Bichromatic 2-center Problem *

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Abstract

Given two sets of points which can be separated by a straight line, we want to find two points in the plane whose bisector separates the sets, representing each, one of the two sets. The quality of the chosen representant is measured as the radius of the circle centered at the new point containing its corresponding set. In other words, how can we find two circles of minimal equal size so that the bisector of the centers separates the point set into two predefined clusters and each circle encloses one cluster? In this paper we characterize the solution and show an algorithm for this problem.

Keywords: 1-center, 2-center, facility location, classifiers.

1. Introduction

We are given two sets S_1 and S_2 of points of two different colors, such that all points in each set share the same color, and such that the two sets can be separated by a straight line. We want to find two new points v_1 and v_2 whose Voronoi Regions include all the points of a single color. The radius of the enclosing circle of each set with center at the corresponding new point

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should be minimized, that is, we want to minimize the maximal radius of the two circles. Even though in our case the two sets are given, there is a strong relationship between this problem and the well known facility location *p-center* problem (see [1], [2] or [3]). We will see that, as in [2] and [3], it is interesting to divide the problem into two cases: when the two sets are "well" separated and when they are not. In the first case, the 1-center solution for one of the two sets, leads already to a solution to our problem. Otherwise, we will see that it is not a trivial task to place the two center points.

If the separating bisector b is given, it is easy to compute the two centers. Take $S_1(b)$ to be the symmetric set of S_1 with respect to b. Then, compute the minimal enclosing circle of $S_1(b) \cup S_2$ (see Figure 1). The center of this circle and its reflected image with respect to b are the two solution points. But, what can we do in the general case in order to find a bisector that separates both sets and minimizes this enclosing circle?

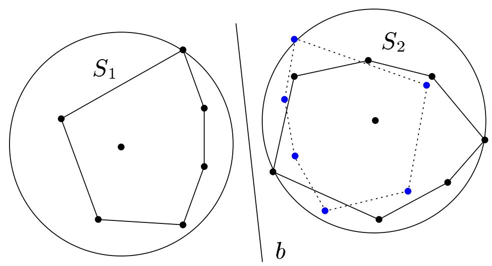

Figure 1: Computing the solution for a given bisector b.

2. One of the two 1-centers is a solution

Let the circle $C_1(c_1, r_1)$ with center at c_1 and radius r_1 (resp. $C_2(c_2, r_2)$) be the smallest enclosing circle for the set S_1 (resp. S_2). Without loss of generality, we will suppose that $r_2 \geq r_1$. In this section we will show how to decide wether $C_2(c_2, r_2)$ gives a solution with $v_2 = c_2$ and how to find a region of valid solutions for the point v_1.

The most simple case is when the circles $C_1(c_1, r_1)$ and $C_2(c_2, r_2)$ do not intersect. If the bisector of the two center points v_1 and v_2 separates the two sets, then $c_1 = v_1$ and $c_2 = v_2$ is already a solution. Otherwise, this bisector must intersect the circle C_2. Let b be the line parallel to the bisector, tangent to C_2 and to the same side of C_1. Since the two circles do not intersect, b separates the two sets. Furthermore, the circle symmetric to C_2 with respect to b contains C_1 and, hence, the point symmetric to v_2 with respect to b is a solution for c_1 and $c_2 = v_2$.

Now that we have discarded two simple cases, let us see how to deal with more complicated situations, that is, when the minimum enclosing circles for the two sets intersect. Let us define the r-region of the set S_1 to be the locus of the points such that a circle centered at that point with radius r contains the set S_1. If we take the Farthest Point Voronoi Diagram of the set S_1 (from now on $FPVD(S_1)$), the boundary of the r-region(S_1) is the intersection of the circles with radius r centered at the points in S_1 with their corresponding Farthest Point Voronoi Regions. If r is bigger than the radius of the smallest enclosing circle of S_1, then the r-region(S_1) is not empty. Particularly, the r_2-region(S_1) is not empty and has linear complexity on the points of the convex hull of S_1 (see Figure 2).

