GEOMETRIC TOOLS FOR COMPUTER GRAPHICS (MIRI)

Comparing Foley - Van Dam's and the quaternions methods: Rotating about the line through the origin with direction (Phi,Theta)

Vera Sacristán

In general, we can assume that the rotation axis has direction
(Sin[phi] Cos[th], Sin[phi] Sin[th], Cos[phi])

Foley - Van Dam' s method

```
MatrixForm[rotationOZTheta = {{Cos[th], -Sin[th], 0}, {Sin[th], Cos[th], 0}, {0, 0, 1}}]
```

$$\begin{pmatrix} \text{Cos}[\text{th}] & -\text{Sin}[\text{th}] & 0 \\ \text{Sin}[\text{th}] & \text{Cos}[\text{th}] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
rotationOZThetaNeg = Transpose[rotationOZTheta];
```

```
MatrixForm[rotationOYPhi = {{Cos[phi], 0, Sin[phi]}, {0, 1, 0}, {-Sin[phi], 0, Cos[phi]}}]
```

$$\begin{pmatrix} \text{Cos}[\text{phi}] & 0 & \text{Sin}[\text{phi}] \\ 0 & 1 & 0 \\ -\text{Sin}[\text{phi}] & 0 & \text{Cos}[\text{phi}] \end{pmatrix}$$

```
rotationOYPhiNeg = Transpose[rotationOYPhi];
```

```
MatrixForm[rotationOZAlpha = {{Cos[a], -Sin[a], 0}, {Sin[a], Cos[a], 0}, {0, 0, 1}}]
```

$$\begin{pmatrix} \text{Cos}[\text{a}] & -\text{Sin}[\text{a}] & 0 \\ \text{Sin}[\text{a}] & \text{Cos}[\text{a}] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
rotationOZAlphaNeg = Transpose[rotationOZAlpha];
```

```
point = {x, y, z};
```

```
rotation =
  rotationOZTheta.rotationOYPhi.rotationOZAlpha.rotationOYPhiNeg.rotationOZThetaNeg;
```

```
newPoint = rotation.point
```

$$\begin{aligned}
&\{z\,(\text{Cos}[\text{phi}]\,\text{Cos}[\text{th}]\,\text{Sin}[\text{phi}] - \text{Sin}[\text{phi}]\,(\text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Cos}[\text{th}] - \text{Sin}[\text{a}]\,\text{Sin}[\text{th}])) + \\
&\quad x\,\big(-\text{Sin}[\text{th}]\,(-\text{Cos}[\text{phi}]\,\text{Cos}[\text{th}]\,\text{Sin}[\text{a}] - \text{Cos}[\text{a}]\,\text{Sin}[\text{th}]) + \\
&\qquad \text{Cos}[\text{th}]\,\big(\text{Cos}[\text{th}]\,\text{Sin}[\text{phi}]^2 + \text{Cos}[\text{phi}]\,(\text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Cos}[\text{th}] - \text{Sin}[\text{a}]\,\text{Sin}[\text{th}])\big)\big) + \\
&\quad y\,\big(\text{Cos}[\text{th}]\,(-\text{Cos}[\text{phi}]\,\text{Cos}[\text{th}]\,\text{Sin}[\text{a}] - \text{Cos}[\text{a}]\,\text{Sin}[\text{th}]) + \\
&\qquad \text{Sin}[\text{th}]\,\big(\text{Cos}[\text{th}]\,\text{Sin}[\text{phi}]^2 + \text{Cos}[\text{phi}]\,(\text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Cos}[\text{th}] - \text{Sin}[\text{a}]\,\text{Sin}[\text{th}])\big)\big), \\
&z\,(\text{Cos}[\text{phi}]\,\text{Sin}[\text{phi}]\,\text{Sin}[\text{th}] - \text{Sin}[\text{phi}]\,(\text{Cos}[\text{th}]\,\text{Sin}[\text{a}] + \text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Sin}[\text{th}])) + \\
&\quad x\,\big(-\text{Sin}[\text{th}]\,(\text{Cos}[\text{a}]\,\text{Cos}[\text{th}] - \text{Cos}[\text{phi}]\,\text{Sin}[\text{a}]\,\text{Sin}[\text{th}]) + \\
&\qquad \text{Cos}[\text{th}]\,\big(\text{Sin}[\text{phi}]^2\,\text{Sin}[\text{th}] + \text{Cos}[\text{phi}]\,(\text{Cos}[\text{th}]\,\text{Sin}[\text{a}] + \text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Sin}[\text{th}])\big)\big) + \\
&\quad y\,\big(\text{Cos}[\text{th}]\,(\text{Cos}[\text{a}]\,\text{Cos}[\text{th}] - \text{Cos}[\text{phi}]\,\text{Sin}[\text{a}]\,\text{Sin}[\text{th}]) + \\
&\qquad \text{Sin}[\text{th}]\,\big(\text{Sin}[\text{phi}]^2\,\text{Sin}[\text{th}] + \text{Cos}[\text{phi}]\,(\text{Cos}[\text{th}]\,\text{Sin}[\text{a}] + \text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Sin}[\text{th}])\big)\big), \\
&z\,\big(\text{Cos}[\text{phi}]^2 + \text{Cos}[\text{a}]\,\text{Sin}[\text{phi}]^2\big) + \\
&\quad x\,(\text{Cos}[\text{th}]\,(\text{Cos}[\text{phi}]\,\text{Sin}[\text{phi}] - \text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Sin}[\text{phi}]) - \text{Sin}[\text{a}]\,\text{Sin}[\text{phi}]\,\text{Sin}[\text{th}]) + \\
&\quad y\,(\text{Cos}[\text{th}]\,\text{Sin}[\text{a}]\,\text{Sin}[\text{phi}] + (\text{Cos}[\text{phi}]\,\text{Sin}[\text{phi}] - \text{Cos}[\text{a}]\,\text{Cos}[\text{phi}]\,\text{Sin}[\text{phi}])\,\text{Sin}[\text{th}])\}
\end{aligned}$$

