# Orbits and bichromatic bottleneck matchings of points in convex position

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Joint work with Miloš Stojaković

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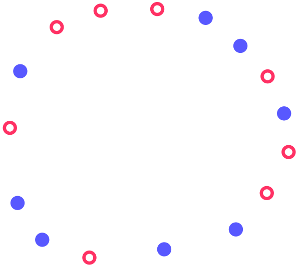

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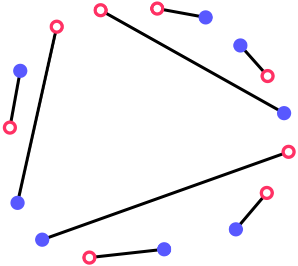

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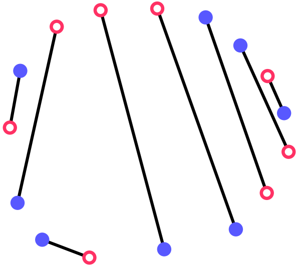

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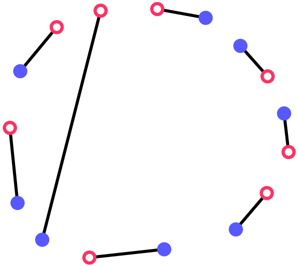

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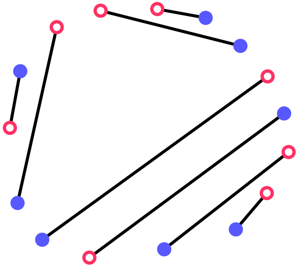

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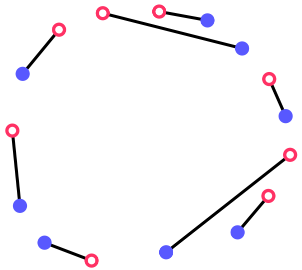

Bottleneck

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- (2014 Abu-Affash, Carmi, Katz, Trablesi)
  - General position is NP-hard, no PTAS
  - Factor $2\sqrt{10}$ approximation algorithm
  - Convex position in $O(n^3)$
- (2015 Abu-Affash, Biniaz, Carmi, Maheshwari, Smid)
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Bichromatic

- (2010 Carlsson, Armbruster)
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- (2018+ Savić, Stojaković)
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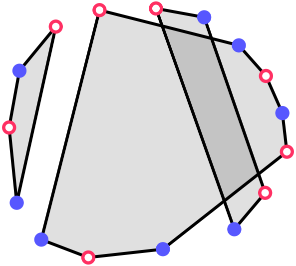

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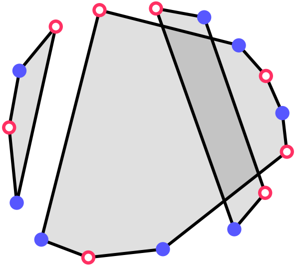
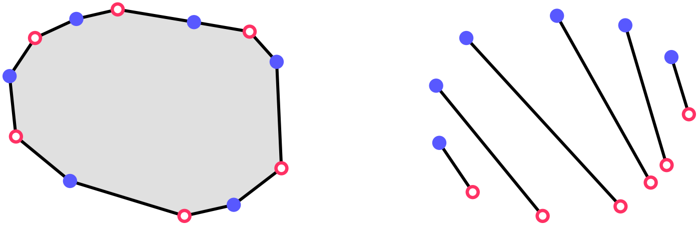


- Orbits can be computed in $O(n)$ time.

# Orbits - examples



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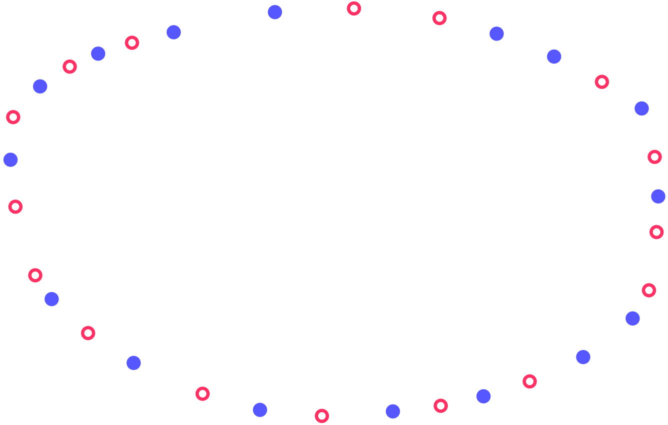

