Topological Graph Theory

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Abstract

The course will outline fundamental results about graphs embedded in surfaces. It will briefly touch on obstructions (minimal non-embeddable graphs), separators and geometric representations (circle packing). Time permitting, some applications will be outlined concerning homotopy or homology classification of cycles, crossing numbers and Laplacian eigenvalues.

The course will follow the combinatorial approach to graphs embedded in surfaces after the monograph written by Carsten Thomassen and the lecturer [MT01].

1 Planar graphs

Connectivity, drawings, Euler's Formula

- Tutte's definition of connectivity.
- Blocks, ear-decomposition, contractible edges in 3-connected graphs.

  Proposition 1. *If G is a 2-connected graph, then it can be obtained from a cycle of length at least three by successively adding a path having only its ends in common with the current graph.*

  The decomposition of a 2-connected graph in a cycle and a sequence of paths is called an *ear decomposition* of the graph.

  Proposition 1 has an analogue for 3-connected graphs.

  Theorem 2 (Tutte [Tu61, Tu66])**.** *Every 3-connected graph can be obtained from a wheel by a sequence of vertex splittings and edge-additions so that all intermediate graphs are 3-connected.*

  Thomassen [Th80] showed that the following "generalized contraction" operation is sufficient to reduce every 3-connected graph to K_4. If G is a graph and $e \in E(G)$, let $G /\!\!/ e$ be the graph obtained from the edge-contracted graph G/e by replacing all multiple edges by single edges joining the same pairs of vertices.

  Lemma 3. *Every 3-connected graph G of order at least five contains an edge e such that the graph $G /\!\!/ e$ is 3-connected.*

- Polygonal drawings, Jordan curve theorem (polygonal version).

  The Jordan Curve Theorem. *Any simple closed curve C in the plane divides the plane into exactly two arcwise connected components. Both of these regions have C as the boundary.*

  This result is a special case of the Jordan-Schönflies Theorem.

If C and C' are simple closed curves, and Γ and Γ' are 2-connected plane graphs whose exterior faces are bounded by C and C', respectively, then Γ and Γ' are said to be *plane-isomorphic* if there is an isomorphism γ of Γ to Γ' which maps C onto C' such that a cycle in Γ bounds a face of Γ if and only if the image of the cycle is a face boundary in Γ'. The isomorphism γ is said to be a *plane-isomorphism* of Γ and Γ'.

The Jordan-Schönflies Theorem. *If f is a homeomorphism of a simple closed curve C in the plane onto a closed curve C' in the plane, then f can be extended to a homeomorphism of the entire plane.*

- Facial cycles in 2-connected plane graphs, Euler's formula.

Proposition 4. *If Γ is a 2-connected plane graph, then Γ has exactly $|E(\Gamma)| - |V(\Gamma)| + 2$ faces. Each face has a cycle of Γ as its boundary.*

Theorem 5. *If G is a 2-connected plane graph, then each face is bounded by a cycle of G. Conversely, if G is a plane graph in which each face is bounded by a cycle of G, then G is 2-connected.*

- 3-connected graphs, the role of peripheral cycles.

A cycle C in a graph G is *nonseparating* if $G - V(C)$ is connected. Tutte called induced nonseparating cycles *peripheral* and proved that such cycles play an important role in 3-connected graphs.

Theorem 6 (Tutte [Tu63]). *Let G be a 3-connected graph. Then every edge e of G is contained in two induced nonseparating cycles having only e and its ends in common. Moreover, the induced nonseparating cycles of G generate the cycle space $Z(G)$ of G.*

Theorem 7 (Tutte [Tu63]). *Let G be a 3-connected planar graph. A cycle C of G is a facial cycle of some planar representation of G if and only if it is induced and nonseparating. In this case, C is a facial cycle in every planar representation of G.*

Corollary 8 (Whitney [Wh33]). *A 3-connected planar graph G has only one planar representation in the sense that the facial cycles are uniquely determined.*

- Duality, connectivity invariance.

