Geometric Tools for Computer Graphics - MIRI
Problems list 1
Year 2016-2017 Q1

1. Parameterize:
   (a) The line through point $P = (2, 3)$ with direction $\vec{u} = (1, -2)$.
   (b) The line through points $P = (1, -2)$ and $Q = (3, 7)$.
   (c) The line of equation $2x - 3y + 7 = 0$.

2. Given two points P and Q in dimension 2 or 3, prove that the expression

   $$X = (1 - t)P + tQ$$

   is a parametrization of

   - the line through P and Q, if $t \in \mathbb{R}$;
   - the halfline through Q with endpoint in P, if $t \in [0, +\infty)$;
   - the segment $\overline{PQ}$, if $t \in [0, 1]$.

   How can we compute the midpoint of P and Q?

3. Given two points P and Q in dimension 2, find a parametrization of the perpendicular bisector of segment $\overline{PQ}$. What happens in dimension 3?

4. Find the line through point $Q = (1, 1, 0)$ lying in the plane $\pi : \ 2x - y + z - 1 = 0$ and intersecting line $s : \begin{cases} x = 2 - t \\ y = 2 + t \\ z = t. \end{cases}$

5. Find an equation of the line intersecting lines r and s and parallel to line t:

   $$r : \begin{cases} x + y = -1 \\ 2x + z = 0 \end{cases} \qquad s : \begin{cases} x + 3y - z = 1 \\ y + z = 2 \end{cases} \qquad t : \frac{x-1}{2} = \frac{y}{-1} = \frac{z-1}{3}.$$

6. Prove the following facts:
   (a) The diagonals of a parallelogram mutually intersect in their midpoints.
   (b) The four diagonals of a parallelepiped intersect in one point and that they mutually bisect each other.
   (c) The segments connecting one vertex of a parallelogram with the midpoints of its opposite edges trisect the diagonal of the parallelogram not containing the initial vertex.

7. Centers of a triangle
   (a) A median of a triangle is any line connecting one of its vertices with the midpoint of the opposite edge. Prove that the three medians of any triangle intersect in a point. This point is called centroid of the triangle. Find its coordinates as a function of the coordinates of the vertices of the triangle.
   (b) Compute the intersection if two internal angular bisectors of a triangle. Use the result to prove that all three angular bisectors of a triangle intersect in a point (called incenter of the triangle).

(c) How would you compute the circumcenter of a triangle? Does the circumcircle coincide with the minimum enclosing circle?

(d) Did you ever heard of the ortocenter of a triangle? Where is it located?

8. Prove Pythagoras' theorem:

$$\vec{u} \perp \vec{v} \Leftrightarrow \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2.$$

Prove the cosinus theorem: for any triangle with edges a, b, c,

$$c^2 = a^2 + b^2 - 2ab \cos \alpha,$$

where α es the angle formed by edges a and b.

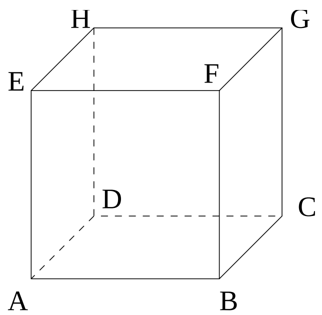
9. Use the fact that the angle inscribed in a semicircle is $\pi/2$, to compute an equation of a sphere if a diametral pair of points is given.

10. A few curious angles

(a) Prove that in any regular tetrahedron, any two opposite edges are perpendicular.

(b) From one vertex of a square, draw two lines bisecting its opposite edges. Compute the angle they form.

(c) For a cube like the one in the figure, compute the angle between diagonal AG and edge AE.



11. (a) Given a point $C = (a, b, c)$ and a positive real number r, the sphere centered at C with radius r is the geometric locus of all points in space at distance r from C. Obtain the equation of the sphere.

(b) Given a point $C = (a, b)$ in the plane and a positive real number r, the circle centered at C with radius r is the geometric locus of all points in the plane at distance r from C. Obtain the equation of the circle. Given a point $C = (a, b, c)$ and a positive real number r, which is the equation of the circle centered at C with radius r in the plane $z = c$?

(c) Given a positive real number r, the circular straight cylinder having axis Oz and radius r is the geometric locus of all points in space at distance r from Oz. Obtain the equation of this cylinder. Which is the equation of the cylinder of the same radius, whose axis is parallel to Oz and goes through point $C = (a, b, c)$?

12. Prove the parallelogram law: the summation of the squared diagonals equals twice the summation of the squared sides.

13. Some 2D CAD software offers, as incorporated functions, several possible constructions, such as the following:

(a) drawing a circle through tree points (i.e., the circumcircle of a given triangle),

(b) drawing the incircle of a given triangle.

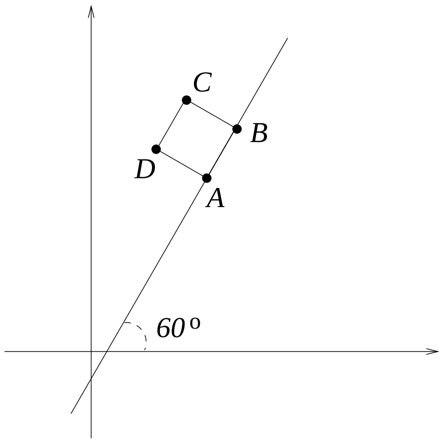
Do an analytical study of the problem, detect all possible degeneracies, and come up with the equations needed to implement these graphical options in a program.

14. Given a tetrahedron $ABCD$, consider the coordinate system $S = (A; \{\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\})$.

    (a) Write in system S the equations of the planes of the faces of the tetrahedron.

    (b) Consider now the coordinate system $S_0 = (O; \{\vec{e}_1, \vec{e}_2, \vec{e}_3\})$, and suppose that, in this system, the given vertices are $A = (3, 3, 4)$, $B = (1, 7, 0)$, $C = (4, 5, 0)$, and $D = (2, 2, 0)$. Use the previous result to obtain the equations of the planes of the faces in system S_0.

15. Consider the cartesian coordinate system $S = (O; \{\vec{e}_1, \vec{e}_2\})$ and the unit square with vertices $ABCD$ (given in counterclockwise order along the boundary), where $A = (2, 3)$ and AB forms a $60°$ angle with the positive x-axis. Use the formulae to change coordinates, and obtain the coordinates in S of the vertices of the given square.



16. Consider a truncated elliptic cylinder, with axis Oz, semi-axes a and b, and height h, standing on the plane $z = 0$. How can an identical copy of it be placed on the plane $x + 2y + z = 4$, centered at point $(1, 1, 1)$?

17. Obtain the truncated straight circular cylinder with radius R and height h, standing on one of its circular basis on the plane $x + y + z - 3 = 0$, and lying in the half-space opposite to the origin. Its axis, which must be perpendicular to the plane, should contain point $(1, 1, 1)$.