Take B_1 (resp. B_2) to be the set of supporting lines of the convex hull of S_1, $CH(S_1)$ (resp. $CH(S_2)$) which separate S_1 and S_2. We call the c_2-umbrella(S_1, S_2) the locus of the reflected images of c_2 for all lines separating the sets. If we take the points of $S_1 \cup S_2$ which are in some line of $B_1 \cup B_2$, each point p contributes to the boundary of the c_2-umbrella(S_1, S_2) with a circular arc with center at p and radius $|pc_2|$, i. e. the distance between p and c_2. Hence, the boundary of this region has linear complexity on the number of points of $CH(S_2)$ and $CH(S_1)$ between the two common inner tangent lines (see Figure 2).

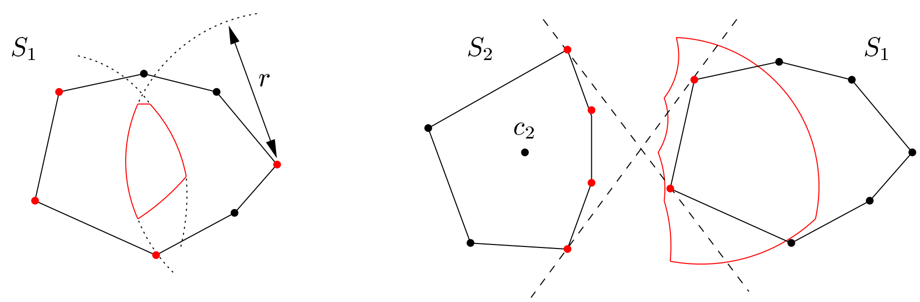

Figure 2: The r_2-region(S_1) (left) and the c_2-umbrella(S_1, S_2) (right).

Theorem 1 *If the r_2-region(S_1) and the c_2-umbrella(S_1, S_2) intersect, then $v_2 = c_2$, and any point v_1 in the intersection is a solution with optimal radius r_2.*

Proof. Since the c_2-umbrella(S_1, S_2) contains all possible reflections of c_2 and the r_2-region(S_1) contains all center points for circles containing S_1 with radius r_2, whenever these two regions intersect, all points in this intersection are solutions for v_1. This solution is optimal since no solution can show a smaller radius than the biggest 1-center solution for each set. $\square$

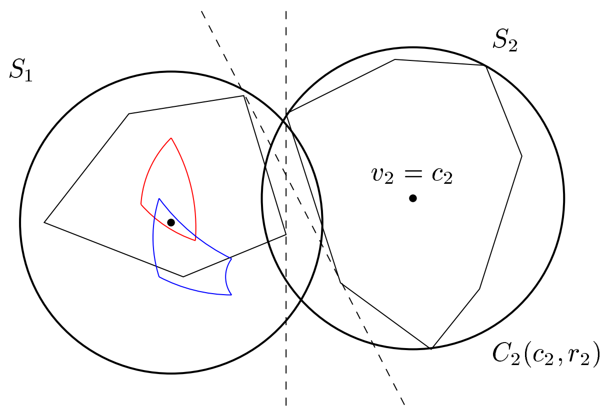

Figure 3: The r_2-region(S_1) (red) intersects the c_2-umbrella(S_1, S_2) (blue).

3. None of the two 1-centers is a solution

Let us see what happens if the r_2-region(S_1) and the c_2-umbrella(S_1, S_2) do not intersect. First of all, let us prove that not all separating lines must be considered, but only those which are supporting lines of some of $CH(S_1)$ or $CH(S_2)$.

Lemma 2 *If $c_2 = v_2$ is not a solution for the radius r_2 then the bisector for the optimal solutions touches some of the two sets S_1 or S_2.*

*P*roof. Suppose that the r_2-region(S_1) and the c_2-umbrella(S_1, S_2) do not intersect. This means that the solution needs a radius bigger than r_2. We consider the dual of the separating lines of the two sets S_1 and S_2, which is a convex set of points. On the other hand, we can compute the r-regions for S_1 and for S_2 for a given radius r and all bisectors of points in those sets. The dual of this set of lines gives also a connected region.