Quaternions method

```
Needs["Quaternions`"]
```

```
pointQ = Quaternion[0, x, y, z];
```

```
rotationAlphaQ = Quaternion[Cos[a / 2],
   Sin[phi] Cos[th] Sin[a / 2], Sin[phi] Sin[th] Sin[a / 2], Cos[phi] Sin[a / 2]];
```

```
newPointQ = rotationAlphaQ ** pointQ ** Conjugate[rotationAlphaQ]
```

$$\begin{aligned}
&\text{Quaternion}\Big[\\
&\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\left(y\cos\left[\frac{a}{2}\right]+x\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-z\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\right)\sin[\text{th}]+\\
&\cos[\text{phi}]\sin\left[\frac{a}{2}\right]\left(z\cos\left[\frac{a}{2}\right]+y\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-x\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)+\\
&\cos\left[\frac{a}{2}\right]\left(-z\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-x\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-y\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)+\\
&\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\left(x\cos\left[\frac{a}{2}\right]-y\cos[\text{phi}]\sin\left[\frac{a}{2}\right]+z\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right),\\
&-\cos[\text{phi}]\sin\left[\frac{a}{2}\right]\left(y\cos\left[\frac{a}{2}\right]+x\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-z\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\right)+\\
&\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\left(z\cos\left[\frac{a}{2}\right]+y\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-x\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)-\\
&\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\\
&\left(-z\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-x\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-y\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)+\\
&\cos\left[\frac{a}{2}\right]\left(x\cos\left[\frac{a}{2}\right]-y\cos[\text{phi}]\sin\left[\frac{a}{2}\right]+z\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right),\\
&\cos\left[\frac{a}{2}\right]\left(y\cos\left[\frac{a}{2}\right]+x\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-z\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\right)-\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\\
&\left(z\cos\left[\frac{a}{2}\right]+y\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-x\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)-\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\\
&\sin[\text{th}]\left(-z\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-x\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-y\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)+\\
&\cos[\text{phi}]\sin\left[\frac{a}{2}\right]\left(x\cos\left[\frac{a}{2}\right]-y\cos[\text{phi}]\sin\left[\frac{a}{2}\right]+z\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right),\\
&\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\left(y\cos\left[\frac{a}{2}\right]+x\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-z\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\right)+\\
&\cos\left[\frac{a}{2}\right]\left(z\cos\left[\frac{a}{2}\right]+y\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-x\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)-\cos[\text{phi}]\\
&\sin\left[\frac{a}{2}\right]\left(-z\cos[\text{phi}]\sin\left[\frac{a}{2}\right]-x\cos[\text{th}]\sin\left[\frac{a}{2}\right]\sin[\text{phi}]-y\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)-\\
&\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\left(x\cos\left[\frac{a}{2}\right]-y\cos[\text{phi}]\sin\left[\frac{a}{2}\right]+z\sin\left[\frac{a}{2}\right]\sin[\text{phi}]\sin[\text{th}]\right)\Big]
\end{aligned}$$

Comparing the results

```
Simplify[newPointQ[[1]]]
```

```
0
```

```
Table[Simplify[newPointQ[[i + 1]] - newPoint[[i]]], {i, 1, 3}]
```

```
{0, 0, 0}
```