpha = 2\gamma + 2\delta \Rightarrow \alpha = \gamma + \delta \Rightarrow \widehat{p} = \widehat{q} = \alpha$$



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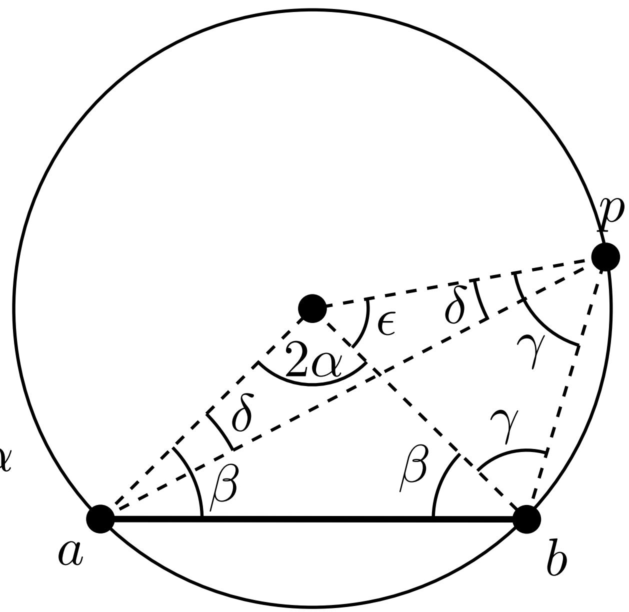
Second case:

$$\left.\begin{array}{r} 2\alpha + \epsilon + 2\delta = \pi \\ 2\gamma + \epsilon = \pi \end{array}\right\} \Rightarrow 2\alpha + 2\delta - 2\gamma = 0 \Rightarrow \alpha = \gamma - \delta \Rightarrow \widehat{p} = \widehat{q} = \alpha$$



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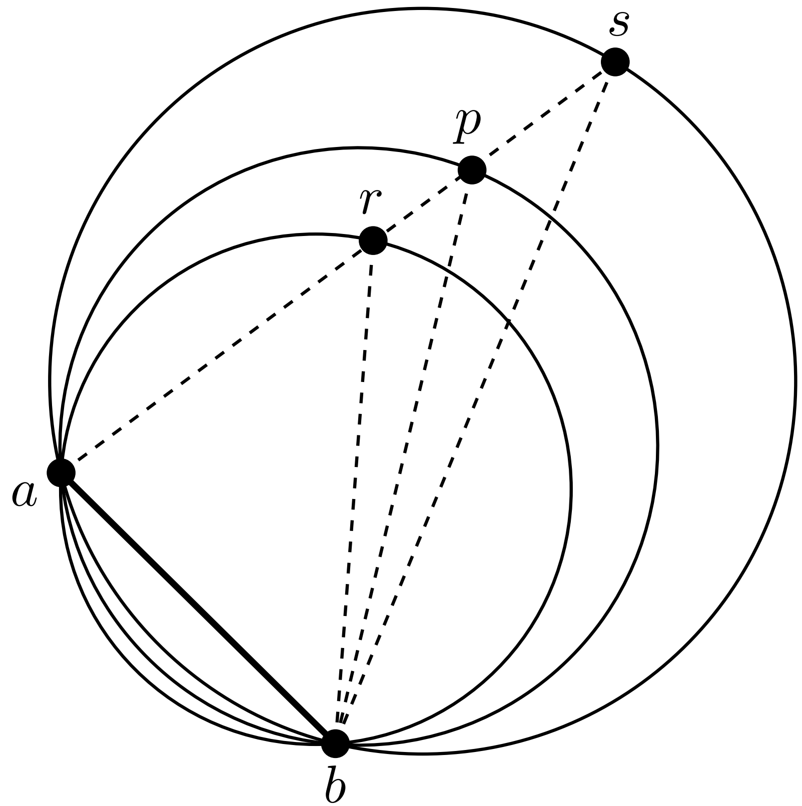

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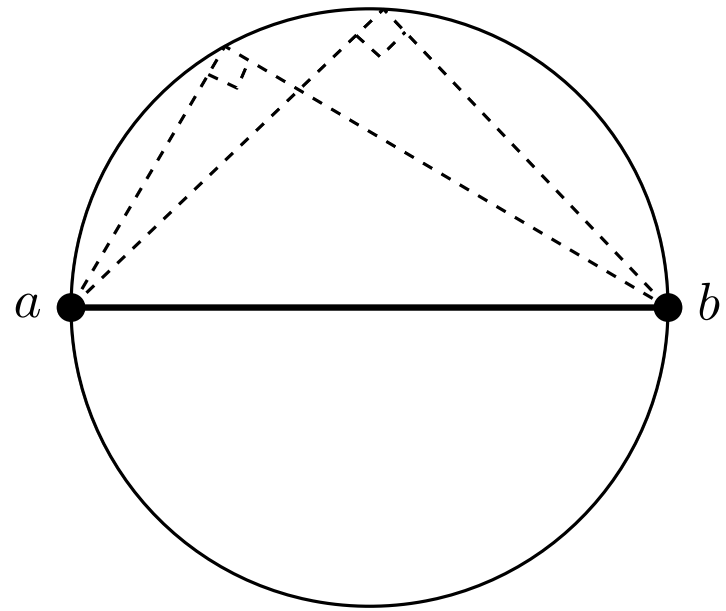

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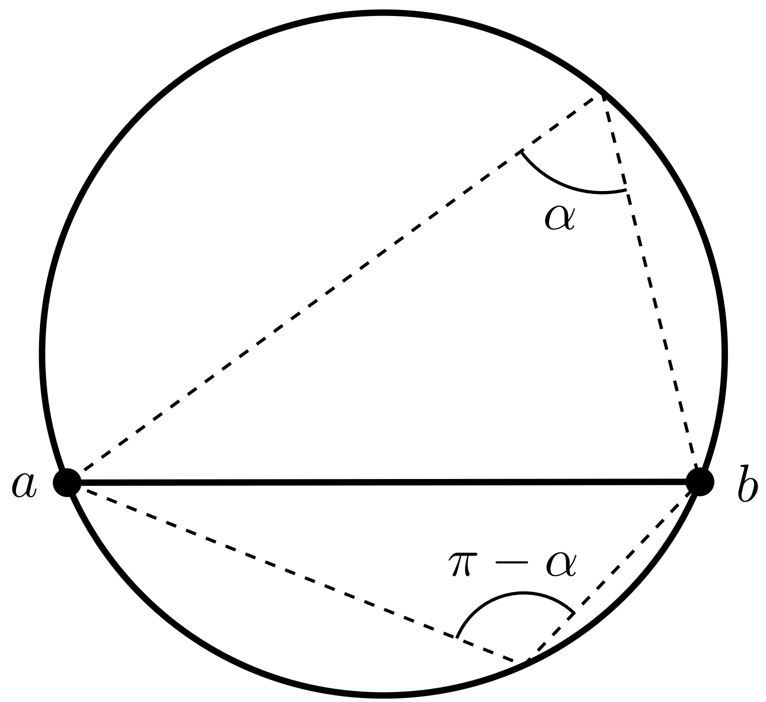

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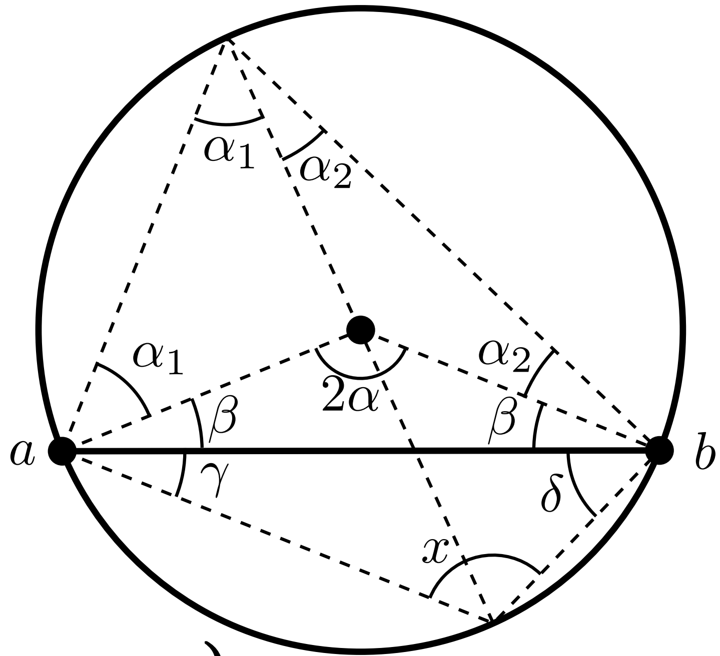

$$\left.\begin{array}{r} \left.\begin{array}{r} \alpha_1 + \beta + \gamma = \frac{\pi}{2} \\ \alpha_2 + \beta + \delta = \frac{\pi}{2} \end{array}\right\} \Rightarrow \alpha + 2\beta + \gamma + \delta = \pi \\ x + \gamma + \delta = \pi \end{array}\right\} \Rightarrow x = \alpha + 2\beta$$

