Dealing with numbers, points, and lines

Vera Sacristán

Geometric Tools for Computer Graphics
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

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$\mathbb{N}$ and $\mathbb{Z}$ are easy to deal with.

$\mathbb{Q}$ is $\mathbb{Z} \times \mathbb{Z}/\sim$, where $(a, b) \sim (a', b') \iff ab' = a'b$.

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DEALING WITH NUMBERS, POINTS, AND LINES

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Recommended library

The Computational Geometry Algorithms Library: http://www.cgal.org/

Homogeneous coordinates

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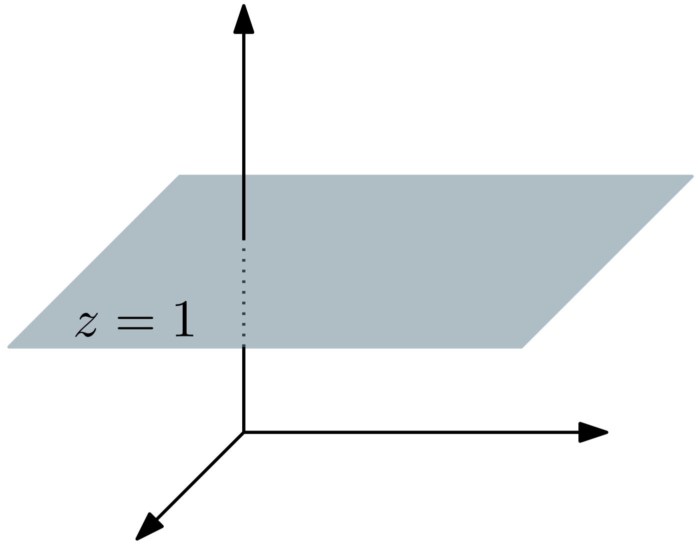
Embed the plane in 3-dimensional space as plane $z = 1$.



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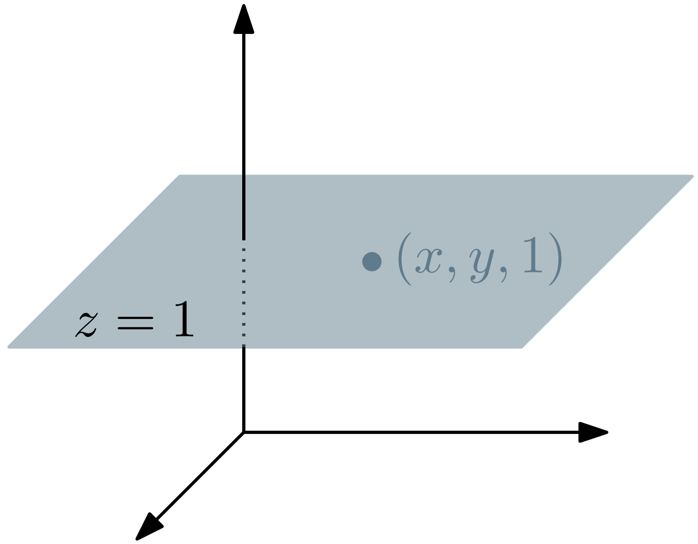

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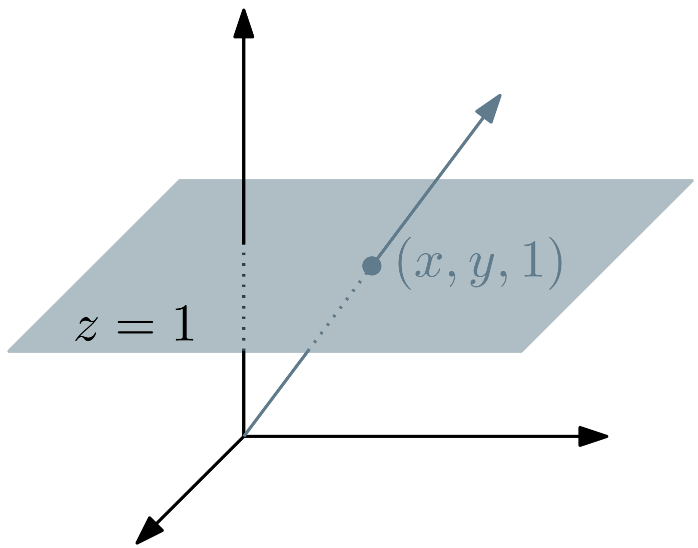
Then identify point $(x, y, 1)$ with all the points in the ray connecting it to the origin:

$(x, y) \sim (x, y, 1) \sim (\lambda x : \lambda y : \lambda)$, where $\lambda > 0$.



