# Lower bounds for some geometric problems

*V. Sacristán*

Universitat Politècnica de Catalunya

Technical Report MA2-IR-98-0034

July 1998 (Revised June 1999)

# Lower bounds for some geometric problems

Vera wrote it, but many did it

June 1999

**Abstract**



## 1 Introduction

I wrote this report to clarify my own ideas and knowledge about a subject that was a real mess for me. Let's hope that I learned something while writing it, and that it may be useful to someone else too.

The first person that made me an observation about the correctness of a lower bound proof that appears in my PhD thesis was Godfried Toussaint. I didn't realize how serious it was, because I didn't know enough about decision tree models, algebraic tree models, algebraic computation models, real RAM, and so on. Also Ferran Hurtado kept saying that I should confirm the model in which those things could be done. On the other hand, I had read many serious authors who had done the same kind of things, and I thought that the bug could not be so big. Anyway, the good fact is that Godfried and Ferran still talk to me, even if I didn't fix the error in time.

The third person that made me notice that I was making a mistake was Raimund Seidel, and I had to take notice of what he said to me, given that he did it in public in a conference. So, I sarted working more seriously on all that staff, and this for-myself-report is a first abstract of what I've found out so far.

Most of the results are due to other people, either because I read them in some published papers (this goes specially for sections 2 through 5, and partially for section 6) or because they helped me in proving new lower bounds (this goes specially for sections 6 through 9). Ferran Hurtado, Pedro Ramos, Jesús García López, Francisco Gómez, Godfried Toussaint, and all the people from the Universidad Politécnica de Madrid, did a lot of this work.

All together, the result is that our lower bounds were correct, even though our proofs were not. Let's hope that now they are right.

# 2 Models of computation and Ben-Or's theorem

The first general result about lower bounds for an entire class of problems was given in 1975 by Dobkin and Lipton [3] and can be considered as the ancestor of Ben-Or's theorem, for it is based in topological arguments. Dobkin and Lipton's theorem applies to linear decision trees.

A **decision tree** is a binary tree in which

- every internal vertex has the form of a comparison $f(x_1, \ldots, x_n) : 0$,
- and the leaves correspond to an output of the form "accept" or "reject",

where $f$ is some function from a class of allowed functions; $x_1, \ldots, x_n$ are the input elements; and $:$ stays for $>$ or $=$. Dobkin and Lipton refer to these trees as "search programs".

A **linear decision tree** is a decision tree in which the allowed functions are all linear.

Given any set $W \subset \mathbb{R}^n$, we say that the tree solves the membership problem for $W$ if the answer returned is correct for every imput $x \in \mathbb{R}^n$, i.e. if the answer is "accept" when $x \in W$ and "reject" otherwise. The result of Dobkin and Lipton bounds the depth of any linear decision tree that solves the memebership problem of a set $W$ in terms of the topology of $W$, more precisely, in terms of the number $\#W$ of path-connected components of $W$.

**Theorem 1 (Dobkin-Lipton)** *Any linear decision tree that solves the membership problem for $W \subseteq \mathbb{R}^n$ must have depth at least $\log_2 \#W$.*

*Proof:* Let $T$ be any such tree. Let $\ell_1, \ldots \ell_r$ be its leaves. For each leave $\ell_i$, consider the set $L_i$ of all the inputs $x \in \mathbb{R}^n$ that exit the program through $\ell_i$. The sets $L_i$ form a partition of $\mathbb{R}^n$ into convex regions (for all the tests of the tree are linear). The goal is to prove that the number of sets $L_i$ (that is, the number of leaves of $T$) must be greater or equal to the number of connected components of $W$: $r \geq \#W$. Hence, the depth of $T$ will have to be greater of equal to $\log_2 \#W$.

In order to prove that $r \geq \#W$, one must only notice that each accepting leave $\ell_i$ of $T$ corresponds to a set $L_i$ enclosed in $W$, and that each such set $L_i$ cannot intersect more than one connected component of $W$, because of its own convexity. □

Dobkin and Lipton were able to apply their result to several problems. Probably the most relevant are the $\Omega(n^2)$ lower bound for the generalized knapsack problem and the $\Omega(n \log n)$ lower bound for the element uniqueness problem. It should be noticed that the proof of their general result strongly relies on the linearity of the functions involved in the "linear search program". Extending the result to more general models was a difficult task, but necessary. The first to suceed were Steele and Yao [16] in 1980. Their result applies to fixed-order algebraic decision trees.

A **$d$-th order algebraic decision tree** is a decision tree where the functions allowed are polynomials of degree at most $d$. The result of Steele and Yao is of the same kind as that of Dobkin and Lipton.

**Theorem 2 (Steele-Yao)** *Let $h$ be the height of any $d$-th order algebraic decision tree that solves the membership problem for a set $W \subset \mathbb{R}^n$. Then,*

$$2^h \beta(hd, n) \geq \#W,$$

*where $\beta(m, n)$ is the maximum number of connected components of any set defined as the complement of the set of roots of a polynomial of degree $m$ on $n$ variables.*

*Proof:* We will only sketch the proof. Consider an accepting leaf $\ell$ of a $d$-th order algebraic decision tree $T$ that solves the problem for $W$, and assume that all the tests that an input $x$ passes through, before getting to $\ell$, are of the kind $p(x) < 0$. The set $L$ associated to $\ell$ is not necessarily convex, as it was for the linear decision trees. Nevertheless, it may be noticed that

$$\begin{aligned} L &= \{x \in \mathbb{R}^n \mid x \text{ ends in } \ell\} \\ &= \{x \in \mathbb{R}^n \mid p_1(x) < 0, \ldots, p_k(x) < 0\} \\ &\subset \{x \in \mathbb{R}^n \mid \textstyle\prod_{i=1}^k p_i(x) \neq 0\}. \end{aligned}$$

This last set is defined as the complement of the set of roots of a polynomial of degree $\leq hd$ on $n$ variables. Hence, the number of connected components of $L$ is bounded by $\beta(hd, n)$. As the number of leaves of the tree is at most $2^h$, the result follows. □

With this result in hand, Steele and Yao were able to obtain lower bounds on the height $h$ of any $d$-th order algebraic decision tree, for several sets $W$. They used a bound on the Betti numbers $\beta(m, n)$ based on a theorem due to Milnor [13] and Thom [17]:

$$\beta(m, n) \leq (m+1)^n.$$

The main problem of Steele and Yao's result is that it needs the number of connected components of $W$ to be $\Omega(n^n)$ in order to obtain nonlinear lower bounds for the memebership problem for $W$. This particularly means that their method was powerful enough to prove an $\Omega(n \log n)$ bound for the knapsack problem, but it was not for the element uniqueness or the extreme points problems, for example. The reason for that was double: first, their bound of the number of connected components of each leave-set $L$ of the tree was not tight enough; second, the bound for the Betti numbers was also too loose (Milnor's theorem bounds the sum of the Betti numbers of all dimensions). The authors themselves in their paper say "... and a better determination in the future may result in an improved bound."

This is exactly what Ben-Or did in 1983, in a paper [2] which I think is really clear. In fact, Ben-Or's result applies to an even more extended model: the algebraic computation tree.

An **algebraic computation tree** is a binary tree $T$ with a function that assigns:

- to any simple vertex $v$ (any vertex with exactly one son) an operational instruction of any of the following forms:

- $f_v := f_{v_1} \circ f_{v_2}$,
- $f_v := c \circ f_{v_1}$,
- $f_v := \sqrt{f_{v_1}}$;

where $v_i$ is an ancestor of $v$ in the tree $T$, or $f_{v_i}$ is one of the initial input elements, $x_1, \ldots, x_n$; the symbol $\circ$ means any of $+, -, \cdot, /$; and $c$ is a constant.

- to any braching vertex $v$ (any vertex with exactly two sons) a test instruction of any of the forms:
  - $f_{v_1} > 0$,
  - $f_{v_1} \geq 0$,
  - $f_{v_1} = 0$;

  where $v_1$ is an ancestor of $v$ in the tree $T$, or $f_{v_1}$ is one of the initial input elements, $x_1, \ldots, x_n$.

