Crossing numbers and related topics in combinatorial geometry

Gelasio Salazar *

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1 Overview

The field of crossing numbers deals with the problem of minimizing the number of edge intersections in a drawing of a graph in some surface (usually the plane). In order to investigate this question, we first must agree on exactly what is a drawing of a graph, and how intersections are counted. In most (but not all) models of graph drawing, the vertices are represented by points, the edges are represented by arcs (homeomorphic images of $[0, 1]$), and the endpoints of an arc are the two vertices incident with the corresponding edge. It is implicitly assumed that every two edges intersect each other a finite number of times.

The issue of how intersections are counted is a very subtle one, and in fact it has given rise to some of the most fundamental and interesting open questions in the field. At the simplest and perhaps more natural level, one could propose to count the number of points that are intersections between edges (at a point other than a common vertex), avoiding drawings in which more than two edges intersect at a common point. The minimum number of such intersections over all drawings of a graph G is the most common definition of the *crossing number* $\mathrm{cr}(G)$ of G. The term *crossing* comes up because, as it is not difficult to verify, in a drawing with the minimum total number of intersections every intersection is a crossing, rather than tangential.

There are, however, many other possible ways to define the crossing number of a graph, depending on the context and motivation. There are so many reasonable ways to propose a definition for the crossing number of a graph that this is the subject of a long, thorough monograph by Marcus Schaefer [25]. In this course we will mostly deal with crossing numbers

*Instituto de Física, Universidad Autónoma de San Luis Potosí. San Luis Potosí, Mexico. E-mail: gsalazar@ifisica.uaslp.mx.

under the definition given above, although we will devote some time to the subtle (and difficult) questions that arise when we consider alternative definitions of this parameter.

Talking about parameters, let us bring up an issue that we heard a long time ago in a conversation with Neil Robertson. It seems to be (almost) universally accepted that a mathematical result is more “serious” if it deals with a question of a general, structural character, rather than with a particular instance of a problem. This matter of what is a question of a structural character is of course open to interpretation, and it is not our purpose to initiate a fruitless debate on this issue. However, it is difficult to argue that the statement “the crossing number of this particular graph is 5” is almost surely less interesting and deep that the statement “every crossing-critical graph has bounded path-width”, regardless of the space and time devoted to prove either of these statements.

When we started to work in the field of crossing numbers, most results in the literature were of a parametric character, investigating the crossing number of a graph (or family of graphs), rather than showing, for instance, which structural properties of a graph force a large crossing number. There were of course notable exceptions, but it seems fair to say that those constituted a minority. In the last 20 years or so, the field of crossing numbers has seen a remarkable development, and now we have a rich collection of theorems (and many open questions) of a structural character.

Having expressed this preference for the flavour of structural results, we now look at the other side of the coin. Isn’t it true that most theories are motivated from trying to understand better some specific examples, that bring up issues that eventually mold the theory? Isn’t it true that as part of the natural evolution of a theory one must develop tools that allow us, in particular, to calculate the relevant parameters for the “usual suspects”?

In graph theory the usual suspects, perhaps the most natural graphs one can think of, are the complete graphs K_n and the complete bipartite graphs $K_{m,n}$. How many “natural” graph theory parameters does the reader know for which we do not know their values for K_n or $K_{m,n}$? For classical parameters such as the chromatic number or the independence number, their values for these graphs are totally trivial. We cannot even remotely say the same for the crossing number.

Continuing in this theme, as far as we know the earliest crossing number problem in the literature seems to be the “three castles and three gates” puzzle from Luca Pacioli in the early 1500s. In modern graph theory terminology, this is translated to the question of whether or not $K_{3,3}$ is planar. Although it is easy to get convinced of the nonplanarity of $K_{3,3}$, especially for a profesional mathematician, showing this formally depends on the Jordan curve theorem, a statement that could hardly be regarded as trivial. Having solved this problem (the crossing number of $K_{3,3}$ is 1, because it cannot be drawn with 0 crossings, but it can be drawn with 1 crossing), it seems natural to step up to the question of the crossing number of $K_{m,n}$. To this day, we only know the exact value of $K_{m,n}$ for a handful of values of m and n, and when $\min\{m, n\} \geq 7$, the few results we have make essential use of a computer. The problem of determining $\text{cr}(K_{m,n})$ was posed by Turán in the early 1940’s, and remains open to this day. We have a nice conjecture, usually known as Zarankiewicz’s Crossing Number Conjecture. This conjecture has a colourful story, as Zarankiewicz did not pose it as a conjecture, but published a paper claiming to have proved it. About 15 years later a gap was discovered in his proof, a gap that has remained stubbornly unpluggable. It seems to be a safe bet (or is it?) that Zarankiewicz's conjecture is true. Now that we have so many interesting, deep crossing number results of a structural character, shouldn't we be able to apply some of these results to settle this conjecture?