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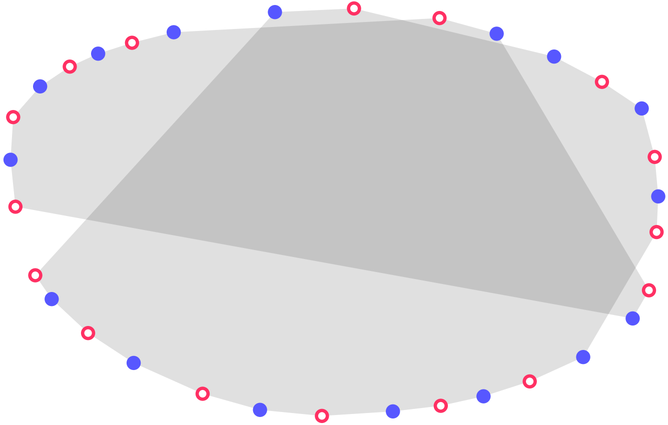

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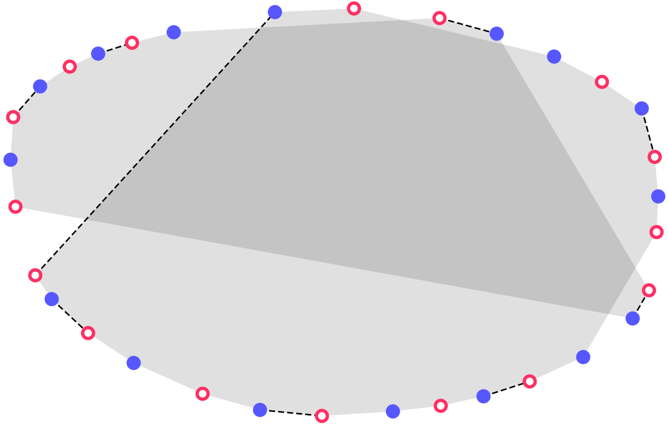

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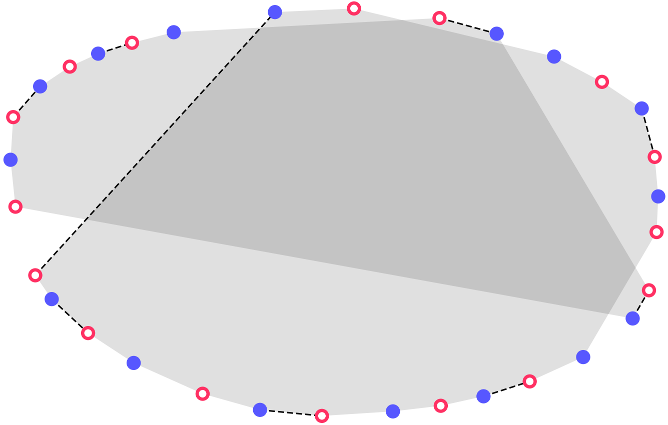

Edges (red-blue, blue-red)

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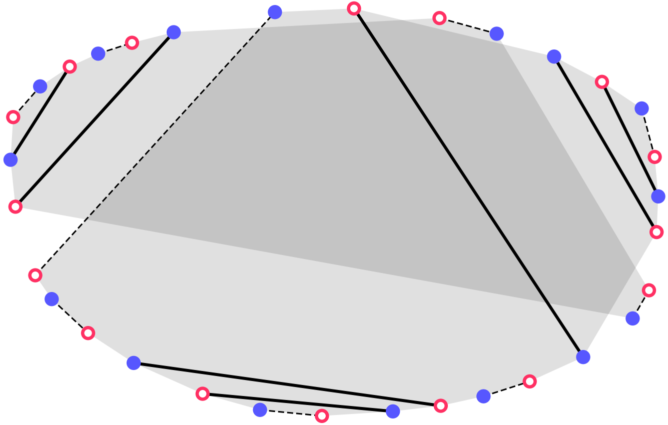

Diagonals

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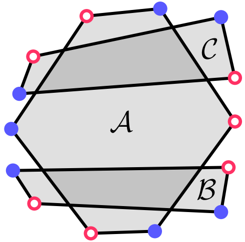

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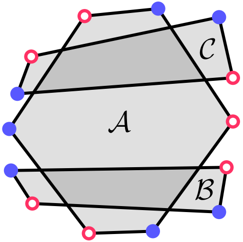


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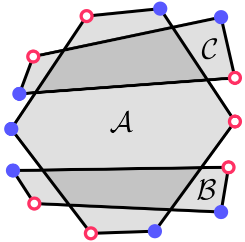


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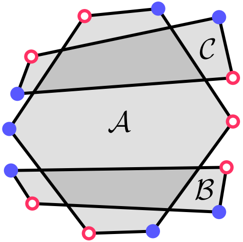


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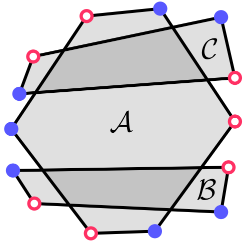


