CONVEX HULLS IN ARBITRARY DIMENSION

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An **affine combination** of points in S is a linear combination such that $\sum_{i=1}^{k} \lambda_i = 1.$

- The set of all affine combinations of S is called **affine hull** of S.
- The affine hull of two points p and q is the line through them.
- $k+1$ points are called **affinely independent** if their affine hull is an affine space of dimension k.

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- The convex hull of two affinely independent points p and q is the line segment pq.
- The convex hull of three affinely independent points p, q and r is the triangle pqr.
- The convex hull of four affinely independent points p, q, r and s is the tetrahedron $pqrs$.

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A set is called **convex** if it is stable under convex combinations.

CONVEX HULLS IN ARBITRARY DIMENSION

Proposition: A set S is convex iff it contains segment pq for all $p, q \in S$.

Proof: If S is convex, it is stable under convex combinations and, in particular, it contains all segments with endpoints in S. The reciprocal is proved by induction on the number of points k of the convex combination. For $k = 2$, we have the hypothesis. Consider now a convex combination with $k + 1$ points. We have

$$\sum_{i=0}^{k} \lambda_i p_i = \lambda_0 p_0 + \sum_{i=1}^{k} \lambda_i p_i = \lambda_0 p_0 + (1 - \lambda_0) \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_0} p_i \in S.$$

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Corollary: The intersection of convex sets is convex.

Corollary: The convex hull $ch(S)$ of a set S is the smallest convex set containing S.

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The **dimension** of a convex set is defined as the dimension of its affine hull.

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- The convex hull of a finite set of points is called **polytope**.
- A polytope of dimension k is called k-**polytope**.
- The convex hull of $k+1$ affinely independent points is a k-polytope called k-**simplex**.

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Proposition: Polytopes are closed and bounded.

Let P be a polytope and H be a hyperplane in E^d:

- H **supports** P if $P \cap H \neq \emptyset$ and $(P \subset \overline{H^+}$ or $P \subset \overline{H^-})$.
- If H supports P, then we call $P \cap H$ a **face** of P.
- The 0-faces are called **vertices** of P.
- The 1-faces are called **edges** of P.
- The $(d-1)$-faces are called **facets** of P.

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Incidence

Two faces are called **incident** if one of them is a subset of the other one.

Adjacency

- Two vertices are **adjacent** if they are incident to the same edge.
- Two facets are **adjacent** if they are incident to the same $(d-1)$-face.

PROPERTIES OF POLYTOPES

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1. The boundary of a polytope is the union of all its faces.
2. Every polytope has a finite number of faces, and each one of them is a polytope.
3. Every polytope is the convex hull of its vertices.
4. Every polytope is the intersection of a finite set of closed halfspaces, namely, one for each of its $(d-1)$-faces.
5. The intersection of a finite number of closed halfspaces, if bounded, is a polytope.
6. Every face of a polytope P is a face of a $(d-1)$-face of P. Reciprocally, every face of a face of P is a face of P.
7. If P is a polytope, then
   (a) The intersection of any family of faces of P is a face of P.
   (b) Every $(d-2)$-face of P is the intersection of two $(d-1)$ faces of P.
   (c) If $j, k \in \mathbb{N}$, and $0 \leq j \leq k < d$, every j-face is the intersection of all the k-faces containining it.

COMBINATORICS OF POLYTOPES

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Let P be a d-polytope.
Let n_k be its number of k-faces, for $-1 \leq k \leq d$ (the (-1)-face being $\emptyset$ and the d-face being P itself).

Euler's relation

$$\sum_{k=0}^{d-1} (-1)^k n_k = 1 - (-1)^d \qquad \text{or, equivalently,} \qquad \sum_{k=-1}^{d} (-1)^k n_k = 0.$$

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The Dehn-Sommerville relations

If P is simple (i.e., each of its vertices belongs exactly to d facets), then $\sum_{j=0}^{k}(-1)^j \binom{d-j}{d-k} n_j = n_k$.

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The upper bound theorem

Any d-polytope with n vertices (or n facets) has at most $O(n^{\lfloor d/2 \rfloor})$ faces of all dimensions and $O(n^{\lfloor d/2 \rfloor})$ pairs of incident faces of all dimensions.

