Minimal Set of Constraints for 2D Constrained Delaunay Reconstruction

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Abstract

Given a triangulation T of n points in the plane, we are interested in the minimal set of edges in T such that T can be reconstructed from this set (and the vertices of T) using constrained Delaunay triangulation. We show that this minimal set consists of the non locally Delaunay edges, and that its cardinality is less than or equal to $n + i/2$ (if i is the number of interior points in T), which is a tight bound.
Keywords: Triangulation, Delaunay, 2D, Reconstruction, Minimal Constraints Set

1 Introduction

1.1 Motivations

In the very active field of geometric compression, the works that deal with meshes compression give generally a method to encode the whole topology of the geometric object [7, 8, 2]. In some cases the topology can be computed from the geometry, for example some terrain models or some finite elements meshes are obtained by using the Delaunay triangulation; in these cases, alternative methods coding only the geometry can be used [6, 3] saving the cost of coding the topology. Unfortunately, not all triangulations are Delaunay triangulations, and coding a non Delaunay triangulation must include some topology, although in practice many edges look like Delaunay edges. A method consists in coding only few constrained edges and then reconstruct the topology using the constrained Delaunay triangulation. This idea is exploited in particular by Kim et al. to achieve terrain models compression [4].

This leads to the two theoretical problems that are addressed in Sections 2 and 3:

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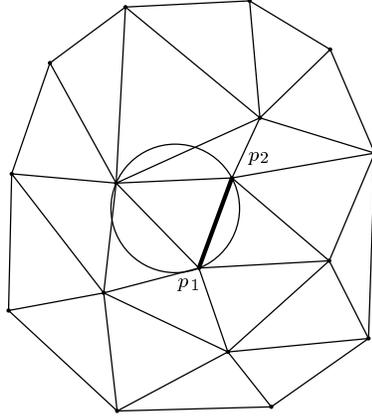[§]Univ. de Alcalá, Aptdo. de Correos 20, 28871 Alcalá de Henares, Madrid, Spain

Figure 1: Delaunay criterion

- given a 2-dimensional triangulation T, compute the minimal set of edges E such that T is the constrained Delaunay triangulation of E,
- find the theoretical worst-case bound for this minimal set.

We also study the practical efficiency of this approach: we give in Section 4 statistics on the number of non Delaunay edges in some geographic databases.

1.2 Basic definitions

Definition 1 (Delaunay criterion) *Let $p_1 p_2$ be an edge in a 2-dimensional triangulation T. We say that $p_1 p_2$ is a Delaunay edge if there exists a circle going through p_1 and p_2 empty of points of T (see Figure 1).*

Definition 2 (Local Delaunay criterion) *Let $p_1 p_2$ be an edge in T, and let $\{p_1 p_2 p_3\}$ and $\{p_1 p_2 p_4\}$ be the triangles adjacents to $p_1 p_2$. We will say that $p_1 p_2$ is a locally Delaunay edge if the circle $(p_1 p_2 p_3)$ does not contain p_4 or equivalently if the circle $(p_1 p_2 p_4)$ does not contain p_3 (see Figure 2).*

Remark 3 *In particular, if the quadrilateral $\{p_1 p_2 p_3 p_4\}$ is non convex with reflex angle in p_1 or p_2, then $p_1 p_2$ is locally Delaunay (see Figure 3). Convex hull edges can also be considered as locally Delaunay.*

Definition 4 (Edge Flip) *Let $p_1 p_2$ be an edge in T, and let p_3 and p_4 be the vertices of its adjacent triangles. We say that $p_1 p_2$ is flipped when it is replaced by $p_3 p_4$ in T. This is possible only if $\{p_1, p_2, p_3, p_4\}$ is a convex quadrilateral (see Figure 4).*
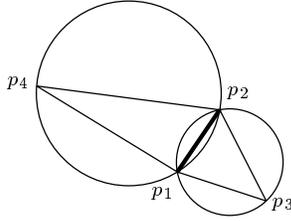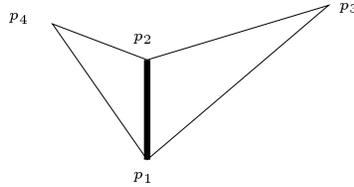
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Figure 2: Local Delaunay criterion

Figure 3: Non convex quadrilateral

Definition 5 (Constrained Delaunay triangulation) *Given a set of points P and a set of edges E in the plane, the constrained Delaunay triangulation $CD(P, E)$ is the unique triangulation such that each of its edges is either in E or locally Delaunay.*

Remark 6 *This definition is equivalent to the classical definition used for example by Chew [1]: $CD(P, E)$ is the unique triangulation containing E and such that for each remaining edge e of $CD(P, E)$, there exists a circle c with the following properties:*

(1) *The endpoints of edge e are on the boundary of c,*

(2) *If any vertex v of E is in the interior of c then it cannot be "seen" from at least one of the endpoints of e.*

This equivalence is shown in particular in an article by Lee and Lin [5].