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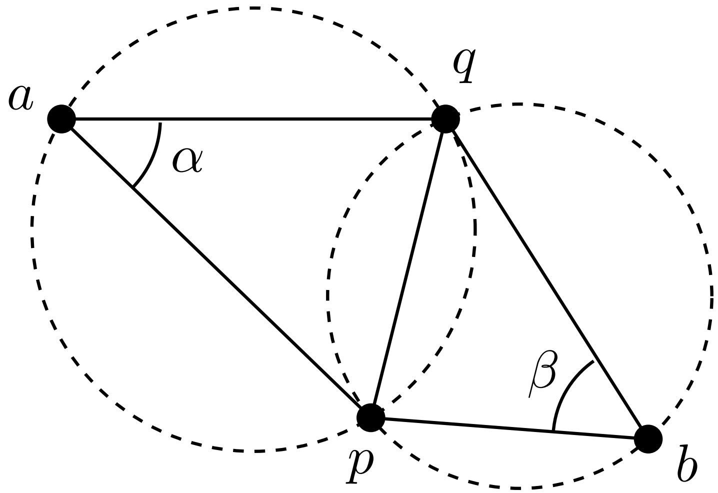

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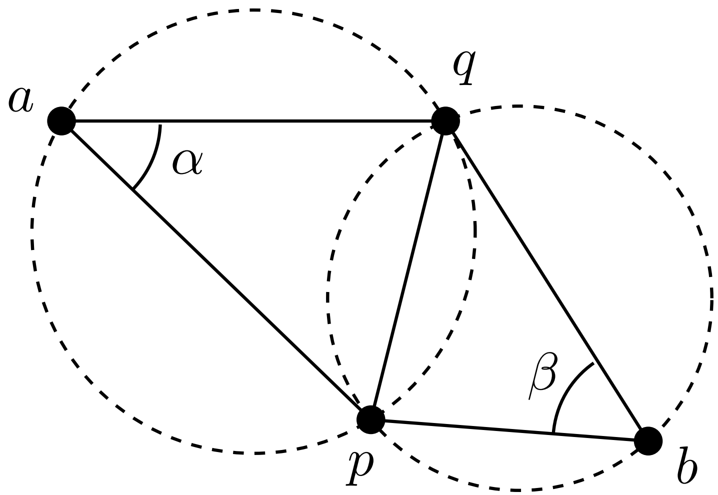

$$a \in ext(C_{pqb}) \Longleftrightarrow \alpha < \pi - \beta \Longleftrightarrow \beta < \pi - \alpha \Longleftrightarrow b \in ext(C_{pqa})$$

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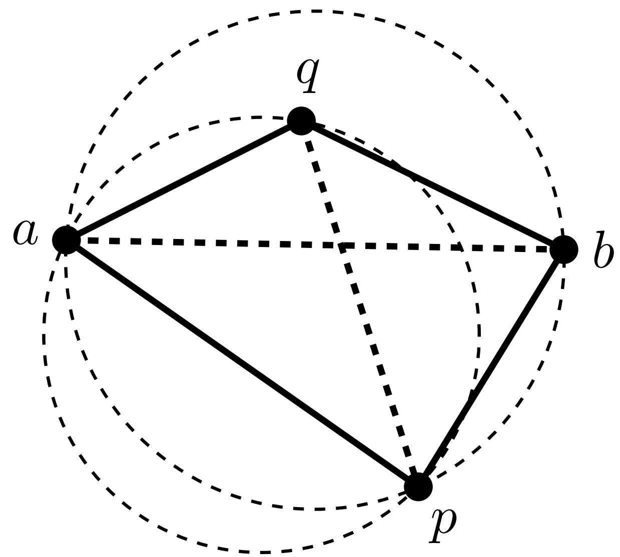

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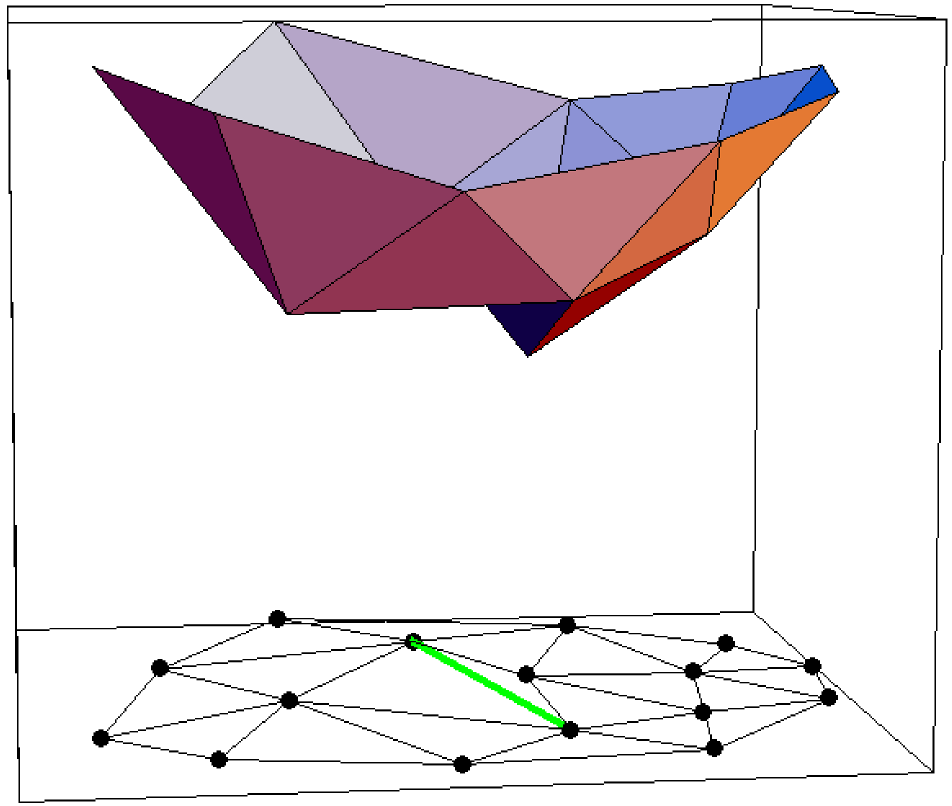

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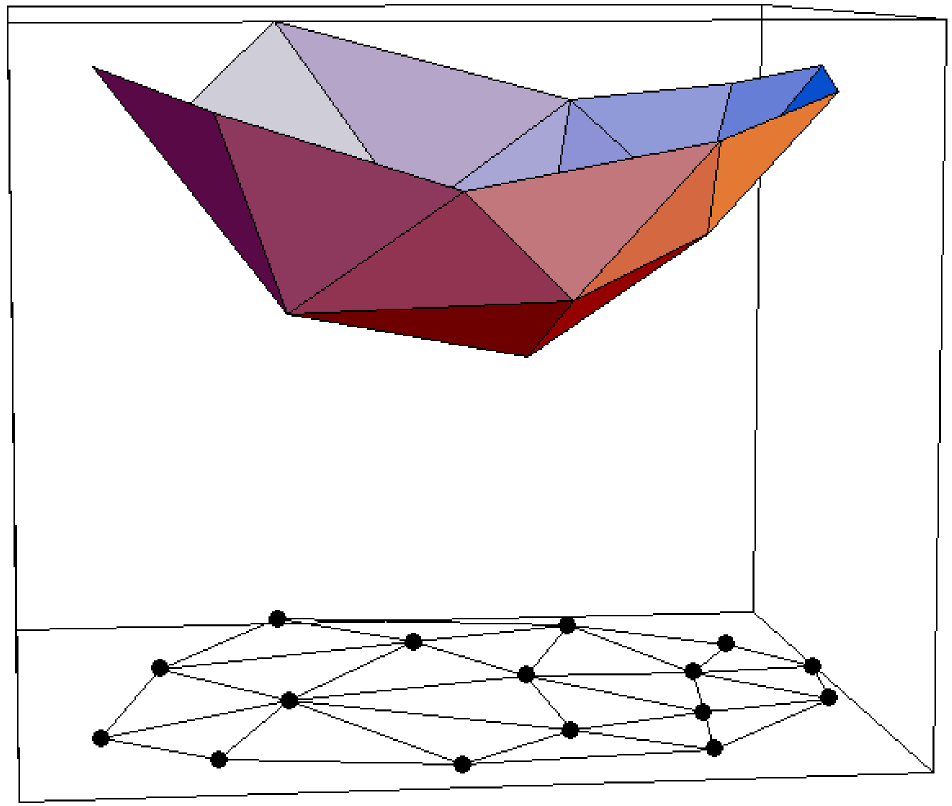

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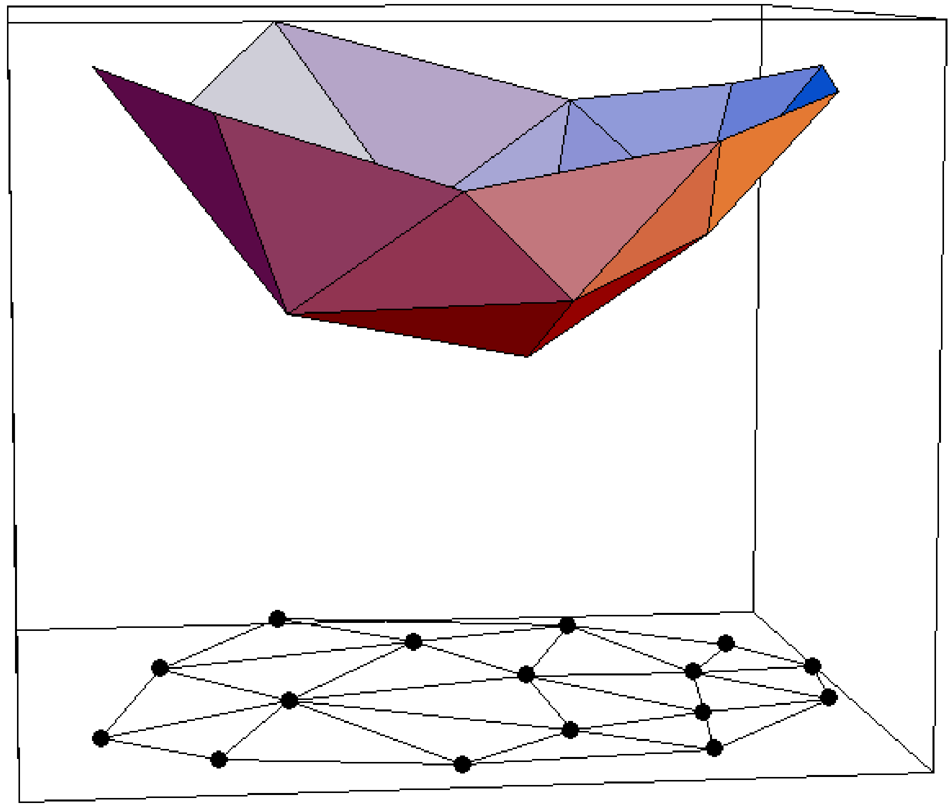
Once flipped, the quatrilateral is locally Delaunay: the fourth point lies in the exterior of the circumcircle of the triangle.