- to any leaf (any vertex without any son) an output "accept" or "reject".

Given any set $W \subset \mathbb{R}^n$, we say that the computation tree $T$ solves the membership problem for $W$ if the answer returned is correct for every imput $x \in \mathbb{R}^n$. Let $C(x, T)$ denote the number of vertices that $x$ passes through. The complexity of $T$, $C(T)$, is given by the maximum of $C(x, T)$ for any $x$. The complexity of the membership problem for $W$ in the algebraic computation tree model is, then,

$$C(W) = \min_{T \text{ solves } W} C(T) = \min_{T \text{ solves } W} \max_{x \in \mathbb{R}^n} C(x, T).$$

**Theorem 3 (Ben-Or)** *Let $W \subseteq \mathbb{R}^n$ be any set, and $T$ be a computation tree that solves the membership problem for $W$. If $\#W$ is the number of connected components of $W$, and $h$ is the height of $T$, then*

$$2^h 3^{n+h} \geq \#W.$$

Taking logarithms in the previous result, Ben-Or obtains:

**Theorem 4 (Ben-Or)** *For any $W \subseteq \mathbb{R}^n$,*

$$C(W) \geq \frac{\log N}{1 + \log 3} - \frac{\log 3}{1 + \log 3} n = \Omega(\log N - n),$$

*where $N = \max\{\#W, \#(\mathbb{R}^n - W)\}$.*

*Proof:* It's impossible to reproduce here the details of the proof (see theorem 3 in [2]). Let us try to sintetize what its essential elements are.

As before, one can partition $W$ into the sets $L$ corresponding to the leaves $\ell$ of the kind "accept". Hence,

$$\#W \leq 2^h \max_L \#L,$$

and the goal is to bound $\#L$.

A first clever point is Ben-Or's way of counting $\#L$. He produces a set of equations and inequations that represent the path that leads to the leave $\ell$ in the following way: every time that a node is encountered of the kind $f_v = \sqrt{f_{v_i}}$, for example, a new equation of the kind $f_u^2 = f_{u_i}$ is introduced, as well as the new variable $f_u$. And so on for every kind of node. In this way, the set $L$ is represented by a set of $s$ inequalities together with a certain number of equalities, all in $\mathbb{R}^{n+r}$, such that $s+r \leq h$. All the polynomials involved in both the equalities and the inequalities are of degree 2. Notice that the main difference with Steele and Yao's way of counting $\#L$ is the introduction of the new variables, wich allows to formulate the problem in higher dimension, while mantaining a very low degree of the polinomials involved.

A second important point is the bounding of the number of connected components of such a set. Let $V \subset \mathbb{R}^n$ be a set defined by a certain number (that will be irrelevant) of equalities of the kind $q(x) = 0$, and a certain number $h$ of inequalities of the kind $p(x) > 0$ and $p(x) \geq 0$. If all the polinomials $p$ and $q$ are of degree at most $d$, Ben-Or proves that the number of connected components of $V$ is bounded by

$$\#V \leq d(2d-1)^{n+h-1}.$$

This result follows, as the bound in Steele and Yao's paper, from Milnor's theorem. The difference lies in the fact that Ben-Or applies Milnor's theorem to a set related to $V$, in dimension $n+h$.

Finally, the fact that his set is defined by polynomials of degree 2 gives the bound $\#L \leq 3^{n+h}$. □

Comparing the two results, one can notice that Steele and Yao bounded $\#L$ by

$$\#L \leq (hd+1)^n,$$

while Ben-Or manages to bound it by

$$\#L \leq d(2d-1)^{n+h-1}$$

and get $d$ to be $d = 2$. In this way, Ben-Or solves the open problems of Steele and Yao's paper by means of a clever use of the ideas that were first proposed by Dobkin and Lipton (bounding the heigth of the decision tree is a matter of counting the connected components of the set $W$) and by Steele and Yao (in the algebraic case, the counting of the number of connected components associated to each leave of the tree can be done using the bounds to the Betti numbers given Milnor and Thom).

Let us see now an **extension** of the previous model. It is based on a detailed examination of the proof of Ben-Or's theorem (see theorem 3 in [2]). One can see that all is needed for the proof to work is that the degree of the operations will not be higher than 2. Thus, one can allow new operations, by associating a cost to them:

- addition, substraction and multiplication by constants have cost 0;
- multiplication, division, taking square roots, any bilinear operation and comparison, all have cost 1;
- taking $k$-th roots costs $O(\log k)$;

- solving a polynomial (of any degree) has the cost of the complexity of evaluating it at a given point.

Given a computation tree $T$ (with or without the new operations), let $M(x, T)$ denote the sum of the costs of the operations applied to $x$ along its path. The multiplicative complexity of $T$, $M(T)$, is the maximum of $M(x, T)$ for any $x \in \mathbb{R}^n$. The (multiplicative) complexity of the membership problem for $W$ is, then,

$$M(W) = \min_{T \text{ solves } W} M(T) = \min_{T \text{ solves } W} \max_{x \in \mathbb{R}^n} M(x, T).$$

**Theorem 5 (Ben-Or)** *For any $W \subseteq \mathbb{R}^n$,*

$$M(W) = \Omega(\log N - n),$$

*where $N = \max\{\#W, \#(\mathbb{R}^n - W)\}$.*

Something similar can be done while dealing with complex numbers, instead of real numbers, by representing each $z \in \mathbb{C}$ as $x + iy$:

**Theorem 6 (Ben-Or)** *Let $W \subseteq \mathbb{C}^n$ be any set, and let $T$ be an algebraic computation tree that solves the memebership problem for $W$, using the functions $Re$ and $Im$ with cost 0 and comparisons on real numbers with cost 1. Then,*

$$M(T) = \Omega(\log N - n),$$

*where $N = \max\{\#W, \#(\mathbb{C}^n - W)\}$.*

By continuity arguments, Ben-Or extends his method to deal also with rational numbers:

**Theorem 7 (Ben-Or)** *Let $W \subseteq \mathbb{Q}^n$ be any set, and let $T$ be an algebraic computation tree that solves the memebership problem for $W$. Then,*

$$M(T) = \Omega(\log N - n),$$

*where $N$ is the number of connected components of $\overline{W}$ in $\mathbb{R}^n$ with non empty interior.*

Ben-Or's algebraic computation trees are an extension of the **algebraic decision trees**. Let us denote by $C_d(W)$ the minimum height for any $d$-th order decision tree that solves the membership problem for $W$:

$$C_d(W) = \min_{T \text{ solves } W} \text{height}(T).$$

The lower bound also applies in this case:

**Theorem 8 (Ben-Or)** *For any $W \subseteq \mathbb{R}^n$, and any fixed $d$,*

$$C_d(W) = \Omega(\log N - n),$$

*where $N = \max\{\#W, \#(\mathbb{R}^n - W)\}$.*

With all these results, Ben-Or was able to prove an $\Omega(n \log n)$ bound for many problems for wich $\#W = \Omega(n!)$ (notice the improvement with respect the $\#W = \Omega(n^n)$ needed by Steele and Yao). Most of his results, that were published in his paper [2] are reported in the following sections. Nevertheless, Ben-Or explains what the fundamental limitations of his method are:

1. "... it is worthwile noting that the bounds provided by Milnor and Thom actually bound the sum of the Betti numbers of algebraic varieties and not only the number of connected components. Thus it may be possible to use the dimension of higher cohomology groups to establish lower bounds on straight line computations."

2. "A basic limitation to our method is the fact that it is a "degree" based method. Given a polynomial $p$ of degree $d$ in $n$ variables, the best lower bound that can be derived by our method to the complexity of evaluating $p$ is of order $n \log d$. [...] Any general method that can pass this limitation would be of great interest".

