On the minimum size of visibility graphs *

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Abstract

In this paper we give tight lower bounds on the size of the visibility graph, the contracted visibility graph, and the bar-visibility graph of n disjoint line segments in the plane, according to their vertex-connectivity.

Keywords: Computational geometry, combinatorial geometry, visibility graph, lower bound.

1 Introduction and preliminary definitions

The *visibility graph* $VG(S)$ of a set S of n disjoint line segments in the plane has a vertex for every endpoint of a segment in S, two of them being adjacent when the corresponding points see each other, i.e. the segment they define does not cross any segment in S. Visibility graphs have attracted a lot of attention and many properties of these graphs have been established (see [6] for a summary of results). In particular, Shen and Edelsbrunner [8] proved that the minimum size of the graph is $5n-4$ and gave an example showing that the bound is tight.

The visibility graph $VG(P)$ of a simple polygon P with n sides, has a vertex for every vertex of P, and an edge for every pair of vertices u, v which see each other internally: the diagonal (or boundary edge) uv does not cross the boundary of P. Trivially the minimum size of $VG(P)$ is $2n-3$, because the boundary gives n edges and any triangulation of P gives $n-3$ internal edges. This lower bound is tight, because some polygons have a unique triangulation. A sharper result was given by Hurtado and Noy in [4]: if the number of concave vertices is k, the minimum size of $VG(P)$ is

$$k + s\binom{\lceil \frac{n+k}{k+1} \rceil}{2} + (k+1-s)\binom{\lfloor \frac{n+k}{k+1} \rfloor}{2},$$

where s is the residue of the division of $n+k$ by $k+1$, and this bound is tight.

For a simple polygon P it is possible to consider simultaneously the internal and external visibility: two vertices u and v see each other when the segment they define does not cross the boundary of P, it does not matter whether uv is internal or external. In this case,

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Bagga, Gewali and Ntafos [2], and Urrutia [9], proved independently that the number of visibility edges is at least $n + \lfloor \frac{3n-1}{2} \rfloor - 4$.

In [1], Bagga, Dey, Gewali, Emert and McGrew introduced the *contracted visibility graph* $CVG(S)$ of a set S of n disjoint line segments. $CVG(S)$ has a vertex for every segment in S, two of them, s_1 and s_2, being adjacent when *some* endpoint of s_1 sees *some* endpoint of s_2. They proved that the minimum size of $CVG(S)$ is $n-1$, which is tight. In the example showing the tightness of the bound, the set S of segments has chords (a chord is a segment with both endpoints in the convex hull $CH(S)$ but not being an edge of $CH(S)$), and the authors state the conjecture that the minimum size might be $2n-3$ when S contains no chords.

As the example in [8] giving the tightness of the lower bound for the minimum size of $VG(S)$ also had chords (in fact $n-2$ chords), it is natural to ask for a similar conjecture about the minimum size of $VG(S)$ when S contains no chords.

Let S be a set of n disjoint horizontal line segments in the plane without two endpoints having equal abscissa. The *bar-visibility graph* $BVG(S)$ has a vertex for every segment of S, and two segments s and t are adjacent if there is a vertical segment joining s and t and touching no other segment of S. These graphs have been widely studied [6] and they were characterized in [5].

In this paper we prove that, when S has no chords, the minimum size of $VG(S)$ is $6n-6$ and the minimum size of $CVG(S)$ is $2n-3$, hence extending the result in [8] and settling the conjecture in [1]; moreover, both bounds are tight. As a corollary, it is easy to see that in the presence of c chords, the tight lower bounds for $VG(S)$ and $CVG(S)$ are $6n-c-6$ and $2n-c-3$, respectively. The bound $6n-6$ can be deduced from a result in [7] based on previous work on pseudo-triangulations; our proof follows an elementary approach as in [8].

The absence of chords is equivalent to $VG(S)$ and $CVG(S)$ being a 3–connected graph and a 2–connected graph respectively. Additional results in this paper regard the case in which stronger assumptions about the vertex–connectivity of the graphs are made. In particular, for bar-visibility graphs we give tight lower bounds on the size for all possible values of the connectivity.

2 Segment endpoint visibility graphs

Let S be a set of n disjoint line segments in the plane without chords. The main result in this section is that the size of $VG(S)$ is at least $6n-6$. This bound is achieved, for example, by the set of segments in Figure 1. Before proving the bound, we need some lemmas.

