Incidence Angle Constrained Visibility

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March 26, 1996

Abstract

We present the first part of a study on what we call *quality pictures*, where we introduce a quality parameter α that indicates the minimum incidence angle allowed between the vision direction and a seen surface. We solve here some combinatorial and algorithmic problems about both external and internal quality pictures of polygons.

1 Introduction

Since 1973, when V. Klee proposed the problem of determining the minimum number of points (guards or cameras) that would always suffice to see any n-polygon, many variations of this problem have been studied [20], [22]. Various studies have considered restricting the class of polygons to be guarded (orthogonal, etc.), new kinds of guards have been introduced (edge-guards, mobile guards,...) and even watchman routes have been studied.

Traditionally, it is assumed that guards can see in any direction. If we imagine cameras, instead of guards, this would imply a 360° field of aperture. More recently, more realistic approaches have been proposed in which the angle of visibility is optimized (aperture angle optimization problems [14], [5]) or restricted (floodlight problems [4], [8], [23], [9]); or in which the distance to the seen object is bounded (bounded reach visibility [10], [1]).

We introduce the concept of *quality pictures* in terms of a lower bounded incidence angle visibility, and solve certain combinatorial and algorithmic classical problems with this quality requirement.

1.1 Definitions

A *point* p lying in a line l will be said to be seen with quality $\alpha \in [0, \pi/2)$ from a viewpoint v if it can be seen from v and the incidence angle of vision that forms the line segment $\overline{vp}$ with l is strictly greater than α (see Figure 1).

A *line segment* $\overline{pq}$ will be said to be seen with quality α from a viewpoint v if all the points in $\overline{pq}$ can be seen from v with quality α (see Figure 2).

So, the region of the halfplane from where $\overline{pq}$ can be seen with quality α is a wedge limited by two rays, one from each endpoint of the segment, that form an angle of α whith it. We call them the right and the left α-rays from the segment (see Figure 3).

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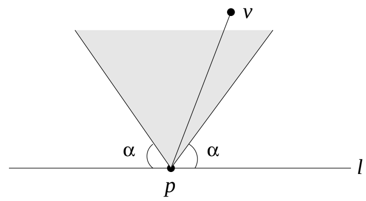

Figure 1: Point v sees p with quality α.

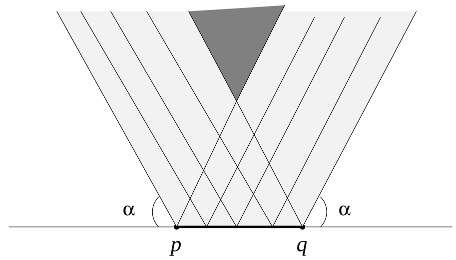

Figure 2: Segment $\overline{pq}$ can be seen with quality α from the dark region.

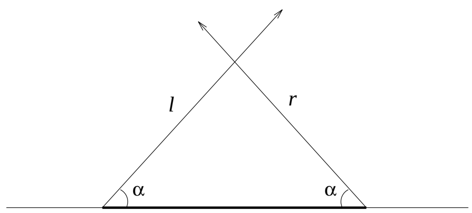

Figure 3: The α-visibility region of a segment is determined by its left and right α-rays.

A *polygon* P will be said to be seen with quality α from a set C if there is a finite subset V of C such that each edge of P can be seen with quality α from a viewpoint in V. One can imagine that we are photographying P from viewpoints in C and requiring every edge to appear in at least one picture with quality α. External (resp. internal) visibility of P with quality α refers to the case in which C is the exterior (resp. interior) of P (see figure 4).

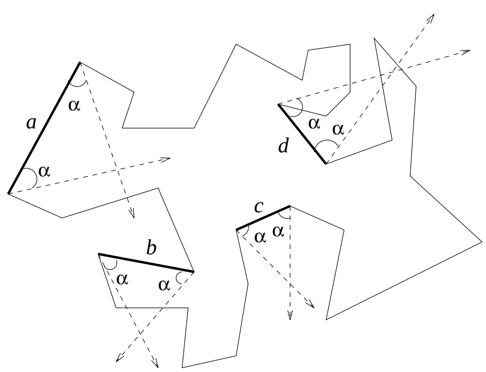

Figure 4: Edge a (resp. c) can be internally (externally) seen with quality α, while b (d) cannot.

