Geometric Tools for Computer Graphics - MIRI

Introduction to conics

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Conics are all around us

Figura 1: These images show conic curves (ellipses, parabolas and hyperbolae). They also show some cylinders and surfaces of revolution which use conics as generatrices and directrices.

As illustrated by the images in Figure 1, these curves don't only appear in nature, but also in human manufactured objects and buildings. This is due to the many convenient geometric properties conics have.

Focal properties of conics

- An ellipse can be defined as the locus of all points X in the plane such that $d(X, F_1) + d(X, F_2) = k$, where F_1, F_2 (called focus of the ellipse) are two points in the plane, and k is any real number greater than the Euclidean distance between them $k > d(F_1, F_2)$. Notice that a circle of radius $r > 0$ is an ellipse such that $F_1 = F_2$.

- A hyperbola can be defined as the locus of all points X in the plane such that $|d(X, F_1) - d(X, F_2)| = k$, where F_1, F_2 (called focus of the hyperbola) are two points in the plane, and k is any positive real number smaller than the Euclidean distance between them $0 < k < d(F_1, F_2)$.
- A parabola can be defined as the locus of all points X in the plane such that $d(X, F) = d(X, \ell)$, where F (called focus of the parabola) is a point in the plane, and ℓ is a line (called directrix of the parabola) not containing F.

These properties are illustrated in Figure 2.

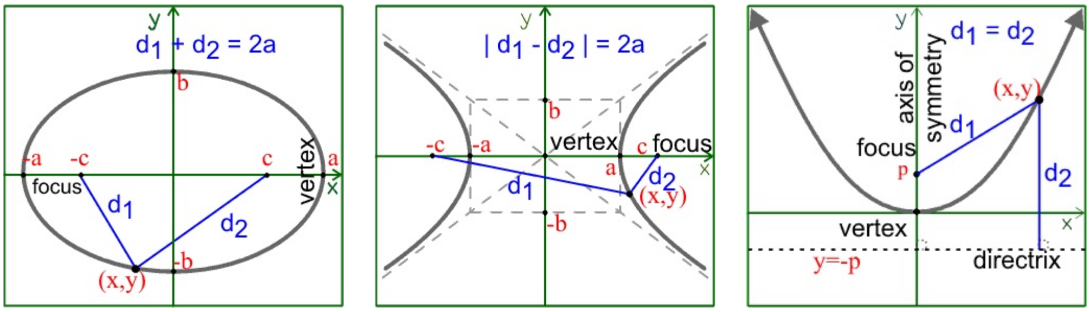

Figura 2: Focal properties of conics.

Reflective properties

- Any ray sent from one of the focus of an ellipse gets reflected by the curve into a ray towards the other focus.
- Any ray sent towards one of the focus of a hyperbola gets reflected by the curve into a ray towards the other focus.
- Any ray sent perpendicularly to the directrix of a parabola gets reflected by the curve into a ray towards its focus.

These properties (illustrated in Figure 3) lie underneath the use of conics in parabolic antennas, solar heat pipes, lithotripsy machines, electric heaters,...

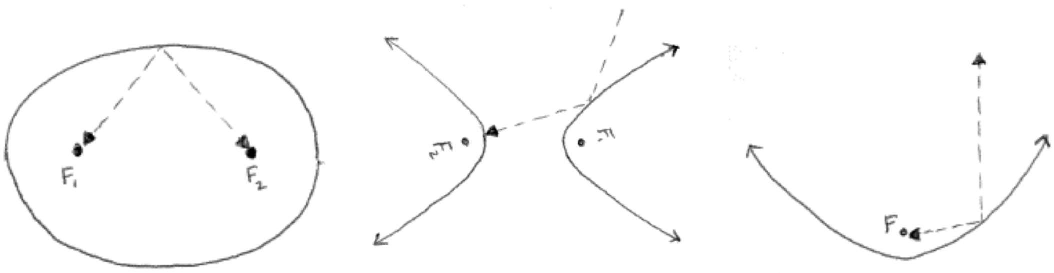

Figura 3: Reflective properties of conics.