There have been several people involved in extending and/or improving these results. In the first place, we have Yao himself, who studied in 1991 the problem of how the complexity changes when one restricts the input elements to be integers [18]. His general results states that if $W \subset \mathbb{R}^n$ is scale-invariant and rational-dispersed, then any algebraic computation tree that solves the memebrship problem for $W$ for integer inputs (nothing is said about the behaviour of the tree for non integer inputs) has height $\Omega(\log \widehat{\#W} - n)$, where $\widehat{\#W}$ is the number of connected components of $W$ that are not of measure 0.

Yao also obtained in 1994 some results in which he used higher-dimension Betti numbers [19], as Ben-Or proposed in its open problems, but this is a paper that I haven't read yet.

Some other authors have studied the problem of extending the set of allowed primitives. Nameley, Grigoriev and Vorobjov [6] obtained in 1994 a method for proving an $\Omega(\log n)$ lower bound for testing membership to a convex polyhedron with $n$ facets in all dimensions, in a model that admits as gate functions along with the usual arithmetic operations also algebraic or trascendental functions like exp, log, sin, square root an, in general, Pfaffian functions. In 1995, Ben-Amram [1] extended Ben-Or's results to algebraic random access machines. Depending upon the kind of addressing admitted (integer or real), he obtained different extensions of Ben-Or's theorem, in which the complexity of the memebership problem for a set $W$ is bounded by the logarithm of the number of connected components of either $\mathring{W}$ or $\overline{\mathring{W}}$. But this is something that I still have to work on to understand.

We of course have people who have been interested in working on average analysis, like Ben-Or himself [2], and on lower bounds on probabilistic or nondeterministic algorithms, like Mamber and Tompa [12], who in 1982 obtained interesting results for linear decision algorithms, following the line of Dobkin and Lipton. For probabilistic decision trees, it is proved in their paper that they are not asymptotically faster than ordinary deterministic decision trees, for a collection of natural problems. For nondeterministic decision trees, Mamber and Tompa generalize Dobkin and Lipton's technique. Finally, they show that the success in using decision trees to demonstrate deterministic, probabilistic and nondeterministic lower bounds seems not to be easy to extend for demonstrating lower bounds on the parallel time required to solve sorting-like problems using alternating decision trees.

Parallel algorithms have been studied by Sen [15], who obtained in 1994 a result analogous to that of Ben-Or for the parallel case: any fixed-degree algebraic decision tree using $kn$ ($k \geq 1$) processors must use $\Omega(\log \#W/n \log k)$ steps in order to solve the memebership problem for $W \subset \mathbb{R}^n$.

# 3 Problems on sets

The following problems are known to have non linear lower bounds. For each of them, the precise bound and computation model where it holds is stated. All of them come from Ben-Or's paper [2], where more details can be found, such as which were the previous known results about a concrete problem, or some proofs that I omit here.

**Element uniqueness**
*Given $x_1, \ldots, x_n \in \mathbb{R}$, determine whether or not there exists a pair $i \neq j$ such that $x_i = x_j$. For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* This is the membership problem for the set

$$W = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \prod_{i \neq j} (x_i - x_j) \neq 0 \right\}.$$

It is easy to prove that $\#W = n!$, each of the connected components being of the form $W_\sigma = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(n)}\}$ for $\sigma \in \mathfrak{S}_n$. More precisely, consider the $n-1$ functions $f_i(x_1, \ldots, x_n) = \operatorname{sgn}(x_{i+1} - x_i)$. All these functions are constant over each $W_\sigma$. On the other hand, if $\sigma \neq \tau$, at least one of the functions $f_i$ must take different values over $W_\sigma$ and $W_\tau$. Consider two points $P \in W_\sigma$ and $Q \in W_\tau$. Because of continuity arguments, any path connecting $P$ and $Q$ must go through a point in which the value of at least one of the $f_i$ is 0, that is, must go through a point which does not belong to $W$. Hence, $W_\sigma$ and $W_\tau$ belong to different connected components of $W$. This proves that $\#W \geq n!$, which is the result really needed (in fact, $\#W = n!$, but this is not substantial here). □

**Set equality**
*Given two sets $A = \{x_1, \ldots, x_n\}$, $B = \{y_1, \ldots, y_n\}$, determine whether or not $A = B$. For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* Without lost of generality, we can consider $B$ to be $B = \{1, 2, \ldots, n\}$. Set $W = \{(\sigma(1), \sigma(2), \ldots, \sigma(n)) \mid \sigma \in \mathfrak{S}_n\}$. The *set equality* problem $A = B$ is the membership problem $A \in W$, and $\#W = n!$ trivially. □

**Set inclusion**
*Given two sets $A = \{x_1, \ldots, x_n\}$, $B = \{y_1, \ldots, y_n\}$, determine whether or not $A \subseteq B$. For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* Again, let $B = \{1, 2, \ldots, n\}$. Our problem is equivalent to the membership problem $A \in W = \{(x_1, \ldots, x_n) \mid x_i \in \{1, 2, \ldots, n\}\}$, where $\#W = n^n$. □

**Set disjointness**
*Given two sets $A = \{x_1, \ldots, x_n\}$, $B = \{y_1, \ldots, y_n\}$, determine whether or not $A \cap B = \emptyset$.*

*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* Set

$$\begin{aligned} W &= \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid x_i \neq y_j \ \forall\, i,j\} \\ &= \left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid \prod_{i,j}(x_i - y_j) \neq 0\right\}. \end{aligned}$$

Let us see that $\#W \geq (n!)^2$. For each $\sigma, \tau \in \mathfrak{S}_n$, consider the set

$$W_{\sigma,\tau} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid x_{\sigma(1)} < y_{\tau(1)} < x_{\sigma(2)} < y_{\tau(2)} < \ldots < x_{\sigma(n)} < y_{\tau(n)}\}.$$

It is clear that $\cup_{\sigma,\tau} W_{\sigma,\tau} \subset W$. Consider the family of functions $f_{ij}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \operatorname{sgn}(x_i - y_j)$. Because of continuity arguments, at least one of the functions takes value 0 along any path connecting two points of two different sets $W_{\sigma,\tau}$. Hence, each set $W_{\sigma,\tau}$ belongs to a different connected component of $W$, and $\#W \geq \# \cup_{\sigma,\tau} W_{\sigma,\tau} \geq (n!)^2$. □

**Sorting**
*Given a list of $n$ real numbers $x_1, \ldots, x_n$, find a permutation $\sigma \in \mathfrak{S}_n$ such that $x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(n)}$.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from *element uniqueness*: given $n$ real numbers, sorting them allows to know in linear time whether or not two of them are equal, by comparing each number whith its consecutive in the sorted order. □

# 4 Morphology problems

I include here several well known lower bounds for geometric problems relative to the morphology of sets of points and, more concretely, to the convex hull and related problems. Most of them are reported in the book by Preparata and Shamos [14], except for the last one, which is due to F. Hurtado [7]. The basic result is on the *extreme points test*, and is due to Steele and Yao [16] and Ben-Or [2].

**Extreme points test**
*Given $n$ points in the plane, determine whether their convex hull possess $n$ vertices.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* Set

$$W = \{(p_1, \ldots p_n) \in \mathbb{R}^{2n} \mid \exists\, \sigma \in \mathfrak{S}_{n-1}\, CH(\{p_1, p_2, \ldots, p_n\}) = (p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(n-1)}, p_n)\}.$$

Notice that the last point, $p_n$ is fixed (and let's hope that nobody is worried by the funny writing $(p_1, \ldots p_n) \in \mathbb{R}^{2n}$). Clearly, $W = \cup W_\sigma$, where $W_\sigma$ is the "natural" subset that you are imagining. It is easy to prove that if $\sigma \neq \tau$ then $W_\sigma$ and $W_\tau$ belong to different connected components of $W$. Concretely, consider the family of functions $f_{ijk}$ that assign to each point $(p_1, \ldots, p_n) \in \mathbb{R}^{2n}$ the oriented area of the triangle $p_i, p_j, p_k$. Given $\sigma \neq \tau$ and two points $P \in W_\sigma$ and $Q \in W_\tau$, at least one of these functions $f_{ijk}$ takes different values on $P$ and $Q$. Hence, any path connecting $P$ and $Q$ must go through a point $X$ in which the value of the mentioned function $f_{ijk}$ is 0. Such a point $X = (x_1, \ldots, x_n) \in \mathbb{R}^{2n}$

does not belong to $W$, for $x_i, x_j, x_k$ are aligned and $x_j$ is not a vertex of $CH(\{x_1, \ldots, x_n\})$. As a conclusion, if $\sigma \neq \tau$, $W_\sigma$ and $W_\tau$ belong to different connected components of $W$. We have proved that $\#W \geq (n-1)!$. □

Another proof of this result can be obtained by reduction from *element uniqueness*.