1.2 Problems

The questions we will consider are the following ones:

Decision problem: Given a polygon P, a set C and an angle α, is it possible to photograph P from C with quality α?

Optimization problem: Given a polygon P and a set C, find the maximum α so that P can be seen from C with quality α.

Combinatorial maximization problem: Minimize the number of pictures needed in order to photograph any n-gon with a given quality α.

Minimization algorithm: Give an algorithm that locates points from which a minimum number of pictures of a given polygon are to be taken in order to achieve a given quality α.

External pictures of convex polygons being the basic problem to solve, we will study them in Sections 2 and 3 in both non-restricted and restricted version. Internal quality pictures of convex polygons are studied in Section 4. Both questions for simple polygons are solved in Section 5.

2 External quality pictures of convex polygons

Two basic facts need to be noticed while taking quality pictures of a convex polygon P from its exterior.

Fact 1 *Whenever it's possible to photograph simultaneously with quality α two edges, e_i and e_{i+k}, then all the edges belonging to the chain $e_i, e_{i+1}, \ldots, e_{i+k}$ appear in the same picture with quality α (refer to Figute 5).*

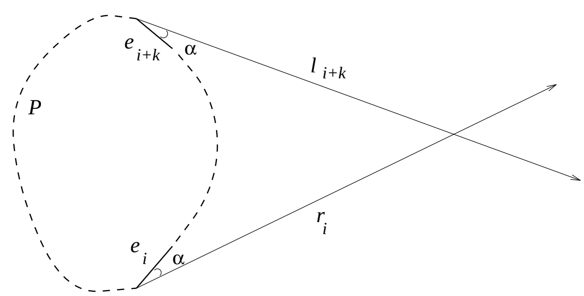

Figure 5: If e_i and e_{i+k} are seen, then all $e_i, e_{i+1}, \ldots, e_{i+k}$ are seen.

Fact 2 *When α becomes small enough ($\alpha \to 0^+$), it's possible to α-photograph any convex polygon with only 2 pictures (Figure 6).*

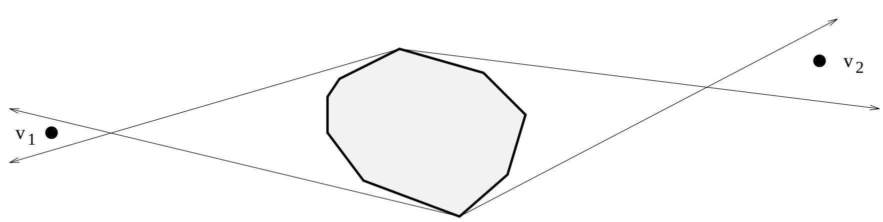

Figure 6: When $\alpha \to 0^+$, only 2 pictures are needed.

When α becomes big enough ($\alpha \to \frac{\pi}{2}^-$), then n pictures are needed to α-photograph a convex n-gon, one for each edge (Figure 7).

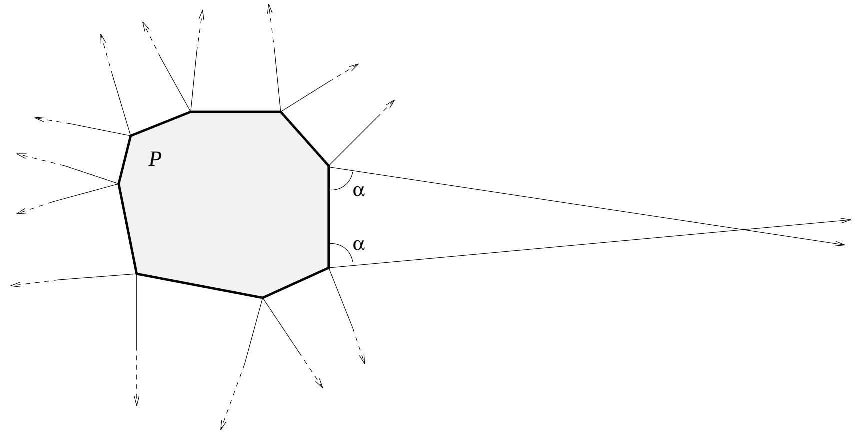

Figure 7: When $\alpha \to \frac{\pi}{2}^-$, then n pictures are needed.