Eccentricity properties

Any conic can be defined as the locus of all points X in the plane such that $\frac{d(X,F)}{d(X,\ell)} = k$, where F (called focus) is a point in the plane, ℓ (called directrix) is a line not containing F, and k os a positive real number. This property is illustrated in Figure 4. Depending of the value of k, different conics are obtained, namely:

- When $k < 1$, the conic is an ellipse.
- When $k = 1$, the conic is a parabola.
- When $k > 1$, the conic is a hyperbola.

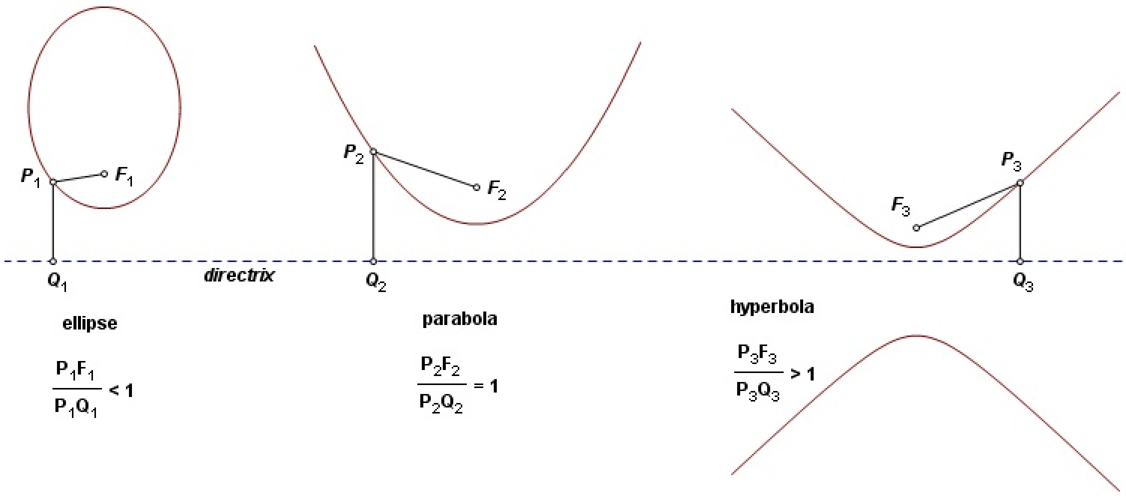

Figura 4: Eccentricity properties of conics.

Conics are plane sections of circular cones

Let C be a right circular cone with axis Oz, apex at the origin, and semi-aperture angle α. Consider a plane with normal vector $\overrightarrow{n}$. The intersection C with the plane is:

- An ellipse if $\overrightarrow{n}$ forms with the axis an angle smaller than α (a circle if the angle is 0, a point if the plane contains the apex).
- A parabola if $\overrightarrow{n}$ forms with the axis an angle equal to α (a line if the plane contains the apex).
- A hyperbola if $\overrightarrow{n}$ forms with the axis an angle greater than α (two intersecting lines if the plane contains the apex).

These possibilities are illustrated in Figure 5.

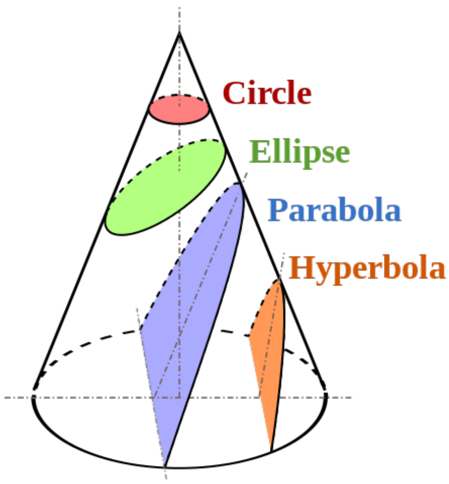

Figura 5: Conics are planar sections of circular right cones.