It is a well known fact that a point external to the convex hull of a set of disjoint segments can see completely at least one of the segments. We have no clear reference for this result, hence we include here a proof for completeness (in fact we prove a slightly stronger result).

Lemma 2.1 *If S is a set of disjoint segments and p is a point such that there is some ray r with origin at p not crossing any segment in S, then p sees completely some segment in S.*

Proof. Take the point p as origin and the ray r as positive x-axis. Sort the endpoints of the segments lexicographically, first by polar angle, second by distance to p. This induces a total order on S by sorting the segments by their first endpoints (the ray r prevents cyclicity).

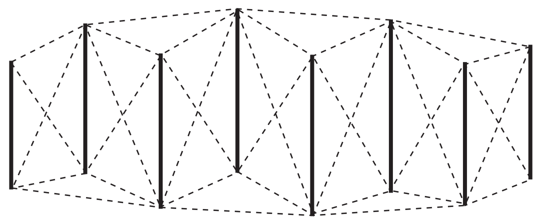

Figure 1: A set S of n segments with $\text{size}(VG(S)) = 6n - 6$.

Let us consider the subset S' of segments in S such that their first endpoint sees p. The first endpoint encountered by r in a counterclockwise rotation around p gives a segment in S', hence S' is non empty. Let s be the last segment in S', by this choice p must see s completely. $\square$

Any triangulation T of S gives a number of edges in $VG(S)$. We prove now that there are always some visibility edges missed by T. The proofs follows the scheme of a similar result in [3].

Lemma 2.2 *Let T be a triangulation of a set S of n segments ($n \geq 3$). Then $VG(S)$ contains at least $n-3$ edges that do not belong to T.*

Proof. Extend the segments, one by one, until the extension hits another segment or a previous extension, otherwise going to infinity. This gives a decomposition of the plane into $n+1$ convex pieces $C_1, \ldots, C_{n+1}$. Let m_i be the number of segment endpoints in C_i for $i = 1 \ldots, n+1$; then $\sum m_i \geq 4n$ because every endpoint belongs to the two faces in each side of the segment.

Take any triangulation T_i of the m_i endpoints in C_i. The triangulation T_i will use m_i edges for the boundary and $m_i - 3$ diagonals internal to $CH(C_i)$, thus $\binom{m_i}{2} - m_i - (m_i - 3) = \binom{m_i}{2} - 2m_i + 3 = \binom{m_i - 2}{2}$ internal visibility edges will be unused. Complete $\bigcup T_i$ to a triangulation $\widehat{T}$ of S. The number of visibility edges not used in $\widehat{T}$ is at least

$$\sum_{i=1}^{n+1} \binom{m_i - 2}{2} \geq (n-3)\binom{2}{2} + 4\binom{1}{2} = n - 3,$$

where the inequality comes from the fact that the sum $\sum \binom{m_i - 2}{2}$ is minimized when the m_i are as even as possible, namely $n-3$ of them equal to 4, the other four of them equal to 3.

Hence the statement has been proved for $\widehat{T}$. But as all triangulations of S use the same number of edges, the result follows. $\square$

Now we are ready for proving the main result of this section:

Theorem 2.1 *If S is a set of $n \geq 2$ segments that does not contain a chord, then $VG(S)$ has at least $6n - 6$ edges, which is tight.*

Proof. The case $n = 2$ is trivial, let us assume that $n \geq 3$. An example achieving the bound appears in Figure 1. Now, let T be any triangulation of S and let B be the boundary of the convex hull of S. Then T has $6n - 3 - r$ edges, where r is the number of endpoints in B. By Lemma 2.2, the graph $VG(S)$ contains at least $7n - 6 - r$ edges. If $r \leq n$ we are done. Otherwise $r > n$ and there must exist some $s = xy$ in S having its two endpoints in B. Since there are no chords, s must be an edge of B.

The set $S - s$ can have chords, suppose it has c chords.

Case 1. $c = 0$. In this case $S - s$ contains no chords, and by induction $VG(S - s)$ has at least $6(n-1) - 6 = 6n - 12$ edges. We have to show that bringing s back produces at least 6 new edges.