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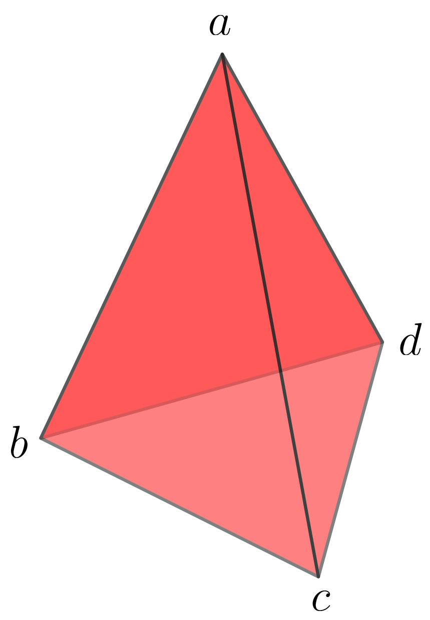
Any d-polytope with n vertices (or n facets) has at most $O(n^{\lfloor d/2 \rfloor})$ faces of all dimensions and $O(n^{\lfloor d/2 \rfloor})$ pairs of incident faces of all dimensions.

Tightness of the bound

There exists a d-polytope with $\Omega(n^{\lfloor d/2 \rfloor})$ faces of all dimensions and $\Omega(n^{\lfloor d/2 \rfloor})$ pairs of incident faces of all dimensions.

STORING A POLYTOPE



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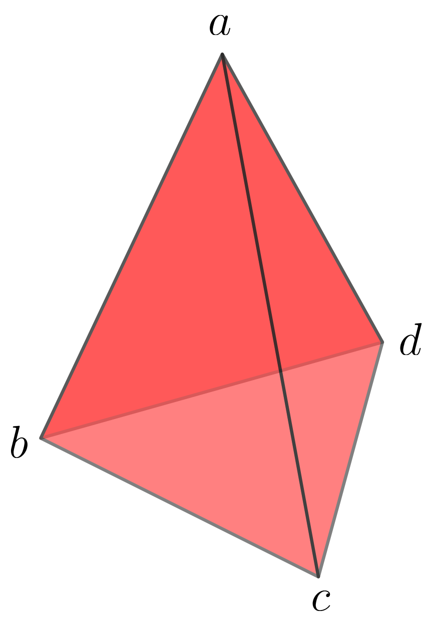
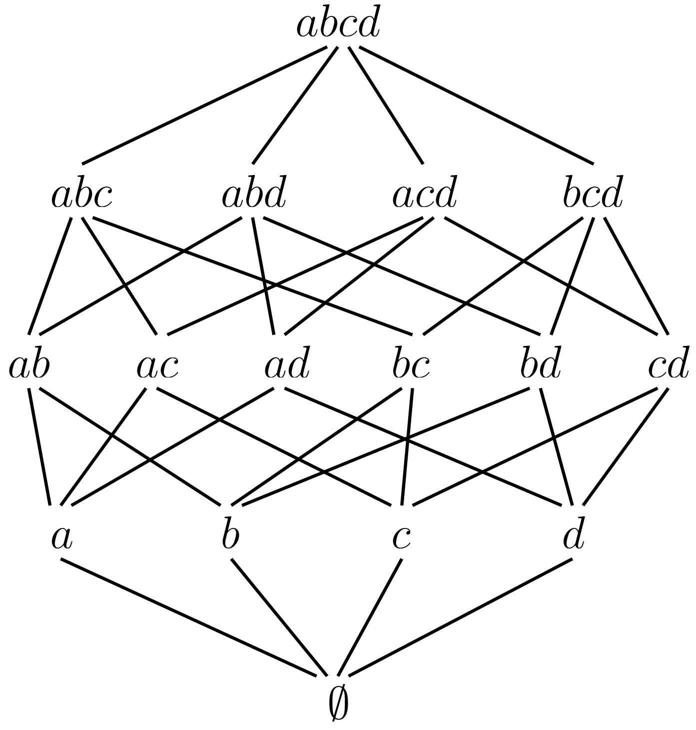
The **incidence graph** stores one node for each face and an edge for each pair of incident faces. The space used is $O(n^{\lfloor d/2 \rfloor})$.





