Real-valued symmetric matrices always diagonalize

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**Definition 1** The sum of two subspaces \( W_1 \) and \( W_2 \) of a vector space \( V \) is defined as
\[
W = W_1 + W_2 = \{ w \in V \mid w = w_1 + w_2, w_1 \in W_1, w_2 \in W_2 \}.
\]
If \( W_1 \cap W_2 = \{0\} \), the sum is called direct, and we write \( W_1 \oplus W_2 \).

**Lemma 1** The sum \( W_1 + W_2 \) of two subspaces of a vector space \( V \) is a subspace of \( V \).

**Proof:** Immediate. □

**Definition 2** Let \( W \) be a subspace of an Euclidean vector space \( V \). The subset of \( V \) orthogonal to \( W \) is defined as
\[
W^\perp = \{ v \in V \mid v \perp w \forall w \in W \}.
\]

**Lemma 2** If \( W \) is a subspace of \( V \), then the set \( W^\perp \) is a subspace of \( V \).

**Proof:** Immediate. □

**Theorem 3** If \( W \) is a subspace of an Euclidean vector space \( V \), then \( V = W \oplus W^\perp \).

**Proof:** The fact that \( W \cap W^\perp = \{0\} \) is easy to prove: if \( u \in W \cap W^\perp \) then \( u \in W \) and \( u \cdot w = 0 \) for all \( w \in W \). In particular, then, \( u \cdot u = 0 \) and we get \( u = 0 \). Therefore, the sum of \( W \) and \( W^\perp \) is direct. Trivially, \( W \oplus W^\perp \subseteq V \). In order to prove that \( V \subseteq W \oplus W^\perp \), consider \( w_1, \ldots, w_r \) an orthonormal basis of \( W \) which can be obtained using Gram-Schmidt method, for example. Let \( w_{r+1}, \ldots, w_n \) be its completion to an orthonormal basis of \( V \), which can also be obtained using Gram-Schmidt method. It is immediate to prove that \( w_{r+1}, \ldots, w_n \) is a basis of \( W^\perp \) due to the orthonormality of the basis \( w_1, \ldots, w_n \). Therefore, every vector \( x \in V \) can be written as
\[
x = \sum_{i=1}^n x_i w_i = \sum_{i=1}^r x_i w_i + \sum_{i=r+1}^n x_i w_i \in W \oplus W^\perp.
\]

□

**Lemma 4** Let \( f \) be an endomorphism in an Euclidean vector space \( V \) whose associated matrix in some orthonormal basis is symmetric. Then \( f(u) \cdot v = u \cdot f(v) \forall u, v \in V \).

**Proof:** Let \( e_1, \ldots, e_n \) be the orthonormal basis in which \( f \) is represented by the symmetric matrix \( A = (a_{ij}) \). Due to the bilinearity of the dot product, we only need to prove that \( f(e_i) \cdot e_j = e_i \cdot f(e_j) \forall i, j \in \{1, \ldots, n\} \):
\[
f(e_i) \cdot e_j = \left( \sum_{k=1}^n a_{ik}^j w_k \right) \cdot w_j = \sum_{k=1}^n a_{ik}^j \delta_{k,j} = a_{ij}^j
\]
\[
e_i \cdot f(e_j) = e_i \cdot \left( \sum_{k=1}^n a_{ik}^j w_k \right) = \sum_{k=1}^n a_{ik}^j \delta_{i,k} = a_{ij}^j
\]

Since \( A \) is symmetric, we obtain \( f(e_i) \cdot e_j = a_{ij}^j = a_{ji}^j = e_i \cdot f(e_j) \forall i, j \in \{1, \ldots, n\} \). □
**Proposition 5** Let $A$ be a symmetric matrix of size $n$ with real coefficients, and let $\lambda_1, \ldots, \lambda_r$ be its eigenvalues, with multiplicities $k_1, \ldots, k_r$, respectively. Then $\lambda_i \in \mathbb{R}$ for all $i \in \{1, \ldots, r\}$ and $\sum_{i=1}^r k_i = n$.

**Proof:** Let us consider $A \in M_n(\mathbb{R}) \subset M_n(\mathbb{C})$. The Fundamental Theorem of Algebra guarantees that the characteristic polynomial of $A$ has complex roots $\lambda_1, \ldots, \lambda_r$, with multiplicities $k_1, \ldots, k_r$ respectively, and $\sum_{i=1}^r k_i = n$. We will prove that $\lambda_i \in \mathbb{R}$ for all $i \in \{1, \ldots, r\}$. Indeed, if $z = (z_1, \ldots, z_n)$ is a (complex) eigenvector of eigenvalue $\lambda$, we have $Az = \lambda z$. Since the coefficients of $A$ are real numbers, when we conjugate the previous equality we obtain that $\overline{\lambda} z = A \overline{z} = \overline{A} \overline{z}$. In other words, $\overline{z}$ is an eigenvector of $A$ with eigenvalue $\overline{\lambda}$. Then,

$$\lambda|z| = \lambda \overline{z}^T z = \overline{z}^T A \overline{z} = \overline{z}^T A z = (z^T A \overline{z})^T = z^T \overline{A} \overline{z} = \overline{z}^T \overline{A} \overline{z} = \overline{x}^T \overline{x} = \overline{\lambda}|z|,$$

where the starred equality holds because $\lambda|z| \in \mathbb{R}$. Hence, $\lambda = \overline{\lambda}$, and $\lambda \in \mathbb{R}$. \hfill $\square$

**Theorem 6** Let $V$ be an Euclidean real vector space $V$, and let $A$ be a symmetric matrix. Then $V$ admits a basis of orthonormal eigenvectors of $A$.

**Proof:** By induction over the dimension of $V$, denoted $n$. The base case corresponds to $n = 1$ and is immediate: each non null vector is an eigenvector of $A$ and can be normalized.

The induction step is proved as follows: let $\lambda \in \mathbb{R}$ be an eigenvalue of $A$, and let $v$ be a unit eigenvector for $\lambda$. We know that $< v > \oplus < v >^\perp = V$. We will prove that $< v >^\perp$ is invariant under the endomorphism $f$ associated to $A$. If $u \in < v >^\perp$, then $u \cdot v = 0$. As a consequence, $f(u) \cdot v = u \cdot f(v) = u \cdot \lambda v = \lambda u \cdot v = 0$. This proves that $f(u) \in < v >^\perp$.

By inductive hypothesis, $< v >^\perp$ has an orthonormal basis made of eigenvectors of $f$ restricted to $< v >^\perp$. Adding $v$ to this basis we obtain an orthonormal basis made of eigenvectors of $f$. \hfill $\square$

**Corollary 7** Let $V$ be an Euclidean real vector space $V$, and let $A$ be a symmetric matrix. Then $A$ diagonalizes in orthonormal basis.

**Proof:** Immediate from Theorem 6. \hfill $\square$