Since r_2 does not lead to a solution, we can increase the radius r in order to make these two sets grow. At a certain time the bisector dual set will hit the dual of the separating lines. This gives a solution and the bisector is on the boundary of the separating lines, that is, it is a supporting line of $CH(S_1)$ or $CH(S_2)$. $\square$

The previous Lemma shows that we can take all separating lines supporting some set, $B_1 \cup B_2$, reflect the set S_1 with respect to them and compute, for each line, the minimum enclosing circle of S_2 and this reflected image. The smallest of these circles is the solution.

§ 3.1. Localy optimal solutions.— From now on we will set S_1 to be to the left of S_2 and we will always reflect S_1 with respect to some bisector $b \in B_1 \cup B_2$. Note that all the properties we prove under this circumstances hold also when reflecting the set S_2.

Take some bisector $b \in B_1 \cup B_2$ and call $S_1(b)$ the reflected image of S_1 for this line. We will denote $MEC(S_1(b), S2)$ the minimum enclosing circle of $S_1(b) \cup S_2$, $c(b)$ the position of its center and $r(b)$ its radius. Since we have discarded the posibility of having the minimum enclosing circle of S_1 or S_2 equal to $MEC(S_1(b), S2)$, there must be three points from different sets on the boundary of the minimum enclosing circle. We will study the posibility of having two points from S_2 and, at least, one from $S_1(b)$ on this boundary, since the symmetric case can be obtained analogously. Under these conditions, we must consider the posibility of having the following three solutions: one point from $S_1(b)$ with b tangent at single point (*type-A*); one point from $S_1(b)$, being b tangent at two points (*type-B*); and two points from $S_1(b)$ (*type-C*).

Definition 3 *We call* type-A local optimum *the position of the bisector $b \in B_1 \cup B_2$ such that*

- *the bisector is tangent at a single point s,*
- *only one point $p(b)$ for $p \in S_1$ lies on the boundary of $MEC(S_1(b), S_2)$,*
- *two points $p_2, q_2 \in S_2$ lie on the boundary of $MEC(S_1(b), S_2)$, one above and one below $p(b)$,*
- *the center $c(b)$, s and $p(b)$ lie on a line.*

Definition 4 *We call* type-B local optimum *the position of the bisector $b \in B_1 \cup B_2$ such that*

- *the bisector is tangent at two points s_1, s_2,*
- *only one point $p(b)$ for $p \in S_1$ lies on the boundary of $MEC(S_1(b), S_2)$,*
- *two points $p_2, q_2 \in S_2$ lie on the boundary of $MEC(S_1(b), S_2)$, one above and one below $p(b)$,*
- *the center $c(b)$, is inside the wedge $s_1, p(b), s_2$.*

Definition 5 *We call* type-C local optimum *the position of the bisector $b \in B_1 \cup B_2$ such that*

- *the bisector is tangent at a single point s,*
- *two points $p(b), q(b)$, for $p, q \in S_1$ lie on the boundary of $MEC(S_1(b), S_2)$,*
- *two points $p_2, q_2 \in S_2$ lie on the boundary of $MEC(S_1(b), S_2)$, one above and one below $p(b)$,*
- *the center $c(b)$, is inside the wedge $p(b), s, q(b)$.*