Fact 2 shows that the decision problem is always solved positively, while Fact 1 is in the basis of the the solution to the combinatorial optimization problem.

Theorem 2.1 $\min(\lceil \frac{2\pi}{\pi-2\alpha} \rceil, n)$ *pictures are always sufficient to photograph any convex n-gon with quality α.*

Proof: By Fact 1, a poligonal chain of edges $e_1, \ldots, e_k$ of a convex n-gon P will appear with quality α in a picture if e_1 and e_k do, that is, if rays r_1 and l_k intersect (see Figure 8).

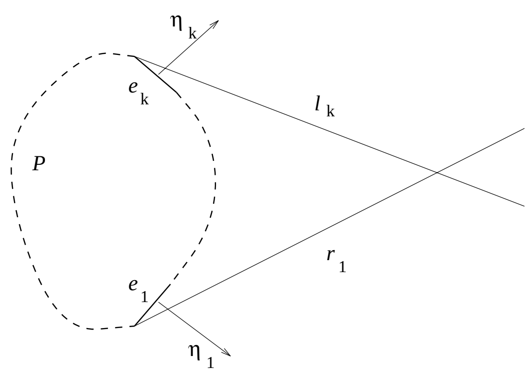

Figure 8: Rays r_1 and l_k intersect.

This condition can be expressed in terms of the external normal vectors of the edges e_1 and e_k: $e_1, \ldots, e_k$ can appear with quality α in the same picture if, and only if, $\angle(\eta_1 \eta_k) < \pi - 2\alpha$.

Let us represent all the external normal vectors of the edges of P in the unit circle. As the circle can be split into $\lceil \frac{2\pi}{\pi-2\alpha} \rceil$ consecutive arcs of length $\leq \pi - 2\alpha$, every normal vector will lie inside a right-open interval that represents a picture, so $\lceil \frac{2\pi}{\pi-2\alpha} \rceil$ pictures always suffice.

Notice that when $\alpha \to \frac{\pi}{2}^+$, the trivial bound of n pictures is preferable. □

This proof leads to a simple $O(n)$ algorithm to find N positions, $N \leq \min(\lceil \frac{2\pi}{\pi-2\alpha} \rceil, n)$, from where P can be α-photographed.

Algorithm 1

Input: A list $P = \{e_1, \ldots, e_n\}$ of the edges of the polygon, in counterclockwise order.

Output: A list of pairs (q'_i, k_i), where q'_i is a shot point for a quality picture of the edges $e_{k_i}, \ldots, e_{k_{i+1}-1}$.

- Find $T = \{\eta_1, \ldots, \eta_n\}$, the list of all external normals of the edges of P.
- Initialize $k = 1$, $j = 2$.
- While $k \le n$ do
  - While $\angle(\eta_k \eta_j) < \pi - 2\alpha$, Set $j = j + 1 \pmod{n}$.
  - Find q, the intersection of r_k and l_{j-1}.
  - Print (q', k), where $q' = q + (\eta_k + \eta_{j-1})$.
  - Set $k = j$, $j = k + 1$.

Theorem 2.2 $\min(\lfloor \frac{2\pi}{\pi - 2\alpha} \rfloor, n)$ *pictures are sometimes needed to photograph a convex n-gon with quality* α.

Proof: Given an ordered list of points in the unit circle, $\eta_1, \ldots, \eta_n$, the construction of a convex polygon that has these values as external normal vectors is immediate. Consider the tangent lines to the circle at those points, the intersection of two consecutive tangents form the vertices of the polygon.