Conics are described by a polynomial equation of degree two

These are the standard forms of the conics equations:

- Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- Parabola: $y^2 = 4ax$.

The geometric meaning of the parameters in the equations are illustrated in Figure 6.

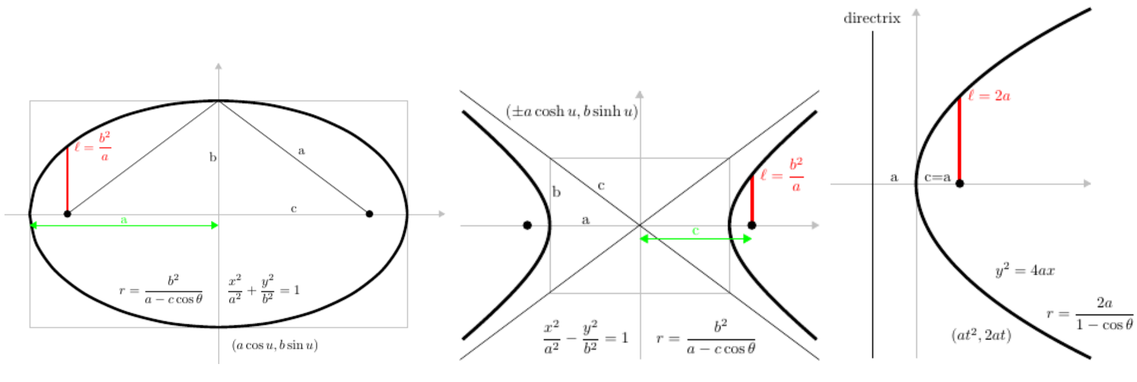

Figura 6: Standard form of conics equations.

Any polynomial equation of degree 2 describes a conic

Degree two polynomial equations have the following expression:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

All these equations can be reduce to one of the following:

Conic	**Implicit equation**	**Parametric equation**
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$(a\cos t, b\sin t),\ t \in [0, 2\pi]$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$(\pm a\cosh t, b\sinh t),\ t \in \mathbb{R}$
Empty set	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	
One point	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	$(0, 0)$
Two intersecting lines	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	$(at, \pm bt),\ t \in \mathbb{R}$
Parabola	$\frac{x^2}{a^2} - y = 0$	$(a^2 t, t^2),\ t \in \mathbb{R}$
Two parallel lines	$\frac{x^2}{a^2} = 1$	$(\pm a, t),\ t \in \mathbb{R}$
Empty set	$-\frac{x^2}{a^2} = 1$	
One (double) line	$\frac{x^2}{a^2} = 0$	$(0, t),\ t \in \mathbb{R}$

How to prove all these things

Definition of conics. There are many ways of defining conics and then proving their properties. One of the easiest consists of defining them by their focal properties. From there, the standard form of their equations is easy to obtain. Once the equations are known, it is easy to prove that conics are plane sections of circular right cones and have reflective and eccentricity properties. Finally, proving that all polynomial equations of degree two describe a (possibly degenerate) conic is done through diagonalization.

Equations. This is done as follows:

- Ellipse: Given F_1 and F_2, locate the origin in their midpoint and let the Ox axis to be the line through them. Then $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Assume that the ellipse is the locus of all points X such that $d(X, F_1) + d(X, F_) = 2a$. Prove that $a \geq c$ (otherwise the ellipse is the empty set) and let $b^2 = a^2 - c^2$. Prove that the equation $d(X, F_1) + d(X, F_2) = 2a$ is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- Hyperbola: As in the previous case, let $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let the hyperbola be the locus of all points X such that $|d(X, F_1) - d(X, F_2)| = 2a$. Prove that in this case $c \geq a$ and let $b^2 = c^2 - a^2$. Prove that the equation $|d(X, F_1) - d(X, F_)| = 2a$ is equivalent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- Parabola: Given F and ℓ, let the Ox axis be the line through F perpendicular to ℓ, O be the midpoint between F and ℓ along Ox, and let Oy be the line through O parallel to ℓ. Then $F = (a, 0)$ and ℓ has equation $x = -a$. Prove that the equation $d(X, F) = d(X, \ell) = 2a$ is equivalent to $y^2 = 4ax$.