The story behind the quest for the crossing number of K_n is also a colourful one. The first serious effort to exhibit drawings of K_n with as few crossings as possible seems to have been undertaken by the British artist Anthony Hill in the late 1950's. Hill produced some natural drawings of K_n, and conjectured that $\mathrm{cr}(K_n)$ is attained by these drawings. Hill's conjecture also remains open to this day. As with $K_{m,n}$, one can prove by hand that $\mathrm{cr}(K_n)$ is as conjectured only for very small values of n, namely $n \leq 10$. There are only two more values for which $\mathrm{cr}(K_n)$ is known ($n = 11$ and 12), and the available proofs also make essential use of a computer. Shouldn't we have reached a point, after so many decades of crossing number research by many talented people, in which we can either verify or disprove Hill's conjecture?

In this course we will alternate between the parametric and the structural approaches to crossing numbers, as well as how the former approach has influenced the remarkable advancement of the latter. We will discuss some natural variants of the crossing number, such as the rectilinear crossing number and the 2-page crossing number, which turn this topological problem into a geometric and a combinatorial problem, respectively. We will survey some of the most interesting results of a structural character, and some of the most fruitful techniques (not only in crossing numbers but in general in combinatorial geometry/topology), which involve results from probabilistic and extremal graph theory. We expect to give a panoramic view of this thriving field, and of several closely related issues in combinatorial and geometrical topology, although due to time constraints the results presented will necessarily reflect the taste and research interests of the speaker. Our purpose is not only to bring up to speed the attendants of this course with some recent developments in this field and in related areas, but we also hope to engage the participants with open questions that, as it happens so often in combinatorics and related fields, are natural and easy to state, but that despite our combined efforts remain stubbornly open.

The coming of age of the field of crossing numbers is witnessed by the very recent publication of Marcus Schaefer's book devoted to the subject [26]. We highly recommend this book both for the many interesting results included, and also as a great source of open problems.

2 Topics to be covered

2.1 On the definition(s) of the crossing number

For someone not acquainted with crossing numbers, it might seem at first strange that there is more than one way to define the crossing number of a graph. Of course one can come up easily with many alternative definitions, but it may still be suspected that such definitions would be artificial, and not naturally motivated in some interesting context. As it happens, the total opposite is true.

In what is perhaps the most natural variant of the definition of the crossing number of a graph, one takes the minimum number of crossings in a *rectilinear* (or *geometric*) drawing of a graph, that is, a drawing in which all edges are drawn as straight segments. The minimum number of crossings in a rectilinear drawing of a graph G is the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ of G. It is not obvious that $\overline{\mathrm{cr}}(G)$ may be different than $\mathrm{cr}(G)$ for some graph G, but it is not too difficult to come up with examples that show this. We will review some of the most important results regarding the rectilinear crossing number, and the close notion of the pseudolinear crossing number [3, 7, 8, 14].

Another variant that arises naturally in some contexts is the 2-*page crossing number* of a graph. In a 2-*page drawing*, the vertices are placed on the x-axis, and every edge lies either completely on the upper halfplane, or completely in the lower halfplane. The 2-*page crossing number* 2-Page(G) of a graph G is the minimum number of crossings in a 2-page drawing of a graph. We will cover some basic material on this parameter. [1, 9, 11]

Another alternative definition for the crossing number of a graph comes up when one looks at the crossing number from a slightly more abstract point of view. Consider a crossing-minimal (under the usual definition) drawing $\mathcal{D}$ of a graph G. The crossing number of G is then the number of edge crossings in $\mathcal{D}$, but it is also the number of pairs of edges that cross each other in $\mathcal{D}$. What happens if we adopt this as the definition of the crossing number? The *pair crossing number* $\mathrm{pcr}(\mathcal{D})$ of a drawing is the number of pairs of edges which cross in the drawing. The *pair crossing number* $\mathrm{pcr}(G)$ of a graph G is the smallest $\mathrm{pcr}(\mathcal{D})$ over all drawings $\mathcal{D}$ of G. Is it true that $\mathrm{pcr}(G) = \mathrm{cr}(G)$ for every graph G? This is one of the most important open questions in the field. We will present some selected results on what is known about $\mathrm{pcr}(G)$ and its relationship to $\mathrm{cr}(G)$ [22, 23].

We will also talk a bit about other related variants, such as the odd crossing number, and the independent odd crossing number [28].