Let u and v be the neighbors of x and y, respectively, in B. If the quadrilateral $uxyv$ is empty we get the five new edges ux, xy, yv, uy and xv. By Lemma 2.1 x sees completely some segment t in $S - s$, and $t \neq uv$, as otherwise uv would be a chord in S. The endpoint of t which is neither u nor v gives the sixth required edge.

If the quadrilateral $uxyv$ is nonempty, it will contain some endpoint w seen from both x and y. We again have five new edges: ux, xy, yv, xw and yw. As x can see completely some segment t_x in $S - s$, if one of its endpoints is none of u, v, w we are done. Otherwise let us assume that $t_x = uw$. Now y can also see completely some segment t_y in $S - s$; if $t_y = t_x$ then yu is a new visibility edge, otherwise the endpoint of t_y different from u, w, v gives the sixth required new visibility edge.

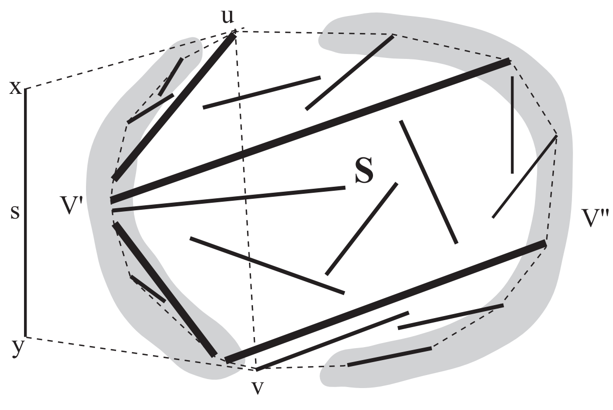

Figure 2: The inductive step for $S - s$. The thick segments are the chords in $S - s$.

Case 2. $c > 0$. The c chords divide the convex hull of $S - s$ into $c + 1$ regions. Let $n_1, n_2, \ldots, n_{c+1}$ be the number of segments in each of these regions, counting for every region the chords in its boundary, so that $\sum n_i = n + c - 1$. By induction, the visibility graph induced in region i has at least $6n_i - 6$ edges, which gives a total of $\sum(6n_i - 6) = 6n - 12$. We have to discount the c chords that have been counted twice. This gives at least

$$6n - 12 - c$$

edges in $VG(S-s)$. We have to show that adding back s we obtain at least $c+6$ new edges.

Let u and v be the neighbors of x and y, respectively, in B. We divide the vertices in the boundary of the convex hull of $S-s$ into four sets: $\{u\}, \{v\}, V', V''$, where V' is the set of vertices between u and v facing s (see Figure 2).

We classify the chords of $S-s$ into three classes. Let c' be the number of those chords with both endpoints in V', let c'' be the number of those with one endpoint in V' and the other one in V'', and finally let $\delta (= 0, 1, 2)$ be the number of chords incident with u or v. Observe that no chord can have both endpoints in V'', and that uv cannot be a chord; hence

$$c = c' + c'' + \delta.$$

Consider the polygon P with boundary $xyvV'ux$. It has at least $2c' + c'' + \delta + 4 = c + c' + 4$ vertices and contains at least $c + c' + 1$ internal diagonals. Together with xy, xu, yv this gives at least $c+4$ new edges. In order to produce the remaining two new edges we treat two subcases separately.

2.1. $c' > 0$ or $\delta > 0$. If $c' > 0$ let ab be a chord with both endpoints in V', appearing in the order $u \ldots a \ldots b \ldots v$, such that there is no other chord between a and b. Then either x or y is to the left of the oriented line ba, assume it is x. There are some segments to the left of ba, and none of them is a chord. By Lemma 2.1 x sees completely one of them and this gives the two new edges required.

If $c' = 0$ and $\delta > 0$ we apply the same reasoning to the corresponding chord ua or vb.