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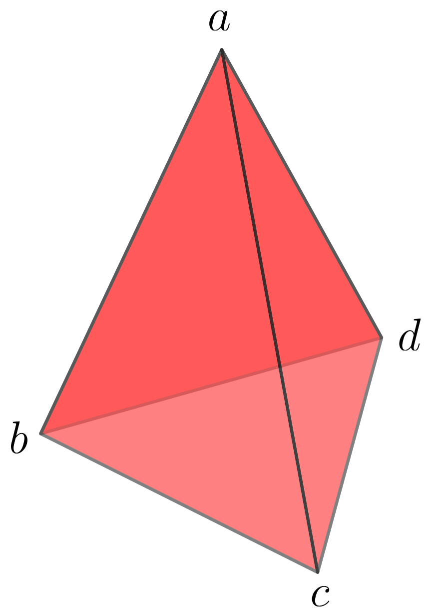
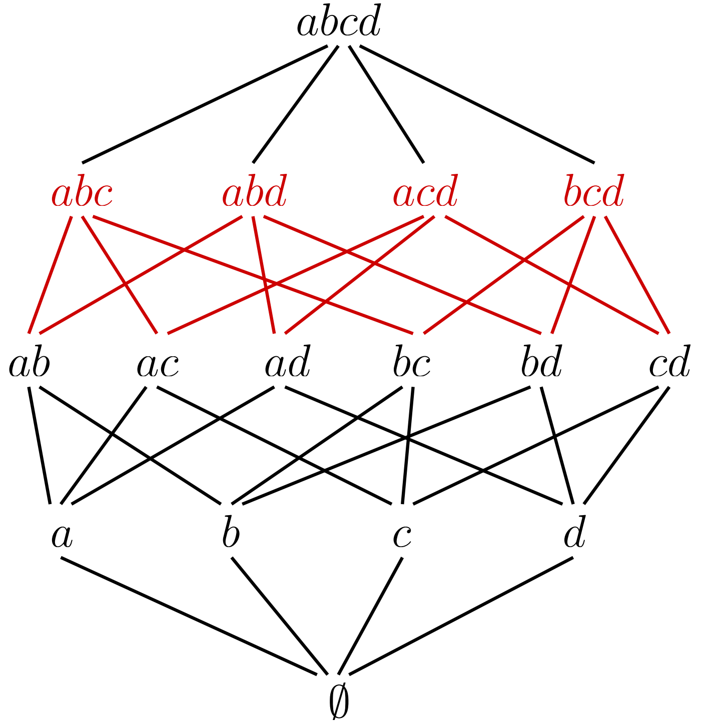
The incidence graph also encodes the **adjacency graph**, which has a node for each facet and an arc for each pair of adjacent facets: the arcs of the adjacency graph are in one-to-one correspondance with the $(d-2)$-faces.





COMPUTING d-DIMENSIONAL CONVEX HULLS

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Lower bound: Computing the convex hull of n points in E^d is $\Omega(n \log n + n^{\lfloor d/2 \rfloor})$.

Proof: If $d \geq 4$, then $\Omega(n \log n + n^{\lfloor d/2 \rfloor}) = \Omega(n^{\lfloor d/2 \rfloor})$, which is the size of the output. If $d = 2, 3$, then $\Omega(n \log n + n^{\lfloor d/2 \rfloor}) = \Omega(n \log n)$, which we know is a lower bound for the problem in dimension 2.

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Notation

- Denote by H_F the supporting hyperplane of each facet F of a given polytope. Among the two closed subspaces determined by H_F, let $\overline{H_F^+}$ be the one containing the polytope, and $\overline{H_F^-}$ the one not containing the polytope.
- Let p be a point exterior to a convex polytope C, and suppose that p does not belong to any hyperplane supporting a facet of C. Then:
  - Facets F such that $p \notin \overline{H_F^+}$ are called red.
  - Facets F such that $p \in \overline{H_F^+}$ are called blue.
  - The color of any remaining face is the intersection (red, blue or purple) of the colors of the facets incident to it.

CONVEX HULLS IN ARBITRARY DIMENSION

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Let C be a convex polytope and p a point in general position with respect to C.

Lemma 1: Every face of $ch(C \cup \{p\})$ is either a blue or purple face of C or the convex hull $ch(G \cup \{p\})$ of p and a purple face G of C.