So, we just need to prove that $\eta_1, \ldots, \eta_n$ can be placed in the circle so that $\min(\lfloor \frac{2\pi}{\pi - 2\alpha} \rfloor, n)$ pictures are needed to cover them.

Write $M = \lfloor \frac{2\pi}{\pi - 2\alpha} \rfloor$, and let's suppose $M < n$. Start placing η_0 in an arbitrary point of the unit circle, then place $\eta_1, \ldots, \eta_M$ so that every arc (η_i, η_{i+1}) has length $\pi - 2\alpha$. The circle is divided that way in M arcs of length $\pi - 2\alpha$ and, possibly, a smaller one (η_M, η_0). If this last one is not empty, place $\eta_{M+1}, \ldots, \eta_{n-1}$ in it, otherwise, place them in any of the other arcs. In both cases, M pictures are needed to α-photograph all the polygon.

If $M \ge n$, just pick n points in $\eta_1, \ldots, \eta_M$. □

3 Restricted external quality pictures of convex polygons

Given two convex polygons, P and R, with respectively n and m vertices, and so that $P \subset R$, the goal is now to take quality external pictures of P from viewpoints lying in R.

As shown in figure 9, it is not always possible to α-photograph P from R, for a given α.

Consider, for each edge of P, the two rays r and l that form angle α with the edge at its extreme points. If the intersection point q of r and l does not lie in the interior of R, no α-picture of the edge can be taken from R.

On the other hand, if q lies in the interior of R, then the intersection of ∂R with the wedge formed by r and l with apex q defines an interval (as in Figure 10) of ∂R that we will call α-factibility interval of the edge.

Theorem 3.1 *Given a convex n-gon P and a convex m-gon R such that* $P \subset R$, *and given an angle* $\alpha \in [0, \frac{\pi}{2})$, *it is possible to detect if P can be photographed with quality* α *from R in* $O(\min(n + m, n \log m))$ *time.*

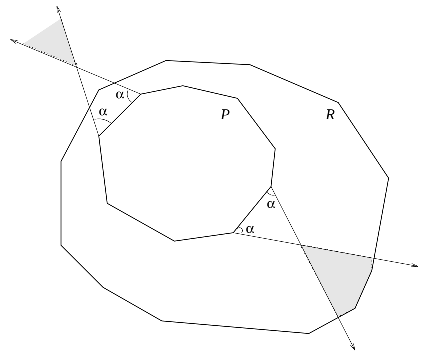

Figure 9: P cannot be α-photographed from R.

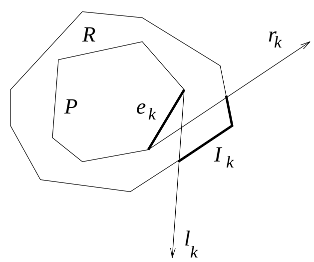

Figure 10: The α-factibility interval of an edge of P.

Proof: For each edge e_i of P, find it's critical shot point q_i, i.e. the intersection of the rays r_i and l_i. Detecting if all points q_i lie in the interior of R can be done in two different ways.

Because of the convexity of P, $q_1, \ldots, q_n$ form a simple n-gon Q. One can decide if $Q \subset R$ by computing the convex hull of Q and deciding whether it is contained in R or not. This process takes $O(n+m)$ time ([11], [15], [21]).

Another option is to determine for each i whether the point q_i is interior or not to R in $O(\log m)$ time. An $O(n \log m)$ algorithm is obtained that way. □

Theorem 3.2 *Given a convex n-gon P and a convex m-gon R such that* $P \subset R$*, the maximum value of* α *such that P can be photographed from R with quality* α *can be found in* $O(\min(n+m, n\log m))$ *time.*

Proof: From each edge e_i of P, trace a ray c_i from its middle point with the external normal direction. Consider the point q_i where c_i intersects ∂R (refer to Figure 11). Each q_i defines the maximal quality for its edge, α_i. Take $\alpha = \min_{i=1\ldots n} \alpha_i$.

Again, this can be done in two ways. First of all, as P is convex, the rays c_i are ordered. So, as R is also convex, the points q_i can be found in $O(n+m)$ time by walking along P and R, advancing now on P, now on R.