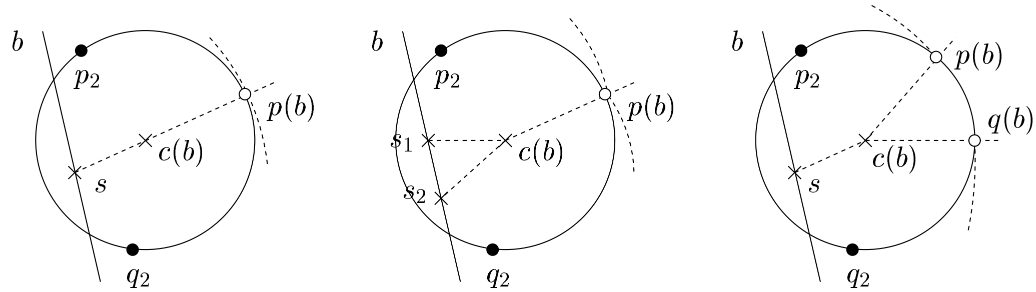

Figure 4: A slight turn on the bisector b increases the radius of the circle for type-A (left), type-B (middle) or type-C (right) local optima.

The following lemma shows that, under these conditions, we reach indeed localy optimal situations.

Lemma 6 *If bisector b reaches a* type-A, type-B*, or* type-C local optimum, *then for all α in a sufficiently small interval, if bisector b' is a CW/CCW turning of b anchored at the tangency points with angle α, then $r(b) < r(b')$.*

Proof. Suppose p_2 and q_2 were both below (resp. above) $p(b)$ in any of the three configurations. A small CW (resp. CCW) turning (b') of b anchored at the tangency points would place $p(b')$ inside $MEC(S_1(b), S_2)$, reducing its radius. With respect to *type-A* configuration, if $p(b)$ is to the left (resp. to the right) of the line through s and $c(b)$, then there is some bisector b' which is a clockwise (resp. CCW) turn of b anchored at s such that $p(b')$ lies inside $MEC(S_1(b), S_2)$, reducing its radius. Similar arguments prove *type-B* and *type-C* local optima. □

Lemma 7 *If bisector b reaches a* type-A, type-B*, or* type-C local optimum *and $s \in S_2$, then this is not the global solution.*

Proof. Let b' be a bisector parallel to b such that b' still separates S_1 and S_2 but very close to b. The point $p(b')$ (and $q(b')$ for *type-C* optimum) is inside $MEC(S_1(b), S_2)$, since $p(b)$ (and $q(b)$ for *type-C* optimum) lies on the opposite halfcircle as s. Hence, this local optimum is not a global optimum. □

§ 3.2. Local and global optimality.— In this section we will discuss the remaining cases for local optimal solutions, showing that, when the bisector is tangent to the *appropriate* set, then the local optimum is also the global optimum.

Theorem 8 *Let b_* be a bisector of B_1. If b_* is a* type-A, B *or* type-C local optimum, *then $MEC(S_1(b_*), S_2)$ is smaller than $MEC(S_1(b), S_2)$, for all $b \in B_1$, with $b \neq b_*$.*

Proof. We will prove that *type-A* and *type-B* are global optimal situations, since *type-C* is a direct consequence *type-B*.

W.l.o.g. we can assume that the convex chain of S_1 within the inner tangents of S_1 and S_2 visible from S_2 is monotone in Y and that $b_* \in B_1$ is tangent to a vertex $s_* \in S_1$. Let $s_*, s_1, \ldots, s_k$ denote the lower part of the convex chain of S_1 visible from S_2 and restricted by the inner tangent. We consider the bundle of bisectors of B_1 by rotating b_* clockwise and changing the rotational centers adequately (CCW rotation can be handled analogously). For *type-A* and *type-B* we will show that the smallest circle containing $p(b), p_2$ and q_2 must be bigger than $MEC(S_1(b_*), S_2)$, for all b CW rotations of b_*.

We say $p(b)$ satisfies condition (I) if it lies out of some of the two circles $C(p_2, 2r(b_*))$, $C(q_2, 2r(b_*))$. If $p(b)$ satisfies condition (I) then the smallest circle containing $p(b)$ and the two fixed points p_2 and q_2 is bigger than $MEC(S_1(b_*), S_2)$, since the distance from p_2 or q_2 to $p(b)$ is bigger than twice the radius of this circle (see Figure 5).