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Proof: If P belongs to C, every face is blue and the result follows.
Otherwise:

- Every blue facet F of C is a facet of $ch(C \cup \{p\})$ because H_F supports $ch(C \cup \{p\})$ and $H_F \cap ch(C \cup \{p\}) = F$. The remaining blue faces being the intersection of blue facets, the result holds for all blue faces.
- If G is a purple face of C, then G must belong at least to one red facet F_1 and one blue facet F_2. Then $p \in H^+_{F_2}$ and $H_{F_2} \cap ch(C \cup \{p\}) = G$. Therefore, G is a face of $ch(C \cup \{p\})$. Since $p \in H^-_{F_2}$, any hyperplane H rotating about $H_{F_1} \cap H_{F_2}$, will eventually hit p. Then H supports $ch(C \cup \{p\})$ and $ch(C \cup \{p\}) \cap H \supset ch(G \cup \{p\})$. Therefore, $ch(G \cup \{p\})$ is a face of $ch(C \cup \{p\})$.
- Any face of $ch(G \cup \{p\})$ that does not contain p must be a blue or purple face of C. Any face of $ch(C \cup \{p\})$ that contains p must be of the form $ch(G \cup \{p\})$, for some face of C that must be purple. (Note: in particular, p itself is a face of $ch(C \cup \{p\})$ because the empty face of C must be purple.)

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Lemma 2: Incidences in $ch(C \cup \{p\})$ are:

– All blue–blue, blue–purple and purple–purple incidences from C.

– G and $ch(G \cup \{p\})$, for all purple G in C.

– $ch(F \cup \{p\})$ and $ch(G \cup \{p\})$, for all purple F and G incident in C.

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Lemma 3: The set of red facets is connected. The set of blue facets is connected too.

Proof: If $d = 2$, this is a well known fact. Otherwise, let r_1 and r_2 be two points in E^d on two different red (blue) facets of C. The plane π spanned by p, r_1, r_2 intersects C in a 2-polytope, where the set of all red (blue) edges is connected. Therefore, there exists a red (blue) path connecting r_1 and r_1 in $\pi \cap C \subset C$.

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Lemma 4: The set of purple faces is isomorphic to a $(d-1)$-politope of at most n vertices.

Proof: Any hyperplane H separating p from C intersects all the faces of $ch(C \cup \{p\})$ containing p (except for the vertex p) and those faces only. The trace in H of these faces is a $(d-1)$-polytope whose incidences correspond to incidences in C.

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Input: $p_1, \dots, p_n \in E^d$
Output: Incidence graph of $ch(\{p_1, \dots, p_n\})$

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Algorithm

1. Lexicographically sort the points.
2. Initialize $C_{d+1} = simplex\,(p_1, \ldots, p_{d+1}) = ch(\{p_1, \ldots, p_{d+1}\})$
3. Construct $C_{i+1} = ch(\{p_1, \ldots, p_{i+1}\})$ from $C_i = ch(\{p_1, \ldots, p_i\})$.
    1. Identify a red facet of C_i as seen from p_{i+1}.
    2. Construct three lists, respectively containing all red facets, all $(d-2)$ red faces, and all $(d-2)$ purple faces.
    3. Construct two more lists, respectively containing all remaining red faces, and all remaining purple faces.
    4. Update the incidence graph.

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       Explore all facets incident to p_i (the last inserted point). Due to the lexicographical order of the points, at least one of them needs to be red.

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       A deptf-first search allows to find all red facets and classify all $(d-2)$-faces into red and purple.

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    Once all red and purple faces have been classified from dimension $d-1$ down to dimension $k+1$, all k-subfaces of purple $(k+1)$-faces are declared purple. Once this done, all unclassified k-subfaces of red $(k+1)$-faces are declared red.

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    4. Update the incidence graph.

       Fist, all red faces are eliminated from the incidence graph (both nodes and incident arcs). Then, starting from $k = 0$, if F is a purple k-face a new node is created for the $(k+1)$-face $ch(F \cup \{p\})$. This new node is connected to F and also to all the k-faces of the form $ch(G \cup \{p\})$, where G is a $(k-1)$-subface of F.

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Analysis

COMPUTING d-DIMENSIONAL CONVEX HULLS

Input: $p_1, \ldots, p_n \in E^d$
Output: Incidence graph of $ch(\{p_1, \ldots, p_n\})$

Algorithm

1. Lexicographically sort the points.
2. Initialize $C_{d+1} = simplex\,(p_1, \ldots, p_{d+1}) = ch(\{p_1, \ldots, p_{d+1}\})$
3. Construct $C_{i+1} = ch(\{p_1, \ldots, p_{i+1}\})$ from $C_i = ch(\{p_1, \ldots, p_i\})$.
    1. Identify a red facet of C_i as seen from p_{i+1}.
    2. Construct three lists, respectively containing all red facets, all $(d-2)$ red faces, and all $(d-2)$ purple faces.
    3. Construct two more lists, respectively containing all remaining red faces, and all remaining purple faces.
    4. Update the incidence graph.