On the other hand, as R is convex, each intersection point q_i can be found in $O(\log m)$ time, and an $O(n \log m)$ algorithm is obtained. □

Theorem 3.3 *Given a convex n-gon P and a convex m-gon R such that* $P \subset R$*, and given an angle* $\alpha \in [0, \frac{\pi}{2})$*, the minimum number of pictures to* α*-photograph P from R and a location of the shot points can be obtained in* $O(\min(n+m, n\log m))$ *time.*

Proof: A complete proof of this theorem can be found in [2]. The essential steps are: First notice that the algorithm must find a minimal α-cover of P, i.e. a minimal set of polygonal subchains

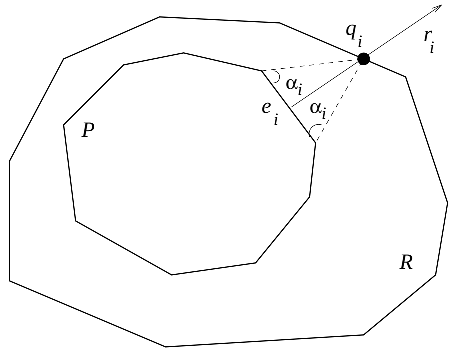

Figure 11: Maximizing α.

of P such that (1) each chain can appear in a single picture from R with quality α and (2) the chains cover P. Then prove that such a minimal α-cover of P can always be achieved by considering only maximal α-visible subchains of P that do not overlap more than once.

The resulting algorithm is now sketched:

Algorithm 2

Input: Two counterclockwise ordered lists of edges of two convex polygons $P = \{e_1, \ldots, e_n\}$, $R = \{r_1, \ldots, r_m\}$ such that $P \subset R$, and a value $\alpha \in [0, \frac{\pi}{2})$.

Output: A list of pairs (q_i, k_i), where q_i is a point in ∂R from where an α-picture of the edges $e_{k_i}, \ldots, e_{k_{i+1}-1}$ can be taken.

1. Find all maximal chains and a shot point for each one:
   (a) For $k = 1, \ldots, n$ find I_k, the α-factibility interval for e_k in ∂R (Figure 10).
   (b) Initialize $k = 1$, $j = 2$.
   (c) While $k \leq n$ do
      - $I = I_k$.
      - While $I \cap I_j \neq \emptyset$ do
         - $I = I \cap I_j$.
         - $j = j + 1 \pmod{n}$.
      - chainend$(k) = j$. $\quad$ ($e_k, \ldots, e_{j-1}$ is maximal)
      - shotpoint$(k) = q$. $\quad$ (any $q \in I$)
2. Find a cicle that α-covers P:
   (a) Initialize $L = \{e_1\}$, $j =$ chainend(1).
   (b) While $e_j \notin L$ do
      - $L = L \cup \{e_j\}$.
      - $j =$ chainend(j).
3. Use it to produce the output:
   (a) Initialize $L = \{j\}$, $k =$ chainend(j).

(b) While $k \notin [j, \text{chainend}(j))$ do

- $L = L \cup \{k\}$.
- $k = \text{chainend}(k)$.

(c) Print all pairs (q_i, k_i) with $q_i = \text{shotpoint}(k_i)$ and $k_i \in L$.

All steps in the algorithm take $O(n)$ time, except 1(a), that can be done in $O(\min(n + m, n \log m))$ time, as in Theorem 3.2. □

4 Internal quality pictures of convex polygons

Taking internal quality pictures of a convex polygon turns out to be a slightly different problem.

First, it can happen that the polygon cannot be α-photographed for a given value of $\alpha \in [0, \frac{\pi}{2})$, as it is shown in Figure 12:

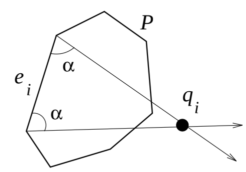

Figure 12: A convex polygon that cannot be internally seen with quality α.

The decision problem for a given convex n-gon P and a given value of $\alpha \in [0, \frac{\pi}{2})$ can be trivially solved in $O(n \log n)$ time, detecting for each edge e_i of P whether or not it's critical point q_i lies in the interior of P.