In the paraboloid, this means that the fourth point lies above the triangular face of the polyhedrization.



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Corollary. Given any triangulation of P, performing locally Delaunay flips is a procedure converging to $Del(P)$.

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4. Incremental algorithm

Compute an enclosing triangle for $\{p_1, \ldots, p_n\}$

Compute $Del(p_1, \ldots, p_{i+1})$ from $Del(p_1, \ldots, p_i)$

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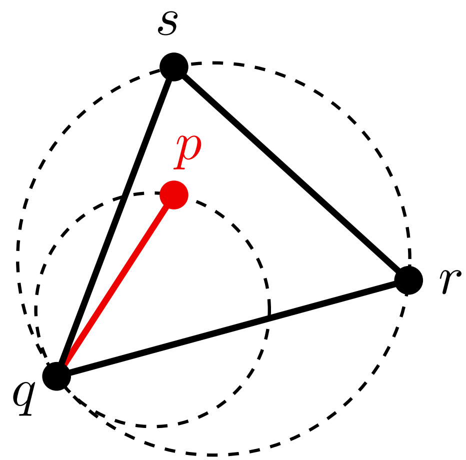
As C_{qrs} is empty, there exist empty circles C_{pq}, such as the circle through p and q tangent to C_{qrs} in q. Similarly for r and s.



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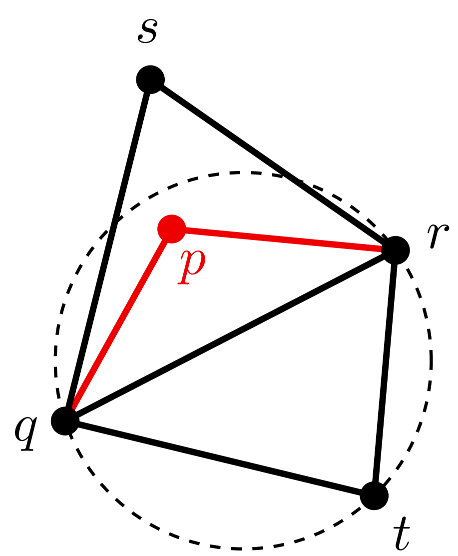
Since p may lie in the interior of C_{qrt}.



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> Obvious, because the property is local: it affects only quadrilaterals formed by two triangles sharing an edge.

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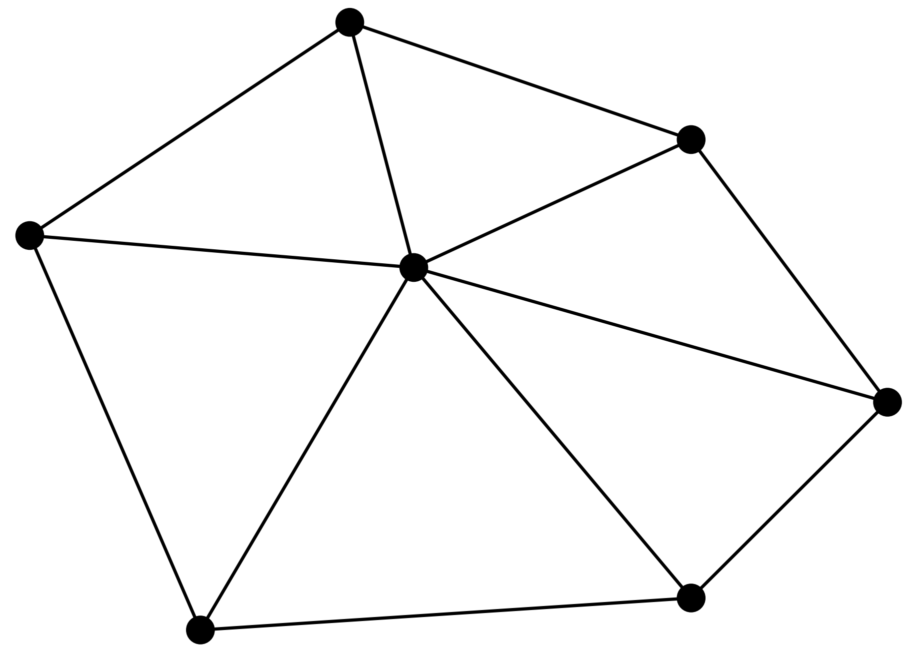

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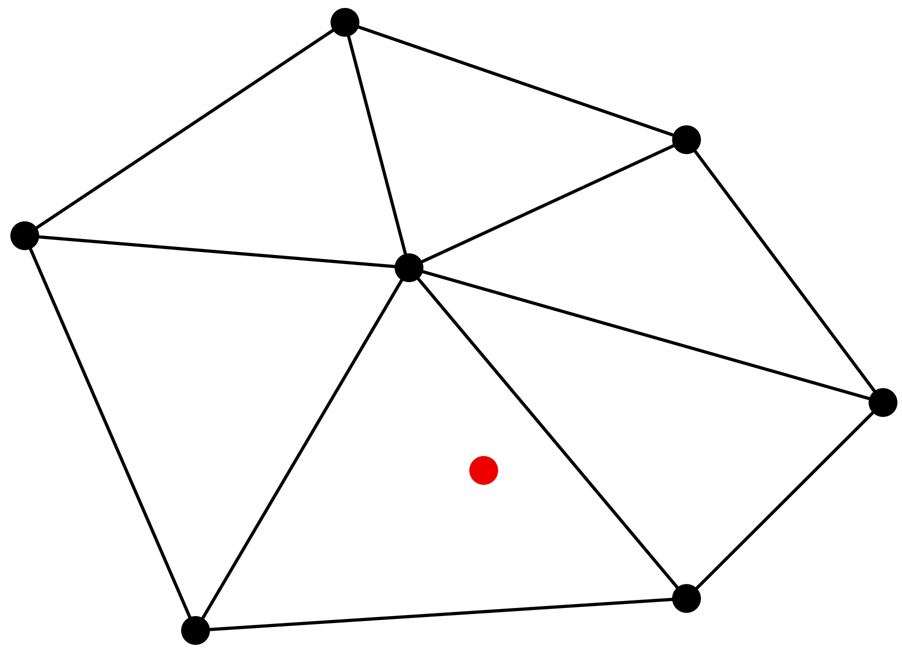

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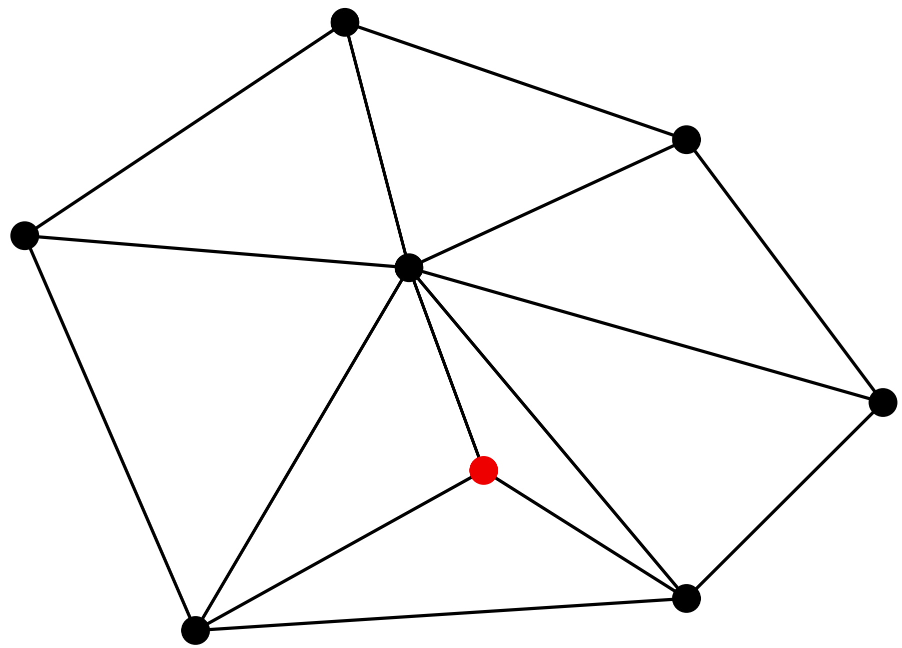