*Proof:* Given $x_1, \ldots, x_n \in \mathbb{R}$, construct the points $p_i = (x_i, x_i^2)$ for $i = 1, \ldots, n$. Answering the *extreme points test*, we will obtain the answer to the *element uniqueness* problem, because of the convexity of the parabola. □

**Extreme points report**
*Given $n$ points in the plane, report those who are vertices of their convex hull.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from the *extreme points test*: if one has the (unordered) list of the extreme points, it takes $O(n)$ time to count them. □

**Convex hull**
*Given $n$ points in the plane, report the ordered list of the vertices of their convex hull.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* Obvious, by reduction from any of the previous problems. Nevertheless, the most famous reduction comes from *sorting*: given $x_1, \ldots, x_n \in \mathbb{R}$, construct the points $p_i = (x_i, x_i^2)$ for $i = 1, \ldots, n$. The convex hull of $p_1, \ldots, p_n$ gives the order of $x_1, \ldots, x_n$. □

Notice that the previous lower bounds have been proved in two dimensions, but they hold in any dimension $d \geq 2$, for we may have $n$ coplanar points as input, and the output will be the same as in dimension 2 (this is not true for dimension 1).

The convex hull is not the only element used to capture the shape of a set that has been studied from the point of view of its lower bounds. The angular maximality is the first generalization that we will consider. The following result is due to Kung, Luccio and Preparata [10]:

**Maximal points**
*Given $n$ points in the plane, report those who are axes-parallel $\pi/2$-maximal.*
*In the comparison tree model, any algorithm that solves this problem requires time $\Omega(n \log n)$.*

*Proof:* By reduction from *sorting*. See [14] for details. □

This result was extended and improved by F. Hurtado in [7]:

**Unoriented $\theta$-maximal points**
*Given $n$ points in the plane, report those who are unoriented $\theta$-maximal, for $\theta \geq \pi/2$.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from *integer element uniqueness*, which was proved to be $\Omega(n \log n)$ by Yao [18]. See [7] for details. □

Another interesting information about the "shape" of a set of points is the depth of each of its elements. The depth of a point $p$ in a set $S$ is defined as the number of convex hulls that have to be stripped from $S$ in order to remove $p$.

**Depth of points**

*Given a set of $n$ points in the plane, find the depth of each point.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from *sorting*. Given $n$ real numbers, think of them as imbedded in the plane (in the $x$ axis, for example). Knowing the depth of each point allows to trivially sort them in linear time. □

Finally, we will consider the diameter of a set of points.

**Diameter**

*Given a set of $n$ points in the plane, find its diameter.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction form *set disjointness*. WLG, consider the two sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ to be in $\mathbb{R}^+$. For each $i \in \{1, \ldots, n\}$ construct the following points $p_i$ and $q_i$:

$$p_i = \left( \frac{1}{\sqrt{a_i^2+1}}, \frac{a_i}{\sqrt{a_i^2+1}} \right), \qquad q_i = \left( \frac{-1}{\sqrt{b_i^2+1}}, \frac{-b_i}{\sqrt{b_i^2+1}} \right).$$

It is easy to see that $p_i$ is the intersection of the line $y = a_i x$ with the first quadrant of the unit circle, while $q_i$ is the intersection of the line $y = b_i x$ with the third quadrant of the same circle.

A similar construction could be done by setting

$$p_i = \left( \frac{1-a_i^2}{1+a_i^2}, \frac{2a_i}{1+a_i^2} \right), \qquad q_i = \left( -\frac{1-b_i^2}{1+b_i^2}, -\frac{2b_i}{1+b_i^2} \right).$$

In this later case, the points $p_i$ (respectively $q_i$) have been obtained by intersecting the first (resp. third) quadrant of the unit circle with the lines that form an angle $2 \arctan a_i$ (resp. $2 \arctan b_i$) with the horizontal axis.

In any of both cases, knowing the diameter of the point set $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ directly allows to solve the set disjointness problem for $A$ and $B$: the diameter equals 2 if, and only if, $A \cap B \neq \emptyset$. □

# 5 Proximity problems

I include here several well known lower bounds that are related to proximity problems such as closest pair, euclidean minimum spanning tree or Voronoi diagram, and also triangulation and gaps. Most of the problems in this section are reported in the book of Preparata and Shamos [14]. Specially relevant for the following sections will be the results on the uniform and the maximum gaps. These results are due to Lee and Wu [11].

**Closest pair**

*Given $n$ points in the plane, determine two whose mutual distance is smallest.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction form *element uniqueness*: if the closest pair in $x_1, \ldots, x_n \in \mathbb{R}$ is at distance 0, two of the values are the same. □

**All nearest neighbors**

*Given n points in the plane, find a nearest neighbor of each.*
*For this problem,* $C(W)$, $M(W)$, $C_d(W)$ *are* $\Omega(n \log n)$.

*Proof:* By (obvious) reduction from *closest pair*. □

**Euclidean minimum spanning tree**

*Given n points in the plane, construct a tree of minimal total length whose vertices are the given points.*
*For this problem,* $C(W)$, $M(W)$, $C_d(W)$ *are* $\Omega(n \log n)$.

*Proof:* By (obvious) reduction from *closest pair*. It can also be done by reduction from *sorting*, because the EMST of $x_1, \ldots, x_n \in \mathbb{R}$ trivially gives the sorted list. □

**Triangulation**

*Given n points in the plane, join them by non intersecting straight line segments so that every region internal to the convex hull is a triangle.*
*For this problem,* $C(W)$, $M(W)$, $C_d(W)$ *are* $\Omega(n \log n)$.

*Proof:* By reduction form *sorting*. Given $x_1 \ldots, x_n \in \mathbb{R}$, consider the points $p_i = (x_i, 0)$ $i \in \{1, \ldots, n\}$, together with $q = (0, -1)$. A triangulation of $\{p_1, \ldots, p_n, q\}$ immediatly gives the order of $x_1, \ldots, x_n$. An alternative proof is a reduction from *convex hull*. □

**Voronoi diagram**

*Given n points in the plane, construct their Voronoi diagram.*
*For this problem,* $C(W)$, $M(W)$, $C_d(W)$ *are* $\Omega(n \log n)$.

*Proof:* By reduction from any of the previous proximity problems. It can be also done by reduction from *sorting*, or from *convex hull*. □

This ends the classical Computational Geometry proximity problems. As for the gaps, I will refer to results obtained by Lee and Wu in 1986. Their paper [11] was the first to state an $\Omega(n \log n)$ lower bound for these problems in the algebraic computation model. Actually, they extended this result to the analogous problems on the circle, but this will be reported in a following section.

**Uniform gap**

*Given n real numbers* $x_1, x_2, \ldots, x_n$, *and a positive value* $\epsilon \in \mathbb{R}^+$, *determine if the gaps between consecutive numbers are uniformly* $\epsilon$.
*For this problem,* $C(W)$, $M(W)$, $C_d(W)$ *are* $\Omega(n \log n)$.