Given a convex n-gon P, the maximum value of $\alpha \in [0, \frac{\pi}{2})$ such that P can be internally seen with quality α can also be found in $O(n \log n)$ time as the minimum of the maxima of α over the edges.

In both cases, a linear algorithm is not likely to exist for the critical points q_i can form a non simple polygon and the bisectors of the edges behave similarly.

On the other hand, for any n and α there always exists a convex n-gon that needs n pictures to be α-photographed, as the scheme of Figure 13 shows.

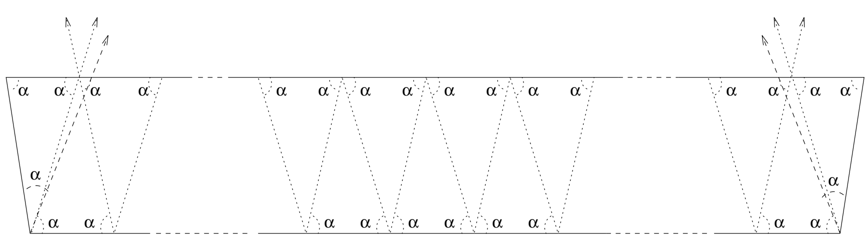

Figure 13: A polygon that needs n photos.

In this case, also the problem of seeing the entire polygon from a single viewpoint can be posed. The decision problem of whether or not a given polygon can be seen with a given quality from a single viewpoint can be solved in linear time by linear programming, because it is just a matter of deciding whether or not an intersection of halfplanes is empty ([16], [17]).

The optimization problem consists now in finding the maximum quality that allows to see the entire polygon from a single viewpoint. As the decision problem is solved by linear programming, it will be possible to solve the optimization problem in $O(n \log n)$ time by parametric search [18].

Finally, notice another difference between internal and external pictures of convex polygons. When taking internal pictures, it may happen that two edges can be simultaneously seen with some quality while some parts of their intermediate chain are seen with a worse quality, as shown in Figure 14.

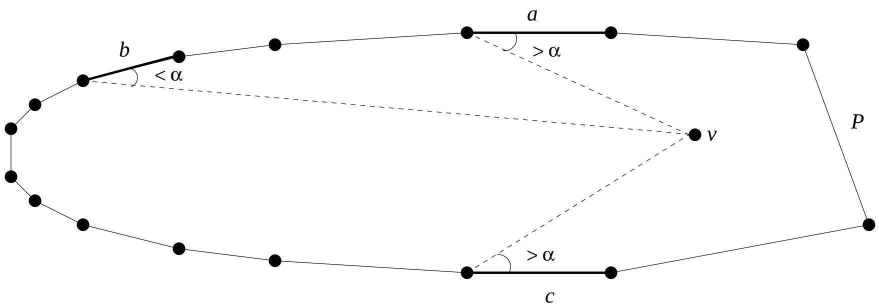

Figure 14: Edges a and c are seen with quality α from v, while b is not.

If we force each photograpy to capture only one connected chain of α-visible edges, the minimization algorithm of Theorem 3.3 can be re-adapted. In this case, calculating the maximal chains does not mean intersecting consecutively overlapping intervals any more. Here the visibility cones of the edges must be intersected. This increases the complexity of the algorithm up to $O(n \log^2 n)$, because the problem can be solved by dynamicly manteining an intersection of halfplanes [19].

5 Quality pictures of simple polygons

Dealing with simple polygons only adds a small difficulty, namely that a point v satisfying the angular quality condition for a point p does not necessarily see p (Figure 15). Internal and external quality vision behave in a very similar way in this case.

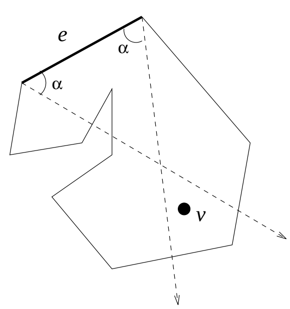

Figure 15: v satisfies the angular condition for e but does not see e.