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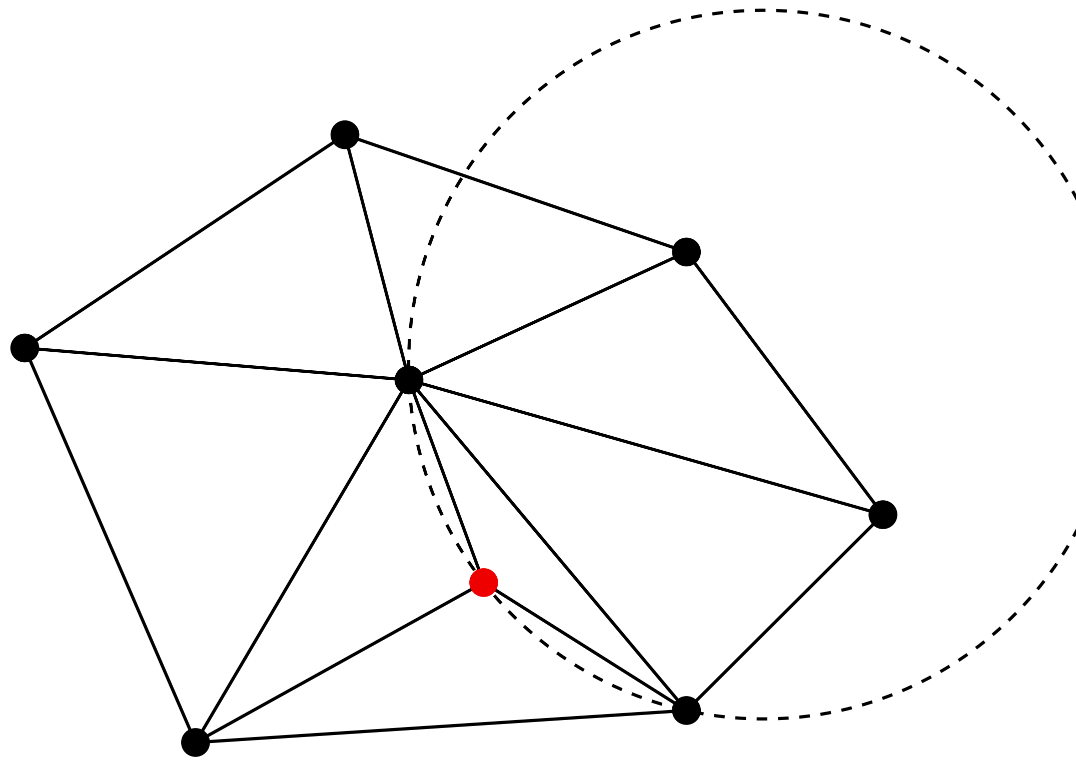

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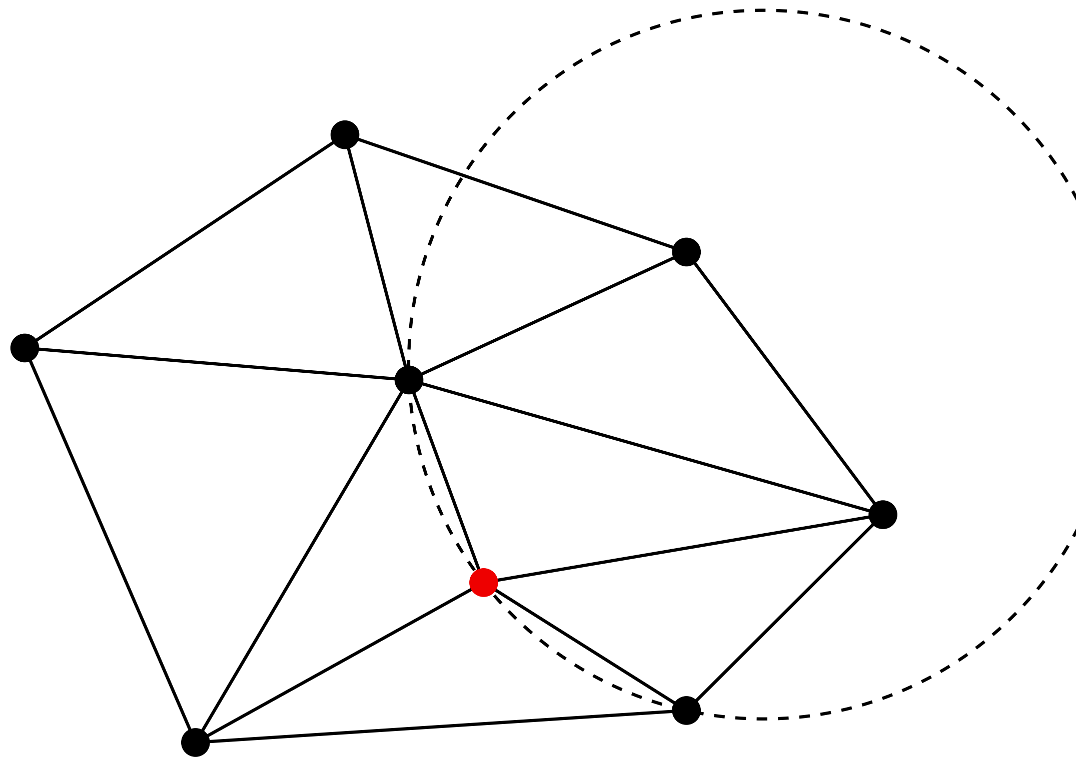

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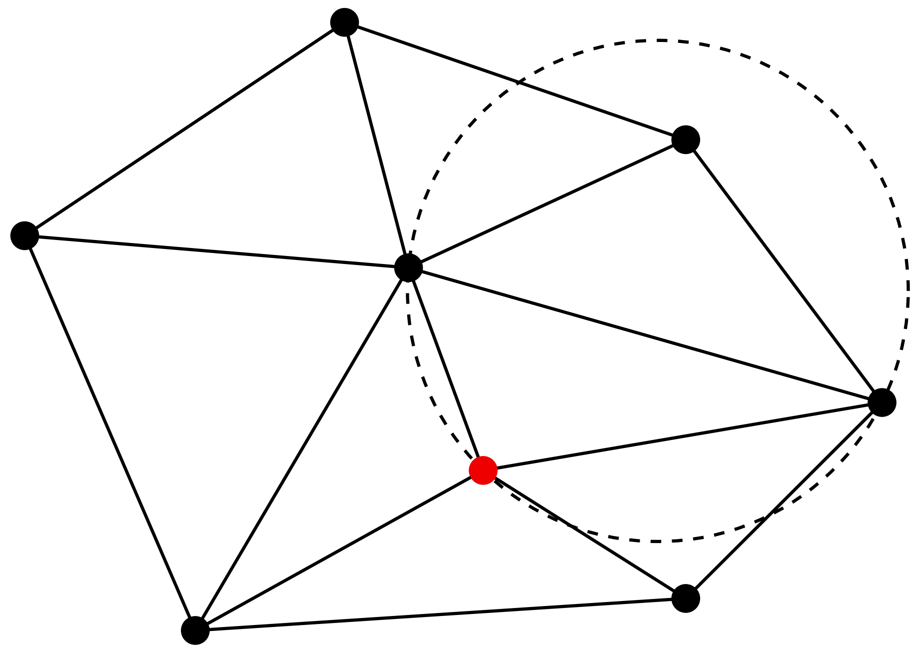

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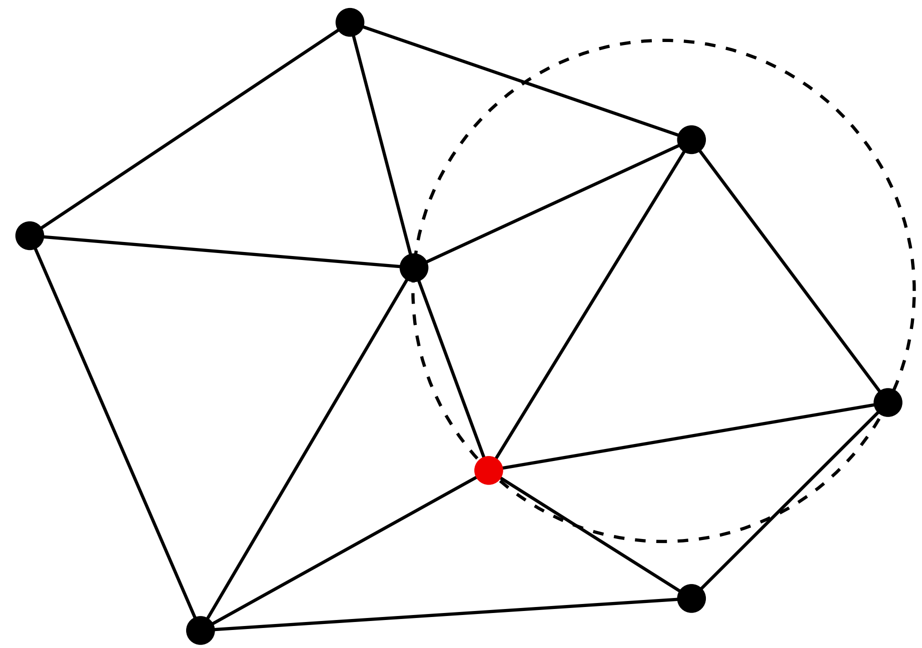