*Proof:* The *uniform gap* problem is the membership problem for $W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \exists \sigma \in \mathfrak{S}_n \; x_{\sigma(i+1)} = x_{\sigma(i)} + \epsilon, \; i = 1, \ldots, n-1\}$. We can easily see that $\#W \geq n!$ in the following way (the details of this proof do not appear in [11], in the present version, they are due to F. Hurtado [8]). Consider the functions $f_{ij}(x_1, \ldots, x_n) = \mathrm{sgn}(x_i - x_j)$. All these functions are constant on each $W_\sigma$. For $\sigma, \tau \in \mathfrak{S}_n$, if $\sigma \neq \tau$ then at least one of the functions $f_{ij}$ must take different values over $W_\sigma$ and $W_\tau$. Hence, any path going from $W_\sigma$ to $W_\tau$ must go through a point of $\mathbb{R}^n$ where the value of that $f_{ij}$ is 0, that is, must go through a point that connot be in $W$. □

The *maximum gap* problem had an established $\Omega(n \log n)$ time bound [12] under the decision tree model. Lee and Wu proved that the lower bound also holds true for the algebraic computation tree model [11]. It is worth to remind that by simply adding the floor function to the model, the *maximum gap* problem can be solved in $O(n)$ time, as was proved by Gonzalez in 1975 [5].

**Maximum Gap**
*Given $n$ real numbers $x_1, x_2, \ldots, x_n$, find the maximum difference between consecutive numbers.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* The *uniform gap* problem can be reduced to the *maximum gap* problem in $O(n)$ time. Given an instance of the *uniform gap* problem, first use the *maximum gap* algorithm to compute the maximum gap $g$ of the $n$ numbers $x_1, x_2, \ldots, x_n$. If $g \neq \epsilon$, reject. Otherwise, compute the maximum $x_{max}$ and the minimum $x_{min}$ of the $n$ numbers. If $x_{max} - x_{min} = (n-1)\epsilon$, accept. Otherwise, reject.

It is obvious that the above process correctly solves the *uniform gap* problem and it can be done in $O(n)$ plus the time spent for computing the maximum gap. Therefore, the *maximum gap* problem also requires $\Omega(n \log n)$ time. □

**Largest empty cicle**
*Given $n$ points in the plane, find the largest circle that contains no points of the set and whose center belongs to the convex hull of those points.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* In dimension $d = 1$, the *largest empty circle* problem is the *maximum gap* problem. The same happens in any dimension, if the points are aligned. □

**$\epsilon$-closeness**
*Given $n$ real numbers and a positive real value $\epsilon$, determine whether any two points are at distance less that $\epsilon$ from each other.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* This is equivalent to the memebership problem for

$$W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \exists \sigma \in \mathfrak{S}_n \; x_{\sigma(i+1)} > x_{\sigma(i)} + \epsilon, \; i = 1, \ldots, n-1\}.$$

It is easy to prove that $\#W \geq n!$ using the functions $f_{ij}(x_1, \ldots, x_n) = \text{sgn}(x_i - x_j)$. □

# 6 Gaps on the circle

In their paper, Lee and Wu proved an $\Omega(n \log n)$ lower bound for what they called the *uniform gap* and the *maximum gap* problems *on the circle*. To be more precise, their version refers only to a *quadrant* of a circle, and cannot be extended to the entire circle in a straight way. This is our version of the first of the two problems, which intends to simplify the version that appears in [11]. The details of the present proofs are due to V. Sacristán and F. Hurtado [8].

**Uniform gap on a quadrant of a circle**
*Given $n$ points $(x_1, y_1)$, $(x_2, y_2), \ldots, (x_n, y_n)$ on the first quadrant of the unit circle, and $\epsilon \in \mathbb{R}^+$, determine if the straight line distances between neighboring points on the unit circle are uniformly $\epsilon$.*

*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* The *uniform gap on a quadrant of a circle* problem is the membership problem for $W \subset \mathbb{R}^{2n}$ where $W = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \mid x_1^2 + y_1^2 = \ldots = x_n^2 + y_n^2 = 1;\ x_1, \ldots, x_n \geq 0;\ y_1, \ldots, y_n \geq 0;\ \exists \sigma \in \mathfrak{S}_n\ x_{\sigma(i)} < x_{\sigma(i+1)},\ i = 1, \ldots, n-1;\ (x_{\sigma(i)} - x_{\sigma(i+1)})^2 + (y_{\sigma(i)} - y_{\sigma(i+1)})^2 = \epsilon^2,\ i = 1, \ldots, n-1\}$. It can be seen that $\#W \geq n!$. The proof is analogous to that of the *uniform gap*. Consider the functions $f_{ij}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \mathrm{sgn}(x_i - x_j)$. All of them are constant over each $W_\sigma$. For $\sigma, \tau \in \mathfrak{S}_n$, if $\sigma \neq \tau$ then at least one of the functions $f_{ij}$ must take different values over $W_\sigma$ and $W_\tau$. Hence, any path from $W_\sigma$ to $W_\tau$ must go through a point of $\mathbb{R}^n$ in which the mentioned function vanishes, that is, a point that does not belong to $W$. $\square$

As for the *maximum gap*, the following result was given by Lee and Wu in [11], I only worked out the details of their proof, and made it apparently longer.

**Maximum gap on a quadrant of a circle**
*Given $n$ points $(x_1, y_1)$, $(x_2, y_2), \ldots, (x_n, y_n)$ on the first quadrant of the unit circle, find the maximum straight line distance between neighboring points on the unit circle.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* The *uniform gap on a quadrant of a circle* problem can be reduced to the *maximum gap on a quadrant of a circle* problem in $O(n)$ time. Given an instance of the *uniform gap on a quadrant of a circle* problem, we do the following:

1. If $\epsilon > \sqrt{2}$, reject.

2. Use the *maximum gap on a quadrant of a circle* algorithm to compute the maximum straight line distance $g$ between neighboring points on the unit circle. If $g \neq \epsilon$, reject.

3. Compute the maximum $x_{max}$ and the minimum $x_{min}$ of the $x$-coordinates of the $n$ points. We need to check whether or not the arc from $(x_{min}, y_{min})$ to $(x_{max}, y_{max})$, let's call it $\beta$, is $(n-1)$ times the maximum arc that we just found (call it $\alpha$). Instead of directly checking whether

$$\beta = (n-1)\alpha,$$

we pose the equation it terms of distances: if we call $d$ the distance between $(x_{min}, y_{min})$ and $(x_{max}, y_{max})$, then $\beta = 2 \arcsin \frac{d}{2}$ and $\alpha = 2 \arcsin \frac{\epsilon}{2}$, and the equality to be checked is:

$$2 \arcsin \frac{d}{2} = (n-1)\, 2 \arcsin \frac{\epsilon}{2},$$

or, equivalently:

$$d = 2 \sin \left( (n-1) \arcsin \frac{\epsilon}{2} \right).$$

Of course there are two evident objections:

- we cannot perform such operations in the algebraic computation model;
- arcsin is not a well defined function.

Nevertheless, this two objections can be solved in the following way:

4. Compute the straight line distance between the two extreme points $(x_{min}, y_{min})$, and $(x_{max}, y_{max})$:
$$d = \sqrt{(x_{max} - x_{min})^2 + (y_{max} - y_{min})^2}.$$
Check $d$ against $d_\epsilon$ (see details below). If $d = d_\epsilon$, accept, else reject.

   Let us see how to compute $d_\epsilon$ in the algebraic computation model. We will use the following trigonometric equalities
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
to recursively compute $s_i = \sin\left(i \arcsin\frac{\epsilon}{2}\right)$ and $c_i = \cos\left(i \arcsin\frac{\epsilon}{2}\right)$, for $i = 1, \ldots, n$. The recurrence is as follows:
$$\begin{cases} s_1 = \frac{\epsilon}{2}, \\ c_1 = \sqrt{1 - s_1^2} = \frac{1}{2}\sqrt{4 - \epsilon^2}, \end{cases} \qquad \begin{cases} s_{i+1} = s_i c_1 + c_i s_1, \\ c_{i+1} = c_i c_1 - s_i s_1, \end{cases}$$
and finally $d_\epsilon = 2s_{n-1}$.

