Geometric Tools for Computer Graphics - MIRI
Problems list 3
Year 2016-2017 Q1

1. Find a parametrization of the right circular cylinder with radius r having as axis the line through the origin with direction $(1, 1, 1)$.

2. Let C be a right circular cone with height h, semi-aperture α, and apex at the origin, lying on the plane $z = 0$ (i.e., tangent to $z = 0$), enclosed in the first octant, and such that the line segment connecting its apex with the center point of its basis orthogonally projects onto $z = 0$ in the line $x = y, z = 0$. Obtain a parametrization of the cone in the standard coordinate system.

3. Obtain a parametrization of the circular cone of height h, semi-aperture α, and apex O, located in the first octant, tangent to the coordinate axis Oz along a generatrix, and such that the axis of the cone orthogonally projects onto the plane $z = 0$ in the line $x = y$, $z = 0$.

4. Consider the plane $\pi : x + y + z - 3 = 0$, and let A, B and C respectively be the intersection points of π with the coordinate axes Ox, Oy and Oz. Let P be the midpoint of segment AC, let Q be the midpoint of segment AB. Consider the cylinder C of radius r lying on the plane. Suppose that the contact generatrix coincides with the line PQ. Among the two half-spaces defined by π, C lies in the one not containing the origin. Find a parametrization of the portion of C limited by two bases respectively containing points P and Q.

5. Obtain a parametrization of the union of all lines parallel to the plane $z = 0$ which intersect axis Oz and line $\ell : x = (1, 1, 0) + \lambda(1, 1, 1)$, $\lambda \in \mathbb{R}$.

6. An *helicoid* is the trace of a moving line, touching a circular helix and its axis, and parallel to a plane perpendicular to the axis of the helix. Assume, for a start, that the axis of the helix is the coordinate axis Oy and that the helix is right-handed.

   (a) Give a parametrization of the helicoid.

   (b) Obtain the implicit equation of the helicoid.

   (c) Find a parametrization of the helicoid corresponding to a right-handed helix with axis Ox, radius 3 and pitch 2.

7. A *generalized cylindrical surface* is created by a moving line (*generatrix*) intersecting a given curve (*directrix*), and parallel to a given line or direction. Second order cylinders and planes are examples of such surfaces.

   (a) Find the directrix and the direction of the cylindrical surface whose equation is $y^2 = 2z + 3$.

   (b) If a parametrization of the directrix curve is known, obtain a parametrization for the cylindrical surface.

   (c) Obtain an equation for the cylindrical surface whose directrix is

   $$\begin{cases} x = t + 1 \\ y = t^2 \\ z = t \end{cases}$$

   and whose generatrices are parallel to the line $x = 2z, y = z$.

8. A *cylindroid* is the surface created by a moving line (*generatrix*), intersecting two fixed given curves (*directrices*), while parallel to a fix given plane (*director*). Supposing that the two directrix curves are parameterized, obtain a parametrization of the cylindroid. Find the equation of the cylindroid determined by the curves $x = \sin z$, $y = 0$, and $(2t, 1 - t^2, t^3)$, if the director plane is $z = 0$.

9. A *generalized conical surface* is created by a moving line (*generatrix*) through a fix point (*apex*) and a point in a fixed given curve (*directrix*).

   (a) Describe a method to obtain a parametrization of a conical surface from a parametrization of its directrix.

   (b) Obtain an equation of the conical surface having apex $(1, 0, -3)$ and directrix

$$\begin{cases} x = 1 + t, \\ y = t^2, \\ z = 0. \end{cases}$$

   (c) Find an equation for the conical surface with apex O and directrix $x^2 = y, x + z - 1 = 0$.

10. A *conoid* is the surface created by a line (*generatrix*) when it moves touching a fixed curve (*directrix*) and a fixed line (*axis*), while staying parallel to a fixed plane (*director*).

    (a) Describe a method to obtain a parametrization of the portion of a conoid created by the line segments determined by its axis and its directrix.

    (b) Is an helicoid a particular case of conoid?

    (c) Parameterize the conoid with director plane xy, axis Oz, and directrix $x = y + 1 = z$.

    (d) Parameterize the conoid with director plane xy, axis Oz, and directrix $x^2 + z^2 = 4$, $y = 3$.

    (e) Parameterize the conoid with director plane yz, axis $x + z = 1$, $y = 0$ and directrix $y^2 = x$, $z = 0$.

    (f) Parameterize the conoid created by a line parallel to the plane xz, intersecting the ellipse $y^2 + 4z^2 = 4, x = 0$ and the line $x = 3, z = 0$.

11. Consider the curve parameterized as $r(t) = (e^t \cos t, e^t \sin t, e^t)$ $t \in [0, \pi]$. Determine the implicit equation of a surface containing the curve and make a schematic drawing of both the surface and the curve.

12. Find an equation of the surface created by rotating the line

$$\begin{cases} x + y - 2z + 4 = 0, \\ x - y - 2 = 0, \end{cases}$$

    about the axis Oz.