Theorem 5.1 *An $O(n \log n)$ algorithm can be given that maximizes the visibility quality $\alpha \in [0, \frac{\pi}{2})$ for a given simple polygon.*

Proof: For each edge of P, the problem of finding the maximum α that allows to see the edge with quality α is equivalent to the problem of finding the isosceles triangle with maximum area that is contained in P and has basis in the edge (see Figure 16).

The procedure to find such a triangle is as follows: Trace a ray c internal to P and bisecting the edge. Find q, its first intersection point with ∂P. Consider the shortest path from the

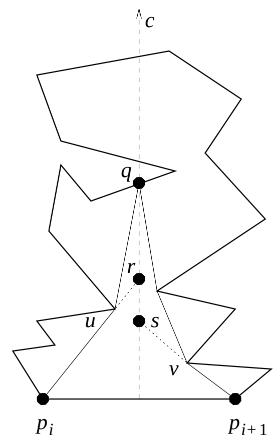

Figure 16: s determines the maximum value of α for edge e.

extreme points of the edge to q. Let u and v be its respective first vertices. Intersect the bisector c with two lines, one through each endpoint, each one passing through the first vertex of the corresponding shortest path. Among the two intersecting points, the closest one to the edge determines the desired triangle, that is, determines the maximum value of α for that edge. So, the maximization problem can be solved with the following algorithm.

Algorithm 3

Input: An ordered list of vertices of a simple polygon $P = \{p_1, \ldots, p_n\}$.

Output: The maximum value of $\alpha \in [0, \frac{\pi}{2})$ that allows to α-photograph P from it's interior.

1. For each edge $\overline{p_i p_{i+1}}$ of P find q_i, the first point in ∂P that intersects the internal bisector ray of $\overline{p_i p_{i+1}}$.
2. For each edge $\overline{p_i p_{i+1}}$ of P, find u, the first vertex of the the shortest path in P' from p_i to q_i and v, the first vertex of the shortest path from p_{i+1} to q_i.
3. For each edge $\overline{p_i p_{i+1}}$ of P find r_i and s_i, the points where the bisector intersects the lines $\overline{p_i u}$ and $\overline{p_i v}$. Take the closest to $\overline{p_i p_{i+1}}$ and calculate α_i.
4. Take $\alpha = \min_{i=1 \ldots n} \alpha_i$.

Step 1 can be done by ray shooting in $O(\log n)$ time for each edge, after a $O(n)$ time preprocess of the polygon ([7], [6], [13]). Step 2 also can be solved in $O(\log n)$ time per edge, after a $O(n)$ time preprocess [12]. The rest of the algorithm only requires linear time.

A small modification of this algorithm allows to calculate the maximum quality in the external vision of simple polygons. Its complexity remains unchanged. □

6 Conclusions and More to come

We have proved that $\min(\lceil \frac{2\pi}{\pi - 2\alpha} \rceil, n)$ photos are always sufficient to photograph a convex n-gon externally with quality α, while $\min(\lfloor \frac{2\pi}{\pi - 2\alpha} \rfloor, n)$ photos are sometimes needed. Our proof leads to a $O(n)$ time algorithm to determine the shot placements.

When restricting the viewpoints to be inside a convex m-gon R ($P \subset R$), we give $O(n+m)$ and $O(n \log m)$ algorithms to solve the decision problem, maximize the quality and find shot optimal locations.

Internal quality pictures of a convex n-gon turn out to be a slightly different problem. The decision and the optimization problems can be solved in $O(n \log m)$ time. The minimization algorithm for external pictures can be adapted if we consider only consecutive chains of edges in our photos, leading to a $O(n \log^2 n)$ algorithm.

Finally, we study simple polygons, where external and internal quality picture problems become very similar, and the previous decision and maximization algorithms can be re-adapted into new $O(n \log n)$ time ones.

These and some other related problems can be found in [2], [3], in which we generalize these questions to the 3D space, taking quality pictures of terrains and polyhedra from restricted zones or trajectories in space.

References

[1] M. Abellanas, J. García and F. Hurtado: Bounded Reach Visibility, in preparation.