2 Minimal set of constraints

Theorem 7 *Let T be a 2-dimensional triangulation, P_T the set of its vertices, and NLD_T the set of its edges that are not locally Delaunay. Then NLD_T is the minimal set S such that $CD(P_T, S) = T$.*

Proof: It is easy to see that $CD(P_T, NLD_T) = T$. Indeed, $NLD_T \subset T$ so we can complete NLD_T to obtain T. But doing that, we add $T \setminus NLD_T$, which, by definition, consists of locally Delaunay edges only. Therefore there is no Delaunay flippable edges and the constrained Delaunay triangulation is over. So NLD_T is a sufficient set of constraints.
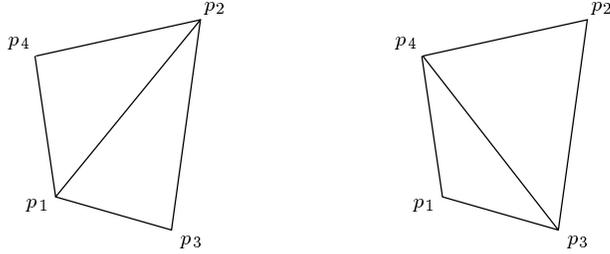
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Figure 4: Edge flip

Reciprocally, let us show that NLD_T is necessary. Let W_T be a subset of the edges in T such that $CD(P_T, W_T) = T$. Let e be an edge in $NLD_T \setminus W_T$, and $\{e_1, e_2, e_3, e_4\}$ the edges and $\{p_1, p_2, p_3, p_4\}$ the vertices of the corresponding quadrilateral (which is convex by remark 3). Since $CD(P_T, W_T) = T$, then $CD(P_T, W_T \cup \{e_1, e_2, e_3, e_4\}) = T$. This implies in particular that $e \in CD(\{p_1, p_2, p_3, p_4\}, \{e_1, e_2, e_3, e_4\})$, which is false since e is non locally Delaunay. Hence such an edge e cannot exist, and NLD_T is minimal. $\qquad\square$

Remark 8 *As a direct consequence, we obtain a linear algorithm to compute the minimal set of constraints of a 2-dimensional triangulation.*

3   Worst case study

In this section, we will show that, given a n points triangulation in the plane, the maximal number of non locally Delaunay edges is half the total number of edges, and that this bound is tight.

Lemma 9 *Given a triangulation T, any open half-plane H whose boundary contains one interior point p of T, contains at least a locally Delaunay edge incident to p.*

Proof: Let us consider $p\,p_1$, the first edge (going counterclockwise from the boundary of H) incident to p in H (it exists because p is an interior point of T). If $p\,p_1$ is the only edge incident to p in H, it is necessarily locally Delaunay (this is clear by remark 3). Else, let $p\,p_2$ be the second edge incident to p in H. Let us assume that none of $p\,p_1$ and $p\,p_2$ is locally Delaunay, and let C be the circle defined by p, p_1 and p_2. The arc A of C bounded by p and p_2 and not containing p_1 is necessarily in H. Since p_2 is non locally Delaunay, it exists a point p_3 lying between the line segment $[p, p_2]$ and the arc A (see Figure 5). By iterating the process, we obtain that $p\,p_i$ non locally Delaunay implies p_{i+1} in H. Now, since p is an interior point of T, there is at least a point p_k in the other half-plane. Therefore, the $(k-1)$th edge — lying in H — is locally Delaunay. $\qquad\square$
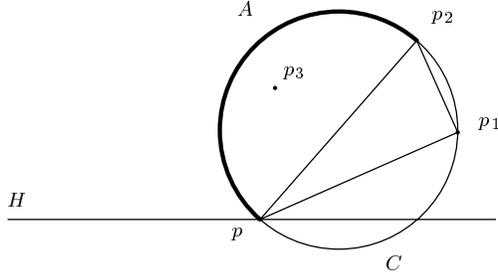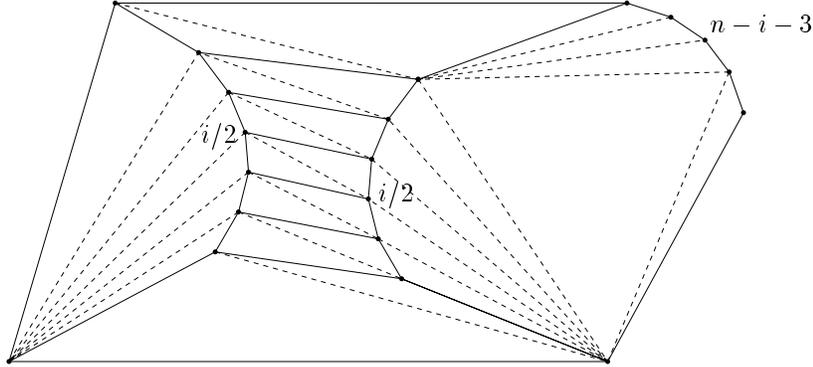
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Figure 5: lemma 9

Figure 6: the $n + \frac{i}{2} - 5$ non locally Delaunay edges are in dashed lines

Theorem 10 *Let P be a set of n points, i of them being interior. Then every triangulation of P contains at least $n+i/2$ locally Delaunay edges. Moreover, this bound is tight, up to a constant.*

Proof: Lemma 9 clearly implies that for any interior point p of T, there are at least 3 edges incident to p in T that are locally Delaunay. When adding the edges on the convex hull, we obtain $\frac{3i}{2} + (n - i) = n + i/2$ locally Delaunay edges in T.