13. Determine an equation of the surface created by rotating the line

$$\begin{cases} x = t - 2, \\ y = t - 1, \\ z = 2t + 1, \end{cases}$$

    about the axis Oy.

14. Find an equation of the torus created by rotating the circle

$$\begin{cases} x^2 - 4x + y^2 + 3 = 0, \\ z = 0, \end{cases}$$

about the axis Oy.

15. Parameterize the surface created by rotating about the axis Oy the curve of the plane xy whose polar equation is $r(t) = \pi/2 - t$, on $-\pi/2 < t < \pi/2$.

16. Consider the first coil of the spiral curve with polar equation $r = \theta$ in the plane xy. Parameterize the surface it creates by rotation about the line $x = y + 9$, $z = 0$.

17. Parameterize the surface created by rotation about the line $x = y = z$ of the curve

$$\left.\begin{array}{l} x(t) = \cos t + \sin t \\ y(t) = \cos t + \sin t \\ z(t) = -2\cos t + \sin t \end{array}\right\}, \qquad t \in [0, \pi/2].$$

18. Find the implicit equation of the parametric curve

$$\begin{cases} x = 4\sin^2 t \\ y = 2\cos t \\ z = 2\sin t. \end{cases}$$

19. Write the curve

$$\left.\begin{array}{l} x = \sin t \\ y = t \\ z = 1 - \cos t \end{array}\right\} \quad t \in [0, 2\pi]$$

as the intersection of two surfaces, and give an idea of their graphical representation.

20. The curve

$$\left.\begin{array}{l} x = 4\cos t \\ y = 4\sin t \\ z = 4\cos t \end{array}\right\} \quad t \in [0, 2\pi]$$

is plane. Find its supporting plane.

21. Can we write a circular helix as the intersection of two surfaces? Which?

22. Parameterize the intersection of the sphere

$$x^2 + y^2 + z^2 - 2x - 2y - 4z - 4 = 0.$$

And the plane $x + y - z = 1$. Hint: vertically project the curve onto the plane $z = 0$.

23. *Conic sections.* Consider the right circular cone $x^2 + y^2 = a^2z^2$, where $a > 0$.

   (a) Prove that the horizontal section of the cone by any plane $z = k$ is a circle (if $k = 0$, the circle degenerates into a point).

   (b) Prove that the vertical section of the cone by any plane $y = k$ is an hyperbola (if $k = 0$, the hyperbola degenerates into a pair of crossing lines). What is the curve obtained by any other vertical section of the cone? Why?

   (c) Prove that the oblique section of the cone by a plane $y = az + k$ is a parabola (if $k = 0$, the parabola degenerates into a line). Which is the result if the section is caused by any other plane parallel to a generatrix of the cone? Why?

(d) Prove that any other oblique section of the cone, i.e., any section by a plane $y = bz + k$, with $b \neq a$ and $b \neq 0$, is an ellipse (if $b > a$), or an hyperbola (if $0 < b < a$) (if $k = 0$, it degenerates into a point). What about the section caused by any other oblique planes?

24. Parameterize the surface of revolution created by rotation about the axis Oy of the intersection curve of the paraboloid $z = x^2 + y^2$ and the plane $z = y + 2$.

25. Consider the cone with apex O, axis Oz, and semi-aperture $\alpha = \pi/6$.

    (a) Find its implicit equation.

    (b) Parameterize the intersection of the cone with the plane $z = y + 3$. What is the perpendicular projection of this curve onto the plane xy?

    (c) Parameterize the portion of the cone limited by planes $z = 0$ and $z = y + 3$. Let C_1 be the resulting surface. Its apex is point $(0, 0, 0)$, its axis is Oz, and its basis is the curve described in question b.

    (d) Parameterize an identical copy C_2 of the surface C_1, located in space as follows:

        i. The basis of C_2 lies in the plane $x + y - z = 0$;

        ii. The axis of C_2 contains point $(0, 0, 0)$;

        iii. The third coordinate of the apex of C_2 is positive.

26. Constructing a torus from a circle. Parameterize the circle Γ intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y = 1$. Find the line tangent to Γ at a generic point of the curve, in terms of the parameter used in the previous parametrization. Parameterize now the circle of radius r centered at a generic point of the curve, perpendicular to Γ at its center.

27. Let (a, b, c) be a point of the paraboloid $z = x^2 + y^2$.

    (a) Prove that the following is the equation of the plane tangent to the paraboloid at the given point: $z = 2ax + 2by - a^2 - b^2$.

    (b) Prove that if we vertically translate the plane by r^2, then the plane intersects the paraboloid in a curve that projects orthogonally onto the plane $z = 0$ in a circle. Which are its center and radius?

28. From point $P = (2, 1, 5)$ a ray is sent in the direction given by vector $u = (0, 0, -1)$. The ray reflects onto the surface $z = \frac{x^2}{6} + \frac{y^2}{3}$. Compute the direction of the reflected ray.