Geometric Tools for Computer Graphics - MIRI
Problems list 2
Year 2016-2017 Q1

1. Propose parameterizations for circular helices with radius a and pitch b having the following axes:

   (a) Axis parallel to Oz, through point $(1, 1, 0)$.

   (b) Axis parallel to Ox, through point $(0, 1, 2)$.

   (c) Axis through point $P_0 = (1, 0, 0)$ with direction $\vec{u} = (1, 1, 1)$.

   How many degrees of freedom do you have for the solutions?

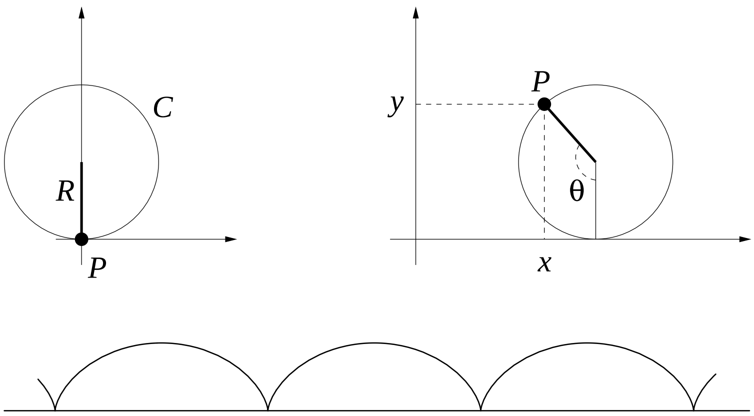
2. Let Γ be the six coils Archimedean spiral with polar equation $r = \theta$. We want to locate a copy of Γ in the plane through point $P = (1, 1, 0)$ and perpendicular to the line through P with direction $v = (1, 1, 1)$. The curve must be positioned as to have its origin Q on the line, and to be tangent in Q to the direction $t = (-1, 1, 0)$. Is this possible? Is the solution unique? Propose parameterizations for all possible solutions.

3. Consider the parameterized line

$$\ell : \begin{cases} x = 1 + \lambda, \\ y = 1 + 2\lambda, \\ z = 1 + \lambda. \end{cases}$$

   We want to draw a circle perpendicular to ℓ, with radius r, centered in a point $C \in \ell$ corresponding to a given value λ_0 of the parameter. Obtain a parametrization of the circle that can be used to produce the desired drawing. What happens as we continuously vary the value of λ_0?

4. A *cycloid* is a curve described by a point P of a circle C, when C rolls over a line ℓ.



   (a) Take the line $y = 0$ as ℓ, and let C have radius r. Suppose that when the parameter θ equals 0, C is centered at $x = 0$, and P coincides with the origin. Prove that when C rolls over ℓ, point P describes a curve which can be parameterized as follows:

$$\left.\begin{array}{l} x(\theta) = r(\theta - \sin\theta) \\ y(\theta) = r(1 - \cos\theta) \end{array}\right\}, \text{ with } \theta \in \mathbb{R}.$$

   (b) Parameterize the cycloid produced a circle of radius 1 rolling over a line of slope 1, starting at point $P = (2, 1)$.

5. Let C be a fixed circle centered at $(0, a)$, tangent to the axis Ox at O. For each line ℓ through O, let Q be the intersection point of ℓ and C, and let A be the intersection point of ℓ and the line parallel to Ox and tangent to C. Finally, let P be the intersection point of the vertical line through A and the horizontal line through Q. The *locus* of all points P obtained through this procedure from all possible lines ℓ is called *Witch of Agnesi (versiera di Agnesi)*.

   (a) Using the polar angle θ of line ℓ, verify that the following is a parametrization of the curve:
   $$\left.\begin{aligned} x &= 2a\frac{\cos\theta}{\sin\theta} \\ y &= 2a\sin^2\theta \end{aligned}\right\}, \quad \theta \in [0, \pi].$$

   (b) Obtain the implicit equation of the curve.

   (c) Prove that point $P = (2a, a)$ belongs to the curve. Which value of the parameter θ corresponds to it? Prove that the tangent vector to the curve at point P, is not perpendicular to the position vector of P.

   (d) Compute the position of the the center of a circle of radius 3 tangent to the curve at point P, and located above the curve.

   (e) Give a parametrization of the *Witch* associated to a circle of radius 2 tangent to line $x + 3y = 1$ at point $(4, -1)$, and located above the line.