   As a comment, it can be noticed that computing $s_i$ and $c_i$ does not involve as many square roots as it could be thought, for it can be proved by induction that, for all $i$,
$$s_{2i} = p(\epsilon)\sqrt{4 - \epsilon^2},$$
$$s_{2i+1} = q(\epsilon),$$
$$c_{2i} = r(\epsilon),$$
$$c_{2i+1} = s(\epsilon)\sqrt{4 - \epsilon^2},$$
where $p$, $q$, $r$ and $s$ are polynomial functions of $\epsilon$.

   This solves the first objection. As for the second, at each stage we check the sign of $c_i$: if at some point $c_i < 0$, reject, else, keep going. □

The previous version of the *max gap* problem is enough to fix our proofs for the implicit polygons, because we can construct the polygons from points placed in the first quadrant of the circle and their diametrally opposite points in the circle (see the following sections). Nevertheless, for our facility location lower bounds, we need another version of the *max gap* problem, that allows to place the points in the entire circle. This does not appear in the paper by Lee and Wu. The proof is due to F. Hurtado [8].

**Uniform gap on a circle**
*Given $n$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ on the unit circle, and $\epsilon \in \mathbb{R}^+$, determine if the straight line distances between consecutive points on the unit circle are uniformly $\epsilon$. For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Comment:* two points $(x_i, y_i)$ and $(x_j, y_j)$ are said to be *consecutive* in any of the following four cases:

1. $x_j \le x_i$; $y_i, y_j > 0$; and $\nexists k$ such that $x_j < x_k < x_i$ and $y_k > 0$;
2. $x_i \le x_j$; $y_i, y_j < 0$; and $\nexists k$ such that $x_i < x_k < x_j$ and $y_k < 0$;
3. $x_i$ is the leftmost point with $y_i > 0$, and $x_j$ is the leftmost point with $y_j < 0$;

4. $x_i$ is the rightmost point with $y_i < 0$, and $x_j$ is the rightmost point with $y_j > 0$.

*Proof:* The *uniform gap on a circle* problem is the membership problem for $W = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 = \ldots = x_n^2 + y_n^2 = 1;\ \exists \sigma \in \mathfrak{S}_n$ with $\sigma(n) = n$, such that $(x_{\sigma(i)}, y_{\sigma(i)})$ and $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$ are consecutive for $i = 1, \ldots, n \pmod n$; $(x_{\sigma(i)} - x_{\sigma(i+1)})^2 + (y_{\sigma(i)} - y_{\sigma(i+1)})^2 = \epsilon^2$, $i = 1, \ldots, n \pmod n\}$. Notice that each $\sigma \in \mathfrak{S}_n$ with $\sigma(n) = n$ can be identified with its restriction to $\{1, \ldots, n-1\}$, so that $W$ is decomposed into $(n-1)!$ sets $W_\sigma$, with $\sigma \in \mathfrak{S}_{n-1}$. We will prove that $\#W \geq (n-1)!$.

For each pair $i, j \in \{1, \ldots, n-1\}$, consider the function

$$f_{ij}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \operatorname{sgn} \begin{vmatrix} x_n & y_n & 1 \\ x_{\sigma(i)} & y_{\sigma(i)} & 1 \\ x_{\sigma(j)} & y_{\sigma(j)} & 1 \end{vmatrix}.$$

All these functions are constant in each $W_\sigma$. Given two permutations $\sigma \neq \tau \in \mathfrak{S}_{n-1}$, at least one of the functions $f_{ij}$ takes different values over $W_\sigma$ and $W_\tau$. Hence, each $W_\sigma$ belongs to a different connected component of $W$, and $\#W \geq (n-1)!$. □

**Maximum gap on a circle**

*Given $n$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ on the unit circle, find the maximum straight line distance between neighboring points on the unit circle.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* The *uniform gap on a circle* problem can be reduced to the *maximum gap on a circle* problem in $O(n)$ time. Given an instance of the *uniform gap on a circle* problem, we do the following:

1. If $\epsilon > 2$, reject.
2. If $\sqrt{2} \leq \epsilon \leq 2$ solve the problem by hand.
3. Use the *maximum gap on a circle* algorithm to compute the maximum straight line distance $g$ between consecutive points on the unit circle. If $g \neq \epsilon$, reject.
4. We need to check whether or not the maximum arc corresponding to the maximum gap found in the previous step (call it $\alpha$), equals $2\pi/n$:

   $$n\alpha = 2\pi.$$

   In terms of $\epsilon$,

   $$n\, 2 \arcsin \frac{\epsilon}{2} = 2\pi,$$

   or, equivalently:

   $$\begin{cases} \sin\left(n \arcsin \frac{\epsilon}{2}\right) = 0, \\ \cos\left(n \arcsin \frac{\epsilon}{2}\right) = -1. \end{cases}$$

   Again, there are two objections:

   - we cannot perform such operations in the algebraic computation model;
   - arcsin is not a well defined function.

   Nevertheless, the first of the two objections can be solved analogously as it was for the *maximum gap on a quadrant of a circle*, and the second can be solved by making sure that $\cos\left(i \arcsin \frac{\epsilon}{2}\right)$ is never negative when $i = 1, \ldots, \lfloor \frac{n}{4} \rfloor$ (assuming that $(x_1, y_1)$ is the rightmost point on the upper halfcircle). □

# 7 Implicit convex poligons (intersection of halfplanes)

We finally come to the problems that caused me to study all the stuff that you found in the previous sections.

Given a polygon $P$ in the plane, there are many elementary problems, such as deciding whether or not a given point $p$ belongs to $P$, computing the supporting lines of $P$ from an external point $p$, or finding the minimum circle that encloses $P$, that, in spite of their apparent simplicity, are fundamental pieces of many different applications. As a consequence, these problems have been largely studied, and the obtained results are well known. They are not all of the same complexity, but most of them can be solved very efficiently, specially when the involved polygons are known to be convex. Most of the solution algorithms strongly rely on the order of the list of vertices of the polygons, and not much has been said for other situations, such as the case in which the polygons under consideration are given as the convex hull of a set of non ordered and possibly redundant points, or the case in which they are given by a set of linear restrictions, i.e. by a possibly redundant and unordered intersection of halfplanes. There is no need of much justification of the claim that these problems are very natural, for they modelize a great number of real problems.

In both cases, one could always "construct" the polygons, that is, obtain the ordered lists of their vertices from the sets of points or halfplanes that define the polygons, and then apply the well known algorithms to solve the problems for the "ordered" polygons. But the complexity of such a procedure would always be $\Omega(n \log n)$, for the reconstruction of the polygons cannot be done more efficiently.

In a joint work with F. Gómez, F. Hurtado, S. Ramaswami and G. Toussaint [4] we proved that many of these problems can be solved in optimal $\Theta(n)$ time. The algorithms that we proposed are based on a prune-and-search strategy. But some of the problems came out to have complexity $\Omega(n \log n)$. Here we offer the proofs relative to the polygons defined as intersection of halfplanes.

**Width**

*Compute the width of the intersection of a set of $n$ halfplanes.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from *set disjointness*: given two sets of $n$ real positive numbers, $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, consider the following points on the unit circle (see Figure 1):

- for each $a_i \in A$, construct $p_i = \left(\dfrac{1-a_i^2}{1+a_i^2}, \dfrac{2a_i}{1+a_i^2}\right)$;
- for each $b_i \in B$, construct $q_i = \left(-\dfrac{1-b_i^2}{1+b_i^2}, -\dfrac{2b_i}{1+b_i^2}\right)$.

Remember, from the construction for *diameter*, that the points $p_i$ (respectively $q_i$) have been obtained by intersecting the first (resp. third) quadrant of the unit circle with the lines that form an angle $2\arctan a_i$ (resp. $2\arctan b_i$) with the horizontal axis.