Plane sections of the cone. In order to prove that the plane sections of a circular right cone are conics, consider the right circular cone with equation $x^2 + y^2 = a^2z^2$, where $a > 0$. Intersect the cone with horizontal planes $z = k$, with oblique planes $y = az + k$ (which are parallel to a generatrix), and with planes $y = bz + k$ where $0 \leq b \neq a$, distinguishing the cases $b < a$ and $b > a$. Finally, use rotational symmetry arguments to extend your results to all remaining planar sections of the cone.

Reflective properties. Reflective properties are proved as follows:

- Ellipse: Assume that $X = \gamma(t)$ is a parametrization of your ellipse. Let $\gamma'(t)$ be the vector tangent to the ellipse at point $X = \gamma(t)$. Let $d_i(t) = d(F_i, \gamma(t))$, and let $\overrightarrow{u_i(t)}$ be the unit vectors $\frac{\overrightarrow{F_iX}}{d_i(t)}$, for $i = 1, 2$. Use the facts that $\gamma(t) = F_1 + d_1(t)u_1(t) = F_2 + d_2(t)u_2(t)$ and $d_1(t) + d_2(t)$ =constant to prove that the angles formed by γ' and u_1 and $-\gamma'$ and u_2 are the same at point X.

- Hyperbola: Use an analogous strategy.
- Parabola: In this case the vectors to be considered are $\overrightarrow{FX}$ and the horizontal.

Eccentricity properties. Eccentricity properties are almost immediate if the appropriate coordinate system is used. Let Ox be line ℓ, and Oy be the line through F perpendicular to ℓ. Write the equality $\frac{d(X,F)}{d(X,\ell)} = k$ in this coordinate system, complete the squares of the resulting equation and classify the conic.

All degree 2 polynomial equations are conic equations. In order to prove that all degree 2 polynomial equations give rise to a conic, rewrite the equation as

$$ax^2 + 2bxy + cy^2 + dx + ey + f.$$

We say that $Q(x,y) = ax^2 + 2bxy + cy^2$ is it quadratic part and $L(x,y) = dx + ey$ is its linear part. The equation can be expressed in terms of matrices as

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f = 0.$$

Matrix Q being symmetric, it diagonalizes in orthonormal basis. Therefore, a coordinates change $X = M\overline{X}$ transforms the equation into the following:

$$\begin{pmatrix} \overline{x} & \overline{y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} + \begin{pmatrix} \gamma & \delta \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} + f = 0,$$

i.e.,

$$\alpha\overline{x}^2 + \beta\overline{y}^2 + \gamma\overline{x} + \delta\overline{y} + f = 0,$$

where the cross product $\overline{xy}$ does not appear. At this point if $\alpha, \beta \neq 0$, completing squares by the appropriate translation $\overline{X} = \overline{\overline{X}} + W$ allows to transform the previous equation into the following:

$$\begin{pmatrix} \overline{\overline{x}} & \overline{\overline{y}} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \overline{\overline{x}} \\ \overline{\overline{y}} \end{pmatrix} + \varepsilon = 0,$$

which is

$$\alpha\overline{\overline{x}}^2 + \beta\overline{\overline{y}}^2 + \varepsilon = 0.$$

If $\beta = 0$, completing squares gives rise to the following equation:

$$\begin{pmatrix} \overline{\overline{x}} & \overline{\overline{y}} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\overline{x}} \\ \overline{\overline{y}} \end{pmatrix} + \begin{pmatrix} 0 & \delta \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} + \varepsilon = 0$$

or, equivalently,

$$\alpha\overline{\overline{x}}^2 + \delta\overline{\overline{y}} + \varepsilon = 0.$$

The table in page 5 reports all possible combinations of positive, negative and null values for the coefficients $\alpha, \beta, \gamma, \delta, \varepsilon$.