Geometric Tools for Computer Graphics - MIRI
Problems list 4
Year 2016-2017 Q1

1. In the plane:
   (a) What is the result of composing three rotations of 60° about three consecutive vertices of an equilateral triangle?
   (b) What is the result of performing 4 reflections across the edges of a square?
   (c) Prove that the composition of two reflections across two parallel axes is a translation, and determine its details.
   (d) Prove that the composition of two reflections across two intersecting lines is a rotation, and determine its details.

2. In the plane:
   (a) Given a regular pentagon with center $(0,0)$, radius 1, and one vertex at point $(1,0)$, how can we obtain a regular pentagon having as one of its edges the segment connecting $(1,0)$ and $(0,1)$?
   (b) Consider the circle $C : (x-2)^2 + (y-3)^2 = 1$ and the l'ellipse E centered at point $D = (4,5)$, with semi-axes $a = 4$ and $b = 1$, such that the positive semi-axis of the abscissae forms a 45° angle with the larger semi-axis of the ellipse. Find a composition of basic affinities transforming C into E.
   (c) Let ℓ_1 and ℓ_2 be two lines through a common point $P_0 = (x_0, y_0)$: ℓ_1 is defined by points P_0 and $P_1 = (x_1, y_1)$, while ℓ_2 is defined by P_0 and $P_2 = (x_2, y_2)$. Which rotation(s) about P_0 transform(s) ℓ_1 into ℓ_2?

3. In the plane, prove that the result of performing a rotation of angle α about the origin, followed by a translation of vector w is, in fact, a rotation of angle α about some center C, and that C is uniquely determined if $\alpha \neq 0$. What if $\alpha = 0$?

4. In the plane, prove that the result of performing a reflection across the axis Ox followed by a translation of vector w is, in fact, a reflection about some axis ℓ followed by a translation of some vector c parallel to ℓ. Precise which are ℓ and c.

5. Consider the following two lines:

$$\begin{cases} 2x + 3y - 3z = 5, \\ y + z = 1, \end{cases} \qquad \begin{cases} x + 2y - z = 3, \\ 2x + 4y - z = 9, \end{cases}$$

   (a) Prove that they intersect in a point, and find an equation of the plane containing the lines.
   (b) Find the mirror reflection of point $(3, 1, -1)$ across the previous plane.
   (c) Find an equation for the mirror reflection of the line $x + y = 1$, $y + 2z = 4$.

   Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.

6. Let ℓ be the intersection line of planes $\pi_1 : y + \sqrt{3}z = 0$ and $\pi_2 : -y + \sqrt{3}z = 0$. Find the image of point $(2, 4, 5)$ after rotating it about line ℓ by twice the angle between π_1 and π_2. Orient the line in the direction of positive x, and the angle in the direction from π_1 towards π_2. Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.

7. Find the equations of the rotation about the intersection line of planes π_1 and π_2, when the rotation angle coincides with the angle between planes π_3 and π_4, where:

$$\begin{array}{ll} \pi_1: \ 4x - y + z = 0; & \pi_2: \ y - z = 0; \\ \pi_3: \ x + 2y - z = 4; & \pi_4: \ y - z = 1. \end{array}$$

Orient the line in the direction of positive z, and the angle in the direction from π_3 towards π_4. Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.

8. Consider line ℓ: $x - y + z = 3$, $y + z = 1$.

   (a) Find the equations of the reflection across ℓ.

   (b) Compute the image of the plane $\pi: \ x + 3y - z = 1$.

   (c) Compute the image of the line $2x + y = 0$, $z = 1$.

   Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.

9. Find the equations of the following rotations:

   (a) Of angle $\pi/3$ about line $y - 1 = x = z + 1$.

   (b) Of angle $\pi/4$ about line $z = 1$, $x - y = 2$.

   In the first case, orient the axis in the direction of positive z. In the second case, orient the axis in the direction of positive y. Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.

10. Compute the image of the following objects:

   (a) sphere $x^2 + y^2 + z^2 - 8x + 4y + 6z + 4 = 0$,

   (b) line $\frac{x+2}{-3} = \frac{y-1}{3} = \frac{z+3}{2}$,

   (c) plane $2x - 3y + 4z = 5$,

   by a central inversion with respect to point $(1, 4, -2)$. Are there many different methods that you can use?

11. Derive the equations of the rotation of angle $\pi/4$ about $\ell: x + y = 0$, $2\sqrt{6}x + \sqrt{6}y + z = 0$. Transform the sphere centered at $(5, 1, 0)$ with radius 2. Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.

12. Let Γ_0 be a spiral curve located in $z \geq 0$ on the cone $x^2 + y^2 = z^2$. The spiral orthogonally projects onto an Archimedean spiral in $z = 0$. Consider a copy Γ of Γ_0 in the following position: *(i)* Spiral Γ "starts" at point $A = (2, 3, 0)$. *(ii)* The axis of spiral Γ is a halfline ℓ through A, forming a 60° angle with the plane $z = 0$. Halfline ℓ entirely lies in the octant $x \geq 0$, $y \geq o$, $z \geq 0$, and orthogonally projects onto $z = 0$ in a line forming a 45° angle with Ox^+. Obtain a parametrization of Γ. Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.

13. Consider the sphere S centered at the origin with radius 5. Its South pole is point $(0, 0, -5)$. Describe the motion taking S to a new position in which its South pole is point $(2, 0, 0)$, and S is tangent to the plane $x - \sqrt{3}y - 2z - 2 = 0$ at such point. Parameterize the resulting sphere. Consider several methods to solve your problem (as many as you can), and compare both the results and the methods.