Consider the line through each point $p_i$ and $q_i$ tangent to the circle. Among the two halfplanes that each of these lines define, take the one enclosing the circle. Apply the

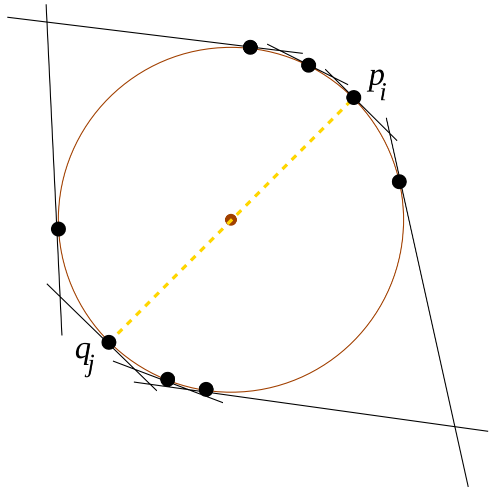


Figure 1: If $width = 2$, then $p_i = q_j$, and $a_i = b_j$. Hence $A \cap B \neq \emptyset$.

*width* algorithm to this intersection of halfplanes. If the width is 2, then $A \cap B \neq \emptyset$, otherwise (i.e., if the width is greater than 2) $A$ and $B$ are disjoint. □

**Diametral pair**
*Compute a diametral pair of points of the intersection of a set of $n$ halfplanes.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from *maximum gap on a quadrant of a circle.* Given $n$ points on the first quadrant of the unit circle, $p_1, \ldots, p_n$, compute their diametrally opposite $q_i = -p_i$, $i = 1, \ldots, n$. Consider the line through each point $p_i$ and $q_i$ tangent to the circle (refer to Figure 2). Among the two halfplanes that each of these lines define, take that enclosing the circle. Add two new lines to the set, namely the line through the rigmost points $p_1$ and $q_1$, together with the line through the leftmost points $p_n$ and $q_n$. Among the two halfplanes that each of these two lines define, take that enclosing all the points $p_i, q_i$.

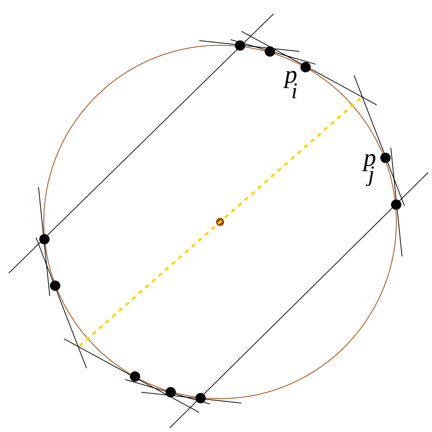


Figure 2: The diametral pair is reached in the intersection of the tangent lines through the two points $p_i$, $p_j$ that define the maximum gap.

Apply the *diametral pair* algorithm to the intersection of all these halfplanes. The diameter is reached in the intersection of the two tangent lines through the points $p_i, p_j$ (and $q_i, q_j$) that define the maximum gap in $p_1, \ldots, p_n$. □

**Minimum enclosing circle**

*Compute the minimum enclosing circle of the intersection of a set of $n$ halfplanes.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from *maximum gap on a quadrant of a circle.* The construction is the same as in the previous reduction (refer to Figure 3).

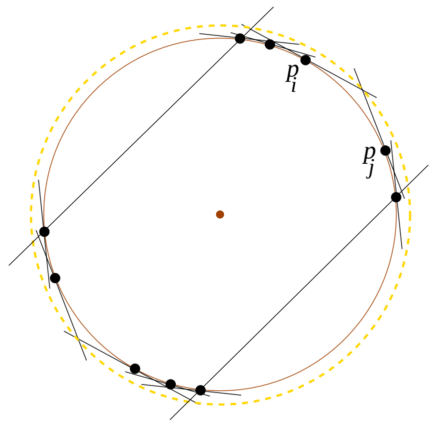


Figure 3: The minimum enclosing circle is determined by the points $p_i$, $p_j$ that define the maximum gap.

Apply the *minimum enclosing circle* algorithm to this intersection of halfplanes. As in the previous reduction, the circle is determined by two diametrically opposite vertices of the implicit polygon, namely the intersection of the two tangent lines through the points $p_i, p_j$ and $q_i, q_j$ that define the maximum gap in $p_1, \ldots, p_n$. □

# 8 Implicit convex polygons (convex hull of points)

In this section you will find the proofs that some problems have complexity $\Omega(n \log n)$ for convex polygons defined as the convex hull of a set of points. The reader will notice the existence of an evident duality with respect to the problems in the previous section.

**Width**

*Compute the width of the convex hull of a set of $n$ points.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* (This proof was done by Lee and Wu in [11].) By reduction from *maximum gap on a quadrant of a circle.* Given $n$ points on the first quadrant of the unit circle, $p_1, \ldots, p_n$, compute their diametrally opposite $q_i = -p_i$, $i = 1, \ldots, n$ (see Figure 4). Add two more points, namely $p_0$ and $q_0$ that are computed in the following way: $p_0$ is the intersection of the lines through the leftmost points $p_i$ and $q_i$ that are tangent to the circle, while $q_0 = -p_0$. To be more precise let us call $p_1 = (x_1, y_1)$ the leftmost point among the $p_i$, and let $p_n = (x_n, y_n)$ be the rightmost point. Then,

$$p_0 = \left( -\frac{y_n + y_1}{x_n y_1 - x_1 y_n}, \frac{x_n + x_1}{x_n y_1 - x_1 y_n} \right), \qquad q_0 = -p_0.$$

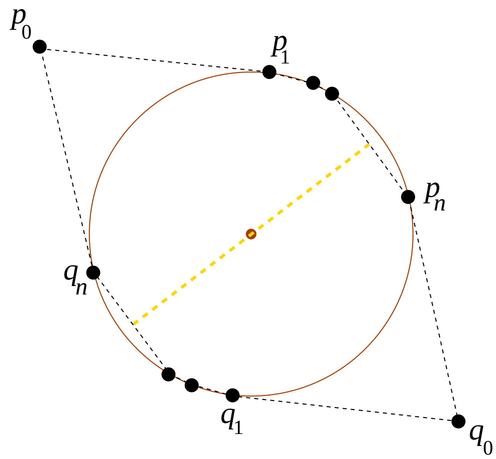


Figure 4: The width detects the maximum gap.

Apply the *width* algorithm to this set of points. The width is reached in the edges of the convex hull through the points $p_i, p_j$ (and $q_i, q_j$) that define the maximum gap in $p_1, \ldots, p_n$. □

**Maximum enclosed circle**

*Compute the maximum circle enclosed in the convex hull of a set of $n$ points.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* By reduction from *maximum gap on a quadrant of a circle*. The construction is analogous to that in the previous problem (see Figure 5).

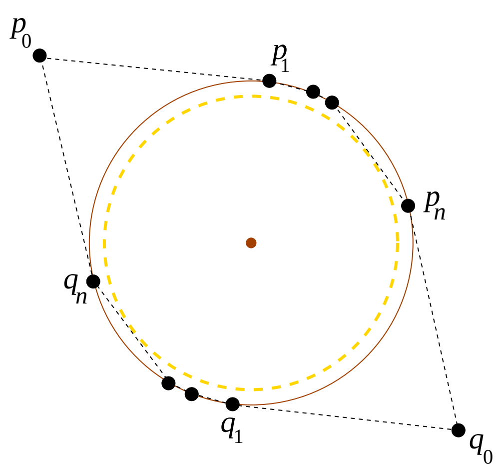


Figure 5: The minimum enclosed circle detects the maximum gap.

Apply the *maximum enclosed circle* algorithm to this set of points. The circle is tangent to the edges of the convex hull through the points $p_i, p_j$ (and $q_i, q_j$) that define the maximum gap in $p_1, \ldots, p_n$. □

**Diameter**

*Compute the diameter of the convex hull of a set of $n$ points.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

This result has been reported in Section 4 (Morphology problems), I only remind it here (together with its proof) in order to make more evident the duality between this set of problems and those in the previous section.