2.2. $c' = \delta = 0$. We now show that x sees some endpoint different from u, v and not being the endpoint of a chord of $S-s$ in V'. This follows from the fact that x must see the two endpoints of a segment in $S-s$ by Lemma 2.1. The same argument applies to y and this finishes the proof. □

Corollary 2.1 *If S is a set of $n \geq 2$ segments that contains exactly c chords, then $VG(S)$ has at least $6n - c - 6$ edges.*

Proof. The chords decompose S into $c+1$ subsets with $n_1, \ldots, n_{c+1}$ segments, respectively, with $\sum n_i = n + c$, as the chords are counted twice, one in each side. We apply now Theorem 2.1 to each subset and $VG(S)$ has at least

$$\left[\sum_{1}^{c+1} (6n_i - 6)\right] - c = 6(n+c) - 6(c+1) - c = 6n - c - 6$$

edges. □

The graph $VG(S)$ is always 2-connected, and in this general situation the best lower bound for the minimum size is $5n-4$. The absence of chords is equivalent to $VG(S)$ being a 3–connected graph; this is because every cut set of size 2 must consist of the two endpoints of a chord, as shown in [3]. In this case we have just proved a $6n-6$ lower bound. It is natural to wonder what would be the bound if we assume stronger connectivity. Nevertheless, we show now that there is not any improvement.

Theorem 2.2 *If S is a set of $n \geq 3$ segments such that $VG(S)$ is a 4-connected graph, then the size of $VG(S)$ is least $6n-6$, which is a tight bound.*

Proof. By Theorem 2.1 we only have to prove tightness. For this, let us consider the set S of segments in Figure 3, left, where the degrees of the vertices in $VG(S)$ are displayed. They add up to $12n-12$, thus the size of $VG(S)$ is $6n-6$.

On the other hand, we are going to prove that $VG(S)$ is a 4-connected graph. We can consider a cycle in $VG(S)$ with the external points (dashed in Figure 3, right), another cycle through the internal points (dotted in the figure) and a Hamiltonian cycle doing a zig-zag between the cycles (solid in the figure). Then it is easy to see that in the union of the three cycles, there are 4 internally disjoint paths between any two vertices, which by Menger's theorem gives the claimed 4-connectivity. □

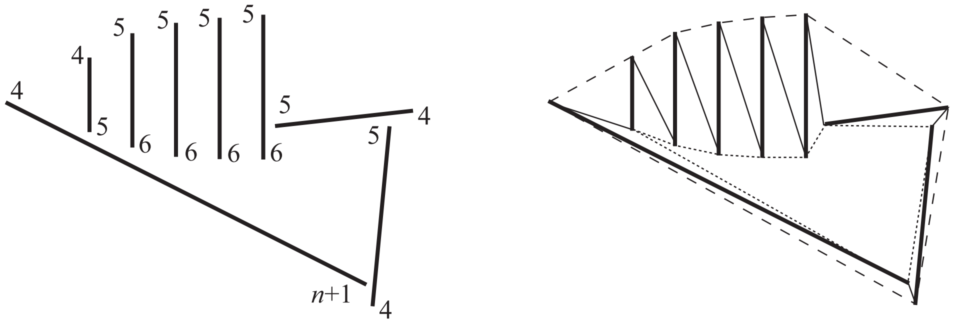

Figure 3: A set S of n segments ($n-3$ of them are vertical) with 4-connected visibility graph and size$(VG(S)) = 6n-6$.

3 Contracted visibility graphs

Let us recall that the *contracted visibility graph* $CVG(S)$ of a set S of n disjoint line segments has a vertex for every segment in S, two of them, s_1 and s_2, being adjacent when *some* endpoint of s_1 sees *some* endpoint of s_2 (Figure 4). $CVG(S)$ is always connected but may be not 2-connected: the presence of a chord gives a cut vertex in $CVG(S)$. In this case the minimum size may be as small as $n-1$. In this section we prove that the minimum size goes up to $2n-3$ when $CVG(S)$ is 2-connected, and that assuming that the graph is 3-connected gives only one additional edge.

Theorem 3.1 *If S is a set of $n \geq 3$ segments that does not contain a chord, then $CVG(S)$ has at least $2n-3$ edges. In addition, if $CVG(S)$ is 3-connected, then $CVG(S)$ has at least $2n-2$ edges. Both bounds are tight.*

Proof. Let T be a triangulation of S and let B the boundary of the convex hull of S. We can see T as a graph $G = (V, E)$ where V is the set of $2n$ endpoints of the n segments and

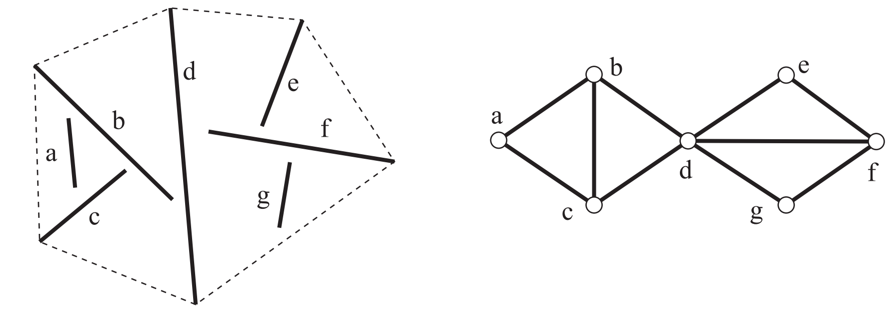

Figure 4: A set of segments and its contracted visibility graph.