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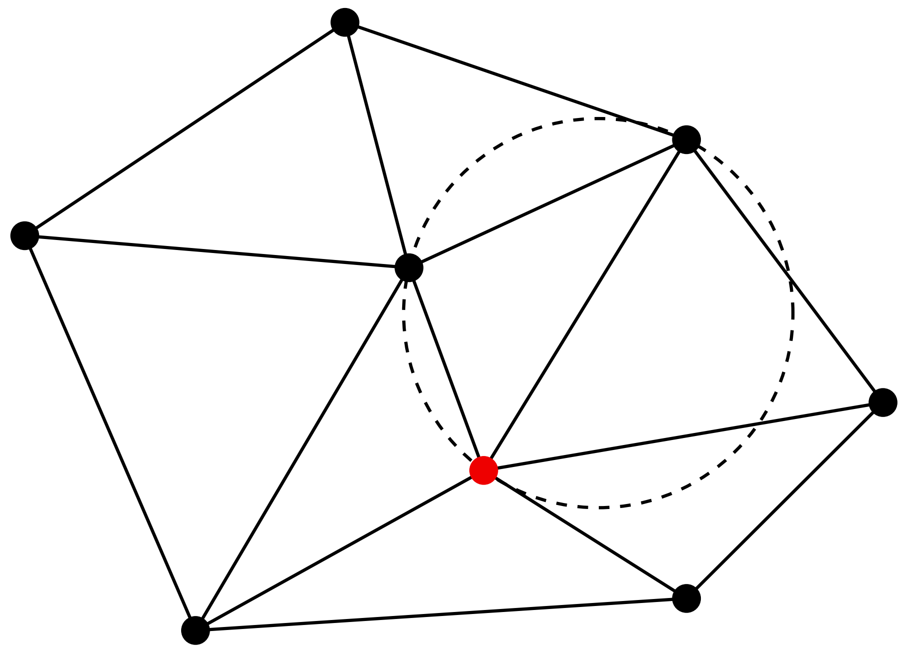

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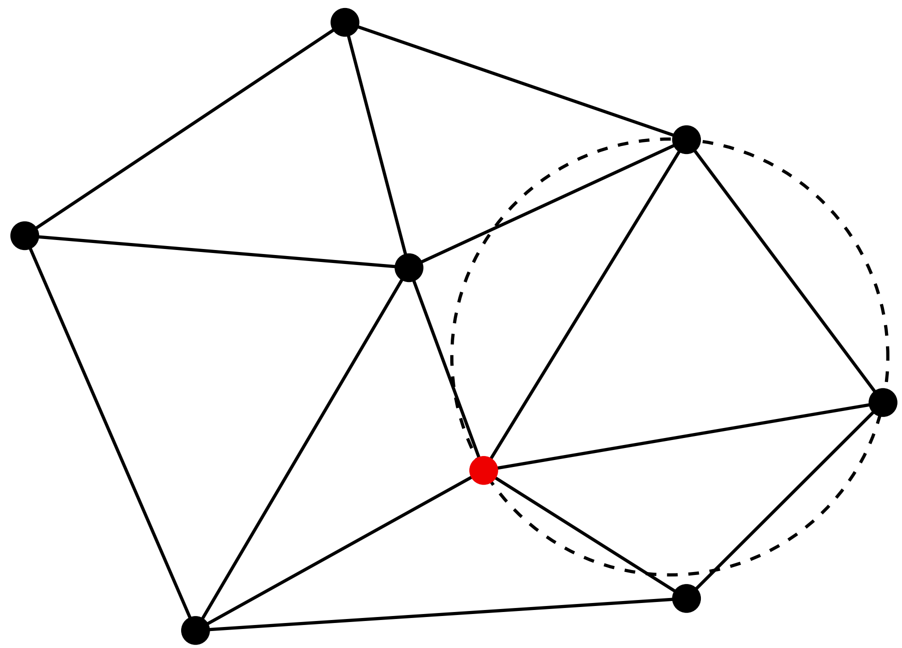

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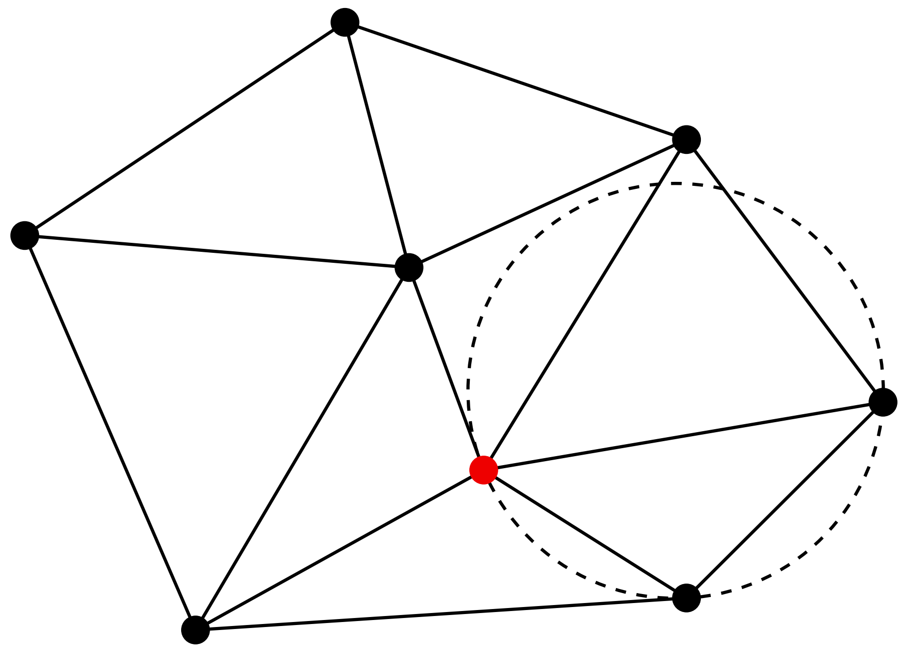

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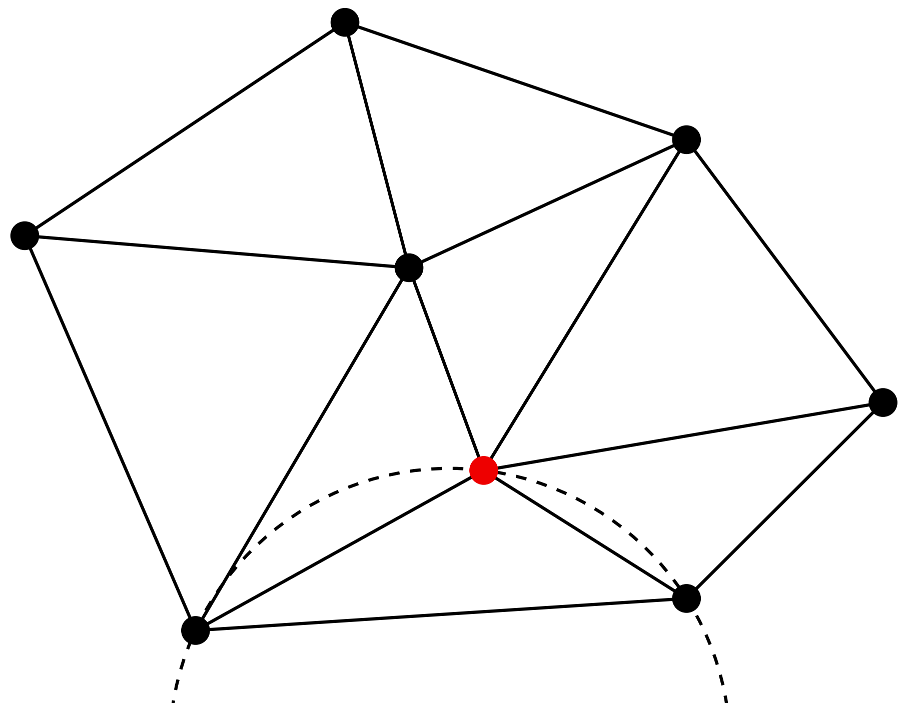

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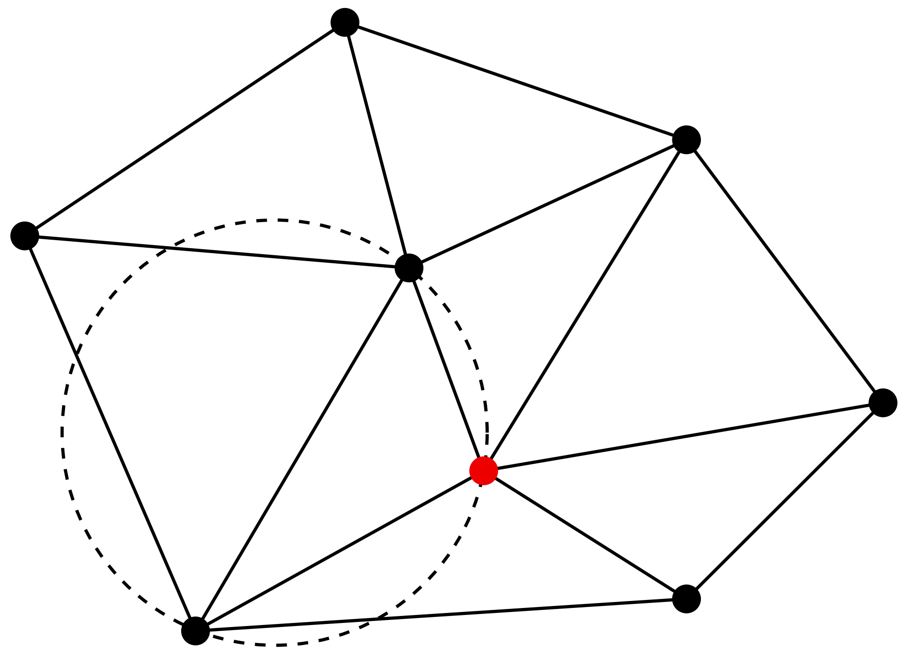

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The added running time of performing the flips when adding p_i is

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As the average order is smaller than 6, the expected added running time is not $O(n^2)$ but simply $O(n)$.