*Proof:* By reduction from *set disjointness*: given two sets of $n$ real positive numbers, $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, consider the following points in the unit circle (see Figure 6):

- for each $a_i \in A$, construct $p_i = \left(\frac{1-a_i^2}{1+a_i^2}, \frac{2a_i}{1+a_i^2}\right)$;
- for each $b_i \in B$, construct $q_i = \left(-\frac{1-b_i^2}{1+b_i^2}, -\frac{2b_i}{1+b_i^2}\right)$.

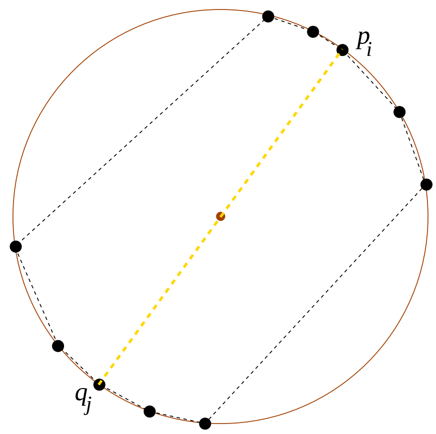


Figure 6: If *diameter* $= 2$ then $p_i = q_j$, and $a_i = b_j$. Hence, $A \cap B \neq \emptyset$.

Apply the *diameter* algorithm to this set of points. If the diameter equals 2, then $A \cap B \neq \emptyset$, otherwise (i.e., if the diameter is smaller than 2) $A$ and $B$ are disjoint. □

## 9 Facility location

In this last section I report some lower bound results for both minimax and maximin location problems. These results are related to a joint paper with F. Hurtado and G. Toussaint [9], in which we considered constrained versions of the classical Euclidean facility location problems. Our main result is an optimal linear time algorith for the case in which the 1-center is constrained to satisfy a set of linear restrictions. Here you have three related problems for which one can prove an $\Omega(n \log n)$ lower bound.

**Minimax in the plane**
*Compute the point that minimizes the maximum distance to all the points that belong to the intersection of a set of $n$ halfplanes.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* Done in section 7 (minimum enclosing circle of the implicitly given polygon). □

**Maximin in the plane**
*Compute the point that belongs to the convex hull of a given set of $n$ points and maximizes the minimum distance to its boundary.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* Done in section 8 (maximum enclosed circle of the implicitly given polygon). □

**Minimax on the sphere**

*Compute the point of the sphere that minimizes the maximum geodesic distance to a given set of $n$ points on the sphere.*
*For this problem, $C(W)$, $M(W)$, $C_d(W)$ are $\Omega(n \log n)$.*

*Proof:* The problem is to find the minimum spherical cap that encloses the set of points. We prove the complexity of this problem by reduction from the *maximum gap on a circle* problem, in $O(n)$ time. Given an instance of the *maximum gap on a circle* problem, $(x_1, y_1), \ldots, (x_n, y_n)$, we do the following:

1. Consider the points $p_i = (x_i, y_i, 0)$, for $i = 1, \ldots, n$, on the unit sphere.

2. Add the two poles of the sphere: $p_0 = (0, 0, 1)$, $p_{n+1} = (0, 0, -1)$.

3. Apply the *minimum enclosing cap* algorithm to $p_0, p_1, \ldots, p_n, p_{n+1}$.

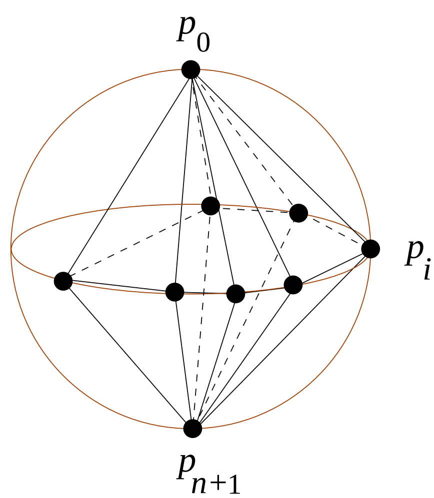


Figure 7: The convex hull triangles.

Finding the minimum cap covering the points $p_0, p_1, \ldots, p_n, p_{n+1}$ is the same as finding the maximum free cap that they determine on the sphere or, equivalently, finding the maximum triangle formed in the convex hull of $p_0, p_1, \ldots, p_n, p_{n+1}$ (Figure 7). This largest triangle is determined by the largest free arc between $p_1, \ldots, p_n$, that is, by the maximum gap between the original points $(x_1, y_1), \ldots, (x_n, y_n)$.

Notice that the maximum empty cap is determined by a plane which is the same that contains the largest triangle on the convex hull, except when $p_1, \ldots, p_n$ lie in a halfsphere. In that case, the edge $p_0 p_{n+1}$ belongs to the convex hull and the maximum empty cap is a halfsphere: its boundary is a maximum circle through $p_0$ and $p_{n+1}$, and is determined by two points, not three. But even in this case the maximum empty cap determines the maximum gap in $(x_1, y_1), \ldots, (x_n, y_n)$. □

# References


[1] A. M. Ben-Amram. Lower bounds on algebraic random access machienes, *Proc. 22nd Internat. Colloq. Automata Lang. Prog.*, LNCS 944, pp. 360-371, 1995.

[2] M. Ben-Or. Lower bounds for algebraic computation trees, *Proc. 15th Annual Symposium on Theory of Computing*, pp. 80-86, 1983.

[3] P. Dobkin and R. J. Lipton. On the complexity of computations under varying sets of primitives, *J. Computer and System Sciences*, Vol. 18, pp. 86-91, 1979.

[4] F. Gómez, F. Hurtado, S. Ramaswami, V. Sacristán and G. Toussaint. Implicit convex polygons, *Proc. 14th European Workshop on Computational Geometry*, pp. 83-84, 1998.

[5] T. Gonzalez. Algorithms on sets and related problems, Tech. Report, Univ. of Oklahoma, 1975.

[6] D. Grigoriev and N. Vorobjov. Complexity lower bounds for computation trees with elementary thanscendental function gates, *Theoretical Computer Science*, Vol. 157, pp. 185-214, 1996.

[7] F. Hurtado. Problemas geométricos de visibilidad, PhD. Thesis, Univ. Politècnica de Catalunya, 1994.

[8] F. Hurtado. Personal communication, 1998.

[9] F. Hurtado, V. Sacristán and G. Toussaint. Some constrained minimax and maximin location problems, *Studies in Locational Analysis*, to appear.

[10] H. T. Kung, F. Luccio and F. P. Preparata, On finding the maxima of a set of vectors, *J. of the ACM*, Vol. 22, N. 4, pp. 469-476, 1975.

[11] D. T. Lee and Y. F. Wu, Geometric complexity of some location problems, *Algorithmica*, Vol. 1, pp. 193-211, 1986.

[12] U. Mamber and M. Tompa, The complexity of problems on probabilistic, nondeterministic and alternating decision trees, *J. of the ACM*, Vol. 32, N. 3, pp. 720-732, 1985.

[13] J. Milnor. On the Betti numbers of real varieties, *Proc. of the AMS*, Vol. 15, pp. 275-280, 1964.

[14] F. P. Preparata and M. I. Shamos, Computational Geometry: An Introduction, Springer-Verlag, 1986.

[15] S. Sen. Lower bounds for parallel algebraic decision trees, complexity of convex hulls and related problems, *Proc. 14th Conf. Found. Soft. Tech. Theoret. Comput. Sci.*, pp. 193-204, 1994.

[16] J. M. Steele and A. C. Yao, Lower bounds for algebraic decision trees, *J. of Algorithms*, Vol. 3, pp. 1-8, 1982.

[17] R. Thom. Sur l'homologie des variétées algébriques réelles, *Differential and Combinatorial Topology*, S. S. Cairns ed., Princeton University Press, 1965.

[18] A. C. Yao. Lower bounds for algebraic computation trees with integer inputs, *SIAM J. Computing*, Vol. 20, pp. 308-313, 1991.

[19] A. C. Yao. Decision tree complexity and Betti numbers, *Proc. ACM Symposium on Theory of Computing*, pp. 615-624, 1994.