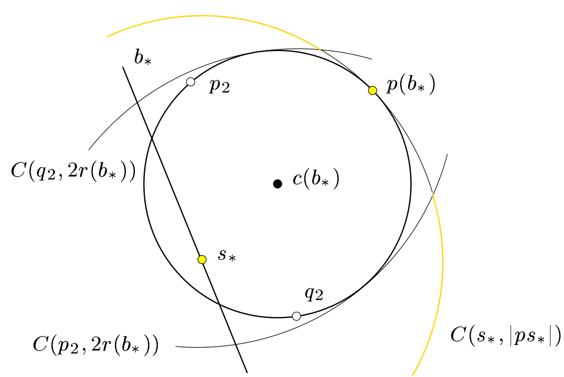

Figure 5: Points satisfying condition (I).

We say $p(b)$ satisfies condition (II) if the wedge $p_2, p(b), q_2$ contains $c(b_*)$ and $p(b)$ is outside $MEC(S_1(b_*), S_2)$. If $p(b)$ satisfies condition (II) then the circumcircle of the triangle $p_2, p(b), q_2$ is the smallest circle containing these three points and, since $p(b)$ is outside $MEC(S_1(b_*), S_2)$, it is bigger than the second (see Figure 6).

Altogether both conditions mean that the smallest circle containing p_2, q_2 and $p(b)$ is always bigger than $MEC(S_1(b_*), S_2)$. We will prove that while following the point $p(b)$ with the rotation of the bisector, one of the two conditions (I) or (II) is fulfilled.

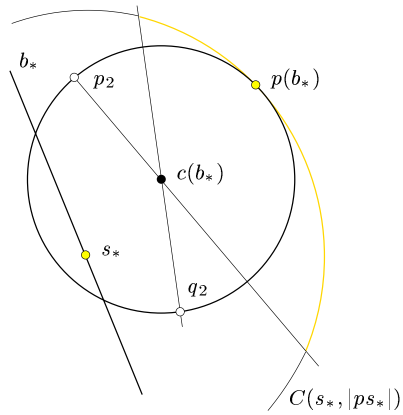

Figure 6: Points satisfying condition (II).

We start with *type-A*. In the beginning $p(b)$ moves along the circle $C(s_*, |ps_*|)$ starting at $p(b_*)$. Let us first assume that the center of rotation s_* will not change. If $s_*, c(b_*)$ and $p(b_*)$ are colinear, then

$$MEC(S_1(b_*), S_2) \cap C(s_*, |ps_*|) = p(b_*).$$

Suppose p_2 to be above q_2 and take the two halflines $p_2, c(b_*)$ and $q_2, c(b_*)$. Let $u(p_2)$ and $u(q_2)$ be the intersection of $C(s_*, |ps_*|)$ with each halfline. Take now the two circles centered at p_2 and q_2 with radius $2r(b_*)$ and let $v(p_2)$ and $v(q_2)$ be the intersection of $C(s_*, |ps_*|)$ with each circle.

We call *left-limit* (resp. *right-limit*) the intersection point of the halfline $q_2 s_*$ (resp. $p_2 s_*$) with $C(s_*, |ps_*|)$. For points in this circle, the following holds.

- Points between the left-limit and $u(p_2)$ satisfy condition (I).
- Points between $u(p_2)$ and $v(p_2)$ satisfy conditions (I) and (II).
- Points between $v(p_2)$ and $v(q_2)$ satisfy condition (II).
- Points between $v(q_2)$ and $u(p_2)$ satisfy conditions (I) and (II).
- Points between $u(p_2)$ and the right-limit satisfy condition (I).

Note that the left and right limits follow from the fact that the bisector is also a separating line and the lines $q_2 s_*$ and $p_2 s_*$ show extreme values for the two inner tangents to S_1 and S_2.