E is the set of edges in T. From G, we define the graph G' with n vertices obtained by contracting each segment s_i to a vertex w_i (see Figure 5) . Notice that an edge of G' is also an edge of $CVG(S)$ and that G' is a planar graph in which all faces but the external one are triangles. A vertex lies on the external face when the corresponding segment has at least one endpoint in B. The graph G' has $3n-3-r$ edges, where r is the number of edges on the external face. Since there are no chords, we have $r \leq n$, thus $CVG(S)$ has at least $2n-3$ edges.

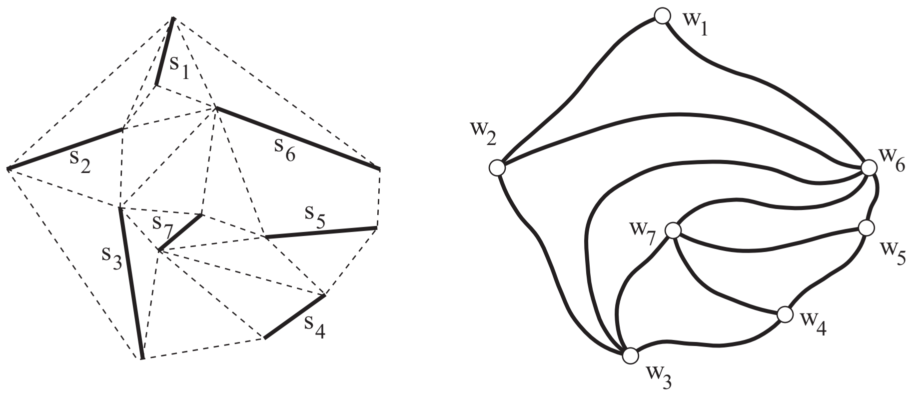

Figure 5: Left: A triangulation T of a set S of n segments. Right: Graph obtained from T by contracting the edges which are segments in S.

In addition, if $CVG(S)$ is 3-connected, we must have $r < n$, because $r = n$ means that all the segments have some endpoint in B, and in this case G' is an outerplanar graph: these graphs can never be 3-connected, as at least two vertices have degree less or equal than 2.

Both bounds are tight, as shown by the segment configurations in Figure 6. $\square$

An easy adaptation of the proof for Corollary 2.1 gives now our last result.

Corollary 3.1 *If S is a set of $n \geq 3$ segments that contains exactly c chords, then $CVG(S)$ has at least $2n-c-3$ edges.*