[2] G. Blanco, J. García, F. Hurtado, P. Ramos and V. Sacristán: Fotografías de calidad (in Spanish), *Sextos Encuentros de Geometría Computacional*, Barcelona, 5-7 July, 1995.

[3] G. Blanco, J. García, F. Hurtado, P. Ramos and V. Sacristán: Fotografías de calidad en 3D (in Spanish), in preparation.

[4] P. Bose, L. Guibas, A. Lubiw, M. Overmars, D. Souvaine and J. Urrutia: The Floodlight Problem, *Fifth Canadian Conference on Computational Geometry*, Waterloo, 5-9 August, 1993, pp. 399-404.

[5] P. Bose, F. Hurtado, E. Omaña and G. Toussaint: Aperture angle optimization problems, *Seventh Canadian Conference on Computational Geometry*, Québec, 10-13 August, 1995, pp. 73-78.

[6] B. Chazelle, H. Edelsbrunner, M. Grigni, L. Guibas, J. Hershberger, M. Sharir and J. Snoeyink: Ray shooting in polygons using geodesic triangulations, *Proc. 18th Internat. Colloq. Automata Lang. Program.*, LNCS 510, 1991, pp. 661-673.

[7] B. Chazelle and L. Guibas: Visibility and intersection problems in plane geometry, *Discrete Comput. Geom.*, 1989(4), pp. 551-581.

[8] J. Czyzowicz, E. Rivera-Campo and J. Urrutia: Optimal Floodlight Illymination of Stages, *Fifth Canadian Conference on Computational Geometry*, Waterloo, 5-9 August, 1993, pp. 393-398.

[9] V. Estivill-Castro and J. Urrutia: Optimal floodlight Illumination of Orthogonal Art Galleries, *Sixth Canadian Conference on Computational Geometry*, Saskatoon, 2-6 August, 1994, pp. 81-86.

[10] J. García: Problemas algorítmico-combinatorios de visibilidad, Ph.D. Thesis (in Spanish), 1995.

[11] R.L. Graham and F.F. Yao: Finding the Convex Hull of a Simple Polygon, *Journal of Algorithmics*, 1984(4), pp. 324-331.

[12] L. Guibas and J. Hershberger: Optimal shortest path queries in a simple polygon, *J. Comput. Syst. Sci.*, 1989(39), pp. 126-152.

[13] J. Hershberger and S. Suri: A Pedestrian Approach to Ray Shooting: {Shoot} a Ray, Take a Walk, *Proc. 4th ACM-SIAM Sympos. Discrete Algorithms*, 1993, pp. 54-63.

[14] F. Hurtado: Looking Through a Window, *Fifth Canadian Conference on Computational Geometry*, Waterloo, 5-9 August, 1993, pp. 234-239.

[15] D.T. Lee: Visibility of a Simple Polygon, *Computer Vision, Graphics and Image Processing*, 1983(22), pp. 207-221.

[16] N. Megiddo: Linear-time algorithms for linear programming in $\mathbb{R}^3$ and related problems, *SIAM J. Computing*, 1983(12), pp. 720-732.

[17] N. Megiddo: Linear programming in linear time when the dimension is fixed, *J. ACM*, 1984(31), pp. 114-127.

[18] N. Megiddo: Applying parallel computation algorithms in the design of serial algorithms, *J. ACM*, 1983(30), pp. 852-865.

[19] F. P. Preparata, An optimal real time algorithm for planar convex hulls, *Comm. ACM*, 22 (1979) pp. 402-405.

[20] J. O'Rourke: Art Gallery Theorems and Algorithms, Oxford U. Press, 1987.

[21] F. Preparata and M. Shamos: Computational Geometry, An Introduction. Springer-Verlag, 1985.

[22] I. Shermer: Recent Results in Art Galleries, *Proceeding of the IEEE*, September, 1992.

[23] W. Steiger and I. Streinu: Positive and Negative Results on the Floodlight Problem, *Sixth Canadian Conference on Computational Geometry*, Saskatoon, 2-6 August, 1994, pp. 87-92.