6. Consider a fixed circle centered at $(a/2, 0)$, and tangent to the axis Oy at the origin, O. For each line ℓ through O, let C be the intersection point of ℓ and the line parallel to Oy and tangent to the circle, and let B be the intersection point of ℓ and the circle. Finally, let P be the point of line ℓ such that $d(O, P) = d(O, C) - d(O, B)$. The *locus* of all points P that can be obtained this way from all possible lines ℓ is known as *cissoid of Diocles*. Point O is its apex.

   (a) Use the polar angle θ of line ℓ to verify that the following is a parametrization of the curve:
   $$\left.\begin{aligned} x &= a\sin^2\theta \\ y &= a\frac{\sin^3\theta}{\cos\theta} \end{aligned}\right\}, \quad \theta \in [-\pi/2, \pi/2].$$

   (b) Obtain the implicit equation of the curve.

   (c) Prove that point $P = (a/2, a/2)$ belongs to the curve, and give its corresponding parameter. Prove that the tangent vector to the curve at point P, is not perpendicular to the position vector of P.

   (d) Compute the position of the center of a circle of radius 5 tangent to the curve at point P, and located above the curve.

   (e) Give a parametrization of the *Cissoid* associated to a circle of radius 4 tangent to line $3x + y = 3$ with apex $(2, -3)$, and located above the line.

7. Compute the Frechet trihedron of $\gamma(t) = (t, t^2, t^3)$ at point $(1, 1, 1)$. Suggestion: use your brain to figure out what needs to be done, and use a computer to do it!

8. Let curve Γ_1 have polar equation $r = 2(3 + \cos 2\theta)$, with $\theta \in [0, 2\pi]$.

   (a) Give a parametrization of Γ_1.

   (b) Obtain a parametrization of Γ_2, which is an identical copy of Γ_1, located in the half-plane $y \geq 0$, going through the origin, and tangent to the axis of abscissae through its point P, corresponding to the value $\theta = \pi/4$ of the parameter.

(c) Obtain a parametrization of Γ_3, which is an identical copy of Γ_1 and Γ_2, positioned as to satisfy the following properties:

i. Γ_3 has point P in the half-line $x = y$, $x \geq 0$;

ii. Γ_3 is tangent to the line of slope -1 whose distance to the origin equals 2;

iii. Γ_3 belongs to the upper half-plane defined by the previous line.

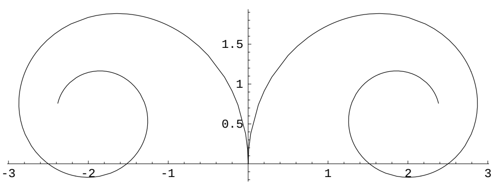
9. Consider the following plane curve in polar form:

$$r(t) = \frac{2\pi}{t}, \qquad t \in [\pi, 4\pi].$$

(a) Find the smallest circle centered at the origin and enclosing the curve, as well as the largest circle, also centered at the origin, not containing any point of the curve.

(b) Compute the general expression of the tangent vector to the curve in an arbitrary point. Compute it in the points corresponding to the following values of the polar angle: $t = \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}$.

(c) Using the previous curve, obtain a parametrization of the curve that you can see in the figure:



10. Let $r(\theta) = a(1 + \cos\theta)$, with $\theta \in [-\pi, \pi]$ be the polar equation of a cardioid.

(a) Prove that this curve is symmetrical with respect to the axis of abscissae.

(b) Obtain a parametrization of the curve.

(c) How can we compute the values of θ for the points of minimum abscissa of the curve? Compute them and verify that the value of the minimum abscissa is $x = -a/4$.

(d) Let ℓ be the line through points $P = (1, 0)$ and $Q = (0, 3)$. We want to place an identical copy of our cardioid such that:

i. It is tangent to line ℓ in the points mentioned in the previous question;

ii. Among the two half-planes defined by ℓ, it is in the one not containing the origin;

iii. Its vertex perpendicularly projects onto ℓ in the midpoint of P and Q.

Propose a parametrization of the resulting curve.

11. Consider the curve parameterized by the function $\gamma : [0, \pi] \longrightarrow \mathbb{R}^2$, defined as follows:

$$\gamma(t) = (x(t), y(t)) = (5\cos(3t)\cos t, 5\cos(3t)\sin t).$$

(a) Compute the distance to the origin of an arbitrary point of the curve. Which is the maximum value of such distances? Find the points of the curve achieving this maximum value.

(b) Compute the vector tangent to the curve at an arbitrary point. Verify that the tangent vector at the point corresponding to parameter $t = 0$ is linearly dependent from the tangent vector to the circle of radius 5 centered at the origin, at point $(5, 0)$.

(c) Find a parametrization of a copy of the given curve, placed in a disc of radius 5 centered at point $(-1, 3)$, so that both curves are tangent at point $(2, -1)$.