Suppose now that the center of rotation s_* changes to s_1 (which must be below it since we are turning clockwise). We call b_1 the bisector through points s_* and s_1. The two circles $C(s_*, |ps_*|)$ and $C(s_1, |ps_1|)$ intersect twice, once at p, and once at $p(b_1)$. Since points in $C(s_*, |ps_*|)$ satisfy conditions

(I) or (II) and after $p(b_1)$ the second circle is fully outside the first, points in this circle also satisfy one of the two conditions and the theorem holds.

For *type-B* and *type-C* we can make similar constructions and argue analogously. □

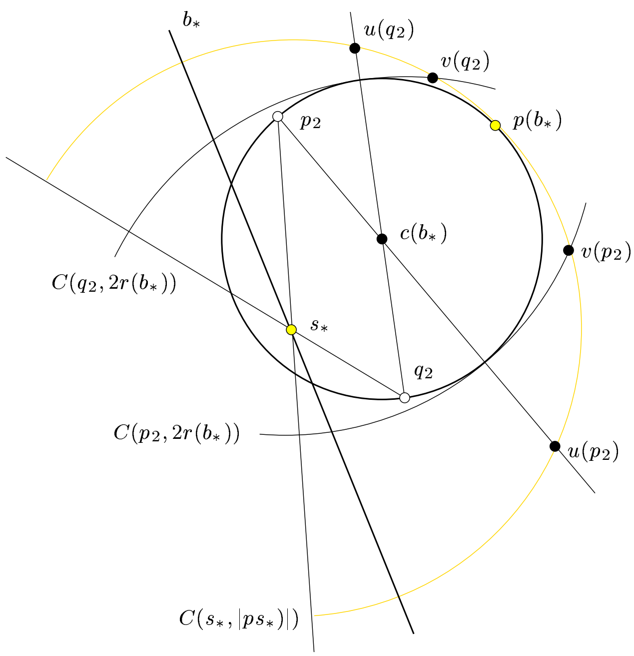

Figure 7: All reflected points $p(b)$ for $p \in S_1$ satisfy conditions (I) or (II).

§ 3.3. Algorithm.— The best algorithm known so far has time complexity $O(n^3)$. Since this is an ongoing work, we expect to lower this bound in the close future. Therefore, we give here only a sketch of the algorithm:

Algorithm 1: Candidates search

1: Construct the farthest Voronoi Diagrams of S_1 and of S_2
2: **for all** pair of points $p_2, q_2 \in S_2$ such that p_2 and q_2 share an edge on the farthest Voronoi Diagram of S_2 **do**
3:     **for all** pair of consecutive points s_1, s_2 of $CH(S_1)$ between the inner tangents of $CH(S_1)$ and $CH(S_2)$ **do**
4:         Compute the minimum enclosing circle of S_2 and the reflected image of S_1 by the line through s_1 and s_2
5:         The center of the computed circle is a candidate to be a type-B solution
6:     **end for**

for all point s of $CH(S_1)$ between the inner tangents of $CH(S_1)$ and $CH(S_2)$ **do**
  for all point $p \in S_1$, compute the line b through s such that the center c of the circle through p_2, q_2 and $p(b)$, the point $p(b)$ and s lie on a line **do**
    If $c(b)$ is on the farthest Voronoi Region of p for the set S_1, then c is a candidate to be a type-A solution
    For each pair of points $p, q \in S_1$ which are neighbours in the farthest Voronoi Diagram of S_1, compute the line supporting $CH(S_1)$ such that p, q, p_2, and q_2 are cocircular
    If the center of the circle through the four points lies on the farthest Voronoi edges of both pairs, then c is a candidate to be a type-C solution.
  end for
end for
end for

Lemma 9 *The time complexity of* ALGORITHM 1 *is* $O(n^3)$.

Note that the whole process will be repeated for every pair of neighbours on the farthest Voronoi Diagram of S_2, which is $O(n)$. Since there are $O(n)$ points between the two inner tangents of the two sets and a linear number of candidates of S_1 to build the solution, altogether $O(n^3)$ steps are required to find all posible candidates.

References

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