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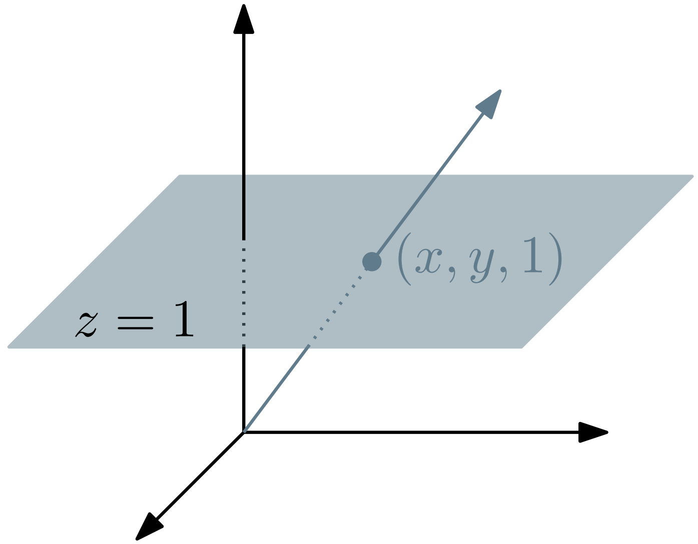

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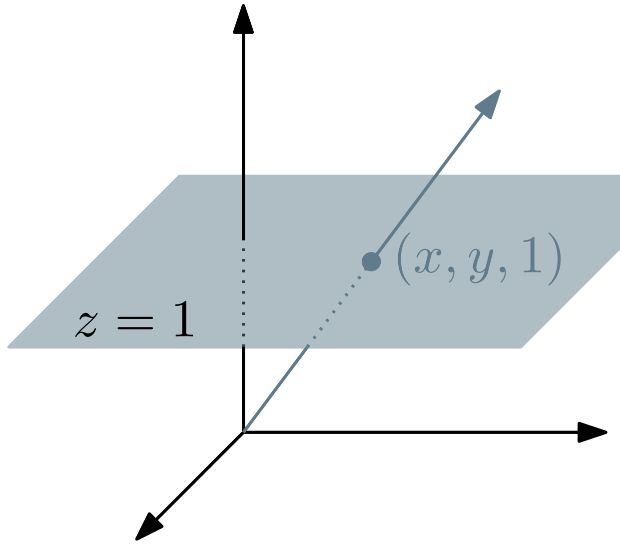

Welcome to **projective geometry**!

What about vectors?

- Vectors have coordinates $(x : y : 0)$
  because they are parallel to the plane $z = 1$.
- They correspond to horizontal rays, i.e., to points at infinity.

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Lines:

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Line through two points

$$\left.\begin{array}{l} p \text{ lies in } \ell \implies p \perp \ell \\ q \text{ lies in } \ell \implies q \perp \ell \end{array}\right\} \implies \ell = p \times q$$

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Intersecting two lines

$$\left.\begin{array}{l} p \text{ lies in } \ell_1 \implies p \perp \ell_1 \\ p \text{ lies in } \ell_2 \implies p \perp \ell_2 \end{array}\right\} \implies p = \ell_1 \times \ell_2$$

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Parallel lines

$$\ell_1 \| \ell_2 \iff \left\{ \begin{array}{l} \ell_1 = (a, b, c) \\ \ell_2 = (\lambda a, \lambda b, d) \end{array} \right\} \iff \ell_1 \times \ell_2 = (x, y, 0)$$

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Identical lines (or points)

$$\ell_1 = \ell_2 \Longleftrightarrow \left\{ \begin{array}{l} \ell_1 = (a, b, c) \\ \ell_2 = (\lambda a, \lambda b, \lambda c) \end{array} \right\} \Longleftrightarrow \ell_1 \times \ell_2 = (0, 0, 0)$$

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If you implement the cross product, you get a code that is:

- **Correct:** it computes exactly what it should.
- **Reusable:** it is used to solve several (apparently different) problems.
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There is no way you can do the same using affine coordinates!