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Let us be more precise:

If $\mathcal{T} = \{T_1, \ldots, T_t\}$ is a triangulation of P, the "fineness" of $\mathcal{T}$ is the increasingly sorted list of the angles of all the triangles T_i of $\mathcal{T}$: $F(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3t})$.

DELAUNAY TRIANGULATION

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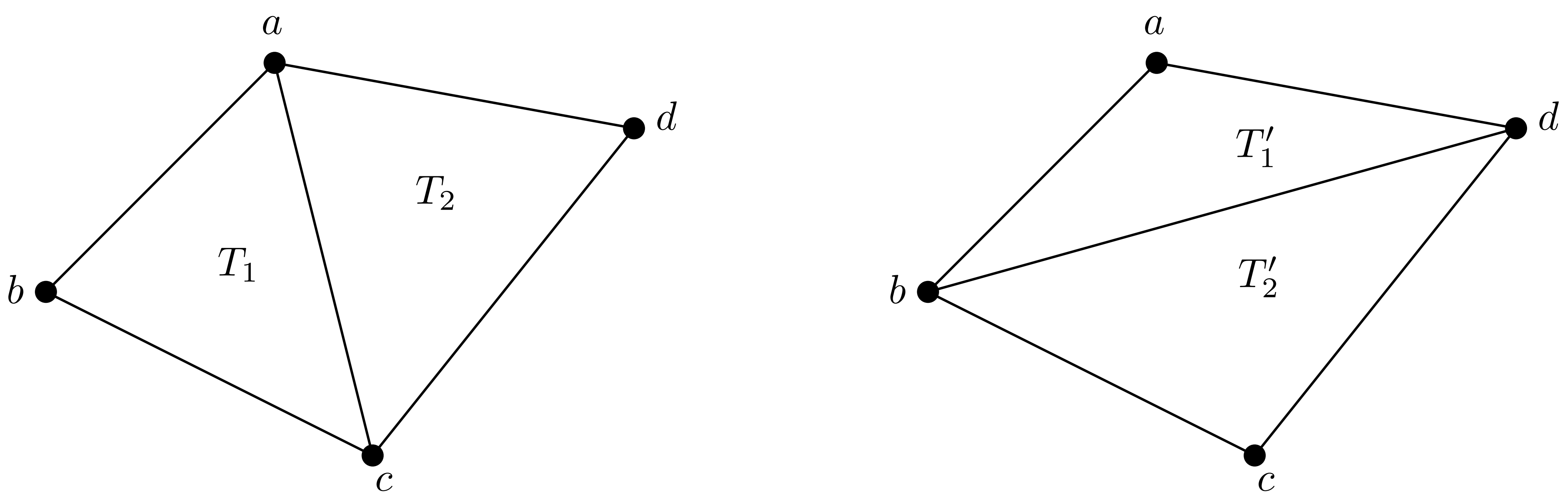
The proof of this statement requires a last lemma.

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Lemma 7. Let a, b, c and d be four points forming a convex quadrilateral, in counterclockwise order. Let $\mathcal{T}$ and $\mathcal{T}'$ be the two possible triangulations of the quadrilateral: $\mathcal{T}$ uses the diagonal $\overline{ac}$ and $\mathcal{T}'$ uses $\overline{bd}$. Let ϵ and ϵ' respectively be the minimum angles of $\mathcal{T}$ and $\mathcal{T}'$. Then:

$$\epsilon > \epsilon' \iff d \in ext(C_{abc})$$
$$\epsilon = \epsilon' \iff d \in \partial(C_{abc})$$
$$\epsilon < \epsilon' \iff d \in int(C_{abc})$$



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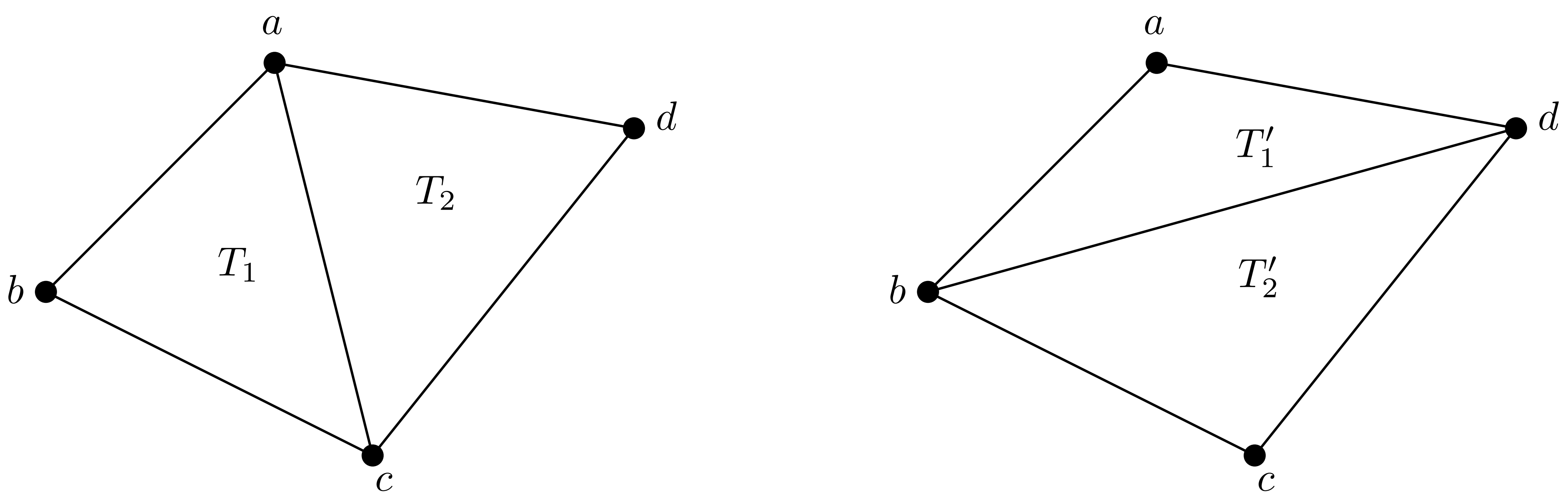
Due to the simmetry of the problem, we only need to prove that $\epsilon > \epsilon' \iff d \in ext(C_{abc})$.



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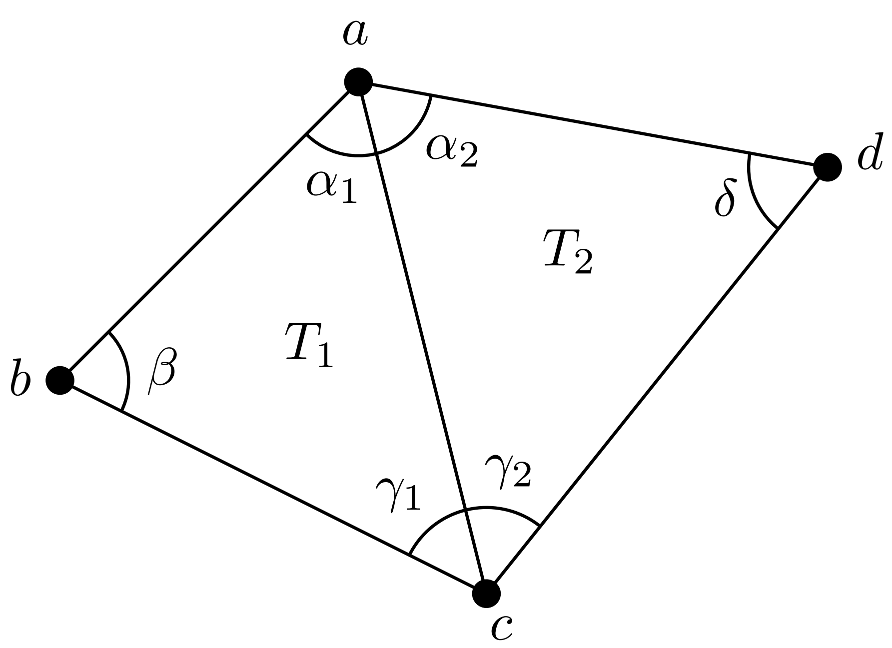