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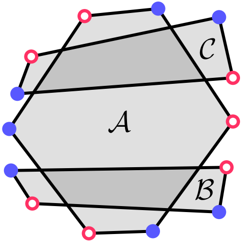


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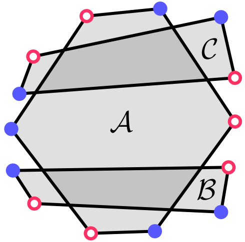


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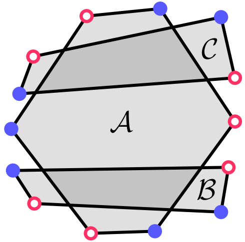


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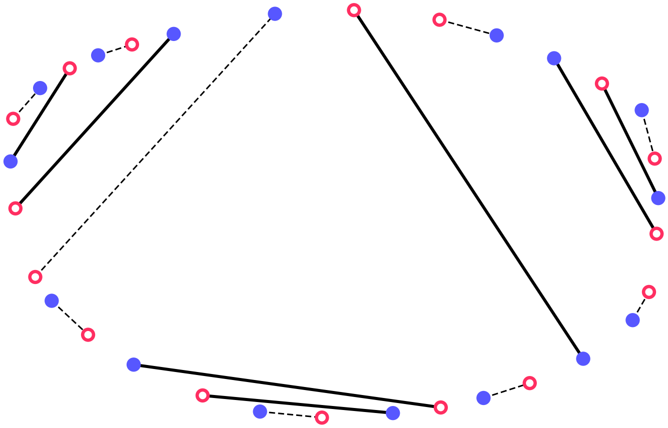

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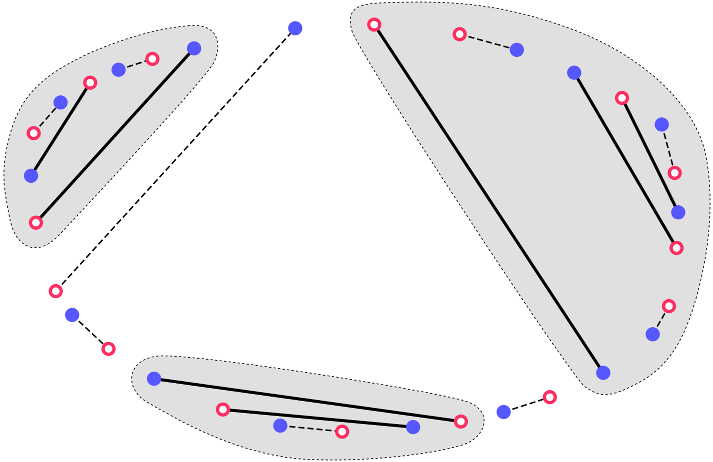

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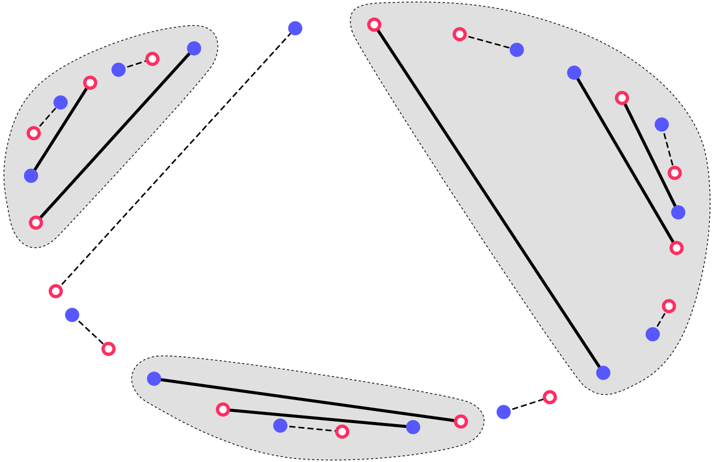

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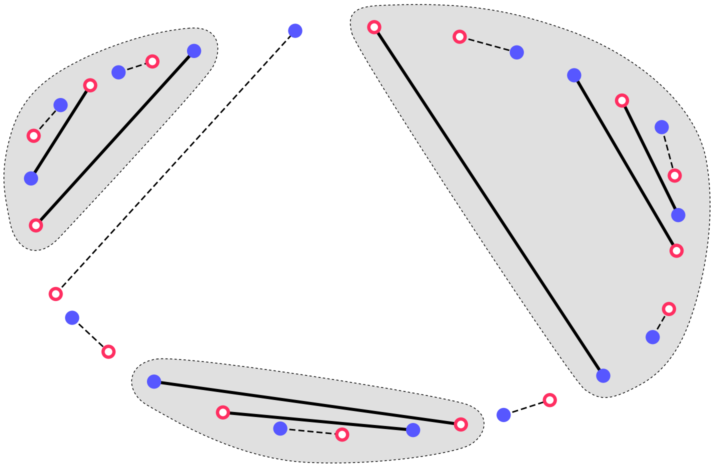