As for tightness, since there are $2\,n - 3 + i$ edges in a triangulation, we just showed that any triangulation contains at most $2\,n - 3 + i - (n + i/2) \approx n + i/2$ non locally Delaunay edges. The Figure 6 shows a triangulation containing $n + i/2 - 5$ non locally Delaunay edges and $n + i/2 + 2$ locally Delaunay edges. $\qquad\square$

4   Experimental results

We propose here some statistical results on the size of the set NLD_T in practice. The first example of the table comes from the viewpoint collection (http://avalon.viewpoint.com/), whereas the other terrain models tested can be found on a web site of the U.S. Environmen-

	number of vertices	number of edges	NLD edges (%)	cocirc. edges (%)	topo. cost (bpv)	geom. cost (bpv)
Crater	5135	14814	0.3	0.6	0.2	19.8
Gatlinburg area	3037	9016	3.4	5.1	1.3	22.8
Great Smoky Mountains	7636	22736	1.0	21.4	1.1	20.4
Mariposa West California	8390	25061	0.5	18.1	0.8	19.7
Sevierville/Pigeon Forge	3516	10447	3.1	9.9	1.4	21.8

Figure 7: statistics on the minimal constraints set of practical models
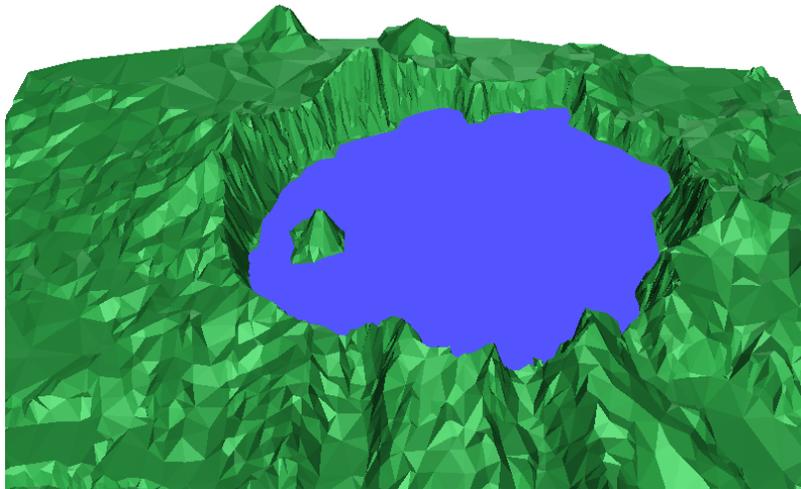

Figure 8: the "crater" terrain model

tal Protection Agency offering several triangulated irregular networks (TIN) in VRML format (http://www.epa.gov/gisvis/vrml/).

Due to the small number of bits used to store the points coordinates, degenerate cases of four cocircular points appears to be frequent. Referring to Definition 2, we call an edge $p_1 p_2$ cocircular if the four points $p_1 p_2 p_3$ and p_4 are cocircular. We count in Figure 7 the percentage of non Delaunay edges, and the percentage of cocircular edges. The last two columns of the table show the compression ratios (in bits per vertex) obtained by the algorithm of Devillers and Gandoin [3]. The topologic ratios include the coding of cocircular edges, achieved by an additional sequence of 1 bit per cocircular edge. The geometric ratios correspond to a quantization of 12 bits per coordinate.

5 Conclusion

In the information needed to represent a triangulation, we distinguish a geometric part (the vertex positions) and a topological part (the edges). We prove in this paper that if only the non locally Delaunay edges of a triangulation are stored, then other edges can be reconstructed by constrained Delaunay triangulation. We also prove that this number cannot exceed half of the total number of edges.

Although this bound is tight on some pathological examples, our experiments on real data sets shows a practical rate of non locally Delaunay edges of less than 3%, which yields to a very effective compression of the topological part of the triangulation.

For three-dimensional meshes, the definition of the constrained Delaunay triangulation (definition 5) generalizes straightforwardly. Unlike the $2D$ case, given a set of constraints, the CDT does not always exist. However, given a triangulation $T(P, E)$ and its non locally Delaunay faces NLD_T, $CD(P, NLD_T)$ exists and the theorem 7 generalizes easily.

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