Proposition 9. *A plane graph G is 2-connected if and only if its dual G^* is 2-connected. It is 3-connected if and only if G^* is 3-connected.*

Main theorems

Theorem 10 (Kuratowski [Ku30])**.** *A graph G is nonplanar if and only if it contains a subgraph isomorphic to a subdivision of $K_{3,3}$ or K_5.*

Theorem 11 (Wagner [Wa37])**.** *A graph G is planar if and only if neither $K_{3,3}$ nor K_5 is a minor of G.*

Theorem 12 (Stein [St51], Tutte [Tu60])**.** *Every 3-connected planar graph has a convex embedding in the plane.*

Theorem 13 (Wagner [Wa36], Fáry [Fa48])**.** *Every planar graph has a straight line embedding in the plane.*

Problems

1. Consequences of Euler's formula (vertex of degree at most 5; at least 4 vertices of degree at most 5; if at most 3 triangles, then a vertex of degree at most 3).

2. Discharging technique: Prove that every 3-connected planar graph contains a *light edge* (an edge uv such that $\deg(u) + \deg(v)$ is small). Is 3-connectivity needed? How small is the sum of degrees in such a light edge? What about light path of length 2? Try proving it for 4-connected planar graphs using their Hamilton cycle. How tight is the outcome?

3. Prove that for every 4-connected planar graph G its dual is 3-connected but not 4-connected. What about weaker connectivity notions like *internally* 4*-connected*?

4. Outerplanar graphs. Forbidden minor characterization of outerplanar graphs.

5. Series-parallel graphs, equivalence of definitions.

6. Wye-delta reducibility.

7. Let $P = (X, <)$ be a *partial order*, i.e., $<$ is a transitive, asymmetric and irreflexive binary relation on X. A partial order is a *linear order* if every pair of distinct elements of X is comparable. The *dimension* $\dim P$ of a partial order P as the minimum number of linear orders on X whose intersection is the relation $<$. If G is a graph, let $X_G = V(G) \cup E(G)$. The incidence relation of G defines a partial order $<_G$

on X_G, i.e. $a <_G b$ if and only if $a \in V(G)$, $b \in E(G)$ and a is an endvertex of b. We define the *order-dimension* of G as the dimension of the partial order $(X_G, <_G)$.

Theorem 14 (Schnyder [Sch89]). *A graph is planar if and only if its order-dimension is at most three. Moreover, for each 3-dimensional representation $\{<_1, <_2, <_3\}$ of $<_G$ there is a straight line embedding of G in the plane, $v \mapsto (v_1, v_2)$ (where $v \in V(G)$ and $(v_1, v_2) \in \mathbb{R}^2$), such that for all $u, v \in V(G)$ we have $u <_i v$ if and only if $u_i < v_i$ $(i = 1, 2)$.*

Geometric consequences. Schnyder trees.

2 Genus of a graph

1. Classification of surfaces.
2. 2-cell embeddings, Euler's formula.
3. The genus problem, Heawood formula ([Ri74]), genus of $K_{m,n}$ ([Ri65a, Ri65b]).
4. Surgery, cutting the surface along a cycle. Contractible and noncontractible cycles.
5. Basic homology: Surface-separating cycles. Cycle space. Cycle space dimension. Different bases of the cycle space.

Problems

1. Face-tracing algorithm examples.
2. Examples of graphs embedded in the torus $(K_5, K_{3,3}, K_7, K_{4,4}, K_{3,6})$ and the projective plane $(K_6, K_{3,4})$.
3. Small genus embeddings of $K_n^{(3)}$, $n = 4, 5, 6$.
4. Genus of $K_4(3)$.
5. Genus of $K_3 \square K_3 \square K_3$ is 7. Can you prove it? Maybe find a decent lower and upper bounds. Upper and lower bound for the genus of $K_3 \square K_3 \square K_3 \square K_3$.

6. Prove and generalize the following result.

   Proposition 15. *A 2-connected plane graph is bipartite if and only if every facial cycle is even.*

   What about the torus?

3 Edge-width and face-width

1. Homotopy and homology classification of cycles and walks, 3-path property.
2. Edge-width and face-width.
3. Locally planar graphs: planarizing cycles, some consequences.
4. Embedding flexibility.
5. Separator theorem.

4 Circle packing

1. Circle packing in the plane. Primal-dual version and its consequences ([BS93, Mo97] and [MT01]).
2. Basic hyperbolic geometry and surfaces of constant negative curvature.

Problems

1. Examples on the torus with essential 3-connectivity and their circle packing.
2. Hyperbolic disc and hyperbolic upper half-plane.
3. Benjamini-Schram convergence and limits of graphs on surfaces.

5 Topic of momentary inspiration

The lecturer leaves the subject of the last lecture open to cover some recent results. Possible topics include: Eigenvalues of planar graphs; algorithmic questions; obstructions.

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