# Approximation by polytopes – An application of $\varepsilon$ nets

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# Picking by the uniform measure

$K$: a fixed **centered** convex body in $\mathbb{R}^d$.

### The general question

**Given** $K$, a positive integer $t \geq d+1$, and $\delta, \vartheta \in (0,1)$.
**Goal:** under some assumptions on $d, t, \delta, \vartheta$ (no assumption on $K$), the convex hull $P$ of $t$ randomly, uniformly and independently chosen points of $K$ satisfies

$$\vartheta K \subseteq P,$$

with probability at least $1-\delta$.

*Example 1:* Rough approximation.

### Brazitikos, Chasapis, Hioni 2016

There are universal constants $c, \beta > 0$ and $\alpha > 1$, such that for $t = \alpha d$, we have $\frac{\beta}{d} K \subseteq P$,
with probability $1-\delta = 1-e^{-cd}$.

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*Example 2:* Fine approximation.

## Giannopoulos, Milman, 2000

Given $\delta, \gamma \in (0,1)$. Let $t = e^{\gamma d}$. Then $c(\delta)\gamma K \subseteq P$, with probability $1-\delta$.

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*Example 3:* General result.

## Giannopoulos, Milman, 2000

Given $\delta, \vartheta \in (0,1)$. Let $t = c(\delta)\left(\frac{c}{1-\vartheta}\right)^d$. Then $\vartheta K \subseteq P$, with probability $1-\delta$.

# Main result

**Theorem (NM)**

$\vartheta \in (0,1)$, $C \geq 2$. Set

$$t := \left\lceil C \frac{(d+1)e}{(1-\vartheta)^d} \ln \frac{e}{(1-\vartheta)^d} \right\rceil .$$

Then for any **centered** convex body $K$ in $\mathbb{R}^d$, if $t$ points $X_1, \ldots, X_t$ of $K$ are chosen randomly, independently and uniformly, then

$$\vartheta K \subseteq \operatorname{conv}\{X_1, \ldots, X_t\} \subseteq K$$

with probability at least $1-\delta$, where $\delta := 4 \left[ 11 C^2 \left( \frac{(1-\vartheta)^d}{e} \right)^{C-2} \right]^{d+1}$.

Substitute $\vartheta = \frac{1}{d}$, $C = 6$ to get [BCH].
Substitute $C = 3$ and $\vartheta = c(\delta)\gamma$ to get [GM].

# Tool #1: Grünbaum

$K$: a fixed **centered** convex body in $\mathbb{R}^d$.

Grünbaum's theorem '60: For any half-space $F_0$ that contains the origin we have $\mathrm{vol}\,(K)\,/e \leq \mathrm{vol}\,(K \cap F_0)$.

Stability version: $0 < \vartheta < 1$. Let $F$ be a half-space supporting $\vartheta K$ from outside. Then $\mathrm{vol}\,(K)\,\frac{(1-\vartheta)^d}{e} \leq \mathrm{vol}\,(K \cap F)$.

# Tool #2: Haussler & Welzl

$\mathcal{F}$: a family of subsets of some set $U$.

*Vapnik–Chervonenkis dimension* (*VC-dimension*) of $\mathcal{F}$: the maximal cardinality of a subset $V$ of $U$ such that $V$ is shattered by $\mathcal{F}$, that is, $\{F \cap V \ : \ F \in \mathcal{F}\} = 2^V$.

A *transversal* of $\mathcal{F}$ is a $Q \subseteq U$ that intersects each member of $\mathcal{F}$.

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Radon's lemma: If $\mathcal{F}$ is a family of half-spaces in $\mathbb{R}^d$, then VC-dim$(\mathcal{F}) \leq d + 1$.

$\varepsilon$-net Theorem: $\mu$ a probability measure on a set $U$, $\mu(F) \geq \varepsilon$ for all $F \in \mathcal{F}$, and VC-dim$(\mathcal{F}) \leq D$.
Pick $t := \left\lceil C \frac{D}{\varepsilon} \ln \frac{1}{\varepsilon} \right\rceil$, elements according to $\mu$.
Then, we get a transversal of $\mathcal{F}$ with high probability.

# Proof of the Theorem

## Theorem (MN)

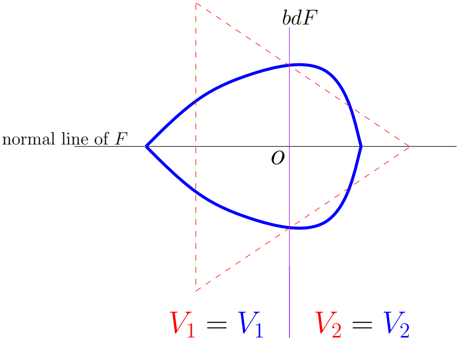
Given $\delta, \vartheta \in (0,1)$. Let $t := \left\lceil C\frac{(d+1)e}{(1-\vartheta)^d} \ln \frac{e}{(1-\vartheta)^d} \right\rceil$, where $C \geq 2$ satisfies some easy-to-satisfy condition.
Then $\vartheta K \subseteq P$, with probability $1-\delta$.

### Combine Tools #1 and #2:

- By Grünbaum, caps are big, ie. $\varepsilon \approx \frac{1}{(1-\vartheta)^d}$.
- By Radon's lemma, caps form a simple family, ie. VC-dim$(\mathcal{F}) \leq d+1$.
- By the $\varepsilon$-net theorem, caps are easy to hit, ie. $t$ is small.

# Proof of Grünbaum's theorem

Apply Steiner's symmetrization about the normal line of $F$ through $o$.
We obtain a convex body $\tilde{K}$ (by Brunn–Minkowski).
Draw a cone $C$.




Compare
the centroids of the two parts $C_1$ and $C_2$ of $C$ to
the centroids of the two parts $\tilde{K}_1$ and $\tilde{K}_2$ of $\tilde{K}$.