2.2 Zarankiewicz's and Hill's conjectures

We will talk briefly about the interesting stories behind Zarankiewicz's and Hill's conjectures. We will survey what cases have been settled, and the best lower bounds known for $\mathrm{cr}(K_n)$ and $\mathrm{cr}(K_{m,n})$.

As we will see, Zarankiewicz's and Hill's conjectures are closely related: if Zarankiewicz's conjecture is asymptotically true, then so is Hill's conjecture. Thus in this sense Zarankiewicz's conjecture may be regarded as "more difficult" to settle than Hill's conjecture. We will also see that a good reason to think that Hill's conjecture is true is that a random spherical geodesic drawing of K_n has (asymptotically) as many crossings as predicted by Hill's conjecture. We will exhibit a somewhat similar result (but even stronger: the word "asymptotically" may be removed) for spherical geodesic drawings of $K_{m,n}$.

For this material we highly recommend the lively survey by Beineke and Wilson [6], and the recent thorough survey by Laci Székely [29].

2.3 The rectilinear and 2-page crossing numbers

Geometric graph theory is a highly developed field and, as expected, there is no lack of interesting results on the rectilinear crossing number of a graph. We will cover a selected collection of results, including the recent breakthrough paper by Fox, Pach, and Suk on approximating the rectilinear crossing number of any graph [13]. We will also talk about the rectilinear crossing number of K_n, for which we have much better estimates than for $\mathrm{cr}(K_n)$ but, paradoxically, as opposed to Hill's conjecture, we do not have any reasonable conjecture for the exact value of $\overline{\mathrm{cr}}(K_n)$ for general n [3]. We will talk about the surprising connection between the $\overline{\mathrm{cr}}(K_n)$ and Sylvester's Four Point Problem from 1864 [27].

The basic technique for bounding the rectilinear crossing number of K_n was independently developed by Lovasz et al. [17] and by Ábrego and Fernández-Merchant [2], and has been the basis of all subsequent important developments on this problem. We will talk about how these techniques get extended to not necessarily rectilinear drawings. In particular, they apply to 2-page drawings (and for *cylindrical* drawings, a notion we will also briefly review), and settle Hill's conjecture for these classes of drawings.

2.4 Techniques from algebra and semidefinite programming

For almost 25 years, the best available general lower bound for $\mathrm{cr}(K_{m,n})$ was the one obtained with a simple counting by using Kleitman's result that $\mathrm{cr}(K_{5,n})$ is as in Zarankiewicz's conjecture [15]. Then in 2004, de Klerk et al. used the technique of rotation systems to set a quadratic optimization problem whose solution gives (asymptotically) a lower bound for $\mathrm{cr}(K_{7,n})$ [10]. These bound improved the best (asymptotic) lower bound for $\mathrm{cr}(K_{m,n})$. Slightly later, de Klerk et al. took this one step further, obtaining a lower bound for $\mathrm{cr}(K_{9,n})$ that further improved the best lower bound for $\mathrm{cr}(K_{m,n})$ [12]. We will explain how these techniques were used and implemented.

2.5 Other graph drawing results and questions

The collection of interesting results in crossing numbers and related areas is evidently too large for a mini-course (and also too large even for a full-year course). We will present a sellected collection of results that not only are interesting on their own, but also make clever use of combinatorial and probabilistic techniques. This part of the course will consist exclusively of a presentation of notions and results, without proofs; the proofs will be given later.

2.6 Techniques from probabilistic graph theory

We will give some selected examples of how probabilistic methods can successfully be applied in geometric and topological graph theory. We expect to cover, for instance, the Crossing

Lemma [4,16], and to give lower bounds related to the Pach-Solymosi-Toth result on unavoidable configurations in drawings of complete graphs [21], and Negami's result on unavoidable configurations in drawings of complete bipartite graphs [19].

2.7 Techniques from extremal graph theory and flag algebras

Ramsey theory is a powerful tool that can be applied to problems of a geometric and topological nature. We will illustrate the use of Ramsey theory in the Pach-Solymosi-Toth's and Negami's results mentioned above, as well as a recent result on unavoidable arrangements of pseudocircles. We will also give a brief introduction to the application of flag algebras [24] to topological and geometric graph theory, including the Norin-Zwols lower bound for $\mathrm{cr}(K_{m,n})$ [20], and a recent result showing that Hill's conjecture is "97.3% true", using Laci Székely's lively expression [5].

2.8 Connections with knot theory

We have recently been involved with some knot theory open questions that, at their heart, are of a strongly combinatorial nature. We will explain a recent result on the number of unknot diagrams of a given shadow [18]. We will talk about the combinatorial nature of the problem of classifying great circle links and, time permitting, we will also bring to the audience's attention an interesting problem on testing whether a given shadow can be given an assignment to produce a diagram of a pre-given knot.

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