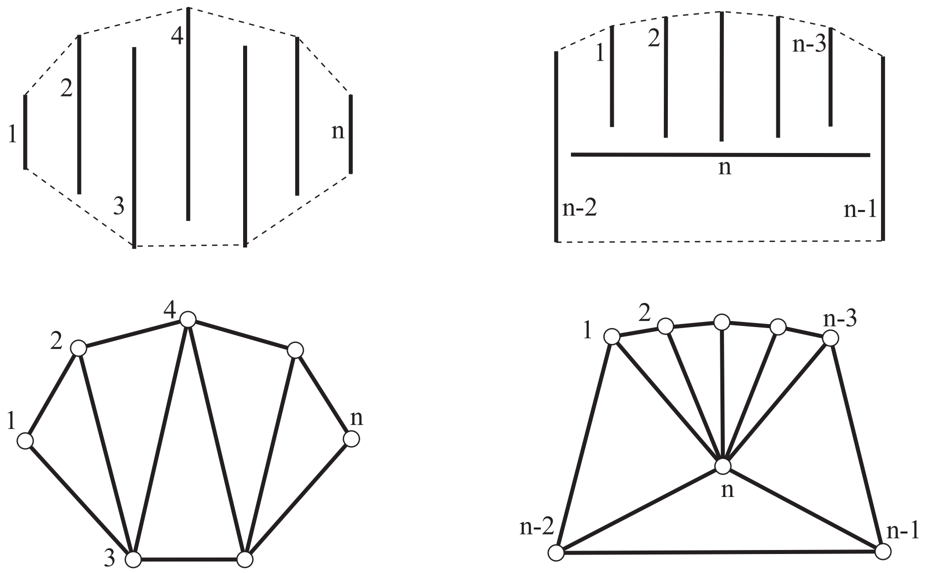

Figure 6: Left: A set S' of n segments with no chord and size$(CVG(S')) = 2n - 3$. Right: A set S'' of n segments for which $CVG(S'')$ is 3-connected and has size $2n - 2$.

4 Bar-visibility graphs

Let S be a set of n disjoint horizontal line segments in the plane without two endpoints having equal abscissa. This assumption is essential for the results in this section to be correct. We give here lower bounds on the size of $BVG(S)$ when the graph is k-connected for $k = 2, 3, 4, 5$. This exhausts all cases as $BVG(S)$ is planar and cannot be 6-connected.

Extend $BVG(S)$ to a multigraph $BVM(S)$ as follows: if the segments s and t see each other, two visibility vertical segments between s and t will define different edges in $BVM(S)$ when they cannot be translated into each other without intersecting another segment from s. In other words, the multiplicity of the edge is equal to the number of visibility zones between s and t. Luccio, Mazzone and Wong [5] gave the following characterization:

Lemma 4.1 *A plane multigraph M is the extended multigraph $BVM(S)$ of a set of disjoint horizontal line segments in the plane without endpoints with equal abscissa if and only if all the internal faces of M are triangular.*

Now we are ready for proving the next theorem.

Theorem 4.1 *Let S be a set of n disjoint horizontal line segments in the plane without endpoints with equal abscissa. Then*

(a) If $n \geq 3$ and $BVG(S)$ is 2-connected then it has at least $2n - 3$ edges.

(b) If $n \geq 4$ and $BVG(S)$ is 3-connected then it has at least $2n - 2$ edges.

(c) If $n \geq 5$ and $BVG(S)$ is 4-connected then it has at least $\lceil (5n/2 - 3) \rceil$ edges.

(d) If $n \geq 6$ and $BVG(S)$ is 5-connected then it has at least $\lfloor (19n - 17)/7 \rfloor$ edges.

Proof.
(a). Let e be the number of edges in $BVG(S)$, let $e + f$ be the number of edges in $BVM(S)$ and let r be the size of the external face of $BVM(S)$. Since the multigraph is triangulated, by Euler's formula we have $e + f = 3n - 3 - r$. For every repeated edge in $BVM(S)$ we must have an interior vertex in order to prevent a face with only two edges. Therefore the number of interior vertices $n - r$ is at least f and

$$e = 3n - 3 - r - f \geq 2n - 3.$$

(b). We know from (a) that $e \geq 2n - 3$. If contrary to the claim $e = 2n - 3$, we must have $r = n - f$. If $f = 0$ then $r = n$ and the graph is outerplanar, a class not containing 3-connected graphs, therefore $f \geq 1$. Let xy be a duplicated edge in $BVM(S)$. Unless the external face is reduced to x and y these two vertices give a 2-cut. If they form the external face then $r = 2$ and $f = n - 2 \geq 2$; this means that there is an internal duplicated edge which also gives a 2-cut, a contradiction.
(c). As in (b) the only possibility for having multiple edges is that $r = 2$, i.e. the external face is a digon, which implies that $BVG(S)$ is also triangulated. Therefore in all cases we have $e = 3n - 3 - r$.

Since all the degrees are at least 4, every external vertex is adjacent to at least two internal vertices. On the other hand, an internal vertex cannot be adjacent to two non-consecutive external vertices, as otherwise there would be a 3-cut. Also, an internal vertex v cannot be adjacent to three or more external vertices, because more than three would give a non-consecutive pair, and exactly three would produce a 3-cut containing v.