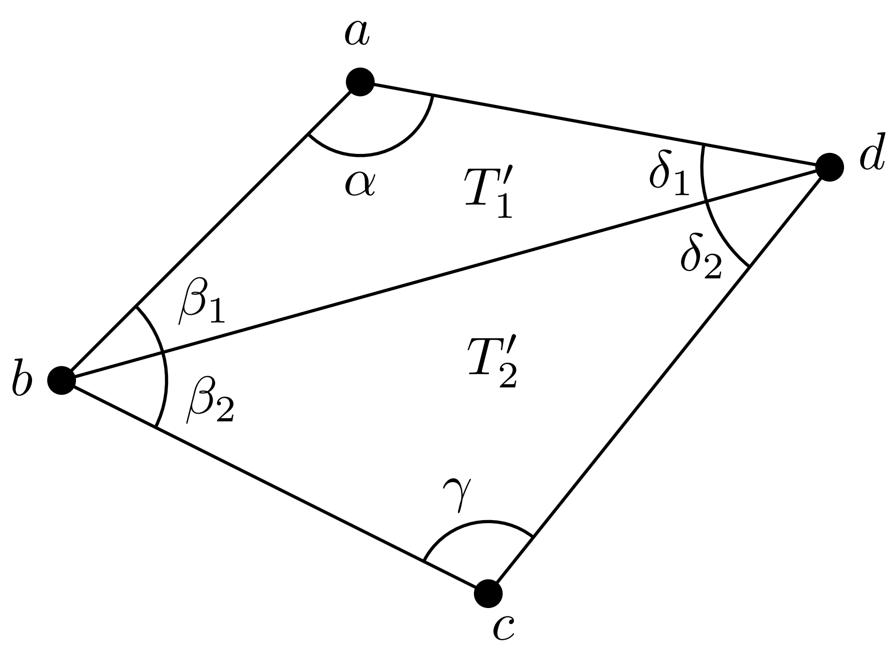

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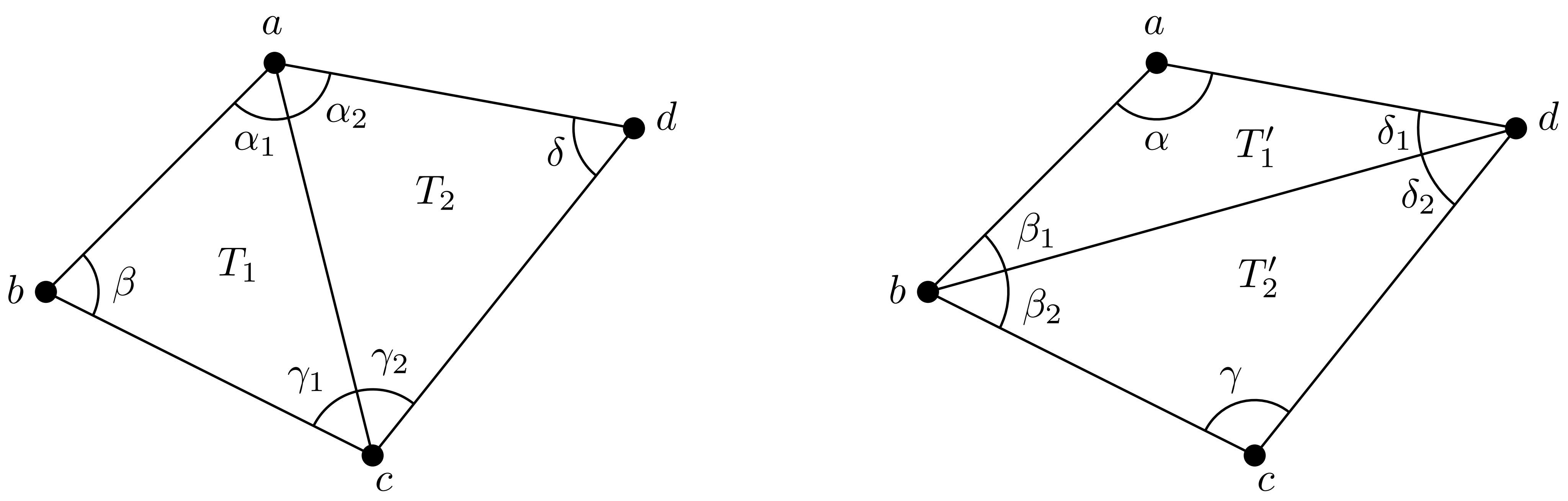
If $\epsilon > \epsilon'$, then ϵ' cannot be α, nor γ.



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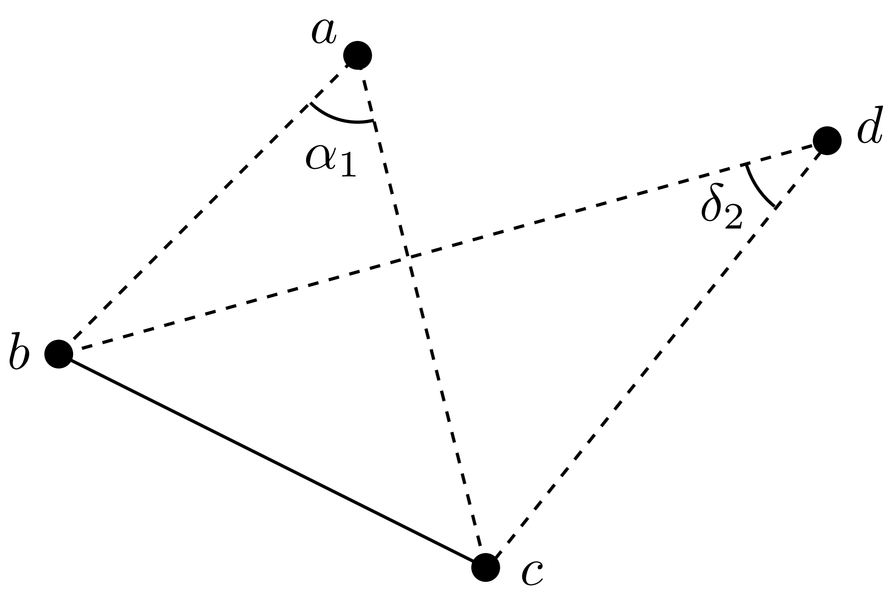
If $\epsilon' = \delta_2$, then $\delta_2 = \epsilon' < \epsilon \leq \alpha_1$ and, therefore, $d \in ext(C_{abc})$.



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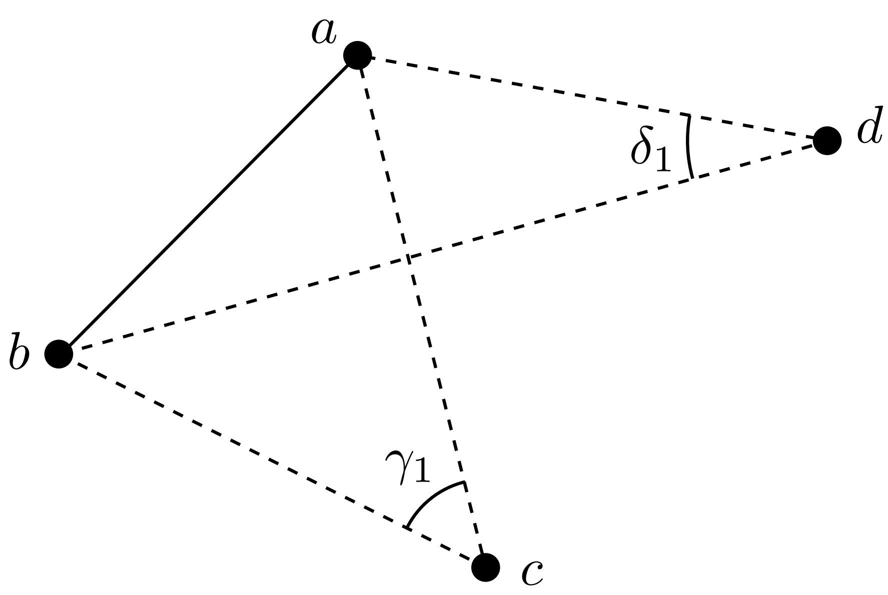
- If $\epsilon' = \delta_1$, then $\delta_1 = \epsilon' < \epsilon \leq \gamma_1$ and, therefore, $d \in ext(C_{abc})$.



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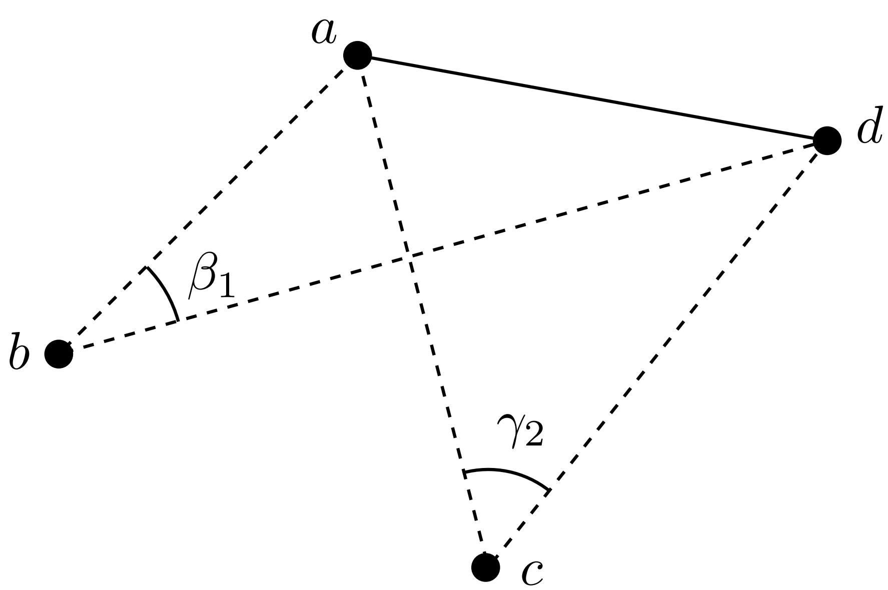
- If $\epsilon' = \delta_1$, then $\delta_1 = \epsilon' < \epsilon \leq \gamma_1$ and, therefore, $d \in ext(C_{abc})$.
- If $\epsilon' = \beta_1$, then $\beta_1 = \epsilon' < \epsilon \leq \gamma_2$ and, therefore, $d \in ext(C_{abc})$.