Analysis

$O(n \log n)$

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Algorithm	Analysis
1. Lexicographically sort the points.	$O(n \log n)$
2. Initialize $C_{d+1} = simplex\,(p_1, \ldots, p_{d+1}) = ch(\{p_1, \ldots, p_{d+1}\})$	$O(1)$
3. Construct $C_{i+1} = ch(\{p_1, \ldots, p_{i+1}\})$ from $C_i = ch(\{p_1, \ldots, p_i\})$.	

1. Identify a red facet of C_i as seen from p_{i+1}.
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CONVEX HULLS IN ARBITRARY DIMENSION

COMPUTING d-DIMENSIONAL CONVEX HULLS

Input: $p_1, \ldots, p_n \in E^d$
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Algorithm	**Analysis**
1. Lexicographically sort the points.	$O(n \log n)$
2. Initialize $C_{d+1} = simplex\,(p_1, \ldots, p_{d+1}) = ch(\{p_1, \ldots, p_{d+1}\})$	$O(1)$
3. Construct $C_{i+1} = ch(\{p_1, \ldots, p_{i+1}\})$ from $C_i = ch(\{p_1, \ldots, p_i\})$.	
1. Identify a red facet of C_i as seen from p_{i+1}.	Linear in #facets created in step i.
2. Construct three lists, respectively containing all red facets, all $(d-2)$ red faces, and all $(d-2)$ purple faces.	
3. Construct two more lists, respectively containing all remaining red faces, and all remaining purple faces.	
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1. Identify a red facet of C_i as seen from p_{i+1}.	Linear in #facets created in step i.
2. Construct three lists, respectively containing all red facets, all $(d-2)$ red faces, and all $(d-2)$ purple faces.	Linear in #red facets of C_i and their adjacencies.
3. Construct two more lists, respectively containing all remaining red faces, and all remaining purple faces.	
4. Update the incidence graph.	

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1. Identify a red facet of C_i as seen from p_{i+1}.	Linear in #facets created in step i.
2. Construct three lists, respectively containing all red facets, all $(d-2)$ red faces, and all $(d-2)$ purple faces.	Linear in #red facets of C_i and their adjacencies.
3. Construct two more lists, respectively containing all remaining red faces, and all remaining purple faces.	Linear in #red-red and red-purple incidences in C_i.
4. Update the incidence graph.	

CONVEX HULLS IN ARBITRARY DIMENSION

COMPUTING d-DIMENSIONAL CONVEX HULLS

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Algorithm	**Analysis**
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1. Identify a red facet of C_i as seen from p_{i+1}.	Linear in #facets created in step i.
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3. Construct two more lists, respectively containing all remaining red faces, and all remaining purple faces.	Linear in #red-red and red-purple incidences in C_i.
4. Update the incidence graph.	Linear in #red faces and their incidences and purple faces and their purple incidences.

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    4. Update the incidence graph.

Analysis

$O(n \log n)$

$O(1)$

Proportional to the number of faces and incidences created along the algorithm.

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Analysis

$O(n \log n)$

$O(1)$

Proportional to the number of faces and incidences created along the algorithm.

$$\sum_{i=1}^{n} O(i^{\lfloor \frac{d-1}{2} \rfloor}) = O(n^{\lfloor \frac{d+1}{2} \rfloor})$$

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    4. Update the incidence graph.

Analysis

1. $O(n \log n)$
2. $O(1)$
3. Proportional to the number of faces and incidences created along the algorithm.

$$\sum_{i=1}^{n} O(i^{\lfloor \frac{d-1}{2} \rfloor}) = O(n^{\lfloor \frac{d+1}{2} \rfloor})$$

And the space used is $O(n^{\lfloor \frac{d}{2} \rfloor})$

CONVEX HULLS IN ARBITRARY DIMENSION

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Analysis

$O(n \log n)$

$O(1)$

Proportional to the number of faces and incidences created along the algorithm.

$$\sum_{i=1}^{n} O(i^{\lfloor \frac{d-1}{2} \rfloor}) = O(n^{\lfloor \frac{d+1}{2} \rfloor})$$

And the space used is $O(n^{\lfloor \frac{d}{2} \rfloor})$

This is optimal when n is even.

FURTHER READING

J.-D. Boissonat. M. Yvinec, **Algorithmic Geometry**, Cambridge University Press, 1998.