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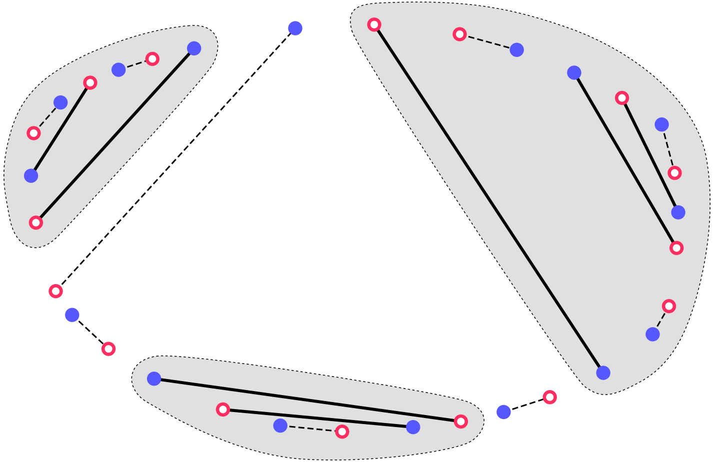

- There is a bottleneck matching with at most 3 cascades.
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  - All subproblems can be solved in $O(n^2)$ total time.

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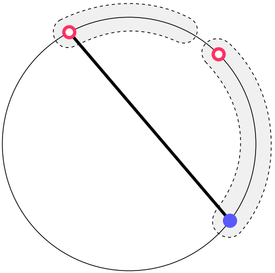

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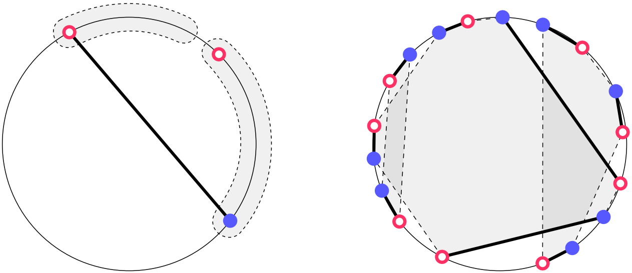

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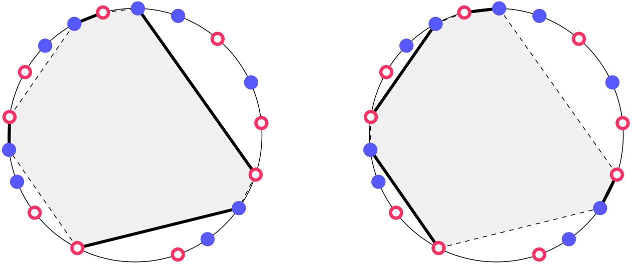

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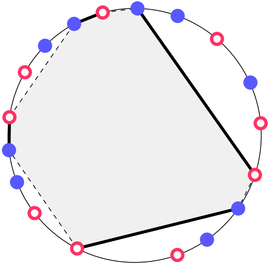

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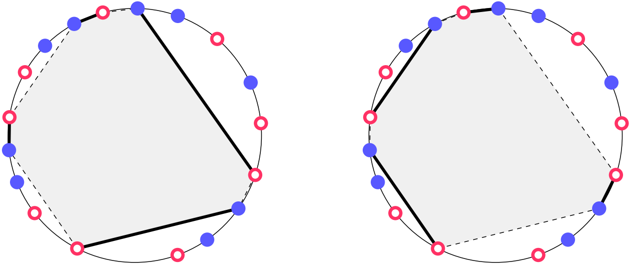

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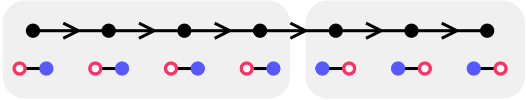

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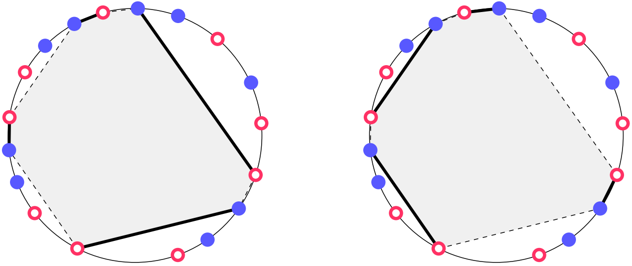

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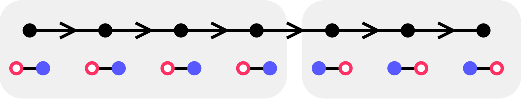

- $O(n)$ time!

Thank you!