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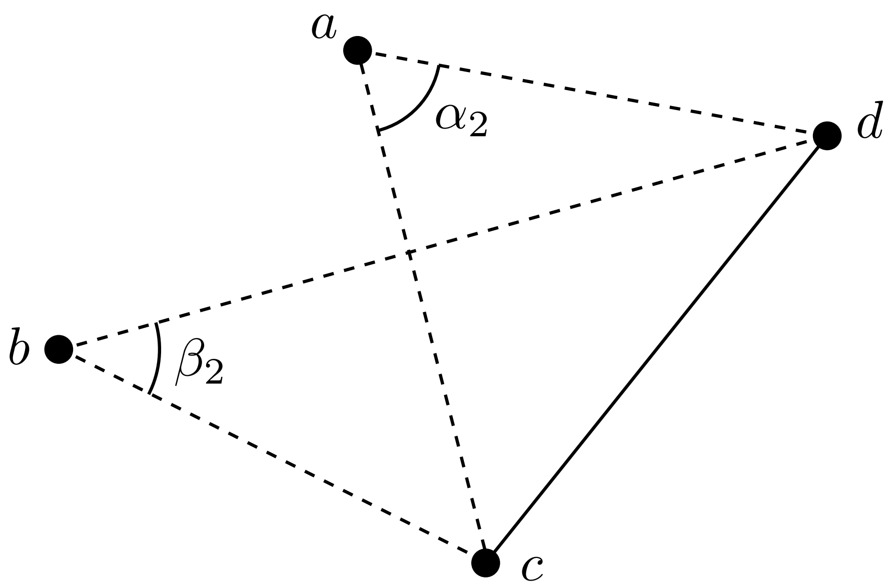

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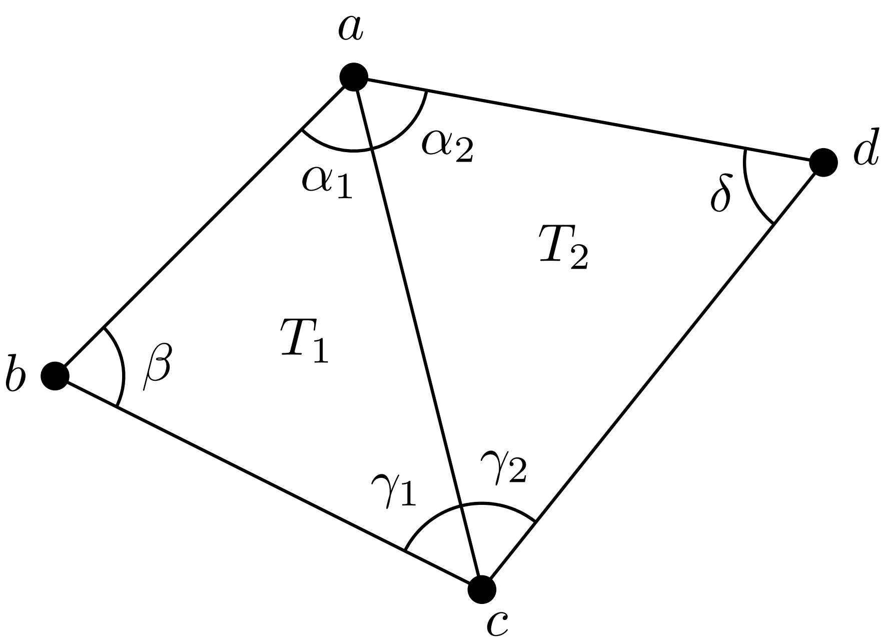
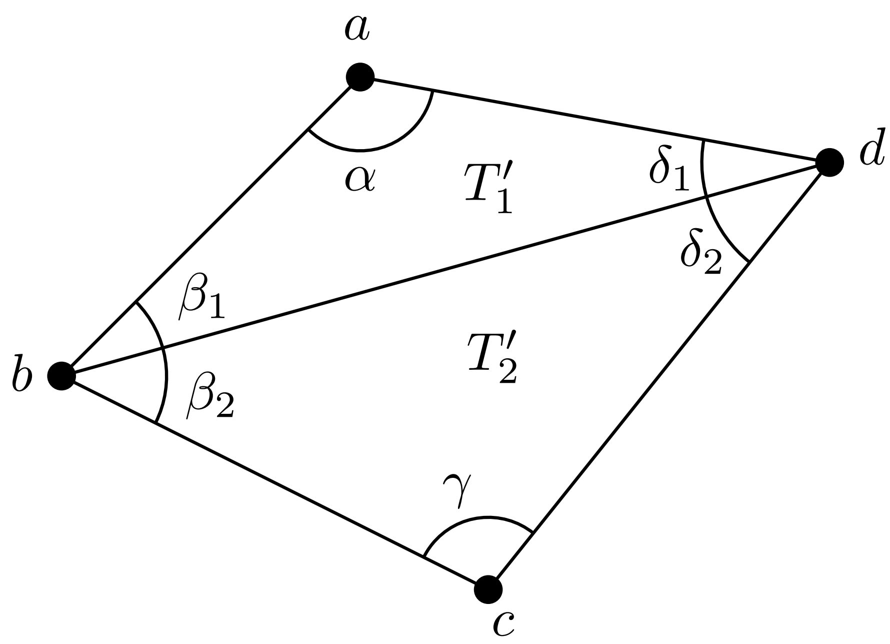
- If $\epsilon = \beta$, then $\epsilon = \beta > \beta_1 \geq \epsilon'$
- If $\epsilon = \delta$, then $\epsilon = \delta > \delta_1 \geq \epsilon'$





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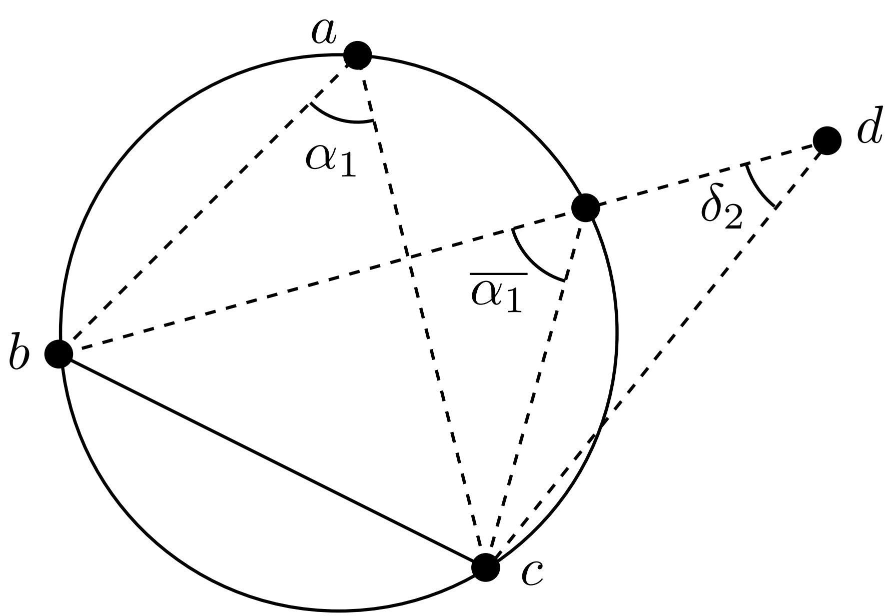
If $\epsilon = \alpha_1$, then $\epsilon = \alpha_1 = \overline{\alpha_1} > \delta_2 \geq \epsilon'$



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- If $\epsilon = \gamma_1$, then $\epsilon = \gamma_1 = \overline{\gamma_1} > \delta_1 \geq \epsilon'$
- If $\epsilon = \gamma_2$, then $\epsilon = \gamma_2 = \overline{\gamma_2} > \beta_1 \geq \epsilon'$

DELAUNAY TRIANGULATION AND EQUIANGULARITY

Corollary. The Delaunay triangulation is the most equiangular among all triangulations of a given set of points.

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Corollary. The Delaunay triangulation is the most equiangular among all triangulations of a given set of points.

If P does not contain four or more concyclic points, it follows from the previous lemma.

If P contains four or more concyclic points, $Del(P)$ contains a polygon inscribed in a circle which can be triangulated in several ways. Nevertheless, Lemma 1 (on the geometrical locus of all the points from which a segment is seen under a given angle) guarantees that every triangulation of a polygon inscribed in a circle has the same fineness, since each edge of the polygon belongs to a triangle, and every possible triangle gives rise to the same angle.

DELAUNAY TRIANGULATION AND INTERPOLATION

The Delaunay triangulation is used to interpolate terrains, because it also minimizes the roughness of the terrain, in other words, the integral of the square of the L_2-norm of the terrain's gradient.

It is important to notice that this property is independent from the data, in other words, it is independent from the values of the z-coordinates of the input points.

DELAUNAY TRIANGULATION

SOME ADDRESSES TO PLAY WITH DELAUNAY TRIANGULATIONS

http://www.cs.cornell.edu/Info/People/chew/Delaunay.html

http://web.informatik.uni-bonn.de/I/GeomLab/VoroGlide/index.html.en

http://www.dma.fi.upm.es/recursos/aplicaciones/geometria_computacional_y_grafos/

http://www.cs.unc.edu/∼snoeyink/terrain/Demo.html

AND TWO BOOKS WITH MUCH MORE INFORMATION

A. Okabe, B. Boots, K. Sugihara, S. N. Chiu
Spatial Tessellations
2nd ed., J. Wiley & Sons, 2000.

F. Aurenhammer, R. Klein, D.-T. Lee
Voronoi Diagrams and Delaunay Triangulations
World Scientific, 2013.