Therefore every edge on the external face belongs to a triangle with a different interior vertex, hence $r \leq n/2$ and

$$e = 3n - 3 - r \geq 5n/2 - 3.$$

(d). All the considerations in (c) also apply. We divide the vertices into three sets: E, N and I, corresponding respectively to external vertices, their neighbors not external, and the remaining interior vertices. We have $|E| = r$ and, since the minimum degree is 5, every external vertex is adjacent to at least three internal vertices and $|N| = 2r + j$ for some $j \geq 0$. Let $i = |I|$, so that $n = 3r + j + i$. We assume $i > 3$ becasuse when $i = 3$ it is easy to see that $BVG(S)$ is the 1-skeleton of the icosahedron, and the claimed inequality holds.

The number of triangles in $BVG(S)$ is $2n - r - 2 = 5r + 2j + 2i - 2$. On the other hand, there are $3r + j$ triangles with at least one vertex in the external face. Since there is no 4-cut, there is no triangle with the three vertices in N, and therefore there are $2r + j$ triangles with two vertices in N and one in I. Also there are at least r triangles with one vertex in N and two in I. The subgraph G_I induced by I is connected and all vertices have degree at least 2, otherwise we would have a 4-cut in $BVG(S)$. Therefore, either G_I contains a triangulated cycle of length greater or equal than 4, or it contains at least two triangles connected by a path. In either case, $BVG(S)$ contains at least two triangles with three vertices in I. Summing up

$$5r + 2j + 2i - 2 \geq (3r + j) + (2r + j) + r + 2,$$

giving $i \geq r/2+2$ and $n \geq 7r/2+2$. Finally the number of edges is $3n-3-r \geq (19n-17)/7$.

The two graphs in Fig. 6 show that the bounds in (a) and (b) are tight. For the cases (c) and (d) the tight examples are shown in Fig. 7. All these graphs are BVG because of Lemma 4.1. □

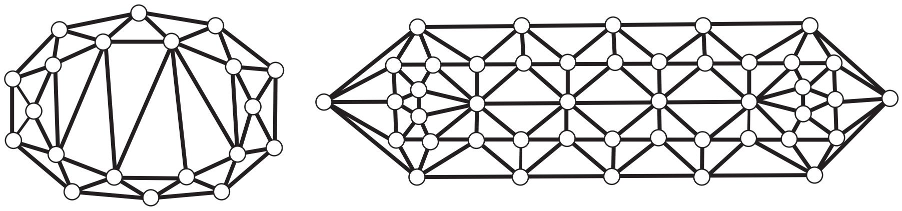

Figure 7: Left: a 4-connected BVG with $5n/2-3$ edges. Right: a 5-connected BVG with $19n/7-3$ edges.

References

[1] J. Bagga, S. Dey, L. Gewali, J. Emert and J. McGrew, "Contracted Visibility Graphs of Line Segments", *Proc. of the 9th Canadian Conf. on Comput. Geom.*, pp. 76-81, 1997.

[2] J. Bagga, L. Gewali, and S. Ntafos, "Visibility edges, mixed edges and unique triangulation", *Proceedings of the 13th European Conf. on Comput. Geom.*, pp. 11-11, 1997.

[3] H. Everett, C. T. Hoàng, K. Kilakos and M. Noy, "Planar Segment Visibility Graphs", *Proceedings of the 9th Canadian Conf. on Comput. Geom.*, pp. 65-69, 1997.

[4] F. Hurtado and M. Noy, "Triangulations, visibility graph and reflex vertices of a simple polygon", *Computational Geometry: Theory and Applications*, Vol. 6, pp. 355-369, 1996.

[5] F. Luccio, S. Mazzone and C. K. Wong, "A note on visibility graphs", *Discrete Mathematics* 64, pp. 209-219, 1987.

[6] J. O'Rourke, "Computational geometry column 18", *International J. Comput. Geom. Appl.*, Vol. 3, pp. 107-113, 1993. Also in *SIGACT NEWS*, Vol. 24, pp. 20-25, 1993.

[7] M. Pocchiola and G. Vegter, "Minimal tangent visibility graphs", *Computational Geometry: Theory and Applications*, Vol. 6, pp. 303-314, 1996.

[8] X. Shen and H. Edelsbrunner, "A tight lower bound on the size of visibility graphs", *Information Processing Letters*, Vol. 26, pp. 61-64, 1987/88.

[9] J. Urrutia, "On the number of internal and external visibility edges of polygons", *Proceedings of the 9th Canadian Conf. on Comput. Geom.*, pp. 17-20, 1997.